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Algebraic Characterization of Petri Nets

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ALGEBRAIC CHARACTERIZATION OF PETRI NETS

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ABSTRACT

Let C^n be the direct product of the bicyclic monoid C, taken n times, where n is a positive integer.

It is shown that (1) every Petri net with n places can be represented by a finite subset of C^n, (2) every finite subset of C^n represents a Petri net with n places, and (3) the firing rule of Petri nets can be defined as a faithful representation of C^n by the inverse hull of the additive semigroup N^n, where N is the set of natural numbers.

A generalization of C, called 'link semigroup', is defined, and the above results on Petri nets are derived as a special case of a general property of the link semigroup.

CR Categories: 5.20, 5.23

Key Words and Phrases: Petri Net, Bicyclic Semigroup, Inverse Hull, Link Semigroup.

0. Introduction

This work establishes the connection between the bicyclic monoid and the Petri net. The connection is tight in the sense that the class of direct products of the bicyclic monoid completely characterize the class of Petri nets and their behaviors.

The bicyclic monoid C is the set of all ordered pairs of natural numbers with the following operation:

$$(a,b)\cdot(c,d) \overline{\overline{\Delta}} (a + c - \min\{c,b\}, b + d - \min\{b,c\})$$

for any natural numbers a,b,c, and d.

To illustrate the connection, consider the Petri net of Figure 1.

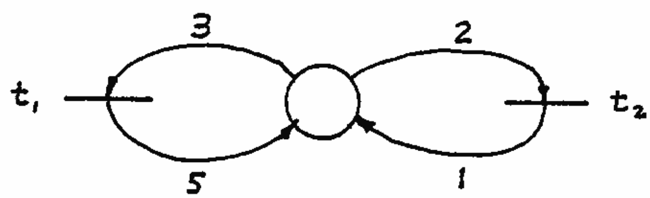

Figure 1. Example of Petri Net

A firing of t_1 consumes 3 and produces 5 tokens, and a firing of t_2 consumes 2 and produces 1 token. Question: How many tokens does a firing of t_1 then t_2 or t_2 then t_1, consume and produce?

Answer: t_1t_2 consumes 3 and produces 4 tokens, and

t_2t_1 consumes 4 and produces 5 tokens, because,

$t_1t_2 = (3,5)\cdot(2,1) = (3+2 - \min\{2,5\}, 5+1 - \min\{5,2\}) = (3,4)$,

$t_2t_1 = (2,1)\cdot(3,5) = (2+3 - \min\{3,1\}, 1+5 - \min\{1,3\}) = (4,5)$

The bicyclic monoid has been known to algebraists for more than twenty-five years [1] [10]. It was imported into computer science by Gorn [2] in his study of prefix languages and the algebra of processor-linkage [3] [4]. Nivat [11] and Kimura [6] introduced a generalization of the bicyclic monoid, called the 'polycyclic monoid' by Nivat and the 'Z-monoid' by Kimura, for investigating the syntactic monoids of context-free languages. See also [12] [14].

Considering (1) the known fact that the bicyclic monoid is homomorphic to the syntactic monoid of the Dyck language of two letters [6] [11], (2) the close relationship between the Dyck language and semaphore operations [7] [5], and (3) the similarity between semaphore array operations and Petri nets [8] [9]; it is not difficult to see the connection between the bicyclic monoid and the Petri net. The question was how close they are.

In the next section we introduce the definitions, notations and some facts on Petri nets, inverse semigroups, and the bicyclic monoid. Nothing is new in this section. Terminologies and notations for algebraic concepts are taken from either Clifford & Preston [1] or Ljapin [10].

In Section 2, we generalize the bicyclic monoid into what we call 'link semigroup' following the original intention of Gorn. A link semigroup is constructed from another monoid (the base monoid) with certain property. The bicyclic monoid C, C^n, and polycyclic monoids are all link semigroups with different monoids as the bases.

In Section 3, we present the main result that any link semigroup is isomorphic to the inverse hull of its base monoid. For C^n, the isomorphism represents the firing rule of the Petri net.

1. Definitions and Notations

Petri Nets

We define a Petri net as an ordered pair of two finite sets (π,Σ) such that $\Sigma \subsetneq N^{\pi} \times N^{\pi}$, where N is the set of natural numbers and N^{π} is the set of total functions from π to N. Let $M \underset{\Delta}{=} N^{\pi}$, then M is called the markings, π is the places and Σ is the transitions. [13]

Let $t \underset{\Delta}{=} (I_t, O_t) \in M \times M$ be a transition in Σ. Then $I_t, O_t : \pi \rightarrow N$ are the numbers of tokens to be consumed (I_t) and to be produced (O_t) by a firing of t. The effects of the firing is representable by the following binary relation on M :

$$\{(m_1,m_2) \in M \times M \mid \exists m \in M;\ m_1 = I_t + m \text{ and } m_2 = O_t + m\}$$

where + is the functional addition; i.e. for any $a,b,c \in M$,

$a + b = c$ iff $a(p) + b(p) = c(p)$ for all $p \in \pi$. In general, the firing of Petri nets is representable by a total function θ from $M \times M$ to the set of binary relations on M such that

$$\theta : M \times M \rightarrow P\,(M \times M)$$

$$\theta\,(a,b) \underset{\Delta}{=} \{(m_1, m_2) \in M \times M \mid \exists m \in M;\ m_1 = a + m \text{ and } m_2 = b + m\}$$

Inverse Semigroups

A semigroup S is an inverse semigroup if for any $a \in S$ there exists a unique $\bar{a} \in S$ such that $a\bar{a}a=a$ and $\bar{a}a\bar{a}=\bar{a}$, where $\bar{a}$ is called the inverse of a. For any set X, the set of all one-to-one partial transformations on X form an inverse semigroup under the relational composition. It is called the symmetric inverse semigroup on X and denoted by I_X. It is known that for any inverse semigroup S there exists a homomorphism from S into the symmetric inverse semigroup I_X for some X. Such a homomorphism is called a representation of S by I_X. A representation is faithful if it is an isomorphism.

Inverse Hull

Let S be an arbitrary cancellative semigroup. With each a ε S, we associate transformations ρa and λa on S, defined by: $\rho a(x) \underset{\Delta}{=} xa$, $\lambda a(x) \underset{\Delta}{=} ax$ for all x ε S. Let ρa^{-1} and λa^{-1} be the inverse transformations of ρa and λa. Then the following hold for all $a,b \varepsilon S$:

(1) $\rho a, \rho a^{-1}, \lambda a, \lambda a^{-1} \varepsilon \mathcal{I} s$

(2) $\rho a\ \rho a^{-1} = \lambda a\ \lambda a^{-1} = \iota$ (the identity of $\mathcal{I}s$)

(3) $\rho ab = \rho a\ \rho b$, $\rho ab^{-1} = \rho b^{-1}\ \rho a^{-1}$, $\lambda ab = \lambda b\ \lambda a$, $\lambda ab^{-1} = \lambda a^{-1}\ \lambda b^{-1}$,

(4) $\phi r : a \mapsto \rho a$, $\overline{\phi}\ell : a \mapsto \lambda a^{-1}$ are isomorphisms, and $\overline{\phi}r : a \mapsto \rho a^{-1}$, $\phi\ell : a \mapsto \lambda a$ are anti-isomorphisms.

The <u>inverse hull</u> $\mathcal{H}s$ of S is the inverse subsemigroup of $\mathcal{I}s$ generated by $\phi r(S)$; i.e. the set of all finite products of elements of $\phi r(S)$ and inverses of elements of $\phi r(S)$. In another notation, $\mathcal{H}s = \langle \phi r(s) \cup \overline{\phi} r(s) \rangle$. Note that $\mathcal{H}s$ is isomorphic to $\langle \phi \ell(s) \cup \overline{\phi} \ell(s) \rangle$.

Direct Product of Bicyclic Monoid

Let S and T be arbitrary semigroups. The <u>direct product</u> of S and T is the set S x T with the operation defined by: $(s,t)(s', t') \underset{\Delta}{=} (ss', tt')$ for all $s,s' \varepsilon s$ and $t,t' \varepsilon T$.

The bicyclic monoid C is defined in the previous section. The associativity of the key operation:

$(a,b)(c,d) \underset{\Delta}{=} (a + c - \min\{c,b\}, b + d - \min\{b,c\})$ is shown in Ljapin [10]. We will prove the associativity later in more general form. C is the inverse semigroup with identity (0,0) and (b,a) being the inverse of (a,b).

It is known that the following semigroups are isomorphic to C:

(1) $\langle p,q \rangle$: the subsemigroup of an arbitrary semigroup S with the identity element e, generated by such elements p and q in S that $pq = e$ and $qp \neq e$.

(2) HN : the inverse hull of the additive semigroup of natural numbers.

(3) $\{a,\bar{a}\}^*/\{a\bar{a} = \lambda\}$: the semigroup generated by $\{a,\bar{a}\}$ subject to the single generating relation $a\bar{a} = \lambda$, where λ is the identity of the free monoid $\{a,\bar{a}\}^*$.

An element $(m,n)\varepsilon C$ is represented by (1) $q^m p^n$, (2) $\{(m + x,\ n + x) \mid x \in N\}$, and (3) $[\bar{a}^m\ a^n]_{a\bar{a} = \lambda}$ in the above three semigroups.

Remark: The element $[\lambda]_{a\bar{a} = \lambda}$ of the semigroup (3) is the Dyck language generated by the grammar:

$\{S \rightarrow \lambda,\ S \rightarrow aS\bar{a},\ S \rightarrow SS\}$.

2. Link Semigroups

We will generalize the bicyclic monoid based on the following observation on its operation.

Let $\dot{-}$ be a binary operation on natural numbers defined by: for any $m, n \in N$,

$$m \dot{-} n \underset{\Delta}{=} m - n \qquad \text{if } m \geq n,$$
$$\underset{\Delta}{=} 0 \qquad \text{otherwise.}$$

This operation, which is called the 'monus' operation by Gorn, is <u>almost</u> the inversion of the addition. With this operation, we can define the bicyclic monoid operation as:

$$(a,b)\cdot(c,d) \underset{\Delta}{=} (a + (c \dot{-} b),\ d + (b \dot{-} c)).$$

(End of observation).

First, we postulate a class of semigroups that carry a binary operation similar to the monus operation on natural numbers. We call them 'quasi invertible semigroups'. Then, we will define the <u>link semigroup</u> $\mathcal{L}s$ of a quasi invertible semigroup S as the set of all ordered pairs of S with the operation defined by: for any a, b, c, d ε S,

$$(a,b)\cdot(c,d) \underset{\Delta}{=} (a*(c \dot{-} b),\ d*(b \dot{-} c))$$

where * is the semigroup operation of S, and $\dot{-}$ is the almost inversion of *.

<u>Definition</u>:

A <u>quasi invertible</u> semigroup $(S, *, e, \dot{-})$ is a cancellative monoid $(S, *, e)$ with a binary operation $\dot{-}$ satisfying the following axioms:

for all a, b, c ε S,

A1. $a \dot{-} a = e$ (the identity of S)

A2. $a \dot{-} e = a,\ e \dot{-} a = e$

A3. $a\ (b \dot{-} a) = b\ (a \dot{-} b)$

A4. $a \dot{-} (bc) = (a \dot{-} b) \dot{-} c$

A5. $(ab) \dot{-} c = (a \dot{-} c)\ (b \dot{-} (c \dot{-} a))$

(End of Definition)

Example 1: $(N, +, 0, \dot{-})$

where $\dot{-}$ is the monus operation.

Verification: A1, A2 are trivial.

A3. (i) $b \geq a$. $a + (b \dot{-} a) = a + (b - a) = b$

$b + (a \dot{-} b) = b + 0 = b$

(ii) $b < a$. $a + (b \dot{-} a) = a + 0 = a$

$b + (a \dot{-} b) = b + (a - b) = a$

A4. (i) $a \geq b + c$ i.e. $a - b \geq c \geq 0$.

$a \dot{-} (b + c) = a - (b + c)$

$(a \dot{-} b) \dot{-} c = (a - b) - c$

(ii) $a < b + c$ and $a \leq b$

$a \dot{-} (b + c) = 0$

$(a \dot{-} b) \dot{-} c = 0 \dot{-} c = 0$

(iii) $a < b + c$ and $a \leq c$ i.e. $a - b \leq c - b \leq c$

$a \dot{-} (b + c) = 0$

$(a \dot{-} b) \dot{-} c = 0 \quad \because \quad a \dot{-} b \leq a - b \leq c.$

A5. (i) $a + b \geq c$ and $a \geq c$

$(a + b) \dot{-} c = (a + b) - c$

$(a \dot{-} c) + (b \dot{-} (c \dot{-} a)) = (a - c) + (b - 0)$

(ii) $a + b \geq c$ and $a < c$

$(a + b) \dot{-} c = (a + b) - c$

$(a \dot{-} c) + (b \dot{-} (c \dot{-} a)) = 0 + (b - (c - a)) \quad \because \quad b \geq c - a.$

(iii) $a + b < c$

$$(a + b) \dot{-} c = 0$$

$$(a \dot{-} c) + (b \dot{-} (c \dot{-} a)) = 0 + 0 \quad \because\ a < c \text{ and } b < c - a.$$

(End of Example 1)

<u>Example 2</u>: $(N^X, +, 0, \dot{-})$

where X is an arbitrary set,

N^X is the set of all functions from X to N,

$+$ is the functional addition, i.e. for any $f, g, h \in N^X$, $f + g = h$ iff $f(x) + g(x) = h(x)$ for all $x \in X$,

0 is the constant function, i.e. $0(x) = 0$ for all $x \in X$,

$\dot{-}$ is the functional monus operation defined by:

for any $f, g, h \in N^X$, $f \dot{-} g = h$ iff $f(x) \dot{-} g(x) = h(x)$ for all $x \in X$.

Verification: similar to Example 1.

(End of Example 2)

Let $(S, *, e, \dot{-})$ be a quasi invertible semigroup. By definition, S is cancellative and S has no zero element. Define an <u>extended quasi invertible</u> semigroup, denoted by S^0, as $(S \cup \{0\}, *, e, \dot{-})$ such that for all $a \in S$, $0 * a \underset{\Delta}{=} a * 0 \underset{\Delta}{=} 0$, $0 \dot{-} a \underset{\Delta}{=} a \dot{-} 0 \underset{\Delta}{=} 0$.

The following example is an extended quasi invertible semigroup.

<u>Example 3</u>: $A^*_0 = (A^* \cup \{0\}, \cdot, \lambda, \dot{-})$

where A^* is the free monoid generated by a finite alphabet A,

$\cdot$ is the concatenation,

λ is the identity of A^*, and

$$\alpha \dot{-} \beta \underset{\Delta}{=} \begin{cases} \gamma & \text{if } \exists \gamma \in A^*;\ \alpha = \beta\gamma, \\ \lambda & \text{if } \exists \gamma \in A^*;\ \beta = \alpha\gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Note: $\dot{-}$ is the deconcatenation of a prefix word.

Verification: Define a partial order $\leq$ on A^* by $\beta \leq \alpha$ iff $\exists \gamma \varepsilon A^*; \alpha = \beta\gamma$.
Then, the argument goes similarly to Example 1.

(End of Example 3)

Lemma1: Let S be a quasi invertible semigroup with identity element e, and let a, b,c, x, y ε S. Then,

(1) $a = b$ iff $(b \dot{-} a) = (a \dot{-} b) = e$

(2) $ab \dot{-} a = b$

(3) $ab = c$ iff $c \dot{-} a = b$ and $a \dot{-} c = e$

(4) $ax = by$ iff for some $z \varepsilon S$, $x = (b \dot{-} a)\ z$ and $Y = (a \dot{-} b)\ z$.

Proof:

(1) Only-if part is trivial from A1. Since S is cancellative, if $(b \dot{-} a) = (a \dot{-} b)$, then from A3, $a = b$.

(2) $ab \dot{-} a = (a \dot{-} a)\ (b \dot{-} (a \dot{-} a))$ by A5,

$= e\ (b \dot{-} e) = b$ by A1 and A2.

(3) Assume that $ab = c$, then

$c \dot{-} a = ab \dot{-} a = b$ by Lemma1(2), and

$a \dot{-} c = a \dot{-} ab = (a \dot{-} a) \dot{-} b = e \dot{-} b = e$ by A4, A1 and A2.

Assume that $c \dot{-} a = b$ and $a \dot{-} c = e$, then

$ab \dot{-} c = (a \dot{-} c)\ (b \dot{-} (c \dot{-} a))$ by A5,

$= e\ (b \dot{-} b)$ from the assumption,

$= e$ by A1.

$c \dot{-} ab = (\ c \dot{-} a) \dot{-} b$ by A4

$= b \dot{-} b$ from the assumption

$= e$ by A1

Therefore, by Lemma1(1), $ab = c$.

(4) Assume that $ax = by$, then by Lemma(2) and A5,

$x = ax \dot{-} a = by \dot{-} a = (b \dot{-} a)\ (y \dot{-} (a \dot{-} b))$,

$y = by \dot{-} b = ax \dot{-} b = (a \dot{-} b)\ (x \dot{-} (b \dot{-} a))$.

Let $u \underset{\Delta}{=} y \dot{-} (a \dot{-} b)$ and $v \underset{\Delta}{=} x \dot{-} (b \dot{-} a)$,

then $x = (b \dot{-} a)\ u$ and $y = (a \dot{-} b)v$.

By Lemma1(3), from $x = (b \dot{-} a)\ u$; $x \dot{-} (b \dot{-} a) = u$, therefore $u = v$

and $\quad x = (b \dot{-} a)\ u$,

$\quad y = (a \dot{-} b)\ u$, for some $u \in S$.

Assume that $x = (b \dot{-} a)\ u$ and $y = (a \dot{-} b)\ u$, then

$ax = a\ (b \dot{-} a)\ u$

$= b\ (a \dot{-} b)\ u \quad$ by A3

$= by$. Q.E.D.

<u>Definition</u>:

Let $(S, *, e, \dot{-})$ be a quasi invertible semigroup. The <u>link semigroup</u> Ls of S is the set of all ordered pairs of S with the operation defined by: for any $a, b, c, d \in S$,

$$(a,b)\cdot(c,d) \underset{\Delta}{=} (a * (c \dot{-} b),\ d * (b \dot{-} c)).$$

<u>Notation</u>: To avoid excess parentheses, we denote an ordered pair (a,b) by $\frac{a}{b}$. For example, the above operation can be defined as: $\frac{a}{b} \cdot \frac{c}{d} \underset{\Delta}{=} \frac{a*(c \dot{-} b)}{d*(b \dot{-} c)}$.

(End of Notation)

<u>Theorem 1</u>: Ls is an inverse semigroup.

<u>Proof</u>: (1) Associativity. We want to show that

$$\left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f} = \frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) \quad \text{for any } \frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in Ls.$$

By definition, $\left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f} = \frac{a\ (c \dot{-} b)\ (e \dot{-} d\ (b \dot{-} c))}{f\ (d(b \dot{-} c) \dot{-} e)}$,

$$\frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right) = \frac{a\ (c\ (e \dot{-} d) \dot{-} b)}{f\ (d \dot{-} e)\ (b \dot{-} c\ (e \dot{-} d))} .$$

By A4, $\quad e \dot{-} d\ (b \dot{-} c) = (e \dot{-} d) \dot{-} (b \dot{-} c)$, and

$\quad b \dot{-} c\ (e \dot{-} d) = (b \dot{-} c) \dot{-} (e \dot{-} d)$.

By A5, $c(e \dot{-} d) \dot{-} b = (c \dot{-} b)\ ((e \dot{-} d) \dot{-} (b \dot{-} c))$

$$= (c \dot{-} b)\ (e \dot{-} d\ (b \dot{-} c)),\text{ and}$$

$$d\ (b \dot{-} c) \dot{-} e = (d \dot{-} e)\ ((b \dot{-} c) \dot{-} (e \dot{-} d))$$

$$= (d \dot{-} e)\ (b \dot{-} c\ (e \dot{-} d).$$

(2) Inverse. We want to show that $\frac{b}{a}$ is the inverse of $\frac{a}{b}$;

$$\frac{a}{b} \cdot \frac{b}{a} \cdot \frac{a}{b} = \frac{a\ (b \dot{-} b)\ (a \dot{-} a\ (b \dot{-} b))}{b\ (\ a\ (b \dot{-} b) \dot{-} a)} = \frac{a}{b}\text{, and similarly,}$$

$$\frac{b}{a} \cdot \frac{a}{b} \cdot \frac{b}{a} = \frac{b}{a}$$

Q.E.D.

Note that $\frac{e}{e}$ is the identity of Ls.

Obviously, $C = L$N, when N is the quasi invertible semigroup of Example 1. The link semigroup $L_{A_0^*}$ where A_0^* is defined in Example 3, is called the polycyclic monoid by Nivat [11], and it represents the activity linkage through a pushdown mechanism with A being the pushdown symbols.

The link semigroup LN^X of N^X in Example 2 represents the marking linkage through firings of transitions of a Petri net with X as the set of places, as we show in the next section, and LN^X is isomorphic to $C^{|X|}$.

Theorem 2:

Let $X = \{x_1, x_2, \ldots, x_n\}$, then LN^X is isomorphic to C^n.

Proof: Denote by $(\frac{ai}{bi})$ an element of C^n, $1 \leq i \leq n$.

Let ϕ: $LN^X \to C^n$ be defined by:

$$\phi\ (\frac{f}{g}) \underset{\Delta}{=} (\frac{f(x_i)}{g(x_i)})\text{ for all }\frac{f}{g} \in LN^X.$$

Then obviously ϕ is one-to-one onto.

Let $\frac{f1}{g1}$, $\frac{f2}{g2} \in LN^X$, then

$$\phi\ (\frac{f1}{g1} \cdot \frac{f2}{g2}) = \phi\ (\frac{f1 + (f2 \dot{-} g1)}{g2 + (g1 \dot{-} f2)})$$

$$= \left(\frac{f1(xi) + (f2\ (xi) \dot{-} g1(xi))}{g2(xi) + (g1\ (xi) \dot{-} f2(xi))}\right)$$

$$= \left(\frac{f1(xi)}{g1(xi)} \cdot \frac{f2(xi)}{g2(xi)}\right)$$

$$= \left(\frac{f1(xi)}{g1(xi)}\right) \left(\frac{f2(xi)}{g2(xi)}\right)$$

$$= \phi \left(\frac{f1}{g1}\right) \phi \left(\frac{f2}{g2}\right)$$

Q.E.D.

3. Main Result

In this section we will show that any link semigroup $\mathcal{L}s$ is isomorphic (onto) to the inverse hull $\mathcal{H}s$.

Let S be an arbitrary quasi invertible semigroup.

Lemma 2: Let a, b ε S and $\phi\ell$ be the regular anti-representation of S. Then, $\phi\ell\ (a)\ \bar{\phi}\ell\ (b) = \bar{\phi}\ell\ (b \dot{-} a)\ \phi\ell\ (a \dot{-} b)$, i.e. $\lambda_a \lambda_b^{-1} = \lambda^{-1}_{b \dot{-} a}\ \lambda_{a \dot{-} b}$.

Proof: By definition,

$$\phi\ell\ (a) = \lambda_a = \{\tfrac{x}{ax} \mid x \in S\}\ ,$$

$$\bar{\phi}\ell\ (b) = \lambda_b^{-1} = \{\tfrac{by}{y} \mid y \in S\},\ \text{and}$$

$$\begin{aligned}\lambda_a \lambda_b^{-1} &= \{\tfrac{x}{y} \mid x,\ y \in S \text{ and } ax = by\} \\ &= \{\tfrac{x}{y} \mid x = (b \dot{-} a)Z \text{ and } y = (a \dot{-} b)Z,\ Z \in S\} \text{ by Lemma 1 (4)} \\ &= \{\tfrac{(b \dot{-} a)Z}{(a \dot{-} b)Z} \mid Z \in S\} \\ &= \{\tfrac{(b \dot{-} a)Z}{Z} \mid Z \in S\}\ \{\tfrac{Z}{(a \dot{-} b)Z} \mid Z \in S\} \\ &= \lambda^{-1}_{b \dot{-} a}\ \lambda_{a \dot{-} b}\end{aligned}$$

Q.E.D.

Theorem 3: $\mathcal{L}s$ is isomorphic onto $\mathcal{H}s$.

Proof: We show that $\mathcal{L}s$ is isomorphic to $\langle\phi\ell(s) \cup \phi\ell(s)\rangle$ which is isomorphic onto $\mathcal{H}s$.

Let σ: $\mathcal{L}s \to \langle\phi\ell\ (s) \cup \bar{\phi}\ell\ (s)\rangle$

$$\sigma\ (\tfrac{a}{b}) \underset{\Delta}{=} \lambda_a^{-1}\ \lambda_b \quad \text{for all a, b } \varepsilon \text{ S.}$$

(1) σ is a homomorphism.

$$\begin{aligned}\sigma\ (\tfrac{a}{b} \cdot \tfrac{c}{d}) &= \sigma\ (\tfrac{a\ (c \dot{-} b)}{d\ (b \dot{-} c)}) = \lambda^{-1}_{a\ (c \dot{-} b)}\ \lambda_{d\ (b \dot{-} c)} \\ &= \lambda_a^{-1}\ \lambda^{-1}_{c \dot{-} b}\ \lambda_{b \dot{-} c}\ \lambda_d \quad \text{Note: } \lambda_x \lambda_y = \lambda_{yx} \\ &\qquad\qquad\qquad\qquad\qquad\quad \lambda_x^{-1} \lambda_y^{-1} = \lambda_{xy}^{-1}\end{aligned}$$

$= \lambda_a^{-1} \lambda_b \lambda_c^{-1} \lambda_d$ by Lemma 2,

$= \sigma (\frac{a}{b}) \sigma (\frac{c}{d})$ by definition.

(2) σ is one-to-one.

Assume that $\sigma (\frac{a}{b}) = \sigma (\frac{c}{d})$ then $\lambda_a^{-1} \lambda_b = \lambda_c^{-1} \lambda_d$.

Since $\lambda_a^{-1} \lambda_b = \{\frac{ax}{bx} \mid x \in S\}$,

$\lambda_c^{-1} \lambda_d = \{\frac{cy}{dy} \mid y \in S\}$,

$a = cy$ and $b = dy$ for some $y \in S$

$c = ax$ and $d = bx$ for some $x \in S$.

By substitution, $a = a\,x\,y$.

Since S is cancellative, $e = x\,y$.

By Lemma 1 (3) and A2, $y = e \dot{-} x = e$, therefore $a = ce = c$ and $b = de = d$, i.e. $\frac{a}{b} = \frac{c}{d}$.

(3) σ is onto.

Let $\alpha_1 \alpha_2 \dots \alpha_n \in \langle \phi\ell (S) \cup \overline{\phi\ell} (S)\rangle$.

where $\alpha_i = \lambda_{ai}$ or λ_{ai}^{-1} for some $ai \in S$, $1 \leq i \leq n$.

Let $\beta o \underset{\Delta}{=} \lambda e$ and $\beta i \underset{\Delta}{=} \alpha_1 \alpha_2 \dots \alpha_i$ $1 \leq i \leq n$.

By induction, we will show that for any $0 \leq i \leq n$,

$\beta i = \lambda_x^{-1} \lambda_y$ for some $x, y \in S$.

(i) $\beta o = \lambda e = \lambda_e^{-1} \lambda_e$, and $e \in S$.

(ii) Assume that $\beta i = \lambda_x^{-1} \lambda_y$ for some $s, y \in S$.

If $\alpha_{i+1} = \lambda_a$ for some $a \in S$, then $\beta_{i+1} = \beta_i \alpha_{i+1} = \lambda_x^{-1} \lambda_y \lambda_a = \lambda_x^{-1} \lambda_{ay}$, and $a\,y \in S$.

If $\alpha_{i+1} = \lambda_a^{-1}$ for some $a \in S$, then $\beta_{i+1} = \beta_i\ \alpha_{i+1}$

$= \lambda_x^{-1} \lambda_y \lambda_a^{-1}$

$= \lambda_x^{-1} \lambda_{a \dot{-} y}^{-1} \lambda_{y \dot{-} a}$ by Lemma 2,

$= \lambda_{x (a \dot{-} y)}^{-1} \lambda_{y \dot{-} a}$, and

$x (a \dot{-} y), (y \dot{-} a) \in S$.

Therefore, for some $x, y \in S$.

$$\alpha_1 \alpha_2 \ldots \alpha_n = \beta_n = \lambda_x^{-1} \lambda_y = \sigma \left(\frac{x}{y}\right).$$

Q.E.D.

Note that σ is a faithful representation of L_S by $H_S \subsetneq I_S$.

Let (π, Σ) be a Petri net, and let $M \overset{=}{\Delta} N^{\pi}$. The firing rule θ is defined by: For any $t \overset{=}{\Delta} (I_t, O_t) \in \Sigma$

$$\theta(t) \overset{=}{\Delta} \{(m_1, m_2) \in M \times M \mid \exists\, m \in M;\ m_1 = I_t + m \text{ and } m_2 = O_t + m\}$$

$$= \left\{\frac{I_t + m}{O_t + m} \mid m \in M\right\}.$$

Since M is a quasi invertible semigroup (Example 2), $M \times M$ is a link semigroup of M and $\Sigma \subseteq LM$. Furthermore, the firing rule θ is a faithful representation of LM by HM.

Note that $\theta(t) = \sigma\left(\frac{I_t}{O_t}\right) \in HM$ and $\frac{I_t}{O_t} \in LM$.

On the other hand, let X be an arbitrary finit set, and let $Y \overset{=}{\Delta} N^X$, where N is the set of natural numbers. Then, any finite subset Z of L_Y forms a Petri net (X,Z) whose firing rule is a faithful representation σ of L_Y by the inverse hull H_Y. Therefore, we can conclude:

Theorem 4:

(π, Σ) is a Petri net iff Σ is a finite subset of the link semigroup of

N^{π}, where the firing rule is a faithful representation of the link semigroup by the inverse hull of N^{π}.

Corollary: A finite set Σ is a Petri net iff Σ is a subset of C^{n} for some $n \geq 1$, where the firing rule is a faithful representation of C^{n} by the inverse hull of $N^{n} \subseteq \mathbb{N}^{n}$.

Proof: In the theorem, without loss of generality, we can let $\pi \triangleq \{0, 1, 2, \ldots, n-1\} = n$. Then by Theorem 2, LN^{n} is isomorphic to C^{n}.

Q.E.D.

Example:

$$\text{Let } \Sigma \triangleq \{t_1 = (\tfrac{2}{1}, \tfrac{0}{2}),\ t_2 = (\tfrac{0}{3}, \tfrac{4}{1}),\ t_3 = (\tfrac{0}{0}, \tfrac{5}{3})\} \subseteq C^2.$$

Then Σ represents a Petri net of Figure 2.

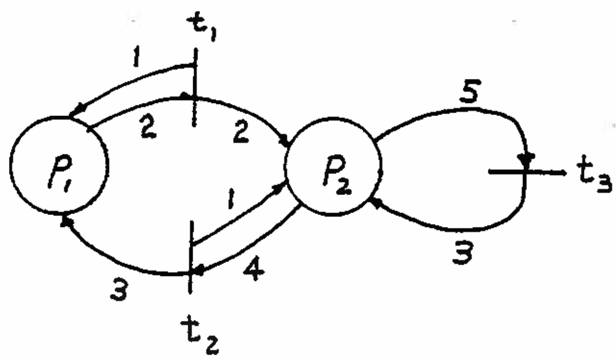

Figure 2: Pet Net of Σ

The firing of $t_1\ t_2\ t_3$ in this order will consume and produce the number of tokens computed as follows:

$$t_1\ t_2\ t_3 = (\tfrac{2}{1}, \tfrac{0}{2})\ (\tfrac{0}{3}, \tfrac{4}{1})\ (\tfrac{0}{0}, \tfrac{5}{3})$$

$$= (\tfrac{2}{1} \cdot \tfrac{0}{3} \cdot \tfrac{0}{0},\ \tfrac{0}{2} \cdot \tfrac{4}{1} \cdot \tfrac{5}{3}) = (\tfrac{2}{4}, \tfrac{6}{3}).$$

Namely, t_1 t_2 t_3 consumes 2 and 6 tokens and produces 4 and 3 tokens from and into P_1 and P_2, respectively. If P_1 and P_2 had less than 2 and 6 tokens initially, the firing sequence t_1 t_2 t_3 is illegal in Σ.

(End of Example)

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References

[1] Clifford, A. H., and Preston, G. B. (1961), The Algebraic Theory of Semigroups, Vol. I, American Mathematical Society, Surveys No. 7, Providence, R.I.

[2] Gorn, S. (1960), Common Programming Language Task, Part I, Rep. AD60UR1, U. S. Army Signal Research and Development Laboratories, Fort Monmouth, N.J.

[3] Gorn, S. (1962), An axiomatic approach to prefix languages, in Symbolic Languages in Data Processing, Gordon and Breach, New York.

[4] Gorn, S. (1968), The identification of the computer and information sciences: Their fundamental semiotic concepts and relationships, Foundation of Languages, 4, D. Reidel, Holland.

[5] Hack, M. (1975), Petri net languages, MIT Project MAC memo 124.

[6] Kimura, T. (1971), Completion problem and its solution for context-free languages (Algebraic approach), Ph.D. Thesis, University of Pensylvania.

[7] Kimura, T. (1976), An algebraic system for process structuring and interprocess communication, 8th. ACM SIGACT Symposium on Theory of Computing.

[8] Kosaraju, S. R., (1973), Limitations of Dijkstra's semaphore primitives and Petri nets, 4th ACM Symposium on OS principles.

[9] Lipton, R. J., Snyder, L., Zalostein, Y., (1974), A comparison study of models of parallel computation, 15th ACM-IEEE Symposium on Switching and Automata Theory.

[10] Ljapin, E. S. (1963), Semigroup, Translations of Mathematical Monographs, vol. 3, Americal Math. Soc., Providence, R. I.

[11] Nivat, M and Perrot, J. (1970), Une generalisation du monoide bicyclique, C. R. Acad. Sc. Paris, t271.

[12] Perrot, J., and Sakarovitch, J., (1977) A theory of syntactic monoids for context-free languages, Information Processing 77, IFIP, North-Holland.

[13] Peterson, J. L., (1977) Petri nets, Computing Survey, vol. 9, No. 3.

[14] Smith, J. M. (1972), Monoid acceptors and their relations to formal languages, Ph.D. Thesis, University of Pennsylvania.