Piezoelectric Control of Structures Prone to Instabilities

Sunjung Kim

Follow this and additional works at: https://openscholarship.wustl.edu/etd

Recommended Citation
Kim, Sunjung, "Piezoelectric Control of Structures Prone to Instabilities" (2010). All Theses and Dissertations (ETDs). 183.
https://openscholarship.wustl.edu/etd/183
WASHINGTON UNIVERSITY IN ST. LOUIS

School of Engineering and Applied Science

Department of Mechanical Engineering and Materials Science

Dissertation Examination Committee:

Srinivasan Sridharan, Chair
Philip Bayly
Kenneth Jerina
David Peters
Ramesh Agarwal
Daren Chen

Piezoelectric Control of Structures Prone to Instabilities

by

Sunjung Kim

A dissertation presented to the School of Arts and Sciences
of Washington University in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
August 2010
Saint Louis, Missouri
Thin-walled structures such as stiffened panels fabricated out of high strength materials are ubiquitous in aerospace structures. These are prone to buckle in a variety of modes with strong possibility of adverse interaction under axial compression and/or bending. Optimally designed stiffened panels, at an appropriate combination of axial compression and suddenly applied lateral pressure undergo large amplitude oscillations and may experience divergence. Under aerodynamic loading, they can experience flutter instability with the amplitudes of oscillations attaining a limit (LCO) or escalating without any limit. Control of structures prone to these forms of instability using piezoelectric actuators is the theme of this dissertation.

Issues involved in the control of stiffened panels under axial compression and liable to buckle simultaneously in local and overall modes are studied. The analytical approach employs finite elements in which are embedded periodic components of local buckling including the second order effects. It is shown that the adverse effects of mode interaction can be counteracted by simply controlling the overall bending of the stiffener by piezoelectric actuators attached its tips. Control is exercised by self-sensing actuators by direct negative feedback voltages proportional to the bending strains of the stiffener. In a
dynamic loading environment, where vibrations are triggered by suddenly applied lateral pressure, negative velocity feedback is employed with voltages proportional to the bending strain-rate. The local plate oscillations are effectively controlled by a piezo-electric actuators placed along the longitudinal center line of the panel.

The problem of flutter under aerodynamic pressure of stiffened panels in the linear and post-critical regimes is studied using modal analysis and finite strips. The analysis, control and interpretation of the response are facilitated by identification of two families of characteristic modes of vibration, viz. local and overall modes and by a classification of the local modes into two distinct categories, viz., symmetric and anti-symmetric modes respectively. The symmetric local modes interact with overall modes from the outset, i.e. in the linear flutter problem whereas both the sets of local modes interact with overall modes in the post-critical range via cubic terms in the elastic potential. However the effects of interaction in the flutter problem are far less dramatic in comparison to the interactive buckling problem unless the overall modes are activated, say by dynamic pressure on the plate. Control of the panel is exercised by piezo-electric patches placed on the plate at regions of maximum curvature as well as on the stiffener.

Two types of control strategies were investigated for the panel subject to fluttering instability. The first is the direct negative velocity feedback control using a single gain factor for each of the sets of plate patches and stiffener patches respectively. A systematic method of determining the gains for the patches has been developed. This is based on the application of LQR algorithm in conjunction with a linearized stiffness matrix of the uncontrolled structure computed at a set of pre-selected times. This type of control was successful till the aerodynamic pressure coefficient reaches up to about six times its critical value, where after it simply failed. The second type of control is the multi-input and multi-output full state feedback control. The LQR algorithm and the linearized stiffness matrix are invoked again, but the gain matrix is computed at the beginning of every time step in the analysis and immediately implemented to control the structure. This type of control proved very effective the only limitation stemming from the maximum field strength that can be sustained by the piezo-electric material employed.
Acknowledgments

Special thanks to Dr. Srinivasan Sridharan for the constant support and guidance that encouraged me to successfully complete this dissertation. I would like to thank to MASE department for giving me financial support and opportunities to have valuable experiences by way of Teaching Assistantship. Also I would like to thank to my dissertation committee members, Dr. Philip Bayly, Dr. Ken Jerina, Dr. David Peters, Dr. Ramesh Agarwal, and Dr. Daren Chen. Their time and insight are greatly appreciated.

Sunjung Kim

Washington University in St. Louis
August 2010
Dedicated to my parents.
# Contents

Abstract .................................................................................................................................................. ii
Acknowledgments....................................................................................................................................... iv
Contents .................................................................................................................................................. vi
List of Tables ............................................................................................................................................ xi
List of Figures .......................................................................................................................................... xiii
1. Introduction ........................................................................................................................................ 1
   1.1. Piezoelectricity: the phenomenon – Basic Equations .............................................................. 1
      1.1.1. Equations of Piezoelectricity ............................................................................................ 2
          1.1.1.1. Conservation of Energy .............................................................................................. 2
          1.1.1.2. Piezo-electric constitutive Equations ............................................................................ 4
          1.1.1.3. Variational Principle ..................................................................................................... 6
          1.1.1.4. Effect of Crystal Symmetry .......................................................................................... 7
          1.1.1.5. Constitutive Relations for Lamina in Plane Stress ...................................................... 8
          1.1.1.6. Electric Enthalpy and Its First Variation for the Lamina ............................................ 10
   1.2. Applications: Control, Energy Harvesting, Health Monitoring ............................................ 11
      1.2.1. Control ................................................................................................................................ 11
      1.2.2. Energy Harvesting .............................................................................................................. 12
      1.2.3. Health Monitoring .............................................................................................................. 12
   1.3. Piezoelectric Materials ............................................................................................................... 13
   1.4. Literature Review ....................................................................................................................... 16
      1.4.1. Applications to Dynamics of Beams and Plates ................................................................. 16
      1.4.2. Nonlinearities and Instabilities ............................................................................................ 18
      1.4.3. Optimal Control ................................................................................................................... 21
   1.5. Objectives .................................................................................................................................... 23
2. Methodology ...................................................................................................................................... 26
   2.1. Analysis and Modeling Issues ................................................................................................... 27
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1.1.</td>
<td>Finite Element Formulation</td>
<td>28</td>
</tr>
<tr>
<td>2.1.2.</td>
<td>Sensor and Actuator Equations</td>
<td>34</td>
</tr>
<tr>
<td>2.1.2.1.</td>
<td>Sensor Response</td>
<td>35</td>
</tr>
<tr>
<td>2.1.2.2.</td>
<td>Feedback Control by Actuator</td>
<td>37</td>
</tr>
<tr>
<td>3.</td>
<td>Column under Conservative Loads</td>
<td>41</td>
</tr>
<tr>
<td>3.1.</td>
<td>Description of problem</td>
<td>42</td>
</tr>
<tr>
<td>3.2.</td>
<td>Equations of Motion</td>
<td>43</td>
</tr>
<tr>
<td>3.3.</td>
<td>Scope of Control Strategies</td>
<td>51</td>
</tr>
<tr>
<td>3.4.</td>
<td>Examples and Discussion</td>
<td>54</td>
</tr>
<tr>
<td>3.4.1.</td>
<td>Two Modes, Continuous patch problem</td>
<td>55</td>
</tr>
<tr>
<td>3.4.2.</td>
<td>Enhancement of $P_s$</td>
<td>55</td>
</tr>
<tr>
<td>3.4.3.</td>
<td>Enhancement of $P_d$</td>
<td>56</td>
</tr>
<tr>
<td>3.4.4.</td>
<td>Comparison with 10 Mode Solutions (Continuous Patch)</td>
<td>61</td>
</tr>
<tr>
<td>3.4.5.</td>
<td>Control with Discrete Patches</td>
<td>62</td>
</tr>
<tr>
<td>3.4.6.</td>
<td>Failure of Patch Control</td>
<td>66</td>
</tr>
<tr>
<td>3.5.</td>
<td>Conclusions</td>
<td>69</td>
</tr>
<tr>
<td>4.</td>
<td>Column under Non-conservative Loads</td>
<td>71</td>
</tr>
<tr>
<td>4.1.</td>
<td>Description of Problem</td>
<td>72</td>
</tr>
<tr>
<td>4.2.</td>
<td>Solution Methodology</td>
<td>73</td>
</tr>
<tr>
<td>4.3.</td>
<td>Validation by reference</td>
<td>78</td>
</tr>
<tr>
<td>4.4.</td>
<td>Results and discussion</td>
<td>81</td>
</tr>
<tr>
<td>4.4.1.</td>
<td>Numerical Study of Controlled Column Response</td>
<td>81</td>
</tr>
<tr>
<td>4.4.1.1.</td>
<td>Geometry and Material Properties</td>
<td>81</td>
</tr>
<tr>
<td>4.4.1.2.</td>
<td>Perturbation to trigger dynamics</td>
<td>82</td>
</tr>
<tr>
<td>4.4.1.3.</td>
<td>Solution procedure</td>
<td>83</td>
</tr>
<tr>
<td>4.4.2.</td>
<td>Behavior and Control of Column with Linear Spring</td>
<td>83</td>
</tr>
<tr>
<td>4.4.2.1.</td>
<td>Small Damping Effect</td>
<td>83</td>
</tr>
<tr>
<td>4.4.2.2</td>
<td>Critical Load Enhancement</td>
<td>84</td>
</tr>
<tr>
<td>4.4.2.3.</td>
<td>System Performance</td>
<td>86</td>
</tr>
<tr>
<td>4.4.2.4.</td>
<td>Significant Role of the Spring</td>
<td>89</td>
</tr>
<tr>
<td>4.4.2.5.</td>
<td>Effect of Loading Sequence</td>
<td>90</td>
</tr>
</tbody>
</table>
4.4.2.6. Nonlinear Case ................................................................. 92
4.4.2.7. Patch Control ................................................................. 96
4.5. Conclusion ............................................................................ 105
5. Piezoelectric Control of Stiffened Panel: Finite Element Approach ........ 107
  5.1. Theory .............................................................................. 109
    5.1.1. Displacement, Strain, and Stress Vectors ......................... 109
    5.1.2. Solution of the Local Buckling Problem ......................... 111
      5.1.2.1. Linear Stability Analysis ............................................ 111
      5.1.2.2. Second Order Field .................................................... 115
      5.1.2.3. Modification of Local Buckling Deformation under Interaction ..... 117
    5.1.3. Finite Element Formulation ............................................ 118
      5.1.3.1. B- Matrix and Current Stress ..................................... 119
      5.1.3.2. Internal Virtual Work Contribution from a Typical Element ...... 121
      5.1.3.3. External Virtual Work .................................................. 131
    5.1.4. Boundary Conditions of the Panel .................................... 133
    5.1.5. Analysis of Stiffened Panel ............................................ 134
  5.2. Case Study .......................................................................... 135
    5.2.1. Geometry, Material and Buckling Data .................................. 135
    5.2.2. Selection of piezo-electric material properties ...................... 136
  5.3. Results and Discussion ....................................................... 137
    5.3.1. Static Problem ............................................................... 137
    5.3.2. Dynamic Problem ........................................................... 140
    5.3.3. Scaled up Panel ............................................................... 146
      5.3.3.1. Static Case ............................................................... 147
      5.3.3.2. Dynamic Case ........................................................... 147
  5.4. Conclusions ........................................................................ 150
6. Flutter of Axially Compressed Stiffened Panels with Edges Free to Move In-plane ................................................................. 152
  6.1. Nonlinear Modal Analysis of Axially Compressed Stiffened Plates .... 155
    6.1.1. General Considerations .................................................... 155
    6.1.2. Eigen-value Problem for Modes of Vibration ...................... 158
6.1.2.1. Shape Functions ................................................................. 162
6.1.2.2. Linear Flutter Problem ...................................................... 163
6.1.2.3. Post-critical Analysis ......................................................... 167
6.1.2.4. Evaluation of Second Order Field Functions ...................... 169
6.1.2.5. Higher Order Strain Energy Contributions ....................... 175
6.1.4.6. Equations of Motion .......................................................... 179
6.2. Investigation of Model Accuracy .................................................. 182
6.2.1. Example of a Plate Flutter ..................................................... 182
6.2.1.1. Linear Flutter ................................................................. 182
6.2.1.2. Limit Cycle Oscillations ................................................... 183
6.2.2. Post-buckling Response of Plate ............................................. 185
6.2.3. Interactive Buckling of Tvergaard Panel ............................... 187
6.2.4. Flutter Response of Tvergaard Panels: The Linear Flutter Problem ... 188
6.2.5. Post-critical Response: Anti-symmetric Case ......................... 190
6.2.5.1. Source of Instability: Edge Movements .............................. 195
6.2.5.2. Effect of Axial Compression .............................................. 196
6.2.5.3. Effect of Initial Imperfections in The Presence of Axial Compression .. .............................................................. 197
6.2.5.4. Effect of Suddenly Applied Pressure .................................. 203
6.2.5.5. Role of Interaction Between Local and Overall Action ........... 203
6.2.6. Post-critical Response: Symmetric case ................................. 204
6.2.6.1. Linear Flutter Problem ..................................................... 204
6.2.6.2. Post-critical Response of Panel ......................................... 205
6.2.6.3. Post-critical Response of Plate .......................................... 209
6.2.6.4. Role of Local and Overall Interaction ................................... 212
6.3. Conclusions ............................................................................. 212
7. Control Methodologies for Stiffened Panels .................................... 214
7.1. Introduction .............................................................................. 214
7.2. Theory .................................................................................... 215
7.2.1. Equation of Motion ............................................................. 215
7.2.2. Piezo-electric contribution .................................................. 217
List of Tables

Table 1.1 Material properties of piezoelectric materials \[^{29}\] ......................................................... 15

Table 3.1. Maximum load and corresponding maximum voltage (volts) .............. 56

Table 3.2. Load capacity for gain and corresponding maximum voltages .......... 57

Table 3.3. Maximum and final voltages for various load cases ......................... 57

Table 3.4. Variation of voltages and the settling times with axial load .......... 59

Table 3.5. Example I: 3 patch case ................................................................. 62

Table 3.6. Results of Examples ................................................................. 64

Table 4.1. Relationship of non-dimensional linear spring coefficient and its critical limit ................................................................. 80

Table 4.2. Dimensions and material properties of cantilever beam .................. 82

Table 4.3. Gain vs. maximum load and voltage .............................................. 86

Table 4.4. Maximum Loads ............................................................................ 94

Table 4.5. Description of Patches for Case (i) ................................................. 102

Table 4.6. Description of Patches for Case (ii) ................................................. 102

Table 5.1. Geometry of the stiffened panel .................................................. 135
Table 5.2. Material properties of the panel (host structure) ........................................ 135

Table 6.1. Comparison with benchmark results .......................................................... 183

Table 6.2. Critical Non-dimensional Aerodynamic Pressure, $\lambda_{cr}$ values ................. 189

Table 6.3. Critical Non-dimensional Aerodynamic Pressure, $\lambda_{cr}$ .......................... 205

Table 7.1. Case 1 ........................................................................................................... 237

Table 7.2. Relationships between $R_d^{lo}$, $\lambda$, Voltages and Gains.......................... 241

Table 7.3. Relationships between $R_d^{lo}$, $\lambda$, Voltages and Gains.......................... 242

Table 7.4. Case (i): Relationships between $R_d^{lo}$, $\lambda$, Voltages and Gains.............. 245

Table 7.5. Case (i): Relationships between $R_d^{lo}$, $\lambda$, Voltages and Gains (Stiffener-plate patch active) $R_d^{ov} = 1000$ ................................................................. 246
List of Figures

Figure 1.1. PZT Materials \[20\] ........................................................................................................... 14

Figure 1.2. MFC (NASA Langley, Smart Material Inc.) ........................................................................ 14

Figure 1.3. MFC types (Smart Material, Inc.) .................................................................................... 16

Figure 2.1. Closed feedback loop used for the control of the column ........................................ 38

Figure 3.1. The simply supported column resting on an elastic foundation .......................... 42

Figure 3.2. Host column with continuous piezo-electric patches at the top and bottom surfaces ........................................................................................................ 46

Figure 3.3. A typical arrangement of the host column with discrete piezo-electric actuators ........................................................................................................ 47

Figure 3.4. Cross-section of the beam illustrating the senses of the electric field at the piezo-electric patches at top and bottom inducing respectively tension and compression ................................................................. 47

Figure 3.5. Time history of \( P = 0.5 \, P_{cr} \) with \( G_d = 2500 \) ........................................... 58

Figure 3.6. Voltage distributions along the length ................................................................. 60

Figure 3.7. Displacement (non-dimensional) time history for \( \bar{v}_1 \) and \( \bar{v}_2 \) with \( N=10 \) .......................... 61

Figure 3.8. Voltage time history for 3 patch case \( (P = 0.5 \, P_{cr}, \text{case (i)}) \)........................... 65
Figure 3.9. Voltage time history for 4 patch case ($P = 0.5 \ P_{cr}$, case (i))................. 66

Figure 3.10. Displacement (nondimensional) time history of Case(i): $P = 0.5 \ P_{cr}$.......... 68

Figure 4.1. The Cantilever column with a spring under follower force together with a cross-section of the column ............................................................... 72

Figure 4.2. Non-dimensional flutter and buckling capacity vs. spring coefficient ......... 77

Figure 4.3. Non-dimensional frequencies vs. spring coefficient................................. 79

Figure 4.4. Variation of maximum $P/P_E$ with $\beta_1$ ................................................. 85

Figure 4.5. Critical load enhancement with Gain for a cantilevered column with $\beta_1 = 0$. ........................................................................................................... 85

Figure 4.6. Time histories of $\ddot{v}, \dot{v}, v$ at $x = L$ and voltage at $x = 0$ with $G = 2.5E4, \beta_1 = 0.0$, LP-1, $P = 2/3 \ P_{cr}$ ................................................................. 89

Figure 4.7. Time histories of $\ddot{v}, \dot{v}, v$ at $x = L$ and voltage at $x = 0$ with $G = 1.E4, \beta_1 = 0.0$, LP-1, $P = 2/3 \ P_{cr}$ ................................................................. 91

Figure 4.8. Time histories of $\ddot{v}, \dot{v}, v$ at $x = L$ and voltage at $x = 0$ with $G = 1.E4, \beta_1 = 10.0$, LP-2, $P = 2/3 \ P_{cr}$ ................................................................. 91

Figure 4.9. Time histories of $\ddot{v}, \dot{v}, v$ at $x = L$ and voltage at $x = 0$ with $G = 1.E4, \beta_1 = 10.0$, LP-2, $P = 2/3 \ P_{cr}$ ................................................................. 92

Figure 4.10. Effect of Nonlinearity and Gain on the load carrying capacity ............. 93

Figure 4.11. Time histories of $\ddot{v}, \dot{v}, v$ at $x = L$ and voltage at $x = 0$ with $G = 1.E4, \beta_1 = 10.0, \beta_3 = -2000$, LP-1, $P = 2/3 \ P_{cr}$ ........................................................................ 95

Figure 4.12. Voltage distribution along the length (@ $t = 0.011$ sec) ......................... 96
Figure 4.13. Time histories of $\bar{v}, \dot{v}, \ddot{v}$ at $x = L$ and voltage at $x = 0$ with $G = 1.0 \times 10^3$, $\beta_i = 10.0$, LP-1, $P = 2/3 P_{cr}$ ................................................................. 97

Figure 4.14. Voltage distribution along the length of the piezo patch @ 0.08 sec (half piezo patch with $G = 1.0 \times 10^3, \beta_i = 10.0, P = 2/3 P_{cr}$) .................................................. 98

Figure 4.15. Critical limits using half patch (0 to $L/2$) ................................................... 98

Figure 4.16. One third piezo patch: Time histories of $\bar{v}, \dot{v}, \ddot{v}$ at $x = L$ and voltage at $x = 0$ with $G = 2.0 \times 10^3, \beta_i = 10.0$, LP-1, $P = 2/3 P_{cr}$ .................................................. 99

Figure 4.17. Voltage distribution along the length of one third patch @ 0.09 sec .......... 99

Figure 4.18. Case (i): Time histories of $\bar{v}, \dot{v}, \ddot{v}$ at $x = L$ with $G = 2.5 \times 10^4, \beta_i = 10.0$, LP-1, $P = 2/3 P_{cr}$ .................................................................................. 101

Figure 4.19. Case (i): Voltage time histories of each patch (Voltages correspond to middle of each patch) ................................................................. 101

Figure 4.20. Case (ii): Time histories of $\bar{v}, \dot{v}, \ddot{v}$ at $x = L$ with $G = 2.5 \times 10^4, \beta_i = 10.0$, LP-1, $P = 2/3 P_{cr}$ .................................................................................. 103

Figure 4.21. Voltage time histories of each patch for Case (ii) ................................. 103

Figure 4.22. Two patch case: Time histories of $\bar{v}, \dot{v}, \ddot{v}$ at $x = L$ with $G = 2.5 \times 10^4, \beta_i = 10.0$, LP-1, $P = 2/3 P_{cr}$ .................................................................................. 104

Figure 4.23. Voltage time histories for each patch Case (ii) without patch 3 .......... 104

Figure 5.1. Geometry of the cross-section of a stiffened panel .................................... 113

Figure 5.2. Cross-sectional view of local buckling of stiffened panel ........................ 114
Figure 5.3. “Wide” stiffened plate and a typical panel ............................................... 114

Figure 5.4. Local coordinate axes for a typical plate and stiffener ............................. 115

Figure 5.5. The location of piezo-electric patches ...................................................... 122

Figure 5.6. Axial compression versus maximum displacement ............................... 138

Figure 5.7. Axial compression versus Stiffener patch voltage at mid-span ............... 139

Figure 5.8. Axial compression versus maximum local buckling amplitude .............. 139

Figure 5.9. Vibration amplitude at plate center under suddenly applied lateral load versus time ................................................................................................ 140

Figure 5.10. Maximum stiffener displacement time history under Stiffener (only) control ................................................................................................................ 141

Figure 5.11. Variation of voltage across the piezo patch at the center of stiffener with time ........................................................................................................... 142

Figure 5.12. Maximum local buckling amplitude versus time under Stiffener only control ........................................................................................................... 142

Figure 5.13. Maximum stiffener displacement time history under stiffener-panel control ........................................................................................................... 143

Figure 5.14. Variation of voltage (V/mm) across the piezo patch at the center of stiffener with time .............................................................................................. 144

Figure 5.15. Maximum local buckling amplitude versus time under stiffener-panel control ........................................................................................................... 144

Figure 5.16. Overall component of voltage across the panel patch at the center of the panel versus time................................................................. 145
Figure 5.17. Maximum amplitude of the sinusoidal component of voltage across the panel patch versus time................................................................. 146

Figure 5.18. Displacement time history for the scaled up panel with gains scaling with geometry ........................................................................... 148

Figure 5.19. Stiffener patch voltage for the scaled up panel (gains scaling with geometry) ................................................................................. 149

Figure 5.20. Displacement time history for the panel ................................................................. 149

Figure 5.21. Stiffener patch voltage time history for the panel (geometry scaled up by 3 and gains scaled up by 10) ......................................................... 150

Figure 6.1. Cross-sectional view of (a) anti-symmetric local mode and (b) symmetric local mode ................................................................. 161

Figure 6.2 \( \lambda \) vs. displacement (verification with Dowell’s) .............................................. 184

Figure 6.3 Natural loading paths for a plate ........................................................................ 186

Figure 6.4. Total downward deflection vs. non-dimensional axial compression.............. 187

Figure 6.5. Time histories of \( \lambda = 9650, \sigma_o = 0, C_a = 0.1, \xi_m = 0.1/m; X_m = 0.1/m \) ............ 191

Figure 6.6. Time history of maximum deflection when \( \lambda = 10600 \) ................................. 193

Figure 6.7. Variation of maximum deflection with aerodynamic pressure ................. 193

Figure 6.8 Panel response at subcritical \( \lambda = 9300 \) with high initial values destabilizing the structure ................................................................. 194

Figure 6.9 \( \lambda \) vs. maximum deflections when edge movements are allowed .......... 195
Figure 6.10. Variation of maximum deflection with aerodynamic pressure \((\sigma_0 = 0.4\sigma_{cr})\).
........................................................................................................................................ 196

Figure 6.11. Time history of max. deflection for \(\lambda = 3430, \xi_m = 0.01/\text{m}; X_m = 0.01/\text{m}\).
........................................................................................................................................ 197

Figure 6.12. Variation of maximum deflection with aerodynamic pressure.............. 198

Figure 6.13. Time histories of \(\lambda = 3250\) with imperfections.............................. 199

Figure 6.14. Time histories at \(\lambda = 2500, \sigma_x = 0.4 \sigma_{cr}, \) with imperfection and \(p_o\) .... 200

Figure 6.15. Time histories at \(\lambda = 2600, \sigma_x = 0.4 \sigma_{cr}, \) with imperfection and \(p_o\) .... 201

Figure 6.16. Local and overall interaction effects comparison(\(\lambda=2500, C_a=0.1, \sigma_x = 0.4*\sigma_{cr}, \) no vimp, \(p_o=0.001, \xi_m = 0.01/\text{m}; X_m = 0.01/\text{m}\)).................. 202

Figure 6.17. Time histories of \(\lambda = 8400; C_a =0.1, \sigma_x = 0.0, \xi_m = 0.001/\text{m}; X_m = 0.001/\text{m}\)
........................................................................................................................................ 206

Figure 6.18. Time histories of \(\lambda = 8500, C_a =0.1, \sigma_x = 0.0, \xi_m = 0.001/\text{m}; X_m = 0.001/\text{m}\)
........................................................................................................................................ 207

Figure 6.19. Time histories of \(\lambda = 8000; C_a =0.1, \sigma_x = 0.0, \xi_m = 0.5 /\text{m}; X_m = 0.5 /\text{m}\) ...
........................................................................................................................................ 208

Figure 6.20. \(\lambda\) vs. maximum displacements for symmetric plate.............................. 209

Figure 6.21. Examples of three stages............................................................................ 210

Figure 6.22. Local-overall interaction comparisons in symmetric stiffened panel ...... 211

Figure 7.1. Piezoelectric sensor/actuator patches location........................................... 218
Figure 7.2. Time histories of first overall mode and stiffener voltage for $R_{d}^{lo} = R_{d}^{ov} = 10$ ($G_{p} = 361.3$; $G_{st} = 31.82$) ................................................................. 238

Figure 7.3. Time histories of first overall mode and stiffener voltage for $R_{d}^{lo} = 10$, $R_{d}^{ov} = 1000$ ($G_{p} = 361.3$; $G_{st} = 0.46$) ............................................................................. 239

Figure 7.4. Maximum plate patch voltage time history ............................................... 239

Figure 7.5. Time history of center plate voltage for $\lambda = 12000$ ($R_{d}^{lo} = 10$, $R_{d}^{ov} = 1000$, $G_{p} = 389.7$; $G_{st} = 0.42$) ......................................................................................... 240

Figure 7.6. Time histories of maximum local plate displacement and first overall mode for $\lambda = 52000$ ($R_{d}^{lo} = 0.1$, $R_{d}^{ov} = 1000$, $G_{p} = 3257.8$; $G_{st} = 12.05$) .......... 242

Figure 7.7. Time histories of maximum local plate displacement and first overall mode for $\lambda = 52000$ ($G_{p} = 4000$; $G_{st} = 4000$) ................................................................................. 243

Figure 7.8. $G_{p} vs. \lambda$ relationship for symmetric modes........................................... 247

Figure 7.9. $G_{p} vs. \lambda$ relationship for anti-symmetric modes.................................... 247

Figure 7.10. Scenario 1: Example 1 .............................................................................. 251

Figure 7.11. Scenario 1: Example 2 .............................................................................. 252

Figure 7.12. Scenario 2: Example ............................................................................... 254

Figure 7.13. Scenario 1: Example 2 .............................................................................. 255

Figure 7.14. Panel time histories of symmetric local mode under $\lambda = 50000$, MIMO, maximum plate voltage = 80.2355................................................................. 256
Chapter 1

Introduction

The theme of the research reported in this dissertation is the control of structures prone to static/dynamic instability using piezoelectric sensors/actuators. This chapter introduces the general concepts of piezoelectricity together with the basic equations and properties of typical piezoelectric materials currently in use; this is followed by review of literature on the subject, a statement of objectives of the research and a description of the contents of this dissertation.

1.1. Piezoelectricity: the phenomenon – Basic Equations

Piezoelectricity is the ability of crystals to generate a voltage in response to applied mechanical stress. The piezoelectric effect is reversible, in that piezoelectric crystals can change their shape in response to an externally applied voltage. The piezoelectric effect was discovered in 1880 by the brothers Jacques and Pierre Curie. In a piezoelectric crystal, the positive and negative electrical charges are separated, but symmetrically distributed, so that the crystal is electrically neutral overall. When a mechanical stress is applied, this symmetry of charges is disturbed, and the charge asymmetry generates a voltage across the material in a certain direction (say, $X$). As noted above, piezoelectric
materials also show the opposite effect, called converse piezoelectricity, where the application of an electrical field in a certain direction ($X$) creates mechanical deformation in the crystal [3]. The direction $X$ is called the poling direction of the crystal, and the charges become separated and aligned during the application of the electric field.

Many materials exhibit the piezoelectric effect, including quartz analogue crystals. As an example, the polymer polyvinylidene fluoride, PVDF, exhibits piezoelectricity several times larger than quartz. Bone also exhibits mild piezoelectric properties, due to the apatite crystals: it has been hypothesized that piezoelectricity is part of the mechanism of bone remodeling in response to stress, as electric fields generated by the apatite crystals stimulate further bone growth. Also, tourmaline, quartz, topaz, cane sugar, and Rochelle salt (sodium potassium tartrate tetrahydrate) generate electrical polarization in response to mechanical stress [3]. Elements of the theory of piezo-electricity are well expounded by Tiersten in his classic treatise [33] and in the American National Standard on Piezo-electricity, IEEE std. 176-1987 [3]. These sources are followed closely in the following summary of the equations of piezo-electricity.

### 1.1.1. Equations of Piezoelectricity

#### 1.1.1.1. Conservation of Energy

The principle of conservation of energy for a piezoelectric medium may be stated as given below.

In any volume, $V$, bounded by a surface, $s$ (with unit outward normal denoted by $\hat{n}$), the rate of increase of energy is equal to the rate at which work is done on the surface tractions minus the generated flow of electrical energy outward across $s$. Thus we have

$$\frac{\partial}{\partial t} \int_V \left( \frac{1}{2} \rho \dot{u}_i \dot{u}_i + U \right) dV = \int_s \left( t_i \dot{u}_i - n_i \phi \dot{D}_i \right) ds + \int_s f_i \dot{u}_i dV,$$

(1.1)
where $\rho$ is the mass density, $u_i$ are the displacement components, $t_i$ are the traction vector components, $D_i$ are the electric displacements, and $\phi$ is the electric field potential and the dot denotes differentiation with respect to time. Note that $U$ is the internal energy, and $f_i$ is the body force component per unit volume. This is valid for an arbitrary volume $V$ inside the body. The left hand side represents the rate of change of kinetic energy, the internal energy, and the energy from the body force per unit volume.

Noting that $t_i = T_{ij} n_j$ ($T$ is the Cauchy stress tensor) and applying the divergence theorem, we have

$$
\rho \ddot{u}_i + \dot{U} = \left(T_{ij} \dot{u}_i\right)_{,j} - (\phi \dot{D}_i)_{,j} + f_i \dot{u}_i ,
$$

Thus,

$$
\dot{U} = (T_{ij} - \rho \ddot{u}_i + f_i) \dot{u}_i + T_{ij} \dddot{u}_i - \phi \dddot{D}_i - \phi_j \dddot{D}_i .
$$

By invoking $T_{ij} + f_i = \rho \ddot{u}_i$ (the equilibrium equation), $\dot{D}_{ij} = 0$ (the charge equation of electro-statics), and $E_i = -\phi_j$ (the electric field potential relation), we obtain

$$
\dot{U} = T_{ij} \dot{u}_{i,j} + E_i \dot{D}_i .
$$

Let $S$ be the symmetric strain tensor for small deformation given by

$$
S_{ij} = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right).
$$

Thus,

$$
\dot{U} = T_{ij} \dot{S}_{ij} + E_i \dot{D}_i .
$$
1.1.1.2. Piezoelectric constitutive Equations

Consider a function $H$ in the form

$$H = U - E_i D_i,$$  \hspace{1cm} (1.6)

where $H$ has dimensions of energy per unit volume. Differentiating $H$ with respect to time, we obtain

$$\dot{H} = \dot{U} - E_i \dot{D}_i - \dot{E}_i D_i.$$  \hspace{1cm} (1.7)

Substituting eq. (1.5) in eq. (1.7),

$$\dot{H} = T_{ij} \dot{S}_{ij} - \dot{E}_i D_i.$$  \hspace{1cm} (1.8)

Since we anticipate constitutive relations expressing $T$ and $D$ in terms of $S$ and $E$, we write

$$H = H(S, E).$$  \hspace{1cm} (1.9)

Differentiating w.r.t. time,

$$\dot{H} = \frac{\partial H}{\partial S_{ij}} \dot{S}_{ij} + \frac{\partial H}{\partial E_i} \dot{E}_i.$$  \hspace{1cm} (1.10)

Comparing eq. (1.10) with eq. (1.8),

$$T_{ij} = \frac{\partial H}{\partial S_{ij}} \quad \text{and} \quad D_i = -\frac{\partial H}{\partial E_i}.$$  \hspace{1cm} (1.11.a-b)

Since we seek to construct a linear theory, $H$ is sought in the form of a homogeneous quadratic.
The only possible form of $H$ is

$$H = \frac{1}{2} c_{ijkl} S_{ij} S_{kl} - e_{ijk} E_i S_{jk} - \frac{1}{2} \varepsilon_{ij} E_i E_j ,$$

(1.12)

where $e_{ijk}$ is the set of piezoelectric constants (electro-mechanical coupling constants), $c_{ijkl}$ is the 4th order material tensor, and $\varepsilon_{ij}$ is the dielectric tensor.

Here,

$$c_{ijkl} = c_{ijkl} = c_{jiki},$$
$$e_{ijk} = e_{ijk}, \text{ and}$$
$$\varepsilon_{ij} = \varepsilon_{ji}.$$ (1.13.a-c)

Thus we have 21 independent elastic constants, 18 independent piezoelectric constants, and 6 independent dielectric constants in the most general case.

Differentiation of eq. (1.12) yields the constitutive relations

$$T_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k , \text{ and}$$
$$D_i = e_{ikl} S_{kl} + \varepsilon_{ik} E_k .$$ (1.14.a-b)

Substituting eq. (1.14.a-b) in eq. (1.12) and invoking eq. (1.6), we have

$$U = \frac{1}{2} c_{ijkl} S_{ij} S_{kl} + \frac{1}{2} \varepsilon_{ij} E_i E_j ,$$

(1.15)

which is a positive definite function (being a sum of two positive definite functions). There is no piezo-electric interactive term in the expression for $U$, and the positive definiteness of $U$ places restrictions on $c_{ijkl}$ and $\varepsilon_{ij}$ and not on $e_{kij}$. 
1.1.1.3. Variational Principle

We next seek a variational principle which will encapsulate all the equilibrium equations and natural boundary conditions. A simple form of such a principle is the principle of virtual work, which equates the external virtual work to the internal virtual work.

External virtual work may be written in the form

$$\delta W_{\text{ext}} = \int_V \left[ \int_S \{ t_i \delta u_i + Q \delta \phi \} dS - \int_V \rho \dot{u}_i \delta u_i \, dV + \int_V f_i \cdot \delta u_i \, dV \right],$$

where \( t_i \) are surface tractions prescribed over a part of the surface \( A_t \), \( Q \) is the surface charge per unit area prescribed over a part of the surface \( A_q \), \( \delta u_i \) and \( \delta \phi \) are arbitrary virtual quantities over \( A_t \) and \( A_q \), respectively and are zero, where \( u_i \) and \( \phi \) are prescribed. \( f_i \) is body force distributed over the volume, \( V \), of the solid.

By virtue of eq. (1.4.a-b), the first two terms vanish. Using eq. (1.4.c-d), we have

$$\delta W_{\text{ext}} = \int_V \left[ \int_S \left( T_{ik,l} + f_i - \rho \dot{u}_i \right) \delta u_i \right] dV + \int_V D_{ik,l} \delta \phi \, dV$$

$$+ \int_V \left[ T_{ik} \delta u_{i,k} + D_{ik} \delta \phi_{,k} \right] dV.$$  

By virtue of eq. (1.4.a-b), the first two terms vanish. Using eq. (1.4.c-d), we have

$$\delta W_{\text{ext}} = \int_V \left[ T_{ik} \delta S_{kl} - D_{ik} \delta E_{,k} \right] dV.$$  

From eq. (1.11.a-b),

$$\delta W_{\text{ext}} = \int_V \left[ \frac{\partial H}{\partial S_{kl}} \delta S_{kl} + \frac{\partial H}{\partial E_{,k}} \delta E_{,k} \right] dV = \int_V \delta H \, dV.$$  

(1.19)
The right hand side is interpreted as the internal virtual work, and \( H \) is the electrical enthalpy of the system.

Introducing the standard single subscripted notation for the stress and strain, i.e.,

\[
T_1 = T_{11} \; ; \; T_2 = T_{22} \; ; \; T_3 = T_{33} \; ; \; T_4 = T_{23} \; ; \; T_5 = T_{31} \; ; \; T_6 = T_{12}, \quad \text{and} \\
S_1 = S_{11} \; ; \; S_2 = S_{22} \; ; \; S_3 = S_{33} \; ; \; S_4 = 2S_{23} \; ; \; S_5 = 2S_{31} \; ; \; S_6 = 2S_{12}.
\]

The virtual work equation may be stated as

\[
\int_V \left[ T_m \delta S_m - D_i \delta E_i \right] dV = \int_V t_i \delta u_i dA + \int_A Q \delta \psi dA + \int_V (f_i - \rho \ddot{u}_i) \delta u_i dV
\]

\[
( m = 1, \ldots 6; \; i = 1, \ldots 3).
\]

\[1.20\]

1.1.1.4. **Effect of Crystal Symmetry**

We first write the constitutive equations (1.14.a-b) in the form

\[
T_p = C_{pq} S_q - e_{kp} E_k, \quad \text{and} \\
D_i = e_{iq} S_q + \varepsilon_{ik} E_k
\]

\[1.21.a-b\]

\[(p = 1, \ldots 6; \; i = 1, \ldots 3); \; (q = 1, \ldots 6; \; k = 1, \ldots 3).\]

where \( C \) is a 2 dimensional matrix giving the strain and stress relationship.

The relationships are simplified for crystals exhibiting symmetry. If there is a center of symmetry (all the three axes are simultaneously reversible without affecting the constitutive relations), then the piezo-electric effect vanishes \((e_{kp} = 0 \text{ for all } k \text{ and } p)\).

Consider an orthorhombic system, a crystal with three mutually perpendicular axes.

The following types of symmetry are possible:

(i) Three twofold rotation axes, designated as 222 \((\text{A two-rotation symmetry implies that one of the axis is held fixed, with the other two turned through } 180^\circ \text{ without affecting the constitutive relationship matrix})\).
(ii) Two mutually perpendicular planes of reflection symmetry, designated typically as \( \text{mm2} \). (A reflection plane implies that the axis normal to it can be reversed without affecting the constitutive relationship matrix. The presence of two planes of symmetry automatically guarantees the two fold rotation axis – the poling direction – the line of intersection of the planes of reflection symmetry.)

(iii) Three planes of reflection symmetry (\( \text{mmm} \)).

While the last case does not exhibit any piezoelectric effect, the piezo-ceramics exhibit \( \text{mm2} \) symmetry or variations thereof. The constitutive relationship with poling direction coinciding with \( x_3 \) axis takes the form

\[
\begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4 \\
S_5 \\
S_6
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & 0 & e_{31} \\
0 & 0 & 0 & e_{32} \\
0 & 0 & 0 & e_{33} \\
0 & 0 & e_{24} & 0 \\
e_{15} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix}
\]

(1.22.a-b)

\[
\begin{bmatrix}
D_1 \\
D_2 \\
D_3
\end{bmatrix} = 
\begin{bmatrix}
0 & 0 & 0 & 0 & e_{15} & 0 \\
0 & 0 & 0 & e_{24} & 0 & 0 \\
e_{31} & e_{32} & e_{33} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4 \\
S_5 \\
S_6
\end{bmatrix}
= 
\begin{bmatrix}
\varepsilon_{11} & 0 & 0 \\
0 & \varepsilon_{22} & 0 \\
0 & 0 & \varepsilon_{33}
\end{bmatrix}
\begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix}
\]

1.1.1.5. Constitutive Relations for Lamina in Plane Stress

Written in matrix form, eq. (1.22.a-b) reads

\[
\{T\} = [C] \{S\} - [e]^T \{E\}
\]

\[
\{D\} = [e] \{S\} + [e] \{E\}.
\]
This can be transformed to read

\[
\{S\} = [\bar{S}] \{T\} + [d]^T \{E\}, \text{ and}
\]

\[
\{D\} = [d] \{T\} + [\varepsilon^*] \{E\}.
\]

(1.24.a-b)

where \([\bar{S}] = [C]^{-1}\); \([d] = [e][\bar{S}]\) or \([d]^T = [\bar{S}][e]^T\) and \([\varepsilon^*] = [e][d]^T + [\varepsilon]\).

Consider a plane stress situation with \(T_{33} = T_{23} = T_{31} = 0\). From eq. (1.24.a), we have

\[
\begin{bmatrix}
S_1 \\
S_2 \\
S_6
\end{bmatrix} =
\begin{bmatrix}
\bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\
\bar{S}_{21} & \bar{S}_{22} & \bar{S}_{26} \\
\bar{S}_{61} & \bar{S}_{62} & \bar{S}_{66}
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
T_6
\end{bmatrix} +
\begin{bmatrix}
\bar{d}_{11} & \bar{d}_{12} & \bar{d}_{13} \\
\bar{d}_{21} & \bar{d}_{22} & \bar{d}_{23} \\
\bar{d}_{31} & \bar{d}_{32} & \bar{d}_{33}
\end{bmatrix}^T
\begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix}.
\]

(1.25)

This can be rearranged as

\[
\begin{bmatrix}
T_1 \\
T_2 \\
T_6
\end{bmatrix} =
\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{61} & \bar{Q}_{62} & \bar{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
S_1 \\
S_2 \\
S_6
\end{bmatrix} -
\begin{bmatrix}
\bar{d}_{11} & \bar{d}_{12} & \bar{d}_{13} \\
\bar{d}_{21} & \bar{d}_{22} & \bar{d}_{23} \\
\bar{d}_{31} & \bar{d}_{32} & \bar{d}_{33}
\end{bmatrix}^T
\begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix},
\]

(1.26)

where \([\bar{Q}] = [\bar{S}]^{-1}\).

For an orthorhombic \((mm2\) system) with poling direction coinciding with \(x_3\) this reduces to

\[
\begin{bmatrix}
T_1 \\
T_2 \\
T_6
\end{bmatrix} =
\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & 0 \\
\bar{Q}_{21} & \bar{Q}_{22} & 0 \\
0 & 0 & \bar{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
S_1 \\
S_2 \\
S_6
\end{bmatrix} -
\begin{bmatrix}
0 & 0 & \bar{d}_{31} \\
0 & 0 & \bar{d}_{32} \\
0 & 0 & 0
\end{bmatrix}^T
\begin{bmatrix}
E_1 \\
E_2 \\
E_3
\end{bmatrix}.
\]

(1.27)
1.1.1.6. Electric Enthalpy and Its First Variation for the Lamina

With prescribed voltages, the electric enthalpy density (eq. (1.12)) with single subscripted notation for stress and strain takes the form

\[ H = \frac{1}{2} Q_{ij} S_i S_j - e_{mj} E_m S_j \quad (i,j = 1, 2, 6), (m = 1, \ldots, 3) \]  

(1.28)

The first variation of \( H \), may be written as

\[ \delta H = Q_{ij} S_i \delta S_j - e_{mj} E_m \delta S_j \]  

(1.29)

Expressed in matrix notation,

\[ \delta H = \{\delta S\}^T \{Q\} \{S\} - \{\delta S\}^T \{e\}^T \{E\} \]  

(1.30)

Note the second term gives the first variation associated with piezo-electric effects.

Letting \([e]^T = [Q][d]^T\), we have

\[ \delta H = \{\delta S\}^T \{Q\} \{\varepsilon\} - \{\delta S\}^T \{Q\}[d]^T \{E\} \]  

(1.31)

\[ = \{\delta \varepsilon\}^T \{Q\} \{\varepsilon\} - \{\delta \sigma\}^T \{d\}^T \{E\} \]

Note in the kinematic strain and the corresponding stress are indicated by the more familiar notation of \( \varepsilon \) and \( \sigma \) respectively. For the case under consideration,
With voltage applied only in the $x$-3 direction, and letting $d_{31} = d_{32} = e_p$, the electro-mechanical coupled term in eq. (1.33) denoted by $\delta H^P$ takes the form

$$
\delta H^P = -\{\delta \sigma_1 + \delta \sigma_2\} e_p E_3
$$

(1.33)

### 1.2. Applications: Control, Energy Harvesting, Health Monitoring

Piezoelectric materials are used in numerous ways due to their good characteristics as high-voltage sources, sensors, actuators, and in equipment for reduction of vibrations, noise control, precision position control, health monitoring systems, etc. Use of piezoelectric crystals has many advantages such as easy manufacturing technique, rapid electro-mechanical response characteristics, light weight, and flexible design [3], [32]. Even though piezoelectricity was discovered a long time ago, the applications of sensing and actuating using piezoelectric materials for structures are relatively new.

#### 1.2.1. Control

An advantage of using the piezoelectric material to control structural behavior is that it can be used as self-sensing actuator, since it generates electricity by mechanical strain. Due to its dual structural functioning, it can be used as embedded actuator that responds
to electric loads and generates strains, deformation, and forces. It also functions as an integrated part of the structural skeleton and contributes to the mechanical load carrying mechanism. This advantage is even more significant in subscale aircraft such as unmanned aerial vehicles, small missiles, guided munitions, and projectiles [7]. In these cases, this active structural skeleton saves space required and reduces the overall weight of the structure. The piezo-laminated layer can be embedded inside of composite or attached as outer layer (surface bonded).

1.2.2. Energy Harvesting

Power harvesting devices capture normally lost energy, and this can produce devices less dependent on finite energy sources. Since piezoelectric materials generate electricity by mechanical forces with high efficiency of conversion, it can produce electricity in many portable and small electric devices such as MP3 players, mobile phones, etc [35]. Current technology provides miniaturized self-powered generators in many shapes including flexibility in shapes. Wireless sensors in many civil structures can use ambient vibration from the host structure to generate its own power to function [35]. When a structure is under bending, a self-sensing actuator with self power generation function can be used as a shape control device [34]. Piezoelectric materials respond to almost any type and magnitude of physical stimulus, including but not limited to pressure, tensile force, and torsion. Wearable applications have embedded piezoelectric materials into shoes to generate power from walking [36].

1.2.3. Health Monitoring

Structural health monitoring or crack/damage detection involves using structural measurements that characterize the condition or state of a structure and thus diagnose the existence of damage. Piezo-electric sensors can be effective instruments of non-destructive evaluation structural health [18]. The damage localization can be also achieved
by piezoelectric sensors. Furthermore, the self-sensing piezoelectric actuator can function as vibration suppression device in many structures.

1.3. Piezoelectric Materials

For the piezoelectric sensors and actuators, piezoelectric materials are fabricated in many ways. Traditional piezoelectric material is usually manufactured from lead zirconate titanate (PZT) or polyvinylidene fluoride (PVDF) piezoelectric ceramics. During manufacturing process, piezoelectric properties are induced in the ceramics using the poling process to create high electrical field in a specific direction.

New piezoelectric materials with ever increasing values of $e_p (= d_{31})$ are being currently introduced, e.g., $180 \times 10^{-12}$ mV$^{-1}$ for PZT5A to $370 \times 10^{-12}$ mV$^{-1}$ for PZT5K (MorganElectroCeramics Inc.). The value of $E_p$ may be taken as 63 GPa for this family of materials. The product $E_p e_p$ (with units of NV$^{-1}$mm$^{-1}$) takes on values of 0.0113 and 0.0233 respectively. In the present study the PZT material is used for the most part and a bench mark value $E_p e_p$ as 0.0283 is used – to be consistent with earlier work [22]. Apparently this value is on the higher end of the spectrum. If the actual value of $E_p e_p$ is smaller, say, one half of this value (0.01415), the results quoted here will still be applicable provided the gain $G$ and the corresponding voltages are doubled (it is assumed the piezo-patches are sufficiently thin in comparison to thickness/depth of the host structural element). As we shall see this would, in some cases call for a higher thickness of the piezo-electric patch as there is a limit to the field strength the material can be subjected to.

Recently, anisotropic piezoelectric actuators/sensors are manufactured such as the active fiber composites (AFC) and the micro fiber composites (MFC). Both are constructed of unidirectional aligned piezoceramic fibers surrounded by a polymer matrix. Both are film types that are extremely flexible, durable, light, and have the advantage of higher electromechanical coupling effects granted through the interdigitated electrodes as shown
in Figure 1.2. AFC uses round cross-sectional piezoelectric fibers embedded in glass rods, while MFC uses fibers of rectangular cross section [43]. Due to the manufacturing process and cost, MFC is most popular in industry. Fiber spacing, fiber diameters, fiber orientations, and their variations are numerous but 1-3 or 3-3 poling directions are most common in industry as shown in Figure 1.3.

Figure 1.1. PZT Materials [19]

Figure 1.2. MFC (NASA Langley, Smart Material Inc.)
(Note that ‘3’ in $d_{33}$ denotes the longitudinal direction.)
Table 1.1  Material properties of piezoelectric materials \(^{[29]}\)
(Graphite-Epoxy included for comparison)

<table>
<thead>
<tr>
<th>Materials</th>
<th>Graphite-Epoxy</th>
<th>PZT5A</th>
<th>MFC</th>
</tr>
</thead>
</table>
| Young’s modulus, N/m\(^2\) (psi) | \(E_1 = 15.5e10\)  
\(22.50e6\)  
\(E_2 = 8.07e9\)  
\(1.17e6\) | \(E_p = 6.21e10\)  
\(9.00e6\) | \(E_{p1} = 6.51e10\)  
\(5.29e6\)  
\(E_{p2} = 7.58e9\)  
\(1.10e6\) |
| Shear modulus, N/m\(^2\) (psi) | \(G_{12} = 4.55e9\)  
\(0.66e6\) | \(G_p = 2.39e10\)  
\(3.46e6\) | \(G_{p12} = 1.46e10\)  
\(2.12e6\) |
| Poisson’s ratio | \(\nu_{12} = 0.22\)  
\(\nu_{21} = 0.011\) | \(\nu = 0.30\) | \(\nu_{p1} = 0.25\)  
\(\nu_{p2} = 0.05\) |
| Density, kg/m\(^3\) (lb*s\(^2\)/in\(^4\)) | \(\rho = 1550\)  
1.458e-4 | \(\rho_p = 7582\)  
7.10e-4 | \(\rho_{p23} = 7552\)  
7.07e-4 |
| Charge constant, m/V (in./V) | -  
-  
-  
- | \(d_{31} = -1.91e-10\)  
\((-7.51e-9)\) | \(d_{11} = 5.31e-10\)  
\(2.09e-8\) | \(d_{12} = -2.10e-10\)  
\((-8.27e-9)\) |
| Maximum voltage, V/mm | - | | | 820 | 2000 |
1.4. Literature Review

1.4.1. Applications to Dynamics of Beams and Plates

In many structures including aerospace, sustained vibration is considered detrimental as it can lead to fatigue failure which can be sudden and catastrophic. Even if the failure does not occur, vibration itself and the resulting noise are not desirable. Techniques of suppressing vibrations using collocated piezo-electric sensors and actuators are getting popular as seen in the current literature which is vast and varied in scope. Numerous applications of piezo-electric control of beams and plates can be found in literature. Finite element modeling of structures actuated by piezo-electric patches has been pursued by a number of investigators. Here only a few typical contributions are mentioned.

Surface bonding of actuator patches results in damping of vibrations in laminated plates subject to bending since it provides largest moment arm for the piezoelectric forces about the laminate mid-plane [25],[38],[39],[57]. Piezo-electric patches when properly
constructed and bonded to the structure can perform the dual role of sensing the local strains or strain rates and providing feedback control for vibration suppression.

Lam et al. study the issues involved in the feedback control of laminated composite cantilevered plates with various stacking sequences and boundary conditions using piezoelectric devices [25]. Their analysis is based on the classical laminated plate theory and Hamilton’s principle and they employ a simple negative feedback control to suppress the vibration.

Tan et al. consider dynamic characteristics of a beam system with alternative configurations of piezo-electric actuators [38]. Piezo-electric fibers are embedded in a visco-elastic matrix to provide active damping. The modal analysis in the presence of feedback control proportional to rate of charge accumulated in the sensor layer leads to a set of coupled linear differential equations. Both axial and flexural vibrations are studied. Wang and Tang use an accurate model of a piezoelectric composite beam employing Reddy’s high order theory (the third order displacement theory) to model the displacement field through the thickness of the beam [57]. They found that the constant electric field model, i.e. the linear potential model, is a good approximation for piezoelectric actuator with electric potential applied through the thickness direction, but not accurate enough for sensors in bending mode.

Qiu et al. present a theoretical analysis and experimental results of vibration suppression of a flexible cantilever plate with bonded PZT sensors and actuators [39]. They propose optimal locations of collocated sensors and actuators for flexural and torsional vibrations respectively.

Jiang et al. propose a finite element model of piezo-thermo-elastic composite beams with distributed piezoelectric sensor and actuator layers [20]. They employ higher order shape functions for the electric field. They studied issue involved in active vibration control on the composite beam under mechanical impulse and thermal excitations. They found that
the deflection induced due to thermal ingredient cannot be controlled by the negative velocity feedback control, which is effective only for dynamic control.

1.4.2. Nonlinearities and Instabilities

There have been many attempts to increase stability limits of the structures using piezoelectricity as seen in the current literature.

Meressi et al. study buckling control of flexible beams using piezoelectric actuators [31]. It is shown that the buckling capacity of a flexible beam can be enhanced beyond the first critical load by means of feedback using piezoelectric actuators and strain gauges. In a similar study involving experimental work, Thompson and Loughlan demonstrate the potential application of piezoelectric actuation in eliminating the effect of imperfections and enhancing the critical load of a composite column [37].

Chase et al. propose an optimal design of actuator scheme which prevents a composite laminated plate from buckling when loaded above the critical buckling load [12]. The static output – displacements and velocities from the strain sensors – is simply multiplied by gain and used as feedback to the system in their study.

Wang et al. investigated the buckling enhancement of a very thin column by surface bonding of a pair of piezoelectric layers [58]. The possibility of piezoelectric actuation in enhancing the critical load as well as its effectiveness in damping out the vibrations below the critical load is investigated. They consider the problem of a column clamped at one end and acted upon by a follower compression force and supported by a linear spring at the other end. Piezoelectric layers carrying pre-tension of varying amounts are used to enhance the critical load corresponding to flutter.

Rabinovitch studies geometrically nonlinear response of piezo-electrically actuated cantilevered plates employing Kirchhoff plate theory in conjunction with von-Karman
strain-displacement relations [40]. Cantilevered plates with various kinematic boundary conditions are considered. The plates carry piezoelectric layers at the top and bottom and are actuated in bending or twisting by an application of voltages across the piezo-electric layers. The plates show stiffening nonlinearity under piezo-electric actuation.

Nonlinear dynamic response of plates is studied by Oh. He concludes that the possibility of snap-through should be considered in any attempt at shape control in these problems [41]. The study deals with the piezoelectric-elastic laminated plates under thermal loading which causes dynamic instability. The deflection caused by thermal load can be reduced by piezoelectricity, but the plate undergoes snap-through to an inverted profile. The major lacuna of the paper is that the plate is controlled by a fixed voltage increment which is initially chosen and does not vary as the deflections change. This is what probably causes the snap-through phenomenon.

Schultz et al. demonstrate the use of a piezo-electric actuator to achieve a snap through of an unsymmetric laminate from one stable configuration to another using finite element method and experiments [42]. The plate model involves piezo-ceramic actuator layer only on one side of laminate. Using Rayleigh-Ritz method, they show that the predicted deformation and actuation voltage are reasonably close to the experimental results.

Giannopoulos et al. [16] have investigated the possibility of stabilizing an inherently unstable state of a column by causing a snap-through to a stable state by piezo-electric actuation.

Sridharan and Kim discuss the issues of piezoelectric control of structures prone to nonlinear static and dynamic instabilities by a simple example of simply supported imperfect column on an elastic softening foundation [44]. Using the voltage proportional to the strain rate as feedback, the increase in critical limit and effective controllability near the critical limit is demonstrated. In further work, the problem of stiffened plates subject to interaction of local and overall buckling is considered [45].
Flutter is a form of self-excited oscillation of structural elements exposed to airflow at sufficiently high velocities. It is the result of an adverse interaction of aerodynamic pressure exerted on the structure and the modes of vibration of the structure. There is a critical velocity at which vibrations are triggered and these build up in amplitude as the velocity increases. In plate panels this takes the form of limit cycle oscillations (LCO), i.e. the oscillations do not escalate but attain a finite value for velocities exceeding the critical value. However if there are softening nonlinearities in the system the oscillations may become unbounded. The classic treatise of Dowell [13] gives a succinct introduction to the subject and offers some valuable benchmark results. Since late 1950s, there have been many publications addressing linear and nonlinear panel flutter. Linear and nonlinear panel flutter theories and analysis are reviewed by Dowell in early 1970s [15]. In 1990s, researches on active control of flutter are pursued by several investigators. Dongi et al. considered the problem of suppression of flutter of flat and slightly curved panels in high supersonic flow using von Karman plate model [11]. They propose a control approach, using output feedback from a pair of collocated self-sensing actuator, based on active compensation of aerodynamic stiffness.

Piezo-electric control of plates undergoing flutter due to aerodynamic pressure acting on them has been the subject of intense and in depth investigation by Mei and his co-workers [26],[27],[29],[59]. They address the issues in the control of nonlinear flutter of plates undergoing limit cycle oscillations (LCO). Optimal control techniques are used to identify the best possible locations of piezo-electric actuators. Lai et al. study the control of nonlinear flutter of a simply supported isotropic plate using piezoelectric actuators attached using the optimal control theory [26],[27]. The optimal control theory is used in the simulation. They concluded that the bending moment induced by piezoelectric actuators is more effective than in-plane forces for flutter suppression. Zhou et al. [59] investigate the problem of enhancing the critical velocity corresponding to flutter of a plate panel and suppression of limit cycle oscillations. They attempt to use linear optimal control technique to select the gains of the piezo actuators. Because nonlinearities are neglected in their optimal control methodology, it was found to be ineffective in some cases. Motagaly studies the active control of nonlinear panel flutter of composite plates
using finite element analysis [1],[2]. Employing the first-order piston theory of aerodynamics and optimal control algorithms, he proposed successful way of suppressing flutter. Some aspects of optimal control are discussed in the following section.

1.4.3. Optimal Control

There have been several attempts to employ optimal control techniques to design the piezo-electric sensor-actuator configuration for plates liable to flutter. Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is satisfied. Settling time of vibration, maximum required voltage, energy consumed, cost of material, etc. are examples of factors in optimality in control. Early attempts at optimal control problems involved the application of linear quadratic regulator (LQR) [1],[2],[59].

A standard state space model is developed from the governing equations in the form

\[
\dot{X} = \bar{A}X + BU, \quad \text{where} \quad X = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}.
\]  

(1.34)

Every attempt is made to reduce the number of degrees of freedom, \( q \) using a modal approach if necessary. The matrix \( \bar{A} \) is nonlinear, but may be linearized in some appropriate manner. A linear quadratic regulator (LQR) may be employed to optimize the system.
This approach seeks a solution for the linear full-state feedback problem wherein the objective is to find

\[
U = -KX
\]

(1.35)

, which minimizes a quadratic performance index.
This index is may be chosen as the cost functional of the system and control effort

\[ J = \int_0^\infty [X^T Q X + U^T R U] dt \]  

(1.36)

where \( X \) and \( R \) are appropriately chosen to be positive definite matrices.

Minimizing \( J \), the controller gain matrix \( K \) may be obtained as

\[ K = R^{-1} B^T P \]  

(1.37)

where \( P \) is a positive definite matrix obtained from the following Riccati equation

\[ A^T P + PA - PB R^{-1} B^T P = -Q \]  

(1.38)

In practical situations, there are is no knowledge of \( \{X\} \) \textit{a priori} and these must be obtained from sensors. This underlines the importance of the placement of sensors. The sensor estimates are also subject to noise in actual practice. An appropriately designed Kalman filter is employed to obtain the best possible estimate of the \( \{X\} \). LQR and Kalman filter are combined to form the more practical LQG compensator where the controller output is based on estimated states. The limitation of linearity in earlier Kalman filter applications was overcome by the use of extended Kalman filter (EKF) by Motagaly and Mei [2]. That is based on linearization with respect to a trajectory that is continuously updated with the estimated states.

Kumar and Narayanan study the effectiveness of optimal control strategies applied to piezoelectric vibration control of beams. They use constant gain negative velocity feedback and the LQR optimal control scheme coupled with genetic algorithms to find optimal locations of piezoelectric actuators and sensors [24]. They demonstrate that the optimal locations are in the regions of high modal strain energies and the LQR optimal control offers an effective control with less peak actuator voltages.
In the present study the use of optimal control strategy would be restricted to finding gains for actuator placed in pre-selected locations.

### 1.5. Objectives

The main goal of the research is to study the issues of piezoelectric control of stiffened structures subjected to axial compression and/or aerodynamic pressure and liable to nonlinear modal interaction. The questions of interest are: feasibility, effectiveness, and optimality. A related objective is to develop a novel methodology based on a combination of perturbation technique, finite element and modal analysis which would effectively handle the complexities of local (shortwave) plate bending and modal interaction. Thus the objectives of the research may be spelled out as:

1. Examine the issues in the piezo-electric control of slender columns, and stiffened panels subject to nonlinear mode interaction in both static and dynamic loading environments. These issues pertain to:
   
   (i) the feasibility of control of instabilities,
   (ii) effectiveness of the control in terms of energy consumption and time of taken to settle the structure, and
   (iii) the simplicity of sensor/actuator configuration.
   (iv) optimality of selecting gain for the control

2. Examine the role of piezo-electric control in the context flutter of longitudinally stiffened plate structures which would involve nonlinearities due to local plate bending and interaction of local and overall bending.

3. Examine the efficacy and optimality of strategically placed piezo-electric actuator patches – along the stiffener tips and longitudinal lines of symmetry of plate panels.
4. Develop appropriate analytical tools for the study using a combination of asymptotic approach and finite elements/modal analysis.

**General remarks on the findings of the present research**

One of the issues in the control of structural vibrations is the selection of location of piezoelectric patches in the structure. In axially compressed stiffened plates the primary focus is on controlling the overall buckling. The local buckling of plate elements by itself turns out to be relatively innocuous once the overall action is suppressed. Thus effective control that stabilizes the structure up to the critical axial compression ($P_{cr}$) can be achieved by appropriate piezoelectric self-sensing actuators attached to the tips of the stiffener. In a dynamic loading environment, for effective control of local oscillations additional patches along the longitudinal centerline of the panel were found to be necessary. In the flutter problem, the situation is reversed. The overall oscillations are damped out fairly quickly because of aerodynamic damping present and the major focus of the control effort is to suppress the local vibrations of the plate. This situation continues till the non-dimensional aerodynamic pressure, $\lambda$, attains a value corresponding to flutter of the panel in the overall mode.

The study demonstrates the effectiveness of strategically located piezo-electric patches, i.e. self-sensing actuators attached to the tips of the stiffeners and along the longitudinal centerlines of panels to control overall and local bending respectively.

The other issue is the selection of the type of feedback control. In a situation where vibration suppression is the goal, direct negative velocity feedback appears to be the logical choice. The present study proposes a simple methodology for selecting gains for the self-sensing piezo-electric patches based on the application of a LQR algorithm appropriately modified to account for the geometric nonlinearity. The study also reveals
the limitation of this type of control in the context of a flutter problem. A more general MIMO control is warranted beyond a certain value of $\lambda$.

The following is the arrangement of the contents of this study:

Chapter 2 summarizes the basic equations of the analysis and assumptions made in the present work and the numerical solution procedure. Chapter 3 and 4 illustrate a number of points pertaining to piezo-electric control of structural elements subjected to conservative and non-conservative loading respectively. The former deals with the control issues of an axially compressed simply supported column resting on a softening elastic foundation whereas the latter deals with a cantilever column propped by a spring and carrying a compressive follower force. Chapter 5, 6, and 7 explore a number of aspects of behavior and control of a stiffened panel. Axially compressed stiffened panels under interaction of local and overall buckling and subjected to lateral disturbances are studied in chapter 5 using a special finite element method. In chapter 6, stiffened panels under aerodynamic pressure with potential interaction of local and overall modes of vibration are analyzed extensively using modal approach. Chapter 7 studies in detail the optimal control strategies of stiffened panel liable to flutter. Chapter 8 summarizes the major findings and contributions of this research.
Chapter 2

Methodology

In this chapter key aspects of the methodology employed in the present study will be outlined. The treatment is kept at a conceptual level with actual applications taken up in the succeeding chapters. The following aspects are discussed: (i) statement of virtual work equation for the entire structure (ii) expression of the variables in terms of degrees of freedom associated with shape functions (iii) the formulation of governing differential equations in time domain in incremental form together with the solution procedure and (iv) piezo-electric contributions to the governing equations from sensor and actuator patches respectively and (v) simplifications adopted in the present study.

The solution methodology can employ finite elements, Raleigh-Ritz approach or modal analysis. The meaning of the degrees of freedom, shape functions and generalized forces vary depending upon the approach selected, but all the approaches lend themselves to a common description with some variations. Section 2.1 develops the formulation assuming that finite elements are used. The structure in question undergoes large deflections and carries sensors and actuators in specific locations. A set of nonlinear differential equations in time are derived there from. The numerical procedure employed for the solution is then described.

The equations though expressed in a general form may be viewed as an ensemble of three different types of equations.
(i) Those applicable to the host structure: Apart from the usual elastic forces these would contain terms representing generalized forces associated with the piezo-electric sensor and actuator patches via shared degrees of freedom.

(ii) Sensor equations: Those involve degrees of freedom associated with sensor patches only. These relate the sensor degrees of freedom to the voltages developed across the patch or sum of the charges collected over the surfaces of the patch.

(iii) Actuation equations: The voltages prescribed across the actuator patches are related to sensor degrees of freedom by a chosen feedback control mechanism. These relations must be substituted in the equations of the host structure.

2.1. Analysis and Modeling Issues

Equilibrium equations can be developed using virtual work statement in eq. (2.1). In the present study, the displacement components $u_i$ and electric potential $\phi$ are selected as the variables. Appropriate shape functions associated with degrees of freedom are selected to describe these variables and the respective variations. The equations obtained are nonlinear in so far as geometric nonlinearities are duly considered.

In a static problem the equilibrium equations are solved using an incremental and iterative procedure such as the Newton - Raphson method. This is coupled with an arc-length procedure to facilitate tracing highly nonlinear ranges of behavior past the limit point.

In a dynamic problem the nonlinear equations of motion are solved in time domain selecting appropriate time increments. First an implicit procedure such as the Newmark beta method is used to express the current accelerations and velocities in terms of the known past values and current incremental, as yet unknown, values of the degrees of freedom. The resulting nonlinear equations are solved using the Newton-Raphson method.
2.1.1. Finite Element Formulation

The virtual work equation for a three dimensional body takes the form (vide eq. (1.20)) as

\[
\int_V \left[ T_m \delta S_m - D_r \delta E_r \right] dV = \int_A t_r \delta u_r dA + \int_A Q \delta \phi dA + \int_V (f_r - \rho \ddot{u}_r) \delta u_r dV
\]  
\( (m = 1\ldots6; r = 1\ldots3). \)  

(2.1)

From now on \( T_m \) is taken as the second Piaho-Kirchoff stress which for the case of small strains has a clear physical meaning as the true stress co-rotational with the element as it deforms and rotates. \( S_m \) is its conjugate, viz. Green-Lagrange strain. The prescribed body forces, surface tractions and charges appear on the r. h. s of the equations. \( V \) is the volume of the body, and \( A_t \) and \( A_q \) are those parts of the surface area over which tractions and electric charges are prescribed respectively. Each displacement component is taken in the form

\[
u_r = N_k q_r^{(r)} \ (k = 1,\ldots n; r = 1,\ldots 3),
\]

(2.2)

where \( N_k \) are the shape functions and \( q_r^{(r)} \) are the corresponding degrees of freedom.

Taking the three displacements components together, we may write

\[
\{u\} = [N] \{q\}, \text{ and } \{\delta S\} = [B] \{\delta q\}
\]

(2.3.a-b)

\( B_{mk} \delta q_k \ (m = 1,\ldots 6; k = 1,\ldots N) \),

where \( \{\delta S\} \) is the “virtual strain”, numerically equivalent to the first variation of strain. Note \( [B] \) involves not only the shape functions but the current values of the displacement degrees of freedom.
Likewise

\[ \phi = N_k^\phi \varphi_k, \quad (2.4) \]

where \( N_k^\phi \) are the shape functions and \( \varphi \) 's are the electrical degrees of freedom.

Then,

\[ \{ \delta E \} = -[A]\{ \delta \varphi \}, \quad (2.5) \]

where \([A]\) is obtained by appropriate differentiation.

A simple form of \([A]\)-matrix used in the present study is presented below:

Consider the case of a patch with poling along the thickness direction. We may assume that the patches are sufficiently thin so that the electric potential varies linearly across the patch leading to constant field strength in the thickness direction. In this case the \([A]\) matrix takes the diagonal form

\[ [A] = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{h} \end{bmatrix}, \quad (2.6) \]

where ‘h’ is the thickness of the patch and \( x_3 \) refers to the thickness direction.

Substituting eq. (2.2 – 2.5) in eq. (2.1) and letting the virtual quantities to be arbitrary, we have the following sets of equations.

\[ M_{ij} \ddot{q}_j + \int_V B_{mi} \sigma_m dV = P_i + F_i, \text{ and} \quad \quad (2.7.a-b) \]

\[ \int_V A_{ik} D_k dV = \sigma_k, \]
On the right hand side of eq. (2.7.b-c) ‘p’ identifies the particular traction/body force component corresponding to \( \delta q_i \), the \( i^{th} \) virtual displacement. On the right hand side of eq. (2.7.a), ‘p’ indicates that the shape functions, \( N_i \) and \( N_j \) must both correspond to the same displacement component as \( \delta q_i \). These equations consist of contributions from the host structure as well as sensor and actuator patches. From these general equations, those pertaining to a sensor or an actuator involving only the corresponding degrees of freedom can be readily isolated.

Note eq. (2.7) is not complete and used here for illustrative purposes only. Actual problems would involve damping forces proportional to \( \{q\} \), non-conservative aerodynamic pressure and feedback voltages proportional \( \{q\} \) and/or \( \{\dot{q}\} \). Since these are linear terms and the following treatment implicitly covers their treatment too.

- **Solution Procedure**

(i) **Equations in incremental form in time**

Let us assume that at time \( t \), these equations are satisfied and we need to find incremental values, \( \Delta q \) and \( \Delta \phi \) corresponding to a time increment of \( \Delta t \).

As a first step we express \( \{q\}_{t+\Delta t} \) and \( \{\dot{q}\}_{t+\Delta t} \) in terms of the quantities known at time \( t \) and the incremental values of \( \Delta q \). Recalling the following relations of the Newmark beta method

\[
M_{q} = \rho \int_{V} N_i^p N_j^p \, dV, \\
P_i = \int_{A_i} N_i t_p \, dV, \\
F_i = \int_{V} N_i f_p \, dV, \text{ and} \\
\sigma_k = \int_{A_k} N_k^s Q \, dA.
\]
\[
\{\ddot{q}_{t+\Delta t}\} = \{\ddot{q}_{t}\} + \left[(1-\delta)\{\ddot{q}_{t}\} + \delta\{\ddot{q}_{t+\Delta t}\}\right]\Delta t, \text{ and} \\
\{q_{t+\Delta t}\} = \{q_{t}\} + \{\dot{q}_{t}\}\Delta t + \left[\left(\frac{1}{2} - \beta\right)\{\dot{q}_{t}\} + \beta\{\dot{q}_{t+\Delta t}\}\right](\Delta t)^2.
\] (2.9.a-b)

Or alternately,

\[
\{\ddot{q}_{t+\Delta t}\} = a_1\{\Delta q\} + a_2\{\dot{q}_{t}\} + a_3\{\ddot{q}_{t}\}, \text{ and} \\
\{\dot{q}_{t+\Delta t}\} = b_1(\Delta q) + b_2\{\dot{q}_{t}\} + b_3\{\ddot{q}_{t}\},
\] (2.10.a-b)

where

\[
a_1 = \frac{1}{\beta(\Delta t)^2}, \quad a_2 = -\frac{1}{\beta\Delta t}, \quad a_3 = -\frac{1}{\beta}\left(\frac{1}{2} - \beta\right), \\
b_1 = \frac{1}{2\beta(\Delta t)}, \quad b_2 = 1 - \frac{1}{2\beta}, \quad \text{and} \quad b_3 = \frac{1}{2}\left(1 - \frac{1}{\beta}\left(\frac{1}{2} - \beta\right)\right)(\Delta t).
\] (2.11.a-b)

In calculations, we set \(\beta = \frac{1}{4}\) and \(\delta = \frac{1}{4}\).

Equation (2.9.a) is used to express \(\{\ddot{q}_{t+\Delta t}\}\) in terms of quantities known at time \(t\) and \(\{\Delta q\}\).

(ii) **Breaking down the geometric nonlinearity**

Noting that the term \([B]^T\{T\}\) is nonlinear, we linearize it in order to predict an increment of the d.o.f. over the time interval using the Newton-Raphson approach. This will be followed by iterations till a converged solution is obtained satisfying the equations exactly.

The stress \(\{T\}\) at time \(t + \Delta t\) can be expressed as

\[
\{T\}_{t+\Delta t} = \{T\}_t + \{\Delta T\},
\] (2.12)
where
\[
\Delta T_m = C_{mn} \Delta S_n - e_{mr} \Delta E_r. \tag{2.13}
\]

As an initial approximation,
\[
\{B_m T_m\}_{t+\Delta t} \approx \{B_m T_m\}_t + \{B_m\}_t, \Delta T_m + \{\bar{B}_{mij} T_m\}_t \{\Delta q_j\},
\]
\[
\text{where } \bar{B}_{mij} = \frac{\partial B_{mi}}{\partial q_j}. \tag{2.14}
\]

This is independent of time as \(B\) is only linear in \(q\).

Substituting for \(\Delta T_m\) from eq. (2.12), we have
\[
\{B_m T_m\}_{t+\Delta t} \approx \{B_m T_m\}_t + \{B_m\}_t, C_{mn} \Delta S_n - \{B_m\}_t e_{mr} \Delta E_r + \{\bar{B}_{mij} T_m\}_t \{\Delta q_j\}. \tag{2.15}
\]

Approximating incremental strains as \(\Delta S_m = \{B_m\}_t \Delta q_i\), and substituting \(\Delta E_r = -A_{rk} \Delta \phi_k\),
\[
\{B_m T_m\}_{t+\Delta t}
\]
\[
\approx \{B_m T_m\}_t + \{B_m\}_t, C_{mn} \{B_{nj}\}_t \Delta q_i + \{B_m\}_t e_{mr} A_{rk} \Delta \phi_k + \{\bar{B}_{mij} T_m\}_t \{\Delta q_j\}. \tag{2.16}
\]

The predictive equations take the form
\[
\{a_i M_{ij} + K_{ij}^u + K_{ij}^\sigma\} \Delta q_j + K_{ik}^{\phi\phi} \Delta \phi_k = \Delta P_i + \Delta F_i - M_{ij} \left(a_i \dot{q}_j + a_i \ddot{q}_j\right) \tag{2.17.a-b}
\]
\[
, \text{ and } \{K_{ij}^{\phi\phi} \Delta q_j + K_{ki}^{\phi\phi} \Delta \phi_k\} = \Delta \sigma_k,
\]
where

\[
\begin{align*}
K_{ij}^u &= \int Y \left\{ B_{mi} \right\}_i C_{mi} \left\{ B_{nj} \right\}_j dv \\
K_{ij}^\sigma &= \int Y \left\{ B_{mn} T_m \right\}_i dv \\
K_{ik}^{\sigma u} &= \int Y \left\{ B_{mk} \right\}_i e_m A_{rk} dv \\
K_{kj}^{\phi u} &= \int Y A_{rk} e_m \left\{ B_{mj} \right\}_j dv \\
K_{ki}^{\phi \phi} &= \int Y A_{rk} e_r A_{se} dv
\end{align*}
\] (2.18.a-e)

Solution of eq. (2.16) gives the first approximation of the incremental d.o.f's, viz. \( \{\Delta q\}^a, \{\Delta \phi\}^a \), and these in turn give the updated stresses and electric displacements as shown below.

\[
\begin{align*}
T_m &= (T_m)_I + \Delta T_m \\
D_r &= (D_r)_I + \Delta D_r
\end{align*}
\] (2.19.a-b)

(iii) Correction by iterations

Let us take

\[
\begin{align*}
\{q\}_{t+\Delta t} &= \{q\}_I + \{\Delta q\}^a + \{\Delta q\}^c \\
\{\phi\}_{t+\Delta t} &= \{\phi\}_I + \{\Delta \phi\}^a + \{\Delta \phi\}^c
\end{align*}
\] (2.20.a-b)

where \( \{\Delta q\}^c, \{\Delta \phi\}^c \) are corrections to be determined. Reverting back to eq. (2.7.a-b) and linearizing them once again, we have

\[
\begin{align*}
\left\{a_i M_{ij} + K_{ij}^u + K_{ij}^\sigma \right\}_{t+\Delta t} + \int V \left\{ B_{mj} T_m \right\}_i dv &= \left\{ \int V \left\{ B_{mj} T_m \right\}_i dv \right\}^a \\
K_{kj}^{\phi u} \Delta q_j^c + K_{kj}^{\phi \phi} \Delta \phi_j^c &= \left\{ \sigma_k \right\}_{t+\Delta t} - \left\{ \int V A_{rk} D_r dV \right\}^a.
\end{align*}
\] (2.21.a-b)
where the superscript ‘a’ indicates current approximate values. The $K$-matrices on the left hand side need not be updated for the modified Newton-Raphson procedure. The solution of the foregoing equations yields the corrections $\{\Delta q\}^c$, $\{\Delta \phi\}^c$. The process is continued till convergence is achieved as seen from the right hand side becoming vanishingly small.

### 2.1.2. Sensor and Actuator Equations

**Sensor, Actuator and Host structure relationship**

Consider a piezo-electric patch bounded by surfaces $S^{(1)}_p$ and $S^{(2)}_p$, the former being the outer surface of the patch and the latter the surface of the structural component. Because the material of the structural component is insulated from the patch, we may prescribe either charges or voltages on the exterior surface $S^{(1)}_p$ as well as on interior surface $S^{(2)}_p$. The only connection between the patch and the structural component is via compatibility of strains at the interface. Note further the sensor and actuator patch voltage degrees of freedom are fully decoupled from each other. $\{q\}$, however, includes all the mechanical degrees of freedom. The sensor and actuator degrees of freedom form subsets of $\{q\}$, i.e. $\{q\} = \{q^{(h)} q^{(s)} q^{(a)}\}^T$, where $q^{(h)}$ are the d.o.f. of the host structure exclusively, $q^{(s)}$ and $q^{(a)}$ are the d.o.f. of the sensor and actuator respectively, each of which include those shared with the host structure.

We consider the following scenarios.

(i) The mechanical d.o.f. of the actuator and sensor patches are fully decoupled from each other, e.g. the patches constitute two distinct elements attached at different locations to the host structure. Thus the matrices $[K^{u\phi}]$, $[K^{u\mu}]$ and $[K^{\phi\phi}]$
become block diagonal, and the sensor and actuator equations are fully decoupled in eq. (2.17. a-b).

(ii) They share the same set of mechanical degrees of freedom (“self-sensing” actuator concept). In this case, \( q^{(s)} = q^{(a)} \), the electromechanical contributions appear together in the eq. (2.17.a) and decoupled in eq. (2.17b).

2.1.2.1. Sensor Response

- **Voltage Sensor**

For a voltage sensor patch, we prescribe \( \phi \) (the values of \( \phi \)'s) at the surface \( S_{p}^{(2)} \) to be zero and the charges \( Q \) at the surface \( S_{p}^{(1)} \) to be zero. The value of \( \phi \) at \( S_{p}^{(1)} \) is then the voltage across the patch. Consider the eq. (2.1) derived from virtual potentials applied to sensors. Since the integrand \( Q \delta \phi \) on the left hand side of eq. (2.1) on the sensor surfaces vanishes, the \( \{ \sigma \} \) term on the right hand side of eq. (2.6.b) vanishes as well.

Thus these equations take the form

\[
\int_{V} A_{rk} D_{r} dV = 0, \quad (2.22)
\]

which in the predictor equations take the form

\[
[K_{kj}^{\phi u}]_{(s)} \Delta q^{(s)}_{j} + [K_{kl}^{\phi \phi}]_{(s)} \Delta \phi^{(s)}_{l} = 0, \quad (2.23)
\]

where the superscript and subscript \( (s) \) refer to a quantity pertaining to the sensor patch.
Equation (2.23) leads to

$$\{\Delta \phi^{(s)}\} = -[K_{\phi\phi}^{-1}]_{l(s)} [K_{\phi u}]_{l(s)} \{\Delta q^{(s)}\}$$  \hspace{1cm} (2.24)$$

For a piezo-electric material with two of its principal axes aligned with the surface of the structural component, the voltage sensed across the sensor patch can be shown to be a linear combination of the strains in the principal directions averaged over the volume of the patch.

- **Charge Sensor**

For a charge sensor we prescribe $\phi$ to be zero at both the surfaces $S_p^{(1)}$ and $S_p^{(2)}$ and thus all the $\phi$’s are taken as zero.

The internal electric displacement is related to the surface charges as in

$$\int_{V_r} A_r \, D_r \, dV = \sigma_k,$$  \hspace{1cm} (2.25)$$

which can be simplified to read as

$$\int_{V_r} A_r \, e_{rm} \, S_m \, dV = \sigma_k,$$  \hspace{1cm} (2.26)$$

and in incremental form

$$[K_{\phi u}]_{l(s)} \Delta q_j^{(s)} = \Delta \sigma_k.$$  \hspace{1cm} (2.27)$$

The term on the right hand side is proportional to the incremental charges collected on the two surfaces of the sensor patch and the equations involve only the sensor degrees of freedom. For a piezo-electric material with two of its principal axes aligned with the
surface of the structural component, the accumulated charges can be shown to be proportional to the sum of in-plane strains along the principal directions.

### 2.1.2.2. Feedback Control by Actuator

- **Actuation using a Voltage Sensor**

If velocity feedback control law implemented is based on voltage sensed by the patches (Figure 2.1), then

\[
\{\Delta \phi^{(a)}\} = -G_c \{\Delta \phi^{(s)}\} = G_c \left[K^{\phi\phi}\right]_{(s)}^{-1} \left[K^{\phi\mu}\right]_{(s)} \{\Delta \dot{q}^{(s)}\}.
\]

(2.28)

where the superscript \((a)\) refers to a quantity pertaining to the actuator patch and \(G_c\) is the actuator gain. Substituting in the general equations eq. (2.16.a) for the actuator voltage degrees of freedom, we get

\[
\begin{align*}
\{a_1[M] + [K^u] + [K^\sigma]\} \{\Delta q\} + G_c \left[K^{u\phi}\right]_{(a)} \left[K^{\phi\phi}\right]_{(s)}^{-1} \left[K^{\phi\mu}\right]_{(s)} \{\Delta \dot{q}^{(s)}\} = \\
\{\Delta P\} + \{\Delta F\} - [M]\{a_2 \{\dot{q}\} + a_3 \{\ddot{q}\}\},
\end{align*}
\]

(2.29)

where the subscript \(a\) and \(s\) refer to actuator and sensor quantities respectively.

Let the sensor and actuator patches are placed together in the same locations and have the same geometric and material properties or consider the case of “self-sensing” actuators. In this case the actuator and sensor degrees of freedom are identical and so the active damping term, (the second term on the left hand side (l.h.s.) in eq. (2.29)), takes the form

\[
[D] = \left[K^{u\phi}\right]_{(a)} \left[K^{\phi\phi}\right]_{(a)}^{-1} \left[K^{\phi\mu}\right]_{(a)}.
\]

(2.30)

Note that the matrix \([D]\) is symmetric.
• Actuation using a Charge Sensor

The negative velocity feedback voltages are based on rate of charge sensed. Thus the incremental voltage across the actuator patches takes the form

\[
\{\Delta \varphi\}_a = -G_c\left(\frac{\partial \{\Delta \sigma\}}{\partial t}\right) = G_c[K^\phi u_l]_s \{\Delta \dot{q}^{(s)}\}
\]  
(2.31)

so that the incremental equations read

\[
\{a_1[M] + [K^u] + [K^\sigma]\} \{\Delta q\} + G_c[K^{u\phi}]_a[K^\phi u_l]_s \{\Delta \dot{q}\}_s = \\
\{\Delta P\} + \{\Delta F\} - [M] \{a_2 \{\dot{q}\} + a_3 \{\ddot{q}\}\}.
\]
(2.32)

For a self-sensing actuator, the damping term (2\text{nd} on the l.h.s)

\[
[D] = G_c[K^{u\phi}]_a[K^\phi u_l]_a \{\Delta \dot{q}^{(a)}\}
\]  
(2.33)
Note the passive sensor terms, being sufficiently small have been neglected in eq. (2.29) and (2.32)

- **Simplifications in the Treatment of Piezo-electric Patches**

In the present work a few simplifying assumptions are made regarding the role of piezo-electric patches with a view to capture with facility the essentials of the response of the structure under control.

(i) The piezo-electric patches are sufficiently thin in comparison to the host plate
\[
\left( \frac{h}{t} \right) \ll 1 \text{ or beam } \left( \frac{h}{d} \right) \ll 1
\]
where \( h, t \) and \( d \) are respectively the thickness of the piezo-electric patch, plate thickness and depth of the beam respectively. Upper limit of this ratio is taken as 0.1.

(ii) The mass and stiffness contribution of the piezo-electric patches are neglected.

(iii) Identical piezo-electric patches are used in pairs and bonded to the top and bottom surfaces respectively of the host plate or beam. The voltages are assumed to be equal and opposite across these patches.

Assumption (i) makes it possible to assume a linear variation of electric potential across the patch as already mentioned.

With regard to assumption (ii), the inclusion of mass and stiffness contributions does not call for significant additional effort in formulation or computation, but neglecting those makes for significant simplicity in understanding and interpretation of the results. If this assumption is made, thickness \( h \) gets cancelled in the coupled electro-mechanical term and disappears from the calculation, leaving the formulation in terms of the voltage across the patch.
Consider the key electro-mechanical coupled term in eq. (2.34).

$$K_{ik}^{\mu \phi} = \int_V \{B_{mi}\}_t e_{rm} A_{rk} \, dv$$

(2.34)

Because of the constancy of the $B$ - matrix and the piezo-electric properties in the thickness direction in the foregoing integral, the integration over the thickness is tantamount to multiplication by $h$. Since the $A$-matrix involves thickness ($h$) of the patch in the denominator (eq. (2.34)), $h$ gets cancelled out. As a result it is possible to proceed with the analysis without knowledge a priori of the thickness of the patch. The thickness can be designed subsequently based on the voltage needed for control which can then be incorporated into the analysis using a layered plate theory taking the plate and the patches together or layered beam theory likewise.

Assumption (iii) is merely a control strategy used in the study. The proposed arrangement directly controls the bending deformation which is linear in the lateral displacement. This leads to control terms which are linear in applied voltages or if a feedback control scheme is employed, linear in the degrees of freedom or the rates thereof.

The examples considered in the following chapters will make the implications of these assumptions more explicit.
Chapter 3

Column under Conservative Loads

In this chapter, the problem of a simply supported column under axial compression and resting on a nonlinear elastic softening foundation is studied in some detail. This is an example of a structural element under conservative loading. Several pertinent questions are addressed, such as feasibility and practicality of enhancing the buckling load, the effectiveness of negative velocity feedback control and the relative advantages of piezoelectric actuators in the form of continuous and patch elements attached to the faces of the column. It is shown that while the buckling capacity of the column can be increased by a feedback voltage proportional to the strain at the top/bottom faces (or curvature) of the column, there is a premium price to pay in terms of energy that must continue to be supplied to sustain loads higher than the limit point load of the uncontrolled column.

The dynamic problem is studied by disturbing the compressed column by a suddenly applied pressure. As long the axial compression $P < P_d$ the dynamic instability load, the column settles always to its underlying static equilibrium configuration. However to enhance the dynamic instability load, the field strengths required are prohibitively high. This chapter concludes by pointing out the difficulties of controlling extremely short-wave modes, should they happen to govern the response of the column.
3.1. Description of problem

A column is simply supported and resting on a nonlinearly elastic and softening foundation as shown in Figure 3.1. Let $P_{cr}$ the critical load as computed by a linear stability analysis. In the presence of imperfections, the column buckles under static conditions at a value of $P (= P_s)$ smaller than $P_{cr}$. Under small dynamic disturbances such as a suddenly applied lateral pressure, the column experiences divergence form of instability at the dynamic buckling load $P_d < P_s$. As $P$ approaches $P_d$, the column experiences large amplitude vibrations. Effectiveness of Piezo-electric control in (i) enhancing the static buckling capacity, (ii) in damping out the large amplitude oscillations as $P \rightarrow P_d$, and (iii) increasing the dynamic buckling load will be investigated.

Figure 3.1. The simply supported column resting on an elastic foundation
3.2. Equations of Motion

Equations of motion are derived using virtual work principle. The internal virtual work due to elastic forces is numerically equal to the first variation of the strain energy which is the sum of bending energy of the host column and the energy stored in the foundation. To this we add the electro-mechanical contribution of the piezo-electric patches. The external virtual work is due to inertial forces, the prescribed axial load and lateral load.

- **Internal virtual work from the host column and foundation**

The bending energy of the Euler-Bernoulli column may be written as

\[
U_{beam} = \frac{E_I I_H}{2} \int_0^L \left( \frac{d^2 v}{dx^2} \right)^2 \, dx
\]  

(3.1)

where \(E_I\) and \(I_H\) are respectively the young’s modulus of the material of the host column and the moment of inertia of the cross-section with respect to the axis of bending. ‘\(v\)’ is the lateral displacement of the column.

Consider a nonlinearly elastic foundation characterized by the following force per unit length (\(f\)) versus displacement (\(v\)) relationship,

\[
f = K_1 v - K_3 v^3
\]  

(3.2)

where \(K_1\) and \(K_3\) are both positive with units of F/L^2 and F/L^4 respectively.

The strain energy stored in the elastic foundation is

\[
U_{foundation} = \frac{1}{2} K_1 \int_0^L v^2 \, dx - \frac{1}{4} K_3 \int_0^L v^4 \, dx
\]  

(3.3)
The internal virtual work of elastic forces is then

\[ \delta W_{\text{int}} = EI \int_0^L \frac{d^2 v}{dx^2} \frac{d^2 (\delta v)}{dx^2} \, dx + \int_0^L \left[ K_1 v - K_3 v^3 \right] (\delta v) \, dx \]  

(3.4)

- **External Virtual work**

The centroidal axis of the column does not stretch under small finite displacements. The end-shortening of the column may then be expressed in terms of the lateral deflection \( v \) and the initial imperfection \( v^o \) in the form

\[ \Delta = \frac{1}{2} \int_0^L \left( \frac{dv}{dx} \right)^2 \, dx + \int_0^L \left( \frac{dv^o}{dx} \right) \left( \frac{dv}{dx} \right) \, dx \]  

(3.5)

External virtual work of the axial load \( P \) takes the form

\[ \delta W_{\text{ext}}^{(P)} = P \int_0^L \frac{dv}{dx} \frac{d(\delta v)}{dx} \, dx - P \int_0^L \frac{dv^o}{dx} \frac{d(\delta v)}{dx} \, dx \]  

(3.6)

The contribution of lateral load of intensity \( q(x) \) on the column may be written as

\[ \delta W_{\text{ext}}^{(q)} = \int_0^L q(x) \cdot \delta v \, dx \]  

(3.7)

The contribution of inertial forces may be written in the form
\[ \delta W_{\text{ext}}^{(M)} = - \int_{0}^{L} (\bar{m} \ddot{v}) \delta v \, dx \]  

(3.8)

where \( \bar{m} \) is the mass per unit length of the column.

- **Piezo-electric Contribution:**

A beam of rectangular cross-section is considered; piezo-electric patches are attached to the top and bottom surfaces of the beam (as shown in Figure 3.2 and 3.3). Voltages are applied across the thickness and the electric field is deemed constant across the thickness; the voltages at the top and bottom patches are equal and opposite to be able to control the bending of the beam (Figure 3.4). Note \( t_p \) and \( b_p (= b) \) are respectively the thickness and width of the patch.

The electro-mechanical contribution of electric enthalpy density (eq. (1.35)) for the case of uniaxial stress, \( \sigma_x \), in the piezo-electric patch is given by

\[ \delta H^P = - \{ \delta \sigma_x \} e_p E_3 \]  

(3.9)

Noting that:

\[ \delta \sigma_x = - E_p \frac{h}{2} \left( \frac{d^2 v}{dx^2} \right) \]  

(3.10)

and

\[ E_3 = - \frac{V}{t_p} \]  

(3.11)
Considering patches at the top and bottom (with voltages reversed), the internal virtual work contribution of the piezo-electric patches is obtained by integration over the volume of the patch

\[
\delta W_{\text{int}}^p = 2 E_p e_p \frac{h}{2} b_p t_p \int_x V(x) \frac{d^2(\delta v)}{dx^2} \, dx
\]  

(3.12)

Note that \( V \) varies with \( x \).

Figure 3.2. Host column with continuous piezo-electric patches at the top and bottom surfaces
Figure 3.3. A typical arrangement of the host column with discrete piezo-electric actuators

Figure 3.4. Cross-section of the beam illustrating the senses of the electric field at the piezo-electric patches at top and bottom inducing respectively tension and compression
• **Virtual work equation**

Equating the internal virtual work to the external virtual work as explained in Chapter 2, we have

\[
\int_{x=0}^{L} \left\{ \bar{m} \dot{\bar{v}} \right\} \delta v \, dx + EI \int_{0}^{L} \frac{d^2 v}{dx^2} \frac{d^2 (\delta v)}{dx^2} \, dx - P \int_{0}^{L} \frac{dv}{dx} \frac{d(\delta v)}{dx} \, dx
\]

\[
+ \int_{0}^{L} \left[ K_1 v - K_3 v^3 \right] \delta v \, dx - P \int_{0}^{L} \frac{dv^o}{dx} \frac{d(\delta v)}{dx} \, dx
\]

\[
+ 2E_P e_P \frac{h}{2} b_P \int_{x}^{L} V(x) \frac{d^2 (\delta v)}{dx^2} \, dx = \int_{0}^{L} q(x) \cdot \delta v \, dx
\]

\[ (3.13) \]

• **Differential Equations of Motion in Time domain**

Let \( v \) be expressed in the form of a Fourier series, satisfying the boundary conditions

\[
v = \sum_{m=1}^{N} v_m \sin \left( \frac{m \pi x}{L} \right).
\]

\[ (3.14) \]

The initial imperfections and the lateral load are taken in similar forms

\[
v^o = \sum_{m=1}^{N} v_m^o \sin \left( \frac{m \pi x}{L} \right), \text{ and } q = \sum_{m=1}^{N} q_m \sin \left( \frac{m \pi x}{L} \right)
\]

\[ (3.15.a-b) \]
Let the voltage across the continuous patches $V$ be taken in the form

$$ V = \sum_{m=1}^{N} V_m \sin \left( \frac{m \pi x}{L} \right) $$  \hspace{1cm} (3.16)$$

For discrete patches, we take it to be constant over each patch, equal to $V_i$ for the $i^{th}$ patch. Virtual work equation, now takes the form

$$ \sum_{m=1}^{N} \frac{L^3}{2} \ddot{v}_m (\dot{\delta v}_m) + C_1 \sum_{m=1}^{N} \left\{ m^4 + \gamma_1 - \frac{P}{P_e} m^2 \right\} v_m (\dot{\delta v}_m) $$

$$ - C_1 \sum_{m=1}^{N} \left\{ \frac{P}{P_e} m^2 v_m^o + \frac{q_m L^2}{2C_1} \right\} (\delta v)_m $$

$$ - \frac{1}{3!} C_1 \sum_{m=1}^{N} \sum_{n=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \left\{ 12 \gamma_3 I_{mnpq} \right\} v_m v_n v_p v_q (\delta v)_m $$

$$ - \sum_{m=1}^{N} E_p e_p h b_p m^2 \pi^2 c_m (\dot{\delta v}_m) = 0 $$

(3.17)

where, $v_m = \frac{v_m}{L}$, $v_m^o = \frac{v_m^o}{L}$; $\gamma_1 = \frac{K_1 L^4}{\pi^4 E I}$, $\gamma_3 = \frac{K_3 L^6}{\pi^4 E I}$; $C_1 = \frac{\pi^4 E I}{2L}$, $P_e = \frac{\pi^2 E I}{L^2}$,

$$ I_{mnpq} = \frac{1}{L} \int_0^L \sin \left( \frac{m \pi x}{L} \right) \sin \left( \frac{n \pi x}{L} \right) \sin \left( \frac{p \pi x}{L} \right) \sin \left( \frac{q \pi x}{L} \right) \, dx $$

The last term represents the piezo-electric contribution. For the case of a continuous patch,

$$ c_m = \frac{V_m}{2} $$  \hspace{1cm} (3.18.a)
For a column carrying $n_p$ number of patches where the $i^{th}$ patch is of width $d_i$, centered at $x = c_i$, and carries a voltage $V_i$ we have

$$ c_m = \frac{1}{L} \sum_{i=1}^{n_p} V_i \int_{x=c_i-d_i/2}^{x=c_i+d_i/2} \sin \left( \frac{m \pi x}{L} \right) dx $$

(3.18.b)

Note that $\gamma_1$ and $\gamma_3$ are dimensionless stiffness parameters of the elastic foundation, and $P_E$ is the Euler critical load. The critical load under quasi-static conditions from the linear stability analysis (Thompson and Hunt, 1973 [54]) can be shown to be

$$ P_{cr} = P_E \left( m^2 + \frac{\gamma_1}{m^2} \right) $$

(3.19)

with $m$ appropriately chosen to minimize $P_{cr}$.

- **Control by direct feedback**

For direct feedback based on displacement and velocity, voltages are taken proportional to the locally sensed linear combination of strains and strain-rates. Thus for a continuous patch, we have

$$ V = \sum_{m=1}^{N} \left( -\frac{m^2 \pi^2}{L^2} \right) \frac{h}{2} \left( G_s v_m + G_d \dot{v}_m \right) \sin \left( \frac{m \pi x}{L} \right) $$

$$ V_m = \left( -\frac{m^2 \pi^2}{L^2} \right) \frac{h}{2} \left( G_s v_m + G_d \dot{v}_m \right) $$

(3.20.a-b)

where $G_s$ and $G_d$ are appropriately chosen static and dynamic gains.
For discrete patches the voltage is taken proportional to the strain–rate sensed at the middle of each patch and a typical \( V_i \) may then be written as

\[
V_i = \sum_{n=1}^{N} \left( -\frac{n^2 \pi^2}{L^2} \right) h \left( G_s v_n + G_d \dot{v}_n \right) \sin \left( \frac{n \pi c_i}{L} \right)
\]  
(3.21)

Substituting for \( V_i \) in eq. (3.21) to eq. (3.18b) and then in turn in eq.(3.17), it is seen that the piezo-electric effects are represented by a fully populated matrix in the case of discrete patches, in contrast to the continuous patch case.

The final form of the equations of motion is

\[
m_{ij} \ddot{v}_j + (a_{ij} - \tilde{p} b_{ij} + s^*_{ij}) \ddot{v}_j \ddot{v}_j + \frac{1}{3!} a_{ijkl} v_j v_k v_l = \tilde{p} b_{ij} \ddot{v}_j + q_i
\]  
(3.22)

\( (j,k,l = 1,\ldots,N); \ (i = 1,\ldots,N) \),

where \( m_{ij}, a_{ij}, b_{ij} \) are diagonal matrices and \( s^*_{ij}, d^*_{ij} \) are generally asymmetric and fully populated matrices and \( \tilde{p} = \frac{P}{P_{cr}} \).

### 3.3. Scope of Control Strategies

The following questions appear pertinent.

- Is it possible, by exercising piezo-electric control, to enhance the limit point load \( P_S \) (under static conditions) and the load at which there is the onset of dynamic instability, \( P_d \)?
• What kind of feedback control is needed or would be effective to achieve this -
displacement and/or velocity feedback?

• Would the voltage required significantly shoot up as the load approaches these
benchmark values?

Consider the first question. A quick answer is possible to this question if nonlinearities
and imperfections are negligible. (In this case, \( \frac{P_s}{P_{cr}} \rightarrow 1 \)).

For a quantitative answer to all the questions an in-depth parametric study may be needed.
In this chapter, we examine a few typical scenarios in a subsequent section.

Assume a perfect column carrying an axial compression. Let the parameter \( \gamma_1 \) be so
chosen that only single mode, consisting of \( m \) half-waves is dominant. Consider the case
of a continuous patch so that all the equations are uncoupled.

The governing differential equation for \( v_m \) takes the form:

\[
\ddot{v}_m + d_m^* \dot{v}_m + (1 - \bar{p} + s_m) v_m = 0 \quad \text{(no } \sum \text{ on } m) \quad (3.23)
\]

where \( \bar{m} = \frac{mL^3}{2C_m} \), \( d_m^* = m^4 \pi^4 \frac{E_p e G_d h^2 b_p}{C_m \frac{4L}{4}} \), \( s_m = m^4 \pi^4 \frac{E_p e G_d h^2 b_p}{C_m \frac{4L}{4}} \), \( \bar{p} = \frac{P}{P_{cr}} \), and \( C_m = \frac{1}{2} m^2 \pi^2 L P_{cr} \). Note that both \( d_m^* \) and \( s_m \geq 0 \).

Seeking the solution in the standard form \( X = A e^{\lambda t} \), the characteristic exponents are obtained as

\[
\lambda_{1,2} = \frac{-d_m^*}{2\bar{m}_m} \pm \frac{1}{2} \sqrt{\left( \frac{d_m^*}{\bar{m}_m} \right)^2 - \frac{4(1 - \bar{p} + s_m)}{\bar{m}_m}}. \quad (3.24)
\]
The following observations can be made.

(i) If \( s_m = 0 \), it is impossible to control the system once \( \tilde{p} > 1 \) whatever the value of \( d_m \), for then one of the roots (eq. (3.15)) will be positive.

(ii) If \( \frac{(d_m^*)^2}{4m_m} > (1 - \tilde{p} + s_m) > 0 \), the roots has no positive real part; system is asymptotically stable.

(iii) If \( (1 - \tilde{p} + s_m) > \frac{(d_m^*)^2}{4m_m} \), the roots are conjugate complex with a negative real part, the system is again asymptotically stable, but approaches equilibrium while performing oscillations of decreasing amplitude.

Thus it is possible, at least in theory, to enhance the critical load by exercising feedback control proportional to displacement by ensuring satisfaction of (ii) above.

Note that all this assumes that the nonlinearities are negligible. If control is exercised from the beginning as \( \tilde{p} \) increases from zero, the deflections tend to remain small so that nonlinearity has a relatively small influence. But in the presence of imperfections, the softening nonlinearity and adverse interaction of several modes, instability will occur at \( \tilde{p} = p_s = \frac{P_s}{P_{cr}} < 1 \) in the absence of control (In the presence of dynamic disturbances, this will be further reduced, i.e. \( \tilde{p} = p_d = \frac{P_d}{P_{cr}} \), and the foregoing conclusions (ii) and (iii) will be in need of modification. In fact, it will be found necessary to set \( s_m \) to be a sizable fraction to ensure stable behavior as the load approaches \( P_{cr} \).
3.4. Examples and Discussion

The following is the list of pertinent data for the examples studied.

- **Geometry:**
  - Length of the column, \( L = 1000 \text{ mm} \);
  - Cross-section: width, \( b = L/20 \); thickness = \( 0.4b \); width of piezo-electric patch \( b_p = b \)

- **Material properties:**
  - \( E_H \) (of host beam) = \( E_P \) (of piezo-electric material) = \( E = 63 \text{ GPa} \),
  - Piezoelectric constant, \( E_P^e = E_p e = 0.0283 \)
  - Safe operating electric field strength = 600 V/mm.

- **Foundation stiffness parameters:**
  - \( \gamma_1 = 4 \) (modes with \( m = 1 \) and \( m = 2 \) are coincident) or 144 (modes with \( m = 3 \) and \( m = 4 \) are coincident); \( \gamma_3 = 20000 \) in all the examples.
  - Combined mass density of the column material: \( 2700 \text{ kg/m}^3 \)

- **Imperfection in \( m^{th} \) mode (with \( m \) half waves) = \( \frac{0.5}{m} \text{ mm} \).

- **Lateral loading components:** \( q_m = \frac{0.01P_{cr}}{mL} \)

- **Loading sequence for the determination of \( P_d \):**
  - The following sequence of loading is adopted to investigate dynamic instability. First the column is subjected to a given static compression given by \( \tilde{p} = P/P_{cr} \). A suddenly applied lateral load of intensity, \( q \) (directed in the positive direction) is
applied and the column response studied. The value of \( p \) at which column experiences divergence gives the dynamic buckling load, i.e. \( \bar{p} = \frac{P_d}{P_{cr}} \).

3.4.1. Two Modes, Continuous patch problem

First consider the case with \( \gamma_1 = 4 \). This leads to coincident buckling, for the modes with \( m = 1 \) and 2. Only these two modes are considered to be active in the first set of examples. The patches are assumed to exist from end to end. We take \( s_1 = s_2 > 0 \), so that in the linear case \( (\gamma_3 = 0) \) in the absence of imperfections, the maximum load that can be attained would be \( (1 + s_1)P_{cr} \).

3.4.2. Enhancement of \( P_s \)

First consider static problem of the imperfect uncontrolled column with \( \gamma_3 = 20,000 \). The maximum load \( (P_s) \) that is attainable under these conditions is \( 0.84P_{cr} \). The maximum loads that are attainable with increasing values of \( s_1 \) and the corresponding voltage amplitudes are given in Table 3.1. The voltages are equivalent to field strength over a millimeter (mm) of piezo-electric material.

It is seen that while it is possible to increase the maximum load beyond \( P_s \) by displacement feedback, the voltages required are enormous. For an increase of 10% beyond \( P_s \) (case with \( s_1 = s_2 = 0.1 \)) the maximum voltage amplitudes take values respectively of 1346 and 538.5 (vide Table 3.1). This would call for a piezo-electric patch of thickness greater than 2 mm. This is certainly not an attractive proposition unless piezo-electric materials with significantly increased ‘\( e_p \)’ values become available, in which case it is possible to reduce the patch thickness. The piezo-control does result in an increased stiffness at load levels smaller than \( P_s \).
Table 3.1. Maximum load and corresponding maximum voltage (volts)

<table>
<thead>
<tr>
<th>$s_1$</th>
<th>$P_{max}/P_{cr}$</th>
<th>$V_1$(max)</th>
<th>$V_2$(max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.85</td>
<td>128.8</td>
<td>51.9</td>
</tr>
<tr>
<td>0.10</td>
<td>0.93</td>
<td>1346</td>
<td>538.5</td>
</tr>
<tr>
<td>0.20</td>
<td>1.02</td>
<td>2925</td>
<td>1137</td>
</tr>
</tbody>
</table>

3.4.3. Enhancement of $P_d$

Let us next consider the control of dynamic response. As mentioned already, the analysis consists of two steps: an initial static compression of the column followed by the application of $q$. In all the calculations we set $q_1L = 0.01P_{cr}$ and $q_2 = q_1/2$. A question of interest is whether under appropriate control, the column can be made capable of carrying loads higher than $P_d$; also of interest is the effectiveness of piezo-electric control below $P_d$ in damping out the oscillations. For the imperfect uncontrolled column with $\gamma_3 = 20,000$, we have $P_d/P_{cr} = 0.60$.

Case 1: Let us consider the case $s_1 = s_2 = 0$. (Note that the values $d_1$ and $d_2$ differ.) Using feedback velocity control, i.e., we consider various values for $G_d$. Thus voltage distribution is the gain multiplied by strain rate and the constant value of $G_d$ results in differing values of $d_m$. Table 3.2 gives the values of the loads beyond which dynamic instability occurs. It is clear that increase in dynamic instability load under purely velocity feedback is marginal and the field strengths needed increase rather sharply with the increase in the $P_{max}$. 
Table 3.2. Load capacity for gain and corresponding maximum voltages

<table>
<thead>
<tr>
<th>$G_d$</th>
<th>$P_{\text{max}}/P_{\text{cr}}$</th>
<th>$V_1(\text{max})$</th>
<th>$V_2(\text{max})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2500</td>
<td>0.61</td>
<td>240.43</td>
<td>173.7</td>
</tr>
<tr>
<td>5000</td>
<td>0.62</td>
<td>473.97</td>
<td>272.59</td>
</tr>
<tr>
<td>10000</td>
<td>0.63</td>
<td>867.12</td>
<td>363.67</td>
</tr>
</tbody>
</table>

**Case 2**: Consider next the case where both the displacement and velocity feedback are applied in concert. We take $s_1 = s_2 = 0.1$, and $G_d = 10000$ (resulting in differing values of $d_m$).

Table 3.3. Maximum and final voltages for various load cases

<table>
<thead>
<tr>
<th>$P/P_{\text{cr}}$</th>
<th>$V_1(\text{max})$</th>
<th>$V_2(\text{max})$</th>
<th>$V_1(\text{final})$</th>
<th>$V_2(\text{final})$</th>
<th>Settling time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>1560</td>
<td>429</td>
<td>791</td>
<td>155</td>
<td>0.12 sec</td>
</tr>
<tr>
<td>0.63</td>
<td>2004</td>
<td>497</td>
<td>1104</td>
<td>228</td>
<td>0.12 sec</td>
</tr>
<tr>
<td>0.65</td>
<td>2107</td>
<td>511</td>
<td>1177</td>
<td>244</td>
<td>0.12 sec</td>
</tr>
<tr>
<td>0.70</td>
<td>2467</td>
<td>550</td>
<td>1423</td>
<td>293</td>
<td>0.12 sec</td>
</tr>
<tr>
<td>0.72</td>
<td>2715</td>
<td>568</td>
<td>1570</td>
<td>319</td>
<td>0.12 sec</td>
</tr>
<tr>
<td>0.73</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>On-set of dynamic instability</td>
</tr>
</tbody>
</table>

Table 3.3 gives values of peak voltages that develop under increasing values of axial load until the onset of dynamic instability. Also are given the settling times and the corresponding voltages. The settling time is that time beyond which the variations in displacements are smaller than 1% of the peak value. It is seen that the dynamic stability load is enhanced by 20% by the application of the control strategy. But the voltages or
the field strengths required increase rather sharply for loads in excess of $P_d$. This may not be a feasible proposition as the piezo-electric material thickness could be of the order of 4 mm in a column of 20 mm thickness. It is possible to reduce $G_d$ to a much smaller value (e.g. 2500) and control the column with an increased settling time, but the final voltages would not change as they are determined by the value of $s_1$ chosen. Figure 3.5(a-b) give respectively the displacement, velocity/voltage histories for mode I and II for the case with $G_d = 2500$.

(a) *Displacement (non-dimensional) time history*

(b) *Voltage time history*

Figure 3.5. Time history of $P = 0.5 \, P_{cr}$ with $G_d = 2500$
**Case 3**: Consider next a case where the priority is not to enhance the buckling load but to damp out the oscillations within a stipulated time with minimal energy consumption. This issue is approached not as a formal optimization problem but examining at once the settling times and maximum voltages needed for various values of gain, $G_d$. Since it is not our objective to enhance the critical load, we may set $s_m = 0$ ($m = 1, 2$) and depend solely on $d_a^*$ to control the vibrations. Table 3.4 gives the maximum voltages developed and the settling times (time beyond which the amplitude of oscillations are less than 1% of the initial maximum). It turns out that the settling times depend directly on the gain and not on $P/P_{cr}$. It is seen that effective control is feasible with voltages that can be sustained by piezoelectric material of thickness which is a fraction of a millimeter, by an appropriate choice of the gain. Figure 3.6 (a-b) show the variation of the voltage distribution along the length of the column at times when $V_1$ and $V_2$ attain their maximum values respectively. It is clear that the voltage distribution continues to change until the oscillations significantly die down, when they approach zero.

<table>
<thead>
<tr>
<th>$P/P_{cr}$</th>
<th>$G_d$</th>
<th>$V_1$(max)</th>
<th>$V_2$(max)</th>
<th>Settling time</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50</td>
<td>10,000</td>
<td>744.8</td>
<td>340.9</td>
<td>0.12 sec</td>
</tr>
<tr>
<td></td>
<td>5000</td>
<td>405.8</td>
<td>250.06</td>
<td>0.23 sec</td>
</tr>
<tr>
<td></td>
<td>2500</td>
<td>212.38</td>
<td>161.07</td>
<td>0.45 sec</td>
</tr>
<tr>
<td>0.55</td>
<td>10,000</td>
<td>783.4</td>
<td>348.75</td>
<td>0.12 sec</td>
</tr>
<tr>
<td></td>
<td>5,000</td>
<td>429.3</td>
<td>258.5</td>
<td>0.23 sec</td>
</tr>
<tr>
<td></td>
<td>2,500</td>
<td>225.5</td>
<td>168</td>
<td>0.45 sec</td>
</tr>
<tr>
<td>0.60</td>
<td>10,000</td>
<td>831.45</td>
<td>357.59</td>
<td>0.12 sec</td>
</tr>
<tr>
<td></td>
<td>5,000</td>
<td>459.17</td>
<td>268.05</td>
<td>0.23 sec</td>
</tr>
<tr>
<td></td>
<td>2,500</td>
<td>242.5</td>
<td>176.24</td>
<td>0.45 sec</td>
</tr>
<tr>
<td>0.63</td>
<td>10,000</td>
<td>867.12</td>
<td>363.67</td>
<td>0.12 sec</td>
</tr>
<tr>
<td></td>
<td>5,000</td>
<td></td>
<td>Onset of dynamic instability</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2,500</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
(a) Voltage distribution when \( V_1 \) is maximum (\( t = 3.7 \times 10^{-3} \) sec.)

(b) Voltage distribution when \( V_2 \) is max. (\( t = 1.6 \times 10^{-3} \) sec.)

Figure 3.6. Voltage distributions along the length
3.4.4. Comparison with 10 Mode Solutions (Continuous Patch)

In order to examine the accuracy of the solution and the voltage distribution in the context of nonlinear interaction with higher harmonics, the response of the column is investigated under the following conditions: We include up to 10 harmonics \( (N = 10) \).

Imperfections are assumed in the form as mentioned earlier:

\[
v^o = \sum_{m=1}^{N} v_m^o \sin \left( \frac{m \pi x}{L} \right), \quad v_m^o = \frac{v_1^o}{m} \quad \text{with} \quad v_1^o = 0.5 \quad \text{as before.}
\]

The lateral load is taken in the form as \( q = \sum_{m=1}^{N} q_m \sin \left( \frac{m \pi x}{L} \right) \) with \( q_m = \frac{q_1}{m} \) and \( q_1 L = 0.01 P_{cr} \) as before. The column response was studied under the following conditions that \( \frac{P}{P_{cr}} = 0.5 \) with \( G_d = 2500 \).

![Graph showing displacement history](image)

**Figure 3.7.** Displacement (non-dimensional) time history for \( \bar{v}_1 \) and \( \bar{v}_2 \) with \( N = 10 \)

The displacement history of the first two modal amplitudes as obtained using \( N = 10 \) are shown in Figure 3.7. These are remarkably close to those shown in Figure 3.5 (a-b)
obtained by just considering two modes, i.e. taking $N = 2$. Time history of maximum voltages and voltage distributions for $N = 2$ and $N = 10$, also bear the remarkable resemblance to each other (not shown). The settling time was found to be 0.45 sec in both analyses.

### 3.4.5. Control with Discrete Patches

As mentioned earlier, we consider discrete patches actuated by constant voltages over their length. This makes for simplicity and facility in implementation.

Details of the examples studied are as follows. As before, $\gamma_i = 4$, leading to coincident buckling with modes corresponding to $m = 1$ and 2 (uncontrolled column) and $\gamma_3 = 20,000$. The number participating harmonics ($N$) is set to 10. Imperfection and lateral load parameters ($\psi_m$, $q_m$) are taken as the same as in the 10-mode example case. Two levels of axial loads are considered, viz. $P/P_{cr} = 0.5$ and 0.6. The latter load pushes the column close to dynamic instability.

The following two examples are studied.

**Example I:** The column has three pairs of patches, a central pair with two other pair patches symmetrically located with respect to the center. The central pair has twice the width and twice the gain of the off-center ones. Two cases are considered with differing patch widths. (Refer Table 3.5(a-b).)

<table>
<thead>
<tr>
<th>Patch location ($c_i$)</th>
<th>Patch width ($d_i$)</th>
<th>Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.1</td>
<td>2000</td>
</tr>
<tr>
<td>0.50</td>
<td>0.2</td>
<td>4000</td>
</tr>
<tr>
<td>0.75</td>
<td>0.1</td>
<td>2000</td>
</tr>
</tbody>
</table>
Table 3.5. (b) Example I: 3 patch case, Case (ii)

<table>
<thead>
<tr>
<th>Patch location ((c_i))</th>
<th>Patch width ((d_i))</th>
<th>Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>0.08</td>
<td>2500</td>
</tr>
<tr>
<td>0.50</td>
<td>0.16</td>
<td>5000</td>
</tr>
<tr>
<td>0.75</td>
<td>0.08</td>
<td>2500</td>
</tr>
</tbody>
</table>

Example II: The column has 4 pairs of patches, symmetrically located with respect to the center; all the patches are of the same width and actuated by the same gain. Once again two cases are considered with differing patch widths. (Refer Table 3.5(c-d).)

Table 3.5 (c) Example II: 4 – patch case, Case (i)

<table>
<thead>
<tr>
<th>Patch location ((c_i))</th>
<th>Patch width ((d_i))</th>
<th>Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.1</td>
<td>2400</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1</td>
<td>2400</td>
</tr>
<tr>
<td>0.6</td>
<td>0.1</td>
<td>2400</td>
</tr>
<tr>
<td>0.8</td>
<td>0.1</td>
<td>2400</td>
</tr>
</tbody>
</table>

Table 3.5 (d) Example II: 4 patch case, Case (ii)

<table>
<thead>
<tr>
<th>Patch location ((c_i))</th>
<th>Patch width ((d_i))</th>
<th>Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.08</td>
<td>3000</td>
</tr>
<tr>
<td>0.4</td>
<td>0.08</td>
<td>3000</td>
</tr>
<tr>
<td>0.6</td>
<td>0.08</td>
<td>3000</td>
</tr>
<tr>
<td>0.8</td>
<td>0.08</td>
<td>3000</td>
</tr>
</tbody>
</table>
Table 3.5 (a-d) gives all the necessary details. 10 modes were considered in the simulations, with imperfection and lateral pressure Fourier coefficients assumed in the same manner as in the previous example dealing with continuous patches. Maximum voltages developed in the patches as well as the settling times are studied in comparison for the four cases. These results will be compared also with the continuous (end-to-end) patch case with similar gains.

Table 3.6. (a) Results of Example I, Case (i)

<table>
<thead>
<tr>
<th>$P / P_{cr} = 0.5$</th>
<th>$P / P_{cr} = 0.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V^{(1)}$</td>
<td>$V^{(2)}$</td>
</tr>
<tr>
<td>242</td>
<td>352</td>
</tr>
</tbody>
</table>

Settling time = 0.41 sec.  Settling time = 0.45 sec.

Table 3.6 (b) Results of Example I, Case (ii)

<table>
<thead>
<tr>
<th>$P / P_{cr} = 0.5$</th>
<th>$P / P_{cr} = 0.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V^{(1)}$</td>
<td>$V^{(2)}$</td>
</tr>
<tr>
<td>306</td>
<td>443</td>
</tr>
</tbody>
</table>

Settling time = 0.44 sec.  Settling time = 0.47 sec.

Table 3.6 (c) Results of Example II, Case (i)

<table>
<thead>
<tr>
<th>$P / P_{cr} = 0.5$</th>
<th>$P / P_{cr} = 0.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V^{(1)}$</td>
<td>$V^{(2)}$</td>
</tr>
<tr>
<td>244</td>
<td>295</td>
</tr>
</tbody>
</table>

Settling time = 0.66 sec  Settling time = 0.70 sec.
Table 3.6 (d) Results of Example II, Case (ii)

<table>
<thead>
<tr>
<th></th>
<th>$P / P_{cr} = 0.5$</th>
<th></th>
<th>$P / P_{cr} = 0.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V^{(1)}$</td>
<td>303</td>
<td>$V^{(1)}$</td>
<td>315</td>
</tr>
<tr>
<td>$V^{(2)}$</td>
<td>369</td>
<td>$V^{(2)}$</td>
<td>404</td>
</tr>
<tr>
<td>$V^{(3)}$</td>
<td>312</td>
<td>$V^{(3)}$</td>
<td>353</td>
</tr>
<tr>
<td>$V^{(4)}$</td>
<td>251</td>
<td>$V^{(4)}$</td>
<td>272</td>
</tr>
<tr>
<td></td>
<td>Settling time = 0.58 sec</td>
<td></td>
<td>Settling time = 0.75 sec</td>
</tr>
</tbody>
</table>

Table 3.6 (a-b) summarizes the results for example I. It shows the maximum voltages $V_1$, $V_2$, $V_3$ for the patches 1, 2 and 3 respectively for the cases (i) and (ii) for two levels of $P/P_{cr}$, viz. 0.5 and 0.6. Table 3.6 (c-d) does the same for example II which is the 4-patch case. The patches of greater width and smaller gain (Cases (i)) are seen to lead to solutions which can be implemented without stretching the limits of current technology; field strengths (Voltage/mm) developed are moderate, making it feasible to use piezoelectric patches of about 0.5 mm thickness. Even though a larger settling time is required, but this is still less than 1 sec. which is acceptable.

![Figure 3.8](image-url)

Figure 3.8. Voltage time history for 3 patch case ($P = 0.5 P_{cr}$, case (i))
Figure 3.8 and Figure 3.9 give respectively voltage time histories of the patches for the case (i) for examples I and II respectively. No significant diminution of performance is seen in comparison to the continuous patch case. Voltages and settling times are similar for similar gains in the two cases. Since the patch gains are proportional to the strain rates at the midpoint of the patch, the modes that have nodes coinciding with these points cannot be controlled. Thus modes with \( m = 4 \) and 8 are uncontrollable in Example I and those with \( m = 5 \) and 10 are uncontrollable in Example II. However the corresponding displacements, (though not the velocities) remain small and are probably of little concern.

3.4.6. **Failure of Patch Control**

Consider the case with \( \gamma_1 = 144 \) (and \( \gamma_3 = 20000 \) as before). This results in simultaneous buckling in two modes, with \( m = 3 \) and \( m = 4 \) respectively. It turns out that 3-patch control is incapable of controlling the mode with \( m = 4 \), as its nodes are located at the midpoints of the three patches. Similar situation exists for all the modes with \( m \) equal to multiples of 4, but these are higher order modes which are not directly triggered. In so far as the imperfections and load components are likely to be relatively small for these higher modes, they play a relatively minor role in the dynamics of the problem.
An example was run using the same parameters as the case (i) with 3-patches.

Figure 3.10 (a – j) give time-histories of the amplitudes of the 10 modes considered in the analysis. As expected modes 4 and 8 are completely uncontrollable. Further because of nonlinear mode interaction represented by the quartic terms $a_{ijkl}$, certain modes such as 1, 7 and 9 are not damped out easily. Because of the high value of imperfection and the component of lateral load and the presence of the nonlinear term $a_{4411}$, the mode 1 tends to play a significant role, takes a longer time (0.5 sec versus 0.1 sec for mode 2) to get damped out. Because of the non-vanishing quartic terms $a_{4471}, a_{4491}$, modes 7 and 9 too tend to be triggered and sustained by the modes 1 and 4.
(c) mode 5 and 6

(d) mode 7 and 8

(e) mode 9 and 10

Figure 3.10. Displacement (nondimensional) time history of Case(i): $P = 0.5 \cdot P_{cr}$
3.5. Conclusions

The control strategies of an imperfect column carrying axial compression and supported on a nonlinear softening foundation are investigated. The initial stiffness of the foundation is so prescribed as to cause simultaneous buckling in the two lowest critical loads. In the first part of the investigation, piezoelectric patches are taken to be continuous from end to end and attached to the top and bottom faces of the column. In subsequently examples, control is exercised by discrete piezoelectric patches of specified widths at specified locations.

It is shown that it is possible to enhance the buckling capacity of the column by negative feedback control proportional to the displacement, but such an increase comes with a steeply rising field strengths and consumption of energy.

The column response is studied in a dynamic context by the application of a suddenly applied small lateral pressure. It was found the dynamic instability load can be enhanced by about 20% by a combination of negative feedback proportional to the displacement and velocity. However such enhancements require inordinately high field strengths or alternatively warrant piezoelectric patch actuators to be heavy. It was found that feedback velocity control by itself was not effective in enhancing the dynamic instability load. A 5% increase was achieved in the example studied, but this was at the expense of considerably high field strengths on the piezoelectric material. However below the dynamic instability load, the velocity feedback control was very effective in damping out the vibrations with voltages that are less than the capacity of a 1mm thick patch.

Feedback control with discrete patches proved to be equally successful while operating under dynamic instability load in damping out the oscillations. In comparison to continuous patch actuators, the voltage increases are seen to be marginal for more or less the same settling times. These would probably be more practical in that the patches are actuated by voltages which do not vary spatially and are proportional to the respective strains sensed locally.
One of the problems associated with patch control is that there are always extremely localized modes that can become uncontrollable; this can potentially lead to structural failure if these modes happen to be the principal ones governing the structural response.
Chapter 4

Column under Non-conservative Loads

A cantilever column carrying a compressive follower force and propped by a spring can become dynamically unstable by flutter or divergence depending upon spring stiffness. In this chapter issues involved in the control of such a column by means of piezoelectric patches at the top and bottom surfaces of the column are studied. Attention is focused on negative feedback control with patch voltages proportional to the locally sensed bending strain-rates. It is found such control is capable of significantly enhancing the critical load in the flutter range, such enhancement being limited often only by the limits on voltage developed across the patch of given thickness. In the range of spring stiffness corresponding to buckling failure, this control strategy is simply ineffective. A relatively ‘light’ spring not only enhances the critical force significantly but also makes the control more effective. There is an optimal spring stiffness which corresponds to the maximum of critical load consistent with the limits on voltages and this lies in the flutter range of the spring stiffness. Softening nonlinearity reduces the critical loads in the range of transition from flutter to divergence and rounds off the sharpness of the drop in this range seen in the linear case.

It will be shown that a partial patch spanning over half the length of the column from the fixed end is significantly more effective than a full patch in the flutter range. Patches spread out in the column from one end to the other were found to be poor in their performance or simply unable to control the column.
4.1. Description of Problem

The cantilevered beam is fixed at one end \((x = 0)\) while the other end \((x = L)\) it is free to rotate but constrained transversely by a spring. It is acted upon by a follower force, \(P\) as shown in Figure 4.1. (The follower force by definition remains tangential to the beam as the beam deflects while the axial force always remains axial no matter what the deflection is.) Instability of such a cantilever is by flutter [7]. In the present context may be defined as flutter is a phenomenon where the amplitude of vibration of the beam due to an initial disturbance grows without limit [14]. When there is no spring attached to the free end, the critical value of the follower compression force is

\[
p = \frac{\pi^2 EI}{(0.699L)^2} \approx 2.045\pi^2 \frac{EI}{L^2} \quad [16],[17].
\]

If there is a spring attached to the free end as shown in Figure 4.1, the cantilever will undergo either buckling or flutter at a certain value of the follower force. The experienced behavior will be dependent on the spring stiffness. The cantilever will flutter when the spring stiffness is small, while it will buckle with a sufficiently stiff spring. Softening nonlinearity in the spring will precipitate flutter earlier for a given initial disturbance. Piezoelectric feedback control will be applied to suppress flutter.

![Figure 4.1](image)

Figure 4.1. The Cantilever column with a spring under follower force together with a cross-section of the column
Also the enhancement of its flutter or buckling capacity, which is one of the advantages of piezoelectricity, will be examined. It is well known that damping, if it is sufficiently small may actually reduce the critical values of the follower force [12]. Since piezoelectricity may be considered as a form of damping, this effect will be examined and discussed in this chapter. Some limitations of piezoelectric control will also be pointed out.

### 4.2. Solution Methodology

We invoke the familiar assumptions of the Euler Bernoulli beam theory which neglects shear deformation. Since we consider a shallow beam, the rotary inertia is neglected. Assuming the displacements be small but finite, the centroidal axis of the column does not stretch.

Figure 4.1 shows the column AB clamped at A (x = 0) and supported by a spring at B (x = L). The deformation of the column under foregoing assumptions is defined by \( \nu \), the lateral displacement of the column. The spring is deemed to be elastic and nonlinear and has the following force displacement relationship.

\[
F = K_1 \delta + K_3 \delta^3 \quad \text{where} \quad \delta = \nu \bigg|_{x=L} \quad (4.1)
\]

In order to facilitate further discussion, the following non-dimensional parameters are introduced.

\[
p = \frac{P}{P_b} \quad ; \quad \beta_1 = \frac{K_1}{K_o} \quad ; \quad \beta_3 = \frac{K_3 L^2}{K_o} \quad (4.2)
\]

where \( P_b = \frac{\pi^2 EI}{L^2} \) and \( K_o = \frac{EI}{L^3} \).
The equation of motion for the beam is written using the principle of virtual work.

The equation of motion is derived by the principle of virtual work theorem. The virtual work equation takes the same form as in Chapter 3, except for the following changes:

- The contribution from the foundation is replaced by that of the spring at $x = L$.
- The follower force adds a stabilizing term evaluated at $x = L$.
- Initial imperfections are not considered.

Equating the internal virtual work to the external virtual work, we have

$$\int_{x=0}^{L} \left[ \frac{d^2v}{dx^2} \delta v \right] dx + EI \int_{0}^{L} \frac{d^2v}{dx^2} \frac{d^2(\delta v)}{dx^2} dx - P \int_{0}^{L} \frac{dv}{dx} \frac{d(\delta v)}{dx} dx$$

$$+ \frac{EI}{L^2} \left[ \beta_1 \left( \frac{v}{L} \right) + \beta_3 \left( \frac{v}{L} \right)^3 \right] (\delta v) \bigg|_{x=L} + P \left( \frac{dv}{dx} (\delta v) \right) \bigg|_{x=L}$$

$$+ 2E_p e_p \frac{h}{2} b_p \int_{x} V(x) \frac{d^2(\delta v)}{dx^2} dx = \int_{0}^{L} q(x) \cdot \delta v \ dx$$

where $\beta_1$ and $\beta_3$ are dimensionless quantities that $\beta_1 = K_1 \frac{L^3}{EI}$ and $\beta_3 = K_3 \frac{L^5}{EI}$.

Selecting appropriate shape functions for $v$ for the cantilever problem, we have

$$v = v_i \phi_i(x) \quad (i = 1, \ldots N) \quad (4.4)$$

The voltage distribution is taken in the form

$$V = V_i \psi_i(x) \quad (i = 1, \ldots N) \quad (4.5)$$

The lateral load is considered to be uniformly distributed and equal to $q_o$ per unit length.
Thus the equations of motion take the form

\[ M_{ij} \ddot{v}_j + \left[ A_{ij} - PB_{ij} + PC_{ij} \right] v_j + A_{ijkl} v_j v_k v_l + E_{ij} V_j = q_i \]  \hspace{1cm} (4.6)

where

\[ M_{ij} = \int_0^L \phi_i \phi_j \, dx \]
\[ A_{ij} = EI \int_0^L \phi_i^* \phi_j^* \, dx + \frac{EI}{L^2} \beta_i \left( \phi_i \phi_j \right)_{x=L} \]
\[ B_{ij} = \int_0^L \phi_i \phi_j' \, dx \]
\[ C_{ij} = \left( \phi_i \phi_j' \right)_{x=L} \]  \hspace{1cm} (4.7.a-g)
\[ E_{ij} = 2E_p e_p \frac{h}{2} b_p \int_0^L \phi_i^* \psi_j \, dx \]
\[ q_i = q_o \int_0^L \phi_i \, dx \]
\[ A_{ijkl} = \frac{EI}{L^5} \beta_3 \left( \phi_i \phi_j \phi_k \phi_l \right)_{x=L} \]  \hspace{1cm} (i, j, k, l = 1, ..., N)

Note that the matrix \( C_{ij} \) is asymmetric. The shape functions \( \phi_m(x) \) are selected to be

\[ \phi_m(x) = \frac{1}{2} \left( 1 - \cos \left( \frac{m \pi x}{L} \right) \right) \]  \hspace{1cm} (m = 1, 3, ..., N)  \hspace{1cm} (4.8)

Thus we have

\[ \varepsilon \left( z = \pm \frac{h}{2} \right) = \frac{h}{2} \frac{d^2 v}{dx^2} = \frac{h}{2} v_i \phi_i'' = \frac{h}{2} \sum_{i=1,3}^N v_i \left( \frac{i \pi}{2L} \right)^2 \cos \left( \frac{i \pi x}{2L} \right) \]  \hspace{1cm} (4.9)

Letting the feedback voltage be proportional to the flexural strain rate, we have

\[ V = \frac{G h}{2} \frac{d^2 \dot{v}}{dx^2} = \frac{G h}{2} \sum m \left( \frac{m \pi}{2L} \right)^2 \cdot \cos \left( \frac{m \pi x}{2L} \right) \]  \hspace{1cm} (4.10)
where $G$ is the gain. Thus,

$$V = \sum_{m=1}^{N} V_m \cos\left(\frac{m\pi x}{2L}\right)$$  \hspace{1cm} (4.11)

where

$$V_m = \frac{h}{2} G \left(\frac{m\pi}{2L}\right)^2 \dot{V}_{m}$$  \hspace{1cm} (no sum on $m$) \hspace{1cm} (4.12)

Then the piezoelectric term $E_y V_j$ on the l.h.s. of eq. (4.6) can be written (suspending the index notation) in the form

$$E_y V_j = 2 E_p e_p G b_p \left(\frac{h}{2}\right)^2 \left(\frac{i\pi}{2L}\right)^2 \sum_{j=1,3,\ldots}^{N} \left(\frac{j\pi}{2L}\right)^2 \dot{V}_j \int_{z} \cos\left(\frac{i\pi x}{2L}\right) \cos\left(\frac{j\pi x}{2L}\right) dx$$  \hspace{1cm} (4.13)

which may be abbreviated as $D_{ij} \dot{V}_{ij}$.

Note if the patches run from end to end $D$-matrix is diagonal. The final equations take the form

$$M_{ij} \ddot{V}_j + D_{ij} \dot{V}_j + [A_{ij} - PB_{ij} + PC_{ij}] V_j + A_{ijkl} v_j v_k v_l = q_i$$  \hspace{1cm} (4.14)

Note that the piezo-electric term (eq. (4.13)) in the governing equations may be viewed as controlled by a single system parameter $E_p e_p G$, i.e. the system response to given external loading is unaffected as long this is kept constant. Voltages, of course, will scale according to $G$. The eq. (4.14) is solved in time-domain taking sufficiently small time increments by a combination of Newmark $\beta$ - method [53] and Newton-Raphson iterative solution [10] of nonlinear equations (similar to chapter 3).
The critical values of the follower force can be obtained incrementing $P$ in small steps and solving the linear Eigen value problem obtained by taking $v_i = A_i e^{\lambda t}$. The lowest critical values of $P$ obtained there from may correspond to flutter (a root for $\lambda$ has a positive real part and a non-zero imaginary part) or divergence (a root is real and
positive). These results obtained for a spectrum of spring stiffness $\beta_1$ ranging from 0 to 140 are shown in Figure 4.2(b). These results are in very close agreement with those obtained by ref. [57].

There are three ranges of spring stiffness each of which corresponds to a characteristic mode of column response viz. flutter range $(0 < \beta_1 < 36)$, transition range $(36 < \beta_1 < 40)$, and divergence (buckling) range $(\beta_1 > 40)$. The remarkable features of the $p$ versus $\beta_1$ relationship (Figure 4.2 (b)) are the abruptness of the transition and the drop in the critical load with increasing spring stiffness.

### 4.3. Validation by reference

The formulation has been checked and validated by comparing results from Wang and Quek [56]. As shown in Figure 4.2 and Figure 4.3, our formulation (vide eq. (4.8 – 14)) gives exactly same result as Wang’s paper, which used the exact method.

We can observe that there is a transition range of spring stiffness below which the cantilever fails by flutter and above which it buckles. For $\beta_1 < 36$, flutter is the mode of instability. (It is seen that the lowest two frequencies approach each other without passing through zero [56]). Beyond the transition range $\beta_1 > 40$, the critical load is still affected by “follower” nature of the compressive force though the mode of instability is pure buckling. As $\beta_1$ increases further $P_{cr}/P_E$ approaches asymptotically the value corresponding to rigid support at the spring end.

Figure 4.3 is the example of the case that the column flutters but does not buckle. The transition range from flutter to buckling is in between non-dimensional spring stiffness of 36 to 40. The critical limits of the cantilever for non-dimensional linear spring coefficients are shown in Table 4.1.
(a) from Wang’s paper \cite{56}

(b) from our formulation

Figure 4.3. Non-dimensional frequencies vs. spring coefficient
Table 4.1. Relationship of non-dimensional linear spring coefficient and its critical limit

<table>
<thead>
<tr>
<th>spring coefficient</th>
<th>flutter and buckling capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2.04 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>5</td>
<td>$2.23 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>10</td>
<td>$2.46 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>15</td>
<td>$2.74 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>20</td>
<td>$3.04 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>25</td>
<td>$3.34 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>30</td>
<td>$3.63 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>34</td>
<td>$3.84 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>35</td>
<td>$3.9 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>36</td>
<td>$3.93 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>38</td>
<td>$4.02 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>40</td>
<td>$2.83 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>45</td>
<td>$2.68 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>60</td>
<td>$2.46 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>80</td>
<td>$2.33 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>100</td>
<td>$2.27 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>120</td>
<td>$2.23 \pi^2 EI / L^2$</td>
</tr>
<tr>
<td>140</td>
<td>$2.2 \pi^2 EI / L^2$</td>
</tr>
</tbody>
</table>
4.4. Results and discussion

The following aspects of stability and control of the cantilevered column are studied.

(i) Relationship between the control gain and the critical load over the entire range of spring stiffness
(ii) Effect of loading sequence
(iii) Influence of spring nonlinearity

The relative merits partial versus full patch control.

4.4.1. Numerical Study of Controlled Column Response

The Eigen-value analysis does not give us any hint of the actual performance of the column and the control system as the critical load is approached. In particular it does not give the voltages that must develop across the piezo patches for a given program of loading. In order to illustrate the effectiveness of piezo-electric control and the voltages developed in the context of a given disturbance we consider a column having specific geometric and material properties as shown in Table 4.2. Analysis in time domain would give us a quantitative feel for the column behavior and control voltages.

4.4.1.1. Geometry and Material Properties

The geometric and material properties of the host column are given in Table 4.2.

For piezo-electric material a standard value of \( E_p^C = E_p e_p \) as 0.0283 is selected (vide chapter 1). To compensate for this relatively high value in comparison to manufactured
materials on date (cf. 0.0233 for PZT5K), the maximum field strength is appropriately reduced to 600 V/mm (cf. 820 V/mm) in evaluating the results.

Table 4.2. Dimensions and material properties of cantilever beam

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Length ($L$)</td>
<td>1000 mm</td>
</tr>
<tr>
<td>Breath ($B$)</td>
<td>$L/20 = 50$ mm</td>
</tr>
<tr>
<td>Depth ($h$)</td>
<td>$B/2.5 = 20$ mm</td>
</tr>
<tr>
<td>Cross sectional area</td>
<td>1000 mm$^2$</td>
</tr>
<tr>
<td>Mass per unit length</td>
<td>2.7 E-6</td>
</tr>
<tr>
<td>Young’s Modulus ($E$)</td>
<td>63.0E3</td>
</tr>
</tbody>
</table>

4.4.1.2. Perturbation to trigger dynamics

Consider a perturbation in the form of uniformly distributed force q/unit length (The manner of application is discussed later). The magnitude of $q$ is selected such that it causes the maximum dynamic deflection of 1% of the column length (10mm) for the column with no spring ($\beta_1 = 0$) in the absence of $P$, the follower force. $q$ is varied in proportion to the critical load for columns carrying springs and its intensity is taken, in general as $q_0 = 2.055\frac{P_{cr}}{L}10^{-3}$, where $P_{cr}$ is the critical load of the uncontrolled column with a given $\beta_1$. 
4.4.1.3. Solution procedure

The set of eq. (4.2) are solved using Newmark - $\beta$ method in conjunction with Newton Raphson iterative procedure when nonlinearity is present [54]. Accuracy of the solution is checked varying the value of $N$ and the time step; it was found that sufficiently accurate results could be obtained taking $N = 9$. Time step needs to be a small fraction of the smallest of the time periods of modes of vibration involved. The accuracy of the numerical results was also checked by recovering the results of the Eigen-value analysis for vanishingly small disturbances.

4.4.2. Behavior and Control of Column with Linear Spring

The linear problem is considered first setting $\beta_3 = 0$. $\beta_1$ varies from 0 to 140. The column is actuated by a double piezo-electric patch covering the entire span. Three values of gain, $G$ (with $E_p \epsilon_p = 0.0283$) are selected for investigation:

Case (i): $G = 0$, i.e. Uncontrolled (un-damped) column
Case (ii): $G = 10$, a very small gain
Case (iii): $G = 2.5 \times 10^4$, representative of values likely to be used in practical applications.

4.4.2.1. Small Damping Effect

First we consider the column propped by a linear spring ($\beta_1 \geq 0$, $\beta_3 = 0$, $q = 0$) with a feed back control given by $G = 10$, (case (ii)). This is a case of “small gain” as it will be shown that with this gain the control voltages across the patches turn out to be very small in comparison to case (iii), under the selected disturbance. The corresponding linear
Eigen-value problem is solved repeatedly with the follower force, $P$ incremented in small steps, till the threshold instability is reached. The results are displayed in Figure 4.4.

It is important to note that in this case, the critical load in fact decreases significantly from that corresponding to case (i) in the flutter range. This is consistent with the well known effect of small viscous damping in linear circulatory systems [7],[17]. Such systems must always be analyzed with some damping taken into account to avoid un-conservative results. Thus the results for the case of gain, $G = 10$ should be taken as truly representative of the uncontrolled column.

The case with $\beta_1 = 0$ is particularly interesting. The drop in the critical load from that for the case (i) is significant (42%). Significant drops in the critical loads are seen in the flutter range up to $\beta_1 \approx 10$. This effect becomes less significant as the spring stiffness increases and vanishes as the buckling range is approached.

4.4.2.2 Critical Load Enhancement

Figure 4.4 also plots the critical load attainable when the gain, $G$ is increased to $2.5 \times 10^4$. Significant enhancements (of about 50% in some cases) in the critical loads are seen throughout the flutter range, but this would come at the expense of significant increases in the voltages across the patches in the context of a specified disturbance – a point that is discussed in the sequel.

Figure 4.5 plots the critical load attainable for various values of gain for the case with $\beta_1 = 0$. The load carrying capacity increases abruptly as the gain is increased from $10^4$ to $10^5$. However in the transitional range (Figure 4.3) and beyond ($\beta_1 > 30$) the velocity dependent feedback control – considered in the present study - cannot and does not enhance the critical load [44]. With $G = 2.5 \times 10^4$ the maximum critical load over the entire range of $\beta_1$ occurs at a value of $\beta_1 = 20$ and this corresponds to an enhancement of the critical load by about 35% from case (i) and 43% from case (ii).
Figure 4.4. Variation of maximum $P/P_E$ with $\beta_1$

Figure 4.5. Critical load enhancement with Gain for a cantilevered column with $\beta_1 = 0$
4.4.2.3. System Performance

The following loading sequences are considered:

- Loading program LP-1: The follower force and lateral loading are ramped up simultaneously in duration of 0.07 sec to reach their respective chosen values, viz., $P$ and $q$.

- Loading program LP-2: The follower force is preset at the specified value of $P$ and the lateral force $q$ is then suddenly applied – a Heaviside step function of time.

LP-1 is the preferred loading program in the present study as it will be seen to more adverse – a point discussed in the sequel. The dynamic performance is studied by systematically increasing $P$ till a preset value is attained. The analysis is repeated increasing $P$ in steps till the onset of dynamic instability.

Table 4.3. (a) $\beta_i = 0$

<table>
<thead>
<tr>
<th>GAIN</th>
<th>Maximum load attained</th>
<th>Maximum Voltage Recorded, Volt</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P/P_E$</td>
<td>$P/P_{cr}$</td>
</tr>
<tr>
<td>0.0</td>
<td>2.04</td>
<td>1.0</td>
</tr>
<tr>
<td>10.0</td>
<td>1.12</td>
<td>0.5490</td>
</tr>
<tr>
<td>1.0E4</td>
<td>1.15</td>
<td>0.5637</td>
</tr>
<tr>
<td>2.5E4</td>
<td>1.36</td>
<td>0.6667</td>
</tr>
<tr>
<td>10.0E4</td>
<td>4.79</td>
<td>2.3480</td>
</tr>
</tbody>
</table>
Table 4.3 (b) $\beta_1 = 30$

<table>
<thead>
<tr>
<th>GAIN</th>
<th>Maximum load attained</th>
<th>Maximum Voltage Recorded, Volt</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P/P_E$</td>
<td>$P/P_{cr}$</td>
</tr>
<tr>
<td>0.0</td>
<td>3.63</td>
<td>1</td>
</tr>
<tr>
<td>10.0</td>
<td>3.46</td>
<td>0.9532</td>
</tr>
<tr>
<td>1.0E4</td>
<td>3.65</td>
<td>1.0055</td>
</tr>
<tr>
<td>2.5E4</td>
<td>3.87</td>
<td>1.0661</td>
</tr>
<tr>
<td>10.0E4</td>
<td>3.98</td>
<td>1.0964</td>
</tr>
</tbody>
</table>

Table 4.3 (c) $\beta_1 = 36$

<table>
<thead>
<tr>
<th>GAIN</th>
<th>Maximum load attained</th>
<th>Maximum Voltage Recorded, Volt</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P/P_E$</td>
<td>$P/P_{cr}$</td>
</tr>
<tr>
<td>0.0</td>
<td>3.93</td>
<td>1</td>
</tr>
<tr>
<td>10.0</td>
<td>3.1</td>
<td>0.788804</td>
</tr>
<tr>
<td>1.0E4</td>
<td>3.1</td>
<td>0.788804</td>
</tr>
<tr>
<td>2.5E4</td>
<td>3.1</td>
<td>0.788804</td>
</tr>
<tr>
<td>10.0E4</td>
<td>3.1</td>
<td>0.788804</td>
</tr>
</tbody>
</table>

The maximum voltages (recorded at the clamped end) just prior to dynamic instability are given in the Table 4.3 (a-d) for values of $\beta_1$, viz. 0, 30, 36 and 120 respectively. The cases with $\beta_1 = 0$ and 30 are of particular interest in that the voltage increases are accompanied by significant increases in the follower force that can be carried...
cases such enhancement of the critical load is either marginal or nonexistent. The maximum voltages recorded across the piezo-electric patches for all the cases increase rapidly with the gains selected. Note that the case (ii) with $G = 10$ is associated with very small voltages while significant reductions occur in the load carrying capacity.

From the point of view of limiting the thickness of piezo-electric patches, it is desirable to keep the maximum voltages as small as possible while at the same time enhancing the load carrying capacity. Since these voltages are attained as soon as the disturbance is felt by the column, it is possible to minimize these by choosing a smaller gain in the first few milliseconds and ramping up the gains rapidly beyond that. However this was not tried in the present study.

Below the critical load the column is completely controllable and this is illustrated in several cases (figure 4.6 – 9) for cases with $\beta_1 = 0, 10$ and under LP-1 and LP-2.

Table 4.3 (d) $\beta_1 = 120$

<table>
<thead>
<tr>
<th>GAIN</th>
<th>Maximum load attained</th>
<th>Maximum Voltage Recorded, Volt</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P/P_E$</td>
<td>$P/P_{cr}$</td>
</tr>
<tr>
<td>0.0</td>
<td>2.23</td>
<td>1</td>
</tr>
<tr>
<td>10.0</td>
<td>2.23</td>
<td>1</td>
</tr>
<tr>
<td>1.0E4</td>
<td>2.23</td>
<td>1</td>
</tr>
<tr>
<td>2.5E4</td>
<td>2.23</td>
<td>1</td>
</tr>
<tr>
<td>10.0E4</td>
<td>2.23</td>
<td>1</td>
</tr>
</tbody>
</table>
4.4.2.4. Significant Role of the Spring

The addition of a relatively light spring significantly enhances the load carrying capacity of the column and this enhancement increases with the gain, $G$ selected. Considering the case of $G = 10$, the addition of a relatively light spring ($\beta_1 = 10$) enhances the load carrying capacity significantly-a 90% increase occurs (figure 4.4). Also for the standard perturbing force selected, the column with $\beta_1 = 10$ requires a smaller gain and a drastically reduced voltage demand for a given settling time. This is illustrated in figure 4.6 and figure 4.7 where the time history responses respectively of a cantilever without the spring and that with a spring with $\beta_1 = 10$ under LP-1 are shown.

![Figure 4.6](image_url)  
**Figure 4.6.** Time histories of $\bar{V}$, $\dot{V}$, $\ddot{V}$ at $x = L$ and voltage at $x = 0$ with $G = 2.5E4$, $\beta_1 = 0.0$, LP-1, $P = 2/3 P_{cr}$. 
Note that here and in subsequent figures, the plots are for \( \vec{v}, \dot{v}, \ddot{v} \), which are dimensionless quantities defined by:

\[
\{ \vec{v}, \dot{v}, \ddot{v} \} = \left\{ \frac{v}{L}, \frac{\dot{v}}{L}, \frac{\ddot{v}}{L} \right\}_{x=L}
\]

A value of \( P \) equal to \( 2/3 \) \( P_{cr} \) (\( P_{cr} \) = critical value of \( P \) for the uncontrolled, un-damped column) is selected in each case to represent a working condition with 50% margin of safety. However it turns out that for the case with \( \beta_i = 0 \), this exceeds the maximum load unless a gain of 2.5E4 is selected. For this gain, control is found to be effective with a settling time of 0.2 sec. Now compare this with the column having \( \beta_i = 10 \). For a settling time less than 0.2 sec, the gain required (1.0E4 versus 2.5E4) and the voltage demand are both 40% of those required for \( \beta_i = 0 \).

### 4.4.2.5. Effect of Loading Sequence

The maximum values of the displacements, velocities and voltages developed depend upon the sequence of application of the follower force, \( P \) and lateral load \( q \).

The time histories for columns with \( \beta_i = 0 \) and \( \beta_i = 10 \) subjected to LP-2 are presented in figure 4.8 and figure 4.9 respectively. These are compared with respectively with figure 4.7 and figure 4.8 which plot time histories for LP-1. It is seen that under LP-2 the maximum displacements as well as the maximum voltages developed are significantly smaller than in LP-1. The reason for this is not far to seek. The follower force has a transverse component which tends to reduce the column displacements. This effect is available in full at the instant the column is disturbed for LP-2 and hence the aforementioned reductions. Thus LP-1 is seen to be more adverse and is used in the present study in order to be conservative.
Figure 4.7. Time histories of $\ddot{V}, \dot{V}, V$ at $x = L$ and voltage at $x = 0$ with $G = 10$, $\beta_1 = 10.0$, LP-1, $P = 2/3 P_{cr}$

Figure 4.8. Time histories of $\ddot{V}, \dot{V}, V$ at $x = L$ and voltage at $x = 0$ with $G = 1.0E4$, $\beta_1 = 10.0$, LP-2, $P = 2/3 P_{cr}$
Next consider the problem where the spring has a softening nonlinearity given by $\beta_1 = -2000$. The spring nonlinearity is such as to cause a reduction of the force carried by it by 20% from that sustained by a linear spring when the spring undergoes a deflection of 10 mm ($\delta = L/100$). The effect of nonlinearity on the load carrying capacity is illustrated in figure 4.10 which shows a relationship between the maximum follower force $P$ that can be carried by the column and the spring stiffness $\beta_1$ for three differing values of gain under LP-1.
The relationship for the nonlinear spring differs significantly from that of the linear spring (Figure 4.4) in one significant respect. The steep increase of $P/P_E$ with $\beta_1$ in the flutter range and its subsequent steep drop in the transition range are replaced by a more rounded characteristic with the critical loads attained significantly reduced from those of the linear case. As spring stiffness increases beyond this range, the nonlinearity has a minimal effect as the deflections suffered by the spring become smaller and smaller. Table 4.4 (a-c) lists the maximum loads that can be attained with these gains and the maximum voltages that develop for three values of $\beta_1$, viz. 30, 36 and 120. Piezoelectric actuation with the higher gain ($G = 2.5E4$) once again is seen to enhance the critical loads from the values corresponding to small gain case (with $G = 10$).
Table 4.4. Maximum Loads

(a) $\beta_1 = 30$

<table>
<thead>
<tr>
<th>GAIN</th>
<th>Maximum load attained</th>
<th>Maximum Voltage Recorded, Volt</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P/P_E$</td>
<td>$P/P_{cr}$</td>
</tr>
<tr>
<td>0.0</td>
<td>3.63</td>
<td>1</td>
</tr>
<tr>
<td>10.0</td>
<td>2.7225</td>
<td>0.75</td>
</tr>
<tr>
<td>1.0E4</td>
<td>2.9766</td>
<td>0.82</td>
</tr>
<tr>
<td>2.5E4</td>
<td>3.0129</td>
<td>0.83</td>
</tr>
<tr>
<td>10.0E4</td>
<td>3.4122</td>
<td>0.94</td>
</tr>
</tbody>
</table>

(b) $\beta_1 = 36$

<table>
<thead>
<tr>
<th>GAIN</th>
<th>Maximum load attained</th>
<th>Maximum Voltage Recorded, Volt</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P/P_E$</td>
<td>$P/P_{cr}$</td>
</tr>
<tr>
<td>0.0</td>
<td>3.93</td>
<td>1</td>
</tr>
<tr>
<td>10.0</td>
<td>2.6724</td>
<td>0.68</td>
</tr>
<tr>
<td>1.0E4</td>
<td>2.751</td>
<td>0.7</td>
</tr>
<tr>
<td>2.5E4</td>
<td>2.8296</td>
<td>0.72</td>
</tr>
<tr>
<td>10.0E4</td>
<td>2.8296</td>
<td>0.72</td>
</tr>
</tbody>
</table>

(c) $\beta_1 = 120$

<table>
<thead>
<tr>
<th>GAIN</th>
<th>Maximum load attained</th>
<th>Maximum Voltage Recorded, Volt</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$P/P_E$</td>
<td>$P/P_{cr}$</td>
</tr>
<tr>
<td>0.0</td>
<td>2.23</td>
<td>1</td>
</tr>
<tr>
<td>10.0</td>
<td>2.0962</td>
<td>0.94</td>
</tr>
<tr>
<td>1.0E4</td>
<td>2.1408</td>
<td>0.96</td>
</tr>
<tr>
<td>2.5E4</td>
<td>2.1408</td>
<td>0.96</td>
</tr>
<tr>
<td>10.0E4</td>
<td>2.1408</td>
<td>0.96</td>
</tr>
</tbody>
</table>
However significant increases are seen only in the flutter and transitional range and the percentage enhancements are similar to that seen in the linear case. The optimal spring stiffness is still around 20, though the corresponding maximum load that can be attained with control suffers a reduction because of nonlinearity and the voltages required for control are higher. Thus while the addition of the spring does improve the system performance, it is important to consider carefully the nonlinearity, if any that may exist.

Figure 4.11 and 4.12 give time histories of a column and voltage distribution along the length respectively with $\beta_1 = 10$, $\beta_3 = -2000$, $P = \frac{2}{3} P_{cr}$, $G = 1.0 \cdot 10^4$ under LP-1. A comparison with figure 4.6 which plots the same for the case with $\beta_3 = 0$ (linear case) brings out the influence of nonlinearity. It is seen that the maximum displacement and velocity at the propped end as well as the maximum voltage at the fixed end are significantly higher.

Figure 4.11. Time histories of $\bar{V}, \dot{V}, \ddot{V}$ at $x = L$ and voltage at $x = 0$ with $G = 1.0 \cdot 10^4$, $\beta_1 = 10.0$, $\beta_3 = -2000$, LP-1, $P = \frac{2}{3} P_{cr}$
4.4.2.7. Patch Control

Figure 4.11 illustrates the variation of voltage along the column length ($\beta_1 = 0$) with $P = 2/3 P_{cr}$ subjected to LP-1 at the instant when the maximum voltage is recorded at the fixed end. It is clear that the voltage distribution is non-uniform; the highest intensity of voltages occur in the portion of the column given by $0 < x < L/2$, i.e. from the fixed end to mid-span ($x = L/2$) and tends to taper off in the right half of the column. Such variation of voltage is typical for columns with relatively light springs with $\beta_1$ falling within the flutter range. Given this voltage distribution it appears that the column can be controlled by a double piezo patch covering a fraction of the column length from the fixed end. In the present study, the column is sought to be controlled with such piezo patches spanning one half, one third and one quarter of the length of the column.

4.4.2.7.1. Half Piezo Patch

Consider the column with $\beta_1 = 10$ carrying a follower force of $2/3 P_{cr}$ subjected to LP-1. It was seen from figure 4.5 that to attain this load, a gain of 1.E04 was required (also refer to figure 4.8 for the time histories). Figure 4.13 plots the time histories for the same
column but with half piezo patch. With half patch a smaller gain of 1000 is found to be adequate. The maximum voltage attained is much smaller (23.5 Volts versus 75 volts of the full patch case) and the settling time is smaller as well (0.18 sec vs. 0.2 sec). Voltage distribution at an instant when the highest value is attained is shown in figure 4.14. Once again the voltage is concentrated in the region closed to clamped end.

With a view to probe further into the relative efficacy of half-patch control over the full-patch control, the critical load capacities were obtained for the entire range of $\beta_t$ values with two representative values of gain, viz. 10 and 1.E3. The results are shown in figure 4.15. Comparing these results with those in figure 4.3, the critical limits are higher with half patch $\beta_t < 25$ indicating that the half patch control is more effective in the flutter range. In the buckling range however, it is ineffective, being a velocity dependent control strategy. Further the half patch control may be counterproductive in this range; in fact for $\beta_t > 247$, the column becomes uncontrollable under half patch control (not shown).

Figure 4.13.  Time histories of $\bar{V}$, $\dot{V}$, $\ddot{V}$ at $x = L$ and voltage at $x = 0$ with $G = 1.0$, $\beta_t = 10.0$, LP-1, $P = 2/3 \frac{P_{cr}}{P}$
Figure 4.14. Voltage distribution along the length of the piezo patch @ 0.08 sec (half piezo patch with $G = 1.0 \times 10^3$, $\beta_i = 10.0$, $P = 2/3 P_{cr}$).

Figure 4.15. Critical limits using half patch (0 to $L/2$)
Figure 4.16. One third piezo patch: Time histories of $\ddot{v}, \dot{v}, \ddot{v}$ at $x = L$ and voltage at $x = 0$ with $G = 2.10^3$, $\beta_I = 10.0$, LP-1, $P = 2/3 \ P_{cr}$

Figure 4.17. Voltage distribution along the length of one third patch @ 0.09 sec.
4.4.2.7.2. Controls with Patches of Length $L/3$ and $L/4$

The same example illustrated in figure 4.13 ($\beta_1 = 10$ and $G = 1.1E3$, $P = 2/3 P_{cr}$, LP-1) is studied with a piezo patch applied from fixed end to one third of its length. The results are shown in figure 4.16. The maximum voltage required is 25.3 V, which is slightly higher than the half patch case. The settling time for this case is about 0.5 second. This is better than the full patch case where no control is possible for $G = 2.1E3$, but certainly is less efficient than the half-patch control. Voltage distribution when the peak value is attained is shown in figure 4.17. When the patch length is reduced to $L/4$, no control is possible for the parameters chosen.

4.4.2.7.3. Three Piezo Patches

In order to gain further insight into the most advantageous distribution of patches, the column was sought to be controlled with three patches.

The following two cases were studied:

Case (i): First, three equal sized patches are tried with the following parameters: Patch width 200 mm (= $L/5$), $G = 2.5E4$, $\beta_1 = 10$, LP- 1. Table 4.5 gives the width and the location of patches.

The time histories of $v$, $\dot{v}$, $\ddot{v}$ at $x = L$ are shown in figure 4.18 and the histories of voltage at the center of patches are shown in figure 4.19. The oscillations become steady with no hint of getting damped as do the patch voltages at the end of 2 sec. It is clear that the strategy has failed.
Figure 4.18. Case (i): Time histories of $\bar{V}, \hat{V}, \ddot{V}$ at $x = L$ with $G = 2.5 \times 10^4$, $\beta_1 = 10.0$, LP-1, $P = \frac{2}{3} P_{cr}$

Figure 4.19. Case (i): Voltage time histories of each patch (Voltages correspond to middle of each patch)
Table 4.5. Description of Patches for Case (i)

<table>
<thead>
<tr>
<th>Patch number</th>
<th>Patch width (mm)</th>
<th>X-coordinate of the center of patch(mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>200</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>200</td>
<td>500</td>
</tr>
<tr>
<td>3</td>
<td>200</td>
<td>900</td>
</tr>
</tbody>
</table>

Case (ii): Next attempt involved varying the widths of the patches and locating the higher width patches closer to the clamped end. The patch widths and the corresponding distances from the clamped end (x-coordinate of the center of patch) are given in table 4.6.

Table 4.6. Description of Patches for Case (ii)

<table>
<thead>
<tr>
<th>Patch number</th>
<th>Patch width (mm)</th>
<th>X-coordinate of the center of patch(mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>300</td>
<td>150</td>
</tr>
<tr>
<td>2</td>
<td>200</td>
<td>500</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
<td>950</td>
</tr>
</tbody>
</table>

The time histories corresponding to \( x = L \) are shown in figure 4.20. Though there is a gradual reduction of the displacements, velocities and accelerations, these take an inordinately long time to get damped out. The controlling maximum voltages in patches are too high (Figure 4.21) relative to half patch case. All the three patches record roughly the same maximum voltage of 253 Volts.
Figure 4.20. Case (ii): Time histories of $\ddot{V}, \dot{V}, \ddot{V}$ at $x = L$ with $G = 2.5E4, \beta_1 = 10.0, \text{LP-1}, P = \frac{2}{3} P_{ct}$

Figure 4.21. Voltage time histories of each patch for Case (ii) (Voltages correspond to middle of each patch)

4.4.2.7.4. Two Piezo Patches

In order to examine the role of the third patch away from the clamped end, case (ii) is studied with the third patch removed. As we can see in figure 4.22 and figure 4.23 in
comparison to figure 4.20 and figure 4.21, there is no significant change in the behavior of the column. The time histories are similar for the case with 3 patches. Thus the third patch does not appear to perform any useful role.

Figure 4.22. Two patch case: Time histories of $\bar{V}, \dot{V}, \ddot{V}$ at $x = L$ with $G = 2.5E4$, $\beta_1 = 10.0$, LP-1, $P = 2/3 P_{cr}$

Figure 4.23. Voltage time histories for each patch Case (ii) without patch 3 (Voltages correspond to middle of each patch)
4.5. Conclusion

(1) The critical loads of a column carrying a follower force can be enhanced significantly by piezo-electric control with feedback voltages proportional to the bending strain-rates. The feasibility of employing such control is demonstrated by computing the voltages that must develop across the piezo patches when the column is subjected to a typical disturbance. While selecting a high gain can ensure a desired enhancement in the critical load, the voltages developed across the piezo patches may turn out to be unacceptably high.

(2) The critical loads obtained with zero gain (no piezo-electric control or no viscous damping) have limited practical significance in the flutter range. A small (but finite) value of gain must be incorporated in numerical analyses to obtain realistic estimates of critical load.

(3) Provision of an elastic support (spring) at the free end of a clamped column can dramatically enhance the performance of both the structure and the control system. There is an optimal spring stiffness which results in the highest critical load for a given gain and this stiffness lies in the range where flutter in contrast to divergence (buckling) is the mode of instability.

(4) Careful consideration must be given to the sequence of loading of the follower force and lateral loading as this can affect the maximum displacements and voltages developed across the patches.
(5) A moderate amount of softening nonlinearity has the effect of drastically rounding off the abrupt transition in the relationship between $P_{cr}$ and $\beta_1$ (the linear part of spring stiffness).

(6) Control using a double piezo patch spanning only one half of the column from the clamped end proved much more effective in terms the higher critical loads that can be attained and the reduction in the voltages across the patches in comparison to control using piezo patches running from end to end. There is a limit to how short the patches can be for control to be effective or even possible.

(7) Control schemes with patches distributed over the entire column were found to be not efficient in controlling flutter. Patches located away from the clamped end are completely ineffective in flutter control.
Chapter 5

Piezoelectric Control of Stiffened Panel: Finite Element Approach

Stiffened plates are ubiquitous in aircraft structures and constitute a problem of practical importance and technical interest. In this chapter we consider a simple form of stiffened panel under axial compression. There are two characteristic modes of buckling of stiffened plates under axial compression, viz. overall buckling associated with a long wave mode and local buckling described by a number of half waves. The stiffener undergoes significant in-plane displacements under overall buckling whereas plate buckles between stiffeners under local buckling mode is dominant. In the prediction of the actual load carrying capacity adverse nonlinear interaction of these modes must be considered. Significant reductions in the load carrying capacity would occur if the local buckling were to occur first. Optimally designed stiffened plates have either coincident or near-coincident critical stresses corresponding to these buckling modes. However stiffened plates designed thus are apt to be imperfection-sensitive as a result of modal interaction. This phenomenon is termed nonlinear as it is controlled by cubic and/or quartic terms of the potential energy function.

This chapter addresses the issues involved in the piezo-electric control of an “optimally designed” stiffened panel. The example chosen for study is however a simple one, a panel consisting of slender plate and relatively stocky stiffener- designated in the literature as Tvergaard panel-1 (Tvergaard, 1973 [55], Sridharan et al., 1994 [46]). It is shown that
feedback voltages across patches at the stiffener tips, proportional to the bending strain have a salutary effect in stiffening the structure at loads that exceed the capacity of the uncontrolled structure under static conditions. In this case local buckling deflections are allowed to occur, but they are seen to be innocuous as long as the overall bending is effectively controlled. Next the feasibility of damping out of large amplitude oscillations liable to be triggered at loads smaller than the dynamic buckling load is studied. As before the control is exercised using piezo-electric actuators attached at the stiffener tips only. The feedback gains are now proportional to the strain rates sensed at the stiffener tips. This has the effect of damping out overall oscillations fairly quickly, but local mode vibrations tend to linger on for a long duration. In an attempt to damp out the local (plate) vibrations additional control is exercised via piezo-electric actuator patches placed at upper and lower surfaces at the middle of each plate panel. The feedback gains are proportional to the sum of the strain rates sensed in the longitudinal and transverse directions. This was found to be very effective in damping out the plate vibrations. Thus by selective use of piezo-electric patch actuators at key locations it was possible to maintain the stiffness of the stiffened plate and damp out the oscillations. Finally the control of a panel with scaled up geometry is studied with practical applications in view with encouraging results.

The analysis of the stiffened plate employs an approach in which the interaction is accounted for by embedding the local buckling deformation (Sridharan et al., 1994 [46]). This approach isolates the local buckling deformation, together with the second order effects, its variation spatially over the panel and the corresponding components of feedback voltage, from the overall effects. This affords a greater insight into the response of the stiffened panel than conventional finite element schemes and makes possible a more focused control strategy. These aspects of analysis are reviewed briefly in the following section. Nevertheless, the specific contribution of the study is in the realm of establishing a viable strategy of piezo-electric control of stiffened panels, with an eye on the practical application.
5.1. Theory

In this section, the theory and formulation of the present finite element model is outlined. Figure 5.1 shows a “wide” stiffened plate and a typical panel consisting of plate elements on either side of a stiffener.

5.1.1. Displacement, Strain, and Stress Vectors

The displacement variables are

\[
\{u\}^T = \{u, v, w, \alpha, \beta\}
\] (5.1)

where \(u\), \(v\), and \(w\) are the displacement components in the axial (\(x\)-), transverse (\(y\)-) and outward normal (\(z\)-) directions respectively at any point on the middle surface plate or stiffener (Figure 5.2) and \(\alpha\) and \(\beta\) are the rotations of the normal in the \(xz\) and \(yz\) planes respectively (Sridharan et al., 1992 [48]).

The generic strain vector \(\{\varepsilon\}\) may now be defined as in Reissner-Mindlin theory

\[
\{\varepsilon\}^T = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}, \chi_x, \chi_y, \chi_{xy}, \gamma_{xz}, \gamma_{xz}\}
\] (5.2)

where

\[
\{\varepsilon\} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}\}
\] (5.3.a)

are the inplane strain components at the plate mid-surface,
\[ \{ \chi \} = \{ \chi_x, \chi_y, \chi_{xy} \} \quad (5.3.b) \]

are the curvature components, and

\[ \{ \gamma \} = \{ \gamma_{xz}, \gamma_{yz} \} \quad (5.3.c) \]

are the transverse shearing strain components. The generic stress vector \( \{ \sigma \} \) conjugate with \( \{ \varepsilon \} \) consist of stress resultants. These consist of the force resultants \( \{ N \} = \{ N_x, N_y, N_{xy} \} \), moment resultants \( \{ M \} = \{ M_x, M_y, M_{xy} \} \), and transverse shear forces \( \{ Q \} = \{ Q_x, Q_y \} \).

The generic stress-strain relations are taken in the standard form

\[
\begin{align*}
N_i &= A_{ij} \bar{\varepsilon}_j + B_{ij} \bar{\chi}_j & (j = 1, 2, 6); (i=1, 2, 6) \\
M_i &= B_{ij} \bar{\varepsilon}_j + D_{ij} \bar{\chi}_j \\
Q_i &= k\overline{G} \tau_{\gamma_i} & (i=1, 2)
\end{align*}
\]

(5.4.a-c)

where \( [A], [B], [D] \) are well known matrices in the literature on layered composites, \( \overline{G} \) is the averaged transverse shear modulus, \( k \) is the shear correction factor (\( = 5/6 \)) and \( t \) is thickness of the plate element.

These equations may be written in the abbreviated form

\[ \sigma_i = H_{ij} \varepsilon_j \quad (5.5) \]

The following strain-displacement relations are used for the plate structure.
These are but von Karman plate equations modified to account for transverse shear deformation and the large in-plane movements of stiffeners such as occur under overall buckling/bending. The strain-displacement relations can be expressed in the abbreviated form

\[
\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right)
\]

\[
\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \left( \frac{\partial w}{\partial y} \right)^2 \right)
\]

\[
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \quad \text{(5.6.a-h)}
\]

\[
\chi_x = \frac{\partial \alpha}{\partial x}; \chi_y = \frac{\partial \beta}{\partial y}; \chi_{xy} = \frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x}
\]

\[
\gamma_{xz} = \alpha + \frac{\partial \psi}{\partial x}; \gamma_{yz} = \beta + \frac{\partial \psi}{\partial y}
\]

where \( L_1 \) stands for linear differential operators and \( L_2 \) for a quadratic operators implicit in eq. (5.6.a-h).

### 5.1.2. Solution of the Local Buckling Problem

#### 5.1.2.1. Linear Stability Analysis

The following notation will be employed: A superscript (1) indicates a first order local buckling quantity (Eigen mode), a superscript (2) indicates a second order quantity and a superscript of ‘o’ indicates a quantity associated with the unbuckled state. Note in the
present treatment, only the axial stress is recognized in the pre-buckling stress state \( \{ \sigma^0 \} \).

The local buckling field will be denoted by

\[
\{ u^{(1)} \} = \{ u^{(1)} \ v^{(1)} \ w^{(1)} \ \alpha^{(1)} \ \beta^{(1)} \}
\]  \hspace{1cm} (5.8)

In the asymptotic analysis parlance this will be the first order field and hence the
superscript (1). The potential energy function for the local buckling can be written in the
form

\[
\Pi^{(1)} = \frac{1}{2} \left[ H_{ik} (u_k^{(1)}) \bullet L_{ij} (u_j^{(1)}) + \sigma_i^{\circ} \bullet L_{ij} (u_j^{(1)}) \right]
\]  \hspace{1cm} (5.9)

where \( \bullet \) denotes multiplication and integration over the entire structure.

For a uniformly compressed stringer stiffened shell made of specially orthotropic
material, the displacement functions that satisfy the differential equations of equilibrium
are of the form

\[
\{ u^{(1)}, \alpha^{(1)} \} = \{ u_i^{(1)}, \alpha_i^{(1)} \} \psi_i(y) \cos(m \pi \eta)
\]
\[
\{ v^{(1)}, \omega^{(1)}, \beta^{(1)} \} = \{ v_i^{(1)}, \omega_i^{(1)}, \beta_i^{(1)} \} \psi_i(y) \sin(m \pi \eta)
\]  \hspace{1cm} (i = 1, \ldots, p+1)  \hspace{1cm} (5.10.a-b)

where \( \eta = \frac{x}{L} \) and \( u_i^{(1)}, \ldots, \beta_i^{(1)} \) are the degrees of freedom (designated as \( q^{(1)} \)) and
\( \psi_i(y) \) are appropriate shape functions of \( y \), \( p \) refers to the highest degree of the set of
\( p+1 \) shape functions, the transverse coordinate of the plate element.

As long as \( m \) is large, the local buckling phenomenon may be assumed to be truly “local”
being free of the end effects.

The potential energy function associated with the buckling problem \( \Pi^{(1)} \) can be
expressed in the form:
where $\lambda$ is the loading parameter (axial compression in the x-direction in the present context). The coefficients $a^{(i)}_{ij}$ and $b^{(i)}_{ij}$ are derived from the integrals of products of shape functions entering into the first and second set of terms of potential energy function in eq. (5.9). The corresponding equations of equilibrium constituting the Eigen value problem take the form:

$$\{a^{(i)}_{ij} - \lambda b^{(i)}_{ij}\}q^{(i)}_j = 0$$  \hspace{1cm} (5.12)

For a purely local buckling problem, the buckling mode involves principally out of plane bending of each plate element and little or no in-plane action. Figure 5.1 gives the geometry a typical panel and Figure 5.2 the cross-sectional view of the local buckling mode.

![Figure 5.1. Geometry of the cross-section of a stiffened panel](image)
Figure 5.2. Cross-sectional view of local buckling of stiffened panel

Figure 5.3. “Wide” stiffened plate and a typical panel
5.1.2.2. Second Order Field

The second order local buckling field will be denoted by

\[
\{u^{(2)}\} = \{\nu^{(2)} \quad w^{(2)} \quad \alpha^{(2)} \quad \beta^{(2)}\}
\]

(5.13)

The potential energy function associated with the second order field problem can be written in the form

\[
\Pi^{(2)} = \frac{1}{2} \left[ H_y L_{1ik} (u_k^{(2)}) \bullet L_{1,ij} (u_i^{(2)}) + \sigma_j^{\circ} \bullet L_{2,ij} (u_j^{(2)}) \\
+ H_{ij} L_{1,ik} (u_k^{(1)}) \bullet L_{1,ij} (u_i^{(1)}, u_j^{(2)}) + 2L_{1,ik} (u_k^{(2)}) \bullet L_{2,ij} (u_j^{(1)}) \right]
\]

(5.14)
where $L_{1,1}$ is the bilinear operator given by: 

$$L_2(u + v) = L_2(u) + 2L_{1,1}(u,v) + L_2(v)$$

The term involving this operator becomes negligible in the context of local buckling of plate structures, since $u^{(1)}$ is a field out of plane deformation (involving ‘$w$’) and $u^{(2)}$ is a field of in-plane deformation (involving ‘$u$’ and ‘$v$’). A part of the particular solution of the corresponding non-homogeneous differential equations governing the second order field problem is of the form

$$\left\{ u^{(2)}, \alpha^{(2)} \right\} = \left\{ \mu_i^{(2)}, \alpha_i^{(2)} \right\} \psi_i(y) \sin(2m\pi\eta)$$

$$\left\{ v^{(2)}, w^{(2)}, \beta^{(2)} \right\} = \left\{ \nu_i^{(2)}, w_i^{(2)}, \beta_i^{(2)} \right\} \psi_i(y) \cos(2m\pi\eta) \quad (i = 1, \ldots, p+1) \quad (5.15)$$

In addition to these “rapidly” varying functions, the solution admits of “slowly” varying contributions to the second order field consistent with the boundary conditions.

Consider the Tvergaard panel-1 [55] shown in Figure 5.3.

Under uniform axial compression, the plate buckles locally with the stiffener participation being minimal. As a result of such local buckling the panel as a whole suffers an end-shortening and deflects downward with the end sections rotating accordingly. These effects are part of the second order field, but because of their ‘slowly’ varying nature, are decoupled from the solution of the harmonic part of the field ($m >> 1$); we choose not to evaluate them as part of the second order field. These will be modeled with facility by the degrees of freedom of the finite elements in which the local buckling fields will be embedded. The uncoupled potential energy function $\Pi^{(2)}$ can be expressed in terms of the generic degrees of freedom $q^{(1)}$ and $q^{(2)}$ defining the first and the second order field respectively as,

$$\Pi^{(2)} = \frac{1}{2} \left\{ a_{ij}^{(2)} - \lambda b_{ij}^{(2)} \right\} q_i^{(2)} q_j^{(2)} - c_{irs} q_i^{(2)} q_r^{(1)} q_s^{(1)} \quad (5.16)$$
The corresponding equations for the second order field take the form

\[
\{a_y^{(2)} - \lambda b_y^{(2)}\} q_j^{(2)} = c_{ir}^s q_i^{(1)} q_s^{(1)}
\]  

(5.17)

The coefficients \(a_y^{(2)}\) and \(b_y^{(2)}\) are derived from the integrals of products of shape functions entering into the first and second terms of potential energy function in eq. (5.17) and the coefficients \(c_{ir}^s\) are obtained from the remainder of the term thereof. For a plate structure undergoing local buckling, this is essentially an in-plane field with little or no out of plane action \((w, \alpha, \beta)\). Thus the solution is unaffected by the potential threat of singularity posed by the destabilizing term associated with \(\lambda\) and turns out to be robust.

5.1.2.3. Modification of Local Buckling Deformation under Interaction

In an asymptotic procedure for post-buckling analysis, the local buckling displacement field is taken in the form

\[
\{u\}^\ell = \{u^{(1)}\}_\xi + \{u^{(2)}\}_{\xi^2} + \ldots
\]  

(5.18)

where \(\xi\) is the scaling factor of the buckling mode, and the superscript \(\ell\) denotes the local buckling field. However, in the context of interaction with an overall mode, one must anticipate modification of the local buckling deformation from that given by eq. (5.18).

Consider the Tvergaard panel once again. In the absence of imperfections, the overall buckling would only reinforce the downward bending tendency caused by local buckling.
This means the plate will suffer additional compression which in turn will accentuate the local buckles at mid-span where the compression due to bending is most severe. This will result in the phenomenon of “amplitude modulation” in the longitudinal direction. The stiffener being stocky and subjected to tension due to overall bending will, if anything, have its local buckling deformation alleviated. So it is necessary to let the amplitude modulation function be different for the different plate elements constituting the structure. These two features, viz. the variation of the amplitude in the longitudinal direction and the freedom of the local buckling pattern to vary in the cross-sectional plane account implicitly for secondary local modes liable to be triggered by the interaction [46],[47],[48],[49]. This approach also obviates the need to evaluate and incorporate the mixed second order fields in the formulation.

Thus for a typical plate element

$$\{u\}_{ij}^{\ell} = \left\{u^{(1)}\right\}_{ij}f_{ij}(x)\xi_{ij} + \left\{u^{(2)}\right\}_{ij}(x)f_{ij}(x)\xi_{ij}\xi_{ij} \quad (i,j = 1,\ldots,n) \quad (5.19)$$

where $\xi_{ij}$ are the degrees of freedom which together with the ‘$n$’ shape functions $f_{ij}$ account for amplitude modulation. We anticipate these functions to be a low order polynomial (up to 2$^{nd}$ degree) because of the ‘slowly’ varying nature of the amplitude modulation.

### 5.1.3. Finite Element Formulation

To the amplitude modulated local field given by eq. (5.19), we add the displacement functions describing the overall bending field (denoted by ‘$ov$’) in the form

$$\{u\}^{ov} = \{u_{kl}\}_{ij}^{\ell}\phi_{k}(x)\psi_{l}(y) \quad (k = 1,\ldots,n_{p}; \; l = 1,\ldots,n_{q}) \quad (5.20)$$
where \( n_p, n_q \) are the number of functions in the \( x \)- and \( y \)-directions respectively.

Thus we have

\[
\{u\} = \{u_{kl}\} \phi_k(x) \psi_l(y) + \{u^{(1)}\} f(x) \xi_i + \{u^{(2)}\} f_i f_j \xi_i \xi_j
\]  

(5.21)

5.1.3.1. \textbf{B-} Matrix and Current Stress

In view of the “slowly varying” nature of amplitude modulation, local buckling strains take the form

\[
\{\varepsilon\} = \{\varepsilon^{(1)}\} f_i(x) \xi_i + \{\varepsilon^{(2)}\} f_i(x) f_j(x) \xi_i \xi_j
\]  

(5.22)

where

\[
\begin{align*}
\{\varepsilon^{(1)}\} &= L_1 \{u^{(1)}\} \\
\{\varepsilon^{(2)}\} &= \left[ L_1 \{u^{(2)}\} + \frac{1}{2} L_2 \{u^{(1)}\} \right]
\end{align*}
\]  

(5.23.a-b)

The incremental local strains can be written in the form

\[
\{\Delta \varepsilon\} = \left[ \{\varepsilon^{(1)}\} f_i(x) + 2 \{\varepsilon^{(2)}\} f_i(x) f_j(x) \xi_j \right] (\Delta \xi_j)
\]  

(5.24)
This can be arranged in the form

$$\{\Delta \varepsilon\}^i = \begin{bmatrix} [B_o] + [B_1^c] \cos(\alpha_m x) + [B_1^s] \sin(\alpha_m x) + \\
[B_2^c] \cos(2\alpha_m x) + [B_2^s] \sin(2\alpha_m x) \end{bmatrix} \{\Delta \xi\}$$  \hspace{1cm} (5.25)$$

where $\alpha_m = \frac{m\pi}{L}$. The incremental strains associated with the overall field are written in the form

$$\{\Delta \varepsilon\}^{ov} = [B]^{ov} \{\Delta q^{ov}\}$$  \hspace{1cm} (5.26)$$

where the superscript $ov$ indicates on the $B$-matrix is developed from overall shape functions and $\{q^{ov}\}$ refers to the set of overall degrees of freedom.

The combined local and overall incremental strain may therefore be expressed in the form

$$\{\Delta \varepsilon\} = \begin{bmatrix} [B_o] + [B_1^c] \cos(m\pi \eta) + [B_1^s] \sin(m\pi \eta) + \\
[B_2^c] \cos(2m\pi \eta) + [B_2^s] \sin(2m\pi \eta) \end{bmatrix} \{\Delta q\}$$  \hspace{1cm} (5.27)$$

where $\{q\}$ represents the combined set of overall and local (amplitude modulating) degrees of freedom, numbering $N$.

Note $[B_o]$ contains terms that describe the local buckling and the overall action respectively. The current stresses $\{\sigma\}$ are obtained incrementally from the initially imperfect stress free state and can be arranged in the form shown below

$$\{\sigma\} = \{\sigma_o\} + \{\sigma_1^c \cos(m\pi \eta) + \{\sigma_1^s \sin(m\pi \eta) + \{\sigma_2^c \cos(2m\pi \eta) + \{\sigma_2^s \sin(2m\pi \eta) \}$$  \hspace{1cm} (5.28)$$
Note that initial imperfections in the sense of the local mode is accommodated by prescribing values of $\xi_i$ in eq. (5.21) to reflect uniform amplitude at the start. Likewise overall imperfections in the form of the governing overall buckling mode are inducted by prescribing appropriate initial values in eq. (5.22).

### 5.1.3.2. Internal Virtual Work Contribution from a Typical Element

- **Mechanical Contribution**

\[
\delta W_{int}^E = \int_S \{\delta \varepsilon\}^T \{\sigma\} dA
= \{\delta q\}^T \int_S \left[ \left( B_i \right)^T \{\sigma_i\} \right] dA
\]

\[
= F_i \delta q_i \\
\quad (i = 1, \ldots N)
\]

where the concept of “slowly” varying functions has been employed to decouple the integration of trigonometric terms.

Here we consider the contribution to virtual work from piezo-electric patch actuators which consist of two types:

(i) Those attached to the stiffener tips controlling mainly the overall action, and

(ii) Those attached to the top and bottom surfaces of the plate midway between the stiffeners to control local plate deformations. These are illustrated in Figure 5.5.
- **Piezo-electric Contribution**

With prescribed voltages, the electric enthalpy density (eq. (1.28)) with single subscripted notation for strain takes the form

\[
H = \frac{1}{2} Q_{ij} \varepsilon_i \varepsilon_j - e_{mj} E_m \varepsilon_j \quad (i,j =1,2,6); \quad (m = 1,..3) \quad (5.30)
\]

The first variation of \(H\) may be written as

\[
\delta H = Q_{ij} \varepsilon_i \delta \varepsilon_j - e_{mj} E_m \delta \varepsilon_j \quad (5.31)
\]

Expressed in matrix notation,

\[
\delta H = \{\delta \varepsilon\}^T [Q] \{\varepsilon\} - \{\delta \varepsilon\}^T [e]^T \{E\} \quad (5.32)
\]
Note the second term gives the first variation associated with piezo-electric effects.

Letting: \([e] = [Q][d]^{T}\) we have

\[
\delta H = \{\delta \varepsilon\}^{T}[Q]\{\varepsilon\} - \{\delta \sigma\}^{T}[d]^{T}\{E\}
\]  

(5.33)

The second, electro-mechanical term is

\[
\delta H_2 = -\delta \sigma_i d_{ik} E_k
\]  

(5.34)

where \(\{\sigma\}\) is the in-plane stress due to kinematic strains (i.e. stress exclusive of piezo-electric effects).

(i) Stiffener patches

Consider patches of width equal to the stiffener width \((t_s)\) attached to the stiffener at its top (plate surface) and bottom. For illustration, consider a panel with a stocky stiffener so that the local buckling strains in the stiffener are negligible compared to the overall counterparts, and the only significant stress and strain are those occurring in the longitudinal direction. The stiffener may be treated as a generic beam so that we may assume that the longitudinal strain variation along the depth is linear.

Thus:

\[
\varepsilon_{\text{top}} = \varepsilon_o + \frac{d_s}{2} \chi
\]  

(5.35)

where \(\varepsilon_o\) is the strain at the centroid of the stiffener, \(d_s\) is the depth of the stiffener (approximately equal to the center to center distance between the patches at top and
bottom), and \( \chi \) is the curvature of the stiffener in the global \( XZ \) plane. This curvature must be the same as that of the plate at its junction with the stiffener, i.e.

\[
\chi(x) = \left. \frac{\partial \alpha}{\partial x} \right|_J = \alpha_{ji} \phi_i'(x) \quad (i = 1, \ldots, n_p) \tag{5.36}
\]

where the subscript \( J \) denotes stiffener plate junction (see Figure 5.1) and \( \alpha_{ji} \) denote the degrees of freedom that characterize ‘\( \alpha \)’ along the junction and a prime denotes differentiation with respect to \( x \).

Letting the field strength applied at top and bottom be equal and opposite, we have:

\[
\tilde{E}^{\text{top}}_{\text{bottom}} = \pm \frac{V}{s} \quad (5.37)
\]

where ‘\( s \)’ is the electrode spacing. The electrode spacing equals the thickness of the piezoelectric patch, \( t_p \) if the voltage is applied across the thickness as in a PZT patch. For MFC patch it is measured longitudinally between consecutive interdigitated electrodes.

Longitudinal stress, \( \sigma \) (corresponding to kinematic strains) at the top and bottom can be written in the form

\[
\sigma \big|^{\text{top}}_{\text{bottom}} = \pm E_s \left[ \frac{d_s}{2} \left( \chi - \chi^0 \right) \right]^{\text{top}}_{\text{bottom}} \quad (5.38)
\]

where \( E_s \) is the Young’s modulus of the piezoelectric material of the stiffener patch and \( e_s \) is the relevant piezoelectric constant, \( \chi^0 \) is the initial overall curvature of the stiffener.
Using eq. (5.38), we have

\[
\delta\sigma_{bottom}^{top} = \mp E_s \frac{d_s}{2} \phi_k'(x) \delta\alpha_{jk}
\]  
(5.39)

Piezoelectric contribution to the internal virtual work from the two patches taken together per unit length at any location is the electro-mechanical part of the internal virtual work as,

\[
\delta W_{int}^{\text{piezo}} = 2 \int_A \delta\sigma \cdot E_s \cdot dA = 2E_s e_s \frac{d_s}{2} \int_{[\phi_k']}^2 V(x) \frac{t_s}{s} t_p \delta\alpha_{jk}
\] 
(k = 1, \ldots, n_p)  
(5.40)

Here we distinguish between two cases, viz. PZT and MFC patches respectively. For PZT patches, we have \(i = 3, e_s = d_{31}, s = t_p\). (The voltage is applied across the patches and deformation is longitudinal). For MFC patches, we have \(i = 1, e_s = d_{11}, s = nt_p\) (The voltage is applied in the longitudinal direction and deformation too develops in the longitudinal direction). Note for MFC, the electrode spacing is taken as \(nt_p\), a multiple of \(t_p\). As a result, \(t_p\) gets cancelled out in the expression for piezoelectric contribution (vide eq. (5.40)). The voltage developed is recovered in the form of \(V/n\). Neither \(t_p\) nor \(n\) needs to be selected \textit{a priori}.

Since the voltage would be proportional to the strain or strain-rate sensed at the stiffener tips

\[
V(x) = V_k \phi_k'(x) \quad (k = 1, \ldots, n_p)
\]  
(5.41)

The total piezoelectric contribution may be written in the form

\[
\delta W_{int}^{\text{piezo}} = E_s e_s d_s t_p \int_{\phi_k'}(x) \phi_k'(x) dx \frac{V_f}{s} \delta\alpha_{jk} = c_{kl} E_s e_s d_s t_p t_s \frac{V_f}{s} \delta\alpha_{jk}
\]  
(5.42)
where \( c_{kl} = \int_{x} \phi_k^l(x) \phi_k^l(x) \, dx \) and the integration is taken along the length of the stiffener element. For negative feedback with gains proportional to the strains or strain-rates sensed at the patches, we have

\[
V_k = G_s \frac{d}{2} \bar{\alpha}_{jk} \quad \text{or} \quad V_k = G_d \frac{d}{2} \dot{\alpha}_{jk}
\]

(5.43)

where \( G_s \) and \( G_d \) are the gains respectively in static and dynamic problems in the present study, \( \bar{\alpha} = \alpha - \alpha^0 \), where \( \alpha^0 \) is due to initial imperfection, a dot denotes differentiation with respect to time.

Thus,

\[
\delta W_{\text{int}}^{\text{piezo}} = \frac{d_s^2 t_p}{2s} \left\{ E_s \varepsilon_s G_s \right\} c_{kl} \bar{\alpha}_{jl} \delta \alpha_{jk} \quad \text{or} \quad \frac{d_s^2 t_p}{2s} \left\{ E_s \varepsilon_s G_d \right\} c_{kl} \dot{\alpha}_{jl} \delta \alpha_{jl}
\]

\((i, j = 1, \ldots, n_p)\)

(5.44.a-b)

Note that this term is to be added only for the stiffener element. Again note that the purely mechanical contribution from the piezo-electric patches is not considered in a preliminary calculation as the thickness is as yet undetermined, but can be included properly in subsequent runs.

\[
\delta W_{\text{int}}^{\text{piezo}} = C_{ij} \bar{\alpha}_{ji} \delta \alpha_{jl} \quad \text{or} \quad C_{ij} \dot{\alpha}_{ji} \delta \alpha_{jl} \quad (i, j = 1, \ldots, n_p)
\]

(5.45)
(ii) Double patch on the plate:

Consider the piezo-electric patches running longitudinally at the middle of the plates, i.e mid-way between the stiffeners. The curvature components in the longitudinal and transverse directions of the plate elements are denoted by $\chi^c_x$ and $\chi^c_y$ respectively. The superscript $c$ refers to the center of the plate (A and B in Figure 5.1).

Over the relatively small patch, we shall assume that the strains are uniform in the transverse direction and can be represented by the value at the center of the plate given by at $y = y_c$. For simplicity consider only the bottom patch. For the top patch simply reverse the signs of both the bending strain and the voltage.

The piezo-electric contribution of internal work takes the form

$$\delta H_z = \delta \sigma_t d_{ik} E_k = \left(\delta \sigma_x + \delta \sigma_y\right) e_p \frac{V}{t_p}$$

where we have assumed $d_{13} = d_{23} = e_p$ and that voltage applied across the thickness.

Variation of the overall bending strain component in the $x$-direction is

$$\delta e^b_x = \frac{t}{2} \left[ \frac{\partial (\delta \alpha)}{\partial x} \right]_{y = y_c} = \frac{t}{2} \left\{ \left( \delta \alpha_{kl}(x) \right) \phi'_k(y_c) \right\} \quad (k, l = 1, \ldots, n_p)$$

The foregoing expression may be contracted to the form as,

$$\delta e^b_x = \frac{t}{2} \left\{ \left( \delta \alpha_{ck}(x) \right) \phi'_k \right\} \quad (k = 1, \ldots, n_p)$$
Likewise, the variation of overall bending strain component in the $y$-direction is

$$
\delta e_y^b = \frac{t}{2} \frac{\partial (\delta \beta_y)}{\partial y} \bigg|_{y=y_c} = \frac{t}{2} \left( (\delta \beta_y) \bar{\phi}_j(x) \bar{\phi}_j(y_c) \right)
$$

(5.49)

where a bar indicates differentiation with respect to $y$. Equation (5.49) can be contracted to the form

$$
\delta e_y^b = \frac{t}{2} \left( (\delta \beta_k) \phi_k(x) \right) \quad (k = 1, \ldots, n_p)
$$

(5.50)

Consider next the local bending contribution due to a single dominant mode. The variation of the bending strain component in the $x$ direction is

$$
\delta e_{x,i}^b = \frac{t}{2} \frac{\partial (\delta \alpha_i)}{\partial x} \bigg|_{y=y_c} = -\frac{t}{2} \frac{m \pi}{L} \left\{ \alpha_j^j \phi_j(y_c) \sin \left( \frac{m \pi x}{L} \right) \right\} \delta \xi_i f_i(x)
$$

(5.51)

\[ j = 1, \ldots, n_p; \ i = 1, \ldots, n \]

The foregoing expression may be contracted to the form

$$
\delta e_{x,i}^b = \delta \xi_i \frac{t}{2} \left\{ \alpha_j^j f_i(x) \sin \left( \frac{m \pi x}{L} \right) \right\}
$$

(5.52)

\[ i = 1, \ldots, n \]

Likewise,

$$
\delta e_{y,i}^b = \frac{t}{2} \frac{\partial (\delta \beta_y)}{\partial y} \bigg|_{y=y_c} = \frac{t}{2} \left\{ \beta_j^j \phi_j(y_c) \sin \left( \frac{m \pi x}{L} \right) \right\} \delta \xi_i f_i(x)
$$

(5.53)

\[ j = 1, \ldots, n_p; \ i = 1, \ldots, n \]
which may be contracted to the form

\[
\delta \varepsilon_{y,i}^b = \delta \tilde{\varepsilon}_i \frac{t}{2} \left\{ \beta_i' \phi_i(x) \right\} \sin \left( \frac{m \pi x}{L} \right) \quad (i = 1, \ldots, n) \tag{5.54}
\]

The combined bending stresses on the top piezo-electric patch take the form

\[
\sigma_x^b = \frac{E_p t}{2(1 - \nu^2)} \left\{ \bar{\alpha}_{ck} \phi_k(x) + v \bar{\beta}_{ck} \phi_k(x) + \bar{\varepsilon}_i \left\{ \alpha_i' + v \beta_i' \right\} f_i(x) \sin \left( \frac{m \pi x}{L} \right) + \frac{V}{t} e_p \right\} \quad (k = 1, \ldots, n_p; i = 1, \ldots, n) \tag{5.55}
\]

and

\[
\sigma_y^b = \frac{E_p t}{2(1 - \nu^2)} \left\{ \tilde{\beta}_{ck} \phi_k(x) + v \tilde{\alpha}_{ck} \phi_k(x) + \tilde{\varepsilon}_i \left\{ \beta_i' + v \alpha_i' \right\} f_i(x) \sin \left( \frac{m \pi x}{L} \right) + \frac{V}{t} e_p \right\} \quad (k = 1, \ldots, n_p; i = 1, \ldots, n) \tag{5.56}
\]

where \( E_p \) is the young’s modulus of the piezo-electric material of the panel patch, \( V(x) \) is the voltage across the patch and \( e_p = d_{31} = d_{32} \) assumed to be the same in \( x \) and \( y \) directions. Only the case of voltage applied across the thickness is considered for panel patches.

The internal virtual work contribution, with both patches accounted for, takes the form

\[
\delta W_{\text{piezo}}^{\text{ini}} = \frac{E_p e_p t b_p}{(1 - \nu)} \int_0^L \left[ \delta \tilde{\varepsilon}_i \left\{ \alpha_i' + \beta_i' \right\} f_i(x) \sin \left( \frac{m \pi x}{L} \right) \right] V(x) \, dx \tag{5.57}
\]
Taking the voltage at any location to be proportional to the sum of the bending strains (static problem) or strain-rates (dynamic problem) at that location, we have - for the static problem

\[
V_s(x) = \frac{G_s}{2} t \left[ \alpha_c \phi'_l(x) + \beta_c \phi_i(x) + \tilde{\alpha}_c \left\{ \alpha^{i'}_c + \beta^{i'}_c \right\} f_i(x) \sin \left( \frac{m \pi x}{L} \right) \right]
\] (5.58.a)

where a bar indicates a quantity accumulated from an initially imperfect state.

For the dynamic problem

\[
V_d(x) = \frac{G_d}{2} t \left[ \alpha_c \phi'_l(x) + \beta_c \phi_i(x) + \tilde{\alpha}_c \left\{ \alpha^{i'}_c + \beta^{i'}_c \right\} f_i(x) \sin \left( \frac{m \pi x}{L} \right) \right]
\] (5.58.b)

where \( G_s \) and \( G_d \) are gains in the respective problems. The internal work contribution takes the form

\[
\delta W_{\text{piezo}}^{\text{int}} = \frac{E_p e_p t^2 b_p}{2(1-\nu)} \left[ \delta \alpha_c \left( \alpha_c b_{kl} + \beta_c c_{kl} \right) + \delta \beta_c \left( \tilde{\beta}_c a_{kl} + \tilde{\alpha}_c c_{kl} \right) \right] + \frac{1}{2} \delta \alpha_i \tilde{\alpha}_j \left\{ \alpha^{i'}_c + \beta^{i'}_c \right\} a_{ij}
\] (5.59.a)

or

\[
\delta W_{\text{piezo}}^{\text{int}} = \frac{E_d e_d t^2 b_d}{2(1-\nu)} \left[ \delta \alpha_c \left( \alpha_c b_{kl} + \beta_c c_{kl} \right) + \delta \beta_c \left( \tilde{\beta}_c a_{kl} + \tilde{\alpha}_c c_{kl} \right) \right] + \frac{1}{2} \delta \alpha_i \tilde{\alpha}_j \left\{ \alpha^{i'}_c + \beta^{i'}_c \right\} a_{ij}
\] (5.59.b)

where \( \overline{a}_{ij} = \int_x f_i f_j dx \); \( a_{kl} = \int_x \phi_k \phi_l dx \); \( b_{kl} = \int_x \phi'_k \phi_l dx \); \( c_{kl} = \int_x \phi'_k \phi'_l dx \).

Note once again \( E_p e_p G_s \) or \( E_p e_p G_d \) may be deemed as a single parameter representing panel patch control.
5.1.3.3. External Virtual Work

There are two contributions to external virtual work: the inertial terms, and external forces acting on the structure.

Considering a typical plate element, the out of plane displacement, $w$ is the most significant, arising as it does from both the overall action and local buckling. For a typical stiffener element, the in-plane displacement in the transverse direction, $v$, is the most significant with some minor contribution coming from out of plane displacement due to local buckling. The second order local buckling effects are neglected. The relevant displacement components are written in the form

$$
\begin{align*}
    w &= w_{kl} \phi_k(x) \phi_l(y) + \xi_i \left( w_{ij}^{\ell} \phi_j(y) \sin \left( \frac{m \pi x}{L} \right) \right) f_j(x) \\
    v &= v_{kl} \phi_k(x) \phi_l(y)
\end{align*}
$$

In general, the external virtual work contribution of inertial forces per unit surface area is

$$
- \bar{m} \left( \dot{w} \delta w + \dot{v} \delta v \right),
$$

where $\bar{m}$ is the mass per unit area, assumed constant over the element considered. Integrating over the element area, we obtain

$$
\delta W_{ext}^{(i)} = -\bar{m} \left\{ d_{kr} d_{ls} \left( \ddot{w}_r \delta w_{kl} + \ddot{v}_r \delta v_{kl} \right) + \frac{1}{2} \left( w_{i}^{\ell} w_{j}^{\ell} d_{kl} \ddot{\xi}_j \delta \xi_i \right) \right\}
$$

where

$$
d_{ij} = \int_x \int_y \phi_i(x) \phi_j(y) dy dx.
$$

The external loads on the structure are (i) the uniform axial compression, $\sigma$ applied over the end section and (ii) a disturbance in the form of a Heaviside step function applied at time, $\tau = 0$, and a uniformly distributed force applied directly on the stiffener to trigger a dynamic response. Since we anticipate the stiffener to deflect downwards under the
interaction (putting the plate under compression), the load is also applied in the same sense.

The virtual work of load (i) for an element of thickness $h_e$ takes the form

$$\delta W_{\text{est}}^{(ii)} = \sigma h_e g_{kl} \delta u_{kl}$$  \hspace{1cm} (5.62)

where $g_{kl} = \phi_k (x = 0) \int_y \phi_l (y) dy$ and $x = 0$ refers to the loaded edge and integration is taken over the width of the element.

Virtual work of load (ii) comes from the stiffener element and takes the form

$$\delta W_{\text{est}}^{(iii)} = p_o a_k \delta v_{jk}$$  \hspace{1cm} (5.63)

where $p_o$ is the intensity of the line load applied at the stiffener plate junction, $v_J$ is the overall stiffener displacement at the junction given by $v_J = v_{jk} \phi_k (x)$ and $a_k = \int_x \phi_k (x) dx$.

A system of nonlinear equilibrium equations is generated by equating the internal virtual work to the external virtual work summed up over all the elements; this requires that the matrices associated with linear terms be transformed to obtain relations in a global coordinate system.

Finally letting the virtual displacements be arbitrary, we obtain the forms of the equations as follows.

In the static problem,

$$[D]_r \{q\} + \int_A [B]^T \{\sigma\} dA = \{f\}$$  \hspace{1cm} (5.64.a)
Or, in the dynamic problem,

\[
[M][\ddot{q}] + [D]_d \dot{q} + \int_A [B]^T \{\sigma\} dA = \{f\} \tag{5.64.b}
\]

Piezo-electric contributions are given by the \([D]_s\) and \([D]_d\) matrices respectively.

Note that since the external voltages are deemed to be prescribed, the charge equation, i.e., the relationship between electrical displacements on the one hand and strains and the field strengths on the other becomes defunct (eq.18, [3]). Electrical charges are prescribed to be zero on the surfaces of the actuator and it is assumed that the charges developed within the piezoelectric actuators do not interfere with their behavior. The mechanical contributions of thin piezoelectric patches are subsumed in those of the host structure.

### 5.1.4. Boundary Conditions of the Panel

Figure 5.3 and Figure 5.4 show a wide stiffened plate carrying longitudinal compression. The plate is deemed to be wide so that a typical panel is representative of the entire plate (Figure 5.3). The panel center lines are assumed to be lines of symmetry of plate deformation. This implies that the dominant local buckling mode is assumed to be symmetric with respect to the center lines – an assumption that needs to be verified as an anti-symmetric mode is always a theoretical possibility. The stiffened panel is subjected to uniform uniaxial compression in the pre-buckling state. This is achieved by ensuring that the plate and the stiffener are free to undergo the Poisson expansion in the transverse direction. Thus the lines of symmetry are free to move in the \(y\)-direction, but constrained to remain straight; the stiffener edge is free to move to accommodate the Poisson expansion; end-sections are assumed to be simply supported, i.e., \(w = 0\) for each plate element at the ends.
5.1.5. Analysis of Stiffened Panel

The nonlinear analysis is based on p-version finite elements which have local buckling deformation (both the first order and second order fields) embedded in them, as explained already.

The shape functions, $\phi_i(x) \ (i=1, \ldots, p+1)$ chosen for the local buckling problem and the second order field problem are hierarchical polynomials going up to the fifth degree $p = 5$ providing $C_0$ continuity needed in the analysis. For representing the overall action in the finite elements they are again chosen to be fifth degree polynomials in both $x$ and $y$ direction. ($n_p = n_q = 5$). The precise forms of these functions are given by Szabo and Babuska [51]. The amplitude modulating functions, being “slowly” varying in concept, have $p = 2$. Only three elements are used in all, one for each half of the plate and the third for the stiffener. These have shown to be adequate in previous studies.

However, in the view of the fact the amplitude modulating functions of the plate elements are decoupled from those of the stiffener, there is greater flexibility imported into the present model.

For the analysis of the static problem, arc length method is used to trace the solution in order to negotiate the limit points and/or the phases of the response where load increases sluggishly as the deflections escalate.

For the dynamic problem, loading is taken in two steps.

First, a static axial compression is applied and the solution is traced till a given axial stress $\sigma$ is attained. At this point a suddenly applied line load is applied on the stiffener triggering the dynamic response. The solution is traced with appropriately chosen time increments. Newmark’s $\beta$ method is employed to replace the time dependent terms in terms of unknown incremental displacements for the current time increment and known
quantities at the end of previous time step; this is followed by Newton-Raphson iterations till convergence is achieved [55], [44].

5.2. Case Study

5.2.1. Geometry, Material and Buckling Data

The geometry and material properties of the panel (illustrated in Figure 5.4) is defined as shown in Table 5.1 and Table 5.2 respectively. Note the geometric proportions are the same as first used by Tvergaard [55] and subsequently investigated by several others.

Table 5.1. Geometry of the stiffened panel

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>The length of the plate</td>
<td>454.4 mm</td>
</tr>
<tr>
<td>$b$</td>
<td>The width of the panel</td>
<td>$L/4$</td>
</tr>
<tr>
<td>$t_p$</td>
<td>Thickness of the plate</td>
<td>1 mm</td>
</tr>
<tr>
<td>$d_s$</td>
<td>The depth of the stiffener</td>
<td>$b/10$</td>
</tr>
<tr>
<td>$t_s$</td>
<td>Thickness of the stiffener</td>
<td>$0.4d_s$</td>
</tr>
<tr>
<td>$A$</td>
<td>The cross sectional area</td>
<td>$b t_p + d_s t_s$</td>
</tr>
<tr>
<td>$\bar{h}$</td>
<td>The averaged thickness of the panel</td>
<td>1.4544 mm</td>
</tr>
</tbody>
</table>

Table 5.2. Material properties of the panel (host structure)

<table>
<thead>
<tr>
<th>Host Structure (Aluminum Alloy)</th>
<th>$E$</th>
<th>63 GPa</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\nu$</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>$\rho$</td>
<td>2500 kg/m$^3$</td>
</tr>
</tbody>
</table>
Buckling data and imperfections

The panel is designed to be optimal in that the overall critical stress \( (\sigma_1) \) and the local critical stress \( (\sigma_2) \) are nearly the same with \( \frac{\sigma_1}{E} = 0.469 \times 10^{-3} \) and \( \frac{\sigma_2}{E} = 0.471 \times 10^{-3} \) - \( (\sigma_1 = 29.55 \text{ MPa}, \sigma_2 = 29.67 \text{ MPa}) \) something that makes the structure imperfection-sensitive. \( \sigma_{cr} \), the critical stress is the lower of these two \( (\sigma_1) \), and \( P_{cr} \), the critical load/ \( A\sigma_{cr} \). The overall mode consists of a single half-sine wave in the longitudinal direction with some cross-sectional distortion whereas the local mode consists of 6 half-waves \( (m = 6) \). Cross sectional view is shown in Figure 5.2. Imperfections are assumed to be: 0.1 \( h \) both in the local and overall modes. The overall imperfection is taken on the stiffener side (downwards in the Figure 5.4) which is the preferred side for the panel to deflect under the interaction in the absence of imperfection.

5.2.2. Selection of piezo-electric material properties

All calculations were performed setting \( E_s = E_p = E = 63 \text{ GPa} \) (note that the subscript \( s \) and \( p \) indicates stiffener and plate patches respectively) and \( E_s e_s = E_p e_p = 0.0283 \text{ N/mV} \). However, the results of such a calculation may be interpreted for specific materials provided the actuator gains are deemed to be appropriately adjusted so that the chosen control parameters \( E_s e_s G_s / E_s e_s G_d \) and \( E_p e_p G_s / E_p e_p G_d \) are maintained constant. Voltages, of course, do scale with the gains.

The following materials used in previous research (Li et al. 2008 [29]) are cited in this discussion:

(i) MFC (Micro-Fiber Composite)

This is a high performance material selected for stiffener control. MFC actuator employs the inter-digitated electrodes (Azzouz et al., 2001 [4]) with the direction of the field
coinciding with the longitudinal direction and utilizes the high value of $d_{11}$ for the piezoelectric constant, $e$. The magnitude of $Ee$ from Li, et al., 2008 is $0.03457 \text{ N/mmV}$ [29]. The ratio $Ee / E_s e_s = 1.22$ (refer Table 1.1). The electrode spacing is taken as 1 mm.

(ii) PZT5A

A traditional monolithic isotropic piezo-ceramic composite - which responds to field strength across the thickness in terms of $d_{31} (= d_{32} = e)$. $Ee = 0.01186 \text{ N/mmV}$ [29]. This material may be used for panel control where the voltage demand is evidently less severe. The ratio $Ee / E_p e_p = 0.42$.

5.3. Results and Discussion

5.3.1. Static Problem

First consider the static uncontrolled behavior. The maximum displacement as obtained at the center of the panel, (Point A at mid-span section, Figure 5.1) is plotted against the axial stress in Figure 5.6 below. This excludes the local buckling contribution which vanishes as $m = 6$. It is seen that around 78% of the critical load the panel loses its stiffness and deflections increase without any limit. These results are in close agreement with those obtained from a fully fledged nonlinear analysis using ABAQUS.

Next the panel is sought to be controlled by piezoelectric patches attached to the top and bottom (1 and 2 in Figure 5.5) of the stiffener actuated with voltages ($G_s = 10^7$) proportional to the bending strain (equal to the difference in the strains recorded at top and bottom divided by 2). The width of patches is equal to the thickness $t_s$ of the stiffeners.
Figure 5.6 plots the applied compressive stress against central panel displacement. It is seen that with this scheme stiffener deflections are effectively controlled, the structure stiffens up and the load carrying capacity (in terms of stress) exceeds 0.80 $\sigma_{cr}$ ($\sigma_{cr} = \sigma_l = 29.55$ MPa). But the maximum voltage ($V_{max}$) of the piezo-electric patches at mid-span builds up rapidly as the stress approaches $\sigma_{cr}$ and attains a value of 600 volts at about 0.8 $\sigma_{cr}$ and continues to escalate at an increasing rate as shown in Figure 5.7. With this result in hand, one can select the appropriate piezo-electric material. These results are applicable for MFC piezo electric material with gain $G_s$ and the voltage stepped down by the factor 1.22.

Given the maximum field strength sustainable by MFC is 2000 volts/mm [16], a patch thickness of 0.25 mm should prove adequate. The control exercised on the stiffener has the effect of mitigating local buckling deflections as well, as illustrated by Figure 5.8. Thus piezo-electric control proves effective in neutralizing the adverse effects of mode interaction.
Figure 5.7. Axial compression versus Stiffener patch voltage at mid-span

Figure 5.8. Axial compression versus maximum local buckling amplitude (at the plate center)
5.3.2. **Dynamic Problem**

Next consider the scenario when the stiffened panel carrying an axial compression is of 10 MPa ($= 0.33 \sigma_{cr}$) is perturbed by a suddenly applied load. The axial stress considered is $0.33 \sigma_{cr}$ and the disturbing force is a uniformly distributed line load equal to $0.01 P_{cr}$ applied along the stiffener. The uncontrolled response typified by the plate central deflection time history is illustrated in Figure 5.9.

Control is exercised by voltages are proportional to the bending strain – rate. We present two scenarios:

(i) **Stiffener control**: In the first case control is exercised, as in the static case, by actuators running along the stiffener tips with field strengths proportional to bending strain rate of the stiffener. Though no attempt is made to control plate vibrations directly, a high gain is selected for the patches with a view to influence the local buckling displacements of the plate.
(ii) **Stiffener-Panel control**: In the second case, additional piezo-electric patches are placed on either side of the plate, centered along the lines of symmetry. The width of patch is set equal to $0.2b$ ($b =$ spacing of the stiffeners). A relatively low gain is selected for these patches.

Consider first the “stiffener control” scenario. A high gain of $G_d = 10000$ is selected. Figure 5.10 plots the overall components of displacements at the middle of the panel with time for the case where the control is exercised from the stiffener patches only. It is seen that the oscillations of overall displacements are controlled within 0.02 second. Figure 5.11 shows the corresponding maximum voltages recorded at stiffener midspan. The maximum voltage attained is relatively high (about 1100 volts) and (by the same reasoning used in the static case), calls for an MFC patch of 0.45mm thickness. Figure 5.12 shows the maximum local buckling oscillations. These do not die down but continue to linger for a long time. If the local vibration of the plate elements is not an issue, then this type of control is adequate.
Figure 5.11. Variation of voltage across the piezo patch at the center of stiffener with time.

Figure 5.12. Maximum local buckling amplitude versus time under Stiffener only control.
Next we consider the scenario of “Stiffener-Panel control”. In this case the gains are small for both the stiffener patches and panel patches. These are 1000 and 100 for the stiffener patch ($G_d$) and the panel patch ($\bar{G}_d$) respectively. Figure 5.13 shows the displacement history at the center of the panel. The deflections are controlled within duration of 0.2 sec. Voltages attained are modest (refer Figure 5.14): a maximum of about 160 Volts. This with appropriate scaling of gain ($G_d$) and voltage (divide by 1.22) can be carried by an MFC patch of 0.066mm thickness.

If on the other hand PZT5A is selected, the gain and the maximum voltage are scaled up (by division by 0.44) to be 364 volts and given the capacity of PZT5A is 820 V/mm, a thickness of 0.5mm should suffice. Local buckling displacements too are well controlled as illustrated by Figure 5.15.

![Figure 5.13. Maximum stiffener displacement time history under stiffener-panel control](image)
Figure 5.14. Variation of voltage (V/mm) across the piezo patch at the center of stiffener with time

Figure 5.15. Maximum local buckling amplitude versus time under stiffener-panel control
Panel patch voltage histories are illustrated in. Figure 5.16 and Figure 5.17 show the time history of the maximum overall component recorded at the center of the panel and it is seen that it is less than 8 volts. Figure 5.17 illustrates the time history of maximum amplitude of the sinusoidal component of the voltage across the panel patches and this too is very modest. These must be taken together and thus we estimate the maximum voltage to be 14 V.

Since PZT5A is the chosen piezo-electric material for the panel patches, the voltage and the gain \( \bar{G}_d \) have to be scaled up (by division by 0.44) and, in view of the voltage capacity of PZT5A the thickness patch need not (theoretically) be higher than 0.04 mm.

Figure 5.16. Overall component of voltage across the panel patch at the center of the panel versus time
5.3.3. Scaled up Panel

The panel investigated thus far appears to be of miniature size in comparison to what may be used in typical aircraft structures. The results obtained there from leave open the question of feasibility of piezo-electric control in practical situations. In order to be able to pronounce on the feasibility of piezo-electric control, we consider a panel with dimensions scaled up by a factor of 3. (Thus $t = 3\text{mm}$, $L = 1363.2\text{ mm}$). This panel is examined with respect to the requirements of piezo patch thickness and the corresponding voltage whilst carrying the same axial compressive stress as the original panel.
5.3.3.1. Static Case

First consider the static case. Dimensional analysis indicates that both the scaled up and original panels perform in an identically same manner considering strains, stresses and electrical field strengths. Given that the patch thickness must scale in the same manner as the geometry (a factor of three), voltage too must scale by the same factor. Since in our model, the piezo-electric patches are subsumed in the host panel and the patch thickness does not explicitly figure in the analysis input, the response of the original panel can be reproduced by scaling the gain by a factor of three. In any case, the voltages across the patches must be amplified three fold. Referring to Figure 5.11, the maximum voltage of 600 volts for the original panel attained at $0.8 P_{cr}$, must be amplified as 1800 volts. For the MFC patch thickness needed works out to be 0.75 mm. This appears feasible given the advances in the manufacturing technology of piezo-electric materials.

5.3.3.2. Dynamic Case

Next consider the dynamic response of the scaled up panel. Here the mass density of the material of the host structure remains the same. From dimensional analysis it is seen that frequencies are scaled inversely proportional to the length scale $L_s$. Let the gains be chosen to be proportional to $L_s$ to mimic the presence of patches of thickness scaled by $L_s$. However since the strain-rate is proportional to the frequency (scaled down by a factor of $L_s$) and the voltages are obtained by multiplying the gains (scaled up by a factor of $L_s$) by strain rates, the voltages across the patches must be the same for both the models.

The foregoing observations are borne out by a comparison of Figure 5.13 - Figure 5.17 and Figure 5.18 - Figure 5.19. Figure 5.18 shows the displacement history at the center of the panel for a $G_d = 3000$ and $G_d = 300$. Comparing this to Figure 5.13, it is seen that the displacement scale up by a factor of $L_s$ and as does the time period of oscillations. Figure 5.19 plots the voltage history at the stiffener patch at mid-span. Comparing this to that in Figure 5.14, the maximum voltage is seen to roughly the same in the two models.
Next consider how one may estimate the settling time for the scaled up model, given the response of the original model. Once again, from dimensional analysis it is seen that the inertial forces are proportional to $L_s^4$, elastic forces (stiffness terms) and the piezo-electric damping forces (with the scaled up gain) are both proportional to $L_s^2$. Thus it follows the damping ratio $\xi$ is inversely proportional to $L_s$.

Figure 5.18. Displacement time history for the scaled up panel with gains scaling with geometry
Figure 5.19. Stiffener patch voltage for the scaled up panel (gains scaling with geometry)

Figure 5.20. Displacement time history for the panel (geometry scaled up by 3 and gains scaled up by 10)
5.4. Conclusions

Piezo-electric control of the static and dynamic responses of a stiffened panel subject to interactive buckling was studied with a view to assess the effectiveness of the strategy and feasibility. Considering first the static response, it was found that the adverse effects of interaction such as imperfection-sensitivity and the loss of stiffness well before the critical load is approached can be counteracted with relative ease using piezo-electric feedback control. The piezo-electric patches are placed along the tips of the stiffener with feedback voltages proportional to the bending strains thereof. With this arrangement, it is possible to stiffen up the structure and attain critical loads obtained by linear stability
analysis. However the voltage across the patches may escalate and reach unacceptably high values at values of load close to the critical.

Dynamic response is triggered by the application of a lateral disturbance to the panel carrying axial compression. The strategy of controlling the stiffener alone by feedback voltages proportional to the bending strain-rate had a salutary effect on the overall response, but could not control the local buckling oscillations. Thin piezo-electric strips attached respectively to the top and bottom surfaces along the longitudinal center line of the panel were employed to control the panel deflections. The feedback voltages were proportional to the sum of the bending strains in the longitudinal and transverse directions. This form of control proved very effective and resulted in a minimal voltage demand for damping out the oscillations.
Chapter 6

Flutter of Axially Compressed Stiffened Panels with Edges Free to Move In-plane

The behavior of ‘optimally’ designed stiffened panels under axial compression and control thereof was the subject of chapter 5. Under static loading they tend to fail by an interaction of overall and local buckling; at certain combinations of axial compression and dynamic disturbances such as lateral acoustic pressure these structures can undergo large amplitude oscillations and could experience divergence at loads smaller than the static buckling load.

This chapter considers flutter of stiffened panels under aero-dynamic loading. As seen in chapter 4, flutter refers to gradually escalating oscillations whose amplitudes may attain limits or increase without any limit. Plates with their edges restrained from moving and subjected to increasing aerodynamic loading ($\lambda$) would experience flutter beyond a critical value of $\lambda$, ($\lambda > \lambda_{cr}$) and execute stable limit cycle oscillations (LCO). The stabilization is due to in-plane stretching and shearing caused by the bending of the plate and thus the plate is under the influence of hardening nonlinearities. Flutter of aircraft wings is a multi-modal phenomenon in that many vibration modes are required to represent the deformations that develop. In stiffened panels these can take the form of a combination of a number of ‘local’ patterns of small wave plate bending on the one hand and long wave ‘overall’ bending with stiffener participation on the other. Though a
significant amount of work has been done in the study of phenomenon of nonlinear flutter of plates and control thereof, the authors are not aware of any work on stiffened panels prone to flutter.

In all the previous work on flutter of plates (vide Literature Survey, section 1.4.2 in chapter 1), it has been assumed that the plate is restrained against in-plane movements. This assumption has been maintained even for plates carrying in-plane compression. As a result of this assumption, it is seen that the plates never lose their dynamic stability, i.e. they undergo stable LCO under aerodynamic pressure of any magnitude as long as they remain elastic. The present study, on the other hand considers plate panels in which the plate edges are allowed to move but constrained to remain straight. These boundary conditions completely change the dynamic behavior of plate subjected to aerodynamic pressure.

Under these relaxed boundary conditions, the stiffened panel may become dynamically unstable as soon as a critical value of air velocity (represented by the critical value of non-dimensional pressure coefficient, $\lambda = \lambda_{cr}$) is reached; or it may exhibit stable LCO for a range of values of $\lambda$ beginning from $\lambda_{cr}$ and ending with a certain limiting value ($\lambda_{lim}$) at which the panel becomes dynamically unstable. In either case a stable limit cycle is unavailable for $\lambda$ greater than maximum of $\lambda_{cr}$ or $\lambda_{lim}$. This situation can precipitate failure as this would correspond to oscillations of rapidly escalating amplitudes. It is also seen that under a suddenly applied pressure or impulse the plate may lose stability for values of $\lambda < \lambda_{lim}$. In so far as edge movements are unavoidable under working conditions, the phenomena reported herein should be of paramount importance in the design, operation and control of aerospace vehicles.

- **Aerodynamic pressure**

The aerodynamic forces on an object are obviously related to the relative speed of the object. Often this is represented by Mach number ($M$). Mach number stands for the relative speed of an object moving in the air divided by the speed of sound and it is used to express the speed of an aircraft or a missile in high speed. It commonly represents the
speed of an aircraft or missile in high speed. $M \gg 1$ for supersonic and hypersonic speeds. The supersonic aerodynamic pressure loading is expressed using the first order piston theory (Bisplinghoff and Ashley, 1962 [6]; Dowell, 1975 [13]). This theory relates the aerodynamic pressure and panel transverse deflection as follows.

\[
\Delta p = -\frac{2q_a}{\beta} \left[ \frac{\partial w}{\partial x} + \frac{M^2 - 2}{M^2 - 1} \frac{1}{\gamma^2} \right] \frac{\partial w}{\partial t} \quad (6.1)
\]

wherein \( q_a = \frac{1}{2} \rho_a V^2 \) and \( \beta = \sqrt{M^2 - 1} \), \( \rho_a \) is the air density, \( V_\infty \) is the air flow velocity. Note that the second term corresponds to aerodynamic damping.

It is assumed that airflow is parallel to \( x \)-direction (the longitudinal direction) on the plate surface. Using non-dimensional parameters, eq. (6.1) may be written in the form (Abdel Motagaly et al., 2005 [2]) as follows.

\[
\Delta p_a = - \left[ \frac{2q_a L^3}{\beta D_{11}} \frac{\partial w}{\partial x} + \frac{g_a D_{11}}{L^4} \frac{\partial w}{\partial t} \right] \quad (6.2)
\]

where:

\[
\lambda = \frac{2q_a L^3}{\beta D_{11}}, \quad g_a = \frac{\rho_a V_\infty (M^2 - 2)}{\rho h \omega \beta^3}, \quad C_a = \frac{g_a}{\lambda}, \quad \omega = \sqrt{\frac{D_{11}}{\rho h L^4}} \quad (6.3.a-c)
\]

are the non-dimensional pressure, non-dimensional aerodynamic damping parameter, and panel reference frequency respectively. \( D_{11} \) is a reference bending stiffness, the first entry in the D-matrix of the composite plate (Jones, 1999 [21]) having a fiber orientation along the \( x \)-axis, \( \rho, L, \) and \( h \) are the density of the material of the panel, the length of the plate and thickness of the plate respectively.
6.1. Nonlinear Modal Analysis of Axially Compressed Stiffened Plates

6.1.1. General Considerations

A modal analysis is employed and the participating modes fall under two distinct categories, viz. “local” plate bending modes and “overall” beam bending modes with their nonlinear modal interaction duly accounted for. Such delineation offers a clear insight into panel response in general and mode interaction in particular. The panel is simply supported at its ends and so a finite strip method is employed for the analysis.

- Potential Energy Function

As a preliminary, we develop the potential energy function for the elastic forces and conservative externally applied forces in the following manner:

\[ \Pi = \frac{1}{2} \{\sigma_0 + \tilde{\sigma}\} \bullet \{\varepsilon_0 + \tilde{\varepsilon}\} + t_0 \circ (u_0 + \tilde{u}) \]  \hspace{1cm} (6.4)

where \(\sigma_0, \varepsilon_0\) are generic stress and strain “vectors” of the ground/unbuckled state \(\tilde{\sigma}, \tilde{\varepsilon}\) are the same evolving there from; \(u_0, \tilde{u}\) are the displacements of the ground state and those evolving there from; and \(t_0\) is the prescribed traction vector at the surface. The heavy dot \(\bullet\) indicates scalar multiplication and integration over the entire volume of the body considered and the light dot \(\circ\) indicates scalar multiplication and integration over the surface of the body. Here and in the sequel we have employed Budiansky’s notation with minor modifications (Budiansky, 1966 [8]).
This expression may be simplified using the following relationships:

(i) By symmetry of the generic stress-strain relationship matrix, viz., $\sigma = H \varepsilon$

\[ \sigma \propto = \varepsilon \sigma \] \hspace{1cm} (6.5)

and discarding terms that depend only on the ground/unbuckled state, we have

\[ \Pi = \frac{1}{2} \sigma \cdot \tilde{c} + \sigma \cdot \varepsilon - t \circ \tilde{u} \] \hspace{1cm} (6.6)

(ii) The strain-displacement relations take the form

\[ \tilde{c} = L_1(\tilde{u}) + \frac{1}{2} L_2(\tilde{u}) \] \hspace{1cm} (6.7)

where $L_1$ and $L_2$ are linear and quadratic differential operators respectively, and

(iii) Invoking divergence theorem:

\[ \tilde{c} \cdot L_1(\tilde{u}) = t \circ \tilde{u} \] \hspace{1cm} (6.8)

we obtain

\[ \Pi = \frac{1}{2} \left( \tilde{c} \cdot \tilde{c} + \sigma \cdot L_2(\tilde{u}) \right) \] \hspace{1cm} (6.9)
**Generic Stress and Strain Vectors**

Considering the coordinate system of stiffened panel as shown in Figure 5.3, the generic strains of the Kirchhoff plate theory are of the form

\[
\{\varepsilon\}^T = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}, \chi_x, \chi_y, \chi_{xy}\}
\]  
(6.10)

where \{\varepsilon\} = \{\varepsilon_x, \varepsilon_y, \gamma_{xy}\} are mid-surface strains, \{\chi\} = \{\chi_x, \chi_y, \chi_{xy}\} are curvature components.

Note

\[
\{\chi\}^T = \left\{-\frac{\partial^2 w}{\partial x^2}, -\frac{\partial^2 w}{\partial y^2}, -2\frac{\partial^2 w}{\partial x \partial y}\right\}
\]  
(6.11)

The conjugate stress quantities are in-plane forces (N) and moments (M) per unit length across the plate thickness as shown below.

\[
\{\sigma\}^T = \{N_x, N_y, N_{xy}, M_x, M_y, M_{xy}\}
\]  
(6.12)

The stress strain relation for a mid-plane symmetric laminate is given by (Jones, 1998 [21]):

\[
\{\sigma\} = [H]\{\varepsilon\} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\{\varepsilon\}
\]  
(6.13)
6.1.2. Eigen-value Problem for Modes of Vibration

As a first step, the characteristic modes of vibration (in the absence of aerodynamic pressure) are determined. The problem of vibration of a plate structure in a state of uniform in-plane stress in general can be formulated in terms of a potential function,

\[
\Pi = \frac{1}{2} \bigg[ HL_i(u) \cdot L_i(u) - \sigma_o \cdot L_{11}(u) - \omega^2 \rho u \cdot u \bigg]
\]  

(6.14)

where \( \cdot \) denotes multiplication and integration over the entire structure, is the pre-buckling stress assumed to be uniform in the \( x \)-direction, \( H \) denotes the generic stress-strain relationship matrix, \( \rho \) stands for mass density of the material and \( \omega \) stands for the frequency. For a given axial compression, an Eigen-value problem for \( \omega \) results upon rendering this function stationary. The equations governing the problem are:

\[
\delta \Pi = \left[ HL_i(u) \cdot L_i(\delta u) - \sigma_o \cdot L_{11}(u, \delta u) - \omega^2 \rho u \cdot \delta u \right] = 0
\]  

(6.15)

Here we consider a uniformly compressed stringer stiffened plate made of a specially orthotropic material.

The \( x \)-variation of the displacement functions are so selected as to satisfy the differential equations of equilibrium and the simply supported end conditions.

Thus

\[
\sigma_o = \begin{bmatrix} -N_x^o & 0 & 0 & 0 & 0 \end{bmatrix}^T
\]  

(6.16)
A description of the mode characterized by \( m \) half-waves in the longitudinal direction within a strip takes the form

\[
\begin{align*}
    u &= u_{mi} \psi_i(y) \cos(m \pi \eta) \\
    v &= v_{mi} \psi_i(y) \cos(m \pi \eta) \quad (i = 0, \ldots, 4) \quad \text{(no sum on ‘m’)} \\
    w &= w_{mi} \phi_i(y) \cos(m \pi \eta)
\end{align*}
\]

where \( \eta = \frac{x}{L} \) and \( u_{mi}, v_{mi}, w_{mi} \) are the degrees of freedom assumed to be four in number and \( \psi_i(y) \) and \( \phi_i(y) \) are appropriate shape functions of \( y \), the transverse coordinate of the plate element (ref. eq. (6.20 - 6.21)). The modes thus obtained constitute the first order field in the nonlinear problem discussed subsequently.

The flutter problem considered here is formulated in terms of the following characteristic modes of deformation:

(i) The “local” or plate bending modes

These modes are such that the stiffener-plate junction undergoes negligibly small displacements and each plate element bends out of plane. In the context of buckling these would be “short-wave” modes characterized by large number of half-waves, but this will not necessarily be the case in the context of flutter. Nevertheless such plate bending modes will be designated as “local” in contrast to “overall” modes discussed in the sequel.

The local modes are readily treated using von Karman plate theory and is described in terms of a single variable ‘\( w \)’ for each plate element.
A typical mode consisting of $m$ half-waves in the $x$-direction takes the form:

\[
\begin{aligned}
    u_{lo} &= v_{lo} = 0 \\
    w_{lo} &= w_{mi} \phi_i(y) \sin(m \pi \eta) \quad \text{(no sum on ‘$m$’)} \\
\end{aligned}
\quad (6.18.a-b)
\]

where $\phi_i(y)$ describe the transverse variation of $w$. Over each element, they take the form of cubic polynomials with inter-element $C_1$ continuity and are given in the next section. Note that in this description of the plate bending mode, the normal displacement of each of the plate elements that meet at a junction are prescribed to be zero at the junction while the rotational compatibility at the junction is maintained. The modes of vibration being orthogonal are uncoupled from each other in the Eigen-value problem. For the stiffened panel considered here, this mode can be symmetric or anti-symmetric with respect to the plate-stiffener junction. Figure 6.1 illustrates these modes.

(i) Overall bending modes

The modes involving bending of the plate together with the stiffener as a beam are designated as “overall”. In this case the stiffener undergoes significant in-plane deformation and the role of in-plane displacements must be accounted for in the mass and initial stress matrices for each plate element in the analysis.

Letting $U$, $V$ and $W$ be the displacements in the $x$-, $y$- and $z$- directions of a typical plate element, typical overall mode takes the form:

\[
\begin{aligned}
    U_{ov} &= U_{mi} \psi_i(y) \cos(m \pi \eta) \\
    V_{ov} &= V_{mi} \psi_i(y) \sin(m \pi \eta) \quad \text{(no sum on ‘$m$’)} \\
    W_{ov} &= W_{mi} \phi_i(y) \sin(m \pi \eta) \\
\end{aligned}
\quad (6.19.a-c)
\]
where \( \psi_i(y) \) are shape functions for in-plane displacements, \( U \) and \( V \). Some constraints are placed on the displacement pattern so that cross-sectional distortions are eliminated so that we would recover pure overall bending modes with have no contamination of transverse plate bending action. Thus the axial displacement \( U \) and the out-of-plane displacement \( W \) of the plate are constrained to remain constant, and out of plane displacement of the stiffener are eliminated. No constraints are placed on the in-plane transverse displacement \( V \) of the plate and the stiffener in order to allow for free Poisson expansion in the cross-section. This approach has been preferred in the present study because it facilitates the separation of the total deformation into two distinct modes of deformation, viz. local and overall, and yields terms that encapsulate the nonlinear mode interaction.

Figure 6.1. Cross-sectional view of (a) anti-symmetrical local mode and (b) symmetric local mode
6.1.2.1.  Shape Functions

Since Kirchhoff’s theory is used, shape functions for \( w \) must have \( C_1 \) continuity. For \( u \) and \( w \), we need shape functions of \( C_0 \) continuity, but in order to minimize membrane locking, higher order polynomials rather than linear functions are chosen. Thus for all the displacements, cubic polynomials are selected.

With origin taken the center of the strip, the shape functions are expressed in terms of \( \zeta = \frac{y}{b_e} \), with \(-1 \leq \zeta \leq 1\), as \(-\frac{b_e}{2} \leq y \leq \frac{b_e}{2}\), \( b_e \) being the width of the strip.

- Shape functions for \( y \)-variation of \( w \)

The shape functions \( \phi_i(y) \) appearing in eq. 6.15(b) and 6.16(c) are taken in the form

\[
\phi_1 = \frac{1}{2} - \frac{3}{4} \zeta + \frac{1}{4} \zeta^3; \quad \phi_2 = \frac{b_e}{2} \left( \frac{1}{4} \zeta \right) \left( \frac{1}{4} \zeta \right) \left( \frac{1}{4} \zeta \right) \left( \frac{1}{4} \zeta \right); \\
\phi_3 = \frac{1}{2} + \frac{3}{4} \zeta - \frac{1}{4} \zeta^3; \quad \phi_4 = \frac{b_e}{2} \left( -\frac{1}{4} \zeta \right) \left( -\frac{1}{4} \zeta \right) \left( -\frac{1}{4} \zeta \right) \left( -\frac{1}{4} \zeta \right) \tag{6.20.a-d}
\]

- Shape functions for \( y \)-variation in-plane displacements

In-plane displacement variations need to be assumed for the obtaining the overall modes as well as for the determination of second order fields discussed subsequently. These take the form

\[
\vec{u} = \vec{u}_{mi} \psi_i f_m(x) \tag{6.21.a}
\]
where

\[ \psi_1 = \frac{1}{2}(1 - \zeta) \quad ; \quad \psi_2 = \frac{1}{2}(1 + \zeta) \]

\[ \psi_3 = \frac{\sqrt{6}}{4}(\zeta^2 - 1) \quad ; \quad \psi_4 = \frac{\sqrt{6}}{4}(\zeta(\zeta^2 - 1)) \]  

(6.21.b-e)

The functions, \( \psi_i(y) \), are hierarchical in nature, the first two being linear and associated with values at the ends of the strip and the other two vanishing at the ends and providing quadratic and cubic variations over the strip respectively. \( f_m(x) \) is a \( x \)-variation in the form of a certain harmonic function. The foregoing shape functions (eq. 6.20 and 6.21) are used to obtain a set of distinct local and overall modes of vibration using eq. (6.15) for each \( m \).

- **Modal degrees of freedom and Total displacement components**

The problem of flutter of a compressed panel with possible nonlinear interaction of local and overall bending is formulated in terms of ‘\( M \)’ local modes and \( M_o \) overall modes, suitably normalized. The scalar parameters associated with the local modes, viz. \( \xi_m (m = 1, \ldots, M) \) and overall modes viz. \( X_m (m = 1, \ldots, M_o) \) together constitute \( M + M_o \) degrees of freedom.

### 6.1.2.2. Linear Flutter Problem

First, consider the linear flutter analysis.

For a constituent strip element, the displacement components in the local coordinate system take the form
\[ u^* = X_n \{ U_n \psi_i \cos(n \pi \eta) \} \]
\[ v^* = X_n \{ V_n \psi_i \sin(n \pi \eta) \} \]
\[ w^* = X_n \{ W_n \psi_i \sin(n \pi \eta) \} + \xi_m \{ w_{mi} \psi_i \sin(m \pi \eta) \} \]
\[(m = 1, \ldots M; \ n = 1, \ldots M_o)\] (6.22.a-c)

For a stiffener element constituting the stiffener

\[ u^* = X_n \{ U_n \psi_i \cos(n \pi \eta) \} \]
\[ v^* = X_n \{ V_n \psi_i \sin(n \pi \eta) \} \]
\[ w^* = \xi_m \{ w_{mi} \psi_i \sin(m \pi \eta) \} \]
\[(m = 1, \ldots M; \ n = 1, \ldots M_o)\] (6.23.a-c)

where * indicates the sum of local and overall contributions.

- **Potential Energy**

The total potential energy of a stiffened plate panel under uniform uniaxial compression in the ground/unbuckled state may be written using eq. (6.9) in the form

\[ \Pi = \frac{1}{2} \int_{x=0}^{l} \int_{y=0}^{h} \left[ \dddot{N}_x \dddot{\xi}_x + \dddot{N}_y \dddot{\xi}_y + \dddot{N}_{xy} \dddot{\gamma}_{xy} + M_x \dddot{\xi}_x + M_y \dddot{\xi}_y + M_{xy} \dddot{\gamma}_{xy} \right] dx dy \]
\[ - \frac{N_x^0}{2} \int_{x=0}^{l} \int_{y=0}^{h} \left[ \left( \frac{\partial \dddot{u}}{\partial x} \right)^2 + \left( \frac{\partial \dddot{v}}{\partial x} \right)^2 + \left( \frac{\partial \dddot{w}}{\partial x} \right)^2 \right] dx dy \] (6.24)

For the determination of the critical aerodynamic pressure corresponding to incipient flutter, only the quadratic terms in the potential energy must be considered.
Thus we have

\[
\Pi^{(1)} = \frac{1}{2} \int_{x=0}^{L} \int_{t=0}^{T} \left[ N_{x,L}^* \epsilon_{x,L}^* + N_{y,L}^* \epsilon_{y,L}^* + N_{xy,L}^* \gamma_{xy,L}^* + M_{x}^* \chi_{x}^* + M_{y}^* \chi_{y}^* + M_{xy}^* \chi_{xy}^* \right] dx dy
\]

\[
- \frac{N_y^0}{2} \int_{x=0}^{L} \int_{t=0}^{T} \left[ \left( \frac{\partial u^*}{\partial x} \right)^2 + \left( \frac{\partial v^*}{\partial x} \right)^2 + \left( \frac{\partial w^*}{\partial x} \right)^2 \right] dx dy
\]

(6.25)

where the subscript \(L\) indicates the linear part of the quantity, e.g. \(\epsilon_{x,L} = \frac{\partial u^*}{\partial x}\) and so on.

Note the tilde quantities have been replaced by the starred quantities to indicate their source viz. eq. (6.22 – 6.23). The first integral gives the “linear” stiffness matrix and the second gives the initial stress matrix. The superscript (1) on \(\Pi\) indicates the order of the problem.

- **Kinetic Energy**

The kinetic energy takes the form

\[
T = \frac{\rho}{2} \int_{x=0}^{L} \int_{t=0}^{T} h \left[ \left( \frac{\partial u^*}{\partial t} \right)^2 + \left( \frac{\partial v^*}{\partial t} \right)^2 + \left( \frac{\partial w^*}{\partial t} \right)^2 \right] dx dy = \frac{\rho}{2} \frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial t}
\]

(6.26)

\(i = 1, \ldots, 3\)

where \(\{u\} = \{u^* v^* w^*\}^T\).

The overall modes being symmetric with respect to the junction line are all orthogonal to the antisymmetric local modes; on the other hand there occurs a coupling of these modes with symmetric local modes. In the latter case, the overall mode associated with a certain ‘\(m\)’ will couple with the local mode associated with the same ‘\(m\)’.
Thus, in a modal analysis of a linear free vibration analysis, where both the local and overall modes are considered together, the mass, stiffness and initial stress matrices would not be diagonal. As a result, for each m there will be two coupled modes, a local mode modified by a small junction displacement ($W$ for the plate and $V$ for the stiffener) and an overall mode carrying some cross-sectional distortions.

**Equations of Motion**

The virtual work equation for incipient flutter problem may be written in the form

$$
\delta \Pi^{(i)} = -\rho \frac{\partial^2 \mathbf{u}^*}{\partial t^2} \cdot \delta \mathbf{u}^* + \frac{D_{ll}}{a^3} \left[ \lambda \frac{\partial w^*}{\partial x} + \frac{\sqrt{C_a \lambda}}{\omega \omega_c} \frac{\partial w^*}{\partial t} \right] \circ \delta w \quad (i = 1,.,3) \quad (6.27)
$$

where $\delta \Pi$ is the first variation of the potential energy (which is numerically equal to the sum internal virtual work of elastic forces including the axial compression); the virtual work contributions of inertial forces and aerodynamic pressure are represented by the first and second terms respectively on the right hand side; $C_a = \frac{g_a^2}{\lambda}$ is an often used aerodynamic damping parameter – an alternative to $g_a$ (eq. 6.6.b) and $\circ$ denotes multiplication and integration over the surface over which flow occurs.

$$
\mathbf{u} = \bar{\mathbf{u}} e^{\alpha t} \quad (6.28)
$$

Letting the linear algebraic Eigen-value problem for is solved for the complex roots of $\alpha$ incrementing $\lambda$ in sufficiently small steps till one of the roots acquires a positive real part.

If aerodynamic damping is neglected, the governing equations take the relatively simple form
\[
\left[ A - \sigma_n [B] - \lambda [C] - \omega^2 [M] \right] \begin{bmatrix} \xi \\ X \end{bmatrix} = 0
\] (6.29)

where \( A, B, C, \) and \( M \) are the stiffness, initial stress, aerodynamic pressure and mass matrices respectively and \( \omega \) is the characteristic frequency. Of these \( A, B, \) and \( M \) are symmetric whereas \( C \) is skew-symmetric. For a set of antisymmetric local modes, there is decoupling of the \( \xi \) and \( X \) degrees of freedom in the equations and the matrices \( A, B \) and \( M \) are diagonal.

6.1.2.3. Post-critical Analysis

- Second order local field

The plate deformation in general can be represented as a linear combination of local modes in the form

\[
w = \sum_{m=1}^{M} \xi_m w_m \phi_m(y) \sin \left( \frac{m\pi x}{L} \right) \quad (m = 1, \ldots, M)
\] (6.30)

where \( \xi_m \) is the scalar parameters associated with given mode characterized by a certain \( m \), the number of half-waves in the longitudinal direction. Note in general for a given ‘\( m \)’, there would be multiple local modes with differing transverse description, but if the panel is sufficiently narrow, we may assume that there would be one dominant mode which would adequately describe cross-sectional deformation.

For a plate deformation field given in terms of \( w \)-distribution over the panel, there exists a second order field of in-plane displacement distributions \( (u \text{ and } v) \), denoted by \( \{ u^{(2)} \} \).
Thus

\[
\{ \ddot{\mathbf{u}}^{(2)} \} = \begin{bmatrix} u^{(2)}_x \\ v^{(2)}_x \\ u^{(2)}_y \\ v^{(2)}_y \end{bmatrix} = \pi_{m,n} \begin{bmatrix} \ddot{u}_{m,n}(x,y) \\ \ddot{v}_{m,n}(x,y) \end{bmatrix} \quad (m, n = 1, \ldots, M) \quad (6.31)
\]

Note that \( w^{(2)} = 0 \) which is tantamount to neglecting the effects of minor plate bending needed to ensure displacement compatibility at the plate stiffener junction.

A potential energy function may be written symbolically in terms of \( \ddot{u}^{(2)}, \ddot{v}^{(2)} \) which when rendered stationary yields the necessary equations for the determination of the second order field quantities. For the case of a flat plate structure (no initial curvature) this takes the simple form

\[
\Pi^{(2)} = \frac{1}{2} \left[ H L_1 \left( \ddot{u}^{(2)} \right) \bullet L_1 \left( \ddot{u}^{(2)} \right) - \sigma_o \bullet L_2 \left( \ddot{u} \right) + 2 L_1 \left( \ddot{u}^{(2)} \right) \bullet L_2 \left( \dddot{u} \right) \right] \quad (6.32)
\]

with \( \dddot{u}_i^{(1)} = 0 \) (\( i = 1,2 \)); \( \dddot{u}_3^{(2)} = 0 \).

For a specially orthotropic laminate

\[
\Pi^{(2)} = \frac{1}{2} \left[ \begin{array}{c}
A_{11} \left( \frac{\partial u^{(2)}}{\partial x} \right)^2 + 2 A_{12} \left( \frac{\partial u^{(2)}}{\partial x} \right) \left( \frac{\partial v^{(2)}}{\partial y} \right) + A_{22} \left( \frac{\partial v^{(2)}}{\partial y} \right)^2 + A_{66} \left( \frac{\partial u^{(2)}}{\partial y} + \frac{\partial v^{(2)}}{\partial x} \right)^2 \\
+ A_{11} \left( \frac{\partial u^{(2)}}{\partial x} \right) \left( \frac{\partial w}{\partial x} \right)^2 + A_{12} \left[ \left( \frac{\partial u^{(2)}}{\partial x} \right) \left( \frac{\partial w}{\partial y} \right)^2 + \left( \frac{\partial v^{(2)}}{\partial y} \right) \left( \frac{\partial w}{\partial x} \right)^2 \right] + A_{22} \left( \frac{\partial v^{(2)}}{\partial y} \right)^2 + A_{66} \left( \frac{\partial u^{(2)}}{\partial y} + \frac{\partial v^{(2)}}{\partial x} \right)^2 \\
+ A_{66} \left( \frac{\partial u^{(2)}}{\partial y} + \frac{\partial v^{(2)}}{\partial x} \right) \left( \frac{\partial w}{\partial x} \right) \left( \frac{\partial w}{\partial y} \right) \right] - \frac{1}{2} N^0_v \left[ \left( \frac{\partial u^{(2)}}{\partial x} \right)^2 + \left( \frac{\partial v^{(2)}}{\partial x} \right)^2 \right] \quad (6.33)
\]

where \( A_{11}, A_{12}, A_{22} \) and \( A_{66} \) are in-plane plate stiffnesses (Jones, 1998 [21]).

The equilibrium equations for the second order field are obtained by setting: \( \partial \Pi^{(2)} = 0 \).
Thus

$$HL_1(\delta u^{(2)}) \cdot L_1(\tilde{u}^{(2)}) - \sigma_o \cdot L_2(\tilde{u}^{(2)}) = - HL_1(\delta u^{(2)}) \cdot L_2(\tilde{u}^{(1)})$$

(6.34)

Appropriate shape functions satisfying the second order field equations for a typical plate element are of the form

$$u^{(2)} = u_0^{(2)} x + \sum_{p=1}^{2M} u_p^{(2)} = u_0^{(2)} x + \sum_{p=1}^{2M} u_{p,j}^{(2)} \psi_i(y) \sin(p \pi \eta)$$

$$v^{(2)} = v_0^{(2)} y + \sum_{p=1}^{2M} v_p^{(2)} = v_0^{(2)} y + \sum_{p=1}^{2M} v_{p,j}^{(2)} \psi_i(y) \cos(p \pi \eta)$$

(6.35.a-b)

Note that $u_0^{(2)}$ and $v_0^{(2)}$ are constants.

6.1.2.4. Evaluation of Second Order Field Functions

The expansion of the quadratic terms involving the first derivatives of $w$ in eq. (6.33) consists of two sets of terms:

(i) Those that are generated by a certain harmonic, say the $m^{th}$, and

(ii) Those that arise by an interaction of two distinct harmonics, say the $m^{th}$ and the $n^{th}$.

These would give rise respectively to (i) “diagonal” fields and (ii) “mixed” fields. Thus the interaction of $M$ plate bending modes would give rise to $\frac{1}{2}M (M + 1)$ second order component fields in all.
• **Variation of $u$, $v$ in “diagonal fields”**

Substituting for $w = w_m \phi_i(y) \sin(m \pi \eta)$ in eq. (6.26) and using the trigonometric identities,

$$
\cos^2(m \pi \eta) = \frac{1}{2} (1 + \cos(2m \pi \eta)) ; \quad \sin^2(m \pi \eta) = \frac{1}{2} (1 - \cos(2m \pi \eta)) \quad (6.36)
$$

the solution is seen to consist of two fields, viz. (i) one independent of trigonometric terms, and (ii) the other associated with trigonometric terms with the argument $(2m \pi \eta)$.

Considering (i), the displacement contribution of the $m^{th}$ mode to the field of type (i) may be written in the form

$$
u^0_{m,m} = \xi_m^2 u^0_{m,m} x, \text{ and } v^0_{m,m} = \xi_m^2 v^0_{m,m} y. \quad (6.37.a-b)
$$

The contributions from the $m^{th}$ mode to the field of type (ii) are of the form

$$
u^{(2m)}_{m,m} = \xi_m^2 u^{(2m)}_{m,m} (y) \sin(2m \pi \eta)
$$

$$
v^{(2m)}_{m,m} = \xi_m^2 v^{(2m)}_{m,m} (y) \cos(2m \pi \eta) \quad (6.38.a-b)
$$

• **Variation of $u$, $v$ in “mixed (off-diagonal)” fields**

Here we encounter “mixed” trigonometric terms $(m \neq n)$ which may be expanded in terms of trigonometric terms with arguments: $(m + n) \pi \eta$ and $(m - n) \pi \eta$ as in e.g.,
\[
\cos(m \pi \eta)\cos(n \pi \eta) = \frac{1}{2} \left(\cos((m + n)\pi \eta) + \cos((m - n)\pi \eta)\right)
\] (6.39)

We consider the interaction of pairs of plate bending modes associated with half-waves \(m\) and \(n\) respectively, letting \(m\) to increase from 1 to \(M - 1\) and taking \(n\) to cover the range, \(m < n \leq M\). For a given \(m\) and \(n\), this gives rise to two sub-fields, which are parts of a \(m\) and \(n\) component of the second order field associated with

(i) \[
\begin{align*}
\mathbf{u}^{(m+n)}_{m,n} &= \xi_m \xi_n \, u^{(m+n)}_{m,n}(y) \sin((n + m)\pi \eta) \\
\mathbf{v}^{(2)}_{m+n} &= \xi_m \xi_n \, v^{(m+n)}_{m,n}(y) \cos((n + m)\pi \eta)
\end{align*}
\] (6.40.a-b)

and

(ii) \[
\begin{align*}
\mathbf{u}^{(n-m)}_{m,n} &= \xi_m \xi_n \, u^{(n-m)}_{m,n}(y) \sin((n - m)\pi \eta) \\
\mathbf{v}^{(2)}_{m+n} &= \xi_m \xi_n \, v^{(n-m)}_{m,n}(y) \cos((n - m)\pi \eta)
\end{align*}
\] (6.41.a-b)

- **Second order displacement field**

The sum of all these contributions may be written in the form

\[
\begin{align*}
\mathbf{u}^{(2)} &= \sum_{m=1}^{M} \xi_m \xi_n \left\{ \mathbf{u}^{(0)}_{m,m} + \mathbf{u}^{(2m)}_{m,m}(y) \sin(2m \pi \eta) \right\} \\
&\quad + \sum_{m=1}^{M-1} \sum_{n=m+1}^{M} \xi_m \xi_n \left\{ \mathbf{u}^{(m+n)}_{m,n}(y) \sin((n + m)\pi \eta) + \mathbf{u}^{(n-m)}_{m,n}(y) \sin((n - m)\pi \eta) \right\}
\end{align*}
\] (6.42.a-b)

\[
\begin{align*}
\mathbf{v}^{(2)} &= \sum_{m=1}^{M} \xi_m \xi_n \left\{ \mathbf{v}^{(0)}_{m,m} + \mathbf{v}^{(2m)}_{m,m}(y) \cos(2m \pi \eta) \right\} \\
&\quad + \sum_{m=1}^{M-1} \sum_{n=m+1}^{M} \xi_m \xi_n \left\{ \mathbf{v}^{(m+n)}_{m,n}(y) \cos((n + m)\pi \eta) + \mathbf{v}^{(n-m)}_{m,n}(y) \cos((n - m)\pi \eta) \right\}
\end{align*}
\]
This form of representation avoids double counting of the “mixed” fields. Alternatively, we may recast these in the form

\[
\begin{align*}
\mathbf{u}^{(2)} &= \sum_{m=1}^{M} \sum_{n=1}^{M} u_{mn}(x, y) \xi_m \xi_n; \\
\mathbf{v}^{(2)} &= \sum_{m=1}^{M} \sum_{n=1}^{M} v_{mn}(x, y) \xi_m \xi_n \quad (6.43.a-b)
\end{align*}
\]

where for the mixed fields \((n > m)\) are

\[
\begin{align*}
u_{mn}(x, y) &= \frac{1}{2} \left[ u_{m,n}^{(m+n)}(y) \sin((n+m)\pi \eta) + u_{m,n}^{(n-m)}(y) \sin((n-m)\pi \eta) \right] \\
v_{mn}(x, y) &= \frac{1}{2} \left[ v_{m,n}^{(m+n)}(y) \cos((n+m)\pi \eta) + v_{m,n}^{(n-m)}(y) \cos((n-m)\pi \eta) \right] \quad (6.44.a-b)
\end{align*}
\]

- **Second order strain field**

The in-plane strains may be written sum of contributions from component fields each associated with a certain \(m\) and \(n\).

Thus,

\[
\begin{align*}
\varepsilon_{x}^{(2)} &= \sum_{m=1}^{M} \sum_{n=1}^{M} \left\{ \frac{\partial u_{mn}}{\partial x} + \frac{1}{2} \frac{\partial w_m}{\partial x} \frac{\partial w_n}{\partial x} \right\} \xi_m \xi_n; \\
\varepsilon_{y}^{(2)} &= \sum_{m=1}^{M} \sum_{n=1}^{M} \left\{ \frac{\partial v_{mn}}{\partial y} + \frac{1}{2} \frac{\partial w_m}{\partial y} \frac{\partial w_n}{\partial y} \right\} \xi_m \xi_n; \quad (6.45.a-b)
\end{align*}
\]

\[
\gamma_{xy}^{(m,n)} = \sum_{m=1}^{M} \sum_{n=1}^{M} \left\{ \frac{\partial u_{mn}}{\partial y} + \frac{\partial v_{mn}}{\partial x} + \frac{1}{2} \left( \frac{\partial w_m}{\partial x} \frac{\partial w_n}{\partial y} + \frac{\partial w_m}{\partial y} \frac{\partial w_n}{\partial x} \right) \right\} \xi_m \xi_n
\]
6.1.2.4.1. Evaluation of a Typical m-n Component Field

• Second order field equations

Shape functions for $u_{m,n}(y)$ and $v_{m,n}(y)$ in eq. (6.36) and (6.38) are taken in the forms already indicated in eq. (6.12.a-e). Using eq. (6.34) a set of linear simultaneous equations re generated in terms of the degrees of freedom for the second order field.

• Shape functions

Shape functions for $u_{m,n}(y)$ and $v_{m,n}(y)$ in eq. (6.36) and (6.38) are taken in the forms already indicated in eq. (6.12.a-e).

• Boundary conditions

It is assumed that the end conditions are such as to allow uniform end-shortening of the panel and whole-sale rotation of the end sections; in addition they allow in plane movement in the transverse direction of the end sections of plate elements. The lines of symmetry of the panel (center lines of the plate) are free to move but constrained to remain straight. The longitudinal bottom edge of the stiffener is free to move and wave in-plane.

• Orthogonality conditions

The second order fields must be orthogonal to the first order fields in order to circumvent singularities in their evaluation. First consider orthogonality of the first order plate bending modes and the associated second order fields. In so far as the former involve
only out of displacement and the latter involve only the in-plane displacements, the orthogonality condition is implicitly satisfied.

Next consider the orthogonality of the second order fields with the first order overall modes. Note the overall bending stress-system involves only the stress-resultant $N_{ov}^x$ acting across the plate thickness over the panel, with $N_{ov}^y = N_{ov}^{xy} = 0$. It therefore takes the simple form

$$N_{ov}^x = E_x h \frac{\partial U}{\partial x}$$  \hspace{1cm} (6.46)

where $E_x$ is the effective Young’s modulus and $h$ is the thickness of the plate element.

(Note $E_x h = \frac{1}{S_{11}}$ where $[S] = [A]^{-1}$.)

So the orthogonality condition takes the form

$$\int_{x=0}^L \int_\Gamma E_x h \frac{\partial U}{\partial x} \frac{\partial u^{(2)}}{\partial x} ds dx = 0$$  \hspace{1cm} (6.47)

where $\Gamma$ stands for the center-line profile of the section.

$u^{(2)}$ consists of two terms:

(i) Zero-harmonic- field given by $u_0^{(2)}$ and $v_0^{(2)}$

In order that the second order field in (i) be orthogonal to the overall bending modes, $u_0^{(2)}$ is taken as a constant for the entire panel. Thus this field is the one of uniform end-shortening and is readily seen to be orthogonal to the overall bending modes.
(ii) Fields given by \( u_{p,j}^{(2)} \psi_i \sin(p \pi \eta), v_{p,i}^{(2)} \psi_i \cos(p \pi \eta) \) with \( p = 1 \ldots 2M \).

Noting that for a given overall deformation pattern, \( U \) can be expressed in terms sine harmonics whilst retaining the same cross-sectional variation, the orthogonality condition takes the form

\[
\int \overline{E_x h U_{m,i}^{(2)} u_{m,j}^{(2)} \psi_i \psi_j} \, ds = 0 \quad (6.48)
\]

Expressed as a summation over the elements, this takes the form for a typical harmonic number \( m \),

\[
\sum_{n=1}^{N_p} E_x h U_{m,i}^e u_{m,j}^e a_{ij} = 0 \quad \text{where} \quad a_{ij} = \int_{-b}^{b} \psi_i(y) \psi_j(y) \, dy \quad (6.49)
\]

This condition is implemented by using a Lagrangian multiplier technique in the evaluation of \( u_{m}^{(2)} \).

6.1.2.5. Higher Order Strain Energy Contributions

- Strain energy contribution from the second order terms

Strain energy contribution from a typical strip-element takes the form

\[
U_e = \frac{1}{2} \int_{-1}^{1} \left[ A_{11} \left( e_{x}^{(2)} \right)^2 + 2 A_{12} \left( e_{x}^{(2)} \right) \left( e_{y}^{(2)} \right) + A_{11} \left( e_{y}^{(2)} \right)^2 + A_{66} \left( \gamma_{xy}^{(2)} \right)^2 \right] \frac{b}{2} d\zeta \, dx \quad (6.50)
\]
Substituting the expressions for the strains from eq. (6.39), and summing up over all the strips, we obtain a homogenous quartic of dimension $M^4$ in $\xi$ in the form

$$U = \frac{1}{4!} d_{ijkl} \xi_i \xi_j \xi_k \xi_l \quad (i,j,k,l = 1,\ldots,M) \quad (6.51)$$

Note that the 4-D matrix $d$ is rendered symmetric in the sense that $d$-coefficient has the same value for any permutation of the subscripts.

- **Higher order Strain energy term associated with overall modes**

Even though overall bending is treated using plate strips, the assumptions regarding the deformation are such that the panel bends essentially as an Euler column.

In the context of “small finite” deflections, we may, as for an Euler column assume inextensionality of the centroidal line, as given by

$$\varepsilon_{ov}^{(2)} = \frac{\partial U_{ov}^{(2)}}{\partial x} + \frac{1}{2} \left( \frac{\partial W_{ov}^{(1)}}{\partial x} \right)^2 = 0 \quad (6.52)$$

where $U_{ov}^{(2)}$ and $W_{ov}^{(1)}$s are the axial and lateral displacements respectively at the centroid of the section. Thus as for an Euler column, all the higher order terms associated with the second order field of overall action vanish. Again in the spirit of earlier studies on interactive buckling (Koiter and Pignataro, 1974 [23]), Sridharan et al, 1994 [46]), all the mixed second order strain components arising out of the interaction of first order local and overall fields are assumed to be zero, i.e. the overall action creates no other stresses than the first order flexural stresses in the section. This does not mean, however, mixed
second order displacements are zero. For example the longitudinal mixed second order field strain may be written as

\[ \varepsilon_x^{(1,1)} = \frac{\partial u^{(1,1)}}{\partial x} + \frac{\partial W^{(1)}}{\partial x} \frac{\partial W^{(1)}}{\partial x} \]  

(6.53)

Letting \( \varepsilon_x^{(1,1)} = 0 \),

\[ \frac{\partial u^{(1,1)}}{\partial x} = - \frac{\partial W^{(1)}}{\partial x} \frac{\partial W^{(1)}}{\partial x} \]  

(6.54)

which does not vanish for the plate.

- **Nonlinear modal interaction terms**

From eq. (6.9), the potential energy takes the form

\[ \Pi = \frac{1}{2} \{ \bar{\sigma} \bullet \bar{\varepsilon} + \sigma_o \bullet L_2(\bar{u}) \} \]  

(6.55)

Writing the generic strains and stresses evolving from the ground state in the ordered form

\[ \bar{\varepsilon} = \varepsilon^{(1)} + \varepsilon^{(2)} = \varepsilon^{(1)}_{lo} + \varepsilon^{(1)}_{ov} + \varepsilon^{(2)}_{lo} \]

\[ = L_1(\mu^{(1)}_{lo}) + L_1(\mu^{(1)}_{ov}) + L_1(\mu^{(2)}_{lo}) + \frac{1}{2} L_2(\mu^{(1)}_{lo}) \]  

(6.56)

\[ \bar{\sigma} = \sigma^{(1)}_{lo} + \sigma^{(1)}_{ov} + \sigma^{(2)}_{lo} \]
and making use of the orthogonality of the local and overall modes (first order fields) with the second order local field,

\[
\sigma_{lo}^{(1)} \cdot L_1 \left( u_{lo}^{(2)} \right) = 0; \\
\sigma_{ov}^{(1)} \cdot L_1 \left( u_{lo}^{(2)} \right) = 0; 
\]

(6.57.a-b)

We have finally,

\[
\Pi = \frac{1}{2} \left\{ \sigma_{lo}^{(1)} \cdot \varepsilon_{lo}^{(1)} + \sigma_{ov}^{(1)} \cdot \varepsilon_{ov}^{(1)} + 2\sigma_{lo}^{(1)} \varepsilon_{ov}^{(1)} + \\
\sigma_{ov}^{(1)} \cdot L_2 \left( u_{lo}^{(1)} \right) + L_2 \left( u_{lo}^{(1)} \right) + 2L_{11} \left( u_{lo}^{(1)} , u_{ov}^{(1)} \right) \right\} 
\]

(6.58.a-b)

Note for the axially compressed stiffened panel, \( \sigma_o = \{ -N_x^\alpha \ 0 \ 0 \ 0 \ 0 \} \).

Note the terms in the first and second lines are quadratic. The last term in each is a coupled bilinear term in the local and overall modes arising from the plate. These terms vanish when the local modes are anti-symmetric with respect to the stiffener junction, as already mentioned.

The last is the quartic term involving only local buckling quantities (eq. 6.40). It is seen that the only “nonlinear” term that accounts for interaction is the cubic term given by \( \sigma_{ov}^{(1)} \cdot L_2 \left( u_{lo}^{(1)} \right) \). This is the effect of overall bending stress interacting with plate bending deformation, accentuating (alleviating) the latter if the former is compressive (tensile). In the case of a relatively slender plate and stocky stiffener, such as considered here, compression due to downward overall bending caused say by a suddenly applied load is sustained over a full oscillation of the beam and can be destabilizing in nature.

This term may be written more explicitly in the form

\[
U^{(3)} = \frac{1}{2} \int_0^L \int_0^L E_s h \frac{\partial U^{(ov)}}{\partial x} \left( \frac{\partial W^{(lo)}}{\partial x} \right)^2 ds \ dx 
\]

(6.59)
where $\Gamma$ is cross-sectional profile. In terms of the modal degrees of freedom cubic terms can be expressed in the form

$$U^{(3)} = \frac{1}{3!} c_{pmn} X_p \xi_m \xi_n = \frac{1}{3!} \sum_{i=1}^{ne} \sum_{m=1}^{M} \sum_{j=1}^{M} \sum_{n=1}^{M} \sum_{k=1}^{M} (3 E_i h U_{p_i} W_{m_i} W_{n_k} I_{pmn} J_{ik}) X_p \xi_m \xi_n$$

where

$$I_{pmn} = - \frac{pmn \pi^3}{2} \int_{0}^{1} \sin(p \pi \eta) \cos(m \pi \eta) \cos(n \pi \eta) d \eta$$

$$J_{ik} = \frac{b_c}{2} \int_{-1}^{1} \psi_i(\zeta) \phi_j(\zeta) \phi_k(\zeta) d \zeta$$

where ‘ne’ represents the total number of strip elements constituting the panel.

### 6.1.4.6. Equations of Motion

The potential energy function, in terms of modal d.o.f., may be written in the form

$$\Pi = \frac{1}{2!} \left\{ a_{ij}^{(c)} \xi_i \xi_j + \frac{1}{2!} \left[ a_{pq}^{(ov)} - \sigma_o b_{pq}^{(ov)} \right] X_p X_q + \left[ a_{ip}^{(c)} - \sigma_o b_{ip}^{(c)} \right] \xi_i \xi_p X_p \right\} \xi_j$$

$$+ \frac{1}{3!} C_{pji} X_p \xi_j \xi_i + \frac{1}{4!} d_{ijkl} \xi_i \xi_j \xi_k \xi_l$$

The kinetic energy takes the form

$$T = \frac{1}{2!} \left[ m_{ij}^{(c)} \ddot{\xi}_i \dot{\xi}_j + m_{pq}^{(ov)} \dot{X}_p \dot{X}_q + 2 m_{ip}^{(c)} \dot{\xi}_i \dot{\xi}_p \right]$$
Nonconservative forces due to aerodynamic pressure associated with $i^{th}$ local d.o.f. and the $p^{th}$ overall d.o.f. are respectively

$$ F^{(i)}_i = c^{(i)}_{ij} \dot{\xi}_j + c^{(c)}_{ip} \ddot{X}_p $$

$$ F^{(ov)}_p = c^{(c)}_{pi} \dot{\xi}_i + c^{(ov)}_{pq} \dot{X}_q $$

(6.64.a-b)

Nonconservative forces due to aerodynamic damping associated with $i^{th}$ local d.o.f. and the $p^{th}$ overall d.o.f. are respectively.

So that

$$ D^{(i)}_i = d^{(i)}_{ij} \ddot{\xi}_j + d^{(c)}_{ip} \dddot{X}_p $$

$$ D^{(ov)}_p = d^{(c)}_{pi} \dddot{\xi}_i + d^{(ov)}_{pq} \dddot{X}_q $$

(6.65.a-b)

In the foregoing the superscript $(c)$ stands for the coupled bilinear/linear terms.

Note

(i) $a^{(i)}_{ij} = b^{(i)}_{ij} = m^{(i)}_{ij} = d^{(i)}_{ij} = 0$ unless $i = j$,

(ii) $a^{(ov)}_{pq} = b^{(ov)}_{pq} = m^{(ov)}_{pq} = d^{(ov)}_{pq} = 0$ unless $p = q$, and

(iii) $a^{(c)}_{ip} = b^{(c)}_{ip} = m^{(c)}_{ip} = d^{(c)}_{ip} = 0$ unless $i = p$.

(6.66.a-c)

The two complementary sets of Lagrange equations of motion are

\[
 m^{(i)}_{ij} \dddot{\xi}_j + m^{(c)}_{ip} \dddot{X}_p + \left\{a^{(i)}_{ij} - \sigma_p b^{(i)}_{ij} \right\} \dddot{\xi}_j + \left\{a^{(c)}_{ip} - \sigma_p b^{(c)}_{ip} \right\} \dddot{X}_p + c^{(i)}_{ij} \dddot{\xi}_j + c^{(c)}_{ip} \dddot{X}_p \\
 + d^{(i)}_{ij} \dddot{\xi}_j + d^{(c)}_{ip} \dddot{X}_p + \frac{1}{3} C_{pji} \dddot{X}_p \dddot{\xi}_j + \frac{1}{3!} d^{(i)}_{ijkl} \dddot{\xi}_j \dddot{\xi}_k \dddot{\xi}_l = 0
\]

(6.67.a)
\[ m^{(av)}_{pq} \ddot{X}_p + m^{(c)}_{pq} \dddot{z}_i + \left\{ d^{(av)}_{pq} - \sigma_o b^{(av)}_{pq} \right\} X_q + \left\{ d^{(c)}_{pq} - \sigma_o b^{(c)}_{pq} \right\} \ddot{z}_i + c^{(c)}_{pq} \ddot{z}_i + c^{(av)}_{pq} X_q \]
\[ + d^{(c)}_{pi} \dddot{z}_i + d^{(av)}_{pq} \dddot{X}_q + \frac{1}{3!} C_{pq} \dddot{z}_i \dddot{z}_j = 0 \quad (6.67.b) \]

Note that \( d_{ijkl} \neq d_{pi} \).

- **Effect of initial imperfections**

Imperfections will be inducted in terms of small perturbations of initial geometry in the form of modes participating in the analysis.

Thus two types of imperfections will be considered: Plate with “local” imperfections and overall imperfections.

Local imperfections an order of magnitude smaller than the plate thickness (e.g. \( h/100 \)) can be accounted for by simply adding a linear term in the potential energy function in the form

\[ -\sigma_o b^{(c)}_{pq} \dddot{z}_i \dddot{z}_j - \sigma_o b^{(c)}_{pq} \dddot{z}_i X_p \quad (6.68) \]

If the imperfections are a sizeable fraction of plate thickness, e.g. 0.2\( h \), then we need to account for higher order terms which arise by virtue of the cubic and quartic terms in the potential energy function. These can be systematically generated by replacing \( \xi_m \xi_n \) in eq. (6.45) and eq. (6.60) by \( \xi_m \xi_n + \xi_m \xi_n^0 + \xi_n \xi_m^0 \). This will result in additional terms which are respectively linear and quadratic in imperfection magnitudes.

In the case of the overall imperfections, imperfection terms are always linear in the absence of higher order (cubic and quartic) in the potential energy expression. We may take these imperfections to be a very small fraction of the length, consistent with fabrication tolerances.
• Solution of the equations

The nonlinear equations in time domain are solved by a combination of Newmark’s beta method (Tedesco et al, 1998 [52]) and Newton - Raphson iterative solution technique. Small initial values are assumed to trigger the dynamics. In case of imperfect panels carrying axial compression, a static analysis is first performed to produce the initial values. In case a suddenly applied pressure is applied, initial accelerations thereof trigger the dynamics.

6.2. Investigation of Model Accuracy

Some relatively simple problems are considered to verify the accuracy of the modeling and numerical analysis employed. The examples considered are (i) linear flutter problem of a plate for the determination of $\lambda_{cr}$, (ii) LCO of simply supported plates under aerodynamic pressure, (ii) Postbuckling response of an axially compressed plate with imperfections in two neighboring modes, and (iii) Interactive buckling of a stiffened plate under axial compression.

6.2.1. Example of a Plate Flutter

6.2.1.1. Linear Flutter

This is the problem of finding $\lambda_{cr}$, the critical value of non-dimensional aerodynamic pressure which corresponds to incipient flutter. Examples considered are simply supported rectangular plates made of isotropic material with three different aspect ratios. Aerodynamic damping is neglected.
The geometric and material properties are as follows:

Length (in the direction of flow) = \(a\); Width = \(b\) = 12 in.; Thickness = 0.05 in.

\[E = 10.5 \text{ Msi}, \ \nu = 0.3, \ \rho = 0.2588 \times 10^{-3} \text{ lb-sec}^2/\text{in}^4\]

The results for three aspect ratios, viz. 0.5, 1, and 2, are shown in Table 6.1 along with results based on Dowell’s analytical approach. In each case convergence of the present results is illustrated by considering two levels of discretization,

(i) 6 modes (determined using 3 strips for the half plate, \(n_e=3\)), and
(ii) 10 modes (determined using 5 strips for the half plate, \(n_e=5\)).

The present results are seen to be in excellent agreement with previously established results.

<table>
<thead>
<tr>
<th>a/b</th>
<th>Non-dimensional Critical Aerodynamic Pressure, (\lambda_{cr})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Motagaly’s (^{[1]})</td>
</tr>
<tr>
<td>0.5</td>
<td>385</td>
</tr>
<tr>
<td>1.0</td>
<td>512.6</td>
</tr>
<tr>
<td>2.0</td>
<td>1110</td>
</tr>
</tbody>
</table>

6.2.1.2. **Limit Cycle Oscillations**

Two examples of simply supported square plates are considered. An isotropic square plate with same material properties given above is examined with an aerodynamic damping coefficient, \(C_a = 0.1\).
Boundary conditions used by Dowell could be exactly replicated in the present analysis by (i) setting $u_{m,m}^0 = v_{m,m}^0 = 0$ (ii) eliminating in-plane waving along the longitudinal edges and (iii) by virtue of the chosen trigonometric variation of $u$ which makes it vanish at the ends. Only one half of the plate is considered in view of symmetry. Figure 6.2 plots the relationship between the maximum deflection and the non-dimensional aerodynamic pressure, $\lambda$ as given by two levels of discretization, for two aspect ratios, viz. $r = 1$, and 2. The agreement between the present results and Dowell’s results [14] as quoted by Motagaly (2001) [1] are excellent.
6.2.2. Post-buckling Response of Plate

With a view to examine the accuracy of multimodal analysis further an axially compressed rectangular plate with edges allowed to move in the plane of the plate is studied. Consider a plate having an aspect ratio of 2, which is simply supported with all the edges allowed to move but held straight.

The plate is subjected uniform end compression in the longitudinal direction. The material of the plate isotropic with $\nu = 0.25$. Interaction between two modes one consisting of 2 half-waves (the primary buckling mode, $m = 2$, designated as mode 2) and the other consisting of 3 half-waves ($m = 3$, mode 3) is considered. Let $\xi_2, \xi_3$ be the scalar parameters associated with the modes. These are respectively the displacement amplitudes developed under loading divided by the plate thickness. Different levels of imperfection viz. $\xi_2^0, \xi_3^0$ are considered with a view to examine the accuracy of the present model and numerical analysis.

This problem has been studied by Supple (1970) [50] using an analytical approach to determine the second order field. In the present analysis, one half of the plate is analyzed taking advantage of symmetry with respect to the longitudinal centerline with 3 and 5 strips respectively to study the convergence of results. The cases studied are:

Case (i): $\xi_2^0 = 0.25; \xi_3^0 = 0.20$

Case (ii): $\xi_2^0 = 0.25; \xi_3^0 = 0.125$

Since the imperfections are sizeable fractions of the plate thickness, $h$, it is necessary to consider higher order terms associated with the quartic terms (the cubic terms being absent for an unstiffened plate). Note that the buckling mode starts being coupled in both the cases, but one of the modes finally predominates over the other which loses ground as loading progresses.
Figure 6.3 Natural loading paths for a plate
Figure 6.3 plots the non-dimensional load ($\sigma/\sigma_c$) versus the total displacements given by sum of respectively by the numerical solution with Supple’s analytical solution. The agreement between the two solutions is excellent.

6.2.3. Interactive Buckling of Tvergaard Panel

The geometry and material properties are same as shown in chapter 5 (Table 5.1 and Table 5.2). This problem has been investigated earlier by Sridharan et al. (1994) [46] by a technique of embedding the local buckling fields (the buckling mode and the periodic part of the second order fields) into a finite element and accounting thus for overall action and amplitude modulation. In the present analysis is simpler in conception, where the several neighboring local buckling modes and the overall bending modes account for amplitude modulation and overall bending respectively.

Figure 6.4. Total downward deflection vs. non-dimensional axial compression
However cross-sectional distortions accompanying overall buckling are not accounted for in the present analysis. This makes for a slight discrepancy in the overall buckling load corresponding to $m = 1$, i.e. $\frac{\sigma_{cr}}{E} = 0.5 \times 10^{-3}$ now as against $0.47 \times 10^{-3}$. The local mode still corresponds to $m = 6$ and the critical load is practically the same, i.e. $\frac{\sigma_{cr}}{E} = 0.47 \times 10^{-3}$.

Imperfections are assumed in the form of these two modes and the amplitudes of the imperfections are taken as $-0.1b$ (downward) for the overall mode and $0.1h$ for the local mode. Since the local imperfection is a significant fraction of the plate thickness, the higher order terms involving the imperfections must be accounted for. Since all the local modes other than that corresponding to $m = 6$ play a relatively minor role, these higher order terms stemming from the cubic and quartic terms respectively in the potential energy function are taken to be

\[
\begin{align*}
(i) & \quad \frac{1}{3!} \xi_{166} (2X_1 \xi_6 \xi_0), \\
(ii) & \quad \frac{1}{4!} d_{6666} \xi_6^3 \xi_6^2 \xi_0 + \xi_6^2 \xi_0^2
\end{align*}
\]

Figure 6.4 plots the non-dimensional stress (normalized w.r.t. overall critical stress) with the maximum deflection at the center of the plate. These are compared with those plotted using the earlier analysis using “locally buckled” elements and Abaqus (version 5.8, 1998). The agreement is very good indeed. (Note however, the actual maximum stress reached in the present analysis is higher)

6.2.4. Flutter Response of Tvergaard Panels:

The Linear Flutter Problem

The linear flutter problem, i.e. the determination of critical aerodynamic pressure ($\lambda_{cr}$) is considered under a variety of conditions:

(i) The effect of aerodynamic damping is illustrated by studying two cases with the aerodynamic damping factor $C_a$ set equal to 0 and 0.1 respectively,
(ii) The effect of axial compression is studied varying axial compressive stress carried by the panel. The following three cases are considered, viz. $\sigma_o = 0$, $0.2 \sigma_{cr}$, and $0.4 \sigma_{cr}$ respectively, and

(iii) The type of local modes considered, viz. those that are antisymmetric and symmetric respectively with respect to the plate-stiffener junction. (These are referred henceforth as antisymmetric and symmetric cases respectively).

All calculations are done with $M = 10$ and $M_o = 10$ with the plate discretized into 6 strips and a single element representing the stiffener. Convergence study indicated that the degree of refinement chosen is adequate with maximum discrepancy begin less than 1%. The results of $\lambda_{cr}$ for the cited values of $C_a$ and $(\sigma_{cr})$ are tabulated in Table 6.2 for both anti-symmetric and symmetric cases.

<table>
<thead>
<tr>
<th>Type of Local modes</th>
<th>Anti-symmetric case</th>
<th>Symmetric case</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C_a = 0$</td>
<td>$C_a = 0.1$</td>
</tr>
<tr>
<td>$\sigma_o = 0$</td>
<td>9015</td>
<td>9630</td>
</tr>
<tr>
<td>$\sigma_o = 0.2 \sigma_{cr}$</td>
<td>5944</td>
<td>6514</td>
</tr>
<tr>
<td>$\sigma_o = 0.4 \sigma_{cr}$</td>
<td>2945</td>
<td>3347</td>
</tr>
</tbody>
</table>

It is clearly seen that aerodynamic damping does play a significant role in delaying flutter and must be duly considered in flutter calculations. The effect of axial compression is even more significant in that $\lambda_{cr}$ is reduced by about 65% for an axial compression of 40% of the critical value associated with buckling.
The values of the symmetric cases are smaller than the corresponding values of the anti-symmetric case. Note that for the antisymmetric case the overall modes are fully decoupled from the local modes as the latter is symmetric with respect to the plate-stiffener junction. This is in contrast to the symmetric case where the two types of modes are coupled: A local mode with \( m \) half waves is coupled with that with the overall mode with the same \( m \) half-waves in stiffness, mass and aerodynamic damping matrices; further in the aerodynamic pressure matrix a local mode with an even number of half waves couples with overall modes with odd number of half-waves and vice-versa. Thus the antisymmetric case is controlled only by the local modes whereas both local and overall modes participate in the vibration mode at incipient flutter for the symmetric case. The critical value corresponding to the overall modes acting alone is much higher (= 57153) in the absence of axial compression or aerodynamic damping.

The antisymmetric case the stiffener does participate in deformation in local modes. The pattern of deformation is akin to torsional buckling of T-section columns which occurs with \( m = 1 \). This makes the long-wave modes more pronounced in the response and the result is a higher value of \( \lambda_{cr} \). This is found to be true for all the values of \( \sigma_o \) considered (vide Table 6.2).

The linear analysis does not give any hint as to the post-critical amplitudes of LCO or further instabilities that might ensue.

### 6.2.5. Post-critical Response: Anti-symmetric Case

First consider the case with \( \sigma_o = 0, C_a = 0.1 \).

Initial conditions are set \( \xi_m = 0.1/m; X_m = 0.1/m \ (m = 1, \ldots M) \). Figure 6.5 shows some typical time histories at \( \lambda = 9650 \), just past the critical value. These are respectively: (i) Maximum “local” deflection, (ii) Deflection in the overall mode with \( m = 1 \), (iii) Deflection in the local mode with \( m = 1 \), and (iv) Deflection in the local mode with \( m = 4 \).
Figure 6.5. Time histories of $\lambda = 9650$, $\sigma_0 = 0$, $C_a = 0.1$, $\xi_m = 0.1/m$, $X_m = 0.1/m$
It is clear the limit cycle oscillations are in the local modes, whereas the overall mode gets damped out well before local limit cycle oscillations attain steady amplitudes. At incipient flutter there is complete decoupling of local and overall modes; also the aerodynamic pressure cannot drive the overall modes whose $\lambda_{cr}$ is about 55000. In the presence of aerodynamic damping the overall oscillation amplitudes attain negligibly small values rapidly notwithstanding modal coupling of local and overall modes (as represented by the cubic terms in the potential energy function).

As $\lambda$ is increased the amplitude of LCO of the local modes steadily increases till $\lambda = 10600$. Figure 6.6 shows the maximum displacement history at this value of $\lambda$. Thereafter the panel loses its stability and amplitudes increase without any limit. In order to precisely capture the point of instability initial conditions for the amplitudes were reduced to $0.01/m$, and the limiting value of $\lambda$ (designated as $\lambda_{lim}$) is found to be 10680. The relationship between the maximum deflection over the LCO regime and $\lambda$ is shown in Figure 6.7.

It is evident that there is a turning point in the response of the panel from stable to unstable behavior and the existence of unstable limit cycle regime for values of $\lambda < 10680$. If this is true, a significant push to the panel by prescribing significantly higher initial values can overcome the attraction associated with the stable LCO and destabilize the panel. To verify this hypothesis, the case of $\lambda = 9300$, a subcritical value with initial conditions, $\xi_m = 0.5/m; X_m = 0.5/m$ was investigated. The panel becomes dynamically unstable with escalating oscillations. Figure 6.8 (a-c) illustrates the time histories of the maximum deflection, the amplitude of the first local mode, and that of the overall mode respectively.
Figure 6.6. Time history of maximum deflection when $\lambda = 10600$

Figure 6.7. Variation of maximum deflection with aerodynamic pressure
Figure 6.8 Panel response at subcritical $\lambda = 9300$ with high initial values destabilizing the structure.
6.2.5.1. Source of Instability: Edge Movements

The phenomenon of instability which sets in at a certain value of $\lambda$ in the post-critical range and the existence of a turning point therein reported here has not been seen in earlier studies of flutter of plates. After some investigation, it was seen that the cause of this phenomenon is the boundary conditions used here which allow the edges (both the ends and longitudinal edges) to move.

To demonstrate this, an antisymmetric case with $\sigma_x = 0$, is studied with edge movements constrained, $u = 0$ at the ends and $v = 0$ along the longitudinal edges in the evaluation of the second order local fields. The relationship between the maximum deflection and $\lambda$ is shown in Figure 6.9. It is seen there is no instability for $\lambda >> \lambda_{cr}$.

![Figure 6.9](image-url)
6.2.5.2. Effect of Axial Compression

As already seen the axial compression reduces the critical velocity $\lambda_{cr}$. Considering the case of $\sigma_0 = 0.4\sigma_{cr}$, the critical velocity drops to 3347 from 9630 corresponding to the case with $\sigma_0 = 0$. But the post-critical response is qualitatively same in both the cases. Figure 6.10 shows the relationship between $\lambda$ and maximum deflection.

After a short regime of LCO until $\lambda \approx 3428$, dynamic instability sets in and stable limit cycle is no longer available. Figure 6.11 shows the time history of maximum deflection for a value of $\lambda = 3430$, just exceeding the stability limit.

Figure 6.10. Variation of maximum deflection with aerodynamic pressure ($\sigma_0 = 0.4\sigma_{cr}$)
6.2.5.3. Effect of Initial Imperfections in The Presence of Axial Compression

Consider the case of a panel having a downward initial imperfection of $0.1\, h$ in the first overall mode ($m = 1$) and a small local imperfection of $0.01\, h$ in the mode corresponding to $m = 6$, the mode in which local buckling occurs under axial compression. First we admit axial compression and evaluate the static response. The aerodynamic pressure is admitted subsequently.

Under static conditions, there is slight bending of the panel in the overall mode and some additional compression is thrown on the plate. This reduces slightly the critical value of $\lambda$ which is now 3152. Aerodynamic pressure is now admitted with values of $\lambda > 3152$. 
Initially the panel executes LCO and the dynamic instability sets in at $\lambda \approx 3246$. Figure 6.12 shows the relationship between maximum deflection and $\lambda$.

Figure 6.12. Variation of maximum deflection with aerodynamic pressure with initial imperfections

Figure 6.13 illustrates the responses at a value $\lambda = 3250$, just beyond the limiting value of $\lambda$: (i) time history of maximum deflection, (ii) A typical local mode ($m = 4$), and (iii) the first overall mode. An interesting observation here is that the overall mode which has been a poor participant in the previous instances is activated, but the oscillations have extremely small amplitudes, but soon there occurs a divergence (static escalation) of the deflection. This is similar what might be observed in the dynamic instability of an axially compressed thin walled column under modal interaction subjected to a significant impulse.
Figure 6.13. Time histories of $\lambda = 3250$ with imperfections
Figure 6.14. Time histories at $\lambda = 2500$, $\sigma_x = 0.4 \sigma_{cr}$, with imperfection and $p_o$.
Figure 6.15. Time histories at $\lambda = 2600$, $\sigma_x = 0.4 \, \sigma_{cr}$, with imperfection and $P_0$. 

(a) Displacement time history

(b) Overall mode 1 time history

(c) Local mode 3 time history
with local-overall interaction

![Graph](image1.png)

without local-overall interaction

![Graph](image2.png)

(a) First overall mode time history

(b) First local mode time history

(c) Maximum displacement time history

Figure 6.16. Local and overall interaction effects comparison ($\lambda=2500$, $C_o=0.1$, $\sigma_x = 0.4*\sigma_{cr}$, no vimp, $p_o=0.001$, $\xi = 0.01/\ m$; $X = 0.01/\ m$)
6.2.5.4. **Effect of Suddenly Applied Pressure**

We consider the same problem of an initially imperfect panel under axial compression \((\sigma = 0.4 \sigma_{cr})\) subjected to subcritical aerodynamic pressure. This time we directly trigger the overall mode by the application of a suddenly applied uniform lateral pressure.

As before, axial compression is applied first. Next the panel is subjected simultaneously to a suddenly applied lateral pressure \((p_0 = 0.001\text{MPa})\) and the aerodynamic pressure. The total thrust on the plate is \(\approx 0.0105 P_{cr}\), where \(P_{cr}\) is the axial compression causing buckling of the panel. The local modes being antisymmetric w.r.t. plate stiffener junction, do not sense this load directly and are slow to respond. They become mildly active under the static phase of application of axial compression because of the small initial imperfection in one of the modes \((m = 6)\). They build up due simultaneously to suddenly applied bending compression and aerodynamic pressure.

A highly chaotic response is observed and the oscillations (both overall and local) do not get damped out. This is illustrated for \(\lambda = 2500\) in Figure 6.14. As \(\lambda\) increases, after exhibiting chaotic behavior, the deflection of the panel increases without limit. This is illustrated for \(\lambda = 2600\) in Figure 6.15. This is essentially due to significant axial compression induced in the plate as the panel oscillates in the overall mode.

6.2.5.5. **Role of Interaction Between Local and Overall Action**

For the anti-symmetric case the only source of interaction is the set of cubic terms \(c_{ijk} \epsilon \xi_j \xi_k\) involving the product of derivatives, i.e. \(\frac{\partial U}{\partial x} \frac{\partial w_j}{\partial x} \frac{\partial w_k}{\partial x}\) in the potential energy function. These terms are of paramount importance in capturing static bending of the whole panel due to panel buckling deflections as seen in the interactive buckling problem. Considering equations of motion, we may view the product of the \(w\)-derivatives (local modes) as a forcing function triggering overall bending. Because of the
wide differences in the frequencies of this forcing function, the oscillations induced are minute especially in the presence of aerodynamic damping. Thus overall modes get damped out prior to the onset of rapid escalation of local mode oscillations.

There are cases, however, where the cubic terms play a pivotal role as they are the source of destabilizing action of additional compression caused by overall bending acting on local deflections – the essence of nonlinear local-overall interaction in stiffened plates. This is illustrated by considering the case considered in the last section (axially compressed panel with initial imperfections and acted upon by a suddenly applied pressure) with $\lambda = 2500$. Responses are computed respectively duly considering and neglecting the cubic terms of interaction. The results are shown in Figure 6.16. It is seen that in the absence of cubic terms the oscillations are quickly damped out. But with the cubic terms included, we observe a sustained chaotic response which might lead at some point in time to displacements that increase without limit.

6.2.6. Post-critical Response: Symmetric case

6.2.6.1. Linear Flutter Problem

In this section we consider the linear flutter problem of the panel. As mentioned already, this problem, in contrast to the anti-symmetric case, involves coupling of local and overall modes. In the absence of this coupling, the plate would act as if it is clamped along the stiffener. In order to study the influence of coupling therefore, a companion problem of a plate clamped along the stiffener (thus rendering the stiffener inactive) is studied.

The results of linear flutter analysis of the symmetric panel and plate respectively are shown in Table 6.3. It is seen that in the value of $\lambda_{cr}$ is higher for the panels as the stiffener tends to bend in the long wave modes pulling the plate with it. On the other hand in the plate problem the overall modes are excluded, short-waves modes (those with higher values of $m$) have full play and this results in a lower value of $\lambda_{cr}$. 
Table 6.3. Critical Non-dimensional Aerodynamic Pressure, $\lambda_{cr}$

<table>
<thead>
<tr>
<th>Type of local modes</th>
<th>Symmetric plate</th>
<th>Symmetric panel</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial stress</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_0 = 0$</td>
<td>5706</td>
<td>7596</td>
</tr>
<tr>
<td>$\sigma_0 = 0.2 \sigma_{cr}$</td>
<td>2394</td>
<td>4773</td>
</tr>
<tr>
<td>$\sigma_0 = 0.4 \sigma_{cr}$</td>
<td>103</td>
<td>2608</td>
</tr>
</tbody>
</table>

6.2.6.2. Post-critical Response of Panel

In contrast to the anti-symmetric case which has a short post-critical range of stable limit cycle oscillations, the post-critical response of the panel with symmetric modes is one of unstable limit cycles, i.e., as soon as $\lambda > \lambda_{cr}$ the panel loses stability and oscillations escalate. From Table 6.3 consider the case with $Ca = 0.1$ and $\sigma = 0$ so that $\lambda = 8465$. Two values of $\lambda$ are chosen, viz. 8400 ($\lambda < \lambda_{cr}$) and 8500 ($\lambda > \lambda_{cr}$). Initial conditions are taken as: $\xi_m = 0.001/m; X_m = 0.001/m$ These are indeed minute initial disturbances chosen with a view to precisely identify the nature of dynamic bifurcation. The responses are shown respectively in Figure 6.17 and Figure 6.18. From Figure 6.17 (a-c), it is seen that for $\lambda < \lambda_{cr}$ the oscillations get damped out with time whereas for $\lambda > \lambda_{cr}$ they gradually escalate without any limit, as seen from Figure 6.18(a-c). Thus it is evident that the post-critical response is unstable. In order to gain further insight into the nature of bifurcation, a pre-critical case, viz. $\lambda = 8000$ is studied with initial values of the modal amplitudes increased as follows: $\xi_m = 0.5/m; X_m = 0.5/m$ as shown in Figure 6.19. This results in gradually increasing deflections at first followed by a chaotic phase of oscillations and finally escalating deflections with no limit. It is clear that critical point is associated with unstable LCO and the panel if sufficiently perturbed can become dynamically unstable in the pre-critical range of $\lambda > \lambda_{cr}$. 
Figure 6.17. Time histories of $\lambda = 8400; C_o = 0.1, \sigma_r = 0.0, \xi_m = 0.001/m; X_m = 0.001/m$
Figure 6.18. Time histories of $\lambda = 8500$, $C_a = 0.1$, $\sigma_{\xi} = 0.0$, $\bar{\xi}_{m} = 0.001/m$; $\dot{X}_{m} = 0.001/m$
(a) Maximum displacement time history

(b) First overall mode time history

(c) First local mode time history

Figure 6.19. Time histories of $\lambda = 8000; C_a = 0.1, \sigma = 0.0, \xi_m = 0.5/m; \ X_m = 0.5/m$
6.2.6.3. Post-critical Response of Plate

With a view to study the role of interaction of local and overall modes, we examine the post-critical response of the plate thereby excluding the overall modes. The plate goes through a short phase of stable LCO in the range \( 8361 \lambda_{cr} < 9320 \) where after it goes through a chaotic phase followed by escalating oscillations without limit (Figure 6.20). These three stages are illustrated respectively by Figure 6.21 (a-c) selecting appropriate values of values of \( \lambda \), viz. 9300, 9350 and 9370.

![Figure 6.20. \( \lambda \) vs. maximum displacements for symmetric plate](image)
Figure 6.21. Examples of three stages

(i) $\lambda = 9300$

(ii) $\lambda = 9350$

(iii) $\lambda = 9370$
Figure 6.22. Local-overall interaction comparisons in symmetric stiffened panel
\( (\lambda = 2500, C_o = 0.1, \sigma_o = 0.4 \sigma_{cr}, \text{no imperfection, } p_o = 0.001, \xi_m = 0.01/m, X_m = 0.01/m) \)
6.2.6.4. Role of Local and Overall Interaction

From a comparison of responses respectively of the plate subjected to symmetric plate modes and the stiffened panel subjected to combined action of local and overall modes indicates, the participation of the overall modes have the effect of precipitating instability at the onset of flutter.

This participation occurs via the cubic terms, $c_{ijk} X_i \xi_j \xi_k$, identified already as those responsible for nonlinear modal interaction.

In order to illustrate the crucial role played by this set of terms, we study a case where significant bending is induced by a suddenly applied pressure on the panel as in the antisymmetric case. Consider a perfect panel subjected to axial compression ($\sigma_o = 0.4 \sigma_{cr}$) subjected to a suddenly applied lateral pressure $p_o = 0.001$ and subcritical $\lambda = 2500 < \lambda_{cr}$ with $C_o = 0.1$. Figure 6.22 shows comparisons of time histories with and without local-overall interaction.

It is seen that in the absence of the cubic terms of interaction, the panel settles down in about 0.1 sec whereas with these terms duly included, the response is chaotic with the possibility that the oscillations may suddenly escalate.

6.3. Conclusions

(1) A finite strip technique has been developed for flutter analysis of stiffened plate panel. The method employs two distinct modes of vibration, viz. “local” plate modes with junction displacements arrested and “overall” modes free of cross-sectional distortions.

(2) Axial compression carried by the panel and any bending which tends to cause additional compression in the slender plate reduce the value of $\lambda_{cr}$. 
(3) Plates and stiffened panels subjected to increasing airflow velocity, \( \lambda \) (expressed in terms of non-dimensional pressure coefficient) higher than the critical value become dynamically unstable, unless the edges are restrained from in-plane movements. For a typical panel investigated herein, instability sets in right at the critical value when the panel vibration modes are symmetric with respect to the stiffener; if these are antisymmetric instability sets in at a value higher than but close to \( \lambda_{cr} \).

(4) Under significant initial disturbances as given by the initial conditions or the application of suddenly applied pressure, panel can become unstable for \( \lambda < \lambda_{cr} \).

(5) A coupling of local and overall modes occurs in the linear problem when symmetric local modes are employed to describe the plate vibrations. Nonlinear interaction of local and overall modes is encapsulated by a set of cubic terms of the strain energy function which are linear in overall displacement d.o.f. and quadratic in local d.o.f. These play a crucial role in precipitating instability under subcritical values of \( \lambda \), when the panel is subjected to suddenly applied pressure.
Chapter 7

Control Methodologies for Stiffened Panels

7.1. Introduction

In this chapter the control of stiffened panel liable to flutter under aerodynamic pressure is considered. In chapter 5, the control was exercised using piezo-electric patches along the center lines of the plates as well as along the tips of the stiffeners as shown in Figure 5.1. Here we consider in addition patches located contiguous to the stiffener on the plates on either side of the stiffener. These are designated as “plate-stiffener patches”. As in the earlier work, we shall assume that the plate patches are sufficiently thin in comparison the plate while the stiffener patch thickness is small in comparison to the depth of the stiffener. As a first step the equations of motion of the controlled structure are developed by duly incorporating the contributions of the piezoelectric patch material. These are expressed in the first order state-space format so that an optimal control methodology may be applied. In order to apply the powerful Linear Quadratic Regulator (LQR) algorithm [28] a linearization scheme is introduced.

Two types of control are considered:

Type (i): Negative velocity feedback control is exercised by actuator patches in proportion to the locally sensed strain rates. In this case, a single time-invariant gain for
each type of patch, (i.e. the plate and stiffener patches respectively) is worked out using
the LQR algorithm.

Type (ii): A general Multi-Input Multi-Output (MIMO) type of control based on a gain
matrix obtained at the beginning of each time step is employed. This is a combination of
feedback based on strains and strain-rates sensed at all the actuator-sensor pairs taken
together.

Obviously the type (i) control is easier to implement and effective but fails at values
which are several multiples of the critical value at incipient flutter. Type (ii) control is
more versatile and totally effective, the only limitation being the limit on the electric field
strength that can be sustained by the patches and requires a more intricate technology for
implementation.

7.2. Theory

7.2.1. Equation of Motion

To the equations of motion of the panel developed in chapter 6, we now need to add the
piezo-electric contributions arising from the stiffener patches as well as the plate patches.
To this end we recapitulate the expressions of internal work contribution from the elastic
forces and external work contributions from inertial forces and aerodynamic pressure and
damping. The potential energy function in terms of modal d.o.f. may be written in the
form

\[
\Pi = \frac{1}{2!} \left[ a_{ij}^{(i)} - \sigma_a b_{ij}^{(i)} \right] \xi_i \xi_j + \frac{1}{2!} \left[ a_{pq}^{(ov)} - \sigma_a b_{pq}^{(ov)} \right] X_p X_q + \left[ d_{ip}^{(c)} - \sigma_a b_{ip}^{(c)} \right] \xi_i X_p

+ \frac{1}{3!} C_{pqj} X_p \xi_i \xi_j + \frac{1}{4!} d_{ijkl} \xi_i \xi_j \xi_k \xi_l \]  

(7.1)
As mentioned in chapter 6, \(\xi_m\)\((m = 1\ldots, M)\) represents local modes and \(X_m\)\((m = 1\ldots, M_o)\) represents overall modes. Both modes together constitute \(M+M_o\) degrees of freedom. In the foregoing the superscript \((c)\) stands for the local-overall terms. The internal virtual work of elastic forces takes the form

\[
\delta\Pi = \left\{\begin{array}{l}
\{d^{(i)}_{ij} - \sigma_a b^{(i)}_{ij}\}\dot{\xi}_j \delta\xi_i + \frac{1}{2}\left\{a^{(ov)}_{pq} - \sigma_a b^{(ov)}_{pq}\right\}X_q \delta X_p \\
+ \left\{d^{(c)}_{ip} - \sigma_a b^{(c)}_{ip}\right\}\left(\xi_i \delta X_p + X_p \delta\xi_i\right) \\
+ \frac{1}{3!} C_{pij} \left(2X_p \dot{\xi}_j \delta\xi_i + \xi_i \dot{\xi}_j \delta X_p\right) + \frac{1}{3!} d_{ijkl}\dot{\xi}_j \dot{\xi}_k \dot{\xi}_i \delta\xi_i
\end{array}\right.
\] (7.2)

The external virtual work due to inertial forces takes the form

\[
\delta W_T = -\left[m^{(i)}_{ij} \dot{\xi}_j \delta\xi_i + m^{(ov)}_{pq} \ddot{X}_q \delta X_p + m^{(c)}_{ip}\left(\dot{\xi}_i \delta X_p + \ddot{X}_p \delta\xi_i\right)\right]
\] (7.3)

Non-conservative forces due to aerodynamic pressure associated with \(i^{th}\) local d.o.f. and \(p^{th}\) overall d.o.f. are respectively:

\[
F^{(i)}_i = c^{(i)}_{ij} \dot{\xi}_j + c^{(c)}_{ip} X_p
\] (7.4.a-b)

\[
F^{(ov)}_p = c^{(c)}_{pi} \dot{\xi}_i + c^{(ov)}_{pq} X_q
\]

Non-conservative forces due to aerodynamic damping associated with \(i^{th}\) local d.o.f. and \(p^{th}\) overall d.o.f. are respectively:

\[
D^{(i)}_i = d^{(i)}_{ij} \dot{\xi}_j + d^{(c)}_{ip} \dot{X}_p
\] (7.5.a-b)

\[
D^{(ov)}_p = d^{(c)}_{pi} \dot{\xi}_i + d^{(ov)}_{pq} \dot{X}_q
\]
7.2.2. Piezo-electric contribution

In this study, PZT patches are proposed to be used with prescribed voltages applied in the $x_3$ direction.

The first variation of the electric enthalpy is given by eq. (1.33) of chapter 1.

$$\delta H = \left\{ \delta \epsilon \right\}^T [Q] \left\{ \epsilon \right\} - \left\{ \delta \sigma \right\}^T [d]^T \{ E \}$$

With voltage applied only in the $x$-3 direction, and letting $d_{31} = d_{32} = e_p$, the electro-mechanical coupled term in eq. (1.33) denoted by $\delta H^p$ takes the form given by eq. (1.35).

$$\delta H^p = - \left\{ \delta \sigma_x + \delta \sigma_y \right\} e_p E_3$$

The foregoing expression is numerically equal to the internal virtual work contribution from piezo-electric effects in the virtual work equation.

7.2.3. Locations of actuators

In the present study we consider two types of piezo-electric patch actuators: (i) those attached to the stiffener tips controlling mainly the overall action, and (ii) those attached to the top and bottom surfaces of the plate midway between the stiffeners. These are designated as “Plate-center” patches. (iii) In a later section additional patches are introduced contiguous to the stiffener and these are designated as “Plate-stiffener” patches.
7.2.3.1. Stiffener Control

Consider patches of width equal to stiffener width \((t_s)\) attached to the stiffener at its top (plate surface) and bottom respectively. The Tvergaard Panel-1 considered in the present study consists of a stocky stiffener so that the local buckling strains are negligible compared to the overall counterparts, and the only significant stress and strain are those occurring in the longitudinal direction. The stiffener may be treated as if it is an Euler-Bernoulli beam so that we may assume that the longitudinal strain variation across the depth is linear.

Thus:

\[
\varepsilon_{\text{top}} = \varepsilon_0 \mp \frac{d_s}{2} \chi
\]  

(7.8)
where $\varepsilon_o$ is the strain at the centroid of the stiffener, $d_s$ is the depth of the stiffener (approximately equal to the center to center distance between the patches at top and bottom), and $\chi$ is the curvature of the stiffener in the global $XZ$ plane (exclusive of initial curvature). This curvature must be the same as that of the plate at its junction with the stiffener, i.e.

$$
\chi(x) = \left. \frac{\partial^2 W'}{\partial x^2} \right|_j = \sum_{m=1}^{M} X_m W'_m \left( \frac{-m^2 \pi^2}{L^2} \right) \sin \left( \frac{m \pi x}{L} \right) \quad \text{(7.9)}
$$

where $W'$ denotes the overall stiffener plate junction displacement (Figure 5.1 in chapter 5) in the global Z-direction.

Considering only the case of voltage being applied across the thickness of the patch ($t_p$) and letting the field strength applied at top and bottom be equal and opposite to each other, the field strength is

$$
E_{3_{\text{top}}} = \pm \frac{V}{t_p} \quad \text{(7.10)}
$$

Longitudinal strain, $\varepsilon$ at the top and bottom can be written in the form:

$$
\varepsilon_{\text{top}} - \varepsilon_{\text{bottom}} = \left\{ \varepsilon_o \pm \frac{d_s}{2} \chi \right\}_{\text{top}} - \left\{ \varepsilon_o \pm \frac{d_s}{2} \chi \right\}_{\text{bottom}} \quad \text{(7.11)}
$$

The virtual strain at the top and bottom patches

$$
\varepsilon_{\text{top}} - \varepsilon_{\text{bottom}} = \left. \varepsilon_o + \frac{d_s}{2} \partial \chi \right|_{\text{bottom}} - \left. \varepsilon_o + \frac{d_s}{2} \sum_{m=1}^{M} \partial X_m W'_m \left( \frac{-m^2 \pi^2}{L^2} \right) \sin \left( \frac{m \pi x}{L} \right) \right|_{\text{bottom}}
$$

\text{(7.12)}
Piezo-electric contribution to the internal virtual work from the two patches taken together per unit length at any location from equation (7.18) takes the form

\[ 2 \int \delta \sigma \, e_p \, E_3 \, dA = 2Ee_s \frac{d_s}{2} t_p (x) t_p \, \sum_{m=1}^{M} \delta \chi_m \, W_m^j \left( -\frac{m^2 \pi^2}{L^2} \right) \sin \left( \frac{m \pi x}{L} \right) \]  

(7.13)

Now taking the voltage variation in the x-direction in the same form as longitudinal strain,

\[ V_{st}^j(x) = \sum_{m=1}^{M} V_{st}^j \sin \left( \frac{m \pi x}{L} \right) \]  

(7.14)

the total piezoelectric contribution may be written in the form

\[ \delta W_{\text{piezo}} = E_s e_p d_s t_s \frac{L}{2} \sum_{m=1}^{M} W_m^j \left( -\frac{m^2 \pi^2}{L^2} \right) V_{st}^j \delta \chi_m \]  

(7.15)

which may written in the abbreviated form:

\[ \delta W_{\text{piezo}} = B_{ij}^m V_{st}^j \delta \chi_i \quad (i, j = 1, \ldots, M_o) \]  

(7.16)

If a negative feedback with gains proportional to the strain-rates sensed at the patches, is adopted,

\[ V_{st}^j = -G_{st} \frac{d_s}{2} \left( -\frac{m^2 \pi^2}{L^2} \right) W_m^j \dot{X}_m \quad (m = 1, \ldots, M_o) \]  

(7.17)
where $G_{st}$ is the gain for the stiffener patch. Thus,

$$\delta W_{int}^{piezo} = \frac{(d_s)^2 t_s L}{4} \left( E_s e_p G_{st} \right) \sum_{m=1}^{M} \left( W_m^j \right)^2 \left( \frac{m^4 \pi^4}{L^4} \right) \hat{X}_m \delta X_m \quad (7.18)$$

which may be abbreviated in the form:

$$\delta W_{int}^{piezo} = b_{ij}^{st} \hat{X}_j \delta X_i \quad (i, j = 1, \ldots, M_0) \quad (7.19)$$

Note $E_s e_p G_{st}$ may be viewed as a single parameter representing stiffener control and the purely mechanical contribution from the piezo-electric patches is deemed to be subsumed in that of the stiffener and is not separately shown.

### 7.2.3.2. Double patch on the center of the plate

Consider the piezo-electric patches running longitudinally at the middle of the plates, i.e. mid-way between the stiffeners. The curvature components in the longitudinal and transverse directions of the plate elements are denoted by $\chi_x^C$ and $\chi_y^C$ respectively. The superscript $c$ refers to the center of the plate (A and B in Figure 5.1). Over the relatively small patch, we shall assume that the strains are uniform in the transverse direction and can be represented by the value at the center of the plate given by at $y = y_c$.

For simplicity only the top patch on the left is considered; for the bottom patch simply reverse the signs of both the bending strain and the voltage. Similarly for the top and bottom patches on the right, the voltages are maintained numerically the same, but their
signs are prescribed to be opposite for the antisymmetric local modes and the same if the symmetric modes are considered.

In developing the control strategy, we may assume that contribution of overall bending which adds small curvatures to the plate in the longitudinal direction are negligible. Thus the plate and stiffener patches are designed for controlling local modes and overall modes respectively. Local bending strain component in the $x$ takes the form,

$$
\varepsilon_{x,l}^b = -\frac{t}{2} \frac{\partial^2 w}{\partial x^2} \bigg|_{y=y_c} = -\frac{t}{2} \sum_{m=1}^{M} \frac{m^2 \pi^2}{L^2} w_m \sin \left( \frac{m \pi x}{L} \right) \zeta_m
$$

(7.20)

The foregoing expression may be contracted to the form,

$$
\varepsilon_{x,l}^b = \frac{t}{2} \sum_{m=1}^{M} \chi_{x,m}^c \sin \left( \frac{m \pi x}{L} \right) \epsilon_m
$$

(7.21)

Likewise,

$$
\varepsilon_{y,l}^b = -\frac{t}{2} \frac{\partial^2 w}{\partial y^2} \bigg|_{y=y_c} = \frac{t}{2} \sum_{m=1}^{M} \chi_{y,m}^c \sin \left( \frac{m \pi y}{L} \right) \epsilon_m
$$

(7.22)

where $\chi_{x,m}^c$ and $\chi_{y,m}^c$ are the $m^{th}$ harmonic coefficients of curvature along the center line of the plate in $x$ and $y$ directions respectively.

The bending stresses on the top piezo-electric patch take the form,

$$
\sigma_x^b = \frac{E_p t}{2(1-\nu^2)} \sum_{m=1}^{M} \left( \chi_{x,m}^c + \nu \chi_{y,m}^c \right) \epsilon_m \sin \left( \frac{m \pi x}{L} \right)
$$

(7.23.a)
\[ \sigma_y^b = \frac{E_p t}{2(1 - \nu^2)} \sum_{m=1}^{M} \left( \chi_{y,m}^c + \nu \chi_{x,m}^c \right) \xi_m \sin \left( \frac{m \pi x}{L} \right) \]  

(7.23.b)

The internal virtual work contribution (eq. (7.7)) takes the form with both the pair of patches accounted,

\[ \delta W_{\text{int}}^{\text{piezo}} = \frac{2 E_p e_p t b_p}{(1 - \nu)} \int_{0}^{l} \left[ \sum_{m=1}^{M} \left( \chi_{x,m}^c + \chi_{y,m}^c \right) \delta \xi_m \sin \left( \frac{m \pi x}{L} \right) \right] V^c(x) \, dx \]  

(7.24)

where \( V^c \) is the voltage distribution along the center patch. Taking the voltage in the form of a series as in eq. (7.14),

\[ V^p(x) = \sum_{m=1}^{M} V_m^p \sin \left( \frac{m \pi x}{L} \right) \]  

(7.25)

Then we have,

\[ \delta W_{\text{int}}^{\text{piezo}} = \frac{E_p e_p t b_p L}{(1 - \nu)} \sum_{m=1}^{M} \left( \chi_{x,m}^c + \chi_{y,m}^c \right) V_m^p \, \delta \xi_m \]  

(7.26)

which may be written in the abbreviated form

\[ = B_{ij}^p V_j^P \, \delta \xi_i \hspace{1cm} (i, j = 1, \ldots, M) \]  

(7.27)
**Negative Feedback Control**

As before, we consider negative feedback with gains proportional to the strain rates sensed at the patches. Taking the voltage at any location to be proportional to the sum of the bending strain-rates (designated as ‘effective strain rate’) at that location, we have

\[ V^p(x) = G_p \frac{t}{2} \left[ \sum_{m=1}^{M} (\chi_{x,m} + \chi_{y,m}) \xi_m \sin \left( \frac{m \pi x}{L} \right) \right] \]  

(7.28)

where \( G_p \) is the gain for the plate patches.

The internal work contribution takes the form as,

\[ \delta W_{\text{int}}^{\text{piezo}} = G_p \frac{E_p e_p t^2 b_p L}{2(1-\nu)} \sum_{m=1}^{M} (\chi_{x,m} + \chi_{y,m})^2 \xi_m \delta \xi_m \]  

(7.29)

which may be abbreviated in the form

\[ \delta W_{\text{int}}^{\text{piezo}} = b_{ij} \xi_j \delta \xi_i \quad (i, j = 1, \ldots, M) \]  

(7.30)

7.2.3.3. **Additional Plate Patches on Either Side of Stiffener**

Depending upon the value of \( \lambda \), and the capacity of the piezo-electric patches at the center of the plate, additional patches may be required for effective control of flutter. These are best placed in a region of maximum curvature in the section, second only to the center of the plate. Thus these must be located on either side of the stiffener on the plate (hence called “stiffener plate patches”). These constitute a set of four patches (two on each side), at top and bottom respectively (Figure 7.1).
To account for the contribution of these patches to the internal virtual work, the expression in eq. (7.26) is augmented by another term as shown below.

\[
\delta W^{\text{piezo}}_{\text{int}} = \frac{E_p e_p t b_p L}{(1-v)} \sum_{m=1}^{M} \left[ (\chi^c_{x,m} + \chi^c_{y,m}) + (\chi^s_{x,m} + \chi^s_{y,m}) \right] V_m^p \delta \xi_m \tag{7.31}
\]

where the superscript ‘s’ refers to quantities pertinent to plate-stiffener patches. Equation (7.31) is again abbreviated to the form given in eq. (7.27). Note that, at this stage, the voltages across two sets of plate patches are taken as equal in magnitude (which are not always the case in subsequent development).

Further:

(i) The transverse curvatures \( \chi^c_{x,m} \), \( \chi^c_{y,m} \) are of opposite in sign to those but so are the voltages, and (ii) The term \( \chi^s_{x,m} \) becomes zero as the local modal displacement at the stiffener plate junction is zero.

- **Negative Feedback Control**

In this case the voltages across the center and stiffener plate patches are taken as different but the gains are considered the same.

Taking the voltage distribution for both the patches as the same form as in eq. (7.29), we get

\[
\delta W^{\text{piezo}}_{\text{int}} = G_p \frac{E_p e_p t^2 b_p L}{2(1-v)} \sum_{m=1}^{M} \left[ (\chi^c_{x,m} + \chi^c_{y,m})^2 + (\chi^s_{x,m})^2 \right] v_m \delta \xi_m \tag{7.32}
\]

, which can be abbreviated in the same form as in eq. (7.30).
7.2.4. Equations of Motion

The two complementary sets of Lagrange equations of motion are

\[
\begin{align*}
\dot{m}_{ij}^{(e)} \ddot{z}_j + m_{ip} \ddot{X}_p + \left\{a_{ij}^{(e)} - \sigma_o b_{ij}^{(e)} \right\} \ddot{z}_j + \left\{a_{ip}^{(e)} - \sigma_o b_{ip}^{(e)} \right\} X_p + c_{ij}^{(e)} \dddot{z}_j + c_{ip}^{(e)} X_p \\
+ d_{ij}^{(e)} \dddot{z}_j + d_{ip}^{(e)} \dddot{X}_p + \frac{1}{3} C_{pji} X_p \dddot{z}_j + \frac{1}{3!} d_{ijklj} \dddot{z}_j \dddot{z}_k \dddot{z}_l + B_{qj}^{p} V_{j}^{p} = 0
\end{align*}
\]

\[
\begin{align*}
\dot{m}_{pq}^{(ov)} \ddot{X}_p + m_{ip}^{(ov)} \ddot{z}_i + \left\{a_{pq}^{(ov)} - \sigma_o b_{pq}^{(ov)} \right\} X_q + \left\{a_{ip}^{(ov)} - \sigma_o b_{ip}^{(ov)} \right\} \dddot{z}_i + c_{pi}^{(ov)} \dddot{z}_i + c_{pq}^{(ov)} X_q \\
+ d_{pi}^{(ov)} \dddot{z}_i + d_{pq}^{(ov)} \dddot{X}_q + \frac{1}{3!} C_{pqj} \dddot{z}_i \ddot{z}_j + B_{qj}^{st} V_{j}^{st} = 0
\end{align*}
\]

Note

(i) \(a_{ij}^{(e)} = b_{ij}^{(e)} = m_{ij}^{(e)} = d_{ij}^{(e)} = 0\) unless \(i = j\),

(ii) \(a_{pq}^{(ov)} = b_{pq}^{(ov)} = m_{pq}^{(ov)} = d_{pq}^{(ov)} = 0\) unless \(p = q\), and

(iii) \(a_{ip}^{(e)} = b_{ip}^{(e)} = m_{ip}^{(e)} = d_{ip}^{(e)} = 0\) unless \(i = p\).

The first and set of equations are designated as “plate” and “stiffener” equations indicating their respective source. For negative feedback control the terms \(B_{qj}^{p} V_{j}^{p}\) and \(B_{qj}^{st} V_{j}^{st}\) are replaced by \(b_{qj}^{p} \dddot{z}_j\) and \(b_{qj}^{st} \dddot{X}_j\) in the plate and stiffener equations respectively.

7.2.5. Selection of Gains and Optimality of Control

Control of linear dynamic systems is a well understood problem and there exist well established algorithms for optimal control of these systems. The present problem is nonlinear and as such these control techniques are not directly applicable. In this section a technique of linearizing the governing equations at any given time is introduced. A linear quadratic regulator (LQR) algorithm is then invoked to obtain a gain matrix. Direct
use of this gain matrix implies a multi-input multi-output scheme which though theoretically efficient may prove too intricate for implementation. In the present study therefore the simpler self-sensing negative velocity feedback control scheme is considered and used in most of the calculations.

### 7.2.5.1. Basic Principles of LQR

Consider the standard dynamic state space model in the form

\[
\begin{bmatrix} \dot{X} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} X \end{bmatrix} + \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} U \end{bmatrix}
\]

(7.35)

Note \( \{X\} \) is a column vector of 2n degrees of freedom, \( \{U\} \) is a set of r control parameters, \( \begin{bmatrix} A \end{bmatrix} \) is a 2n x 2n matrix of constants and \( \begin{bmatrix} B \end{bmatrix} \) is a 2n x r matrix of constants. A performance index is selected in the form

\[
J = \int_0^\infty \left[ \{X\}^T \begin{bmatrix} Q \end{bmatrix} \{X\} + \{U\}^T \begin{bmatrix} R \end{bmatrix} \{U\} \right] dt
\]

(7.36)

where \( Q \) and \( R \) are appropriately chosen symmetric positive definite matrices, which provide a balance between the costs associated with ineffective control and control effort. The gain matrix \( [K_{gain}] \) is defined by

\[
\begin{bmatrix} U \end{bmatrix} = -[K_{gain}] \{X\}
\]

(7.37)

Minimizing \( J \), the controller gain matrix \( K_{gain} \) may be obtained in the form
\[
[K_{\text{gain}}] = [R]^{-1} [B]^T [P]
\] (7.38)

where \([P]\) is a positive definite matrix obtained from the solution of the following Riccati equation

\[
\] (7.39)

### 7.2.5.2. Reduction to State Space Model

Consider a second order linear system,

\[
[M] \{\ddot{q}\} + [D] \{\dot{q}\} + [G] \{q\} = [B_1] \{V\}
\] (7.40)

where \(M\), \(D\), and \(C\) are mass, aerodynamic damping, and aerodynamic pressure matrices respectively. \(G\) is the sum of two matrices, viz. \(K_L\) the linearized stiffness matrix which is symmetric and \(C\) the aerodynamic pressure matrix which is a skew-symmetric. \(\{q\}\) are the ‘\(n\)’ \((n = M + M_o)\) modal degrees of freedom and ordered in the form

\[
\{q\} = \begin{bmatrix} \xi \\ X \end{bmatrix}
\] (7.41)

and \(\{V\}\) are the set of control parameters, \(\{V\} = \begin{bmatrix} V^p \\ V^{st} \end{bmatrix}\).
Note the voltage is assumed to be the same for both the plate-center and plate-stiffener patches for a given $x$.

Letting

$$\{X\} = \{\dot{q}\}$$

(7.42)

Equation (7.40) may then be written in the form

$$\begin{bmatrix} \ddot{q} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -[M]^{-1}[D] & -[M]^{-1}[G] \\ [I] & [0] \end{bmatrix} \begin{bmatrix} \dot{q} \\ q \end{bmatrix} + \begin{bmatrix} [B_1] \\ [0] \end{bmatrix} \{V\}$$

(7.43)

This is of the form

$$\{\ddot{X}\} = [A]\{X\} + [B]\{U\}$$

(7.44)

where $[A] = \begin{bmatrix} -[M]^{-1}[D] & -[M]^{-1}[G] \\ [I] & [0] \end{bmatrix}$, $[B] = \begin{bmatrix} [B_1] \\ [0] \end{bmatrix}$, and

$[B_1] = \begin{bmatrix} [B^p] & [0] \\ [0] & [B^{st}] \end{bmatrix}$.

7.2.5.3. Linearization of the Stiffness Matrix

The potential energy of the elastic forces for the entire panel is developed in terms of local and overall modal d.o.f., and it may be written in the form,
\[
\Pi = \frac{1}{2!} \left\{ a^{(t)}_{ij} - \sigma_o b^{(t)}_{ij} \right\} \xi_i \xi_j + \frac{1}{2!} \left\{ a^{(ov)}_{pq} - \sigma_o b^{(ov)}_{pq} \right\} X_p X_q \\
+ \left\{ a^{(c)}_{ip} - \sigma_o b^{(c)}_{ip} \right\} \xi_i X_p + \frac{1}{3!} C_{ij} X_p \xi_i \xi_j + \frac{1}{4!} d_{ijkl} \xi_i \xi_j \xi_k \xi_l
\]

(7.45)

\[
(i, j, k, l = 1, \ldots, M) \\
(p, q = 1, \ldots, M_o)
\]

This can be written in the form in the complete set of \( n \) (= \( M + M_o \)) degrees of freedom \( \{q\} \) that

\[
\Pi = \frac{1}{2!} a_{ij} q_i q_j + \frac{1}{3!} c_{ijk} q_i q_j q_k + \frac{1}{4!} d_{ijkl} q_i q_j q_k q_l
\]

(7.46)

\[
(i, j, k, l = 1, \ldots, M+M_o)
\]

The first variation of \( \Pi \) takes the form,

\[
\delta \Pi = a_{ij} q_j \delta q_i + \frac{1}{2!} c_{ijk} q_j q_k \delta q_i + \frac{1}{3!} d_{ijkl} q_j q_k q_l \delta q_i
\]

(7.47)

At a given time, \( t_o \), when the current values of \( q_i \) are given by \( q_i^o \), a linearized expression of the first variation takes the form,

\[
\delta \Pi_L = a_{ij} q_j \delta q_i + \frac{1}{2!} c_{ijk} q_k^o q_j \delta q_i + \frac{1}{3!} d_{ijkl} q_k^o q_j^o \delta q_i = K_{ij} q_j \delta q_i
\]

(7.48)
The linearized stiffness matrix is then

\[ K_{Lij} = a_{ij} + \frac{1}{2!} c_{ijk} q_k + \frac{1}{3!} d_{ijk} q_k q_i \] (7.49)

This matrix can be set up only when the current values of \( q \) are solved for by an iterative process and is applicable in the immediate vicinity of the selected point in time. Using this locally applicable matrix as \( t \to \infty \) is questionable, but may be justified as being a pointer towards an optimal solution at every step along the trajectory. With this linearized matrix as the key ingredients in the LQR algorithm, we obtain at a best set of sub-optimal values of the control parameters.

A more explicit expression of the linearized stiffness matrix may be developed in terms of the local and overall modal degrees of freedom as follows. The first variation of the potential of the elastic forces may be derived from eq. (7.45) takes the form:

\[
\delta \Pi = \left[ \left( a^{(i)} - \sigma_o b^{(c)} \right) \xi_j + \left( a^{(e)} - \sigma_o b^{(c)} \right) X_p + \frac{1}{3!} C_{pq} X_p \xi_j + \frac{1}{3!} d_{qik} \xi_j \xi_k \right] \delta \xi_i \\
+ \left[ \left( \xi^{(i)} - \sigma_o b^{(c)} \xi_j \right) X_q + \left( \xi^{(e)} - \sigma_o b^{(c)} \xi_j \right) \xi_j + \frac{1}{6} C_{pq} \xi_j \xi_j \right] \delta X_p
\] (7.50)

The linearized first variation may then be written as,

\[
\delta \Pi_L = \left[ \left( a^{(i)} - \sigma_o b^{(i)} \right) + \frac{1}{6} C_{pq} X_p \xi_j + \frac{1}{3!} d_{qik} \xi_j \xi_k \right] \delta \xi_i \\
+ \left[ \left( a^{(e)} - \sigma_o b^{(c)} \xi_j \right) + \frac{1}{6} C_{pq} \xi_j \right] X_p \delta \xi_i \\
+ \left[ \left( \xi^{(i)} - \sigma_o b^{(i)} \xi_j \right) + \frac{1}{6} C_{pq} \xi_j \right] \xi_i \delta X_p + \left[ \left( \xi^{(c)} - \sigma_o b^{(c)} \xi_j \right) \right] X_q \delta X_p
\] (7.51)
which may be abbreviated as

\[
\delta \Pi_l = \left[ K_{l,ij}^{(11)} \right] \delta \xi^i \delta \xi^j + \left[ K_{l,qr}^{(12)} \right] X_p \delta \xi^i + \left[ K_{l,qr}^{(21)} \right] \xi^i \delta X_p + \left[ K_{l,qr}^{(22)} \right] \xi^i \delta X_p
\]

Thus the linearized stiffness matrix takes the form,

\[
[K_L] = \begin{bmatrix}
K_{L}^{(11)} & K_{L}^{(12)} \\
K_{L}^{(21)} & K_{L}^{(22)} 
\end{bmatrix}
\] (7.53)

Note that the matrix is symmetric, i.e. \( [K_L]^{(12)} = [K_L]^{(21)} \).

### 7.2.6. Application of Optimal Control Strategy

Two alternative strategies are pursued in the present study:

- **Type (i) control**: Negative velocity feedback control where sensing and control voltage at any location have a direct one to one relationship to each other, and
- **Type (ii) control**: the control exercised via full state feedback, i.e. using the full gain matrix \( K \).

The former is relatively simple to implement but there is a limiting value of \( \lambda \) beyond which it fails to control. On the other hand the latter is the multi-input multi-output (MIMO) control requiring somewhat intricate technology for its implementation. This type of control appears to be potentially capable of fully suppressing the vibrations for
any $\lambda$, the only limiting factor being the limiting field strength that can be sustained by
the piezo-electric material.

### 7.2.6.1. Selection of Gains

**(i) Negative Velocity feedback control**

A relatively short duration is selected ($\Delta T$) is selected which is significantly greater than the time period of oscillation corresponding to the lowest of the frequencies of the participating modes. For the geometry and material properties of the stiffened panel considered here, generally duration of $10^{-3}$ sec is found to work satisfactorily. This is duration is subdivided into $N$ (say 10) equal intervals. Thus we would have $N$ station points in time. The dynamic flutter analysis is run for the given structure and at each station point the linearized stiffness matrix is computed and LQR algorithm is invoked to find the gain matrix $[K_{gain}]$.

The voltages may be obtained from

\[
\begin{bmatrix}
V^c \\
V^{st}
\end{bmatrix} = -\begin{bmatrix}
K
\end{bmatrix}_{m2n} \begin{bmatrix}
\dot{q} \\
q
\end{bmatrix}
\]  

(7.54)

As already mentioned, a simpler self-sensing negative velocity feedback is preferred here. Furthermore, only a single gain factor for each type of control, i.e. stiffener patch and plate patch are proposed to be used. Thus considering only the diagonal terms, we obtain the voltages, $V^c_m$ ($m = 1 \ldots M$), and $V^{st}_m$ ($m = 1 \ldots M_0$), in the form,

\[
\begin{align*}
V^c_m &= -K(m,m)\ddot{x}_m \\
V^{st}_m &= -K(m+M,m+M)\dot{x}_m
\end{align*}
\]  

(7.55)
First consider the plate patches. Taking the patch at the center of the plate as a reference, the gain is computed as

\[
G_m^P = \frac{V_m}{\frac{t}{2}(\chi_{x,m}^c + \chi_{y,m}^c)} = -\frac{K(m, m)}{\frac{t}{2}(\chi_{x,m}^c + \chi_{y,m}^c)}
\]

(7.56)

The gain for the stiffener patch is found similarly as

\[
G_m^{st} = \frac{V_m^{st}}{\frac{d_s}{2} \left( -\frac{m^2 \pi^2}{L^2} \right) W_m J_m} = \frac{-K(m + M, m + M)}{\frac{d_s}{2} \left( -\frac{m^2 \pi^2}{L^2} \right) W_m J_m}
\]

(7.57)

For a given \( m \) these values turn out to be sensibly constant for all the station points in time \( (t_i) \). The reason for this constancy is not far to seek. Optimal control algorithm, at any time, tends to select voltage parameters \( (V_m^P, V_m^{st}) \) which are proportional to the corresponding effective strain rates (vide eq. (7.56 – 57)). Thus the gain parameters are sensibly constant over the total duration of \( T \). The values of the gain do vary with \( m \) depending upon the relative dominance of the modes involved. Outside of the cluster of dominant modes, both \( G_m^P \) and \( G_m^{st} \) would become steadily smaller as \( m \) increases.

However it would be highly desirable to have a single value for the gain from end to end (i.e., from \( x = 0 \) to \( x = L \)). This would lead to the same gain for all the modes. This single value of gain may be taken as a weighted average of the gains for all the modes.

Here we propose to use the root mean square value of the modal gains. This is considered more representative than arithmetic mean as it tends to minimize the influence of “fringe” modes with lower values of gains.

Thus the gains for the plate and stiffener patches respectively are found from that
\[ G^p(t_i) = \sqrt{\frac{\sum_{m=1}^{M} \{G^p_m\}^2}{M}}, \text{ and } G^{st}(t_i) = \sqrt{\frac{\sum_{m=1}^{M} \{G^{st}_m\}^2}{M_o}} \] (7.58)

These values are sensibly constant over the duration considered.

(ii) Full state feedback control

The assumptions of a single value of gain for all the modes and its constancy over time, the exclusive dependence on velocity (as against displacement) in the computation of gains and the implicit neglect of the influences in the makeup of \( V_m \) of other harmonics makes the negative velocity feedback control as presented in the foregoing, somewhat simplistic and potentially limited in scope. Hence it needs the use of the full gain matrix and recognition of its variation with time.

In this procedure, the voltages \( (V) \) as given by \(-[K_{gain}]\{X\}\) are computed at the end of each time step and directly substituted in eq. (7.33) and used in the calculation in the following step. Thus the control effort is updated continuously.

7.3. Results

Numerical examples illustrating the performances of the two types of control respectively are presented in this section. The cases of anti-symmetric and symmetric local modes acting on the panel are treated separately as before. The geometric and material properties of the panel are the same as in Chapter 6 and the piezo-electric material is the same as in chapter 5 \( (E_p\varepsilon_p = 0.0283) \). Thicknesses of both the plate and stiffener patches are assumed to be 0.1 mm for computing electric field strengths.
7.3.1. Selection of $[Q]$ and $[R]$

We may select $[Q]$ and $[R]$ as diagonal matrices in the absence of pertinent information. As long as the settling of vibration is within a sufficiently short duration, the magnitude of $[X]$ is not a consideration, and we set $[Q] = [I]$, a relatively small value indeed.

On the other hand $[R]$ directly controls the control effort and the peak voltages that do develop in the patches. Selection of $[R]$ matrix therefore requires some careful consideration. Thus we set $[Q] = [I]$, the identity matrix and consider the selection of the diagonal terms in $[R]$.

Greater the value of a diagonal coefficient of the $R$-matrix, smaller will be the corresponding gain and greater the settling time of that particular mode. Since the overall modes do not, generally play a significant role in a purely flutter problem it stands to reason to choose higher $R$-values for the overall modes relative to the local modes. As $\lambda$ increases, higher and higher gains are needed for control and this means the values of the $R$-coefficients decrease with $\lambda$. In the following several example cases are considered which illustrate the influence of the $R$-values chosen.

7.3.1.1. Type (i) Control: Negative velocity feedback control

7.3.1.1.1. Panel under the anti-symmetric local modes

In this section, the panel is deemed to be subjected to the action of anti-symmetric local modes and overall modes. The panel carries an axial compression, $\sigma_o = 0.4 \sigma_{cr}$ . Aerodynamic damping is considered with $C_a = 0.1$. The corresponding $\lambda_{cr} = 3347$. For a given $R$-matrix, the gain matrix $K_{gain}$ is obtained from the LQR algorithm. From this the panel patch gain $G_p$ and stiffener patch gain $G_{st}$ are computed using eq. (7.56 – 57).
These gains are used to study the dynamic response of the panel for the given $\lambda$ with initial conditions assumed as follows: $\xi_m = 0.01/m$; $X_m = 0.01/m$.

The points of interest are whether or not (i) the panel vibrations do get damped out in time and (ii) the magnitudes of the electric field strengths (Voltage/mm) developed across the patches remain within limits of the capacity of the patch material.

### Case 1: Plate stiffener patch inactive

In this case only the center plate patch and the stiffener patch are alone active, i.e. plate stiffener patch is inactive. The R-matrix is selected to be a diagonal matrix with equal values $R_d$ associated with voltage coefficients of all the local and overall modes.

Table 7.1 summarizes the salient results as shown below.

### Table 7.1. Case 1

<table>
<thead>
<tr>
<th>$R_d$</th>
<th>$\lambda$</th>
<th>Max. Volt/mm at plate patch</th>
<th>Max. Volt/mm at stiffener patch</th>
<th>Gain (plate patch)</th>
<th>Gain (stiffener patch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000</td>
<td>5000</td>
<td>2.09</td>
<td>1.21</td>
<td>64.75</td>
<td>0.63</td>
</tr>
<tr>
<td></td>
<td>5800</td>
<td>Uncontrollable</td>
<td></td>
<td>76.46</td>
<td>0.59</td>
</tr>
<tr>
<td>100</td>
<td>5800</td>
<td>3.64</td>
<td>9.54</td>
<td>132.0</td>
<td>5.00</td>
</tr>
<tr>
<td>7000</td>
<td>escalating</td>
<td></td>
<td></td>
<td>149.4</td>
<td>4.64</td>
</tr>
<tr>
<td>20</td>
<td>7000</td>
<td>5.55</td>
<td>36.90</td>
<td>249.1</td>
<td>19.48</td>
</tr>
<tr>
<td></td>
<td>8000</td>
<td>5.59</td>
<td>35.23</td>
<td>260.9</td>
<td>18.60</td>
</tr>
<tr>
<td></td>
<td>9000</td>
<td>5.63</td>
<td>33.85</td>
<td>273.3</td>
<td>17.87</td>
</tr>
<tr>
<td></td>
<td>9500</td>
<td>Uncontrollable</td>
<td></td>
<td>279.6</td>
<td>17.55</td>
</tr>
<tr>
<td>10</td>
<td>9500</td>
<td>6.61</td>
<td>60.67</td>
<td>356.0</td>
<td>32.34</td>
</tr>
</tbody>
</table>
As mentioned earlier, smaller values of $R_d$ are needed as $\lambda$ increases which result in higher gains for the feedback control and higher electric field strengths across the patches. Assuming the capacity of the patches to be 600 volts/mm, a patch thickness of 0.1mm is sufficient for both the stiffener and plate patches. At $\lambda = 9500$, the maximum electric field strengths that develop are 606.7 volts and 66.1 volts/mm for the stiffener and plate patches respectively. An examination of the actual response of the panel indicates that the overall vibrations get damped out within a fraction of a second whatever the gains employed for the stiffener patches.

To illustrate this we compare the overall response for $\lambda = 10000$ for the following two cases:

(i) $R_d = 10$ for all the modes; the corresponding gains are $G_p = 361.3; G_{st} = 31.82$

(ii) $R_d$ for the local modes, $R_d^{lo} = 10$, and $R_d$ for overall modes, $R_d^{ov} = 1000$. The corresponding gains are: $G_p = 361.3; G_{st} = 0.46$

Time histories are shown in Figure 7.2 and Figure 7.3 for case (i) and (ii) respectively.

Figure 7.2. Time histories of first overall mode and stiffener voltage for $R_d^{lo} = R_d^{ov} = 10$ ($G_p = 361.3; G_{st} = 31.82$)
A comparison of these two responses indicates the voltages developed at the stiffener patches in case (i) is higher, the settling time of overall vibrations is smaller than that of case (ii); however both are very small fractions of a second. Setting a high value for $R^{ov}_{d}$ makes it possible to extend the control well beyond $\lambda = 10000$ as it reduces the stiffener patch gain and thus the corresponding voltage which turns out to be the controlling factor. The maximum plate patch voltage on the other hand is $\approx 6.6$ V/mm in both cases, which is well within the capacity of the patch material (vide Figure 7.4).
In view of this observation, panel control will be investigated for values of $\lambda > 10000$ with $R_d^{v}\nu$ kept at a sufficiently high value (=1000). $R_d^{l0}$ will be kept as high as would make control possible. These results are given in Table 7.2.

For example, control breaks down at $\lambda = 12000$, with $R_d^{l0} = 10$ typified by the escalating plate patch voltage (ref. figure 7.5) but is regained with a reduction of $R_d^{l0} =5$. It is seen that control is possible up to $\lambda = 51000$, but this requires increasing the plate patch gains by a continued reduction of $R_d^{l0}$ to a value of 0.1. At $\lambda = 52000$, the panel becomes uncontrollable with the overall deflections suddenly beginning to increase after being quiescent for about 0.1 sec. as shown in Figure 7.6. However the local modes are well controlled as the panel oscillates in the overall modes up to 0.3 sec.

![Figure 7.5](image_url)

**Figure 7.5.** Time history of center plate voltage for $\lambda = 12000$ ($R_d^{l0} = 10$, $R_d^{sr} = 1000$, $G_p = 389.7$; $G_{sr} = 0.42$)
Table 7.2. Relationships between $R_{d}^{lo}$, $\lambda$, Voltages and Gains
(No stiffener-plate patch employed) $R_{d}^{ov}$ = 1000.

<table>
<thead>
<tr>
<th>$R_{d}^{lo}$</th>
<th>$\lambda$</th>
<th>Max. Volt/mm at plate patch</th>
<th>Max. Volt/mm at stiffener patch</th>
<th>Gain (plate patch)</th>
<th>Gain (stiffener patch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>10000</td>
<td>6.55</td>
<td>0.94</td>
<td>361.3</td>
<td>0.46</td>
</tr>
<tr>
<td></td>
<td>11000</td>
<td>6.63</td>
<td>0.89</td>
<td>372.4</td>
<td>0.44</td>
</tr>
<tr>
<td></td>
<td>12000</td>
<td>6.70</td>
<td>Uncontrollable</td>
<td>383.9</td>
<td>0.43</td>
</tr>
<tr>
<td>5</td>
<td>12000</td>
<td>7.79</td>
<td>0.87</td>
<td>494.4</td>
<td>0.43</td>
</tr>
<tr>
<td></td>
<td>13000</td>
<td>7.84</td>
<td>0.83</td>
<td>504.0</td>
<td>0.41</td>
</tr>
<tr>
<td></td>
<td>14000</td>
<td>7.93</td>
<td>0.80</td>
<td>513.8</td>
<td>0.40</td>
</tr>
<tr>
<td>15000</td>
<td>15000</td>
<td>11.10</td>
<td>0.78</td>
<td>1029.0</td>
<td>0.39</td>
</tr>
<tr>
<td>1</td>
<td>15000</td>
<td>11.54</td>
<td>0.74</td>
<td>1042.5</td>
<td>0.37</td>
</tr>
<tr>
<td>20000</td>
<td>12.25</td>
<td>0.69</td>
<td>1057.0</td>
<td>0.35</td>
<td></td>
</tr>
<tr>
<td></td>
<td>22500</td>
<td>13.25</td>
<td>0.67</td>
<td>1072.0</td>
<td>0.34</td>
</tr>
<tr>
<td></td>
<td>25000</td>
<td>14.43</td>
<td>0.65</td>
<td>1087.5</td>
<td>0.33</td>
</tr>
<tr>
<td></td>
<td>27500</td>
<td>16.50</td>
<td>0.64</td>
<td>1486.0</td>
<td>0.33</td>
</tr>
<tr>
<td>0.5</td>
<td>27500</td>
<td>17.89</td>
<td>0.64</td>
<td>1497.8</td>
<td>0.33</td>
</tr>
<tr>
<td></td>
<td>30000</td>
<td>19.42</td>
<td>0.67</td>
<td>1509.7</td>
<td>0.35</td>
</tr>
<tr>
<td></td>
<td>32500</td>
<td>21.06</td>
<td>0.69</td>
<td>1521.7</td>
<td>0.36</td>
</tr>
<tr>
<td></td>
<td>35000</td>
<td>18.19</td>
<td>Uncontrollable</td>
<td>1534.0</td>
<td>0.39</td>
</tr>
<tr>
<td></td>
<td>37500</td>
<td>22.02</td>
<td>0.75</td>
<td>3228.2</td>
<td>0.39</td>
</tr>
<tr>
<td>0.1</td>
<td>37500</td>
<td>23.56</td>
<td>0.84</td>
<td>3233.0</td>
<td>0.44</td>
</tr>
<tr>
<td></td>
<td>40000</td>
<td>25.18</td>
<td>1.02</td>
<td>3237.9</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
<td>42500</td>
<td>26.86</td>
<td>1.32</td>
<td>3243.0</td>
<td>0.69</td>
</tr>
<tr>
<td></td>
<td>47500</td>
<td>28.61</td>
<td>2.05</td>
<td>3248.2</td>
<td>1.07</td>
</tr>
<tr>
<td></td>
<td>50000</td>
<td>30.44</td>
<td>4.89</td>
<td>3253.5</td>
<td>2.56</td>
</tr>
<tr>
<td></td>
<td>51000</td>
<td>31.19</td>
<td>11.69</td>
<td>3255.6</td>
<td>6.14*</td>
</tr>
<tr>
<td></td>
<td>52000</td>
<td>Uncontrollable</td>
<td>3257.8</td>
<td>12.05*</td>
<td></td>
</tr>
</tbody>
</table>

*Found to vary with time; averaged value over the time interval selected.
Figure 7.6. Time histories of maximum local plate displacement and first overall mode for $\lambda = 52000$ ($R_{d}^{lo} = 0.1$, $R_{d}^{ov} = 1000$, $G_{p} = 3257.8$; $G_{st} = 12.05$)

Table 7.3. Relationships between $R_{d}^{lo}$, $\lambda$, Voltages and Gains (Stiffener-plate patch active) $R_{d}^{ov} = 1000$.

<table>
<thead>
<tr>
<th>$R_{d}^{lo}$</th>
<th>$\lambda$</th>
<th>Max. Volt/mm at plate patch</th>
<th>Max. Volt/mm at stiffener patch</th>
<th>Gain (plate patch)</th>
<th>Gain (stiffener patch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>9000</td>
<td>2.8083</td>
<td>0.9236</td>
<td>127.469</td>
<td>0.4821</td>
</tr>
<tr>
<td>15000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>15000</td>
<td>4.6684</td>
<td>0.7419</td>
<td>341.9888</td>
<td>0.3884</td>
</tr>
<tr>
<td>30000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>30000</td>
<td>10.2383</td>
<td>0.6363</td>
<td>1028.8</td>
<td>0.3354</td>
</tr>
<tr>
<td>40000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>50000</td>
<td>18.1765</td>
<td>4.8514</td>
<td>1462.5</td>
<td>2.5848</td>
</tr>
<tr>
<td>0.1</td>
<td>51000</td>
<td>18.6471</td>
<td>11.5511</td>
<td>1464.9</td>
<td>6.1714</td>
</tr>
<tr>
<td>52000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Case 2. Plate-stiffener patch active

A similar study was conducted activating both the central and stiffener plate patches in addition to the stiffener patch. Some sample results are summarized in Table 7.3. The voltages are significantly lower but the limit of $\lambda$ remains the same.

- Limitation of the negative velocity feedback control

One common feature of the results from case(i) and case(ii) is that the values of $G^{ST}$ used are relatively small because of the high value of $R_d^{ov}$ ($= 1000$) taken. With a view to investigate whether this was the cause of the failure of the control at $\lambda = 52000$, we set both the gains at a high value, i.e. $G^{ST} = G^{P} = 4000$ and study the performance of the control process. Setting high gains proved counterproductive as it results in an earlier escalation of both overall and local displacements (Figure 7.7). It is concluded therefore that this failure of control must be attributed to the limitations of the simple control scheme employed in which the control voltage at any location is based on strain-rate sensed at that point. A more comprehensive MIMO scheme based on the full gain matrix is apparently needed as $\lambda$ exceeds a limiting value.

Figure 7.7. Time histories of maximum local plate displacement and first overall mode for $\lambda = 52000$ ($G_p = 4000$; $G_{st} = 4000$)
7.3.1.1.2. **Panel under symmetric local modes**

We next study the panel response now subjected to the action of symmetric local modes using the control methodology. As before we set $\sigma_o = 0.4 \sigma_{cr}$; $C_a = 0.1$; (the corresponding $\lambda_{cr} = 2901$); and the initial conditions: $\xi_m = 0.01/m$, $X_m = 0.01/m$. and $R_d^{ov} = 1000$. It was observed because of the sharpness of curvatures at the center and stiffener associated with the symmetric local modes, the negative velocity feedback becomes clearly less effective than in the case of panels studied under antisymmetric modes. In order to determine the limiting value of $\lambda$ at which panel becomes uncontrollable, the allowable field strength of piezo-electric patches is taken as 100 V/mm in the study.

**Case (i): Plate stiffener patch inactive**

The results for this case are given in Table 7.4. The plate patch voltage capacity is exceeded at $\lambda = 10200$. As before voltages across stiffener patches are too small to be of significance.

**Case (ii): Plate stiffener patch active**

The results for this case are given in Table 7.5. The center plate patch capacity is exceeded at $\lambda = 19400$. At $\lambda = 19800$ the panel ceases to be controllable whatever the gains selected. This once again, illustrates the limitation inherent in the chosen method control.
<table>
<thead>
<tr>
<th>$R_d^{lo}$</th>
<th>$\lambda$</th>
<th>Max. Volt at plate patch</th>
<th>Max. Volt at stiffener patch</th>
<th>Gain (plate patch)</th>
<th>Gain (stiffener patch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>500</td>
<td>4000</td>
<td>19.91</td>
<td>3.68</td>
<td>47.82</td>
<td>1.37</td>
</tr>
<tr>
<td></td>
<td>4200</td>
<td>20.49</td>
<td>3.68</td>
<td>49.43</td>
<td>1.37</td>
</tr>
<tr>
<td></td>
<td>4400</td>
<td>Uncontrollable</td>
<td>51.10</td>
<td>1.37</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>4400</td>
<td>37.36</td>
<td>1.83</td>
<td>93.74</td>
<td>0.69</td>
</tr>
<tr>
<td></td>
<td>4600</td>
<td>37.85</td>
<td>1.83</td>
<td>95.05</td>
<td>0.69</td>
</tr>
<tr>
<td></td>
<td>5000</td>
<td>38.90</td>
<td>1.83</td>
<td>97.82</td>
<td>0.69</td>
</tr>
<tr>
<td></td>
<td>5400</td>
<td>40.00</td>
<td>1.83</td>
<td>100.7</td>
<td>0.69</td>
</tr>
<tr>
<td></td>
<td>5800</td>
<td>41.14</td>
<td>1.83</td>
<td>103.8</td>
<td>0.69</td>
</tr>
<tr>
<td></td>
<td>6000</td>
<td>Uncontrollable</td>
<td></td>
<td>105.4</td>
<td>0.69</td>
</tr>
<tr>
<td>50</td>
<td>6000</td>
<td>53.17</td>
<td>1.34</td>
<td>136.9</td>
<td>0.51</td>
</tr>
<tr>
<td></td>
<td>7000</td>
<td>55.76</td>
<td>1.34</td>
<td>144.1</td>
<td>0.51</td>
</tr>
<tr>
<td></td>
<td>7200</td>
<td>56.29</td>
<td>1.36</td>
<td>145.6</td>
<td>0.52</td>
</tr>
<tr>
<td></td>
<td>7400</td>
<td>Uncontrollable</td>
<td></td>
<td>147.2</td>
<td>0.52</td>
</tr>
<tr>
<td>25</td>
<td>7400</td>
<td>72.59</td>
<td>0.99</td>
<td>192.1</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>7600</td>
<td>73.03</td>
<td>0.99</td>
<td>193.4</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>8000</td>
<td>73.90</td>
<td>0.98</td>
<td>196.0</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>8400</td>
<td>74.83</td>
<td>0.98</td>
<td>198.8</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>8800</td>
<td>75.77</td>
<td>0.98</td>
<td>201.6</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>9000</td>
<td>76.23</td>
<td>0.98</td>
<td>203.0</td>
<td>0.38</td>
</tr>
<tr>
<td></td>
<td>9200</td>
<td>Uncontrollable</td>
<td></td>
<td>204.4</td>
<td>0.38</td>
</tr>
<tr>
<td>15</td>
<td>9200</td>
<td>91.97</td>
<td>0.79</td>
<td>249.4</td>
<td>0.31</td>
</tr>
<tr>
<td></td>
<td>9600</td>
<td>92.75</td>
<td>0.79</td>
<td>251.9</td>
<td>0.31</td>
</tr>
<tr>
<td></td>
<td>10000</td>
<td>93.52</td>
<td>0.79</td>
<td>254.4</td>
<td>0.31</td>
</tr>
<tr>
<td></td>
<td>10200</td>
<td>Uncontrollable</td>
<td></td>
<td>255.7</td>
<td>0.31</td>
</tr>
<tr>
<td>10</td>
<td>10200</td>
<td>108.7</td>
<td>0.66</td>
<td>301.6</td>
<td>0.26</td>
</tr>
</tbody>
</table>
Table 7.5. Case (i): Relationships between $R_{do}^{lo}$, $\lambda$, Voltages and Gains (Stiffener-plate patch active) $R_{d}^{ov}$ = 1000

<table>
<thead>
<tr>
<th>$R_{d}^{lo}$</th>
<th>$\lambda$</th>
<th>Max. Volt at plate patch</th>
<th>Max. Volt at stiffener patch</th>
<th>Gain (plate patch)</th>
<th>Gain (stiffener patch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>7400</td>
<td>29.17</td>
<td>1.32</td>
<td>74.02</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>7800</td>
<td>29.75</td>
<td>1.32</td>
<td>75.58</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>8200</td>
<td>30.03</td>
<td>1.32</td>
<td>77.18</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>8600</td>
<td>30.93</td>
<td>1.32</td>
<td>78.78</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>9000</td>
<td>Uncontrollable</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>9000</td>
<td>39.18</td>
<td>0.97</td>
<td>102.0</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>9400</td>
<td>39.68</td>
<td>0.97</td>
<td>103.4</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>9800</td>
<td>40.21</td>
<td>0.97</td>
<td>104.9</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>10200</td>
<td>40.70</td>
<td>0.97</td>
<td>106.3</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>10600</td>
<td>41.26</td>
<td>0.97</td>
<td>107.9</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td>11000</td>
<td>Uncontrollable</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>11000</td>
<td>52.17</td>
<td>0.72</td>
<td>140.6</td>
<td>0.28</td>
</tr>
<tr>
<td></td>
<td>11400</td>
<td>52.57</td>
<td>0.72</td>
<td>141.8</td>
<td>0.28</td>
</tr>
<tr>
<td></td>
<td>11800</td>
<td>53.07</td>
<td>0.72</td>
<td>143.2</td>
<td>0.28</td>
</tr>
<tr>
<td></td>
<td>12200</td>
<td>53.47</td>
<td>0.72</td>
<td>144.5</td>
<td>0.28</td>
</tr>
<tr>
<td></td>
<td>12600</td>
<td>Uncontrollable</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>12600</td>
<td>68.58</td>
<td>0.53</td>
<td>192.1</td>
<td>0.21</td>
</tr>
<tr>
<td></td>
<td>13000</td>
<td>68.89</td>
<td>0.53</td>
<td>193.2</td>
<td>0.21</td>
</tr>
<tr>
<td></td>
<td>13400</td>
<td>69.19</td>
<td>0.53</td>
<td>194.3</td>
<td>0.21</td>
</tr>
<tr>
<td></td>
<td>13800</td>
<td>69.53</td>
<td>0.53</td>
<td>195.5</td>
<td>0.21</td>
</tr>
<tr>
<td></td>
<td>14200</td>
<td>69.86</td>
<td>0.53</td>
<td>196.7</td>
<td>0.21</td>
</tr>
<tr>
<td></td>
<td>14600</td>
<td>70.20</td>
<td>0.53</td>
<td>197.9</td>
<td>0.21</td>
</tr>
<tr>
<td></td>
<td>15000</td>
<td>Uncontrollable</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>15000</td>
<td>97.70</td>
<td>0.37</td>
<td>295.1</td>
<td>0.15</td>
</tr>
<tr>
<td></td>
<td>15400</td>
<td>97.91</td>
<td>0.37</td>
<td>296.1</td>
<td>0.15</td>
</tr>
<tr>
<td></td>
<td>15800</td>
<td>98.12</td>
<td>0.37</td>
<td>297.1</td>
<td>0.15</td>
</tr>
<tr>
<td></td>
<td>16200</td>
<td>98.35</td>
<td>0.37</td>
<td>298.2</td>
<td>0.15</td>
</tr>
<tr>
<td></td>
<td>16600</td>
<td>98.59</td>
<td>0.37</td>
<td>299.3</td>
<td>0.15</td>
</tr>
<tr>
<td></td>
<td>17000</td>
<td>98.82</td>
<td>0.37</td>
<td>300.4</td>
<td>0.15</td>
</tr>
<tr>
<td></td>
<td>17400</td>
<td>99.08</td>
<td>0.37</td>
<td>301.6</td>
<td>0.15</td>
</tr>
<tr>
<td></td>
<td>18200</td>
<td>99.60</td>
<td>0.37</td>
<td>304.0</td>
<td>0.15</td>
</tr>
<tr>
<td></td>
<td>19000</td>
<td>100.1</td>
<td>0.39</td>
<td>306.4</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>19400</td>
<td>100.4</td>
<td>0.39</td>
<td>307.7</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>19800</td>
<td>Uncontrollable</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>19800</td>
<td>Uncontrollable</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 7.8. $G_p$ vs. $\lambda$ relationship for symmetric modes

Figure 7.9. $G_p$ vs. $\lambda$ relationship for anti-symmetric modes
7.3.1.1.3. \( G_p \) vs. \( \lambda \) relationship for the panel

From a study of the results presented so far, it turns out that the specification of a single parameter, viz. \( G_p \), is sufficient for the control of the entire panel as long as \( \lambda \) is less than a limiting value, e.g. \( \lambda_{\text{max}} \approx 6\lambda_{cr} \) for the symmetric case. The gain may be specified on the basis of a chosen settling time. Once such a relationship is available it should be a relative simple task to tune the controls based on the sensed air velocity. Of the two scenarios discussed here, i.e., the panel under the action of symmetric and anti symmetric modes, the former governs as it demands greater control effort and has a lower limit on \( \lambda \) within which control is possible.

Figure 7.8 shows \( \lambda \) vs. Gain relationship when the panel is subjected to symmetric modes under the following conditions: both the plate patches are deemed to be active with the same gain and the settling time is specified as 0.2 sec. The figure also pinpoints the value of at which the field strength of 600 V develops in the center patch. This corresponds to standard PZT5K material with a patch thickness of 0.1 mm.

A similar relationship for the anti-symmetric case is shown in Figure 7.9. Here again both the patches are active. The highest voltage develops at \( \lambda = 50000 \) and the corresponding field strength is 186, which is well within the capacity of the material.

7.3.1.1.4. Failure of the Type (i) Control: A commentary

In the flutter problem the source of instability is the aerodynamic pressure and is encapsulated by a skew-symmetric matrix. On the other hand the piezo-electric damping forces are given by a diagonal matrix in the self-sensing negative feedback control (vide eq. (7.29) and (7.30)). Thus there is no direct one to one match between the driving forces (associated with a skew symmetric matrix) and damping forces (given by a
diagonal matrix). Thus one should expect the control to sub-optimal and eventually ineffective.

A full state feedback control (type (ii)) gives a fully populated gain matrix and controls both the velocities and displacements at once. Derived rigorously from optimal control theory for any general ‘A’ matrix, it offers itself as a powerful tool for control.

7.3.1.2. Control Type (ii): MIMO control based on the full control gain matrix

In this approach the LQR algorithm is employed in conjunction with linearized stiffness matrix at any given time to produce the gain matrix $[K_{\text{gain}}]$. The gain matrix is computed at the end of each time step using the current values of the displacements to linearize the stiffness matrix and employed in the governing equations for the succeeding step. Thus the control gains vary throughout the time history. Since all the degrees of freedom are included in the control process, the structure is fully controllable. The voltages developed across the patches, however, have to be monitored to ensure that they are within the capacity of the piezo-electric material.

One of the assumptions of the approach is that in practice that the current values of displacements viz. $\xi_m$, $X_p$ are available at any given time.

For the relatively simple case at hand where only a single local and a single overall mode is associated with a given $m$, the bending strain values recorded at the patches at $(M + M_o)$ number of piezo-electric patch locations would yield the necessary equations for the computation of all the $\xi_m$, $X_p$.

The computation of the degrees of freedom as well as $[K_{\text{gain}}]$ at discrete time steps which are of the order of micro-seconds assumes the availability of high speed computing facilities. In order to illustrate the power of this approach we examine the relative performances of the two control methodologies under the following scenarios.
7.3.1.2.1. **Scenario 1: Panel is controllable with either methodology**

Consider the case with \( \lambda \) close to but less than the limiting value for negative velocity feedback control (type (i) Control).

Specifically the following two cases will be considered.

- **Example 1:** Panel under antis-symmetric local modes under \( \lambda = 50000 \), with \( \sigma_o = 0.4 \sigma_{cr} \), \( C_a = 0.1 \); initial conditions \( \xi_m = 0.01/m; X_p = 0.01/p \). Both the center line and stiffener line plate patches are active. The following control parameters are selected: \( R_d^{lo} = 0.1; R_d^{ov} = 1000 \) (vide table 7.3). The computed gains for type (i) control: \( G^p = 1462.5; G^{st} = 2.59 \).

- **Example 2:** Panel under symmetric local modes under \( \lambda = 19400 \), with \( \sigma_o = 0.4 \sigma_{cr} \), \( C_a = 0.1 \); initial conditions \( \xi_m = 0.01/m; X_p = 0.01/p \). Both the center line and stiffener line plate patches are active. The following control parameters are selected: \( R_d^{lo} = 10; R_d^{ov} = 1000 \) (vide table 7.3). The computed gains for type (i) control: \( G^p = 307.6; G^{st} = 0.16 \).

Figure 7.10 and Figure 7.11 give the typical time histories for Example 1 and 2 respectively.
Type (ii) control (MIMO LQR)  

Type (i) control

(a) Plate patch voltage time history

(b) Stiffener patch voltage time history

(c) Maximum (plate) displacement time history

Figure 7.10. Scenario 1: Example 1
Type (ii) control (MIMO LQR)  

(a) Plate patch voltage time history

(b) Stiffener patch voltage time history

(c) Maximum (plate) displacement time history

Figure 7.11. Scenario 1: Example 2
7.3.1.2.2. Scenario 2: Panel is not controllable with Type (i) Control

Consider the case with $\lambda$ close to but greater than the limiting value for negative velocity feedback control (type (i) Control).

Specifically the following two cases will be considered.

- **Example 1**: Panel under antisymmetric local modes under $\lambda = 52000$, with $\sigma_0 = 0.4\sigma_{cr}, C_a = 0.1$; initial conditions $\xi_m = 0.01/m; X_p = 0.01/p$. Both the center line and stiffener line plate patches are active. The following control parameters are selected: $R_d^{lo} = 0.1; R_d^{ov} = 1000$ (vide table 7.3). The computed gains for type (i) Control: $G^p = 1462.5; G^{st} = 2.59$.

- **Example 2**: Panel under symmetric local modes under $\lambda = 19800$, with $\sigma_0 = 0.4\sigma_{cr}, C_a = 0.1$; initial conditions $\xi_m = 0.01/m; X_p = 0.01/p$. Both the center line and stiffener line plate patches are active. The following control parameters are selected: $R_d^{lo} = 10; R_d^{ov} = 1000$ (vide table 7.3). The computed gains for type (i) control: $G^p = 309.0; G^{st} = 0.16$

Figure 7.12 and Figure 7.13 give the typical time histories for Example 1 and 2 respectively.
Type (ii) control (MIMO LQR)

(a) Plate patch voltage time history

(b) Stiffener patch voltage time history

(c) Maximum (plate) displacement time history

Figure 7.12. Scenario 2: Example
Type (ii) control (MIMO LQR)  

(a) Plate patch voltage time history

(b) Stiffener patch voltage time history

(c) Maximum displacement time history

Figure 7.13. Scenario 1: Example 2
7.3.1.2.3. Efficacy of MIMO Control

As illustrated by Figure 7.10 and Figure 7.11, the MIMO control uses less control effort, both the maximum voltage and settling times are smaller for both the examples.

Figure 7.12 and Figure 7.13 show clearly that MIMO has the ability to control the panel vibrations in situations where the negative velocity feedback control fails.

Each figure shows typical response histories obtained under the two types of control side by side. The response histories compared are those of the maximum plate patch voltage (along the center line patch), maximum stiffener patch voltage and the maximum local displacement for the two types of control respectively.

In order to investigate the efficacy of the MIMO methodology, we consider the case of the panel under symmetric modes subjected to aerodynamic pressure given by $\lambda = 50000$, which is about 3 times the limit of $\lambda$ with negative velocity feedback control. The typical time history responses are illustrated in Figure 7.14. It is seen that the control is achieved with precision. Note however the maximum voltages developed across the patches is higher than what is permissible for the PZT5K material and the use of superior material becomes necessary.

Figure 7.14. Panel time histories of symmetric local mode under $\lambda = 50000$, MIMO, maximum plate voltage = 80.2355
7.4. Conclusions

Effectiveness of two types of control methodologies for suppressing the flutter of stiffened panels is investigated. These are respectively (i) Negative velocity feedback control with self-sensing actuator patches and (ii) An MIMO control with full state feedback control.

Of these Type (i) control is easier to implement and is seen to be effective at least up to \( \lambda \approx 6\lambda_{cr} \). As long as \( \lambda \) is not expected to exceed this limit, this type of control is viable and in fact may be preferable. However, beyond a certain limit this simple control methodology fails.

The state feedback control, where the gain matrix is updated at every time step is found to be versatile and totally effective. Its only limitation comes from the limits on the voltage capacities of the piezo-electric patches employed.

The local modes of vibration play a dominant role for a considerable range of \( \lambda \), starting from \( \lambda_{cr} \) (corresponding to incipient flutter of the uncontrolled panel) and overall modes get readily damped out. Thus they can be ignored in the initial post-critical range. However as \( \lambda \) is increased there comes a point where they suddenly become dominant. This is clearly seen in the response of the panel under anti-symmetric local modes. The flutter response when driven by these modes is not controlled by the self-sensing actuator patches (Type (i) control) and demands a more powerful methodology implicit in MIMO control.
Chapter 8

Conclusion

The main objective of the present study has been to study the feasibility and effectiveness of piezo-electric control of structural elements prone to instability. The problem of control in the presence of potential instability has been considered in the present work in a variety of contexts.

Control of columns under conservative and non-conservative axial compression is studied under softening nonlinearities in chapter 3 and 4 respectively; axially compressed stiffened panels under interaction of local and overall buckling and subjected to lateral disturbances are studied in chapter 5. Stiffened panels under aerodynamic pressure with interaction of local and overall modes of vibration are studied in chapters 6 and 7. For the most part negative velocity feedback control by self-sensing actuators is used for vibration suppression in this study. A systematic way of selecting the gain is introduced in chapter 7 using LQR algorithm in conjunction with stepwise linearization of the stiffness matrix. The limits of capability of this simple type of control are also explored. A more comprehensive MIMO control approach proves always effective, but it too may not be feasible for a given field strength capacity of the piezo-electric patches. In this chapter the main conclusions and contributions of this study will be summarized.
8.1. Control Methodology under Conservative Compressive Loading

For an initially imperfect structure with built-in softening nonlinearities, the static buckling load \((P_s)\) can be enhanced by a feedback control proportional to the locally sensed bending strain/displacement. The dynamic buckling load \((P_d)\) under a given disturbance would be smaller than the static buckling load. It was found that negative feedback control with gains which are a linear combination of the (bending strain) displacement and (bending strain rate) velocity can enhance the dynamic buckling load.

However such control comes with a heavy price in terms of the energy consumption due to the high electric field strength required to be applied to the piezo-electric patches. There is also the potential risk of exceeding the maximum voltages the patches are capable of carrying, precipitating failure of the entire system.

On the other hand, if the dynamic buckling load is not exceeded \((P < P_d)\) the control is effective and economical. Negative velocity feedback is preferable as the electric field disappears once the oscillations due to a disturbance have died down.

8.2. Control Methodology under Non-conservative Compressive Loading

With a view to study the control in a structure in which flutter and divergence are competing modes of instability, the control problem of a cantilever column subjected to follower force and carrying a spring (elastic support) at its ‘free’ end is investigated. The spring has a softening nonlinearity incorporated into it.
As the stiffness of the spring increases, the mode of instability transitions from flutter to divergence with the critical load at first increasing and then dramatically decreasing. This indicates the existence of an optimal value of the spring stiffness.

In the flutter range negative velocity feedback control can enhance the critical load by 100%, but it becomes ineffective as soon as divergence becomes the mode of instability.

It was found that a set of piezo-electric patches distributed over only half the length of the column starting from the fixed end was more effective than a set spanning the entire column length. Apparently the control by self-sensing actuators must be focused in that part of the column where the bending curvatures are severe with other parts of the structure left free. The study points to the existence of a double optimality problem: passive mitigation by the use of optimal spring stiffness and active control by the optimal location of piezo-electric patches. The type of control chosen in the present study is somewhat simple and clearly has limitations in that it cannot increase the load carrying capacity of the column beyond a limit, no matter what the properties of the piezo-electric material used.

8.3. Control of Stiffened Panels in the Context of Local-Overall Mode Interaction

The problem of piezo-electric control of axially compressed stiffened panels is studied in chapter 5. The panel is designed to be “optimal” in the sense the two competing modes of buckling, viz. the local and overall correspond to the same critical load. The panel thus becomes imperfection-sensitive, i.e. its maximum load carrying capacity is less than the critical load and it starts losing its stiffness in the pre-critical range. This problem was studied using finite elements in which local buckling deformations were embedded.
The effectiveness and feasibility of the piezoelectric control of static and dynamic responses of a stiffened panel are demonstrated. It was found that the afore-mentioned adverse effects of interaction under static conditions can be counteracted by suppressing the overall action represented by stiffener bending. Control is exercised by piezo-electric patches attached to the tips of the stiffener with voltages proportional to the overall bending strains of the stiffener. This control is found to stiffen up the structure, which can then attain or even exceed critical loads predicted by linear stability analysis. However as in the column control problem, the voltages across the patches may escalate to reach extremely high values when the load approaches the critical value.

Dynamic response of the axially compressed panel is studied under a suddenly applied lateral disturbance. The control exercised on the stiffener alone by feedback voltages proportional to the bending strain rate is effective in suppressing overall vibrations but is ineffectual with regard to local buckling oscillations. So, thin piezo-electric strips were attached to the top and bottom surfaces of the plate along the longitudinal center line of the panel to control the panel deflections. The feedback voltages of the patches are proportional to the sum of the bending strain rates in the longitudinal and transverse directions. This form of control is found to be very effective and resulted in a minimal voltage demand for damping out the oscillations.

The study demonstrates that piezo-electric control may be designed in two parts: stiffener control with patches at the stiffener tips to control the overall action and panel control by patches placed at the center of the plate (between the stiffeners).

8.4. Flutter of Stiffened Panels under Aerodynamic Pressure

The problem of flutter of a stiffened panel under aerodynamic pressure is studied using modal analysis. Finite strips are employed to extract a set of local and overall modes and
to compute second order in-plane displacement fields associated with the local modes. The effects of axial compression as well as aerodynamic damping are accounted for.

The local modes could be symmetric or anti-symmetric in the cross-sectional plane with respect to the stiffener and these are considered separately for clarity even though a coupling between them cannot be ruled out in practice. The stiffener is not excited directly by aerodynamic pressure but indirectly through coupling of the local and overall modes. The symmetric local modes and overall modes are coupled in the linear flutter problem (whereas the anti-symmetric modes are orthogonal to the overall modes). Further there exists a nonlinear coupling between the local modes and overall modes via a set of cubic terms in the potential of elastic forces.

However in the dynamic problem, apparently due to mismatch between the frequencies of the local modes on the one hand and overall modes on the other, the plate flutter is not communicated to the stiffener effectively. Thus the main action comes from the plate with stiffener playing a relatively minor role. However its role in forcing a nodal line in the transverse deformation profile of the plate is critical.

The numerical model was thoroughly tested for convergence and accuracy using the benchmark results for linear flutter and post-critical limit cycle response of plates, post-buckling modal interaction in plates and stiffened panels under interactive buckling.

It is found that when the edge movements are allowed the plate panels have either a limited LCO range beyond which they become dynamically unstable or become unstable with the onset of flutter. This is not only true of stiffened panels but also for plates not carrying stiffeners. The panels were found to become unstable in the pre-critical range when hit with a suddenly applied pressure or significant enough initial displacements are prescribed. This indicates the existence of unstable limit cycle regime proceeding from the critical or a certain postcritical value of $\lambda$. 
8.5. Control of Flutter of Stiffened Panels under Aerodynamic Pressure

Two control strategies are investigated: Type (i) Negative velocity feedback control with self-sensing actuator patches, and Type (ii) An MIMO control with full state feedback control.

As in the earlier part of the study (Chapter 5) piezo-electric patches were attached along the stiffener tips and plate center lines. Because of the severe curvatures associated with symmetric local mode, it was found necessary to add a pair of patches contiguous to the stiffener, to keep the voltages developed within the capacity of the patches.

As explained earlier stiffener vibrations play a relatively minor role for the most part in the flutter response of the panel and control thereof. However when the plate flutter is controlled over a considerable post-critical range of $\lambda$, there comes a point at which the panel can flutter in the overall mode and at this stage the stiffener control becomes critical.

A systematic way of selecting the gains for Type (i) control is introduced for the plate and stiffener patches. This is based on the application of the LQR optimal control algorithm and extraction of the relevant information from the gain matrix so obtained. A scheme of linearization of the stiffness contribution to the governing equations is introduced for use in conjunction with LQR algorithm.

Type (i) control is easier to implement and is seen to be effective at least up to $\lambda \approx 6\lambda_{cr}$. As long as $\lambda$ is not expected to exceed this limit, this type of control is viable and in fact may be preferable. It is shown that for a given panel geometry and piezo-electric patch configuration it is possible to establish *a priori* a relationship between $\lambda$ and plate patch gain for a pre-selected settling time – something that can be of value in practice. However, beyond a certain limit this simple control methodology fails.
Type (ii) control, viz. the full state feedback control, where the gain matrix is updated at every time step is found to be versatile and totally effective. Its only limitation stems from the limits on the voltage capacities of the piezo-electric patches employed. Obviously it requires sensing of bending strains and strain-rates at a number of carefully selected locations, computation on line of a gain matrix and application of the voltages computed there from at patches at key locations. The technology involved is intricate, though feasible.

8.6. A Final Note

From the foregoing summary, it is seen that this study has revealed a number interesting new findings pertaining to the control of structures prone to instability, which should be of immediate value in aerospace structural design and control.

8.7. Future Work

The control strategy proposed here i.e. dual control of plate and stiffener may be applied to stiffened panels with a variety of cross-sectional configurations. A comparative study that evaluates the relative performance of each for anticipated loading is expected to be of value in design.

The problem of cylindrical shell panels under the combined action of axial compression and aerodynamic pressure offers challenges from the point of view of analysis as well as control. The cylindrical panels have a far greater resistance to flutter and buckling but have in general a sharply unstable post-critical response. They may also undergo somewhat sudden snap-through leading to severe geometric distortion- not a desirable scenario in the operation of aerospace vehicles.
The methodology presented does not take into account uncertainties and noise inherent in the sensors in practical situations; further there may not be enough sensors to capture the current state of the structure fully. In order to deal with noise in sensing and uncertainty in initial conditions, a modified LQR approach in present work and an Extended Kalman Filter (EKF, [30]) may be combined suitably to constitute sensing and controlling system which includes noise.

Further extensions of the present work must include inclusion of thermal loading effects, use of discrete patches with their locations determined using an appropriate optimization methodology, e.g. genetic algorithm.

Finally, it must be emphasized that experimental validation is vitally important to establish the conclusions of the present study on a firm footing and give it currency in the literature on control of aerospace structures.
References


[29] Li, Q., Mei, C., and Huang, J-K., Suppression of thermal postbuckling and nonlinear panel flutter motions using piezo-electric actuators, AIAA Journal, 45, 8, 2008, pp. 1861-1873


[40] Oded Rabinovitch, Geometrically nonlinear behavior of piezoelectric laminated plates, Smart Materials and Structures, v.14, 2005, pp 785-798


[56] Q. Wang, S.T. Quek, Enhancing flutter and buckling capacity of column by piezoelectric layers, international journal of solid and structures, 39 (2002), 4167-4180


Vita

Sunjung Kim

EDUCATION

Washington University in St. Louis, MO
Ph.D. Mechanical, Aerospace & Structural Engineering, August 2010
  Dissertation: Piezoelectric control of structures prone to instability
  Advisor: Srinivasan Sridharan
M. S. Civil Engineering, May 2003
  Thesis: Behavior of sandwich structures
  Advisor: Srinivasan Sridharan
B. S. Civil Engineering, May 2002

Yonsei University, Seoul, Korea
B.S. Nutritions / Urban Planning and Engineering, Feb 1996

PUBLICATIONS

- Sunjung Kim, S. Sridharan, Piezoelectric Control of a Partially Propped Cantilever Subjected to a Follower Force, AIAA Journal, 48, n1, 2010, 144-157
- S. Sridharan, S. Kim, Piezoelectric control of columns prone to instabilities and nonlinear modal interaction, Smart materials and structures, 17, n.3, 2008, pp. 1-14

HONORS

- Student representative of the Department of Mechanical, Aerospace and Structural Engineering, Washington University in St. Louis
- Teaching/Research Assistantship in 2008 and 2009 (NSF)
- Member of Washington University Chapter of Chi Epsilon, the National Civil Engineering Honor Society, 2006
- Research Assistantship in 2001 and 2002 (Office of Naval Research)
Piezoelectric Control of Structures, Kim, Ph.D., 2010