Strict singularity of a Volterra-type integral operator on H^p

Santeri Miihkinen, University of Helsinki

IWOTA
St. Louis, 18-22 July 2016

Background and motivation

- Pommerenke (1977): A novel proof of the deep John-Nirenberg inequality using:
- An integral operator of the type

$$T_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta, \ z \in \mathbb{D},$$

  where f and g are analytic (holomorphic) in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane ($f, g \in H(\mathbb{D})$).
- f is a "variable" here and g is fixed, the **symbol** of T_g.
- **Example 1:** $g(z) = z \Rightarrow T_g f(z) = \int_0^z f(\zeta) d\zeta$ (the classical Volterra operator)

- **Example 2:** $g(z) = \log \frac{1}{1-z} \Rightarrow \frac{1}{z} T_g f(z) = \frac{1}{z} \int_0^z \frac{f(\zeta)}{1-\zeta} d\zeta = \sum_{k=0}^{\infty} \left(\frac{1}{k+1} \sum_{n=0}^{k} a_n \right) z^k$ (the Cesàro operator).
- Characterize the properties of T_g in terms of the "function-theoretic" properties of the symbol g?
- Aleman and Siskakis (1995): systematic research on T_g
- Hardy spaces

$$H^p = \left\{ f \in H(\mathbb{D}) : \|f\|_p = \sup_{0 \le r < 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p} < \infty \right\},$$

where $0 < p < \infty$.

- $T_g : H^p \to H^p$, $1 \leq p < \infty$, bounded (compact) iff $g \in BMOA$ ($g \in VMOA$), where

$$BMOA = \left\{ h \in H(\mathbb{D}) : \|h\|_* = \sup_{a \in \mathbb{D}} \|h \circ \sigma_a - h(a)\|_2 < \infty \right\}$$

and

$$VMOA = \left\{ h \in BMOA : \limsup_{|a| \to 1} \|h \circ \sigma_a - h(a)\|_2 = 0 \right\},$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$, $a, z \in \mathbb{D}$.

- Aleman and Cima (2001): scale $0 < p < 1$.

Strict singularity

- A bounded operator L between Banach spaces X and Y is **strictly singular**, if L restricted to any infinite-dimensional closed subspace of X is not a linear isomorphism onto its range.
- A notion introduced by T. Kato in '58 (in connection to perturbation theory of Fredholm operators).
- Compact operators are strictly singular.
- Denote by $S(X)$ the strictly singular operators on X.

- $S(X)$ (norm-closed) ideal of the bounded operators $B(X)$:
  $L \in S(X), U \in B(X) \Rightarrow LU, UL \in S(X)$.
- **Example.** For $p < q$, the inclusion mapping $i \colon \ell^p \hookrightarrow \ell^q, i(x) = x$, is a non-compact strictly singular operator.
- The strict singularity of T_g acting on H^p?

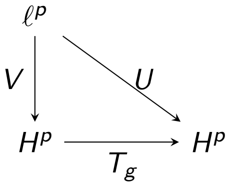
Main result

Theorem

Let $g \in BMOA \setminus VMOA$ and $1 \leq p < \infty$. Then there exists a subspace $M \subset H^p$ isomorphic to ℓ^p s.t. the restriction $T_g|M \colon M \to T_g(M)$ is bounded from below. (i.e. A non-compact $T_g \colon H^p \to H^p$ fixes a copy of ℓ^p.) In particular, T_g is not strictly singular.

Strategy of the proof

Figure: **Operators** U, V **and** T_g



- Strategy: Construct bounded operators U and V:
- U bounded from below when T_g is non-compact (i.e. $g \in BMOA \setminus VMOA$).
- The diagram commutes: $U = T_g V$
- $T_g|M \colon M \to T_g(M)$ bounded from below on $M = V(\ell^p) \approx \ell^p$.

- How to define U and V?
- We utilize suitably chosen standard normalized test functions $f_a \in H^p$ defined by

$$f_a(z) = \left[\frac{1 - |a|^2}{(1 - \bar{a}z)^2} \right]^{1/p}, \quad z \in \mathbb{D},$$

  for each $a \in \mathbb{D}$.
- For $p = 2$, the functions f_a are the normalized reproducing kernels of H^2.

- Define

$$V\colon \ell^p \to H^p,\ V(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_{a_n}$$

and

$$U\colon \ell^p \to H^p,\ U(\alpha) = \sum_{n=1}^{\infty} \alpha_n T_g f_{a_n},$$

where the sequence $(a_n) \in \mathbb{D}$, $|a_n| \to 1$ is suitably chosen and $\alpha = (\alpha_n) \in \ell^p$.

Tools

- V is bounded, if $|a_n| \to 1$ fast enough.
- How to show that U is bounded from below?
- A result by Aleman and Cima (2001):

Theorem

Let $0 < p < \infty$ *and* $t \in (0, p/2)$. *Then there exists a constant* $C = C(p,t) > 0$ *s.t.*

$$\|T_g f_a\|_p \geq C \|g \circ \sigma_a - g(a)\|_t$$

for all $a \in \mathbb{D}$.

- Recall: $g \in BMOA \setminus VMOA \Leftrightarrow \limsup_{|a|\to 1} \|g \circ \sigma_a - g(a)\|_p > 0$ for any $0 < p < \infty$.
- Thus if T_g is non-compact, then there exists a sequence $(a_n) \subset \mathbb{D}$, $a_n \to \omega \in \mathbb{T} = \partial\mathbb{D}$ s.t. $\lim_{n\to\infty} \|T_g f_{a_n}\|_p > 0$.

- A localization result for T_g:

Lemma

Let $g \in BMOA$, $1 \leq p < \infty$, *and* $(a_k) \subset \mathbb{D}$ *be a sequence such that* $a_k \to \omega \in \mathbb{T}$. *Define*

$$A_\varepsilon = \{e^{i\theta} : |e^{i\theta} - \omega| < \varepsilon\}$$

for each $\varepsilon > 0$. *Then*

$$(i) \lim_{k\to\infty} \int_{\mathbb{T}\setminus A_\varepsilon} |T_g f_{a_k}|^p dm = 0 \text{ for every } \varepsilon > 0.$$

$$(ii) \lim_{\varepsilon\to 0} \int_{A_\varepsilon} |T_g f_{a_k}|^p dm = 0 \text{ for each } k.$$

Sketch of the proof of $\|U(\alpha)\|_p \geq C\|\alpha\|_{\ell^p}$ for all $\alpha \in \ell^p$.

- Using conditions (i), (ii) and the previous result by Aleman and Cima, we can define a sequence (ε_n), $\varepsilon_1 > \varepsilon_2 > \ldots > 0$, and a subsequence (b_n) of (a_n) s.t.

$$\text{(i)} \quad \left(\int_{A_n} |T_g f_{b_j}|^p dm\right)^{1/p} < 4^{-n}c, \quad j = 1, \ldots, n-1;$$

$$\text{(ii)} \quad \left(\int_{\mathbb{T}\setminus A_n} |T_g f_{b_n}|^p dm\right)^{1/p} < 4^{-n}c;$$

$$\text{(iii)} \quad \frac{c}{2} \leq \left(\int_{A_n} |T_g f_{b_n}|^p dm\right)^{1/p} \leq 2c$$

for every $n \geq 1$, where

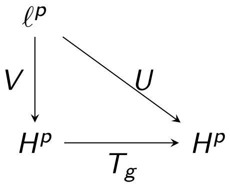
$$A_n = A_{\varepsilon_n} = \{e^{i\theta} : |e^{i\theta} - \omega| < \varepsilon_n\}$$

and $c = \lim_{n\to\infty} \|T_g f_{a_n}\|_p > 0$.

- Functions $|T_g f_{b_n}|$ resemble "disjointly supported peaks in $L^p(\mathbb{T})$" near some boundary point $\omega \in \mathbb{T}$.
- This ensures that $\|U(\alpha)\|_p = \|\sum_n \alpha_n T_g f_{b_n}\|_p \geq C\|\alpha\|_{\ell^p}$ for all $\alpha = (\alpha_n) \in \ell^p$.

$$
\begin{aligned}
\|U\alpha\|_p^p &= \int_{\mathbb{T}} \left| \sum_{j=1}^{\infty} \alpha_j T_g f_{b_j} \right|^p dm \geq \sum_{n=1}^{\infty} \int_{A_n \setminus A_{n+1}} \left| \sum_{j=1}^{\infty} \alpha_j T_g f_{b_j} \right|^p dm \\
&\geq \sum_{n=1}^{\infty} \left| |\alpha_n| \left(\int_{A_n \setminus A_{n+1}} |T_g f_{b_n}|^p dm \right)^{1/p} \right. \\
&\qquad \left. - \left(\int_{A_n \setminus A_{n+1}} |\sum_{j \neq n} \alpha_j T_g f_{b_j}|^p dm \right)^{1/p} \right|^p \\
&\geq \ldots \geq C\|\alpha\|_{\ell^p}^p.
\end{aligned}
$$

Figure: **Operators U, V and T_g**



$$f \in M = V(\ell^p) = \overline{span}\{f_{b_n}\}$$
$$\Rightarrow \|T_g f\|_p = \|U(\alpha)\|_p \geq C\|\alpha\|_{\ell^p} \geq \|V(\alpha)\|_p = \|f\|_p.$$

Strict singularity of T_g on Bergman spaces and Bloch space

- Standard Bergman spaces $A^p_\alpha = L^p(\mathbb{D}, dA_\alpha) \cap H(\mathbb{D})$, where $dA_\alpha(z) = (1-|z|^2)^\alpha dA(z)$, $\alpha > -1$ and $0 < p < \infty$, are isomorphic to ℓ^p.
- $S(\ell^p) = K(\ell^p) \Rightarrow$ the strict singularity and compactness are equivalent for T_g acting on A^p_α.
- Let $\mathcal{B}$ be the Bloch space. Then $T_g \colon \mathcal{B} \to \mathcal{B}$ is strictly singular $\Rightarrow$ $T|\mathcal{B}_0$ is strictly singular. Since the little Bloch space $\mathcal{B}_0$ is isomorphic to c_0 and $S(c_0) = K(c_0)$, the restriction $T|\mathcal{B}_0$ is compact.
- Finally, the biadjoint $(T_g|\mathcal{B}_0)^{**}$ can be identified with $T_g \colon \mathcal{B} \to \mathcal{B}$ and consequently T_g acting on $\mathcal{B}$ is compact.

For further reading

- A. Aleman, *A class of integral operators on spaces of analytic functions,* Topics in complex analysis and operator theory, 3-30, Univ. Málaga, Málaga, 2007.
- A.G. Siskakis, *Volterra operators on spaces of analytic functions-a survey,* Proceedings of the First Advanced Course in Operator Theory and Complex Analysis, Univ. Sevilla Secr. Publ., Seville, 2006, pp. 51-68.
- A. Aleman and A.G. Siskakis, *An integral operator on H^p,* Complex Variables Theory Appl. 28 (1995), no. 2, 149-158.
- A. Aleman and J.A. Cima, *An integral operator on H^p and Hardy's inequality,* J. Analyse Math. 85 (2001), 157-176.
- S. Miihkinen, *Strict singularity of a Volterra-type integral operator on H^p,* Proceedings of the AMS, DOI: http://dx.doi.org/10.1090/proc/13180 (to appear in print).

THANK YOU!