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Bounding Projective Hypersurface Singularities

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by

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Dedicated to my mom, who could not be here today to witness my work, but would undoubtedly be proud of all I have accomplished, who is looking down from above, and my dad who has always supported me and pushed me to be the academic I am today.

ABSTRACT OF THE DISSERTATION

Bounding Projective Hypersurface Singularities

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Professor Matt Kerr, Chair

We first provide background necessary for understanding monodromy and spectra. We then compare several different methods involving Hodge-theoretic spectra of singularities which produce constraints on the number and type of isolated singularities on a projective hypersurface of fixed degree. In particular, we introduce a method based on the spectrum of the nonisolated singularity at the origin of the affine cone on such a hypersurface, and relate the resulting explicit formula to Varchenko's bound. We then provide a purely combinatorial interpretation of our theorems and our conjecture.

1. Background

1.1 Monodromy

Definition 1. Let X, Y, B be topological spaces and let $f : X \to Y$ be a continuous map. Then the triple, $(X, Y; f)$ will be called a *locally trivial fibration* or *fiber bundle* with fiber B, if for any point $y_0 \in Y$, there exists a neighborhood $y_0 \in U \subseteq Y$ and a homeomorphism ν, such that the diagram below commutes:

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\nu} & B \times U \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle \pi_U} & \\ & U & \end{array}$$

Here π_U denotes the projection map onto U, and B is given the discrete topology. The homeomorphism ν is called the *local trivialization* of the fibration.

In our context, $f : X \to Y$ will be a smooth surjective holomorphic mapping of complex manifolds with compact fibers (i.e. a smooth surjective *proper* morphism). Since f is smooth (df has maximal rank at each $x \in X$), all fibers of f are non-singular compact complex analytic submanifolds of X. If we fix a particular fiber, $B = f^{-1}(s_0)$ of f, then it can be shown that $(X, Y; f)$ gives a locally trivial fibration with fiber B, and the trivialization ν can be chosen to be a diffeomorphism of the the smooth manifolds

$f^{-1}(U)$ and $B \times U$. In this particular situation, the triple $(X, Y; f)$ is called a *smooth family* of complex analystic manifolds and the fiber $f^{-1}(s)$ over $s \in Y$ is denoted X_s

Example 2 (Locally trivial fibration/fiber bundle)**.** Let $S^1 \subset \mathbb{C}$ denote the unit circle $\{e^{i\theta} \mid \theta \in [0, 2\pi)\}$, and fix the basepoint $1 \in S^1$. Let $f_n : X = S^1 \to S^1 = Y$ be defined as $f_n(z) \to z^n$. Then $f_n : S^1 \to S^1$ defines a fiber bundle with fiber $B = \{n^{th}$ roots of unity$\}$. For $n = 2$, we may pick our open set:

$$U = \left\{ e^{i\theta} \;\middle|\; \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\} \subseteq Y,$$

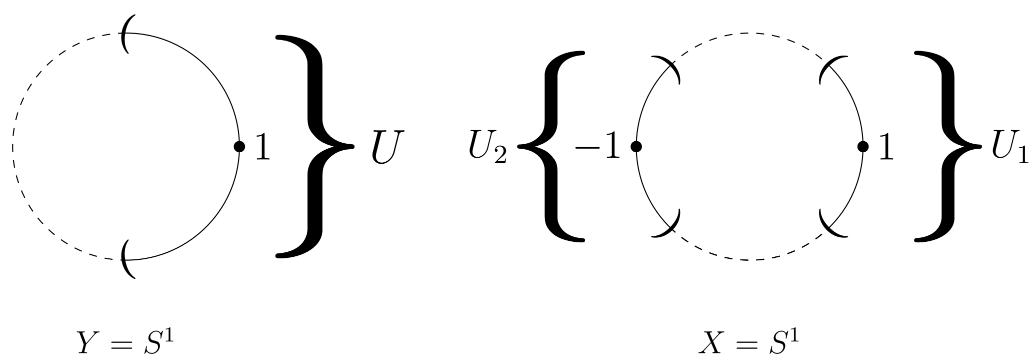
so that we end up with

$$f_2^{-1}(U) = \left\{ e^{i\theta} \;\middle|\; \theta \in U_1 = \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup U_2 = \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right) \right\} \subseteq X.$$

We observe that here $1 \in U$, and so $f_2^{-1}(1) = \{-1, 1\} \subseteq f_2^{-1}(U)$. Furthermore, we note that $f_2^{-1}(U)$ is composed of two disjoint open intervals,each homeomorphic to U and containing only one of the inverse image points. In fact, we may think about the fiber $f_2^{-1}(1) = \{-1, 1\}$ as the set indexing the disjoint components of the inverse image $f^{-1}(U)$. This gives us an intuitive understanding of the fiber bundle. Explicitly:

$$\begin{array}{ccc} f_2^{-1}(U) & \xrightarrow{\quad \nu \quad} & \{-1, 1\} \times U \\ & {}_{f_2}\searrow \quad \swarrow_{\pi_U} & \\ & U & \end{array}$$

Where $\nu : f^{-1}(U) = U_1 \cup U_2 \to \{-1, 1\} \times U$ is given by $\nu(x) = ((-1)^i, f_2(x))$ where $x \in U_i$



Note that, in a similar way, U can be chosen about any other point in Y in a way that patches together, giving the full fiber bundle.

Any locally trivial fibration $(X, Y; f)$ satisfies the *covering homotopy axiom*. Namely, for any homotopy:

$$\gamma_t : K \to Y, \quad t \in [0, 1]$$

of a simiplicial complex K, and any continuous mapping:

$$\beta_0 : K \to X$$

such that $f \circ \beta_0 = \gamma_0$, there exists a homotopy:

$$\beta_t : K \to X, \quad t \in [0, 1]$$

extending β_0 and such that $f \circ \beta_t = \gamma_t$ for all $t \in [0, 1]$.

We restrict to the case that B, the fiber of a locally trivial fibration $(X, Y; f)$ is a simplicial complex and the base Y is path-connected. Consider the arc:

$$\gamma : [0, 1] \to Y, \quad \gamma(0) = y_0, \quad \gamma(1) = y_1$$

Then this curve defines a homotopy $\gamma_t : B \to Y$ defined by the condition that $\gamma_t(b) = \gamma(t)$ for all $b \in B$. Let $\beta_0 : B \to f^{-1}(y_0)$ be a homeomorphism. Then there exists a homotopy $\beta_t : B \to X$ covering γ_t and extending β_0. the mapping:

$$\mu : f^{-1}(y_0) \to f^{-1}(y_1)$$

defined by:

$$\mu(x) = \beta_1(\beta_0^{-1}(x))$$

is a homotopy equivalence of fibers.

It can be deduced from the covering homotopy axiom that the homotopy class of μ depends only on the homotopy type of the path γ joining y_0 and y_1 in Y. The mapping μ, defined up to homotopy equivalence of $\gamma \subseteq Y$ is called the *monodromy transformation* of the fiber $f^{-1}(y_0)$ into the fiber $f^{-1}(y_1)$ defined by the curve γ. That is, if [ ] denotes homotopy class and two paths $\gamma[0,1] \to Y$ and $\alpha : [0,1] \to Y$ with $\gamma(0) = \alpha(0) = y_0, \gamma(1) = \alpha(1) = y_1$ are homotopic ($[\alpha] = [\gamma]$), then $\mu_{[\gamma]} = [\mu_\gamma] = [\mu_\alpha]$ is a well defined map which maps $\mu_{[\gamma]} : f^{-1}(y_0) \to f^{-1}(y_1)$.

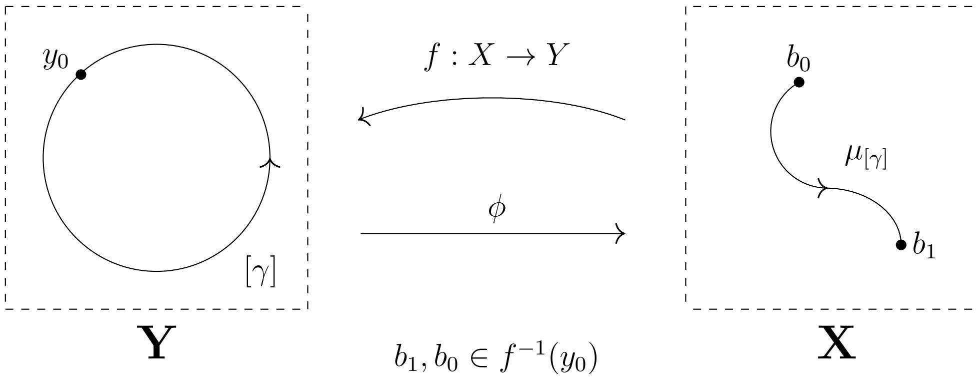
Now we will restrict our attention to only the paths $\gamma : [0,1] \to Y$ which are loops. Fix $y_0 \in Y$ and let $\gamma(0) = \gamma(1) = y_0$. Then $\mu_{[\gamma]}$ is a map from the fiber $f^{-1}(y_0)$ to itself. Therefore, to each element $[\gamma] \in \pi_1(Y; y_0)$ we may associate a monodromy tranformation of the fiber $f^{-1}(y_0)$ (to itself). We may view this correspondence as a mapping:

$$\phi : \pi_1(Y; y_0) \to \{\text{monodromies of } f^{-1}(y_0)\}$$

given by:

$$[\gamma] \to \mu_{[\gamma]}$$

which is in fact a well-defined homeomorphism (viewing the right hand side as a subgroup of automorphisms of the fiber). The image $\phi[\pi_1(Y; y_0)]$ of this homomorphism is called the *monodromy group* of the fiber $f^{-1}(y_0)$. Pictorally:



Note that here our choice of $[\gamma]$ implies that we may choose any loop $\alpha \in [\gamma]$ and get the same $b_0, b_1 \in f^{-1}(y_0)$.

If $\mu : B \to B$ is a continuous map of a simplicial complex $B = f^{-1}(y_0)$, the homotopy class of μ defines endomorphisms of the homology and cohomology groups of B.

Example 3 (Monodromy)**.** Let $f : \Delta^* \to S^*$ be defined $f(z) = s = z^n$ from the punctured unit disk about the origin to itself. Fix any $s_0 \in S^*$. Then the fiber of the locally trivial fibration induced by f is $B = \{z_1, \ldots z_n\}$ where $z_i^n = s_0$ and $z_k = z_1 e^{\frac{2\pi i(k-1)}{n}}$ for $1 \leq i, k \leq n$. We may pick U to be an open wedge around $s_0 = \rho_0 e^{i\phi_0}$, precisely,

$$U = \left\{ \rho e^{i\phi} \middle| \rho \in (0,1), \phi \in \left(\phi_0 - \frac{\pi}{2}, \phi_0 + \frac{\pi}{2} \right) \right\}$$

giving the disjoint union of smaller wedges:

$$f^{-1}(U) = \bigcup_{k=1}^{n} U_k$$

where:

$$U_k = \left\{ \sqrt[n]{\rho} e^{i\phi} \middle| \rho \in (0,1), \phi \in \left(\frac{2\pi(k-1) - \phi_0}{n} - \frac{\pi}{2n}, \frac{2\pi(k-1) + \phi_0}{n} + \frac{\pi}{2n} \right) \right\}$$

We define the fiber bundle by assuming each point in S^* has a corresponding open neighborhood defined the same way. Now let γ be a loop going around the origin once in S^*, and starting at s_0. It is easiest to think about the inverse image of this loop $f^{-1}(\gamma)$ as going n times slower in the pre-image Δ^*. The i^{th} revolution γ makes in S^*, corresponds to a rotation of angle $\frac{2\pi}{n}$ given by $f^{-1}(\gamma)_i$ from z_i to z_{i+1} in Δ^*. Namely, n revolutions of γ, corresponds to one full revolution about the origin made in Δ^*. This relationship in the inverse image sending $z_1 \to z_2 \to \ldots \to z_n \to z_1$, is called the *geometric monodromy.* That is, the geometric monodromy $g : B \to B$ is a map from the fiber of the fibration to itself.

However, the word "monodromy" alone is also used to refer to the *algebraic monodromy* which is the map induced on the reduced homology groups of B by the geometric monodromy map. The algebraic monodromy is therefore defined:

$$T : \widetilde{H}_k(B, \mathbb{Z}) \to \widetilde{H}_k(B, \mathbb{Z}),$$

with $k \in \mathbb{N}$ specified depending on the context. In the above example, B is a set of n-points, therefore the reduced homology groups are:

$$\widetilde{H}_k(B, \mathbb{Z}) \approx \begin{cases} \mathbb{Z}^{n-1} & k = 0 \\ \\ 0 & k \geq 0 \end{cases}$$

In the usual reduced homology complex, $C_0 = \langle \{z_1\}, \ldots \{z_n\} \rangle \cong \mathbb{Z}^n$ the free abelian group on generators $\{z_i\}$, and the map $\varepsilon : C_0 \to \mathbb{Z}$ is given by $\sum_{i=1}^n a_i\{z_i\} \to \sum_{i=1}^n a_i$.

We also know for a set of points $C_1 = 0$, so $\mathrm{Im}(\delta_1) = 0$ Since $\widetilde{H}_0(B, \mathbb{Z})$ is calculated via the quotient:

$$\frac{\mathrm{Ker}(\varepsilon)}{\mathrm{Im}(\delta_1)} = \frac{\{(a_i)_{i=1}^n \in \mathbb{Z}^n | \sum a_i = 0\}}{\langle 0 \rangle} \cong \{(a_i)_{i=1}^n \in \mathbb{Z}^n | \sum a_i = 0\}$$

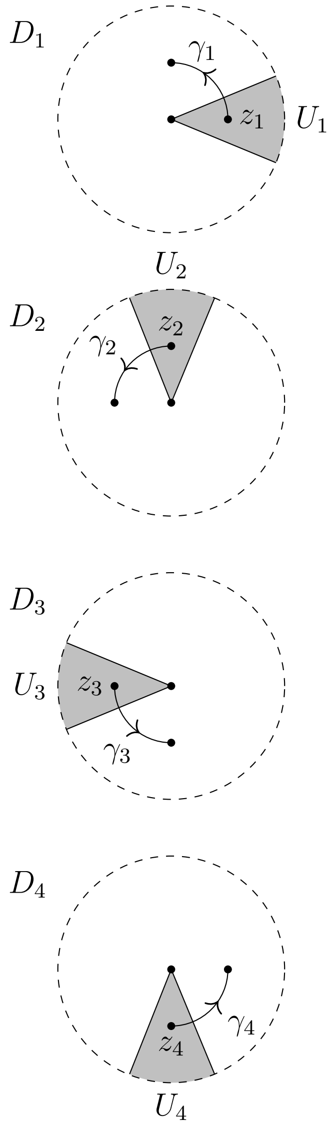
Which as a subgroup of $\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^{n-1}$. We may view $\widetilde{H}_0(B, \mathbb{Z})$ in terms of its $(n-1)$ generators as $\langle \{z_2\} - \{z_1\}, \ldots \{z_n\} - \{z_{n-1}\} \rangle$. The geometric monodromy therefore tells us that T must send the generators:

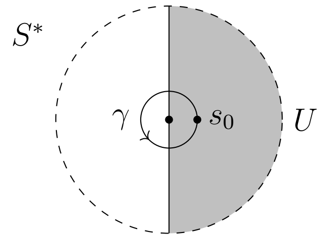
$$\begin{cases} T(\{z_i\} - \{z_{i-1}\}) = \{z_{i+1}\} - \{z_i\} & n-1 \geq i \geq 2 \\ T(\{z_n\} - \{z_{n-1}\}) = \{z_1\} - \{z_n\} & i = n \end{cases}$$

We note that $z_1 - z_n = -[(z_n - z_{n-1}) + (z_{n-1} - z_{n-2}) + \ldots z_2 - z_1)]$ Therefore, T can be represented by the linear $(n-1) \times (n-1)$ matrix in terms of these generators:

$$T = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & -1 \\ 1 & 0 & 0 & \ldots & 0 & -1 \\ 0 & 1 & 0 & \ldots & 0 & -1 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \end{bmatrix}$$

Example 4. In particular, if $n = 4$ in the above construction, we can visualize the monodromy relation via the following picture:





$$f^{-1}(U) = \bigcup_{i=1}^{4} U_i$$

$$f^{-1}(S^*) = \{\bigcup_{i=1}^{4} D_i\}/\{D_j \sim D_k\}$$

$$\gamma_i = [f^{-1}(\gamma)]^i$$

and end up with the 3×3 monodromy matrix below:

$$T = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

[1]

1.2 Horizontal and Vertical Monodromy

We will now consider the case when $f : \mathbb{C}^{n+1} \to \mathbb{C}$ has a 1-dimensional critical locus Σ, and where ℓ is an *admissible* linear form. [2] defines an admissible linear form ℓ to be a linear form $\ell = a_0 x_0 \dots a_n x_n$ with $a_i \in \mathbb{C}$, where $f^{-1}(0) \cap \{\ell = 0\}$ has an isolated singularity. The series of functions $f_N = f + \varepsilon \ell^N$ for $N \in \mathbb{N}$ is referred to as the *Iomdin series* of hypersurface singularities.

The *Milnor fibration* is an important invariant of a singularity. Milnor showed that for $\varepsilon > 0$ chosen small enough, there exists an $\eta > 0$ such that

$$f : B_\varepsilon \cap f^{-1}(S^1_n) \to S^1_n$$

is a locally trival fibre bundle where B_ε is the closed ε-ball in $\mathbb{C}^{n+1}$ about 0 and S^1_n is a circle with radius η in $\mathbb{C}$. A typical fiber $F = f^{-1}(\eta) \cap B_\varepsilon$ (conventially chosen as $\theta = 0$) is called the Milnor fiber of f. We make quick note that the Milnor fibration as a whole is essentially the union $\cup_{\theta \in [0,2\pi)} f^{-1}(\eta e^{i\theta}) \cap B_\varepsilon$.

We denote:

$$\mu_k(f) = \dim \widetilde{H}^k(F)$$

If $\dim \Sigma = 1$, it is well known that $\mu_k(f) = 0$ whenever $k \neq n-1, n$. In the event that f has an isolated singularity we have $\mu_k(f) = 0$ whenever $k \neq n$. In this case we also call $\mu(f) = \mu_n(f)$ the *Milnor number* of f. With this in mind, the Iomdin series is special due to the following:

Theorem 5 (Iomdin)**.** *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ have a 1-dimensional critical locus Σ, and let ℓ be an* admissible *linear form. Then there exists an N_0 such that for $N > N_0$*

1. *$f_N = f + \varepsilon \ell^N$ has an isolated singularity for all $\varepsilon \in \mathbb{C}, \varepsilon \neq 0$*

2. $\mu(f_N) = \mu(f) - \mu_{n-1}(f) + Ne_0(\Sigma)$ *where* $e_0(\Sigma)$ *is the algebraic multiplicity of* Σ *at* $0 \in \mathbb{C}^{n+1}$

In this contextual example of the monodromy described above in the previous section, The geometric monodromy is a diffeomorphism $h : F \to F$, which is a characteristic map for the Milnor fibration over the circle S^1_n of radius η in $\mathbb{C}^{n+1}$. It has the property that there exists a diffeotopy $H : F \times [0, 2\pi] \to B_\varepsilon \cap f^{-1}(S^1_n)$ such that for each $x \in F$:

1. $f(H(x,t)) = \eta e^{it}$
2. $H(x, 0) = x$
3. $H(x, 2\pi) = h(x)$

Which induces the algebraic monodromy map on homology:

$$\Pi : \widetilde{H}_*(F) \to \widetilde{H}_*(F)$$

Since the reduced homology groups of the Milnor fiber F satisfy $\widetilde{H}_k(F) = 0$ for $k \neq n-1, n$ we are only concerned with the nontrivial maps:

$$\Pi : \widetilde{H}_{n-1}(F) \to \widetilde{H}_{n-1}(F)$$

$$\Pi : \widetilde{H}_n(F) \to \widetilde{H}_n(F)$$

For every irreducible branch Σ_i of Σ, we have on $\Sigma_i - \{0\}$ a local system of transversal singularities. That is for $x \in \Sigma_i - \{0\}$, we may take the germ of the generic transversal section. this will give us an isolated singularity whose μ-constant class is well-defined. we may denote a typical Milnor Fiber of a transversal singuarity by F_i'. By Deligne's sheaf of vanishing cycles, we know the only non-vanishing homology group of this is $\widetilde{H}_{n-1}(F_i')$.

Therefore, on the level of homology, we get a local system F_i' with two different monodromies:

(a) (Vertical monodromy):

$$A_i : \widetilde{H}_{n-1}(F_i') \to \widetilde{H}_{n-1}(F_i')$$

Which is the characteristic mapping of the local system over the punctured disc $\Sigma_i - \{0\}$

(b) (Horizontal monodromy):

$$T_i : \widetilde{H}_{n-1}(F_i') \to \widetilde{H}_{n-1}(F_i')$$

Which is the Milnor fibration monodromy when we restrict f to a transversal slice through $x \in \Sigma_i$

We note that our two monodromies A_i, and T_i commute since they are defined on $(\Sigma_i - \{0\}) \times S_n^1$ which is homotopy equivalent to a torus [2].

We now provide an example of what this type of relation looks like.

Example 6. Let $f : \mathbb{C}^2 \to C$ be defined $f(x_0, x_1) = x_0^2 x_1$. Here $n = 1$, and f is homogeneous of degree $d = 3$. Taking the partial derivatives of f, we see that:

$$\partial_0 = 2x_0 x_1$$

$$\partial_1 = x_0^2$$

So $\Sigma = \{f = 0\} \cap \{\partial_0 = 0\} \cap \{\partial_1 = 0\} = [\{x_1 = 0\} \cup \{x_0 = 0\}] \cap [\{x_1 = 0\} \cup \{x_0 = 0\}] \cap [\{x_0 = 0\}] = \{x_0 = 0\}$ which is of dimension 1. here there is only one irreducible component of Σ, namely $\Sigma_1 = \{x_0 = 0\}$.

horizontal monodromy: We must pick a $p_1 \neq (0,0)$ on $\Sigma_1 = \{x_0 = 0\}$, so we pick $p_1 = (0,1)$ and consider the slice U_1 transverse to Σ_1 at p_1. That is, $U_1 = \{(x_0,1) \mid x_0 \in \mathbb{C}\} \subset \mathbb{C}^2$. Define $g_1 = f|_{U_1} = x_0^2$. We may view $g_1\mathbb{C} \to \mathbb{C}$, and calculate the Milnor fiber in this context about the origin. Let $\varepsilon, \eta \in \mathbb{R}_{>0}$ satisfying $0 < |\eta| << |\varepsilon| << 1$. Then the Milnor fiber over $\{\eta\}$ is given by:

$$F_1' = \{g_1^{-1}(\eta)\} \cap B_\varepsilon(0) = \{\eta^{\frac{1}{2}}, -\eta^{\frac{1}{2}}\} \cap B_\varepsilon(0) = \{\eta^{\frac{1}{2}}, -\eta^{\frac{1}{2}}\}$$

Since η is chosen to be small relative to ε. We can easily calculate the relevant reduced homology group of two points:

$$\widetilde{H}_0(F_1') = \widetilde{H}_0(\{\eta^{\frac{1}{2}}, -\eta^{\frac{1}{2}}\}) = \mathbb{Z}$$

Then the Milnor fibration over the circle $\{\eta e^{i\theta} | \theta \in [0,2\pi)\}$ is the union of the milnor fibers of $\eta e^{i\theta}$ at each θ. More precisely, the Milnor fibration is given by:

$$\cup_{\theta\in[0,2\pi)}\{g^{-1}(\eta e^{i\theta}) \cap B_\varepsilon(0)\} = \cup_{\theta\in[0,2\pi)}\{\eta^{\frac{1}{2}}e^{i\theta}, -\eta^{\frac{1}{2}}e^{i\theta}\}$$

We get the following locally trivial fibration over the circle $S^1_n = \{\eta e^{i\theta}\}_{[0,2\pi)}$: About each point $\eta e^{i\theta}$, we take the open set $\{\eta e^{i\phi} | \phi \in (\theta - \pi, \theta + \pi)\} = P_\theta \subset \mathbb{S}^{\Bbbk}_{\ltimes}$. Then:

$$g_1^{-1}(P_\theta) = \left\{\eta^{\frac{1}{2}}e^{i\phi} | \phi \in \left(\frac{\theta}{2}, \frac{\theta + 2\pi}{2}\right)\right\} = P_\theta^1$$
$$\cup \left\{\eta^{\frac{1}{2}}e^{i\phi} | \phi \in \left(\frac{\theta+2\pi}{2}, \frac{\theta + 4\pi}{2}\right)\right\} = P_\theta^2$$

Now we view how the loop $\theta : [0,2,\pi] \to \eta e^{i\theta}$ acts on the Milnor fibration and induces an automorphism of the Milnor fiber.

As θ travels between 0 and 2π, it traverses the circle $\eta e^{i\theta}$, first starting at η, and then traveling counterclockwise and ending back at η. In the inverse image, this corresponds to two half circle paths in the Milnor fibration:

$$\left\{\eta^{\frac{1}{2}} e^{i\phi_1} \mid \phi_1 \in [0, \pi]\right\} \quad \text{and} \quad \left\{\eta^{\frac{1}{2}} e^{i\phi_2} \mid \phi_2 \in [\pi, 2\pi]\right\}$$

which induces:

$$h_* = \begin{cases} \eta^{\frac{1}{2}} \longrightarrow -\eta^{\frac{1}{2}} \\ \\ -\eta^{\frac{1}{2}} \longrightarrow \eta^{\frac{1}{2}} \end{cases}$$

Which is the geometric monodromy map. To find the algebraic monodromy map, we consider the usual chain complex with respect to the reduced homology:

$$\ldots \longrightarrow C_1(F_1') \xrightarrow{\partial_1} C_0(F_1') \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where:

$$\widetilde{H}_0(F_1') = \frac{\ker(\varepsilon)}{\operatorname{Im}(\partial_1)} = \frac{\left\langle \left\{\eta^{\frac{1}{2}}\right\} - \left\{-\eta^{\frac{1}{2}}\right\} \right\rangle}{\langle 0 \rangle} \cong \left\langle \left\{\eta^{\frac{1}{2}}\right\} - \left\{-\eta^{\frac{1}{2}}\right\} \right\rangle \cong \mathbb{Z}$$

Since $\left\{\eta^{\frac{1}{2}}\right\} - \left\{-\eta^{\frac{1}{2}}\right\}$ generates $\widetilde{H}_0(F_1')$, the geometric monodromy map, h_* induces the following map on $\widetilde{H}_0(F_1')$:

$$\Pi : \left\{\eta^{\frac{1}{2}}\right\} - \left\{-\eta^{\frac{1}{2}}\right\} \to \left\{h_*\left(\eta^{\frac{1}{2}}\right)\right\} - \left\{h_*\left(-\eta^{\frac{1}{2}}\right)\right\} = \left\{-\eta^{\frac{1}{2}}\right\} - \left\{\eta^{\frac{1}{2}}\right\}$$

Which sends the generator to it's additive inverse. Therefore the algebraic monodromy

$$\Pi : \widetilde{H}_0(F_1') \to \widetilde{H}_0(F_1')$$

is given by:

$$\Pi : \mathbb{Z} \to \mathbb{Z} = [-1]$$

, the negative linear identity map. We note that by definition, this Π is precisely T_1, the horizontal monodromy map.

vertical monodromy: We now calcluate the vertical monodromy, i.e. the monodromy of the path of an object V_1 transverse to Σ_1 as it revolves around the origin in the slice Σ_1 and stays transverse to Σ_1 throughout. We may imagine this as turning the thin edge of a piece of paper as the center of the bottom edge travels in a circle on a tabletop. In this metaphor, the paper represents V_1 and the tabletop is Σ_1. The spot in the middle of the circle is the origin with respect to the surrounding preimage of the original map f.

We fix the following values: $1 >> \varepsilon >> |t| >> |\eta| >> 0$. We will calculate the monodromy over a circle contained within the slice $\Sigma_1 \cong \mathbb{C}$. Since $\Sigma_1 = \{x_0 = 0\}$, the circle in this slice is $\{(0, \eta e^{i\theta}) | \theta \in [0, 2\pi)\}$. As θ traverses this circle, the slice transverse to Σ_1 changes. That is, unlike choosing a single U_1, as we did in the horizontal monodromy case, at each θ we get the sets:

$$U_{1\theta} = \{(x_0, \eta e^{i\theta})\} \quad \text{for} \quad \theta \in [0, 2\pi)$$

The vertical Milnor fiber is therefore the restriction of the Milnor fiber of f to $U_{1\theta}$, in a way that is consistent with our choice of U_1 when we calculated the horizontal monodromy. Since we chose $U_1 = \{(x_0, 1) | x_0 \in \mathbb{C}\}$, a parallel choice is $\theta = 0 \Rightarrow U_{1,0} = \{(x_0, \eta) | x_0 \in \mathbb{C}\}$. Since the Milnor fiber of f is $\{x_0^2 x_1 = t\} \cap B_\varepsilon$, the vertical Milnor fiber is $\{x_0^2 \eta = t\} \cap B_\varepsilon = \{(v_1, \eta), (v_2, \eta)\}$. Here v_1 and v_2 are simply the two square roots of $\frac{t}{\eta}$.

This leads us to the vertical Milnor fibration which is given by:

$$\bigcup_{\theta\in[0,2\pi)} \left\{ \left(v_1 e^{i\left(\frac{\theta}{2}\right)}, \eta e^{i\theta}\right), \left(v_2 e^{i\left(\frac{\theta}{2}\right)}, \eta e^{i\theta}\right) \right\}$$

Similarly as before, as θ traverses $\{(0, \eta e^{i\theta})$ from 0 to 2π, this induces the geometric monodromy map:

$$h_* = \left(v_i e^{i\left(\frac{\theta}{2}\right)}, \eta e^{i\theta}\right) : \begin{cases} (v_1, \eta) \longrightarrow (v_1 e^{i\pi}, \eta) = (v_2, \eta) \\ \\ (v_2, \eta) \longrightarrow (v_2 e^{i\pi}, \eta) = (v_1, \eta) \end{cases}$$

Since the class $\{(v_1, \eta)\} - \{(v_2, \eta)\}$ generates the reduced homology group $\widetilde{H}_0(F_1') \cong \mathbb{Z}$, the induced algebraic monodromy map (that is the vertical monodromy map A_1 is again $[-1]$.

1.3 Spectra of Hypersurface Singularities

1.3.1 Preliminaries

Let $X = \{\underline{z} : F(\underline{z}) = 0\} \subset \mathbb{P}^n$ be a hypersurface of degree d (that is F is a homogeneous polynomial of total degree d). We first recall that given any $(n-1)-cycle\gamma \in H_{n-1}(X, \mathbb{C})$, we may take a tube $T(\gamma)$, that is locally isomorphic to $\gamma \times S^1$, such that $T(\gamma) \subseteq \mathbb{P}^n - X$. Given any rational n-form ω, we may integrate it over this tube to get a complex number. More precisely, each choice of ω corresponds to a map:

$$\mathrm{Res}_X(\omega) : \gamma \to z \in \mathbb{C}$$

defined by $\gamma \to \int_{T(\gamma)} \omega$ Since this is invariant over the homology class of γ,and is compatible with the cup products it defines a linear transformation on the homology group:

$$\mathrm{Res}_X(\omega) : H_{n-1}(X, \mathbb{C}) \to \mathbb{C}$$

taking each class $[\gamma] \to z \in \mathbb{C}$. Since $H^{n-1}(X, \mathbb{C})$ is just the linear dual of $H_{n-1}(X, \mathbb{C})$, we must then have, $\text{Res}_X(\omega) \in H^{n-1}(X, \mathbb{C})$. We call $\text{Res}_X(\omega)$ the *Poincaré* residue of ω.

We note that if $n = 1$, ω is just a rational function, γ, is just a point in $\mathbb{P}^1$ (allowing points in $\mathbb{C}$ or the point at infinity), and the tube becomes a closed loop about the point, reducing the residue map to the typical case in elementary complex analysis. Computation of higher order residues reduces down to this case through the following algorithm for general n: The residue of any 1-form $\text{Res}\left(\frac{dz}{z} + a\right) = 1$. There exists a chart containing X such that X is precisely the vanishing locus of an n-form ω. Then any meromorphic n-form can be written in the form:

$$\frac{dw}{w^k} \wedge \rho = \frac{1}{k-1}\left(\frac{d\rho}{w^{k-1}} + d\left(\frac{\rho}{w^{k-1}}\right)\right)$$

showing that the cohomology classes:

$$\left[\frac{d\rho}{\omega^k}\right] = \left[\frac{d\rho}{(k-1)\omega^{k-1}}\right]$$

are the same and:

$$\text{Res}_X\left(\alpha \wedge \frac{dw}{w} + \beta\right) = \alpha|_X$$

We will denote by S_n^r the group of homogeneous polynomials in $z_1, \ldots z_n$ of degree r, and let $S_n = \oplus_r S_n^r$ be the ring of homogeneous polynomials. We let $J_F = \left(\frac{\partial F}{\partial z_0}, \ldots, \frac{\partial F}{\partial z_n}\right)$ denote the jacobian ideal, and set R_F to be the quotient:

$$R_F := S_n / J_F = \oplus_r R_F^r$$

which decomposes into the graded pieces (the *Jacobian rings*) $R_F^r = S_n^{r+n-1}/J_F$

Theorem 7. *(Griffiths 1969) Let $n - 1 = p + q$. Then the Poincairé residue map:*

$$Q(\underline{Z}) \to \omega_Q = Res_X\left(\frac{Q(\underline{Z})\sum_j (-1)^j z_j dz_0 dz_1 \wedge \ldots \widehat{dz_j} \ldots \wedge dz_n}{F(\underline{Z})^{q+1}}\right)$$

Induces an isomorphism $R_F^{(q+1)d-n-1} \cong H^{p,q}(X)$.

This theorem allows us to instead work with the quotient rings $R_F^{(q+1)d-n-1}$ to understand the structure of $H^{p,q}(X)$.

We now see how this theory connects to a better setting of the spectrum through the concept of *Griffiths tranversality*. Let $\{X_s\}_{s\in S}$ be a family of smooth projective varieties over a complex manifold S, and let:

$$[\omega_s] \in F^p H^k(X_s, \mathbb{C}) := \bigoplus_{\substack{a+b=k \\ a\geq p}} H^{a,b}(X_s)$$

be a family of cohomology classes within the Hodge filtration. Griffiths transversality is the statement that for any local holomorphic coordinate $t = t(s)$, we must have:

$$\frac{\partial}{\partial t}[\omega_s] \in F^{p-1} H^k(X_s, \mathbb{C})$$

This can be thought of as a differential equation governing the *period map*,

$$\Phi : S \to D/\Gamma,$$

Which uses period integrals to record the Hodge flag $F^\bullet$ as a function of s. Here Γ is the image of the Algebraic monodromy representation $\rho : \pi_1(S) \to \mathrm{Aut}\left(H^k(X_{s_0}, \mathbb{Z})\right)$ discussed above with s_0 some fixed base point of S.

As a consequence of transversality, we get the following theorem:

Theorem 8. *(Local Monodromy Theorem:Griffiths, Landman) Given a family of smooth projective varieties over a punctured disk* Δ^**, let* $T \in Aut(H^k(X_{s_0}, \mathbb{Z}))$ *denote the image of the counterclockwise loop under the monodromy representation* ρ*. Then* T *is quasi-unipotent: that is for some integers* M *and* N*,* $(T^N - I)^M = 0$

An immediate consequence of this is the following corollary which illustrates why the spectrum will be defined so naturally:

Corollary 9. *Given a family of smooth projective varieties over a punctured disk* Δ^**, let* $T \in Aut(H^k(X_{s_0}, \mathbb{Z}))$ *denote the image of the counterclockwise loop under the monodromy representation* ρ*. Then for some* N *all the eigenvalues of* T *are roots of unity of the form* $e^{\frac{2\pi ik}{N}}$*. Furthermore* T *can be written as the product* $T = T_{ss}T_u$ *of a semisimple and unipotent matrix where the eigenvalues of* T *lie on the diagonal of* T_{ss} *up to multiplicity.*

1.3.2 Definition of the Spectrum

Let $\mathscr{S} = \mathbb{Z}^{(\mathbb{Q})}$ be the free abelian group on the generators (α) with $\alpha \in \mathbb{Q}$. An element of $\mathscr{S}$ is denoted as a sum $\sum n_\alpha(\alpha)$. Here $n_\alpha \in \mathbb{Z}$ for all rationals α.

Let $\mathscr{C}$ denote the category with objects defined as $\mathbb{C}[t]$-modules of finite length, each equipped with t-stable decreasing filtrations (i.e. flitrations $\{M_i\}$, $\ldots \supseteq M_i \supseteq M_{i+1} \ldots$ s.t. $M_i \supseteq (t)M_{i+1}$, and for some $s \in \mathbb{N} \quad M_{d+s} = (t)^d M_s \quad \forall d \geq 0$) and such that t acts as an automorphism of finitie order, that is,

$$\gamma : M \to tM$$

is an automorphism for all objects M such that for each M there exists some $n \in \mathbb{N}$ such that $\gamma^n(M) = M$. The morphisms of the category will be $\mathbb{C}[t]$-linear maps compatible with these filtrations. We will denote an object of $\mathscr{C}$ as a triple (H, F, γ), where H is the $\mathbb{C}[t]$ module, F is the filtration, and γ is the automorphism given by the action of t.

In our context, H will be the cohomology group of the Milnor fiber of an isolated hypersurface singularity, F will be the Hodge filtration, and γ will correspond to the action of the semisimple part of the monodromy (note the monodromy itself is not compatible with F here.)

For any three objects $(H, F, \gamma), (H', F, \gamma), (H'', F, \gamma) \in \mathscr{C}$ (Here we abuse notation and write F, γ for each since association is clear in context), we say that a sequence:

$$0 \longrightarrow H' \xrightarrow{\alpha} H \xrightarrow{\beta} H'' \longrightarrow 0$$

is *exact* if the underlying sequence of vector spaces is exact, and if α, and β are strictly compatible with the filtrations (i,e, $\alpha(H') \cap F^p(H) = \alpha(F^pH')$ and $F^pH'' = \beta(F^pH)$ for all p). Here the notation F^pH refers to the filtration:

$$H = F^0H \supseteq F^1H \supseteq F^2H \dots$$

With this concept of exact sequences, we may consider $\mathscr{C}$ to be an *exact category.*

The group $\mathscr{S}$ can be considered the *Grothendieck group* of $\mathscr{C}$ in the following way: Fix $n \in \mathbb{Z}$, and let $(H, F, \gamma) \in \mathscr{C}$. Then γ acts on the quotient $\mathrm{Gr}_F^p(H) = {}^{\mathrm{F}^p}\!/_{\mathrm{F}^{p+1}}$. That is for any $\alpha + F^{p+1} \in \mathrm{Gr}_F^p(H)$, $\gamma \cdot \alpha + F^{p+1} \to \alpha + tF^{p+1} = \beta + F^{p+1}$ for some $\beta \in F^p$ since $(t)F^{p+1} \subseteq F^p$.

We may now define $\mathrm{Sp}_n(H, F, \gamma)$, by finding $\alpha_1, \dots, \alpha_{s(p)} \in \mathbb{Q}$ such that $s(p) = \dim \mathrm{Gr}_F^p(H)$, and the values satisfy:

$$n - p - 1 < \alpha_j \leq n - p$$

$$\det(tI - \gamma; \mathrm{Gr}_F^p(H)) = \prod_{j=1}^{s(p)} (t - e^{-2\pi i \alpha_j})$$

Then $\mathrm{Sp}_n(H, F, \gamma) = \sum_p \sum_{j=1}^{s(p)} (\alpha_j)$

For all $n \in \mathbb{Z}$, the map Sp_n induces an isomorphism between $K_0(\mathscr{C})$ and $\mathscr{S}$. Here, $K_0(\mathscr{C})$ is the zeroth algebraic K-group of the exact category, but in this sense is meant to consider classes of triplets $(H, F, \gamma) \sim (H', F, \gamma)$ via some equivalence. Changing n to $n + j$, or shifting the filtration index by $-j$ corresponds to a shift $(\alpha) \to (\alpha + j)$ in $\mathscr{S}$.

Now let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ define a nonzero holomorphic function germ (i.e. an equivalence class of functions equal in a neighborhood of 0). Then as before, its *Milnor fiber* $X(f)$, is defined:

$$X(f) = \{z \in \mathbb{C}^{n+1} : |z| < \eta \text{ and } f(z) = t\}$$

for $0 < |t| << \eta << 1$.

The cohomology groups $H^*(X(f))$ carry a canonical mixed Hodge structure. If the map:

$$T : H^*(\mathbb{C}^{n+1} \setminus f^{-1}(0)) \to H^*(\mathbb{C} \setminus \{0\})$$

Is the monodromy (or Picard Lefschetz transformation), then the semisimple part T_{ss} acts as an automorphism on the mixed Hodge structures of $X(f)$. In particular, it preserves the Hodge filtration F. We may now define the *spectrum* of f as

$$\sigma_f = \mathrm{Sp}(f) = \sum_{k=0}^{n} (-1)^{n-k} \mathrm{Sp}_n \left(\widetilde{H}^k(X(f)), F, T_{ss} \right)$$

[3]

1.4 Spectra of Quasihomogeneous Isolated Singularities

Steenbrink simplified the explicit calculation of singularity spectra for isolated quasi-homogenous singularities. In particular, his algorithm simplifies the calculation of the spectra of A_k, D_k, E_6, E_7, and E_8 singularities by using their respective normal forms. He proves the following algorithm in [4], which is more explicitly summarized in [3].

Theorem 10. *Let $f \in C[x_0, \dots, x_n]$ be the normal form for a quasihomogenous isolated singularity (at the origin). Let $\langle x^\alpha \rangle_{\alpha \in A}$ be a $\mathbb{C}$-basis for the Artinian ring $Q_f = \mathbb{C}[[x_0, \dots x_n]]/(\partial_0 f, \dots \partial_n f)$, where $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{Z}^{n+1}$ defines the exponents of the monomial $x^\alpha = {x_0}^{\alpha_0} {x_1}^{\alpha_1} \dots {x_n}^{\alpha_n}$. Let $w = (w_0, \dots, w_n)$ define the weight vector with $w_i \in \mathbb{Q}$, chosen in such a way that f has degree 1 (That is, if $f = \sum_{\beta \in B} x^\beta$, then w is chosen so that the dot product $w \cdot \beta = 1$ for all $\beta \in B$). Let $w(\alpha) = \sum_{i=0}^{n} (\alpha_i + 1) w_i - 1$. Then $\sigma_f = Sp(f) = \sum_{\alpha \in A} [w(\alpha)]$.*

We will now view how this theoreom can be used to calculate the spectrum of any A_k singularity.

Example 11. The normal form of an A_k singularity in $\mathbb{C}[x_0, \dots, x_n]$ is given by $f = {x_0}^{k+1} + {x_1}^2 + \dots {x_n}^2$. Calculating the partial derivatives one has:

$$\partial_0 f = (k+1){x_0}^k$$

$$\partial_i f = 2x_i, 1 \leq i \leq n$$

Therefore the Artinian ring Q_f has a $\mathbb{C}$-basis given by the monomials $\langle {x_0}^i \rangle_{i=0}^{k}$. Here we view these monomials in our context as ${x_0}^i = {x_0}^i {x_1}^0 \dots {x_n}^0$. Our set A is given by $A = \{(i, 0, \dots 0)\}_{i=1}^{k-1}$. From f, we can see that our set:
$B = \{(k+1, 0, \dots, 0), (0, 2, 0 \dots, 0), \dots, (0, \dots, 0, 2)\}$. By inspection it is apparent that the weight vector $w = \left(\frac{1}{k+1}, \frac{1}{2}, \dots, \frac{1}{2}\right)$.

Now we calculate $w(\alpha)$ for each $\alpha \in A$. Every $\alpha \in A$ has the form $(i, 0, \dots 0)$, so:

$$w(i, 0, \dots, 0) = \sum_{i=0}^{n} (\alpha_i + 1) w_i - 1 = \left((0+1)\frac{1}{k+1}\right) + \sum_{i=1}^{n} (0+1)\frac{1}{2} = \frac{i+1}{k+1} + \frac{n}{2} - 1$$

Therefore we have:

$$Sp(f) = \sum_{\alpha \in A} [w(\alpha)] = \sum_{i=0}^{k-1} \left[\frac{i+1}{k+1} + \frac{n}{2} - 1\right] = \sum_{i=1}^{k} \left[\frac{i}{k+1} + \frac{n}{2} - 1\right]$$

We may also do the same for the $\widetilde{E}_6$ singularity which is also of interest in our paper later.

Example 12. Let $f = x_0^3 + x_1^3 + x_2^3 - \lambda x_0 x_1 x_2$ with ($\lambda^3 \neq 27$) be the local normal form of an $\widetilde{E}_6$ singularity. Then we have partial derivatives:

$$\partial_0 f = 3x_0^2 - \lambda x_1 x_2$$

$$\partial_1 f = 3x_1^2 - \lambda x_0 x_2$$

$$\partial_2 f = 3x_2^2 - \lambda x_0 x_1$$

$$\Rightarrow Q_f = \frac{\mathbb{C}[x_0, x_1, x_2]}{(3x_0^2 - \lambda x_1 x_2, 3x_1^2 - \lambda x_0 x_2, 3x_2^2 - \lambda x_0 x_1)}$$

For simplicity's sake we will denote the ideal in the denominator as I. We now claim that the following coset representatives serve as a $\mathbb{C}$-basis for Q_f:

$$\{1 \ , x_0 \ , x_1 \ , x_2 \ , x_0 x_1 \ , x_0 x_2 \ , x_1 x_2 \ , x_0 x_1 x_2\}$$

If $\lambda = 0$ this result is immediate. For the remainder, we assume $\lambda \neq 0$

Starting with linear independence, assume:

$$a_1 + a_2 x_0 a_3 x_1 + a_4 x_2 + a_5 x_0 x_1 + a_6 x_0 x_2 + a_7 x_1 x_2 + a_8 x_0 x_1 x_2 \in I$$

For $a_i \in \mathbb{C}$. Then for some $b_i(x_0, x_1, x_2) \in \mathbb{C}[x_0, x_1, x_2]$ we must have that this sum

$$= b_1(x_0, x_1, x_2)[3x_0^2 - \lambda x_1 x_2] + b_2(x_0, x_1, x_2)[3x_1^2 - \lambda x_0 x_2] + b_3(x_0, x_1, x_2)[3x_2^2 - \lambda x_0 x_1]$$

$$\Rightarrow a_1 + a_2x_0 + a_3x_1 + a_4x_2 + [a_5 + b_3(x_0, x_2, x_2)]x_0x_1 + [a_6 + b_2(x_0, x_2, x_2)\lambda]x_0x_2$$

$$+ [a_7 + b_1(x_0, x_2, x_2)\lambda]x_1x_2 + a_8x_0x_1x_2 - b_1(x_0, x_2, x_2)3{x_0}^2 - b_2(x_0, x_2, x_2)3{x_1}^2$$

$$- b_3(x_0, x_2, x_2)3{x_2}^2 = 0$$

We immediately get that $a_1 = a_2 = a_3 = a_4 = 0$. Without loss of generality, we may assume:

$$\lambda b_3(x_0, x_1, x_2) = c_3(x_0, x_1, x_2) - a_5$$

$$\lambda b_2(x_0, x_1, x_2) = c_2(x_0, x_1, x_2) - a_6$$

$$\lambda b_1(x_0, x_1, x_2) = c_1(x_0, x_1, x_2) - a_7$$

where the $c_i(x_0, x_1, x_2)$ have no constant term. This results in the identity:

$$[c_3(x_0, x_2, x_2)]x_0x_1 + [c_2(x_0, x_2, x_2)]x_0x_2 + [c_1(x_0, x_2, x_2)]x_1x_2 + a_8x_0x_1x_2$$

$$-\frac{3}{\lambda}[c_1(x_0, x_2, x_2) - a_7]{x_0}^2 - \frac{3}{\lambda}[c_2(x_0, x_2, x_2 - a_6)]{x_1}^2 - \frac{3}{\lambda}[c_3(x_0, x_2, x_2 - a_5)]{x_2}^2 = 0$$

Since we assumed that the c_i had no constant terms, we get from the coefficients on degree 2 terms that $a_5 = a_6 = a_7 = 0$. From the second line of above, it is clear that the c_i need at least degree 2 to cancel with each other if they cannot have constant terms. Therefore it is impossible for the first line to cancel out if this is the case. This leaves us with the last equality $a_8 = 0$ This proves linear independence.

To show that this collection serves as a generating set over $\mathbb{C}$ in the quotient ideal, we note that $3x_0 - \lambda x_1x_2 \in I \Rightarrow 3{x_0}^2 + I = \lambda(x_1x_2) + I$ and so:

$${x_0}^2 + I = \frac{\lambda}{3}(x_1x_2 + I)$$

and similarly,

$$x_1{}^2 + I = \frac{\lambda}{3}(x_0x_2 + I)$$

$$x_2{}^2 + I = \frac{\lambda}{3}(x_0x_1 + I)$$

We get the third powers $x_i{}^3$ are generated over $\mathbb{C}$ by $x_0x_1x_2$ via relations of the form: $x_0{}^3 + I = (x_0 + I)(x_0{}^2 + I) = \frac{\lambda}{3}(x_0 + I)(x_1x_2 + I) = \frac{\lambda}{3}(x_0x_1x_2 + I)$ And leftover third degree monomial cosets of the form $x_0{}^2x_1 + I$ via relations like:

$$\begin{aligned} x_0{}^2x_1 + I = (x_0{}^2 + I)(x_1 + I) = \frac{\lambda}{3}(x_1x_2 + I)(x_1 + I) = \frac{\lambda}{3}x_1{}^2x_2 + I \\ = \frac{\lambda}{3}(x_1{}^2 + I)(x_2 + I) = \frac{\lambda^2}{9}(x_0x_2 + I)(x_2 + I) = \frac{\lambda^2}{9}x_2{}^2x_0 + I = \frac{\lambda^2}{9}(x_2{}^2 + I)(x_1 + I) \\ = \frac{\lambda^3}{27}(x_1x_0 + I)(x_1 + I) = \frac{\lambda^3}{27}(x_0{}^2x_1 + I) \\ \Rightarrow \left(1 - \frac{\lambda^3}{27}\right) x_0{}^2x_1 \in I \end{aligned}$$

Since $\lambda^3 \neq 27$, this implies $x_0{}^2x_1 \in I$. Additionally we may show powers of 4 are in I via:

$$x_0{}^4 + I = (x_0{}^2 + I)^2 = \left(\frac{\lambda}{3}(x_1x_2 + I)\right)^2 = (x_1{}^2x_2 + I)(x_2 + I) = I$$

Since $x_1{}^2x_2 \in I$ and so $x_0{}^4 \in I$ as well. All this together shows that for each monomial coset $x_0{}^ax_1{}^bx_2{}^c + I = I$ if a, bor $c \geq 4$, $a \geq 2, b$or $c \geq 1$, $b \geq 2, a$or $c \geq 1$, $c \geq 2, a$or $b \geq 1$, and the rest are generated over $\mathbb{C}$, as shown above.

Now we will use this basis to calculate the spectrum. The basis:

$$\{1 \ \ , x_0 \ \ , x_1 \ \ , x_2 \ \ , x_0x_1 \ \ , x_0x_2 \ \ , x_1x_2 \ \ , x_0x_1x_2\}$$

Corresponds to the set:

$$A = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1)\}$$

respectively. Since $f = x_0^3 + x_1^3 + x_2^3 - \lambda x_0 x_1 x_2$, the corresponding weight vector $w = (w_0, w_1, w_2)\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. We note that:

$$w(1,0,0) = w(1,0,1) = w(0,0,1) = \left[(1+1)\frac{1}{3}\right] + \left[(1+0)\frac{1}{3}\right] + \left[(0+1)\frac{1}{3}\right] - 1 = \frac{1}{3}$$

$$w(0,0,0) = \left[(0+1)\frac{1}{3}\right] + \left[(0+1)\frac{1}{3}\right] + \left[(0+1)\frac{1}{3}\right] - 1 = 0$$

$$w(1,1,0) = w(0,1,1) = w(1,0,1) = \left[(1+1)\frac{1}{3}\right] + \left[(1+1)\frac{1}{3}\right] + \left[(0+1)\frac{1}{3}\right] - 1 = \frac{2}{3}$$

$$w(1,1,1) = \left[(1+1)\frac{1}{3}\right] + \left[(1+1)\frac{1}{3}\right] + \left[(1+1)\frac{1}{3}\right] - 1 = 1$$

$$Sp(x_0^3 + x_1^3 + x_2^3 - \lambda x_0 x_1 x_2) = \sum_{\alpha \in A} w(\alpha) = [0] + 3\left[\frac{1}{3}\right] + 3\left[\frac{2}{3}\right] + [1]$$

As we can see, the manual calculations of spectra of even isolated quasihomogeneous hypersurface singularities can be quite laborious, even utilizing Steenbrink's algorithm. In practice, it is often more convenient to use the package [5], which utilizes an programmed algorithm designed by Mathias Schulze based on the work of Brieskorn, Gröebner bases, and other computational theories. This package is capable of calculating the singularity spectrum of any any isolated singularity over a complex polynomial ring given only the local defining equation.

2. Bounding Projective Hypersurface Singularities

2.1 Introduction

The spectrum developed by Steenbrink [6] has served as an invaluable tool in understanding the monodromy about complex singularities, while providing a powerful and easily computable invariant for isolated singularities. It is obtained from a natural mixed Hodge structure on the cohomology of the Milnor fiber of such a singularity, by combining information on the Hodge filtration and eigenvalues of the semisimple part of monodromy. The Saito-Steenbrink formula (conjectured in [3], proved in [7]) expands this theory by relating the spectrum of a *nonisolated* singularity to that of the isolated singularity obtained via its Yomdin deformation by a power of a linear form.

We arrive in this setting by taking the affine cone on a projective hypersurface with isolated singularities, and a separate formula from [3] provides the contributions of the isolated singularities to the spectrum of the (nonisolated) cone point. In the present paper, we show how to combine these results to bound the number of singularities of a particular type (or combination of types) that can be present on the hypersurface. We also reduce the computations of the method down to a bound similar to that of Varchenko [8], and conjecture that this bound is the same.

A projective hypersurface $X \subset \mathbb{P}^n$ of degree d will thus be the main object under discussion. In Section 2.2, we derive a purely combinatorial formula for the Hodge numbers of X. Then in Sections 2.3-2.6 we review several attempts to use Hodge-theoretic methods to bound the number of singularities of any particular deformation class on X in terms of only n and d. Section 2.3 concerns itself with the vanishing cycle sequence method, which utilizes only the exactness of the vanishing cycle sequence, and the properties of cohomological objects, summarized in Prop. 14. However, this method has some limitations, and produces no bound for A_1 singularities when n is even (i.e. when X has odd dimension).

In Section 2.4, we review Varchenko's bounding method [8], which was recently revisited in a beautiful expository article by van Straten [9]. In Section 2.5, we restate and give an initial generalization of the "conical method" worked out by Steenbrink and T. de Jong (cf. [3]) for bounding the number of double points on X, which duplicated Varchenko's bound in that case. The bound is obtained by combining a separate formula of Steenbrink (given here as Theorem 20) with his conjecture in the case (known to him) where the isolated singularities on X are of Pham-Brieskorn type. Steenbrink conjectures in [3] that a version of this latter formula can be proven in much higher generality, which Saito did in [7].

Siersma [2] proved a weaker version of Steenbrink's conjecture by focusing solely on the characteristic polynomials of the monodromy operators (thus dropping the information on the Hodge filtration contained in the spectrum). An attempt to bound the number of singularities based on Siersma's work is given in Section 2.6. Though sometimes better than the "naive" bound from Section 2.3, this bound is still relatively weak; on the other hand, Siersma's results allow us to compute the vertical monodromies required for the more general Saito-Steenbrink formula to work.

In section 2.7, we use this information to generalize the main results of Section 2.5 to arbitrary isolated hypersurface singularities, and we show that in this case the power of the general linear form in the Yomdin deformation of the cone can be chosen to be any $k > d$. This results in our Theorem 29, which we then make completely explicit in Theorems 33 and 36. We conclude the paper by showing that Varchenko's bound, while less discrete, implies Theorem 36, and conjecture that Theorem 36 is in fact equivalent to Varchenko's bound. In any case, it is more explicit and eliminates the guesswork in choosing the spectral interval on which to apply Varchenko.

The main result of this paper is given in Theorem 29 which proceeds as follows; let f be a homogeneous polynomial of degree d, with $X := \widetilde{V}(f) \subseteq \mathbb{P}^n$ having only finitely many isolated singularities P_i. We write $\sigma_{g_i} = \sum_{j=1}^{\mu_i}[\lambda_{i,j}]$ for their spectra, and set $\alpha_{i,j} = d\lambda_{i.j} - \lfloor d\lambda_{i,j} \rfloor$. Then for a suitably general linear form ℓ, and any $k > d$, $f_k = f + \epsilon\ell^k$ has an isolated singularity at zero, and

$$\sigma_{f_k,0} = \gamma_d^{*(n+1)} - \sum_{i,j}\left[\lambda_{ij} - \frac{\alpha_{ij}}{d}\right] * \beta_d + \sum_{i,j}\left[\lambda_{ij} - \frac{\alpha_{ij}}{k}\right] * \beta_k, \tag{2.1}$$

where $\gamma_d^{*(n+1)} = (\sum_{m=1}^{d-1}[-\frac{m}{d}])^{*(n+1)}$ is the spectrum of $\sum_{k=0}^{n} x_k^d$.

In particular, the effectiveness of (2.1) implies that of the right as an element of the free abelian group $\mathbb{Z}[\mathbb{Q}]$. This leads to our "conical bound", which is given in a reduced form in Theorem 36, for isolated singularities on X. Here are a few concrete examples:

- For $X \subset \mathbb{P}^3$ of degree d with only n_6, $\widetilde{E_6}$, n_7 $\widetilde{E_7}$, and n_8 $\widetilde{E_8}$ singularities, the sum $7n_6 + 8n_7 + 9n_8$ is bounded by the polynomial

$$\frac{277}{432}d^3 - \frac{23}{36}d^2 + \frac{53}{12}d + \frac{1}{2},$$

and in the case of only $\widetilde{E_6}$ singularities we have

$$n_6 \leq \frac{31}{378}d^3 - \frac{13}{126}d^2 + \frac{4}{7}d + \frac{1}{14}.$$

This compares favorably to the "naive" Hodge theoretic bound

$$n_6 \leq \frac{1}{9}d^3 - \frac{1}{3}d^2 + \frac{7}{18}d - \frac{1}{6}$$

resulting from the vanishing cycle sequence in Section 2.3.

- If $X \subseteq \mathbb{P}^4$ has degree d with only A_{2m+1} singularities, then the number of these is bounded by

$$r \leq \begin{cases} \frac{1}{2m+1}\left[\frac{115}{192}d^4 - \frac{115}{48}d^3 + \frac{185}{48}d^2 - \frac{35}{12}d + 1\right] & d \equiv 0 \mod 2 \\ \frac{1}{2m+1}\left[\frac{115}{192}d^4 - \frac{115}{48}d^3 + \frac{355}{96}d^2 - \frac{125}{48}d + \frac{45}{64}\right] & d \equiv 1 \mod 2, d > m+1 \end{cases}$$

This easily beats the naive bound, which is asymptotic to $\frac{11}{24m}d^4$.

- Many of the surface singularities "with $K3$ tail" classified in [10, §3], which include for instance the Dolgachev singularities, can occur on a quartic hypersurface in $\mathbb{P}^3$ (Singularities whose simplest form involves powers greater than 4 can have analytically equivalent forms where this is not the case.) The 3rd entry in [op. cit., Table 2], given by $x^2 + y^6 + z^6$, is not ruled out by the "naive" bound; but it is prohibited by the conical bound.

An appendix (Section **??**) contains tables providing additional comparisons of the various bounds.

2.2 A Formula for Hodge Numbers of Smooth Projective Hypersurfaces

In this section, we derive explicit formulas for the Hodge numbers $h^{i,j}(H^{n-1}(X))$ of a smooth projective hypersurface $X \subseteq \mathbb{P}^n$ of degree d in terms of n and d. This is of course very classical and included mainly for reference. Since these numbers are indepedent of X, we denote them by $h^{i,j}_{n,d}$. We will also write

$$[h^{i,j}_{n,d}]' = h^{i,j}_{n,d} - \delta_{i,j}$$

for the primitive Hodge numbers (denoted $h^{i,j}_0(X)$ in [11]). It is well known that $h^{i,j}_{n,d} = \delta_{i,j}$ (Kronecker delta) for $n-1 \neq i+j \leq 2n-2$. However, the middle Hodge numbers $h^{k,n-1-k}_{n,d}$ are much more complicated to calculate.

Theorem 13. *For a smooth hypersurface $X \subseteq \mathbb{P}^n$ of degree d, the middle primitive Hodge numbers $[h^{k,n-1-k}_{n,d}]'$ (where $k \leq \frac{n-1}{2}$) are given by:*

$$[h^{k,n-1-k}_{n,d}]' = (-1)^n \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} \binom{n-(k+1)d+(d-1)i}{n}$$

In particular, if $d > n$, then:

$$[h^{k,n-1-k}_{n,d}]' = \sum_{i=0}^{k} (-1)^i \binom{n+1}{i} \binom{(k+1)d-1-(d-1)i}{n}.$$

The proof makes use of the following

Lemma 1. *We let χ denote the Euler characteristic in the context defined in [11]. Let $X \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree d. Recall, $\Omega^p_X = \Omega^p_X(0)$ and $\Omega^0_X(i) = \mathcal{O}_X(i)$ by definition. Then:*

1. $\chi(\Omega^0_X(i)) = \binom{i+n}{n} - \binom{i+n-d}{n}$

2. $\chi(\Omega^k_{\mathbb{P}}(i)) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{k-j} \binom{n-k+i+j}{n}$

3. $P_d^k(i) = \chi(\Omega_{\mathbb{P}}^k(i)) - \chi(\Omega_{\mathbb{P}}^k(i-d)) = \sum_{j=0}^{k}(-1)^j\binom{n+1}{k-j}\left[\binom{n-k+i+j}{n} - \binom{n-k-d+i+j}{n}\right]$

4. $\chi(\Omega_X^k(i)) = P_d^k(i) - \chi(\Omega_X^{k-1}(i-d))$

5. $\chi(\Omega_X^k) = \sum_{m=0}^{k}(-1)^{k-m}\sum_{j=0}^{m}(-1)^j\binom{n+1}{m-j}\left[\binom{n-m-(k-m)d+j}{n} - \binom{n-m-(k-m+1)d+j}{n}\right]$

Proof: (1)-(4) of the lemma are just [11, Prop 17.3.2] stated in such a way that it is easier to follow the steps of the recurrence relation.

For the proof of (5), note that by (1),

$$\chi\left(\Omega_X^{k-k}(-kd)\right) = \chi\left(\Omega_X^0(-kd)\right) = \binom{-kd+n}{n} - \binom{-kd+n-d}{n}$$

And by (4):

$$\chi\left(\Omega_X^{k-(k-m)}(-(k-m)d\right) = P_d^{k-(k-m)}\left(-(k-m)d\right) - \chi\left(\Omega_X^{k-(k-m+1)}(-(k-m+1)d)\right)$$

These two facts together inductively give us:

$$\chi\left(\Omega_X^k\right) = \left[\sum_{m=1}^{k}(-1)^{k-m}P_d^m(-(k-m)d)\right] + (-1)^k\left[\binom{n-kd}{n} - \binom{n-(k+1)d}{n}\right]$$

And substituting (3) for $P_d^m(-(k-m)d)$, and noting that the second summand is just the case $m=0$, we get (5). □

Now we use the last part of our lemma and the relationship between $[h_{n,d}^{k,n-1-k}]'$ and $\chi(\Omega_X^k)$ to prove our formula.

Proof: (of Theorem) We know from [11, Lemma 17.3.1] that

$$[h_{n,d}^{k,n-1-k}]' = (-1)^{n-1-k}\chi(\Omega_X^k) + (-1)^n$$

Using (5) in the above lemma we easily get a sum for $(-1)^{n-1-k}\chi(\Omega_X^k)$, setting $m-j=i$, and noting that $\binom{n-i}{n} = 0$ for $1 \le i \le k$, we get:

$$[h_{n,d}^{k,n-1-k}]' = (-1)^n\sum_{i=0}^{k}\left[(-1)^i\binom{n+1}{i}\binom{n-(k+1)d+(d-1)i}{n}\right]$$

We know that for $q > 0$, $p < 0$ $\binom{p}{q} = (-1)^q\binom{q-p-1}{q}$. If $d > n$, Then $n-(k+1)d+(d-1)i = (n-d)-kd+(d-1)i \leq -1-kd+(d-1)k \leq -1 < 0$ for $0 \leq i \leq k$. Therefore:

$$[h_{n,d}^{k,n-1-k}]' = \sum_{i=0}^{k}(-1)^i\binom{n+1}{i}\binom{(k+1)d-1-(d-1)i}{n}.$$

□

2.3 The Vanishing Cycle Sequence Method

Let $\pi\colon \mathcal{X} \to \Delta$ be a family of projective hypersurfaces $X_t := \pi^{-1}(t) \subset \mathbb{P}^n$ over a disk about $t = 0$, with smooth total space. We assume that the fibers over $\Delta^* := \Delta \setminus \{0\}$ are smooth of degree d, and X_0 has only isolated singularities. We write $f \in \mathbb{C}[x_0, \ldots, x_n]$ for the homogeneous polynomial of degree d cutting out $X_0\,(= \widetilde{V}(f))$. In this setting, the vanishing cycle sequence is an exact sequence of mixed Hodge structures of the form

$$0 \longrightarrow H^{n-1}(X_0) \longrightarrow H^{n-1}_{lim}(X_t) \longrightarrow H^{n-1}_{van}(X_t) \xrightarrow{\delta} H^n(X_0) \longrightarrow H^n_{lim}(X_t) \longrightarrow 0.$$

(The mixed Hodge structures are induced by the nearby cycles triangle in the derived category of mixed Hodge modules on X_0, cf. [**?**].) In particular, when n is odd, $H^n_{lim}(X_t) = 0$ and the sequence simplifies to

$$0 \longrightarrow H^{n-1}(X_0) \longrightarrow H^{n-1}_{lim}(X_t) \longrightarrow H^{n-1}_{van}(X_t) \xrightarrow{\delta} H^n(X_0) \longrightarrow 0$$

We will use these sequences and the following facts to induce an inequality bounding the number and type of singularities of f.

Proposition 14. *(a) $H^{n-1}(X_t) \neq 0$ has Hodge numbers $h^{p,n-p-1}(H^{n-1}(X_t)) = h^{p,n-p-1}_{n,d}$.*

(b) *Suppose* X_0 *has isolated singularities* $p_1, \dots p_r$ *given locally by polynomials* $g_1, \dots, g_r$ *(in* n *variables), with Milnor fibers* Y_{g_i}*. Then we have an isomorphism* $H^{n-1}_{van}(X_t) \cong \bigoplus_i \tilde{H}^{n-1}(Y_{g_i})$*, where the MHSs on the reduced Milnor fiber cohomologies again come from Saito's theory.*

(c) δ *is a map of pure weight* n *and, as such, can only be nonzero on* (p, q) *parts for* $p + q = n$*.*

(d) *The Hodge numbers* $h^{p,q}_{lim}$ *of* $H^{n-1}_{lim}(X_t)$ *satisfy* $\sum_q h^{p,q}_{lim} = h^{p,n-p-1}_{n,d}$ *for each fixed* p*. Moreover, they are symmetric about the lines* $p = q$ *and* $p + q = n - 1$*.*

(e) *Suppose* U *is a complex algebraic variety of dimension* m*. Then the values of* (p, q) *for which the Hodge numbers* $h^{p,q}(H^k(U)) \neq 0$ *satisfy:*

   (a) $0 \leq p, q \leq k$*;*

   (b) *if* $k > m$ *then* $k - m \leq p, q \leq m$*;*

   (c) *if* U *is smooth then* $p + q \geq k$*; and*

   (d) *if* U *is compact then* $p + q \leq k$*.*

(f) *With* X_0 *the singular fiber above, if* $n = 2m + 1$ *is odd, we have* $h^{m,m}(H^{2m}(X_0)) \geq 1$

Proof:

(a) This follows from the fact that $X_t \subset \mathbb{P}^n$ is a smooth hypersurface of degree d and complex dimension $n - 1$, and therefore has a pure Hodge decomposition.

(b) This follows from [12, Theorem 5.44].

(c) See [13, Prop. 5.5].

(d) By [12, p263,285] $\dim F^m H^k(X_t) = \dim F^m H^k_{lim}(X_t)$ for any $k, m \in \mathbb{N}$, where Steenbrink denotes $H^k_{lim}(X_t) = H^k(X_\infty)$. So $\sum_q h^{p,q}_{lim} = \dim(\mathrm{Gr}^p_F H^{n-1}_{lim}(X_t))$ must equal $\dim(\mathrm{Gr}^p_F H^{n-1}(X_t)) = h^{p,n-p-1}_{n,d}$.

(e) This is just [12, Theorem 5.39]

(f) Let $X_0 \subset \mathbb{P}^{2m+1}$, and let $\widetilde{X_0}$ denote a resolution of singularities: pictorially,

$$\begin{array}{ccc} X_0 & \overset{\imath}{\hookrightarrow} & \mathbb{P}^{2m+1} \\ {\scriptstyle \pi}\uparrow & \nearrow_{\tilde{\imath}} & \\ \widetilde{X_0} & & \end{array}$$

where $\imath$ and π are the usual inclusion and projection maps. This produces a commutative diagram of MHSs

$$\begin{array}{ccccc} H^k(\mathbb{P}^{2m+1}) & \xrightarrow{\imath^*} & H^k(X_0) & \xrightarrow{\quad\imath_*\quad} & H^{k+2}(\mathbb{P}^{2m+1})(1) \\ & & {\scriptstyle \pi^*}\searrow & & \nearrow{\scriptstyle \tilde{\imath}_*} \\ & & & H^k(\widetilde{X_0}) & \end{array}$$

where $\tilde{\imath}_*$ is the Gysin map and $\imath_*$ was defined by the composition $\tilde{\imath}_* \circ \pi^*$. However, the map $\imath_* \circ \imath^*$ is really just the Lefschetz operator $L^k : H^k(\mathbb{P}^{2m+1}) \to H^{k+2}(\mathbb{P}^{2m+1})$ which is an isomorphism (given by cupping with a hyperplane class $[H]$) for all $0 \leq k \leq 4m$ by Hard Lefschetz. This implies that $\imath_*$ is surjective. Taking $k = 2m$, its image has type (m, m), and so $(H^2m(X_0))^{m,m}$ cannot be zero.

□

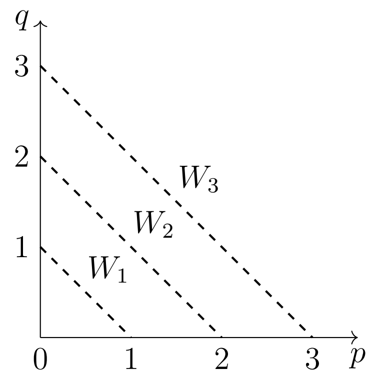
Proposition 15. *Let $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be homogeneous polynomial of degree $d > 3$, and let $\widetilde{V}(f) \subseteq \mathbb{P}^3$ have only isolated $\widetilde{E_6}$ singularities. Then the number r of singular points is bounded by*

$$r \leq \frac{1}{6}(h^{1,1}_{3,d} - 1) = \frac{1}{6}\left[\binom{2d-1}{3} - 4\binom{d}{3}\right] = \frac{1}{9}d^3 - \frac{1}{3}d^2 + \frac{7}{18}d - \frac{1}{6}.$$

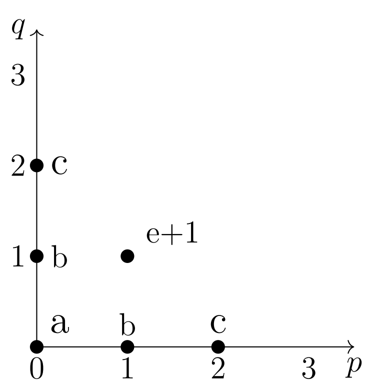
Proof: Since n=3 is odd, we have the following exact sequence of mixed Hodge structures:

$$0 \longrightarrow H^2(X_0) \longrightarrow H^2_{lim}(X_t) \longrightarrow H^2_{van}(X_t) \xrightarrow{\delta} H^3(X_0) \longrightarrow 0$$

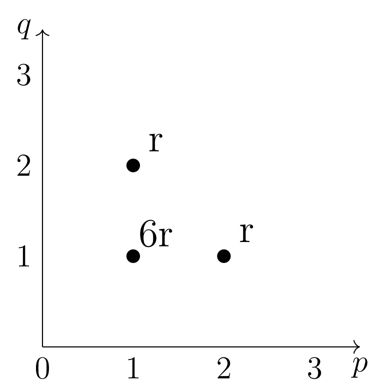
We may visualize the exact sequence in terms of the mixed Hodge numbers using the Hodge-Deligne diagrams, where W_k denotes the vector subspaces with weight k:



Since X_0 is a compact variety of dimension 2, we know that $H^2(X_0)$ has the form below by Prop. 14[(e),(f)]. Additionally, $H^2_{van}(X_t)$ has Hodge numbers given by r times the Hodge numbers of $H^2(Y_g)$, where $g : \mathbb{C}^3 \to \mathbb{C}$ has a single $\widetilde{E_6}$ singularity. These are calculated explicitly in [4] as $h^{1,2}(H^2(Y_g)) = h^{2,1}(H^2(Y_g)) = 1$ and $h^{1,1}(H^2(Y_g)) = 6$, and 0 for the rest, giving the diagram below:

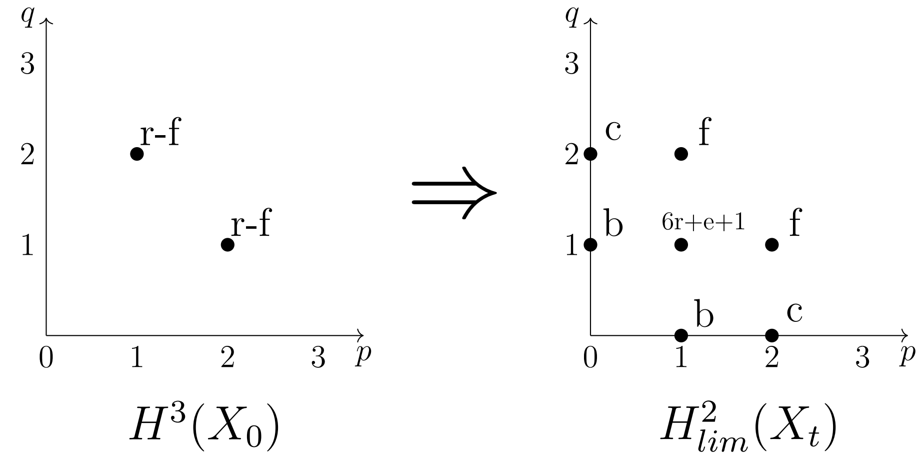


$H^2(X_0)$



$H^2_{van}(X_t)$

The exactness of the above sequence, and the fact that δ has weight 3 then forces the following two forms of the other two diagrams (in order):



Now we may deduce from Prop. 14[(a),(d)] that:

$$c + f = h^{2,0}_{3,d}$$

$$6r + e + b + f + 1 = h^{1,1}_{3,d}$$

$$c + b = h^{0,2}_{3,d} = h^{2,0}_{3,d}$$

So $b = f$ and $6r + 2b + e + 1 = h^{1,1}_{3,d} \Rightarrow r \leq \frac{1}{6}(h^{1,1}_{3,d} - 1)$ $\square$

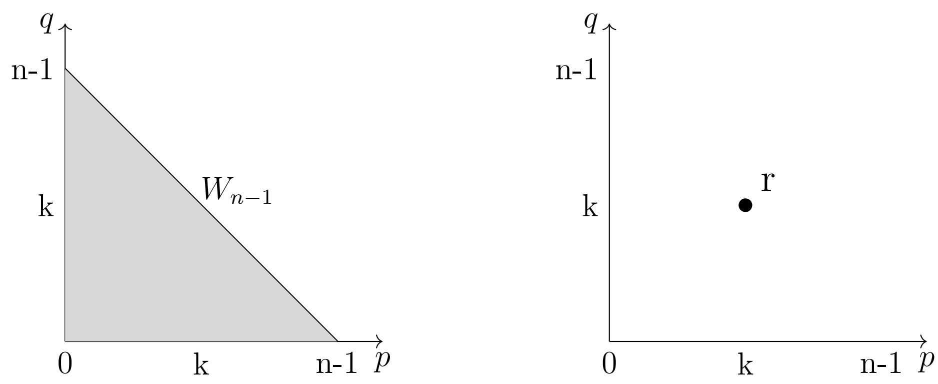
Proposition 16. *Let $f \in \mathbb{C}[x_0, x_1, \ldots x_n]$ be homogeneous polynomial of degree $d > 2$, let $n = 2k+1$, and let $\widetilde{V}(f) \subseteq \mathbb{P}^n$ have only isolated A_1 singularities. Then the number r of singular points is bounded by*

$$r \leq h^{k,k}_{n,d} - 1.$$

Proof: Again we have the following exact sequence of mixed Hodge Structures

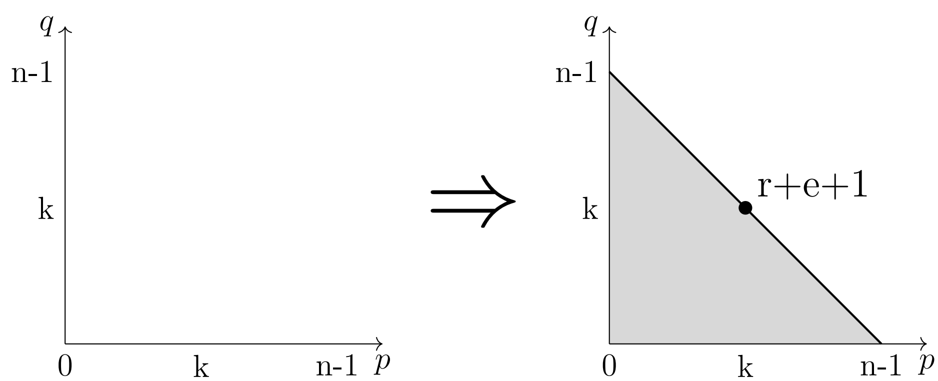
$$0 \longrightarrow H^{n-1}(X_0) \longrightarrow H^{n-1}_{lim}(X_t) \longrightarrow H^{n-1}_{van}(X_t) \xrightarrow{\delta} H^n(X_0) \longrightarrow 0$$

Since $n-1$ is even, we will denote $k = \frac{n-1}{2}$. Since X_0 is a compact variety of dimension $n-1$, we know that $H^{n-1}(X_0)$ has the form below by Prop. 14(e). Additionally, $H^{2k}_{van}(X_t)$ has Hodge numbers given by r times the Hodge numbers of $H^{n-1}(Y_g)$, where $g : \mathbb{C}^n \to \mathbb{C}$ has a single A_1 singularity. These are calculated explicitly using the formula from [4] as $h^{k,k}(H^{2k}(Y_g)) = 1$, and 0 for the rest. giving the diagram below:



$H^{n-1}(X_0)$ $\quad$ $H^{n-1}_{van}(X_t)$

The exactness of the above sequence, and the fact that δ has weight n then forces the following two forms of the other two diagrams (in order):



$H^{n}(X_0)$ $\quad$ $H^{n-1}_{lim}(X_t)$

Now we may deduce from Prop. 14[(a),(d)] that:

$$h_{lim}^{n-1,0} = h_{n,d}^{n-1,0}$$

$$r + e + 1 + \sum_{q \neq k} h_{lim}^{k,q} = h_{n,d}^{k,k}$$

The second equation shows that $r \leq h_{n,d}^{k,k} - 1$, as desired. □

2.4 Varchenko's Bound

Varchenko was the first to integrate the concept of the singularity spectrum in attempt to bound the number of singularities of a projective hypersurface. His original proof can be found in [8], and a further discussion of the proof can be found in [9]. The conical bounding method is able to duplicate these bounds in the case of ordinary double points by means of more advanced properties of the spectrum. A discussion of this process is given in the next section.

We use the convention of the Steenbrink spectrum (denoted σ, and briefly recalled after the theorem below). For notational sake, let $\{\sigma\}$ denote the set of spectral summands *with multiplicity*. That is, for the spectrum $\sigma = \left[\frac{1}{3}\right] + 2\left[\frac{1}{2}\right]$ we would have $\{\sigma\} = \left\{\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right\}$. For any subset $S \subseteq \mathbb{R}$ and spectrum σ, let $S \cap^{\#} \{\sigma\}$ count the number of times the sets intersect (for example if $S = \left\{\frac{1}{2}\right\}$ and σ is the one above $S \cap^{\#} \{\sigma\} = 2$). Varchenko's bound can be summarized as follows.

Theorem 17 ([8,9])**.** *Let $Z \subseteq \mathbb{P}^n$ be a hypersurface of degree d, with only isolated singular points $P_1, \ldots, P_r$. Let $g_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, for $1 \leq i \leq r$ denote the corresponding germs defined locally about P_i. Then for each $\alpha \in \mathbb{R}$, one has:*

$$(\alpha, \alpha+1) \cap^{\#} \{\sigma_{x_1^d+\ldots+x_n^d,0}\} \geq \sum_{i=1}^{r} (\alpha, \alpha+1) \cap^{\#} \{\sigma_{g_i,0}\}$$

Since $\sigma_{x_1^d+\ldots+x_n^d,0} = \gamma_d^{*n}$ *where* $\gamma_d := \sum_{i=1}^{d-1} \left[-\frac{i}{d}\right]$ *this can be restated as:*

$$(\alpha, \alpha+1) \cap^{\#} \{\gamma_d^{*n}\} \geq \sum_{i=1}^{r} (\alpha, \alpha+1) \cap^{\#} \{\sigma_{g_i,0}\}$$

Here we recall that $\sigma_{g,0} = \sum_{q \in \mathbb{Q}} m_q[q] \in \mathbb{Z}[\mathbb{Q}]$ means that

$$m_q = \dim\{\mathrm{Gr}_F^{\lfloor n-q-1 \rfloor}(\tilde{H}^{n-1}(Y_{g,0}))_{e^{-2\pi i q}}\},$$

where the subscript denotes the eigenvalue of the semisimple part T^{ss} of the monodromy operator T. The star notation is defined by $[q] * [q'] = [q + q' + 1]$ on generators, and the spectrum of $\sum_{k=1}^{n} x_k^{d_k}$ is given by $\gamma_{d_1} * \cdots * \gamma_{d_n}$.

Example 18. We will consider the case when $Z \subseteq \mathbb{P}^n$ is a hypersurface of degree d, with only ordinary double points. Then all g_i have the form $g_i = x_1^2 + \ldots + x_n^2$ up to analytic equivalence and spectra $\sigma_{g_i,0} = \left[\frac{n}{2} - 1\right]$. Therefore any choice of $\alpha \in \left(\frac{n}{2} - 2, \frac{n}{2} - 1\right)$ will yield a bound on the number of singularities r possible by the above theorem.

If $n = 3$, then $\sigma_{g_i,0} = \left[\frac{1}{2}\right]$ for all $1 \leq i \leq r$. The Appendix details the spectra γ_d^{*3} of $x_1^d + x_2^d + x_3^d$ (note: in the appendix this will correspond to $n = 2$). It becomes clear from these spectra that our lowest bound will be obtained by choosing $\alpha = -\frac{1}{2} + \frac{1}{2d}$ for d even and $\alpha = \frac{\lfloor \frac{d}{2} \rfloor + 1}{d} - 1$ for general d.

Let α be chosen in this way. Then, in the notation of the theorem, the right hand side of the inequality becomes

$$r\left((\alpha, \alpha+1) \cap^{\#} \{\sigma_{g_i,0}\}\right) = \# \bigcup_{i=1}^{r} \left\{\frac{1}{2}\right\} = r,$$

where the left hand side will be the number of summands in γ_d^{*3} which fall in the interval $(\alpha, \alpha+1))$ inspected from the chart.

For $d = 2$ we get $(\alpha, \alpha+1) = \left(-\frac{1}{4}, \frac{3}{4}\right) \Rightarrow r \leq 1$. For $d = 3$ we get $(\alpha, \alpha+1) = \left(-\frac{1}{3}, \frac{2}{3}\right) \Rightarrow r \leq 1+3 = 4$. For $d = 7$, $r \leq 6+10+15+21+25+27 = 104$. In fact it is the case that the bounds match up with those of the next section, at least in the case of A_1 singularities.

2.5 Conical Bounding Method for Pham-Brieskorn

Throughout this section, $f \in \mathbb{C}[x_0, \ldots, x_n]$ will denote a homogeneous polynomial of degree d. We write as above $\widetilde{V}(f) \subset \mathbb{P}^n$ for the projective hypersurface it defines, and $V(f) \subset \mathbb{C}^{n+1}$ for the affine variety it defines (which is the just the cone on $\widetilde{V}(f)$). If $\widetilde{V}(f)$ has isolated singularities, then the singularity locus $\Sigma \subset V(f)$ has dimension one and consists of lines through the origin. In the neighborhood of an isolated singularity $P \in \widetilde{V}(f)$, we can represent $\widetilde{V}(f)$ in local analytic coordinates (on $\mathbb{P}^n$, about P) by the vanishing of a polynomial $g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. (That is, we are only interested in the intersection of $g = 0$ with a small ball about the origin.)

Regarding the definition of the spectrum for a *nonisolated* singularity p (with local affine equation $F = 0$ and Milnor fiber Y) on a hypersurface of dimension n, the $\tilde{H}^k(Y)$ can be nonvanishing for $n - \dim(\Sigma) \leq k \leq n$ (where Σ is the singularity locus). Accordingly, we define the spectrum as an alternating sum $\sigma_{F,p} := \sum_{j\geq 0}(-1)^j \sigma_{F,p}^{n-j}$ where $\sigma_{F,p}^k$ is derived from the MHS and T^{ss}-action on $\tilde{H}^k(Y)$, see [3]. The main point is that for the cone singularity at the origin of $V(f)$, this takes the form $\sigma_{f,0} = \sigma_{f,0}^n - \sigma_{f,0}^{n-1}$, which may not be effective. In order to circumvent this problem, we relate $\sigma_{f,0}$ to the (effective) spectra of *isolated* singularities in two different ways, given by the next two theorems.

Theorem 19 ([3, Theorem 6.3])**.** *Assume that $\widetilde{V}(f) \subseteq \mathbb{P}^n$ has only isolated singularities, $P_1, \ldots P_r$. Let each germ $g_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be analytically equivalent to a Pham-Brieskorn polynomial (i.e. a polynomial of the form $\sum_{j=1}^{m} {x_j}^{a_j}$, $a_j \in \mathbb{N}$). Let μ_i denote the Milnor number of g_i, and let the values λ_{ij} be defined from the spectra by $\sigma_{g_i,0} = \sum_{j=1}^{\mu_i}[\lambda_{ij}]$. Then there exists a sufficiently general linear form ℓ and sufficiently large $k \in \mathbb{N}$ such that $f_k = f + \epsilon \ell^k$ has an isolated singularity at 0 for $\epsilon \neq 0$ sufficiently small, and*

$$\sigma_{f_k,0} = \sigma_{f,0} + \sum_{i,j} \left[\lambda_{ij} - \frac{\alpha_{ij}}{k}\right] * \beta_k$$

with $\beta_m = \sum_{i=0}^{m-1} \left[\frac{-i}{m}\right]$ and $\alpha_{ij} = d\lambda_{ij} - \lfloor d\lambda_{ij} \rfloor$.

The f_k is called a *Yomdin deformation*. Note that it is *not* necessarily homogeneous.

Theorem 20 ([3, Theorem 6.1])**.** *With the same notation as in Theorem 19, but dropping the Pham-Brieskorn assumption on the isolated singularities, we have*

$$\sigma_{f,0} = \sigma_{h,0} - \sum_{i,j} \left[\lambda_{ij} - \frac{\alpha_{ij}}{d}\right] * \beta_d,$$

where h is a homogeneous polynomial of degree d such that $V(h)$ has an isolated singularity at 0.

Before continuing we record the following facts:

Lemma 2. *(a) The Milnor fiber of an m-dimensional isolated hypersurface singularity is $(m-1)$-connected, so the spectrum of any germ defined locally about this singularity is effective (i.e. all of its summands' coefficients are nonnegative).*

(b) *Let $h \in \mathbb{C}[x_0, \ldots, x_n]$ be homogeneous polynomial of degree d and let h have an isolated singularity at 0. Then $\sigma_{h,0} = \gamma_d^{*(n+1)}$.*

We note that in [3], it was implicitly assumed that $k = d + 1$ is a sufficiently high power of the general linear form ℓ in the context of Theorem 4.1. We later prove the more general Lemma 6.2 using the work of [13], verifying this assumption in greater generality. With this in mind, we arrive at the following bounding argument which expands on an idea of Steenbrink and T. de Jong in the case of A_1 singularities.

Theorem 21 (Conical bounding method)**.** *Assume that $\widetilde{V}(f) \subseteq \mathbb{P}^n$ has only r isolated singularities of a single isomorphism class, describable in local coordinates by a Pham-Brieskorn polynomial g with $\sigma_{g,0} = \sum_j [\lambda_j]$. Define $\alpha_j = d\lambda_j - \lfloor d\lambda_j \rfloor$. Then we have*

$$\sigma_{f_{d+1},0} = \gamma_d^{*(n+1)} - r\left(\sum_j \left[\lambda_j - \frac{\alpha_j}{d}\right] * \beta_d - \sum_j \left[\lambda_j - \frac{\alpha_j}{d+1}\right] * \beta_{d+1}\right),$$

and the effectiveness of the spectrum on the left-hand side bounds the number r.

Proof: By Theorem 19, and the assumption that k=d+1 is sufficient in this case, $f + \epsilon\ell^{d+1}$ has an isolated singularity at the origin, and by Lemma 2(a), the spectrum of $f + \epsilon\ell^{d+1}$ at the origin is effective. By Lemma 2(b), Theorem 19, and Theorem 20 we get

$$\sigma_{f_{d+1},0} - \textstyle\sum_{i,j}[\lambda_{ij} - \frac{\alpha_{ij}}{d+1}] * \beta_{d+1} = \sigma_{f,0} = \gamma_d^{*(n+1)} - \sum_{i,j}[\lambda_{ij} - \frac{\alpha_{ij}}{d}] * \beta_d$$

and thus the desired formula for the spectrum of the Yomdin deformation. □

Corollary 22 (given in [3])**.** *Assume $\widetilde{V}(f) \subseteq \mathbb{P}^n$ has only r isolated A_1 singularities (ordinary double points). Then*

$$r \leq \begin{cases} \text{the coefficient of } \left[\frac{n}{2} - 1 + \frac{1}{d}\right] \text{ in } \gamma_d^{*(n+1)}, & nd \text{ even} \\ \text{the coefficient of } \left[\frac{n}{2} - 1 + \frac{1}{2d}\right] \text{ in } \gamma_d^{*(n+1)}, & nd \text{ odd.} \end{cases}$$

Proof: The local normal form of each ordinary double point is given by germs g_i : $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ given by $z_0^2 + \ldots + z_{n-1}^2$. And so $\sigma_{g_i,0} = \sigma_{g_1,0} = \left[\frac{n}{2} - 1\right]$ for $1 \le i \le r$. This yields:

$$\sigma_{f_{d+1},0} = \gamma_d^{*(n+1)} - r\left(\left[\frac{n}{2} - 1 - \frac{\alpha_{11}}{d}\right] * \beta_d - \left[\frac{n}{2} - 1 - \frac{\alpha_{11}}{d+1}\right] * \beta_{d+1}\right)$$

If nd is even then $\alpha_{11} = d\left(\frac{n}{2} - 1\right) - \left\lfloor d\left(\frac{n}{2} - 1\right)\right\rfloor = 0$ and

$$\sigma_{f_{d+1},0} = \gamma_d^{*(n+1)} - r\left(\left[\frac{n}{2} - 1\right] * \left[\beta_d - \beta_{d+1}\right]\right).$$

As one can calculate, the coefficient of $\left[-\frac{d-1}{d}\right] = \left[\frac{1}{d} - 1\right]$ on the right side side must be r. By the effectiveness of the left hand side we have:

$$r \le \text{the coefficient of } \left[\frac{n}{2} - 1 + \frac{1}{d}\right] \text{ in } \gamma_d^{*(n+1)}$$

If nd is odd then $\alpha_{11} = d\left(\frac{n}{2} - 1\right) - \left\lfloor d\left(\frac{n}{2} - 1\right)\right\rfloor = -\frac{1}{2}$ and

$$\sigma_{f_{d+1},0} = \gamma_d^{*(n+1)} - r\left(\left[\frac{n}{2} - 1 - \frac{1}{2d}\right] * \beta_d - \left[\frac{n}{2} - 1 - \frac{1}{2(d+1)}\right] * \beta_{d+1}\right)$$

The coefficient of $\left[\frac{n}{2} - 1 + \frac{1}{2d}\right]$ in $\left[\frac{n}{2} - 1 - \frac{1}{2d}\right] * \beta_d$ is 1 and the coefficient of $\left[\frac{n}{2} - 1 + \frac{1}{2d}\right]$ in $\left[\frac{n}{2} - 1 - \frac{1}{2(d+1)}\right] * \beta_{d+1}$ is 0

Therefore the coefficient on $\left[\frac{n}{2} - 1 + \frac{1}{2d}\right]$ in

$$r\left(\left[\frac{n}{2} - 1 - \frac{1}{2d}\right] * \beta_d - \left[\frac{n}{2} - 1 - \frac{1}{2(d+1)}\right] * \beta_{d+1}\right)$$

must be r. By the effectiveness of the left hand side we have:

$$r \le \text{the coefficient of } \left[\frac{n}{2} - 1 + \frac{1}{2d}\right] \text{ in } \gamma_d^{*(n+1)}.$$

□

Before continuing, we offer one more application of Theorem 21.

Example 23. Let $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be homogeneous polynomial of degree $d > 3$, and let $\widetilde{V}(f) \subseteq \mathbb{P}^3$ have only isolated $\widetilde{E_6}$ singularities which have normal form $x^3 + y^3 + z^3$. Then it can be shown that the number r of singular points is bounded by

$$r \leq \frac{\text{the coefficient of } \left[1 + \frac{1}{d}\right] \text{ in } \gamma_d^{*4}}{7}$$

in a similar fashion to the proof above with the cases $d \equiv p \mod 3$.

We note that while Theorem 21 technically implies the following better bound for larger d, it cannot be shown without laborious arithmetic, or made apparent without computational facts illustrated later in this paper. If $p = \lfloor \frac{2d}{3} \rfloor + 1$, then

$$r \leq \frac{\text{the coefficient of } \left[\frac{p}{d}\right] \text{ in } \gamma_d^{*4}}{7}.$$

2.6 Eigenvalue Bounding Method

The statements of Section 2.4 might cause one to wonder whether the bound has more to do with the Hodge filtration information in the spectrum of f_k or the multiplicity of eigenvalues of T^{ss}, the semisimple part of monodromy. In this section, we attempt to bound the number of singularities of $\widetilde{V}(f)$ using only the eigenvalue multiplicities in the Milnor fiber cohomology of f_k. We find that while this method produces a decent bound for small values of the degree d, as d increases the bound on the number of singularities is often not nearly as sharp as the methods of Section 2.4. However, we already have the characteristic polynomials of the monodromy for f_k for a much larger group of polynomials f provided in [2].

Let $F \in \mathbb{C}[x_0, \ldots, x_n]$ have at most a single isolated singularity at 0. If Y_F denotes the Milnor fiber of F, then we know that the reduced homology group $\widetilde{H}_k(Y_F)$ can only be nonzero for $k = n$. We will denote by $M[F](\lambda)$ the characteristic polynomial of the algebraic monodromy acting on $\widetilde{H}_n(Y_F)$. In [14], Milnor gives $M[F](\lambda)$ for $F = \sum_{i=0}^{n} z_i^d$. Since any homogeneous polynomial of degree d in $n+1$ variables with only an isolated singularity at 0 is a μ-constant deformation of such a polynomial we have:

Proposition 24. *Let $F \in \mathbb{C}[x_0, \ldots x_n]$ be homogeneous of degree d with only an isolated singularity at 0. Then $M[F](\lambda)$ is given by:*

$$M[F](\lambda) = \begin{cases} (\lambda - 1)^{-1}(\lambda^d - 1)^{\frac{1+(d-1)^{n+1}}{d}} & n \text{ even} \\ (\lambda - 1)(\lambda^d - 1)^{\frac{(d-1)^{n+1}-1}{d}} & n \text{ odd} \end{cases}$$

And since these polynomials are dependent only on n and d, we denote this polynomial by $M_{n,d}^{reg}(\lambda)$.

In [2] this polynomial is denoted by $M_d^{reg}(\lambda)$ since n was assumed to be fixed. One may also verify that the spectrum of such a polynomial $\gamma_d^{*(n+1)}$, stated above and in [3], is consistent with this characteristic polynomial.

Before we proceed with a summary of [2], we make a quick note that Siersma insists that his version of a general linear form ℓ must be *admissible*, that is $\{\ell = 0\} \cap f^{-1}(0)$ has an isolated singularity. However, a close reading of the proof of Lemma 4 detailed later in this paper tells us that our pick of $\ell = y_0$ after a coordinate change will be admissible.

Lemma 3. *Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be homogeneous polynomial of degree $d > 2$ and let $\widetilde{V}(f) \subseteq \mathbb{P}^n$ have only isolated singularities. Then there is a suitable coordinate transformation such that $f(y_0, \ldots, y_n)$ is a homogeneous polynomial of degree d such that y_0 is an admissible linear form. Furthermore there exists an $\varepsilon > 0$ such $f + \varepsilon y_0^k$ has an isolated singularity at $\underline{0}$ for all $k > d$.*

Proof: In the notations of the proof of 4, $f^{-1}(0) \cap \mathrm{sing}(f) \cap \{y_0 = 0\} \subseteq \mathrm{sing}(\pi) \cap \{y_0 = 0\} = \underline{0}$. Therefore $\{y_0 = 0\} \cap f^{-1}(0)$ can have at most an isolated singularity at $\underline{0}$. Since f is homogeneous in our case, for $d > 2$, f must have a singularity at $\{\underline{0}\}$. □

The following is stated in [2, p195]:

Theorem 25. *[2] Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be homogeneous polynomial of degree d such that $\widetilde{V}(f) \subseteq \mathbb{P}^n$ has only isolated singularities, $P_1, \ldots P_r$. Let each germ $g_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be defined locally about P_i, and let μ_i and T_i denote its Milnor number and algebraic monodromy operator on $H_{n-1}(X_{g_i})$ respectively. Then for our choices of ε and an admissible linear form ℓ, we have*

1. $M[f + \varepsilon \ell^d](\lambda) = M^{reg}_{n,d}(\lambda)$

2. *For $k > d$, we have*

$$M[f + \varepsilon \ell^k](\lambda) = \frac{M^{reg}_{n,d}(\lambda)}{(\lambda^d - 1)^{\sum \mu_i}} \cdot \prod_{i=1}^{r} \det(\lambda^k I - T_i^{k-d}).$$

This leads to the following bounding argument which mimics the logic of the conical bounding method above.

Corollary 26. *With the assumptions and notation of Theorem 25, the number of singularities is bounded by the following relation:*

$$(\lambda^d - 1)^{\sum \mu_i} \Big| M^{reg}_{n,d}(\lambda) \cdot \prod_{i=1}^{r} \det(\lambda^{d+1} I - T_i)$$

In particular, if each g_i has the same singularity type, then:

$$(\lambda^d - 1)^{r\mu_1} \Big| M^{reg}_{n,d}(\lambda) \cdot [\det(\lambda^{d+1} I - T_1)]^r$$

Proof: $f + \varepsilon \ell^{d+1}$ has an isolated singularity at $\underline{0}$ by Lemma 3. Therefore $M[f + \varepsilon \ell^k](\lambda)$ must not have any poles (as it must be a polynomial). This implies the denominator of (2) in Theorem 25 must divide the numerator. $\square$

We look at this bound in action with a case we already know from above.

Proposition 27. *Let $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be homogeneous polynomial of degree $d > 3$, and let $\widetilde{V}(f) \subseteq \mathbb{P}^3$ have only isolated $\widetilde{E_6}$ singularities. Then the number r of singular points is bounded by*

$$r \leq \frac{(d-1)^4 - 1}{8d} = \frac{1}{8}d^3 - \frac{1}{2}d^2 + \frac{3}{4}d - \frac{1}{2}$$

Proof: In this case, $n = 3$ is odd, so

$$M^{reg}_{3,d} = (\lambda - 1)(\lambda^d - 1)^{\frac{(d-1)^4 - 1}{d}}$$

Each $\widetilde{E_6}$ singularity corresponds to $\mu_1 = 8$, and $\det(\lambda I - T_1) = \frac{(\lambda^3 - 1)^3}{(\lambda - 1)}$ Which can both be inferred from the spectrum. Therefore our bounding argument gives us:

$$(\lambda^d - 1)^{8r} \Big| (\lambda - 1)(\lambda^d - 1)^{\frac{(d-1)^4 - 1}{d}} \cdot \frac{(\lambda^{3(d+1)} - 1)^{3r}}{(\lambda - 1)^r}$$

We will count the multiplicity of the eigenvalue $e^{\frac{d-1}{d}\cdot 2\pi i}$ on each side. We note that this is not a root of $\lambda^{3(d+1)}-1)^{3r}$ since this would imply $\frac{3(d+1)(d-1)}{d}$ is an integer $\Rightarrow d|3(d^2-1) = 3d^2-3 \Rightarrow d|-3$ which is not possible since $d > 3$. Therefore the multiplicity on the left side is $8r$ and the right side is $\frac{(d-1)^4-1}{d}$. This implies $r \leq \frac{(d-1)^4-1}{8d}$. □

For A_1 singularities, the eigenvalue method gives the following bound:

Proposition 28. *Let $f \in \mathbb{C}[x_0, x_1, x_2, \ldots x_n]$ be a homogeneous polynomial of degree $d > 2$, and let $\widetilde{V}(f) \subseteq \mathbb{P}^n$ have only isolated A_1 singularities. Then the number r of singular points is bounded by*

$$r \leq \begin{cases} \frac{1+(d-1)^{n+1}}{d} & n \text{ even} \\ \frac{(d-1)^{n+1}-1}{d} & n \text{ odd.} \end{cases}$$

Proof: Each local A_1 singularity corresponds to the spectrum $[\frac{n}{2}-1]$, hence to the eigenvalue $(-1)^n$; so the characteristic polynomial of each T_i is $(\lambda-(-1)^n)$. By our corollary this implies:

$$(\lambda^d-1)^r \big| M_{n,d}^{reg}(\lambda) \cdot (\lambda^{d+1}-(-1)^n)$$

Picking out the root $e^{\frac{2\pi i}{d}}$ and counting the multiplicities on each side, we deduce that $r \leq \frac{1}{d}\left((d-1)^{n+1}-(-1)^{n+1}\right)$. □

2.7 A generalization of the conical bound

As we have seen from the last two sections, it is necessary that we consider the spectrum and not just the characteristic polynomial of T^{ss} on $H^n(Y_{f_k})$ to get the sharpest bound on the number of singularities. This has to do with the fact that the Hodge filtration further sorts the eigenvalues of the monodromy, resulting in smaller multiplicities.

The isolated singularities P_i turn out to contribute to the spectrum of the Yomdin deformation in a subtle way, which involves "pairing" the action of both horizontal and vertical monodromies on $H^{n-1}(Y_{g_i})$. Here "horizontal" means that t goes about the origin of the disk, while "vertical" means to go about the cone point on the ith component of Σ.

Fortunately, Saito and Siersma have left us with the necessary tools to generalize Theorem 19 in such a way that we can generalize the conical bounding process. We will start off by giving Steenbrink's Conjecture from [3], which was later proven by Saito in the vast generality of mixed Hodge modules in [7], and was later specified in more detail in a context closer to our own in [10, Thm. 7.5]. We further contextualize this theorem in the case of homogeneous polynomials.

Theorem 29 (SS Formula for Homogeneous Cone Case)**.** *Let $f \in \mathbb{C}[x_0, \dots, x_n]$ be homogeneous polynomial of degree d such that $\widetilde{V}(f) \subseteq \mathbb{P}^n$ has only isolated singularities, $P_1, \dots P_r$. Let each germ $g_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be defined locally about P_i, μ_i denote the Milnor number of g_i, and write $\sigma_{g_i,0} = \sum_{j=1}^{\mu_i} [\lambda_{ij}]$. Put $\alpha_{ij} = d\lambda_{ij} - \lfloor d\lambda_{ij} \rfloor$. Then for a sufficiently general linear form ℓ and small $\varepsilon \neq 0$, $f_k = f + \varepsilon \ell^k$ has an isolated singularity at 0 and*

$$\sigma_{f_k,0} = \gamma_d^{*(n+1)} - \sum_{i,j} \left[\lambda_{ij} - \frac{\alpha_{ij}}{d} \right] * \beta_d + \sum_{i,j} \left[\lambda_{ij} - \frac{\alpha_{ij}}{k} \right] * \beta_k$$

for any $k > d$.

The proof is given later in this section.

Let $U \subseteq \mathbb{C}^{n+1}$ be a ball about 0 and $F \colon U \to \Delta$ a holomorphic germ, with $V_t := F^{-1}(t)$ smooth for $t \neq 0$. Assume that $Z := \mathrm{sing}(V_0)$ has dimension 1, and let Z_i be its irreducible components. We assume that their only intersection point is the origin.

Lemma 4 ([10, §7.2]). *Consider a (sufficiently general) linear form ℓ on $\mathbb{C}^{n+1}$ such that $\{\ell = 0\} \cap Z_i$ is finite for each i. Let $\pi = (F, \ell) : U \to \Delta^2$, and denote the Jacobian matrix by $d\pi$. Let $\mathcal{P} \subseteq U$ denote the set at which $d\pi$ has rank 1; that is, $\mathcal{P}$ is the intersection of the zero loci of all of the 2×2 minor determinants. Assume in addition that $\mathcal{P} \cap \pi^{-1}(\Delta \times \{(s =) \, 0\}) = \{\underline{0}\}$. Writing $\pi_s \colon \mathcal{P} \cap \ell^{-1}(s) \to \Delta \times \{s\}$ for the restriction, every irreducible curve $C_q \subset \cup_{s \in \Delta_s^*} \mathrm{im}(\pi_s)$ has a parametrization of the form $(t_{C_q}(s), s)$, where (for some $r_{C_q} \in \mathbb{Q}_{\geq 0}$ and $\gamma_{C_q} \in \mathbb{C}^*$)*

$$t_{C_q}(s) = \gamma_{C_q} s^{r_{C_q}} + \text{higher order terms.}$$

Furthermore if

$$r := \max_q \{r_{C_q}, 0\} \in \mathbb{Q}_{\geq 0},$$

then for every $a > r$, $F + \ell^a$ has an isolated singularity at 0 (including the vacuous case where it is nonsingular at and near 0).

We make a quick note that in the previous lemma, $Z \subseteq \cup_s (\mathcal{P} \cap \ell^{-1}(s))$ but the reverse inclusion need not hold.

Example 30. Consider the polynomial $F = x_0(x_1^2 + x_2^2) \in \mathbb{C}[x_0, x_1, x_2]$. Then F factors as $x_0(x_1 + ix_2)(x_1 - ix_2)$, and the set $Z = \mathrm{sing}(X_0) = \{(0, z, -iz)\} \cup \{(0, z, iz)\} \cup \{(z, 0, 0)\}$ for $z \in \mathbb{C}$. We choose U to be a ball about $\underline{0}$, and see that in U, Z has dimension 1. We see that $Z = Z_1 \cup Z_2 \cup Z_3$ on U where $Z_1 = \{(0, z, -iz)\}$, $Z_2 = \{(0, z, iz)\}$, $Z_3 = \{(z, 0, 0)\}$. Of course, each Z_i is irreducible of dimension 1.

We choose $\ell = x_0 + x_1$, and note $\{\ell = 0\} \cap Z_i = \{\underline{0}\}$ for $i = 1, 2, 3$. With π as above, we get:

$$d\pi = \begin{bmatrix} x_1^2 + x_2^2 & 2x_0 x_1 & 2x_0 x_2 \\ 1 & 1 & 0 \end{bmatrix}$$

Letting $F_{ij} = |\text{Col i} : \text{Col j}|$ for $i < j$, we have $d\pi$ has rank 1 on $\mathcal{P} = \{F_{12} = 0\} \cap \{F_{13} = 0\} \cap \{F_{23} = 0\}$. Here $\mathcal{P} = Z \cup \{z, 2z, 0\}$. We verify $\mathcal{P} \cap \pi^{-1}(\Delta_t \times \{0\}) = \{Z \cup \{z, 2z, 0\}\} \cap \{(y_1, -y_1, y_2)\} = \{\underline{0}\}$ where $y_i \in \mathbb{C}$. We have $\mathcal{P} \cap \ell^{-1}(s) = \{(\frac{s}{3}, \frac{2s}{3}, 0)\}$, and so $\text{im}(\pi_s) = \{(\frac{4s^3}{27}), s\}$. Therefore our only irreducible curve $C \subset \cup_{s \in \Delta_s^*} \text{Im}(\pi_s)$ is given by $t_C = \frac{4}{27}s^3$, and so $r = 3$. It follows that for every $a > 3$, $F + \ell^a = x_0(x_1^2 + x_2^2) + (x_0 + x_1)^a$ has an isolated singularity at $\underline{0}$.

Example 31. An example of the vacuous case is given by $F = x_0(x_1 - x_2^2) \in \mathbb{C}[x_0, x_1, x_2]$. Here, $Z = \text{sing}(V_0) = \{(0, z^2, z)\}$ for $z \in \mathbb{C}$. Again we choose U to be a ball about $\underline{0}$, and here Z itself is irreducible of dimension 1.

We choose $\ell = x_2$, and note $\{\ell = 0\} \cap Z = \{\underline{0}\}$. With π as above, we get

$$d\pi = \begin{bmatrix} x_1 - x_2^2 & x_0 & -2x_0x_2 \\ 0 & 0 & 1 \end{bmatrix},$$

which has rank 1 on $\mathcal{P} = Z$. We verify $\mathcal{P} \cap \pi^{-1}(\Delta_t, \{0\}) = Z \cap \{(y_1, 0, 0)\} = \{\underline{0}\}$ where $y_i \in \mathbb{C}$. We have $\mathcal{P} \cap \ell^{-1}(s) = \{(0, s^2, s)\}$, and so $\text{Im}\pi_s = \{0, s\}$. Therefore there is no irreducible curve $C \subset \cup_{s \in \Delta_s^*} Im(\pi_s)$ and so $r = 0$. We conclude that for every $a > 0$, $F + \ell^a = x_0(x_1 - x_2^2) + (x_2)^a$ has an isolated singularity at $\underline{0}$. In this case, $x_0(x_1 - x_2^2) + (x_2)^a$ is in fact nonsingular for $a > 0$.

We now apply Lemma 4 to our case of interest.

Lemma 5. *Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be homogeneous polynomial of degree d and let $\widetilde{V}(f) \subseteq \mathbb{P}^n$ have only isolated singularities. Then there exists a sufficiently general linear form ℓ such that $f + \ell^k$ has an isolated singularity at $\underline{0}$ for all $k > d$.*

Proof: Recall that the singular locus $\Sigma := \text{sing}(V(f))$ of the *affine cone* is a union of lines passing through the origin. Then there exists a suitable change of coordinates $(y_0, \ldots, y_n)$ and a ball $B = B_\epsilon(\underline{0})$ such that the components Σ_i of Σ are parametrized by y_0. That is, each Σ_i has the form:

$$\Sigma_i = \cup_{s \in \Delta_s} \{(s, f_i^j(s), \ldots, f_i^n(s)) \mid f_i^j(0) = 0 \quad \forall i\}$$

Furthermore we can set $(y_1, \ldots, y_n) = \underline{y}$ and rewrite $f(y_0, \underline{y}) = \sum_{j=0}^{d} g_j(\underline{y}) y_0^{d-j}$, and since $\widetilde{V}(f)$ has only isolated singularities, we may further assume our coordinates were chosen so that $f(0, \underline{y}) = g_d(\underline{y})$ defines a smooth hypersurface $\widetilde{V}(g_d) \subseteq \mathbb{P}^{n-1}_{(y_0=0)}$. We choose $\ell = y_0$ and let $\pi = (f, \ell) : B \to \Delta^2_{t,s}$. Denote $\partial_i f = \partial_{y_i} f$. Then we have:

$$d\pi = \begin{bmatrix} \partial_0 f & \partial_2 f & \ldots & \partial_n f \\ 1 & 0 & \ldots & 0 \end{bmatrix},$$

which has rank 1 precisely when $\partial_i f = 0$ for every $i > 0$.

We must show that $\text{sing}(\pi) \cap \{y_0 = 0\} = \{\underline{0}\}$. Let $p = (p_0, \underline{p}) \in \text{sing}(\pi) \cap \{y_0 = 0\}$. Then $p = (0, \underline{p})$ and

$$\partial_i f(p) = \partial_i \left[\sum_{j=0}^{d} g_j(\underline{x}) x_0^{d-j} \right] (p) = 0 \text{ for } i > 0.$$

This yields $\partial_i g_d(\underline{p}) = 0$ for $i > 0$ Since $\widetilde{V}(g_d)$ is smooth in $\underline{x}$, this implies $\underline{p} = \underline{\underline{0}} \Rightarrow p = \underline{0}$. Therefore there will exist a k such that for ε small enough $f + \varepsilon y_0^k$ will have an isolated singularity at $\underline{0}$.

In the notation above, $\mathcal{P} = \{\partial_i f = 0 \mid i > 0\} \cap B$. By Euler's identity, we have $d \cdot f = \sum_{i=0}^{n} y_i \partial_i f$. Any curve C given by the image of π_s of points $p_s \in \mathcal{P} \cap \{y_0 = s\} \cap \{f = 0\}$ is just given by $t_C = 0$. So we consider the case when $d \cdot f = x_0 \partial_0 f$. The points where

$f = 0$ just give the curve $t_{C_q} = 0$. We let $p_s \in \mathcal{P} \cap \{f \neq 0\}$. That is, $p_s = (s, \underline{p_s})$. By Euler's homogeneous function theorem,

$$f(p_s) = \frac{1}{d}\sum_{i=0}^{n} p_{s_i}\partial_i f(p_s) f(p_s) = \frac{s}{d}\partial_0 f(p_s) = \frac{s}{d}\sum_{j=0}^{d-1}(d-j)g_j(\underline{p_s})s^{d-j-1} = \frac{1}{d}\cdot\frac{\mathrm{d}}{\mathrm{ds}}f(s, \underline{p_s})$$

Let $h(s) = f(s, \underline{p_S})$. Then by above, $h(s) = \frac{d}{s} \cdot h'(s)$ for $s \neq 0$. Therefore:

$$\int \frac{h'(s)}{h(s)} ds = d \int \frac{ds}{s} \Rightarrow \log(h(s)) = d\log(s) + C \Rightarrow h(s) = As^d$$

And so, by Lemma 4, we have shown that $f + \varepsilon y_0^{d+1}$ must have an isolated singularity at $\underline{0}$. □

The following is detailed in [2, p195]:

Lemma 6. *Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be homogeneous polynomial of degree d such that $\widetilde{V}(f) \subseteq \mathbb{P}^n$ has only isolated singularities, $P_1, \ldots P_r$. Let each germ $g_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be defined locally about P_i. Let μ_i denote the Milnor number of g_i. Let T_i and τ_i be the algebraic horizontal and vertical monodromy operators respectively, each corresponding to g_i. Then we must have:*

$$T_i^{-d} = \tau_i.$$

Proof: [Proof of Theorem 29] We note that our choice of ℓ and $k > d$ yields an f_k with isolated singularity by Lemma 5. Additionally, this lemma was proven in accordance with Lemma 4, which is precisely the condition contained in the preface to [10, Thm. 7.5]. Therefore our choice of d as the bounding exponent r is sufficient to invoke Saito-Steenbrink, but where the α_{ij} are *a priori* given by the eigenvalues of the vertical monodromy operators.

Recall that there exist matrix representations of T_i and τ_i in Jordan normal form, so that they each have a Jordan-Chavalley decomposition into the product of a unipotent and semisimple matrices:

$$T_i = T_i^{ss} T_i^u$$

$$\tau_i = \tau_i^{ss} \tau_i^u$$

Furthermore, these representations can be chosen in such a way that there exists a simultaneous eigenbasis v_{ij} for T_i^{ss} and τ_i^{ss} for which:

$$T_i{}^{ss} v_{ij} = (e^{-2\pi i \lambda_{ij}}) v_{ij}$$

$$\tau_i{}^{ss} v_{ij} = (e^{2\pi i \alpha_{ij}}) v_{ij} \quad \text{ for } \quad \alpha_{ij} \in [0,1)$$

Note that while the values λ_{ij} are the spectral summands above, this relation is what defines the values α_{ij}. By Lemma 6, $T_i^{-d} = \tau_i \Rightarrow (T_i^{ss} T_i^u)^{-d} = \tau_i^{ss} \tau_i^u$, but since the pieces of the Jordan-Chavalley decomposition commute, the LHS is just $(T_i^{ss})^{-d}(T_i^u)^{-d}$. By the uniqueness of the decomposition into semisimple and unipotent parts, we have $(T_i^{ss})^{-d} = \tau_i^{ss}$. This implies our values of $\alpha_{ij} = d\lambda_{ij} - \lfloor d\lambda_{ij} \rfloor$.

Finally, by [10, Thm. 7.5], we must have:

$$\sigma_{f_k,0} = \sigma_{f,0} + \sum_{i,j} \left[\lambda_{ij} - \frac{\alpha_{ij}}{k} \right] * \beta_k$$

where $\beta_m = \sum_{i=0}^{m-1} \left[-\frac{i}{m} \right]$. Combining this with Theorem 20 now gives the desired formula:

$$\sigma_{f_k,0} = \gamma_d^{*(n+1)} - \sum_{i,j} \left[\lambda_{ij} - \frac{\alpha_{ij}}{d} \right] * \beta_d + \sum_{i,j} \left[\lambda_{ij} - \frac{\alpha_{ij}}{k} \right] * \beta_k.$$

□

Considering the case where $k = d+1$ we get the following general bound for multiple singularity types at once:

Theorem 32 (Generalized Conical bounding method)**.** *Let Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d, and let $\widetilde{V}(f) \subseteq \mathbb{P}^n$ have only r isolated singularities given locally by $g_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ having corresponding Milnor numbers μ_i for $1 \leq i \leq r$. Let $\sigma_{g_i,0} = \sum_{j=1}^{\mu_i} [\lambda_{ij}]$ be their corresponding spectra. Then:*

$$\sigma_{f_{d+1},0} = \gamma_d^{*(n+1)} - \left(\sum_{i=1}^{r} \sum_{j=1}^{\mu_i} \sum_{k=1}^{d} \left[\frac{\lfloor d\lambda_{ij} \rfloor + k}{d} \right] - \sum_{i=1}^{r} \sum_{j=1}^{\mu_i} \sum_{k=1}^{d+1} \left[\frac{\lambda_{ij} + \lfloor d\lambda_{ij} \rfloor + k}{d+1} \right] \right)$$

and the effectiveness of this spectrum restricts the set of r singularities which can be present.

Proof: We know from Theorem 29 and Lemma 2 that $k = d + 1$ gives an effective spectrum which satisfies:

$$\sigma_{f_{d+1},0} = \gamma_d^{*(n+1)} - \sum_{i,j} \left[\lambda_{ij} - \frac{\alpha_{ij}}{d} \right] * \beta_d + \sum_{i,j} \left[\lambda_{ij} - \frac{\alpha_{ij}}{d+1} \right] * \beta_{d+1}.$$

Since $\alpha_{ij} = d\lambda_{ij} - \lfloor d\lambda_{ij} \rfloor$, this is just

$$= \gamma_d^{*(n+1)} - \left(\sum_{i,j} \sum_{k=1}^{d} \left[\frac{\lfloor d\lambda_{ij} \rfloor + k}{d} \right] - \sum_{i,j} \sum_{k=1}^{d+1} \left[\frac{\lambda_{ij} + \lfloor d\lambda_{ij} \rfloor + k}{d+1} \right] \right).$$

□

2.8 Some more user-friendly formulas

In this section, we will explain how to adapt Theorem 32 to a formula which serves the purpose of reducing the number of calculations. The caveat is that the formula holds no deeper meaning within greater spectral theory. As one can see, Theorem 32 explicitly describes a relationship between the spectrum $\sigma_{f_{d+1},0}$ and the spectra of local singularities.

If we throw away the concept of spectra all together, we are left with theorems which only describe relationships of elements in $\mathbb{Z}[\mathbb{Q}]$.

This becomes more convenient because it allows us to simply throw things away that don't matter to the arithmetic we need to do to simply bound the possible singularities. We get the following:

Theorem 33. *Let Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d, and let $\widetilde{V}(f) \subseteq \mathbb{P}^n$ have only r isolated singularities given locally by $g_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ having corresponding Milnor numbers μ_i for $1 \leq i \leq r$. Let $\sigma_{g_i,0} = \sum_{j=1}^{\mu_i} [\lambda_{ij}]$ be their corresponding spectra. Then the following sum in $\mathbb{Z}[\mathbb{Q}]$ is effective:*

$$\gamma_d^{*(n+1)} - \left(\sum_{i=1}^{r} \sum_{j=1}^{\mu_i} \sum_{k=1}^{d-1} \left[\frac{\lfloor d\lambda_{ij} \rfloor + k}{d} \right] + \sum_{i=1}^{r} \sum_{\substack{j=1 \\ d\lambda_{ij} \notin \mathbb{Z}}}^{\mu_i} \left[\frac{\lfloor d\lambda_{ij} \rfloor}{d} + 1 \right] \right)$$

and the effectiveness of this sum restricts the set of r singularities which can be present.

We will first need to prove a very short lemma:

Lemma 7. *If the quantities:*

$$\frac{\lfloor d\lambda_{i,j} \rfloor + k}{d} = \frac{\lfloor d\lambda_{gh} \rfloor + \ell + \lambda_{gh}}{d+1}$$

for values $d \in \mathbb{N}$, $1 \leq \ell \leq d+1$, and $1 \leq k \leq d$, then we must have the following:

1. $\ell = d + 1$

2. $d\lambda_{gh} \in \mathbb{Z}$

Proof: Assume:

$$\frac{\lfloor d\lambda_{i,j} \rfloor + k}{d} = \frac{\lfloor d\lambda_{gh} \rfloor + \ell + \lambda_{gh}}{d+1}$$

$\Rightarrow (d+1)(\lfloor d\lambda_{ij}\rfloor + k) = d(\lfloor d\lambda_{gh}\rfloor + \ell + \lambda_{gh})$. Since the left is an integer, the right must be $\Rightarrow d\lfloor d\lambda_{gh}\rfloor + d\ell + d\lambda_{gh} \in \mathbb{Z} \Rightarrow d\lambda_{gh} \in \mathbb{Z} \Rightarrow \lfloor d\lambda_{gh}\rfloor = d\lambda_{gh}$. This restricts the equality to be:

$\Rightarrow (d+1)(\lfloor d\lambda_{ij}\rfloor + k) = d(d\lambda_{gh} + \ell + \lambda_{gh}) = d(d+1)\lambda_{gh} + d\ell$

$\Rightarrow \lfloor d\lambda_{ij}\rfloor + k = d\lambda_{gh} + \frac{d}{d+1}\ell$ Where the left must be an integer so the right must be. Since $d\lambda_{gh}$ is also an integer

$\Rightarrow \frac{d}{d+1}\ell \in \mathbb{Z}$. Our values of ℓ only range $1 \le \ell \le d+1 \Rightarrow \ell = d+1$. □

Proof: [Proof of Theorem 7.1]

By Theorem 32, we have that

$$\sigma_{f_{d+1},0} = \gamma_d^{*(n+1)} - \left(\sum_{i=1}^{r}\sum_{j=1}^{\mu_i}\sum_{k=1}^{d}\left[\frac{\lfloor d\lambda_{ij}\rfloor + k}{d}\right] - \sum_{i=1}^{r}\sum_{j=1}^{\mu_i}\sum_{k=1}^{d+1}\left[\frac{\lambda_{ij} + \lfloor d\lambda_{ij}\rfloor + k}{d+1}\right]\right)$$

is an effective sum in $\mathbb{Z}[\mathbb{Q}]$. By our lemma, the only summands of the right triple sum which may cancel with those of the left triple sum in the subtraction are those that satisfy the properties in the conclusion of the lemma. Therefore

$$\gamma_d^{*(n+1)} + \sum_{\substack{i=1 \\ d\lambda_{ij}}}^{r}\sum_{\substack{j=1 \\ \in\mathbb{Z}}}^{\mu_i}\left[\frac{\lambda_{ij} + d\lambda_{ij}}{d+1} + 1\right] - \sum_{i=1}^{r}\sum_{j=1}^{\mu_i}\sum_{k=1}^{d}\left[\frac{\lfloor d\lambda_{ij}\rfloor + k}{d}\right]$$

$$= \gamma_d^{*(n+1)} + \sum_{\substack{i=1 \\ d\lambda_{ij}}}^{r}\sum_{\substack{j=1 \\ \in\mathbb{Z}}}^{\mu_i}\left[\lambda_{ij} + 1\right] - \sum_{i=1}^{r}\sum_{j=1}^{\mu_i}\sum_{k=1}^{d}\left[\frac{\lfloor d\lambda_{ij}\rfloor + k}{d}\right]$$

is an effective sum in $\mathbb{Z}[\mathbb{Q}]$. But

$$\sum_{i=1}^{r}\sum_{j=1}^{\mu_i}\sum_{k=1}^{d}\left[\frac{\lfloor d\lambda_{ij}\rfloor + k}{d}\right] = \sum_{\substack{i=1 \\ d\lambda_{ij}}}^{r}\sum_{\substack{j=1 \\ \in\mathbb{Z}}}^{\mu_i}\left[\lambda_{ij}+1\right] + \sum_{\substack{i=1 \\ d\lambda_{ij}}}^{r}\sum_{\substack{j=1 \\ \notin\mathbb{Z}}}^{\mu_i}\left[\lambda_{ij}+1\right] + \sum_{i=1}^{r}\sum_{j=1}^{\mu_i}\sum_{k=1}^{d-1}\left[\frac{\lfloor d\lambda_{ij}\rfloor + k}{d}\right]$$

So we conclude that

$$\gamma_d^{*(n+1)} - \left(\sum_{i=1}^{r} \sum_{j=1}^{\mu_i} \sum_{k=1}^{d-1} \left[\frac{\lfloor d\lambda_{ij} \rfloor + k}{d} \right] + \sum_{i=1}^{r} \sum_{\substack{j=1 \\ d\lambda_{ij} \notin \mathbb{Z}}}^{\mu_i} \left[\frac{\lfloor d\lambda_{ij} \rfloor}{d} + 1 \right] \right)$$

is effective in $\mathbb{Z}[\mathbb{Q}]$. □

The power of this form, as opposed to the one in Theorem 32 is immense. The original theorem would have one believe, on first glance, that it were possible to have a set of two types of local singularities g_1 and g_2 such that the values of the summands $\left[\frac{\lambda_{1j}+\lfloor d\lambda_{1j}\rfloor+k}{d+1}\right]$ with positive coefficients corresponding to g_1 give extra wiggle room to the coefficients of $\gamma_d^{*(n+1)}$ to cancel out the summands $\left[\frac{\lfloor d\lambda_{2j}\rfloor+k}{d}\right]$ with negative coefficients corresponding to g_2. With our theorem in this section, we have proven that this possibility is, in fact, irrelevant to our use of the bounding method.

We prove the following statements, which serve to further bridge the gap of how our bound becomes increasingly similar to that of Varchenko in Section 2.4.

Lemma 8. *The number of positive integer solutions $(x_1, \dots, x_k)$ to the equation:*

$$\sum_{i=1}^{k} x_i = N$$

for some positive integer N subject to the constraints $1 \leq x_i \leq \alpha$ for $i = 1, \dots, k$ is given by:

$$\sum_{i=0}^{m} (-1)^i \binom{k}{i} \binom{N - \alpha i - 1}{k - 1}$$

where $m = \left\lfloor \min\left\{k, \frac{N-k}{\alpha}\right\} \right\rfloor$ acts as a truncator.

Furthermore, in the case that $\min\left\{k, \frac{N-k}{\alpha}\right\} = \frac{N-k}{\alpha}$ we may choose any $\left\lfloor \frac{N-k}{\alpha} \right\rfloor \leq m \leq \left\lfloor \frac{N-1}{\alpha} \right\rfloor$, as this simply adds zero terms in the sum.

Proof: This follows from a basic combinatorial argument using a "stars and bars" style proof and inclusion exclusion principles. ☐

Proposition 34. *For any $p \in \mathbb{Z}$ such that $n-d \leq p \leq n(d-1)-d$, we have:*

$$\left\{\frac{p}{d}\right\} \cap^{\#} \{\gamma_d^{*n}\} = \sum_{i=0}^{\left\lfloor n-1-\frac{p}{d-1}\right\rfloor} (-1)^i \binom{n}{i}\binom{d(n-1)-p-1-(d-1)i}{n-1}$$

and this completely determines the spectrum γ_d^{*n}.

Proof: Recall that

$$\gamma_d^{*n} = \left(\sum_{i=1}^{d-1}\left[-\frac{i}{d}\right]\right)^{*n} = \sum_{x_1,\ldots,x_n=1}^{d-1}\left[n-1-\frac{\sum_{i=1}^{n} x_i}{d}\right]$$

So it's clear that every summand of γ_d^{*n} has the form $\frac{p}{d}$ where the numerator $p \in \mathbb{Z}$ is restricted to the range $n-d \leq p \leq n(d-1)-d$. Calculating the coefficient of each summand amounts to counting the number of ways the sum $\sum_{i=1}^{n} x_i = d(n-1)-p$ subject to the constraint $1 \leq x_i \leq d-1$ for $1 \leq i \leq n$. The result then immediately follows from the above lemma. ☐

We make note of the following cute fact, which follows immediately from Theorem 13 and Proposition 34:

Corollary 35. *Let $d > n$ and the values $\left[h_{n,d}^{k,n-1-k}\right]'$ be the primitive hodge numbers of a smooth hypersurface in $\mathbb{P}^n$ of degree d. Then for $k \leq \frac{n-1}{2}$, we have:*

$$\left[h_{n,d}^{k,n-1-k}\right]' = \{n-k-1\} \cap^{\#} \{\gamma_d^{*(n+1)}\}.$$

We state one more lemma comparing the coefficients of γ_d^{*n} and $\gamma_d^{*(n+1)}$:

Lemma 9. *The following is a result of combinatorial arithmetic:*

$$\left\{\frac{p}{d}\right\} \cap^{\#} \{\gamma_d^{*(n+1)}\} = \left\{\frac{p-1}{d}, \frac{p-2}{d}, \ldots, \frac{p-(d-1)}{d}\right\} \cap^{\#} \{\gamma_d^{*n}\}$$

$$= \left(\frac{p}{d} - 1, \frac{p}{d}\right) \cap^{\#} \{\gamma_d^{*n}\}$$

The following is an equivalent statement of the conical bound:

Theorem 36 (Alternative statement of the conical bound)**.** *Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d, and let $\widetilde{V}(f) \subseteq \mathbb{P}^n$ have only r isolated singularities given locally by $g_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ having corresponding Milnor numbers μ_i for $1 \leq i \leq r$. Let $\sigma_{g_i,0}$ be their corresponding spectra. Then for every $p \in \mathbb{Z}$ we must have:*

$$\left\{\frac{p}{d}\right\} \cap^{\#} \{\gamma_d^{*(n+1)}\} \geq \sum_{i=1}^{r} \left(\frac{p}{d} - 1, \frac{p}{d}\right) \cap^{\#} \{\sigma_{g_i,0}\}$$

or equivalently:

$$\left(\frac{p}{d} - 1, \frac{p}{d}\right) \cap^{\#} \{\gamma_d^{*n}\} \geq \sum_{i=1}^{r} \left(\frac{p}{d} - 1, \frac{p}{d}\right) \cap^{\#} \{\sigma_{g_i,0}\}.$$

Proof: Theorem 33 tells us the following sum in $\mathbb{Z}[(\mathbb{Q}]$ is effective:

$$\gamma_d^{*(n+1)} - \left(\sum_{i=1}^{r} \sum_{j=1}^{\mu_i} \sum_{k=1}^{d-1} \left[\frac{\lfloor d\lambda_{ij} \rfloor + k}{d}\right] + \sum_{i=1}^{r} \sum_{\substack{j=1 \\ d\lambda_{ij} \notin \mathbb{Z}}}^{\mu_i} \left[\frac{\lfloor d\lambda_{ij} \rfloor}{d} + 1\right] \right)$$

This is equivalent to a set of statements for every coefficient of $\left[\frac{p}{d}\right]$ in $\gamma_d^{(n+1)}$ and the coefficient on $\left[\frac{p}{d}\right]$ in the summation to the right of it. That is, for every $p \in \mathbb{Z}$, we have:

$$\left\{\frac{p}{d}\right\} \cap^{\#} \{\gamma_d^{*(n+1)}\} \geq \sum_{i=1}^{r} \sum_{j=1}^{\mu_i} \# \left\{[\lambda ij] : \frac{p}{d} = \frac{\lfloor d\lambda_{ij} \rfloor}{d} + 1, d\lambda_{ij} \notin \mathbb{Z}\right\}$$
$$+ \sum_{i=1}^{r} \sum_{j=1}^{\mu_i} \sum_{k=1}^{d-1} \# \left\{[\lambda_{ij}] : \frac{p}{d} = \frac{\lfloor d\lambda_{ij} \rfloor + k}{d}\right\}$$

$$= \sum_{i=1}^{r} \{[\beta] : p - d < d\beta < p\} \cap^{\#} \{\sigma_{g_i,0}\} = \sum_{i=1}^{r} \left(\frac{p}{d} - 1, \frac{p}{d}\right) \cap^{\#} \{\sigma_{g_i,0}\}$$

giving our result. The alternative statement follows immediately from the above lemma.

□

We have now deduced that the bounding argument resulting from our above Theorem 29 mimics the form of a particular case of the Varchenko bound. We note that while Varchenko's bound is stronger than our bound, as proven in the following theorem, it is not the case that Varchenko's bounding argument implies the full scope of Theorem 29 itself. This is due to the fact that we sacrificed many of the structurally important components of the spectra in order to make this bounding argument in the first place.

Theorem 37. *Varchenko's bound Theorem 17 implies the formula given in Theorem 36.*

Proof: Let $Z \subseteq \mathbb{P}^n$ be a hypersurface of degree d, with only isolated singular points $P_1, \ldots, P_r$. Let $g_i : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, for $1 \leq i \leq r$ denote the corresponding germs defined locally about P_i.

Then Theorem 17 tells us that for any α, we must have:

$$(\alpha, \alpha + 1) \cap^{\#} \{\gamma_d^{*n}\} \geq \sum_{i=1}^{r} (\alpha, \alpha + 1) \cap^{\#} \{\sigma_{g_i,0}\}$$

For any $p \in \mathbb{Z}$, let $\alpha = \frac{p}{d} - 1$. Since this can be done for each p the result immediately follows. □

It is our experience that picking $\alpha \neq \frac{p}{d}$ for some $p \in \mathbb{Z}$, always gives a worse bound than either $p = \lfloor d\alpha \rfloor$ or $p = \lfloor d\alpha \rfloor + 1$. However the proof that this is true in general becomes highly technical arithmetic, and so we leave it as the following conjecture:

Conjecture 38. *Our bound, given in Theorem 36, implies the bound given by Varchenko in Theorem 17.*

We will now demonstrate the usefulness of Theorem 36 with several examples, which duplicate or improve known results. In the paper [15], the author gives an explicit example of a projective hypersurface $X \subset \mathbb{P}^4$ of degree 3 with a single A_{11} singularity. In the paper [10], the authors prove that $m = 11$ is in fact the maximal A_m singularity that can be present in such a hypersurface with only isolated singularities. The following example illustrates how one can use Theorem 36 to provide another proof that this is in fact the case:

Example 39. Let $X \subset \mathbb{P}^4$ be a hypersurface of degree 3 with only r isolated singularities. Here we have $n = 4$ and $d = 3$. Assume X has an A_m singularity. Without loss of generality, assume the local equations g_i $1 \leq i \leq r$ corresponding to the r singularities are indexed in such a way that g_1 corresponds to the A_m singularity.

The local normal form of g_1 is given by the equation: $z_0^{m+1} + z_1^2 + z_2^2 + z_3^2$, and the corresponding spectrum is given by:

$$\sigma_{g_1,0} = \sum_{j=1}^{\mu_1} [\lambda_{1j}] = \sum_{j=1}^{m} \left[\frac{j}{m+1} + \frac{1}{2} \right]$$

Therefore by Theorem 36,

$$1 = \left\{ \frac{2}{3} \right\} \cap^{\#} \{\gamma_3^{*5}\} \geq \left(-\frac{2}{6}, \frac{4}{6} \right) \cap^{\#} \{\sigma_{g_1,0}\} = \# \left\{ j : 1 \leq j \leq m, \quad j < \frac{m+1}{6} \right\}$$

However, the last count is greater than 1 whenever $m > 11$, and so we must have $m \leq 11$.

We can easily extend this argument to general n, d. We get that:

$$m \leq \frac{2d + (d-2)(n+1)}{2d - (d-2)(n+1)}.$$

We now show how Theorem 36 can be used to extend and improve the arguments of Proposition 15, and Proposition 16. The Hodge-theoretic bounds rely on the assumption that $n+1$ is even to work. Furthermore, one can verify that the former bound is equivalent to using Theorem 36 with $p = d$, which can be improved if we allow the interval some flexibility.

Proposition 40. *Let $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$ be homogeneous polynomial of degree $d > 3$, and let $\widetilde{V}(f) \subseteq \mathbb{P}^3$ have only n_6 isolated $\widetilde{E_6}$, n_7 isolated $\widetilde{E_7}$, and n_8 isolated $\widetilde{E_8}$ singularities. Define:*

$$b(d,p) = \binom{3d-p-1}{3} - 4\binom{2d-p}{3} + 6\binom{d-p+1}{3}$$
$$= \frac{d^3}{6} + \frac{d^2p}{2} - \frac{p^3}{2} + \frac{dp^2}{2} + -d^2 - 2dp + p^2 + \frac{11d}{6} + \frac{p}{2} - 1$$

Then:

1. $7n_6 \leq b(d,p) \leq \frac{31}{54}d^3 - \frac{13}{18}d^2 + 4d + \frac{1}{2}$ *for* $p = \left\lfloor \frac{2d}{3} \right\rfloor + 1$
2. $7n_6 + 8n_7 \leq b(d,p) \leq \frac{235}{384}d^3 - \frac{11}{16}d^2 + \frac{101}{24}d + \frac{1}{2}$ *for* $p = \left\lfloor \frac{3d}{4} \right\rfloor + 1$
3. $7n_6 + 8n_7 + 9n_8 \leq b(d,p) \leq \frac{277}{432}d^3 - \frac{23}{36}d^2 + \frac{53}{12}d + \frac{1}{2}$ *for* $p = \left\lfloor \frac{5d}{6} \right\rfloor + 1$

Proof: We note that the normal forms and spectra for each singularity type are as follows:

1. $\widetilde{E_6}$: $x^3 + y^3 + z^3$ $\sigma_{\widetilde{E_6}} = [0] + 3\left[\frac{1}{3}\right] + 3\left[\frac{2}{3}\right] + [1]$
2. $\widetilde{E_7}$: $x^2 + y^4 + z^4$ $\sigma_{\widetilde{E_7}} = [0] + 2\left[\frac{1}{4}\right] + 3\left[\frac{1}{2}\right] + 2\left[\frac{3}{4}\right] + [1]$
3. $\widetilde{E_8}$: $x^2 + y^3 + z^6$ $\sigma_{\widetilde{E_8}} = [0] + 1\left[\frac{1}{6}\right] + 2\left[\frac{1}{3}\right] + 2\left[\frac{1}{2}\right] + 2\left[\frac{2}{3}\right] + 1\left[\frac{5}{6}\right] + [1]$

We apply Theorem 36 for the choices of $p = \lfloor \frac{2d}{3} \rfloor + 1$, $\lfloor \frac{3d}{4} \rfloor + 1$, and $\lfloor \frac{5d}{6} \rfloor + 1$, respectively. The $b(d,p)$ are simply calculated using Proposition 34. Since these values of p are increasing, and the values of $\frac{p}{d} - 1 < 0$ in any of these choices, our bound for $8n_7$, duplicates as a bound for $7n_6 + 8n_7$ and our bound for $9n_8$, duplicates as a bound for $7n_6 + 8n_7 + 9n_8$. The polynomials bounding $b(d,p)$, are obtained from inequalities of the form $\frac{2d}{3} \leq \lfloor \frac{2d}{3} \rfloor + 1 \leq \frac{2d}{3} + 1$, and plugging these values into the polynomial form of $b(d,p)$, depending on the sign of p^m in each summand.

□

Example 41. In particular, we can compare the first bound with that of Proposition 2.2. The bounds for the following manually calculated bounds and those of Proposition 2.2 for $4 \leq d \leq 9$

bound/d	4	5	6	7	8	9
$\frac{1}{7}b(d,p)$	2	5	11	17	29	45
$\frac{1}{6}(h^{1,1}_{3,d} - 1)$	3	7	14	24	38	56

for $d \geq 10$, the values $\frac{1}{6}(h^{1,1}_{3,d} - 1) \geq \frac{1}{7}\left(\frac{31}{54}d^3 - \frac{13}{18}d^2 + 4d + \frac{1}{2}\right) \geq \frac{1}{7}b(d,p)$ for $p = \lfloor \frac{2d}{3} \rfloor + 1$ So this bound is always better than the hodge theoretic bound given in Proposition 2.2

Proposition 42. *Let $f \in \mathbb{C}[x_0, x_1, x_2.x_3, x_4]$ be homogeneous polynomial of degree $d > 2$, and let $\widetilde{V}(f) \subseteq \mathbb{P}^4$ have only isolated A_{2m+1} singularities. Then the number r of singular points is bounded by*

$$r \leq \begin{cases} \frac{1}{2m+1}\left[\frac{115}{192}d^4 - \frac{115}{48}d^3 + \frac{185}{48}d^2 - \frac{35}{12}d + 1\right] & d \equiv 0 \mod 2 \\ \\ \frac{1}{2m+1}\left[\frac{115}{192}d^4 - \frac{115}{48}d^3 + \frac{355}{96}d^2 - \frac{125}{48}d + \frac{45}{64}\right] & d \equiv 1 \mod 2, d > m+1 \end{cases}$$

Proof: Let $X \subset \mathbb{P}^4$ be a hypersurface of degree d with only r isolated A_{2m+1} singularities Let the local equations be given by g_i $1 \leq i \leq r$.

The local normal form of g_i for $1 \leq r \leq$ is given by the equation: $z_0^{2m+2} + z_1^2 + z_2^2 + z_3^2$, and the corresponding spectra are given by:

$$\sigma_{g_i,0} = \sum_{j=1}^{\mu_1}[\lambda_{ij}] = \sum_{j=1}^{2m+1}\left[\frac{j}{2m+2} + \frac{1}{2}\right]$$

Pick $p = \frac{3d}{2}$ for d even, and $p = \frac{3d+1}{2}$ for d odd (assuming $d > m+1$). Then by 36,

$$\left\{\frac{p}{d}\right\} \cap^{\#} \{\gamma_d^{*5}\} \geq \sum_{i=1}^{r}\left(\frac{p}{d} - 1, \frac{p}{d}\right) \cap^{\#} \{\sigma_{g_i,0}\} = r(2m+1)$$

A simple calculation of the left hand side using Proposition 34 for $d \equiv 0, 1 \mod 2$ gives the desired result. □

We note, however, that this bound is best for larger m. One can verify, for example, that our bound on the number of A_1 singularities given above does better (this is the case where $m = 0$). This is because the bounds above were chosen to give a convenient polynomial bound that does best for all m. Technically, when working with a particular m, a better bound can be found by choosing $p = \left\lfloor \frac{(3m+2)d}{2m+2} \right\rfloor + 1$. In particular this can be done even for the cases when d is odd and $m + 1 \geq d$.

3. A Purely Combinatorial Interpretation of the Conjecture

Put your first chapter here.

3.1 Spectrums Viewed as Combinatorial Objects

or our purposes, we may simply consider a *spectrum* σ as merely an object $\sigma \in \mathbb{Z}^{(\mathbb{Q})}$, the free abelian group on the generators $[\alpha]$, with $\alpha \in \mathbb{Q}$ such that σ_g must be symmetric, and have ony finitely many nonzero coefficients. Put more formally, we may designate any spectrum as an element $\sigma = \sum_{\alpha\in\mathbb{Q}} n_\alpha[\alpha]$ with $n_\alpha \in \mathbb{Z}$ such that:

1. Finitely many $n_\alpha \neq 0$
2. There exists some $p \in \mathbb{Q}$ such that $\sum_{\alpha\in\mathbb{Q}} n_\alpha[\alpha] = \sum_{\alpha\in\mathbb{Q}} n_{2p-\alpha}[2p-\alpha]$

Where g is a particular isolated singularity, and for our purposes we may consider it as some relevant function dictating the type of singularity, we denote the spectrum associated to g as σ_g.

Example 43. The $\widetilde{E_6}$ singularity $g = x^3 + y^3 + z^3$ has spectrum:

$$\sigma_g = 1[0] + 3\left[\frac{1}{3}\right] + 3\left[\frac{2}{3}\right] + 1[1]$$

The spectrums of particular usefulness are those of *homogeneous pham-brieskorn* functions, that is, those of the form:

$$\sum_{i=1}^{n} x_i^d = x_1^d + x_2^d + \ldots x_n^d$$

Which only vary based on the degree d and the number of variables n. Their spectrums are so combinatorial in structure that we denote them as follows:

$$\sigma_{x_1^d+x_2^d+\ldots x_n^d} = {\gamma_d}^{*n}$$

where:

$${\gamma_d}^{*n} = \sum_{x_1,\ldots x_n=1}^{d-1} \left[n-1-\frac{\sum_{i=1}^n x_i}{d} \right]$$

We note that finding the form $\sum_{\alpha\in\mathbb{Q}} n_\alpha[\alpha]$ of ${\gamma_d}^{*n}$ amounts to counting how many times $[\alpha]$ appears in the above sum, which then amounts to counting the number of ways $\sum_{i=1} x_i = d(n-1-\alpha)$ for $1 \le x_i \le d-1$. It is clear from the notion that $\alpha = n-1-\frac{\sum_{i=1}^n x_i}{d}$, that only α of the form $\frac{p}{d}$ with $p \in \mathbb{Z}$ need be considered. Furthermore, in γ_d^{*n}, the smallest α can be is $n-1-\frac{n(d-1))}{d} = \frac{n}{d}-1$ and the largest α can be is $n-1-\frac{n}{d} = \frac{n(d-1)}{d}-1$, both of which must have a coefficient of 1. A stars and bars style combinatorial proof, along with inclusion exclusion principles yeilds the following lemma:

Lemma 10. *The number of positive integer solutions $(x_1,\ldots,x_k)$ to the equation $\sum_{i=1}^k x_i = N$ for some positive integer N subject to the constraints $1 \le x_i \le \beta$ for $1 \le i \le k$ is given by:*

$$\sum_{i=0}^{m} (-1)^i \binom{k}{i} \binom{N-\beta i-1}{k-1}$$

where $m = \lfloor \min\left\{k, \frac{N-k}{\beta}\right\}$ acts as a truncator of zero value terms.

We now let $\{\sigma\}$ denote the *spectrum set.* That is if $\sigma = \sum_{\alpha\in\mathbb{Q}} n_\alpha[\alpha]$, then $\{\sigma\} = \bigcup_{\alpha\in\mathbb{Q}}\{[\alpha],\ldots,[\alpha]\}$ where the set $\{[\alpha],\ldots,[\alpha]\}$ contains n_α copies of $[\alpha]$.

Example 44. The $\widetilde{E_6}$ singularity $g = x^3+y^3+z^3$ with spectrum:

$$\sigma_g = 1[0] + 3\left[\frac{1}{3}\right] + 3\left[\frac{2}{3}\right] + 1[1]$$

would have spectrum set:

$$\{\sigma_g\} = \left\{[0], \left[\frac{1}{3}\right], \left[\frac{1}{3}\right], \left[\frac{1}{3}\right], \left[\frac{2}{3}\right], \left[\frac{2}{3}\right], \left[\frac{2}{3}\right], [1]\right\}$$

For any set $S \subseteq \mathbb{R}$, let $S \cap^{\#} \{\sigma\}$ denote the number of elements in $\{\sigma\}$ which are contained in S. If g is again the $\widetilde{E_6}$ singularity contained in the above example, then:

$$\left\{\frac{1}{3}\right\} \cap^{\#} \{\sigma_g\} = 3$$

,

$$(0,1) \cap^{\#} \{\sigma_g\} = 6$$

This new notation and the above lemma leads us immediately to the following coralary:

Corollary 45. *For any $p \in \mathbb{Z}$ in the range $n - d \leq p \leq n(d-1) - d$,*

$$\left\{\frac{p}{d}\right\} \cap^{\#} \gamma_d^{*n} = \sum_{i=0}^{\lfloor n-1-\frac{p+1}{d-1} \rfloor} (-1)^i \binom{n}{i} \binom{(n-1)d - p - (d-1)i - 1}{n-1}$$

And this completely determines γ_d^{*n}.

We may give an example verifying the accuracy of this calculation.

Example 46.

$$\gamma_4^{*4} = 1\,[0] + 4\left[\frac{1}{4}\right] + 10\left[\frac{1}{2}\right] + 16\left[\frac{3}{4}\right] + 19\,[1] + 16\left[\frac{5}{4}\right] + 10\left[\frac{3}{2}\right] + 4\left[\frac{7}{4}\right] + 1\,[2]$$

Taking $p = 3$ should give us the coefficient of $\left[\frac{3}{4}\right]$. Plugging it in to the above lemma we get the sum:

$$\sum_{i=0}^{1} (-1)^i \binom{4}{i} \binom{8-3i}{3} = \binom{8}{3} - 4\binom{5}{3} = 56 - 4(10) = 16$$

Which matches what we have above

3.2 Another Purely Combinatorial Way to Determine γ_d^{*n}

Recall that Pascal's triangle is given by the following pyramid:

$n = 0$							1						
$n = 1$						1		1					
$n = 2$					1		2		1				
$n = 3$				1		3		3		1			
$n = 4$			1		4		6		4		1		
$n = 5$		1		5		10		10		5		1	
$n = 6$	1		6		15		20		15		6		1

Where each number is the sum of the two numbers above it. We can similarly view it in a left justified version:

$n = 0$	1						
$n = 1$	1	1					
$n = 2$	1	2	1				
$n = 3$	1	3	3	1			
$n = 4$	1	4	6	4	1		
$n = 5$	1	5	10	10	5	1	
$n = 6$	1	6	15	20	15	6	1

Where each number is instead a sum of the number above it and the one up and to the left. We may instead consider an analogue of this left justified triangle where the number on the bottom is the sum of the three numbers above it, starting with the number above and moving to the left. We consider blank spots to be 0.:

$n=0$	1								
$n=1$	1	1	1						
$n=2$	1	2	3	2	1				
$n=3$	1	3	6	7	6	3	1		
$n=4$	1	4	10	16	19	16	10	4	1

As we can see, the first 16 in the row corresponding to $n = 4$ is given by the sum of the 3 numbers above $16 = 3+6+7$. The last line of numbers in this triangle should seem familiar, as they are the coefficients (in order) of γ_4^{*4} given in the previous subsection. This serves as motivation for exploring whether such a relationship exists between the coefficients of our special spectra and triangles that can be jotted down just as Pascal's can with only simple addition. In order to consider all triangles determined this way, we denote the above triangle as Δ^3, the left justified version of Pascal's triangle will be denoted as Δ^2, and in general, we will denote the left justified triangle determined by summing the n numbers above each spot by Δ^n. As we can see, Δ^4 will start off as follows:

$n=0$	1									
$n=1$	1	1	1	1						
$n=2$	1	2	3	4	3	2	1			
$n=3$	1	3	6	10	12	12	10	6	3	1

We will index their rows i the way we do with $n = i$ starting with row 0, and their columns j again starting with 0. We will denote the i^{th} row of triangle as Δ^n as Δ^n_i, and the number in the $\{i,j\}^{th}$ entry as $\Delta^n_{i,j}$. For example, $\Delta^2_3 = 1,3,3,1$ and $\Delta^4_{2,3} = 4$.

Restating some well-known properties of Pascal's triangle in terms of this new notation, we get:

1. $\Delta^2_{i,j} = \binom{i}{j}$

2. $\sum_j \Delta^2_{i,j} = 2^i$

3. $\Delta^2_{i,j} = \Delta^2_{i-1,j} + \Delta^2_{i-1,j-1}$

Perhaps most interesting is that the second property generalizes to the rest of the triangles. That is, for any natural number $k \geq 2$,

$$\sum_{j=1}^{(k-1)n+1} \Delta^k_{i,j} = k^i$$

For example, $\sum_j \Delta^4_{3,j} = 1 + 3 + 6 + 10 + 12 + 12 + 10 + 6 + 3 + 1 = 64 = 4^3$

As hinted above, the same combinatorial formulas that govern these triangles also govern the coefficients of γ_d^{*n}. Therefore we can use these triangles to completely determine γ_d^{*n}. We recall that the minimum and maximum values of α in $\gamma_d^{*n} = \sum_{\alpha \in \mathbb{Q}} n_\alpha[\alpha]$, are respectively $\frac{n}{d} - 1$ and $\frac{n(d-1)}{d} - 1$ with coefficients 1, and all other $\alpha = \frac{p}{d}$ of which all values of p with $\frac{p}{d}$ between this max and min must be present in the sum. Putting all of this together, it is not a stretch to see that:

$$\gamma_d^{*n} = \sum_{j=0}^{m} n_{\frac{n}{d}-1+\frac{j}{d}} \left[\frac{n}{d} - 1 + \frac{j}{d}\right]$$

where $m = \left(\left(\frac{n(d-1)}{d} - 1\right) - \left(\frac{n}{d} - 1\right)\right) d = n(d-2)$, and $n_{\frac{n}{d}-1+\frac{j}{d}} = \Delta^{d-1}_{n,j}$. Altogether this gives us the following complete determination of γ_d^{*n}:

$$\gamma_d^{*n} = \sum_{j=0}^{n(d-2)} \Delta^{d-1}_{n,j} \left[\frac{n}{d} - 1 - \frac{j}{d}\right]$$

The literature discussing triangles Δ^n are incredibly sparse. The first analogue, Δ^3 is often referred to as the *trinomial triangle*, and the numbers $\Delta^3_{i,j}$ composing it are referred to as *trinomial coefficients*. Several of its properties were discussed in [16]. It is also fairly straightforward to see why the above stated property, $\sum_j \Delta^k_{i,j} = k^i$ holds once one realizes that the same triangles also dictate the coefficients of $(1 + x + x^2 + \ldots + x^n)^d$ ordered by exponents on x. Put more clearly:

$$\frac{(x^k - 1)^n}{(x - 1)^n} = (1 + x + x^2 + \ldots + x^{k-1})^n = \sum_{i=0}^{1+(k-1)n} \Delta^k_{n,i} x^i$$

Plugging in $x = 1$ on the middle and right gives us the result. We give the leftmost equality only by means of quickly calculating the coefficients.

3.3 The Conjecture

In 3.1 we gave the conditions which govern the spectra σ_g we are concerned with as combinatorial objects and elements of $\mathbb{Z}^{\mathbb{Q}}$, the free abelian group with generators in $\mathbb{Q}$ and coefficients in $\mathbb{Z}$. In 3.2, we gave a new way to calculate the coefficients on the bounding element $\gamma_d^{*(n+1)}$. In this section we will state the combinatorial analogue of our conjecture.

Let g be an isolated singularitiy with a normal form represented in n variables. Recall that $\sigma_g \in \mathbb{Z}^{\mathbb{Q}}$ such that σ_g is a sum of finitely many generators and is symmetric about some rational number. More specifically, we know in this case which number about which σ_g must be symmetric, and some information about the coefficients:

1. σ_g is symmetric about $\alpha = \frac{n-2}{2}$

2. All coefficients n_α must be positive in σ_g

We now state the context of our bound:

Theorem 47 (Our Bound)**.** *: Let $g_1, \dots, g_r$ be a set of isolated singularities with normal form represented with n-variables and let this set as the complete set of local isolated singularities about te origin of some homogeneous polynomial f of degree d in $n+1$ variables with singular locus Σ of dimension 1. Then For any $p \in \mathbb{Z}$, we must have:*

$$\left\{\frac{p}{d}\right\} \cap^{\#} \{\gamma_d^{*(n+1)}\} \geq \sum_{i=1}^{r} \left(\frac{p}{d} - 1, \frac{p}{d}\right) \cap^{\#} \{\sigma_{g_i}\}$$

or equivalently:

$$\left(\frac{p}{d} - 1, \frac{p}{d}\right) \cap^{\#} \{\gamma_d^{*n}\} \geq \sum_{i=1}^{r} \left(\frac{p}{d} - 1, \frac{p}{d}\right) \cap^{\#} \{\sigma_{g_i}\}.$$

This gives us the immediate corollary governing bounds on the summands of σ_{g_i}:

Corollary 48. *For $1 \leq i \leq r$ we must have:*

$$\frac{n}{d} - 1 \leq \{\alpha : \{\alpha\} \cap^{\#} \{\sigma_{g_i}\} \neq 0\} \leq \frac{n(d-1)}{d} - 1$$

The Varchenko bound replicates our bound but adds the additional bound that for all $0 < \varepsilon < \frac{1}{d}$,

$$\left\{\frac{p}{d}\right\} \cap^{\#} \{\gamma_d^{*(n+1)}\} + \left\{\frac{p}{d}\right\} \cap^{\#} \{\gamma_d^{*n}\} \geq \sum_{i=1}^{r} \left(\frac{p}{d} - 1 + \varepsilon, \frac{p}{d} + \varepsilon\right) \cap^{\#} \{\sigma_{g_i}\}$$

For simplicity's sake, we may just assume that $\sigma_g = \sum_{i=1}^{r} \sigma_{g_i}$ follows the rules above just as any σ_{g_i}.

Our conjecture essentially boils down to the statement that this extra condition is extraneous. Put more formally:

Conjecture 49. *Assume that σ_g is a finitely generated element of $\mathbb{Z}^{\mathbb{Q}}$ such that:*

1. *σ_g is symmetric about $\alpha = \frac{n-2}{2}$*

2. *All coefficients n_α must be positive in σ_g*

3. *$\frac{n}{d} - 1 \leq \{\alpha : \{\alpha\} \cap^{\#} \{\sigma_g\} \neq 0\} \leq \frac{n(d-1)}{d} - 1$*

and

$$\left\{\frac{p}{d}\right\} \cap^{\#} \{\gamma_d^{*(n+1)}\} \geq \left(\frac{p}{d} - 1, \frac{p}{d}\right) \cap^{\#} \{\sigma_g\}$$

then this implies that for all $0 < \varepsilon < \frac{1}{d}$,

$$\left\{\frac{p}{d}\right\} \cap^{\#} \{\gamma_d^{*(n+1)}\} + \left\{\frac{p}{d}\right\} \cap^{\#} \{\gamma_d^{*n}\} \geq \sum_{i=1}^{r} \left(\frac{p}{d} - 1 + \varepsilon, \frac{p}{d} + \varepsilon\right) \cap^{\#} \{\sigma_g\}$$

This is purely a question of combinatorial arithmetic.

4. Appendix

There is in fact quite a long history of bounding the number of singularities of projective hypersurfaces, most notably bounding the number of nodes (also called an A_1 singularity or ordinary double point). In the tables that follow, "naive" denotes the vanishing cycle sequence method of Section 2.3. Note that it makes no prediction for nodes on a threefold (second table).

A_1 singularities, $n = 3$:

d	Naive	Eigenvalue	Conical	Sharp
Eq	$\frac{2}{3}d^3 - 2d^2 + \frac{7}{3}d - 1$	$d^3 - 4d^2 + 6d - 4$	$\frac{23}{48}d^3 - \frac{9}{8}d^2 + \frac{5}{6}d$, even d $\frac{23}{48}d^3 - \frac{23}{16}d^2 + \frac{78}{48}d + \frac{9}{16}$ odd	
1	0			0
2	1			1
3	6	5	4	4
4	19	20	16	16
5	44	51	31	31
6	85	104	68	65
7	146	185	104	99-104
10	489	656	375	
20	4579	6516	3400	
30	16269	23576	11950	
40	39559	57836	28900	
50	78449	115296	57125	
100	646899	960596	468000	
1,000	664668999	996005996	478042500	

A_1 singularities, $n = 4$:

d	Eigenvalue	Conical	Sharp
Eq	$d^4 - 5d^3 + 10d^2 - 10d + 5$	$\frac{11}{24}d^4 - \frac{19}{12}d^3 + \frac{49}{24}d^2 - \frac{11}{12}d$	
1			0
2			1
3	11	10	10
4	61	45	45
5	205	135	130-135
6	521	320	
7	1111	651	
10	5905	3195	
20	123805	61465	
30	683705	330310	
40	2255605	1075230	
50	5649505	2671725	
100	95099005	44270325	

$\widetilde{E_6}$ singularities, $n = 3$:

d	Naive	Eigenvalue	Conical	Sharp
Eq	$\frac{1}{9}d^3 - \frac{1}{3}d^2 + \frac{7}{18}d - \frac{1}{6}$	$\frac{1}{8}d^3 - \frac{1}{2}d^2 + \frac{3}{4}d - \frac{1}{2}$	$\frac{1}{7}b(d,p)$	
1	0			0
2	0			0
3	1			1
4	3	3	2	1
5	7	6	5	
6	14	13	11	
7	24	23	17	
10	82	82	60	
20	763	815	570	
30	2712	2947	2040	
40	6593	7230	4865	
50	13075	14412	9706	
100	107817	120075	79577	
1,000	110778167	124500750	81764819	

For reference, we give the output of $\gamma_d^{*(n+1)}$ for select values:

n	d	$\gamma_d^{*(n+1)}$
2	2	$1\left[\frac{1}{2}\right]$
2	3	$1[0]+3\left[\frac{1}{3}\right]+3\left[\frac{2}{3}\right]+1[1]$
2	4	$1\left[-\frac{1}{4}\right]+3\,[0]+6\left[\frac{1}{4}\right]+7\left[\frac{1}{2}\right]+6\left[\frac{3}{4}\right]+3\,[1]+1\left[\frac{5}{4}\right]$
2	5	$1\left[-\frac{2}{5}\right]+3\left[-\frac{1}{5}\right]+6\,[0]+10\left[\frac{1}{5}\right]+12\left[\frac{2}{5}\right]$ $+12\left[\frac{3}{5}\right]+10\left[\frac{4}{5}\right]+6\,[1]+3\left[\frac{6}{5}\right]+1\left[\frac{7}{5}\right]$
2	6	$1\left[-\frac{1}{2}\right]+3\left[-\frac{1}{3}\right]+6\left[-\frac{1}{6}\right]+10\,[0]+15\left[\frac{1}{6}\right]+18\left[\frac{1}{3}\right]+19\left[\frac{1}{2}\right]$ $+18\left[\frac{2}{3}\right]+15\left[\frac{5}{6}\right]+10\,[1]+6\left[\frac{7}{6}\right]+3\left[\frac{4}{3}\right]+1\left[\frac{3}{2}\right]$
2	7	$1\left[-\frac{4}{7}\right]+3\left[-\frac{3}{7}\right]+6\left[-\frac{2}{7}\right]+10\left[-\frac{1}{7}\right]+15\,[0]+21\left[\frac{1}{7}\right]+25\left[\frac{2}{7}\right]+27\left[\frac{3}{7}\right]$ $+27\left[\frac{4}{7}\right]+25\left[\frac{5}{7}\right]+21\left[\frac{6}{7}\right]+15\,[1]+10\left[\frac{8}{7}\right]+6\left[\frac{9}{7}\right]+3\left[\frac{10}{7}\right]+1\left[\frac{11}{7}\right]$
3	2	$1[1]$
3	3	$1\left[\frac{1}{3}\right]+4\left[\frac{2}{3}\right]+6\,[1]+4\left[\frac{4}{3}\right]+1\left[\frac{5}{3}\right]$

n	d	$\gamma_d^{*(n+1)}$
3	4	$1[0]+4\left[\frac{1}{4}\right]+10\left[\frac{1}{2}\right]+16\left[\frac{3}{4}\right]+19[1]+16\left[\frac{5}{4}\right]+10\left[\frac{3}{2}\right]+4\left[\frac{7}{4}\right]+1[2]$
3	5	$1\left[-\frac{1}{5}\right]+4[0]+10\left[\frac{1}{5}\right]+20\left[\frac{2}{5}\right]+31\left[\frac{3}{5}\right]+40\left[\frac{4}{5}\right]+44[1]$ $+40\left[\frac{6}{5}\right]+31\left[\frac{7}{5}\right]+20\left[\frac{8}{5}\right]+10\left[\frac{9}{5}\right]+4[2]+1\left[\frac{11}{5}\right]$
3	6	$1\left[-\frac{1}{3}\right]+4\left[-\frac{1}{6}\right]+10[0]+20\left[\frac{1}{6}\right]+35\left[\frac{1}{3}\right]+52\left[\frac{1}{2}\right]+68\left[\frac{2}{3}\right]+80\left[\frac{5}{6}\right]+85[1]$ $+80\left[\frac{7}{6}\right]+68\left[\frac{4}{3}\right]+52\left[\frac{3}{2}\right]+35\left[\frac{5}{3}\right]+20\left[\frac{11}{6}\right]+10[2]+4\left[\frac{13}{6}\right]+1\left[\frac{7}{3}\right]$
3	7	$1\left[-\frac{3}{7}\right]+4\left[-\frac{2}{7}\right]+10\left[-\frac{1}{7}\right]+20[0]+35\left[\frac{1}{7}\right]+56\left[\frac{2}{7}\right]+80\left[\frac{3}{7}\right]+104\left[\frac{4}{7}\right]$ $+125\left[\frac{5}{7}\right]+140\left[\frac{6}{7}\right]+146[1]+140\left[\frac{8}{7}\right]+125\left[\frac{9}{7}\right]+104\left[\frac{10}{7}\right]+80\left[\frac{11}{7}\right]+56\left[\frac{12}{7}\right]$ $+35\left[\frac{13}{7}\right]+20[2]+10\left[\frac{15}{7}\right]+4\left[\frac{16}{7}\right]+1\left[\frac{17}{7}\right]$
4	2	$\left[\frac{3}{2}\right]$
4	3	$1\left[\frac{2}{3}\right]+5[1]+10\left[\frac{4}{3}\right]+10\left[\frac{5}{3}\right]+5[2]+1\left[\frac{7}{3}\right]$
4	4	$1\left[\frac{1}{4}\right]+5\left[\frac{1}{2}\right]+15\left[\frac{3}{4}\right]+30[1]+45\left[\frac{5}{4}\right]+51\left[\frac{3}{2}\right]$ $+45\left[\frac{7}{4}\right]+30[2]+15\left[\frac{9}{4}\right]+5\left[\frac{5}{2}\right]+1\left[\frac{11}{4}\right]$

n	d	$\gamma_d^{*(n+1)}$
4	5	$1\,[0] + 5\left[\frac{1}{5}\right] + 15\left[\frac{2}{5}\right] + 35\left[\frac{3}{5}\right] + 65\left[\frac{4}{5}\right] + 101\,[1] + 135\left[\frac{6}{5}\right] + 155\left[\frac{7}{5}\right]$ $+155\left[\frac{8}{5}\right] + 135\left[\frac{9}{5}\right] + 101\,[2] + 65\left[\frac{11}{5}\right] + 35\left[\frac{12}{5}\right] + 15\left[\frac{13}{5}\right] + 5\left[\frac{14}{5}\right] + 1\,[3]$
4	6	$1\left[-\frac{1}{6}\right] + 5\,[0] + 15\left[\frac{1}{6}\right] + 35\left[\frac{1}{3}\right] + 70\left[\frac{1}{2}\right] + 121\left[\frac{2}{3}\right] + 185\left[\frac{5}{6}\right] + 255\,[1]$ $+320\left[\frac{7}{6}\right] + 365\left[\frac{4}{3}\right] + 381\left[\frac{3}{2}\right] + 365\left[\frac{5}{3}\right] + 320\left[\frac{11}{6}\right] + 255\,[2] + 185\left[\frac{13}{6}\right]$ $+121\left[\frac{7}{3}\right] + 70\left[\frac{5}{2}\right] + 35\left[\frac{8}{3}\right] + 15\left[\frac{17}{6}\right] + 5\,[3] + 1\left[\frac{19}{6}\right]$
4	7	$1\left[-\frac{2}{7}\right] + 5\left[-\frac{1}{7}\right] + 15\,[0] + 35\left[\frac{1}{7}\right] + 70\left[\frac{2}{7}\right] + 126\left[\frac{3}{7}\right] + 205\left[\frac{4}{7}\right]$ $+305\left[\frac{5}{7}\right] + 420\left[\frac{6}{7}\right] + 540\,[1] + 651\left[\frac{8}{7}\right] + 735\left[\frac{9}{7}\right] + 780\left[\frac{10}{7}\right] + 780\left[\frac{11}{7}\right]$ $+735\left[\frac{12}{7}\right] + 651\left[\frac{13}{7}\right] + 540\,[2] + 420\left[\frac{15}{7}\right] + 305\left[\frac{16}{7}\right] + 205\left[\frac{17}{7}\right] + 126\left[\frac{18}{7}\right]$ $+70\left[\frac{19}{7}\right] + 35\left[\frac{20}{7}\right] + 15\,[3] + 5\left[\frac{22}{7}\right] + 1\left[\frac{23}{7}\right]$
5	2	$1\,[2]$
5	3	$1\,[1] + 6\left[\frac{4}{3}\right] + 15\left[\frac{5}{3}\right] + 20\,[2] + 15\left[\frac{7}{3}\right] + 6\left[\frac{8}{3}\right] + 1\,[3]$
5	4	$1\left[\frac{1}{2}\right] + 6\left[\frac{3}{4}\right] + 21\,[1] + 50\left[\frac{5}{4}\right] + 90\left[\frac{3}{2}\right] + 126\left[\frac{7}{4}\right] + 141\,[2]$ $+126\left[\frac{9}{4}\right] + 90\left[\frac{5}{2}\right] + 50\left[\frac{11}{4}\right] + 21\,[3] + 6\left[\frac{13}{4}\right] + 1\left[\frac{7}{2}\right]$

n	d	$\gamma_d^{*(n+1)}$
5	5	$1\left[\frac{1}{5}\right]+6\left[\frac{2}{5}\right]+21\left[\frac{3}{5}\right]+56\left[\frac{4}{5}\right]+120\left[1\right]+216\left[\frac{6}{5}\right]+336\left[\frac{7}{5}\right]$ $+456\left[\frac{8}{5}\right]+546\left[\frac{9}{5}\right]+580\left[2\right]+546\left[\frac{11}{5}\right]+456\left[\frac{12}{5}\right]+336\left[\frac{13}{5}\right]$ $+216\left[\frac{14}{5}\right]+120\left[3\right]+56\left[\frac{16}{5}\right]+21\left[\frac{17}{5}\right]+6\left[\frac{18}{5}\right]+1\left[\frac{19}{5}\right]$
5	6	$1\left[0\right]+6\left[\frac{1}{6}\right]+21\left[\frac{1}{3}\right]+56\left[\frac{1}{2}\right]+126\left[\frac{2}{3}\right]+246\left[\frac{5}{6}\right]+426\left[1\right]$ $+666\left[\frac{7}{6}\right]+951\left[\frac{4}{3}\right]+1246\left[\frac{3}{2}\right]+1506\left[\frac{5}{3}\right]+1686\left[\frac{11}{6}\right]+1751\left[2\right]$ $+1686\left[\frac{13}{6}\right]+1506\left[\frac{7}{3}\right]+1246\left[\frac{5}{2}\right]+951\left[\frac{8}{3}\right]+666\left[\frac{17}{6}\right]+426\left[3\right]$ $+246\left[\frac{19}{6}\right]+126\left[\frac{10}{3}\right]+56\left[\frac{7}{2}\right]+21\left[\frac{11}{3}\right]+6\left[\frac{23}{6}\right]+1\left[4\right]$
5	7	$1\left[-\frac{1}{7}\right]+6\left[0\right]+21\left[\frac{1}{7}\right]+56\left[\frac{2}{7}\right]+126\left[\frac{3}{7}\right]+252\left[\frac{4}{7}\right]+456\left[\frac{5}{7}\right]$ $+756\left[\frac{6}{7}\right]+1161\left[1\right]+1666\left[\frac{8}{7}\right]+2247\left[\frac{9}{7}\right]+2856\left[\frac{10}{7}\right]+3431\left[\frac{11}{7}\right]$ $+3906\left[\frac{12}{7}\right]+4221\left[\frac{13}{7}\right]+4332\left[2\right]+4221\left[\frac{15}{7}\right]+3906\left[\frac{16}{7}\right]+3431\left[\frac{17}{7}\right]$ $+2856\left[\frac{18}{7}\right]+2247\left[\frac{19}{7}\right]+1666\left[\frac{20}{7}\right]+1161\left[3\right]+756\left[\frac{22}{7}\right]+456\left[\frac{23}{7}\right]$ $+252\left[\frac{24}{7}\right]+126\left[\frac{25}{7}\right]+56\left[\frac{26}{7}\right]+21\left[\frac{27}{7}\right]+6\left[4\right]+1\left[\frac{29}{7}\right]$
6	2	$1\left[\frac{5}{2}\right]$
6	3	$1\left[\frac{4}{3}\right]+7\left[\frac{5}{3}\right]+21\left[2\right]+35\left[\frac{7}{3}\right]+35\left[\frac{8}{3}\right]+21\left[3\right]+7\left[\frac{10}{3}\right]+1\left[\frac{11}{3}\right]$

n	d	$\gamma_d^{*(n+1)}$
6	4	$1\left[\frac{3}{4}\right]+7\left[1\right]+28\left[\frac{5}{4}\right]+77\left[\frac{3}{2}\right]+161\left[\frac{7}{4}\right]+266\left[2\right]+357\left[\frac{9}{4}\right]+393\left[\frac{5}{2}\right]$ $+357\left[\frac{11}{4}\right]+266\left[3\right]+161\left[\frac{13}{4}\right]+77\left[\frac{7}{2}\right]+28\left[\frac{15}{4}\right]+7\left[\frac{8}{2}\right]+1\left[\frac{17}{4}\right]$
6	5	$1\left[\frac{2}{5}\right]+7\left[\frac{3}{5}\right]+28\left[\frac{4}{5}\right]+84\left[1\right]+203\left[\frac{6}{5}\right]+413\left[\frac{7}{5}\right]+728\left[\frac{8}{5}\right]+1128\left[\frac{9}{5}\right]$ $+1554\left[2\right]+1918\left[\frac{11}{5}\right]+2128\left[\frac{12}{5}\right]+2128\left[\frac{13}{5}\right]+1918\left[\frac{14}{5}\right]+1554\left[3\right]$ $+1128\left[\frac{16}{5}\right]+728\left[\frac{17}{5}\right]+413\left[\frac{18}{5}\right]+203\left[\frac{19}{5}\right]+84\left[4\right]+28\left[\frac{21}{5}\right]+7\left[\frac{22}{5}\right]+1\left[\frac{23}{5}\right]$
6	6	$1\left[\frac{1}{6}\right]+7\left[\frac{1}{3}\right]+28\left[\frac{1}{2}\right]+84\left[\frac{2}{3}\right]+210\left[\frac{5}{6}\right]+455\left[1\right]+875\left[\frac{7}{6}\right]$ $+1520\left[\frac{4}{3}\right]+2415\left[\frac{3}{2}\right]+3535\left[\frac{5}{3}\right]+4795\left[\frac{11}{6}\right]+6055\left[2\right]+7140\left[\frac{13}{6}\right]+7875\left[\frac{7}{3}\right]$ $+8135\left[\frac{5}{2}\right]+7875\left[\frac{8}{3}\right]+7140\left[\frac{17}{6}\right]+6055\left[3\right]+4795\left[\frac{19}{6}\right]+3535\left[\frac{10}{3}\right]$ $+2415\left[\frac{7}{2}\right]+1520\left[\frac{11}{3}\right]+875\left[\frac{23}{6}\right]+210\left[4\right]+84\left[\frac{25}{6}\right]+28\left[\frac{13}{3}\right]+7\left[\frac{9}{2}\right]+1\left[\frac{29}{6}\right]$
6	7	$1\left[0\right]+7\left[\frac{1}{7}\right]+28\left[\frac{2}{7}\right]+84\left[\frac{3}{7}\right]+210\left[\frac{4}{7}\right]+462\left[\frac{5}{7}\right]+917\left[\frac{6}{7}\right]$ $+1667\left[1\right]+2807\left[\frac{8}{7}\right]+4417\left[\frac{9}{7}\right]+6538\left[\frac{10}{7}\right]+9142\left[\frac{11}{7}\right]+12117\left[\frac{12}{7}\right]$ $+15267\left[\frac{13}{7}\right]+18327\left[2\right]+20993\left[\frac{15}{7}\right]+22967\left[\frac{16}{7}\right]+24017\left[\frac{17}{7}\right]+24017\left[\frac{18}{7}\right]$ $+22967\left[\frac{19}{7}\right]+20993\left[\frac{20}{7}\right]+18327\left[3\right]+15267\left[\frac{22}{7}\right]+12117\left[\frac{23}{7}\right]+9142\left[\frac{24}{7}\right]$ $+6538\left[\frac{25}{7}\right]+4417\left[\frac{26}{7}\right]+2807\left[\frac{27}{7}\right]+1667\left[4\right]+917\left[\frac{29}{7}\right]+462\left[\frac{30}{7}\right]+210\left[\frac{31}{7}\right]$ $+84\left[\frac{32}{7}\right]+28\left[\frac{33}{7}\right]+7\left[\frac{34}{7}\right]+1\left[5\right]$

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