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A remark on the multipliers on spaces of Weak Products of functions

Stefan Richter* and Brett D. Wick

Abstract: If $H$ denotes a Hilbert space of analytic functions on a region $\Omega \subseteq \mathbb{C}^d$, then the weak product is defined by

$$H \odot H = \left\{ h = \sum_{n=1}^{\infty} f_n g_n : \sum_{n=1}^{\infty} \| f_n \|_H \| g_n \|_H < \infty \right\}.$$  

We prove that if $H$ is a first order holomorphic Besov Hilbert space on the unit ball of $\mathbb{C}^d$, then the multiplier algebras of $H$ and of $H \odot H$ coincide.

Keywords: Dirichlet space, Drury-Arveson space, Weak product, Multiplier

MSC: 47B37

1 Introduction

Let $d$ be a positive integer and let $R = \sum_{j=1}^{d} z_i \frac{\partial}{\partial z_i}$ denote the radial derivative operator. For $s \in \mathbb{R}$ the holomorphic Besov space $B_s$ is defined to be the space of holomorphic functions $f$ on the unit ball $B_d$ of $\mathbb{C}^d$ such that for some nonnegative integer $k > s$,

$$\| f \|_{k,s}^2 = \int_{B_d} |(I + R)^k f(z)|^2 (1 - |z|^2)^{2(k-s)-1} dV(z) < \infty.$$  

Here $dV$ denotes Lebesgue measure on $B_d$. It is well-known that for any $f \in \text{Hol}(B_d)$ and any $s \in \mathbb{R}$ the quantity $\| f \|_{k,s}$ is finite for some nonnegative integer $k > s$ if and only if it is finite for all nonnegative integers $k > s$, and that for each $k > s$ $\| \cdot \|_{k,s}$ defines a norm on $B_s$, and that all these norms are equivalent to one another, see [2]. For $s < 0$ one can take $k = 0$ and these spaces are weighted Bergman spaces. In particular, $B_{-1/2} = L^2(\mathbb{B}_d)$ is the unweighted Bergman space. For $s = 0$ one obtains the Hardy space of $B_d$ and one has that for each $k \geq 1 \| f \|_{k,0}^2$ is equivalent to $\int_{\partial B_d} |f|^2 d\sigma$, where $\sigma$ is the rotationally invariant probability measure on $\partial B_d$. We also note that for $s = (d-1)/2$ we have $B_s = H^2_d$, the Drury-Arveson space. If $d = 1$ and $s = 1/2$, then $B_s = D$, the classical Dirichlet space of the unit disc.

Let $H \subseteq \text{Hol}(B_d)$ be a reproducing kernel Hilbert space such that $1 \in H$. The weak product of $H$ is denoted by $H \odot H$ and it is defined to be the collection of all functions $h \in \text{Hol}(B_d)$ such that there are sequences $\{ f_i \}_{i \geq 1}, \{ g_i \}_{i \geq 1} \subseteq H$ with $\sum_{i=1}^{\infty} \| f_i \|_H \| g_i \|_H < \infty$ and for all $z \in B_d, h(z) = \sum_{i=1}^{\infty} f_i(z) g_i(z)$.

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We define a norm on $\mathcal{H} \otimes \mathcal{H}$ by

$$\|h\|_s = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_\mathcal{H} \|g_i\|_\mathcal{H} : h(z) = \sum_{i=1}^{\infty} f_i(z)g_i(z) \text{ for all } z \in \mathbb{B}_d \right\}.$$ 

In what appears below we will frequently take $\mathcal{H} = \mathbb{B}_d$, and will use the same notation for this weak product.

Weak products have their origin in the work of Coifman, Rochberg, and Weiss [5]. In the framework of the Hilbert space $\mathcal{H}$, one may consider the weak product to be an analogue of the Hardy $H^1$-space. 

For example, one has $H^2(\partial \mathbb{B}_d) \circ H^2(\partial \mathbb{B}_d) = H^1(\partial \mathbb{B}_d)$ and $L^2_\mathcal{H}(\mathbb{B}_d) \circ L^2_\mathcal{H}(\mathbb{B}_d) = L^1_\mathcal{H}(\mathbb{B}_d)$, see [5]. For the Dirichlet space $D$ the weak product $D \circ D$ has recently been considered in [1, 3, 6, 7, 9]. The space $H^2_\mathcal{H} \circ H^2_\mathcal{H}$ was used in [10]. For further motivation and general background on weak products we refer to the reader to [1] and [9].

Let $\mathcal{B}$ be a Banach space of analytic functions on $\mathbb{B}_d$ such that point evaluations are continuous and such that $1 \in \mathcal{B}$. We use $M(\mathcal{B})$ to denote the multiplier algebra of $\mathcal{B}$,

$$M(\mathcal{B}) = \{ \varphi : \varphi f \in \mathcal{B} \text{ for all } f \in \mathcal{B} \}.$$ 

The multiplier norm $\|\varphi\|_M$ is defined to be the norm of the associated multiplication operator $M\varphi : \mathcal{B} \to \mathcal{B}$. It is easy to check and is well-known that $M(\mathcal{B}) \subseteq H^\infty(\mathbb{B}_d)$, and that for $s \leq 0$ we have $M(\mathcal{B}_s) = H^\infty(\mathbb{B}_d)$. For $s > d/2$ the space $\mathcal{B}_s$ is an algebra [2], hence $\mathcal{B}_s = M(\mathcal{B}_s)$, but for $0 < s < d/2$ one has $M(\mathcal{B}_s) \subseteq \mathcal{B}_s \cap H^\infty(\partial \mathbb{B}_d)$. For those cases $M(\mathcal{B}_s)$ has been described by a certain Carleson measure condition, see [4, 8].

It is easy to see that $M(\mathcal{H}) \subseteq M(\mathcal{H} \otimes \mathcal{H}) \subseteq H^\infty$ (see Proposition 3.1). Thus, if $s \leq 0$, then $M(\mathcal{B}_s) = M(\mathcal{B}_s \otimes \mathcal{B}_s) = H^\infty$. Furthermore, if $s > d/2$, then $\mathcal{B}_s = \mathcal{B}_s \otimes \mathcal{B}_s = M(\mathcal{B}_s)$ since $\mathcal{B}_s$ is an algebra. This raises the question whether $M(\mathcal{B}_s)$ and $M(\mathcal{B}_s \otimes \mathcal{B}_s)$ always agree. We prove the following:

**Theorem 1.1.** Let $s \in \mathbb{R}$ and $d \in \mathbb{N}$. If $s \leq 1$ or $d \leq 2$, then $M(\mathcal{B}_s) = M(\mathcal{B}_s \otimes \mathcal{B}_s)$.

Note that when $d \leq 2$, then $\mathcal{B}_s$ is an algebra for all $s > 1$. Thus for each $d \in \mathbb{N}$ the nontrivial range of the Theorem is $0 < s \leq 1$. If $d = 1$ then the theorem applies to the classical Dirichlet space of the unit disc and for $d \leq 3$ it applies to the Drury-Arveson space.

## 2 Preliminaries

For $z = (z_1, ..., z_d) \in \mathbb{C}^d$ and $t \in \mathbb{R}$ we write $e^{it}z = (e^{it}z_1, ..., e^{it}z_d)$ and we write $(z, w)$ for the inner product in $\mathbb{C}^d$. Furthermore, if $h$ is a function on $\mathbb{B}_d$, then we define $T_t f$ by $(T_t f)(z) = f(e^{it}z)$. We say that a space $\mathcal{H} \subseteq \text{Hol}(\mathbb{B}_d)$ is radially symmetric, if each $T_t$ acts isometrically on $\mathcal{H}$ and if for all $t_0 \in \mathbb{R}$, $T_t \to T_{t_0}$ in the strong operator topology as $t \to t_0$, i.e. if $\|T_t f\|_\mathcal{H} = \|f\|_\mathcal{H}$ and $\|T_t f - T_{t_0} f\|_\mathcal{H} \to 0$ for all $f \in \mathcal{H}$. For example, for each $s \in \mathbb{R}$ the holomorphic Besov space $\mathcal{B}_s$ is radially symmetric when equipped with any of the norms $\| \cdot \|_{k,s}$, $k > s$.

It is elementary to verify the following lemma.

**Lemma 2.1.** If $\mathcal{H} \subseteq \text{Hol}(\mathbb{B}_d)$ is radially symmetric, then so is $\mathcal{H} \otimes \mathcal{H}$.

Note that if $h$ and $\varphi$ are functions on $\mathbb{B}_d$, then for every $t \in \mathbb{R}$ we have $(T_t \varphi)h = T_t (\varphi T_{-t} h)$, hence if a space is radially symmetric, then $T_t$ acts isometrically on the multiplier algebra. For $0 < r < 1$ we write $f_t(z) = f(rz)$.

**Lemma 2.2.** If $\mathcal{H} \subseteq \text{Hol}(\mathbb{B}_d)$ is radially symmetric, and if $\varphi \in M(\mathcal{H} \otimes \mathcal{H})$, then for all $0 < r < 1$ we have $\|\varphi_t\|_{M(\mathcal{H} \otimes \mathcal{H})} \leq \|\varphi\|_{M(\mathcal{H} \otimes \mathcal{H})}$.

**Proof.** Let $\varphi \in M(\mathcal{H} \otimes \mathcal{H})$ and $h \in \mathcal{H} \otimes \mathcal{H}$, then for $0 < r < 1$ we have

$$\varphi_t h = \int_{-\pi}^{\pi} \frac{1 - r^2}{|1 - re^{it}|^2} \pi (T_t \varphi)h \frac{dt}{2\pi}.$$
This implies
\[
\| \varphi_r h \|_* \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - re^{it}} \| (T_t \varphi) h \|_* \frac{dt}{2\pi} \leq \| \varphi \|_{M(H \odot H)} \| h \|_*.
\]
Thus, \( \| \varphi_r \|_{M(H \odot H)} \leq \| \varphi \|_{M(H \odot H)}. \)

\[\square\]

### 3 Multipliers

The following Proposition is elementary.

**Proposition 3.1.** We have \( M(H) \subset M(H \odot H) \subset H^\infty \) and if \( \varphi \in M(H) \), \( \| \varphi \|_{M(H \odot H)} \leq \| \varphi \|_{M(H)} \).

As explained in the Introduction, the following will establish Theorem 1.1.

**Theorem 3.2.** Let \( 0 < s \leq 1 \). Then \( M(B_s) = M(B_s \odot B_s) \) and there is a \( C_s > 0 \) such that
\[
\| \varphi \|_{M(B_s \odot B_s)} \leq \| \varphi \|_{M(B_s)} \leq C_s \| \varphi \|_{M(B_s \odot B_s)}
\]
for all \( \varphi \in M(B_s) \).

Here for each \( s \) we have the norm on \( B_s \) to be \( \| \cdot \|_{k,s} \), where \( k \) is the smallest natural number \( > s \).

**Proof.** We first do the case \( 0 < s < 1 \). Then \( k = 1 \), and \( \| f \|_{B_s} = \int_{B_s} |(I + R) f(z)|^2 dV_s(z) \), where \( dV_s(z) = (1 - |z|^2)^{1-2s} dV(z) \). For later reference we note that a short calculation shows that \( \int_{B_s} |R f|^2 dV_s \leq \| f \|_{B_s}^2 \).

We write \( \| R \varphi \|_{C_{aB}(B_s)} \) for the Carleson measure norm of \( |R \varphi|^2 \), i.e.
\[
\| R \varphi \|_{C_{aB}(B_s)}^2 \leq \inf \left\{ C > 0 : \int_{B_s} |f|^2 |R \varphi|^2 dV_s \leq C \| f \|_{B_s}^2 \text{ for all } f \in B_s \right\}.
\]
Since \( \| f \|_{B_s}^2 = \int_{B_s} |\varphi(z)(I + R)f(z) + f(z)R\varphi(z)|^2 dV_s(z) \) it is clear that \( \| \varphi \|_{M(B_s)} \) is equivalent to \( \| \varphi \|_{C_{aB}(B_s)} \). Thus, it suffices to show that there is a \( c > 0 \) such that \( \| R \varphi \|_{C_{aB}(B_s)} \leq c \| \varphi \|_{M(B_s \odot B_s)} \) for all \( \varphi \in M(B_s \odot B_s) \).

First we note that if \( b \) is holomorphic in a neighborhood of \( \overline{B_d} \) and \( h = \sum_{i=1}^{\infty} f_i g_i \in B_s \odot B_s \), then
\[
\int_{\overline{B_d}} |(Rh) Rb| dV_s \leq \sum_{i=1}^{\infty} \int_{\overline{B_d}} |(Rf_i)g_i Rb| dV_s + \int_{\overline{B_d}} |(Rg_i) f_i Rb| dV_s
\]
\[
\leq \sum_{i=1}^{\infty} \| f_i \|_{B_s} \left( \int_{\overline{B_d}} |g_i Rb|^2 dV_s \right)^{1/2} + \| g_i \|_{B_s} \left( \int_{\overline{B_d}} |f_i Rb|^2 dV_s \right)^{1/2} \]
\[
\leq 2 \sum_{i=1}^{\infty} \| f_i \|_{B_s} \| g_i \|_{B_s} \| Rb \|_{C_{aB}(B_s)}.
\]

Hence
\[
\int_{\overline{B_d}} |(Rh) Rb| dV_s \leq 2 \| h \|_* \| Rb \|_{C_{aB}(B_s)},
\]
where we have continued to write \( \| \cdot \|_* \) for \( \| \cdot \|_{B_s \odot B_s} \).

Let \( \varphi \in M(B_s \odot B_s) \) and let \( 0 < r < 1 \). Then for all \( f \in B_s \) we have \( f^2, \varphi_r f^2 \in B_s \odot B_s \), hence
\[
\int_{\overline{B_d}} |f|^2 |R \varphi_r f|^2 dV_s = \int_{\overline{B_d}} |R(\varphi_r f^2) - \varphi_r R(f^2)| |R \varphi_r| dV_s
\]
\[
\leq 2(\|\varphi_{r} f^2\|_\infty + \|\varphi\|_\infty \|f^2\|_\infty)\|R\varphi_{r}\|_{C_{0}(B_1)}
\leq 2(\|\varphi\|_{M(B_1 \cap B_2)} \|f^2\|_\infty + \|\varphi\|_\infty \|f^2\|_\infty)\|R\varphi_{r}\|_{C_{0}(B_1)}
\leq 4(\|\varphi\|_{M(B_1 \cap B_2)} \|f\|_{B_2^2}^2 \|R\varphi_{r}\|_{C_{0}(B_1)}).
\]

Next we take the sup of the left hand side of this expression over all \(f\) with \(\|f\|_{B_r} = 1\) and we obtain
\[
\|R\varphi_{r}\|_{C_{0}(B_1)}^2 \leq 4(\|\varphi\|_{M(B_1 \cap B_2)} \|R\varphi_{r}\|_{C_{0}(B_1)}),
\]
which implies that \(\|R\varphi_{r}\|_{C_{0}(B_1)} \leq 4(\|\varphi\|_{M(B_1 \cap B_2)}\) holds for all \(0 < r < 1\). Thus, for \(0 < s < 1\) the result follows from Fatou’s lemma as \(r \to 1\).

If \(s = 1\), then \(\|f\|_{B_1^2}^2 \sim \int_{\partial B_1} |(I + R)f(z)|^2 da(z)\) and the argument proceeds as above. 

\[\square\]

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