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Complete Pick Spaces and Operator Inequalities

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by
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Στους γονείς μου

ABSTRACT OF THE DISSERTATION

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This thesis is concerned with the treatment of three different research topics (each in a separate chapter), all lying at the interface of complex analysis and operator theory.

The first chapter is based on two distinct research projects, both revolving around complete Pick spaces. These are reproducing kernel Hilbert spaces that host an analogue of the Pick interpolation theorem for multipliers. First, we study a generalized inner-outer factorization in the setting of a particular complete Pick space over the annulus. The second project deals with the characterization of interpolating sequences for multipliers between certain pairs of function spaces that enjoy an analogue of the complete Pick property. In particular, we show that a sequence is interpolating for a pair of such spaces if and only if it generates a Carleson measure with respect to the first space and is n-weakly separated by the kernel of the second space, for any $n \geq 2$. We also construct counterexamples to show that n-weak separation cannot, in general, be replaced by weak separation, thus answering a question of Aleman, Hartz, McCarthy and Richter.

The second chapter deals with operator inequalities over the annulus. In particular, we consider three different classes of operators associated with the annulus and offer estimates for the norm of functions of such operators. Each class requires separate treatment: for the first one, we construct a certain "extremal" weighted shift operator, while for the second one, we convert the operator norm into the multiplier norm of a certain Hilbert function space. Further, to handle the third class, we employ a technique due to Crouzeix and Greenbaum that involves the double-layer potential

integral operator (that section is joint work with Michael Jury). Finally, we note that this chapter also contains material that has not been submitted for publication; in Section 2.4, we construct a counterexample to a question of Bello and Yakubovich concerning the class of operators that have the annulus as a spectral set.

Finally, the third chapter, which is joint work with Michael Jury, focuses on the behavior of the iterates

$$F^n := \underbrace{F \circ F \circ \cdots \circ F}_{n \text{ times}}$$

of holomorphic self-maps F of the bidisk that do not have any interior fixed points. It is well-known that, unlike the single-variable case, the sequence $\{F^n\}$ will, in general, diverge. However, it turns out that the limiting behavior of $\{F^n\}$ is heavily influenced by the differentiability properties of F at certain boundary fixed points, which we term Denjoy-Wolff points following the classical setting. In fact, we show that if F possesses Denjoy-Wolff points with particular properties, then $\{F^n\}$ will have to converge. To obtain these results, we employ a certain operator-theoretic representation of holomorphic functions on the bidisk due to Agler.

Chapter 1

Two Problems in Complete Pick Spaces

The material contained in this chapter originates in the following two papers:

Paper I G. Tsikalas. "Subinner-free outer factorizations on an annulus". In: *Integral Equations Operator Theory* 94.4 (2022), Paper No. 40, 15

Paper II G. Tsikalas. "Interpolating sequences for pairs of spaces". In: *J. Funct. Anal.* 285.7 (2023), Paper No. 110059, 43

1.1 Introduction

The starting point of our investigations in this chapter lies in the rich theory of the *Hardy space* H^2, which is the Hilbert space of functions f analytic in the unit disc $\mathbb{D} = \{|z| < 1\}$ and satisfying

$$||f||_2^2 := \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

The Hardy space and its natural generalizations offer a computational and conceptual framework in which operator theory flourishes. The multiplier algebra of H^2, denoted by Mult(H^2), is particularly well-studied. It is the commutative Banach algebra of analytic functions ϕ in the unit disc, such that the induced multiplication operator

$$M_\phi f = \phi \cdot f$$

maps H^2 continuously into itself. It is known that Mult(H^2) can be isometrically identified with H^∞, the algebra of all bounded holomorphic functions on $\mathbb{D}$ equipped with the norm $||\phi||_\infty :=$

$\sup_{z\in\mathbb{D}} |\phi(z)|$. Another well-studied object is the *Szegő kernel*

$$\sigma(z,w) = \sigma_w(z) = \frac{1}{1 - z\overline{w}}, \quad z, w \in \mathbb{D},$$

which is the *reproducing kernel* of H^2, satisfying

$$\langle f, \sigma_w \rangle_{H^2} = f(w), \text{ for every } f \in H^2, w \in \mathbb{D}.$$

H^∞ presents a fertile framework for interpolation problems, one of the most classical ones arguably being the following: given n points $z_1, \dots, z_n \in \mathbb{D}$ and n complex numbers $w_1, \dots, w_n$, determine whether there exists $\phi \in H^\infty$ such that

$$\phi(z_i) = w_i,\ i = 1, \dots, n, \quad \text{and} \quad ||\phi||_\infty \leq 1.$$

Pick's solution [114] of this problem in 1915 impacted the development of function theory throughout the twentieth century, while also having fruitful applications in engineering and, in particular, control theory [19, 95]. Nevanlinna [106] independently arrived at a somewhat different characterization in 1919.

Theorem 1.1.1 (Pick, 1915, Nevanlinna 1919)**.** The above interpolation problem has a solution if and only if the $n \times n$ matrix

$$\left[\frac{1 - w_i\overline{w}_j}{1 - z_i\overline{z}_j}\right]$$

is positive-semidefinite.

It is possible to view the Nevanlinna-Pick theorem as a theorem about H^2: Given n points $z_1, \dots, z_n \in \mathbb{D}$ and n complex numbers $w_1, \dots, w_n$, there exists $\phi \in \mathrm{Mult}(H^2)$ such that

$$\phi(z_i) = w_i,\ i = 1, \dots, n, \quad \text{and} \quad ||\phi||_{\mathrm{Mult}(H^2)} \leq 1$$

if and only if the $n \times n$ matrix

$$\left[(1 - w_i\overline{w}_j)\sigma(z_i, z_j)\right]$$

is positive-semidefinite. This operator-theoretic approach was pioneered by Sarason [123], who actually proved a much more general *commutant lifting* theorem that encodes, unifies and extends a variety of classical interpolation and moment theorems on the disc, Pick's included. That theorem has since been generalized by many authors and the theory of commutant lifting, initiated by Nagy and Foiaş (see [71]), is a widely studied tool in operator theory that derived from Sarason's insights.

The study of the Nevanlinna-Pick interpolation problem for general spaces lead to the birth of *complete Pick spaces*. These are reproducing kernel Hilbert spaces defined in terms of an interpolation property for multipliers that recovers the Nevanlinna-Pick theorem in the case of the Hardy space H^2 (where the multipliers are precisely the H^∞ functions). Further examples include the classical Dirichlet space and standard weighted Dirichlet spaces on the unit disc, the Sobolev space W_1^2 on the unit interval and the Drury–Arveson space H_d^2 on the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$. Following foundational work of McCullough [97, 99], Quiggin [119] and Agler and McCarthy [7], complete Pick spaces have been used to answer several interesting function-theoretical questions on topics such as interpolating sequences [17], invariant subspaces [101], factorization theorems [15], weak product spaces [85], the Column-Row property [79] and the corona theorem [56]. We refer the reader to subsection 1.1.1 for precise definitions and additional examples. A comprehensive treatment of complete Pick spaces can be found in the book [8].

After a brief preliminary section on reproducing kernel Hilbert spaces and the complete Pick property, the rest of this chapter will be split in two parts, each focused on a different problem in the setting of complete Pick spaces. In Section 1.2, we will study a generalized inner-outer factorization (originating in [16, 85]) for functions living in a certain complete Pick space over the annulus. On the other hand, in Section 1.3, we will give a complete characterization of interpolating sequences for multipliers $\phi : \mathcal{H}_s \to \mathcal{H}_\ell$, where ℓ is a reproducing kernel, s is a complete Pick kernel and ℓ/s is also a kernel (e.g. $\mathcal{H}_s$ could be the Hardy space and $\mathcal{H}_\ell$ could be a weighted Bergman space on the disc). Further, we will construct a counterexample to a related question of Aleman, Hartz, McCarthy and Richter [17].

1.1.1 Reproducing Kernel Hilbert Spaces and the Complete Pick Property

Let X be a nonempty set. A function $k : X \times X \to \mathbb{C}$ is called positive semi-definite, if whenever $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$ and $w_1, \dots w_n \in \mathbb{C}$, then $\sum_{i,j=1}^{n} k(x_j, x_i) w_i \overline{w_j} \geq 0$. We also say that k is a *kernel*. For each $x \in X$, define a function $k(\cdot, x)$ on X by $k(\cdot, x)(y) = k(y, x)$. Define also an inner product on the linear span of these functions by

$$\Big\langle \sum_i a_i k(\cdot, x_i), \sum_j b_j k(\cdot, x_j) \Big\rangle = \sum_{i,j} a_i \overline{b_j} k(x_j, x_i).$$

Let $\mathcal{H}_k$ denote the Hilbert space obtained by completing the linear span of the functions $k(\cdot, x)$ with respect to the previous inner product. We may regard vectors f in $\mathcal{H}_k$ as functions on X, with $f(x) = \langle f, k(\cdot, x) \rangle$. $\mathcal{H}_k$ is said to be a *reproducing kernel Hilbert space*. references on the basics of such spaces include the classical paper of Aronszajn [24] and the book [113].

The *multiplier algebra* $\mathrm{Mult}(\mathcal{H}_k)$ is defined as the collection of functions $\phi : X \to \mathbb{C}$ such that $(M_\phi f)(x) = \phi(x) f(x)$ defines a bounded operator $M_\phi : \mathcal{H}_k \to \mathcal{H}_k$. The multipliers ϕ with $||M_\phi|| \leq C$ are characterized by

$$(C^2 - \phi(y)\overline{\phi(x)}) k(y, x) \geq 0, \tag{1.1}$$

since it is equivalent to $||M_\phi^* f||_{\mathcal{H}_k} \leq C ||f||_{\mathcal{H}_k}$, for a dense subset of $\mathcal{H}_k$.

Now, let n be a positive integer, and let $\mathcal{M}_n$ denote the n-by-n complex matrices. We say that k has the *N-point $\mathcal{M}_n$ Pick property* if, for every finite sequence $\lambda_1, \dots, \lambda_N$ of N distinct points in X, and every sequence $W_1, \dots, W_N$ in $\mathcal{M}_n$, positivity of the block matrix

$$\Big[k(\lambda_i, \lambda_j)(I_{\mathbb{C}^n} - W_i W_j{}^*) \Big]_{i,j=1}^{N}$$

implies the existence of a multiplier Φ of $\mathcal{H}_k \otimes \mathbb{C}^n$ of norm at most 1 that satisfies

$$\Phi(\lambda_i) = W_i, \quad \text{for all } 1 \leq i \leq N.$$

When $n = 1$, we say k has the *N-point (scalar) Pick property*. If k has the N-point $\mathcal{M}_n$ Pick property for every n and N, we say the kernel, and the corresponding Hilbert space $\mathcal{H}_k$, have the *complete Pick property*. For brevity, we will also sometimes say that k is a *CP kernel*.

Examples of such kernels and spaces (all proofs can be found in [8]) are the Szegő kernel $\frac{1}{1-z\overline{w}}$ for the Hardy space on the unit disk; the Dirichlet kernel $\frac{-1}{z\overline{w}}\log(1-z\overline{w})$ on the disk; the kernels $\frac{1}{(1-z\overline{w})^t}$ for $0<t<1$ on the disk; the Sobolev space W_1^2 on the unit interval; and the Drury-Arveson space, the space of analytic functions on the unit ball $\mathbb{B}_d$ of a d-dimensional Hilbert space (where d may be infinite) with kernel

$$k(z,w)=\frac{1}{1-\langle z,w\rangle}.$$

A kernel k is said to be *irreducible* if the underlying set X cannot be partitioned into two non-empty disjoint sets X_1, X_2 so that $k(x_1,x_2)=0$ for all $x_1\in X_1, x_2\in X_2$. The kernel k of an irreducible complete Pick space satisfies $k(z,w)\neq 0$ for all $z,w\in X$; see [7, Lemma 1.1]. By Theorem 3.1 of [7], the space $\mathcal{H}_k$ is an irreducible complete Pick space if and only if there exist a function $\delta : X\to\mathbb{C}\setminus 0$, a number $d\in\mathbb{N}\cup\{\infty\}$ and a function $u : X\to\mathbb{B}_d$, where $\mathbb{B}_d$ denotes the open unit ball of a d-dimensional Hilbert space $\mathcal{K}$, so that

$$k_w(z)=\frac{\delta(z)\overline{\delta(w)}}{1-\langle u(z),u(w)\rangle_{\mathcal{K}}}\quad (z,w\in X). \tag{1.2}$$

Finally, a kernel k is normalized at $w_0\in X$ if $k(z,w_0)=1$ for all $z\in X$. One can always *rescale* an irreducible complete Pick kernel (see [8, Section 2.6] for more background on rescaling kernels) to achieve that in (1.2) the function δ is the constant function 1 and $u(w_0)=0$. We point out that working in normalized spaces is merely convenient, not essential for our proofs.

1.2 Subinner-Free Outer Factorizations on the Annulus

1.2.1 Background

Many results originating in the theory of the Hardy space over the unit disc can be extended to the setting of complete Pick spaces. One such result is the following classical factorization theorem. Note that, in this context, an *inner* function is a multiplier $\phi \in H^\infty(\mathbb{D})$ such that the associated multiplication operator M_ϕ is an isometry on $H^2(\mathbb{D})$, while an *outer* function $g \in H^2(\mathbb{D})$ has the property that

$$\text{span}\{\psi \cdot g : \psi \in H^\infty(\mathbb{D})\}$$

is dense in $H^2(\mathbb{D})$.

Theorem 1.2.1 (Inner-outer factorization)**.** Given $f \in H^2(\mathbb{D})$, $f \neq 0$, there exist unique (up to renormalization) functions ϕ, g such that ϕ is an inner multiplier, g is an outer function and

$$f = \phi g.$$

The inner-outer factorization is a valuable tool for both function- and operator-theoretic arguments, as it allows for the replacement of f by the "nicer" functions ϕ and g.

Unfortunately, it is a known fact that in many well-studied function spaces (especially in the multi-variable setting) no analogue of the inner-outer factorization exists. This is, however, not the case with complete Pick spaces. Indeed, assume $\mathcal{H}$ is an arbitrary space with the complete Pick property. In [16], Aleman, Hartz, McCarthy and Richter proved a unique factorization result for functions in $\mathcal{H}$ that involves the two classes of *subinner* and *free outer* functions. In particular, they showed that for every $f \in \mathcal{H} \setminus \{0\}$, there exists a unique (up to unimodular constants) *subinner-free outer pair* (ϕ, h) such that

$$f = \phi h.$$

Here, the subinner factor ϕ is a contractive multiplier of $\mathcal{H}$ such that $||\phi h|| = ||h||$, while the free outer factor h belongs to a special subclass of the cyclic vectors of $\mathcal{H}$ (to be defined below). Note that this factorization is the same (without the uniqueness assertion) as the one considered by Jury and Martin in [85]. In these papers, the authors worked in the setting of the free Fock space $\mathcal{F}_d^2$ in d variables, where a free inner-outer factorization is known to hold (see [23], [64]). Since there exists a natural isometric embedding of $\mathcal{H}$ into $\mathcal{F}_d^2$ for some $d \in \mathbb{N} \cup \infty$ (see subsection 1.2.3), this free inner-outer factorization then also applies to functions in the embedded space $\mathcal{H}$.

Now, in the setting of the Hardy space $H^2(\mathbb{D})$, subinner functions coincide with the classical inner functions. In the general case, subinner functions form a "large" class of contractive multipliers of $\mathcal{H}$; Theorem 1.9 in [16] asserts that every ϕ in the unit ball of the multiplier algebra of $\mathcal{H}$ is a pointwise limit of subinner functions. We note that it may even happen that every function in the space is, up to renormalization, a subinner multiplier! For instance, consider the Sobolev space $\mathcal{W}_1^2$ of those functions on $[0, 1]$ that are absolutely continuous and whose derivatives are square integrable. Agler [11] showed that $\mathcal{W}_1^2$ is a complete Pick space. Theorem 1.5 of the same paper implies that every $f \in \mathcal{W}_1^2$ is a norm-attaining multiplier of $\mathcal{W}_1^2$, hence a constant multiple of a subinner function.

On the other hand, free outer functions are much harder to come by. While they coincide with the usual outer functions in the setting of $H^2(\mathbb{D})$, in general they form a strict subset of the cyclic vectors of $\mathcal{H}$. Also, the product of two free outer functions isn't necessarily free outer (see e.g. [16, Example 11.1]). Another perhaps surprising fact is the existence of nonconstant functions that are simultaneously subinner and free outer. Simple examples of such functions can be found even in the Dirichlet space D; see the remark after the proof of Theorem 14.9 in [16]. Also, in view of of our observation in the last paragraph, we can easily see that suitable scalar multiples of kernel functions in $\mathcal{W}_1^2$ will be both subinner and free outer.

It is fair to say that free outer functions are not well-understood yet. The situation is exacerbated by a striking lack of examples; indeed, besides the familiar Hardy space setting, it seems that only the free outer factors of low-degree polynomials and of projections of kernel functions onto multiplier invariant subspaces (see [16, Example 1.8]) have been worked out explicitly so far. In this note, we enlarge the pool of available examples by investigating subinner-free outer factorizations in the setting of a certain complete Pick space on the annulus $A_r = \{r < |z| < 1\}$, which we now define.

Let $\mathcal{H}^2(A_r)$ denote the Hilbert function space on A_r induced by the kernel

$$k_r(\lambda, \mu) := \frac{1 - r^2}{(1 - \lambda\bar{\mu})(1 - r^2/\lambda\bar{\mu})}.$$

$\mathcal{H}^2(A_r)$ is known to be a complete Pick space (see [20, p. 1137]). Our main result, proved in subsection 1.2.6, is the following:

Theorem 1.2.2. *For every $f \in H^2(\mathbb{D}) \subset \mathcal{H}^2(A_r)$, the classical inner-outer factorization in $H^2(\mathbb{D})$ coincides (up to multiplication by unimodular constants) with the subinner-free outer factorization of f in $\mathcal{H}^2(A_r)$. An analogous result holds for functions in $H^2(\mathbb{D}_0)$*[1] *$\subset \mathcal{H}^2(A_r)$.*

Theorem 1.2.2 tells us that a function in $\mathcal{H}^2(A_r)$ that is also analytic on the unit disk is $\mathcal{H}^2(A_r)$-free outer if and only if it is $H^2(\mathbb{D})$-outer. We show that a similar result holds for $\mathcal{H}^2(A_r)$-subinner functions that are also analytic on $\mathbb{D}$.

Theorem 1.2.3. *Suppose $\phi \in Mult(\mathcal{H}^2(A_r)) \cap Hol(\mathbb{D}) = H^\infty(\mathbb{D})$. Then, ϕ is $H^2(\mathbb{D})$-inner if and only if ϕ is $\mathcal{H}^2(A_r)$-subinner. An analogous result holds for $H^2(\mathbb{D}_0)$-inner functions.*

Finally, Corollary 1.2.14 (see subsection 1.2.6) allows us to obtain new examples of free outer functions in the two-dimensional Drury-Arveson space H^2_2 by using the embedding of $\mathcal{H}^2(A_r)$ into H^2_2.

1.2.2 Preliminaries

Let $\mathcal{H}$ be a separable Hilbert function space on a non-empty set X with reproducing kernel k. Write $\text{Mult}(\mathcal{H}) = \{\phi : X \to \mathbb{C} : \phi f \in \mathcal{H} \text{ for all } f \in \mathcal{H}\}$ for the multiplier algebra of $\mathcal{H}$. Every multiplier ϕ defines a bounded linear operator $M_\phi \in \mathcal{B}(\mathcal{H})$ by $M_\phi(f) = \phi f$. Putting $||\phi||_{\text{Mult}(\mathcal{H})} = ||M_\phi||_{\mathcal{B}(\mathcal{H})}$ turns $\text{Mult}(\mathcal{H})$ into a Banach algebra. If $f \in \mathcal{H}$, write $[f]$ for the

[1] Here, $H^2(\mathbb{D}_0)$ is the Hardy space over the unbounded disk $\mathbb{D}_0 = \{|z| > r\}$ centered at infinity.

multiplier invariant subspace generated by f, i.e. the closure of $\text{Mult}(\mathcal{H})f$ in $\mathcal{H}$. f is called *cyclic*, if $[f] = \mathcal{H}$. Also, define $P_f : \text{Mult}(\mathcal{H}) \to \mathbb{C}$ to be the linear functional $P_f(\phi) := \langle \phi f, f \rangle$ and set

$$\mathcal{E}_f = \{g \in \mathcal{H} : P_f = P_g\}.$$

We can now state the Aleman, Hartz, McCarthy, Richter factorization result. To be precise, their theorems are stated in the setting of a complete Pick space $\mathcal{H}$ with kernel k *normalized* at $z_0 \in X$.

Definition 1.2.4. (a) A function $f \in \mathcal{H}$ is called *free outer*, if

$$|f(z_0)| = \sup\{|g(z_0)| : g \in \mathcal{E}_f\}.$$

(b) A multiplier $\phi \in \text{Mult}(\mathcal{H})$ is called *subinner*, if $||\phi||_{\text{Mult}(\mathcal{H})} = 1$ and if there exists a nonzero $h \in \mathcal{H}$ such that with $||\phi h|| = ||h||$.

(c) A pair (ϕ, f) is called a *subinner/free outer pair*, if ϕ is subinner, f is free outer with $f(z_0) > 0$, and $||\phi f|| = ||f||$.

Theorem 1.2.5. *For every $f \in \mathcal{H} \setminus \{0\}$ there is a unique subinner/free outer pair (ϕ, h) such that $f = \phi h$.*

Theorem 1.2.6. *If $f, g \in \mathcal{H} \setminus \{0\}$, then $P_f = P_g$ if and only if f and g have the same free outer factors.*

Having a distinguished normalization point turns out not to be important.

Corollary 1.2.7. *Let $h \in \mathcal{H}$. The following are equivalent:*

(a) h is free outer,

(b) there is $z \in X$ with $|h(z)| = \sup\{|f(z)| : f \in \mathcal{E}_f\}$,

(c) for all $z \in X$ we have $|h(z)| = \sup\{|f(z)| : f \in \mathcal{E}_f\}$.

1.2.3 The free Fock space

We now briefly discuss preliminaries in regards to the free Fock space. Let $d \in \mathbb{N} \cup \{\infty\}$ and write $\mathbb{F}_d^+$ for the free semigroup on d letters $\{1, 2, \dots\}$; that is, the set of all words $w = w_1 w_2 \cdots w_k$

over all (finite) lengths k, where each $w_j \in \{1, 2, \ldots\}$. We also include the empty word $\emptyset$ in $\mathbb{F}_d^+$, the length of which is defined to be zero. If $w \in \mathbb{F}_d^+$, then $\alpha(w) \in \mathbb{N}_0^d$ is the multi-index associated with w, defined by $\alpha(w) = (\alpha_1, \ldots, \alpha_d)$, where α_j equals the number of times the letter j occurs in w. Also, let $x = (x_1, \ldots, x_d)$ be a freely non-commuting indeterminate with d components. If $w \in \mathbb{F}_d^+$, then the free monomials are defined by $x^w = 1$, if $w = \emptyset$, and $x^w = x_{w_1} \ldots x_{w_k}$, if $w = w_1 \cdots w_k$.

The free Fock space $\mathcal{F}_d^2$ is the space of all power series in d non-commuting formal variables with square-summable coefficients, i.e. $F \in \mathcal{F}_d^2$ if and only if $F(x) = \sum_{w \in \mathbb{F}_d^+} \hat{F}(w) x^w$ and $||F||^2 = \sum_{w \in \mathbb{F}_d^+} |\hat{F}(w)|^2 < \infty$. A distinguished subspace of $\mathcal{F}_d^2$ is the symmetric Fock space $\mathcal{H}_d^2 \subseteq \mathcal{F}_d^2$. An element $F \in \mathcal{F}_d^2$ is in $\mathcal{H}_d^2$ if and only if $\hat{F}(w) = \hat{F}(v)$, whenever $\alpha(w) = \alpha(v)$. Now, for every $z = (z_1, z_2, \ldots, z_d) \in \mathbb{B}_d$ and $\mathbf{n} := (n_1, n_2, \ldots, n_d) \in \mathbb{N}_0^d$, set $z^{\mathbf{n}} := z_1^{n_1} \cdots z_d^{n_d}$. The map $T : H_d^2 \to \mathcal{F}_d^2$ defined by

$$h(z) = \sum_{\mathbf{n} \in \mathbb{N}_0^d} \frac{h_{\mathbf{n}}}{||z^{\mathbf{n}}||_{H_d^2}^2} z^{\mathbf{n}} \mapsto H(x) := \sum_{\mathbf{n} \in \mathbb{N}_0^d} h_{\mathbf{n}} \left(\sum_{w | \lambda(w) = \mathbf{n}} x^w \right) \tag{1.3}$$

is an isometric embedding of H_d^2 into $\mathcal{F}_d^2$ (T identifies H_d^2 with $\mathcal{H}_d^2$, see [130], Section 4). Also, if $k_z(\lambda) = \frac{1}{1 - \langle \lambda, z \rangle}$ denotes the kernel of H_d^2, we set $K_z := T k_z = \sum_{\mathbf{n} \in \mathbb{N}_0^d} \overline{z}^{\mathbf{n}} \sum_{w | \lambda(w) = \mathbf{n}} x^w \in \mathcal{H}_d^2$, for every $z \in \mathbb{B}_d$. For further details about the free Fock space and related non-commutative function theory, see [88], [116], [117], [115], [130] and [122].

Now, consider an arbitrary normalized CP kernel $k_x(y) = \frac{1}{1 - \langle u(y), u(x) \rangle}$, where $u : X \to \mathbb{B}_d$ and $d \in \mathbb{N} \cup \{\infty\}$. We can identify (see [7]) the Hilbert function space $\mathcal{H}_k$ associated with k with the subspace

$$\mathcal{H} = \text{ closed linear span of } \{K_z : z \in \operatorname{ran} u\} \subset \mathcal{H}_d^2 \subset \mathcal{F}_d^2.$$

Furthermore, it can be shown that $\mathcal{H}$ is invariant under adjoints of multipliers of $\mathcal{F}_d^2$ and also the map $U k_x = K_{u(x)}$ extends to be a linear isometry $U : \mathcal{H}_k \to \mathcal{F}_d^2$ with range equal to $\mathcal{H}$. Thus, $UU^* = P_{\mathcal{H}}$ and we have $U^* = C_u$, where $C_u F(x) = F(u(x))$ for all $x \in X$.

Denote by $\mathcal{L}\mathcal{F}_d^\infty$ the algebra of all elements $G \in \mathcal{F}_d^2$ such that the operator $F \mapsto GF$ (multiplication by G from the left) is bounded on $\mathcal{F}_d^2$. The algebra $\mathcal{R}\mathcal{F}_d^\infty$ of bounded right multiplication operators is defined analogously. An element $F \in \mathcal{F}_d^2$ will be called *left-outer* if $\{GF : G \in \mathcal{L}\mathcal{F}_d^\infty\}$

is dense in $\mathcal{F}_d^2$. Similarly, F will be called *right-outer* if $\{FG : G \in \mathcal{RF}_d^\infty\}$ is dense in $\mathcal{F}_d^2$. The following is the content of Theorem 5.3 and Lemma 6.1 in [16].

Theorem 1.2.8. *Suppose $\mathcal{H}_k$ is a CP space and let $U : \mathcal{H}_k \to \mathcal{F}_d^2$ be its natural embedding into $\mathcal{F}_d^2$. If $f \in \mathcal{H}_k$, the following are equivalent:*

a) f is free outer in $\mathcal{H}_k$;

b) Uf is left-outer in $\mathcal{F}_d^2$;

c) Uf is right-outer in $\mathcal{F}_d^2$;

d) $S^{-1}(Uf)$ is free outer in H_d^2, where $S := P_{\mathcal{H}_d^2}T$ is the isometric identification (1.3) of H_d^2 with $\mathcal{H}_d^2$.

1.2.4 A note on normalization

We have already mentioned that Aleman, Hartz, McCarthy and Richter proved their results for normalized CP kernels, i.e. kernels of the form (1.2) that also satisfy $\delta \equiv 1$ and $u(z_0) = 0$ for some point $z_0 \in X$. These normalization assumptions have become the standard setting for research into CP spaces, not, in general, due to necessity but due to the convenience they provide. Still, it could be of interest to note that the subinner-free outer factorization continues to hold, without any modifications, in the more general setting of an irreducible CP kernel (i.e. any kernel of the form (1.2)).

First, assume k is an irreducible CP kernel satisfying $\delta \equiv 1$ and let $\mathcal{H}_k$ denote the associated function space. Fix an arbitrary $z_0 \in X$ (not necessarily one with the property that $u(z_0) = 0$) and put $\lambda_0 = u(z_0) \in \mathbb{B}_d$. Let $\mathcal{H} \subset \mathcal{H}_d^2$ be a $*$-invariant subspace and choose $F \in \mathcal{H} \setminus \{0\}$. Our goal is to show that F is right outer (equivalently, left outer) in $\mathcal{F}_d^2$ if and only if it satisfies $|F(\lambda_0)| = \sup\{|G(\lambda_0)| : G \in \mathcal{H}, P_F = P_G\}$. In [16, Theorem 5.3], this is achieved (for $z_0 = 0$) by making use, among other things, of the fact that any $\Phi \in \mathcal{F}_d^\infty$ satisfying $|\Phi(0)| = ||\Phi||_\infty$ must be constant. This conclusion remains valid under the more general assumption that $|\Phi(z_0)| = ||\Phi||_\infty$.

Indeed, we can look at the restriction $\tilde{\Phi} = \Phi|_{\mathbb{B}_d} \in \text{Mult}(H^2_d)$. Since

$$|\Phi(z_0)| = |\tilde{\Phi}(z_0)| \leq ||\tilde{\Phi}||_{\text{Mult}(H^2_d)} \leq ||\Phi||_\infty = |\Phi(z_0)|,$$

[18, Lemma 2.2] implies that $\tilde{\Phi}$ is constant. Thus, $|\Phi(0)| = |\tilde{\Phi}(0)| = |\tilde{\Phi}(z_0)| = |\Phi(z_0)| = ||\Phi||_\infty$, which implies that Φ is also constant. The rest of the proof of [16, Theorem 5.3] carries over mutatis mutandis and gives us the desired result. Suppose now that $f \in \mathcal{H}_k \setminus \{0\}$. We can define f to be $\mathcal{H}_k$-free outer if $|f(z_0)| = \sup\{|g(z_0)| : g \in \mathcal{E}_f\}$. The proofs of Theorems 1.4-1.5, 5.4, 6.2 and Lemma 6.1 of [16] continue to hold if we adopt this (slightly) more general definition of free outerness, thus giving us a (unique) subinner-free outer factorization for each element of $\mathcal{H}_k$. Corollary 1.2.7 then tells us that our factorization will always be the same (up to unimodular constants), regardless of the z_0 we started with.

For the general case, assume k is a kernel of the form (1.2). Put $k'(x,y) = \frac{k(x,y)}{\delta(x)\overline{\delta(y)}}$. In view of our observations in the previous paragraph, we obtain that Theorems 1.2.5, 1.2.8 continue to hold for functions in $\mathcal{H}_{k'}$. Now, we know that Mult($\mathcal{H}_k$)=Mult($\mathcal{H}_{k'}$), with equality of norms, and also that the linear map $U : \mathcal{H}_k \to \mathcal{H}_{k'}$ defined as $(Uf)(x) = \delta(x)f(x)$ is a unitary (see [8, Section 2.6] for details). These two facts imply that a multiplier is $\mathcal{H}_k$-subinner if and only if it is $\mathcal{H}_{k'}$-subinner and also that a function f in $\mathcal{H}_k$ is $\mathcal{H}_k$-free outer if and only if δf is $\mathcal{H}_{k'}$-free outer. As an immediate consequence, we obtain that Theorems 1.2.5 and 1.2.8 must also be valid in the setting of $\mathcal{H}_k$.

1.2.5 The space $\mathcal{H}^2(A_r)$

We now record a few basic facts about the space $\mathcal{H}^2(A_r)$ that will be needed (see Chapter 2 for the proofs). Let $A_r = \{r < |z| < 1\}$ denote an annulus. $\mathcal{H}^2(A_r)$ is defined as the Hilbert function space on A_r induced by the kernel

$$k_r(\lambda,\mu) := \frac{1-r^2}{(1-\lambda\bar{\mu})(1-r^2/\lambda\bar{\mu})}.$$

Letting $||\cdot||_{\mathcal{H}^2(A_r)}$ denote the corresponding norm, we obtain that

$$||f||^2_{\mathcal{H}^2(A_r)} = \sum_{-\infty}^{-1} r^{2n}|c_n|^2 + \sum_{0}^{\infty} |c_n|^2,$$

for every $f = \sum c_n z^n \in \mathcal{H}^2(A_r)$. Also, every $\phi \in \text{Mult}(\mathcal{H}^2(A_r))$ satisfies

$$||\phi||_\infty \leq ||\phi||_{\text{Mult}(\mathcal{H}^2(A_r))} \leq \sqrt{2}||\phi||_\infty,$$

hence $\text{Mult}(\mathcal{H}^2(A_r))$ is equal (but not isometric) to $H^\infty(A_r)$.

Now, putting

$$u : A_r \to \mathbb{B}_2$$
$$z \mapsto \left(\frac{z}{\sqrt{r^2+1}}, \frac{r}{\sqrt{r^2+1}}\frac{1}{z}\right),$$

and letting k_2 denote the kernel of H^2_2, it can be easily checked that

$$k_r(\lambda, \mu) = \left(\frac{1-r^2}{1+r^2}\right) k_2(u(\lambda), u(\mu)), \ \ \forall \lambda, \mu \in A_r. \tag{1.4}$$

So, $\mathcal{H}^2(A_r)$ is a CP space and, even though k_r is not normalized, dividing it by the constant $(1-r^2)/(1+r^2)$ gives us a kernel of the form (1.2) with $\delta \equiv 1$. Now, it is easy to see that replacing the norm of the base space $\mathcal{H}_k$ by any constant multiple of it does not affect the properties of being subinner or free outer. Hence, working with the original kernel k_r does not make a difference as far as subinner and free outer functions are concerned.

Finally, consider the unitary $V : \mathcal{H}^2(A_r) \to \mathcal{H}^2(A_r)$ defined as $(Vf)(z) = f(r/z)$. Then, $H^2(\mathbb{D}_0)$, which is the function space on $\mathbb{D}_0 = \{|z| > r\}$ induced by $\frac{1}{1-(r/z)(r/\overline{w})}$, will be the image of $H^2(\mathbb{D})$ through V. The map V will prove very useful when passing from $H^2(\mathbb{D})$- to $H^2(\mathbb{D}_0)$-versions of our results.

1.2.6 Main Results

First, we record a helpful lemma; for multipliers of $\mathcal{H}^2(A_r)$ that are analytic either on $\mathbb{D}$ or on $\mathbb{D}_0 = \{r < |z|\}$, the multiplier norm actually coincides with the supremum norm.

Lemma 1.2.9. *i) If $f \in H^2(\mathbb{D})$, then*

$$||f||_{H^2(\mathbb{D})} = ||f||_{\mathcal{H}^2(A_r)}.$$

Similarly, if $f \in H^2(\mathbb{D}_0)$, *then*

$$||f||_{H^2(\mathbb{D}_0)} = ||f||_{\mathcal{H}^2(A_r)}.$$

ii) If either $\phi \in H^\infty(\mathbb{D})$ *or* $\phi \in H^\infty(\mathbb{D}_0)$, *then*

$$||\phi||_{Mult(\mathcal{H}^2(A_r))} = ||\phi||_{H^\infty(A_r)}.$$

Proof. Part i) is obvious so we only show part ii).

WLOG, suppose $||\phi||_{H^\infty(A_r)} \leq 1$. If $\phi \in \mathrm{Hol}(\mathbb{D})$, then $||\phi||_\infty \leq 1$ as a function on $\mathbb{D}$ (by the maximum modulus principle) and so ϕ is a contractive multiplier of $H^2(\mathbb{D})$. By [113, Theorem 5.21], this implies that

$$\begin{aligned}
&(1-\phi(\lambda)\overline{\phi(\mu)})\frac{1}{1-\lambda\bar{\mu}} \geq 0 \text{ on } \mathbb{D}\times\mathbb{D} \\
\implies &(1-\phi(\lambda)\overline{\phi(\mu)})\frac{1}{1-\lambda\bar{\mu}} \geq 0 \text{ on } A_r\times A_r \\
\implies &(1-\phi(\lambda)\overline{\phi(\mu)})\frac{1}{\left(1-\frac{r^2}{\lambda\bar{\mu}}\right)(1-\lambda\bar{\mu})} \geq 0 \text{ on } A_r\times A_r,
\end{aligned}$$

as the Schur product of two positive semi-definite kernels is positive semi-definite (notice that $1/(1-r^2/(\lambda\bar{\mu}))$ is the kernel of the Hardy space on $\mathbb{D}_0$). Hence, we conclude that

$$(1-\phi(\lambda)\overline{\phi(\mu)})k_r(\lambda,\mu) \geq 0$$

$$\implies ||\phi||_{\mathrm{Mult}(\mathcal{H}^2(A_r))} \leq 1.$$

We have shown that $||\phi||_{\mathrm{Mult}(\mathcal{H}^2(A_r))} \leq ||\phi||_{H^\infty(A)}$, which concludes our argument (the reverse inequality $||\phi||_{\mathrm{Mult}(\mathcal{H}^2(A_r))} \geq ||\phi||_{H^\infty(A)}$ is valid in every Hilbert function space) . For the case where $\phi \in \mathrm{Hol}(\mathbb{D}_0)$, the proof proceeds in an analogous manner. □

We proceed to show that the classical inner functions of the Hardy space on $\mathbb{D}$ become subinner functions when viewed as multipliers of $\mathcal{H}^2(A_r)$.

Proposition 1.2.10. *Suppose* $g \in H^\infty(\mathbb{D})$ *is an* $H^2(\mathbb{D})$*-inner function. Then,* g *is a subinner multiplier of* $\mathcal{H}^2(A_r)$. *An analogous result holds if* $g \in H^\infty(\mathbb{D}_0)$ *and* g *is* $H^2(\mathbb{D}_0)$*-inner*[2].

[2] i.e. there exists an $H^2(\mathbb{D})$-inner function h such that $g(z) = h(r/z)$.

Proof. We only consider the case where g is $H^2(\mathbb{D})$-inner. By Lemma 1.2.9, we have

$$||g||_{\mathrm{Mult}(\mathcal{H}^2(A_r))} = ||g||_{H^\infty(A_r)} = ||g||_{H^\infty(\mathbb{D})} = 1.$$

Now, let $h \in H^2(\mathbb{D})$ be arbitrary. Then,

$$||gh||_{\mathcal{H}^2(A_r)} = ||gh||_{H^2(\mathbb{D})} = ||h||_{H^2(\mathbb{D})} = ||h||_{\mathcal{H}^2(A_r)}.$$

Thus, g is a norm-attaining multiplier of $\mathcal{H}^2(A_r)$ with multiplier norm equal to 1, i.e. a subinner multiplier. □

Our next step will be to show that for functions in $H^2(\mathbb{D})$ (similarly, for functions in $H^2(\mathbb{D}_0)$), the property of $\mathcal{H}^2(A_r)$-free outerness coincides with outerness in the classical Hardy space sense. To do this, we will be needing the following two lemmata.

Let Mult$(H^2(\mathbb{D}), \mathcal{H}^2(A_r))$ denote the set of functions $\phi : A_r \to \mathbb{C}$ that multiply $H^2(\mathbb{D})$ (or, more precisely, the restrictions of $H^2(\mathbb{D})$ functions to A_r) boundedly into $\mathcal{H}^2(A_r)$. These multipliers turn out to be equal precisely to all functions $\phi = \sum c_n z^n \in \mathrm{Hol}(A_r)$ such that $\sum_{n>0} c_n z^n \in H^\infty(\mathbb{D})$ and $\sum_{n<0} c_n z^n \in H^2(\mathbb{D}_0)$.

Lemma 1.2.11.

$$Mult(H^2(\mathbb{D}), \mathcal{H}^2(A_r)) = \{f + g : f \in H^2(\mathbb{D}_0), g \in H^\infty(\mathbb{D})\}.$$

Also,

$$||f + g||_{Mult(H^2(\mathbb{D}),\mathcal{H}^2(A_r))} \leq \frac{||f||_{H^2(\mathbb{D}_0)}}{(1-r^2)} + ||g||_{H^\infty(\mathbb{D})}.$$

An analogous result holds for Mult$(H^2(\mathbb{D}_0), \mathcal{H}^2(A_r))$.

Proof. (Note that there is a slight abuse of notation here, as we are identifying $H^2(\mathbb{D})$ with the Hilbert function space consisting of the restrictions of all $H^2(\mathbb{D})$ functions to A_r. The kernel of that space is, of course, the restriction of the Szegő kernel for $\mathbb{D}$ to $A_r \times A_r$.)

Put $M = \mathrm{Mult}(H^2(\mathbb{D}), \mathcal{H}^2(A_r))$. Since $1 \in H^2(\mathbb{D})$, we obtain $M \subset \mathcal{H}^2(A_r)$. Now, let $h \in H^2(\mathbb{D}_0)$. By [113, Theorem 3.11], we obtain

$$h(\lambda)\overline{h(\mu)} \leq ||h||_{H^2(\mathbb{D}_0)} k_{H^2(\mathbb{D}_0)}(\lambda, \mu) \text{ in } \mathbb{D}_0 \times \mathbb{D}_0$$

$$\Rightarrow h(\lambda)\overline{h(\mu)} \leq ||h||_{H^2(\mathbb{D}_0)} k_{H^2(\mathbb{D}_0)}(\lambda,\mu) \text{ in } A_r \times A_r$$

$$\Rightarrow h(\lambda)\overline{h(\mu)} k_{H^2(\mathbb{D})}(\lambda,\mu) \leq ||h||_{H^2(\mathbb{D}_0)} k_{H^2(\mathbb{D}_0)}(\lambda,\mu) k_{H^2(\mathbb{D})}(\lambda,\mu) \text{ in } A_r \times A_r$$

$$\Rightarrow h(\lambda)\overline{h(\mu)} k_{H^2(\mathbb{D})}(\lambda,\mu) \leq \frac{||h||_{H^2(\mathbb{D}_0)}}{(1-r^2)} k_r(\lambda,\mu) \text{ in } A_r \times A_r.$$

By [113, Theorem 5.21], this last positivity condition implies that $h \in M$ and also that $||h||_M \leq ||h||_{H^2(\mathbb{D}_0)}/(1-r^2)$. This gives us $H^2(\mathbb{D}_0) \subset M$.

Now, consider $g \in H^\infty(\mathbb{D})$. For every $f \in H^2(\mathbb{D})$, we have

$$||gf||_{\mathcal{H}^2(A_r)} = ||gf||_{H^2(\mathbb{D})} \leq ||g||_{H^\infty(\mathbb{D})} ||f||_{H^2(\mathbb{D})}.$$

This shows that $g \in M$ and also that $||g||_M = ||g||_{H^\infty(\mathbb{D})}$. We conclude that $H^2(\mathbb{D}_0) + H^\infty(\mathbb{D}) \subset M$. Also, given $h \in H^2(\mathbb{D}_0)$ and $g \in H^\infty(\mathbb{D})$, we obtain

$$||h+g||_M \leq \frac{||h||_{H^2(\mathbb{D}_0)}}{(1-r^2)} + ||g||_{H^\infty(\mathbb{D})}.$$

For the converse, let $f \in M \subset \mathcal{H}^2(A_r)$. Choose any function $f_1 \in H^2(\mathbb{D})$ with the property that $f - f_1 \in H^2(\mathbb{D}_0)$. By our previous observations, we obtain $f - f_1 \in M$. Also, since $f_1 = f - (f - f_1) \in M \cap H^2(\mathbb{D})$ and $H^2(\mathbb{D})$ is contained isometrically in $\mathcal{H}^2(A_r)$, we have $f_1 \in \text{Mult}(H^2(\mathbb{D})) \subset M$ and also $||f_1||_M = ||f_1||_{H^\infty(\mathbb{D})}$. Thus, we obtain $f = (f - f_1) + f_1 \in H^2(\mathbb{D}_0) + H^\infty(\mathbb{D})$ and so we must have $M = H^2(\mathbb{D}_0) + H^\infty(\mathbb{D})$.

Now, we prove the $H^2(\mathbb{D}_0)$-version. Let $f \in H^\infty(\mathbb{D}_0)$ and $g \in H^2(\mathbb{D})$. Then, we have $Vf \in H^\infty(\mathbb{D})$ and $Vg \in H^2(\mathbb{D}_0)$ and so, for every $h \in H^2(\mathbb{D}_0)$, we can write

$$||(f+g)h||_{\mathcal{H}^2(A_r)} = ||(Vf + Vg)Vh||_{\mathcal{H}^2(A_r)}$$

$$\leq \left(||Vg||_{H^2(\mathbb{D}_0)}/(1-r^2) + ||Vf||_{H^\infty(\mathbb{D})}\right)||Vh||_{H^2(\mathbb{D})}$$

$$= \left(||g||_{H^2(\mathbb{D})}/(1-r^2) + ||f||_{H^\infty(\mathbb{D}_0)}\right)||h||_{H^2(\mathbb{D}_0)}.$$

Thus, if $M_0 = \text{Mult}(H^2(\mathbb{D}_0), \mathcal{H}^2(A_r))$, we obtain $||f+g||_{M_0} \leq ||g||_{H^2(\mathbb{D})}/(1-r^2) + ||f||_{H^\infty(\mathbb{D}_0)}$. This shows that $H^2(\mathbb{D}) + H^\infty(\mathbb{D}_0) \subset M_0$.

Conversely, suppose $\phi \in \text{Mult}(H^2(\mathbb{D}_0), \mathcal{H}^2(A_r))$. Then, it is easy to see that $V\phi \in \text{Mult}(H^2(\mathbb{D}), \mathcal{H}^2(A_r))$, hence $V\phi = f + g$, where $f \in H^2(\mathbb{D}_0)$ and $g \in H^\infty(\mathbb{D})$. This implies that $\phi = Vf + Vg \in H^2(\mathbb{D}) + H^\infty(\mathbb{D}_0)$ and our proof is complete. □

Lemma 1.2.12. *Let $f, g \in H^2(\mathbb{D})$ and suppose that*

$$\langle \phi f, f\rangle_{H^2(\mathbb{D})} = \langle \phi g, g\rangle_{H^2(\mathbb{D})}, \textit{ for all } \phi \in H^\infty(\mathbb{D}).$$

Then,

$$\langle \phi f, f\rangle_{\mathcal{H}^2(A_r)} = \langle \phi g, g\rangle_{\mathcal{H}^2(A_r)}, \textit{ for all } \phi \in \textit{Mult}(\mathcal{H}^2(A_r)).$$

Proof. Suppose f, g satisfy the given assumptions and let $\phi \in H^\infty(A_r) = \text{Mult}(\mathcal{H}^2(A_r))$. Write $\phi = \phi_1 + \phi_2$, where $\phi_1 \in H^2(\mathbb{D})$ and $\phi_2 = \sum_{-\infty}^{n=-1} a_n z^n \in \frac{1}{z}H^2(\mathbb{D}_0)$. By the maximum modulus principle, we obtain $\phi_1 \in H^\infty(\mathbb{D}), \phi_2 \in H^\infty(\mathbb{D}_0)$. Now, notice that

$$\lim_{k\to-\infty} \sum_{k}^{n=-1} a_n z^n f = \left(\sum_{-\infty}^{n=-1} a_n z^n\right) f$$

in the $||\cdot||_{\mathcal{H}^2(A_r)}$ norm, as $H^2(\mathbb{D}) \subset \text{Mult}(H^2(\mathbb{D}_0), \mathcal{H}^2(A_r))$, due to Lemma 1.2.11. We can write

$$\langle \phi f, f\rangle_{\mathcal{H}^2(A_r)} = \langle \phi_1 f, f\rangle_{\mathcal{H}^2(A_r)} + \langle \phi_2 f, f\rangle_{\mathcal{H}^2(A_r)},$$

where

$$\begin{aligned} \langle \phi_1 f, f\rangle_{\mathcal{H}^2(A_r)} &= \langle \phi_1 f, f\rangle_{H^2(\mathbb{D})} \\ &= \langle \phi_1 g, g\rangle_{H^2(\mathbb{D})} \text{ (by assumption)} \\ &= \langle \phi_1 g, g\rangle_{\mathcal{H}^2(A_r)} \end{aligned}$$

and also

$$\begin{aligned} \langle \phi_2 f, f\rangle_{\mathcal{H}^2(A_r)} &= \left\langle \left(\lim_{k\to-\infty} \sum_{k}^{n=-1} a_n z^n\right) f, f \right\rangle_{\mathcal{H}^2(A_r)} \\ &= \lim_{k\to-\infty} \sum_{k}^{n=-1} a_n \langle z^n f, f\rangle_{\mathcal{H}^2(A_r)} \\ &= \lim_{k\to-\infty} \sum_{k}^{n=-1} a_n \langle f, z^{-n} f\rangle_{\mathcal{H}^2(A_r)}, \end{aligned}$$

as $f \in H^2(\mathbb{D})$. Since $f, z^{-n}f$ (where $n \leq -1$) are now both in $H^2(\mathbb{D})$, we obtain

$$\langle \phi_2 f, f\rangle_{\mathcal{H}^2(A_r)}] = \lim_{k\to-\infty} \sum_{k}^{n=-1} a_n \langle f, z^{-n}f\rangle_{H^2(\mathbb{D})}$$

$$= \lim_{k\to-\infty} \sum_{k}^{n=-1} a_n \langle g, z^{-n}g\rangle_{H^2(\mathbb{D})},$$

where the last equality is because of our initial assumptions. Working our way in the opposite direction, we can then show that

$$\lim_{k\to-\infty} \sum_{k}^{n=-1} a_n \langle g, z^{-n}g\rangle_{H^2(\mathbb{D})} = \langle \phi_2 g, g\rangle_{\mathcal{H}^2(A_r)}.$$

Hence,

$$\langle \phi f, f\rangle_{\mathcal{H}^2(A_r)} = \langle \phi_1 g, g\rangle_{\mathcal{H}^2(A_r)} + \langle \phi_2 g, g\rangle_{\mathcal{H}^2(A_r)} = \langle \phi g, g\rangle_{\mathcal{H}^2(A_r)},$$

which concludes the proof. □

We can now characterize those functions $f \in H^2(\mathbb{D}) \subset \mathcal{H}^2(A_r)$ that are $\mathcal{H}^2(A_r)$-free outer.

Theorem 1.2.13. *Suppose $f \in H^2(\mathbb{D})$. Then, f is $H^2(\mathbb{D})$-outer if and only if f is $\mathcal{H}^2(A_r)$-free outer. An analogous result holds for $H^2(\mathbb{D}_0)$-outer functions.*

Proof. Fix an arbitrary $z_0 \in A_r$.

First, suppose that $f \in H^2(\mathbb{D})$ is not $H^2(\mathbb{D})$-outer and also let g denote the $H^2(\mathbb{D})$-outer factor of f. Since $|f| = |g|$ on $\partial\mathbb{D}$, we obtain

$$\langle \phi f, f\rangle_{H^2(\mathbb{D})} = \langle \phi g, g\rangle_{H^2(\mathbb{D})}, \text{ for all } \phi \in H^\infty(\mathbb{D}),$$

and also $|g(z_0)| > |f(z_0)|$. But then, Lemma 1.2.12 tells us that

$$\langle \phi f, f\rangle_{\mathcal{H}^2(A_r)} = \langle \phi g, g\rangle_{\mathcal{H}^2(A_r)}, \text{ for all } \phi \in H^\infty(A_r)$$

$$\Rightarrow P_f = P_g \text{ in } \mathcal{H}^2(A_r).$$

Since $|g(z_0)| > |f(z_0)|$, Corollary 1.2.7 implies that f is not $\mathcal{H}^2(A_r)$-free outer.

Conversely, suppose that f is $H^2(\mathbb{D})$-outer. Suppose also that f is not $\mathcal{H}^2(A_r)$-free outer. We will reach a contradiction.

Write $f = gh$ for the subinner-free outer factorization of f in $\mathcal{H}^2(A_r)$. We claim that $h \notin H^2(\mathbb{D})$. Indeed, if $h \in H^2(\mathbb{D})$ we can write

$$P_f = P_h \text{ in } \mathcal{H}^2(A_r) \text{ (as } h \text{ is the free outer factor of } f)$$

$$\Rightarrow \langle \phi f, f \rangle_{\mathcal{H}^2(A_r)} = \langle \phi h, h \rangle_{\mathcal{H}^2(A_r)}, \text{ for all } \phi \in H^\infty(A_r)$$

$$\Rightarrow \langle \phi f, f \rangle_{H^2(\mathbb{D})} = \langle \phi h, h \rangle_{H^2(\mathbb{D})}, \text{ for all } \phi \in H^\infty(\mathbb{D})$$

$$\Rightarrow P_f = P_h \text{ in } H^2(\mathbb{D})$$

$$\Rightarrow |f(z)| \geq |h(z)|, \text{ for all } z \in \mathbb{D},$$

as f is $H^2(\mathbb{D})$-outer. But h is the free outer factor of f and f is *not* $\mathcal{H}^2(A_r)$-free outer, so Corollary 1.2.7 implies that $|h(z_0)| > |f(z_0)|$, a contradiction.

Hence, $h \notin H^2(\mathbb{D})$. This implies the existence of $k < 0$ such that the term $a_k z^k$ in the Laurent expansion of h is nonzero. Thus $||zh||_{\mathcal{H}^2(A_r)} < ||h||_{\mathcal{H}^2(A_r)}$. But then, we obtain

$$||f||_{\mathcal{H}^2(A_r)} = ||f||_{H^2(\mathbb{D})} = ||zf||_{H^2(\mathbb{D})}$$

$$= ||zf||_{\mathcal{H}^2(A_r)} = ||g(zh)||_{\mathcal{H}^2(A_r)} \leq ||zh||_{\mathcal{H}^2(A_r)}$$

$$< ||h||_{\mathcal{H}^2(A_r)} = ||f||_{\mathcal{H}^2(A_r)},$$

a contradiction again. Thus, f must be $\mathcal{H}^2(A_r)$-free outer and our proof is complete.

For the $H^2(\mathbb{D}_0)$-version, note that f is $H^2(\mathbb{D}_0)$-outer if and only if Vf is $H^2(\mathbb{D})$-outer. By our previous result, this is equivalent to Vf (and hence f) being $\mathcal{H}^2(A_r)$-free outer. □

We now show Theorem 1.2.2 from the introduction.

Proof of Theorem 1.2.2. Let $f \in H^2(\mathbb{D})$ and write $f = \phi h$ for the classical inner-outer factorization in $H^2(\mathbb{D})$. By Theorem 1.2.13, h is $\mathcal{H}^2(A_r)$-free outer, while ϕ is subinner by Proposition 1.2.10. Notice also that $||f||_{\mathcal{H}^2(A_r)} = ||h||_{\mathcal{H}^2(A_r)}$ and so $f = \phi h$ coincides (up to multiplication by unimodular constants) with the subinner-free outer factorization of f in $\mathcal{H}^2(A_r)$. The $H^2(\mathbb{D}_0)$-version is entirely analogous. □

We have proved that a function in $\mathcal{H}^2(A_r)$ that is also analytic on the unit disk is $\mathcal{H}^2(A_r)$-free outer if and only if it is $H^2(\mathbb{D})$-outer. Theorem 1.2.3 says that an analogous result holds for $\mathcal{H}^2(A_r)$-subinner functions that are also analytic on $\mathbb{D}$. The key result used in the proof is a modified subinner-free outer factorization for functions in spaces whose reproducing kernel has a CP factor.

Proof of Theorem 1.2.3. Suppose $\phi \in H^\infty(\mathbb{D})$.

If ϕ is $H^2(\mathbb{D})$-inner, then it must also be $\mathcal{H}^2(A_r)$-subinner, by Proposition 1.2.10.

Now, suppose that ϕ is $\mathcal{H}^2(A_r)$-subinner. Thus, $||\phi||_{H^\infty(\mathbb{D})} = 1$ (by Lemma 1.2.9) and also there exists a nonzero $f \in \mathcal{H}^2(A_r)$ such that

$$||\phi f||_{\mathcal{H}^2(A_r)} = ||f||_{\mathcal{H}^2(A_r)}.$$

Letting s denote the classical Szegő kernel on $\mathbb{D}$, it is easy to see that k_r/s is positive semi-definite on $A_r \times A_r$. Hence, by [16, Theorem 1.10], we can find a (unique, up to multiplication by unimodular constants) pair of nonzero functions $\psi \in \text{Mult}(H^2(\mathbb{D}), \mathcal{H}^2(A_r))$ and $h \in H^2(\mathbb{D})$ such that

i) $f = \psi h$,
ii) $||h||_{H^2(\mathbb{D})} = ||f||_{\mathcal{H}^2(A_r)}$,
iii) h is $H^2(\mathbb{D})$-free outer,
iv) $||\psi||_{\text{Mult}(H^2(\mathbb{D}),\mathcal{H}^2(A_r))} \leq 1$.

Since $\phi h \in H^2(\mathbb{D})$, we can write

$$||f||_{\mathcal{H}^2(A_r)} = ||\phi f||_{\mathcal{H}^2(A_r)} = ||\phi\psi h||_{\mathcal{H}^2(A_r)} = ||\psi(\phi h)||_{\mathcal{H}^2(A_r)}$$

$$\leq ||\phi h||_{H^2(\mathbb{D})} \qquad \text{(by item (iv))}$$

$$\leq ||h||_{H^2(\mathbb{D})} \quad \text{(since } ||\phi||_{H^\infty(\mathbb{D})} = 1)$$

$$= ||f||_{\mathcal{H}^2(A_r)} \qquad \text{(by item (ii))}$$

Thus, we must have $||\phi h||_{H^2(\mathbb{D})} = ||h||_{H^2(\mathbb{D})}$, which implies that ϕ is $H^2(\mathbb{D})$-inner.

For the $H^2(\mathbb{D}_0)$-version, note that f is $H^2(\mathbb{D}_0)$-inner if and only if Vf is $H^2(\mathbb{D})$-inner. □

We now combine the embedding (1.4) of $\mathcal{H}^2(A_r)$ into H^2_2 with Theorem 1.2.2 to obtain new examples of free outer functions in H^2_2.

Corollary 1.2.14. *Let $f = \sum_{n=0}^{\infty} a_n z^n$ be an $H^2(\mathbb{D})$-outer function. Then, for every $r \in (0,1)$, the function $F \in H^2_2$ uniquely defined by*

$$\langle F, z_1^i z_2^j \rangle_{H^2_2} = \frac{1-r^2}{1+r^2} \frac{r^j}{\left(\sqrt{r^2+1}\right)^{i+j}} a_{i-j}, \quad (\text{for all } i, j \geq 0)$$

is H^2_2-free outer (put $a_k = 0$ for negative k). Hence, if $T : H^2_d \to \mathcal{F}^2_d$ denotes the natural isometric embedding of H^2_d into $\mathcal{F}^2_d$, $T(F)$ is both left- and right-outer in $\mathcal{F}^2_d$. An analogous result holds for $H^2(\mathbb{D}_0)$-outer functions.

Proof. We only show the $H^2(\mathbb{D})$ version. Suppose $f = \sum_{n=0}^{\infty} a_n z^n$ is $H^2(\mathbb{D})$-outer. By our previous results, f will be $\mathcal{H}^2(A_r)$-free outer. Consider the rescaled kernel defined on $A_r \times A_r$ by

$$k_r'(\lambda, \mu) := \frac{1+r^2}{1-r^2} k_r(\lambda, \mu) = \frac{1+r^2}{(1-\lambda\overline{\mu})(1-r^2/\lambda\overline{\mu})} = k_2(u(\lambda), u(\mu)), \tag{1.5}$$

where k_2 is the kernel of H^2_2. k_r' is now a normalized CNP kernel. It induces the holomorphic function space $\mathcal{H}_{k_r'}$ on A_r, the norm of which is simply the norm of $\mathcal{H}^2(A_r)$ multiplied by $\sqrt{(1-r^2)/(1+r^2)}$. We easily see that f will be $\mathcal{H}_{k'}$-free outer as well.

Now, recall the isometric identification $S = P_{\mathcal{H}^2_d} T$ (see (1.3)) of H^2_d with the symmetric Fock space $\mathcal{H}^2_d$. By Theorem 1.2.8, we obtain that $F := S^{-1} U f$ is H^2_2-free outer, where $U : \mathcal{H}_{k'} \to \mathcal{F}^2_2$ denotes the embedding of $\mathcal{H}_{k'}$ into $\mathcal{H}^2_2$ induced by (1.5). This function satisfies (for all $i, j \geq 0$):

$$\begin{aligned}
\langle F, z_1^i z_2^j \rangle_{H^2_2} &= \langle F, S^{-1} S(z_1^i z_2^j) \rangle_{H^2_2} \\
&= \langle f, U^* S(z_1^i z_2^j) \rangle_{\mathcal{H}_{k'}} \\
&= \left\langle f, \left(\frac{z}{\sqrt{r^2+1}}\right)^i \left(\frac{r}{\sqrt{r^2+1}} \frac{1}{z}\right)^j \right\rangle_{\mathcal{H}_{k'}} \\
&= \frac{1-r^2}{1+r^2} \left\langle f, \frac{r^j}{\left(\sqrt{r^2+1}\right)^{i+j}} z^{i-j} \right\rangle_{\mathcal{H}^2(A_r)}
\end{aligned}$$

$$= \frac{1-r^2}{1+r^2} \frac{r^j}{\left(\sqrt{r^2+1}\right)^{i+j}} a_{i-j},$$

for any $r \in (0,1)$, as desired (for negative k, interpret a_k as being zero). The rest of the Corollary follows immediately from Theorem 1.2.8. □

Example. Consider the function $f(z) = z - \lambda$. This function is

- $\mathcal{H}^2(A_r)$-subinner if and only if $\lambda = 0$ (by Theorem 1.2.3);
- $\mathcal{H}^2(A_r)$-free outer, if $|\lambda| \geq 1$ (by Theorem 1.2.13);
- not $\mathcal{H}^2(A_r)$-cyclic (and hence *not* $\mathcal{H}^2(A_r)$-free outer), if $r < |\lambda| < 1$;
- $\mathcal{H}^2(A_r)$-cyclic but *not* $\mathcal{H}^2(A_r)$-free outer, if $|\lambda| \leq r$ (by Theorem 1.2.13).

To conclude, we have a complete characterization of subinner and free outer functions in $\mathcal{H}^2(A_r) \cap \mathrm{Hol}(\mathbb{D})$. Of course, it would be even more interesting if we were able to describe generic subinner and/or free outer functions in $\mathcal{H}^2(A_r)$. We pose this as a question.

Question 1.2.15. Let $f \in \mathcal{H}^2(A_r)$. What is the subinner-free outer factorization of f?

1.3 Interpolating Sequences for Pairs of Spaces

1.3.1 Background

Let $\mathcal{H}$ denote a reproducing kernel Hilbert space on a nonempty set X. Let Mult($\mathcal{H}$) denote the multiplier algebra of $\mathcal{H}$, that is the set of all functions ϕ on X that multiply $\mathcal{H}$ into itself. A sequence $\{\lambda_i\} \subset X$ is called an *interpolating sequence for* Mult($\mathcal{H}$) ((IM) for short) if, whenever $\{w_i\} \subset \ell^\infty$, there exists a multiplier ϕ such that $\phi(\lambda_i) = w_i$ for all i. Consider also the following *weighted restriction operator* associated to $\{\lambda_i\} \subset X$

$$T : f \mapsto \left(\frac{f(\lambda_i)}{||k_{\lambda_i}||}\right),$$

which maps $\mathcal{H}$ into the space of all complex sequences. $\{\lambda_i\}$ is called an *interpolating sequence for* $\mathcal{H}$ ((IH) for short) if $T(\mathcal{H}) = \ell^2$. In general, the set of all Mult($\mathcal{H}$)-interpolating sequences will be a strict subset of the set of all $\mathcal{H}$-interpolating sequences. However, these two classes turn out to coincide in many well-studied reproducing kernel Hilbert spaces. In particular, a class of spaces that share this property is the class of all *complete Pick spaces*.

Interpolating sequences are often characterized by appropriate separation and Carleson measure conditions. If $\mathcal{H}_k$ is a reproducing kernel Hilbert space with kernel k, then

$$d_k(z, w) = \sqrt{1 - \frac{|\langle k_z, k_w\rangle|^2}{||k_z||^2||k_w||^2}}, \quad z, w \in X,$$

i.e. the distance from $k_z/||k_z||$ to the space spanned by k_w, defines a pseudometric on X (see [8, Lemma 9.9]). Actually, d_k is a metric on X if and only if no two kernel functions k_z, k_w (with $z \neq w$) are linearly dependent. In general, not much is known about d_k and many natural questions remain open (see [21]). In the setting of the Hardy space, d_k is precisely the pseudohyperbolic metric on the unit disk. The sequence $\{\lambda_i\} \subset X$ is said to be *weakly separated by* k if there exists $\epsilon > 0$ such that

$$d_k(\lambda_i, \lambda_j) \geq \epsilon, \quad \text{for all } i \neq j. \tag{WS}$$

We also say that $\{\lambda_i\}$ satisfies the *Carleson measure condition for* $\mathcal{H}_k$ if $\mu = \sum_{i=1}^{\infty} \frac{1}{||k_{\lambda_i}||^2}\delta_{\lambda_i}$ is a Carleson measure for the Hilbert function space $\mathcal{H}_k$. This is equivalent to the existence of $C > 0$ such that

$$\int |f|^2\, d\mu = \sum_{i=1}^{\infty} \frac{|f(\lambda_i)|^2}{||k_{\lambda_i}||^2} \leq C||f||^2_{\mathcal{H}_k}, \quad \forall f \in \mathcal{H}_k. \tag{CM}$$

Carleson [45] and Shapiro-Shields [131] proved that (IM), (IH) and (CM)+(WS) all coincide in the setting of the Hardy space on $\mathbb{D}$. Bishop [36] and Marshall-Sundberg [93] showed that this is still the case if the Hardy space is replaced by the Dirichlet space on $\mathbb{D}$.

As already stated, (IH) and (IM) continue to be equivalent in any complete Pick space. Also, it is not hard to see that the implication (IH)⇒ (CM)+(WS) is valid in every reproducing kernel Hilbert space. The question whether the converse always holds true in a complete Pick space (first formulated by Agler-McCarthy in [8, Question 9.57] and by Seip in [128, Conjecture 1, p. 33]) remained open for more than ten years. It was finally given an affirmative answer by Aleman, Hartz, McCarthy and Richter in the breakthrough paper [17], as a consequence of the positive solution of the Kadison-Singer problem [92]. An alternative proof, using the *Column-Row property* for complete Pick spaces, can be found in [79]. See also [37] for partial progress regarding this problem prior to [17].

We will be concerned with the concept of interpolating sequences for multipliers between spaces, which we now define.

Let k, ℓ be two reproducing kernels on a set X such that $k_z, \ell_z \neq 0$ for all $z \in X$. We will denote the corresponding reproducing kernel Hilbert spaces by $\mathcal{H}_k$ and $\mathcal{H}_\ell$. If $\phi \in \text{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)$, then $\phi \cdot k_z \in \mathcal{H}_\ell$ and so the function ϕ satisfies a growth estimate:

$$|\phi(z)| = \frac{|\phi(z)k_z(z)|}{||k_z||^2} = \frac{\langle \phi k_z, \ell_z\rangle}{||k_z||^2} \leq ||\phi||_{\text{Mult}(\mathcal{H}_k,\mathcal{H}_\ell)}\frac{||\ell_z||}{||k_z||}, \quad \forall z \in X, \tag{1.6}$$

where $|\phi||_{\text{Mult}(\mathcal{H}_k,\mathcal{H}_\ell)}$ denotes the norm of the multiplication operator $M_\phi : \mathcal{H}_k \to \mathcal{H}_\ell$. A sequence $\{\lambda_i\} \subset X$ will be called an *interpolating sequence for Mult*$(\mathcal{H}_k, \mathcal{H}_\ell)$ if, whenever $\{w_i\} \subset \ell^\infty$, there exists a multiplier $\phi \in \text{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)$ such that $\phi(\lambda_i) = w_i \frac{||\ell_{\lambda_i}||}{||k_{\lambda_i}||}$ for all i.

Aleman, Hartz, M^cCarthy and Richter investigated interpolating sequences for pairs of spaces in [17]. For an arbitrary pair (k, ℓ), it can be shown that Mult$(\mathcal{H}_k, \mathcal{H}_\ell)$-interpolating sequences satisfy the Carleson measure condition (CM) for $\mathcal{H}_k$ and are weakly separated by ℓ. One does not expect these two conditions to also be sufficient in general. But what if, in addition, we assume k to be a *complete Pick factor* of ℓ? This is the case, for example, whenever $\mathcal{H}_k$ is the Hardy space on $\mathbb{D}$ and the operator M_z of multiplication by the coordinate function defines a contraction operator on $\mathcal{H}_\ell$.

Question 1.3.1 (Aleman, Hartz, M^cCarthy and Richter [17]). *Let s be a normalized complete Pick kernel on X and let $\ell = \mathrm{g}s$, where g is a kernel on X. Is it true that a sequence $\{\lambda_i\} \subset X$ is interpolating for Mult$(\mathcal{H}_s, \mathcal{H}_\ell)$ if and only if it satisfies the Carleson measure condition for $\mathcal{H}_s$ and is weakly separated by ℓ?*

Aleman, Hartz, M^cCarthy and Richter were able to give a positive answer [17, Theorem 1.3] to Question 1.3.1 under the extra assumption that ℓ is a power of a complete Pick kernel (notice that, by [8, Remark 8.10] and the Schur product theorem, the expression $s_w^t(z)$ defines a reproducing kernel whenever s is a normalized complete Pick kernel and $t > 0$).

Theorem 1.3.2 (Aleman, Hartz, M^cCarthy and Richter [17]). *Let s_1, s_2 be normalized complete Pick kernels on X such that s_2/s_1 is positive semi-definite, and let $t \geq 1$. Then, a sequence is interpolating for Mult$(\mathcal{H}_{s_1}, \mathcal{H}_{s_2^t})$ if and only if it satisfies the Carleson measure condition for $\mathcal{H}_{s_1}$ and is weakly separated by s_2^t (equivalently, by s_2).*

Note that, for $s_1 = s_2 = s$ and $t = 1$, their result recovers the characterization of Mult$(\mathcal{H}_s)$-interpolating sequences in the setting of the complete Pick space $\mathcal{H}_s$.

In subsection 1.3.3, we provide a complete characterization of Mult$(\mathcal{H}_s, \mathcal{H}_\ell)$-interpolating sequences, thus extending Theorem 1.3.2. Surprisingly, the conditions of Question 1.3.1 turn out not to be sufficient, in general, for Mult$(\mathcal{H}_s, \mathcal{H}_\ell)$-interpolation. In particular, a stronger notion of weak separation is required.

Definition 1.3.3. Suppose k is a reproducing kernel on a nonempty set X and $\{\lambda_i\} \subset X$. For any $n \geq 2$, we say that $\{\lambda_i\}$ is *n-weakly separated by k* if there exists $\epsilon > 0$ such that for every n-point subset $\{\mu_1, \dots, \mu_n\} \subset \{\lambda_i\}$ we have

$$\mathrm{dist}\left(\frac{k_{\mu_1}}{||k_{\mu_1}||}, \ \mathrm{span}\left\{\frac{k_{\mu_2}}{||k_{\mu_2}||}, \dots, \frac{k_{\mu_n}}{||k_{\mu_n}||}\right\}\right) \geq \epsilon.$$

Notice that 2-weak separation by k coincides with weak separation by k.

We can now state our first result.

Theorem 1.3.4. *Suppose s is a normalized complete Pick kernel and $\ell = gs$ for some (positive semi-definite) kernel g. Then, a sequence $\{\lambda_i\} \subset X$ is interpolating for Mult$(\mathcal{H}_s, \mathcal{H}_\ell)$ if and only if it satisfies the Carleson measure condition for $\mathcal{H}_s$ and is n-weakly separated by ℓ, for every $n \geq 2$.*

Passing to n-weak separation is a necessity and not merely an artifact of the proof of Theorem 1.3.4, as the following result shows.

Theorem 1.3.5. *There exists a kernel ℓ with a normalized complete Pick factor s and the following property:*
For every $n \geq 2$, there exists a sequence $\{\lambda_i\} \subset X$ that satisfies the Carleson measure condition for $\mathcal{H}_s$ and is n-weakly separated, but not $(n+1)$-weakly separated by ℓ (and hence, not Mult$(\mathcal{H}_s, \mathcal{H}_\ell)$-interpolating).

Thus, the conditions stated in Question 1.3.1 are not, in general, sufficient. A natural line of inquiry then emerges: which conditions do we need to impose on a pair (s, ℓ) for Question 1.3.1 to have a positive answer? We investigate this in subsection 1.3.5. In particular, Theorem 1.3.21 tells us that, at least for "reasonable" pairs (s, ℓ), the issue lies solely with the possible existence of weakly separated sequences that are not n-weakly separated by ℓ (for some $n \geq 3$). In other words, the only obstruction to Question 1.3.1 having a positive answer is that ℓ might not possess the following (rather peculiar) property: for any fixed $n \geq 2$, a kernel $\hat{\ell}_z$ can be "close" to the span of n other kernels $\hat{\ell}_{w_1}, \hat{\ell}_{w_2}, \dots, \hat{\ell}_{w_n}$ if and only if it is "close" to one of them. This implies, perhaps surprisingly, that the answer to Question 1.3.1 is a matter that depends entirely (at least for pairs satisfying the hypotheses of Theorem 1.3.21) on the kernel ℓ; the specific nature of the complete Pick factor s turns out to be irrelevant here. Kernels for which weak separation of a sequence is always equivalent to n-weak separation (for *every* n) will be said to have the *automatic separation property* (also called *AS property* for short).

The question then becomes: which kernels have the automatic separation property? This is explored in subsections 1.3.6-1.3.7. A first class of examples is furnished by kernels satisfying a stronger property, the *multiplier separation property*. These are kernels ℓ such that weak separation by ℓ is always equivalent to weak separation by $\text{Mult}(\mathcal{H}_\ell)$, the latter condition being equivalent to the existence of $\epsilon > 0$ such that for any two points $\lambda_i \neq \lambda_j$, we can find $\phi_{ij} \in \text{Mult}(\mathcal{H}_\ell)$ of norm at most 1 satisfying $\phi_{ij}(\lambda_i) = \epsilon$ and $\phi_{ij}(\lambda_j) = 0$. Examples (to be found in subsection 1.3.6) include products of powers of 2-point Pick kernels (Example 1.3.6) and Hardy spaces on finitely-connected planar domains (Example 1.3.6). In subsection 1.3.7, we give a general criterion for the AS property. The idea here (see Theorem 1.3.35 for a precise statement) is that a kernel ℓ has the AS property if and only if any weakly separated finite union of "sufficiently sparse" sequences forms an $\mathcal{H}_\ell$-interpolating sequence. As a consequence, we discover that an even larger number of well-studied spaces possess AS kernels. These include "large" weighted Bergman spaces (Example 1.3.7) and weighted Bargmann-Fock spaces (Example 1.3.7). Subsection 1.3.7 culminates in Theorem 1.3.39, which describes a large class of pairs (s, ℓ) for which Question 1.3.1 has a positive answer (this includes all pairs (s, ℓ) such that ℓ is one of the kernels from the previous examples and s is a complete Pick factor of ℓ).

Finally, it should be noted that the pair (s, ℓ) constructed in the proof of Theorem 1.3.5, while offering a counterexample to Question 1.3.1, is not a natural setting for the solution of interpolation problems. One might then wonder whether imposing a few weak regularity conditions (like the ones in the statement of Theorem 1.3.35) on (s, ℓ) would always force the pair to behave according to the manner predicted by Question 1.3.1. This doesn't seem to be the case. In particular, subsection

1.3.8 contains the construction of a "nice" holomorphic pair (s, ℓ) on the bidisk which provides us with a more natural counterexample to Question 1.3.1 (however, that construction is not sufficient to establish Theorem 1.3.5 in its entirety).

1.3.2 Preliminaries

Suppose k, ℓ are reproducing kernels on X. Then, $\mathrm{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)$ is the collection of functions $\phi : X \to \mathbb{C}$ such that $(M_\phi f)(z) = \phi(z)f(z)$ defines a bounded operator $M_\phi : \mathcal{H}_k \to \mathcal{H}_\ell$. It is easy to see that for $\phi \in \mathrm{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)$ one has

$$M_\phi^* \ell_w = \overline{\phi(w)} k_w,$$

for all $w \in X$. Moreover, the multipliers ϕ with $||\phi|| \leq M$ are characterized (see [113, Theorem 5.21]) by the positivity of

$$M^2 \ell_w(z) - \phi(z)\overline{\phi(w)} k_w(z). \tag{1.7}$$

We say that the pair (k, ℓ) has the *Pick property* if, for every finite sequence of distinct points $\lambda_1, \dots, \lambda_N \in X$ and every sequence $w_1, \dots, w_N \in \mathbb{C}$, positivity of the matrix

$$\left[\ell_{\lambda_i}(\lambda_j) - w_j \overline{w}_i k_{\lambda_i}(\lambda_j)\right]_{i,j=1}^N \tag{1.8}$$

implies the existence of a multiplier $\phi \in \mathrm{Mult}(\mathcal{H}_k, \mathcal{H}_\ell)$ of norm at most 1 that satisfies

$$\phi(\lambda_i) = w_i, \quad \text{for all } 1 \leq i \leq N.$$

Note that, as observed in [17, Section 4], if the pair (k, ℓ) has the Pick property, then one can solve Pick problems with infinitely many points.

Now, assume that s is an irreducible complete Pick kernel normalized at some point, hence

$$s_w(z) = \frac{1}{1 - \langle b(z), b(w) \rangle},$$

where $b : X \to \mathbb{B}_d$. Assume also that ℓ is another kernel on X such that ℓ/s is positive semi-definite (denoted by $\ell/s >> 0$). Simple examples of such kernels are given by $\ell = s^t$, $t \geq 1$. Note that the positivity condition for ℓ/s is satisfied if and only if $b \in \text{Mult}(\ell \otimes \mathbb{C}^d, \ell)$ with $||M_b|| \leq 1$ (if $d = \infty$, $\mathbb{C}^d$ is treated as ℓ^2), see [15, Lemma 2.2] for a proof. In recent years, kernels with a complete Pick factor have been investigated in regard to invariant subspaces [54], factorization theorems [15], [16] and the Column-Row property [79, Section 3.8].

The following result is very useful in the context of $\text{Mult}(\mathcal{H}_s, \mathcal{H}_\ell)$-interpolating sequences. It appears as Proposition 4.4 in [17], where it is proved as an application of Leech's theorem [8, Theorem 8.57].

Theorem 1.3.6. *Suppose ℓ, s are kernels on X such that s has the complete Pick property and $\ell/s >> 0$. Then, the pair (s, ℓ) has the Pick property.*

An important consequence of Theorem 1.3.6 is:

Theorem 1.3.7 (Aleman, Hartz, McCarthy and Richter [17])**.** *Suppose ℓ, s are kernels on X such that s is a normalized complete Pick kernel and $\ell/s >> 0$.*

(a) A sequence is interpolating for Mult($\mathcal{H}_s, \mathcal{H}_\ell$) if and only if it satisfies the Carleson measure condition (CM) for $\mathcal{H}_s$ and is interpolating for $\mathcal{H}_\ell$.

(b) If a sequence is weakly separated by s, then it is interpolating for Mult($\mathcal{H}_s, \mathcal{H}_\ell$) if and only if it is interpolating for Mult($\mathcal{H}_s$).

Note that, in general, a $\text{Mult}(\mathcal{H}_s, \mathcal{H}_\ell)$-interpolating sequence needn't be weakly separated by s ([17, Example 4.13]).

Let $\mathcal{H}_k$ be a reproducing kernel Hilbert space on a set X, with kernel k. We write $\hat{k}_z = k_z/||k_z||$ for the normalized kernel function at z. Let $\{\lambda_i\}$ be a sequence of distinct points in X. The Grammian, or Gram matrix, associated with the sequence is the (infinite) matrix $G(k) = [G_{i,j}]$, where

$$G_{i,j} = \langle \hat{k}_{\lambda_i}, \hat{k}_{\lambda_j} \rangle = \frac{k(\lambda_j, \lambda_i)}{\sqrt{k(\lambda_j, \lambda_j) k(\lambda_i, \lambda_i)}}.$$

We say that the sequence $\{\lambda_i\} \subset X$ has a bounded Grammian (BG) if the Gram matrix, thought of as an operator on ℓ^2, is bounded; we shall say that it is bounded below (BB) if the Gram matrix is bounded below on ℓ^2. A consequence of Carleson's characterization [45] is that if the Grammian of a sequence for the Szegő kernel $s_w(z) = \frac{1}{1-z\overline{w}}$ is bounded below, then it is bounded above (see e.g. [8, Chapter 9]). This is no longer true in the Dirichlet space; see [36]. Sequences satisfying (BB) have also been called *simply interpolating* and have been studied in [22], [48] and [49] in the setting of the Dirichlet space.

The following lemma is well-known (see [8, Chapter 9] for a proof).

Lemma 1.3.8.

(a) The Grammian is bounded (BG) if and only if the sequence satisfies the Carleson measure condition (CM) for $\mathcal{H}_k$.

(b) The following three conditions are equivalent:

(i) the Grammian is bounded and bounded below (BG)+(BB),

(ii) the functions $\hat{k}_{\lambda_i}$ form a Riesz sequence, i.e. there exist $c_1, c_2 > 0$ such that for all scalars a_i,

$$c_1 \sum_i |a_i|^2 \leq \left|\left| \sum_i a_i \hat{k}_{\lambda_i} \right|\right|^2 \leq c_2 \sum_i |a_i|^2,$$

(iii) the sequence is interpolating for $\mathcal{H}_k$ (IH).

We will also be making crucial use of the following result, which is part of [8, Theorem 9.46]. We use $\{e_i\}$ to denote the standard orthonormal basis for ℓ^2.

Theorem 1.3.9. *Let k be an irreducible complete Pick kernel on X, let $\{\lambda_i\} \subset X$, and let G denote the Grammian associated with $\{\lambda_i\}$. Then, G is bounded if and only if there exists is a multiplier*

$\Psi \in Mult(H_s, H_s \otimes \ell^2)$ *such that*

$$\Psi(\lambda_i) = e_i = \begin{pmatrix} * \\ \vdots \\ * \\ 1 \\ * \\ \vdots \end{pmatrix},$$

for every i.

As noted in [17], a more restrictive definition of irreducibility is used in the statement given in [8], however our more relaxed definition suffices for the proof to go through.

Finally, we record a basic Hilbert space lemma (as seen in [131, Section I]) which will be used repeatedly throughout the paper, often without special mention.

Lemma 1.3.10. *Suppose* $\mathcal{H}$ *is a Hilbert space and* $v_0, v_1, \ldots, v_n \in \mathcal{H}$. *Let* d *denote the distance from* v_0 *to the subspace spanned by* $v_1, \ldots, v_n$. *If* $v_1, \ldots, v_n$ *are also linearly independent, then*

$$d^2 = \frac{\det[\langle v_i, v_j \rangle]_{0 \le i,j \le n}}{\det[\langle v_i, v_j \rangle]_{1 \le i,j \le n}}.$$

1.3.3 A Characterization

Suppose s is a normalized (irreducible) complete Pick kernel defined on a set X. Suppose also that ℓ is another kernel on X, satisfying

$$\ell(z, w) = s(z, w) g(z, w), \quad z, w \in X,$$

where g is a kernel. Let $\{\lambda_i\} \subset X$ and $n \ge 2$. Recall that $\{\lambda_i\}$ is *n-weakly separated by ℓ* if there exists $\epsilon > 0$ such that for every n-point subset $\{\mu_1, \ldots, \mu_n\} \subset \{\lambda_i\}$ we have

$$\text{dist}(\hat{\ell}_{\mu_1}, \text{ span}\{\hat{\ell}_{\mu_2}, \ldots, \hat{\ell}_{\mu_n}\}) \ge \epsilon.$$

Similarly, we say that $\{\lambda_i\}$ is *strongly separated by* ℓ if there exists $\epsilon > 0$ such that for every $i \in \mathbb{N}$ we have

$$\text{dist}\big(\hat{\ell}_{\lambda_i},\ \text{span}_{j\neq i}\{\hat{\ell}_{\lambda_j}\}\big) \geq \epsilon.$$

What is the difference between n-weak separation for all n and strong separation? The former condition allows the use of a different ϵ for each n, while the latter asks for the use of a single ϵ for every n.

Now, the fact that the pair (s, ℓ) satisfies the Pick property allows us to recast weak and strong separation by ℓ in terms of separation by elements of $\text{Mult}(\mathcal{H}_s, \mathcal{H}_\ell)$.

Lemma 1.3.11. *Suppose s is a normalized complete Pick factor of a kernel ℓ on X. Also, let $\{\lambda_i\} \subset X$ and $n \geq 2$.*

(a) $\{\lambda_i\}$ is n-weakly separated by ℓ if and only if there exists $\epsilon > 0$ such that for every n-point subset $\{\mu_1, \mu_2, \ldots, \mu_n\}$ of $\{\lambda_i\}$ there exists a multiplier $\phi \in Mult(\mathcal{H}_s, \mathcal{H}_\ell)$ of norm at most 1 with $\phi(\mu_1) = \epsilon\frac{||\ell_{\mu_1}||}{||s_{\mu_1}||}$ and $\phi(\mu_j) = 0$, for $j = 2, 3, \ldots, n$.

(b) $\{\lambda_i\}$ is strongly separated by ℓ if and only if there exists $\epsilon > 0$ such that for every $i \in \mathbb{N}$ there exists a multiplier $\phi \in Mult(\mathcal{H}_s, \mathcal{H}_\ell)$ of norm at most 1 with $\phi(\lambda_i) = \epsilon\frac{||\ell_{\lambda_i}||}{||s_{\lambda_i}||}$ and $\phi(\lambda_j) = 0$ for every $j \neq i$.

Proof. First, we prove (a). Let $n \geq 2$ and suppose $\{\lambda_i\}$ is n-weakly separated by ℓ. We can then find $\epsilon > 0$ such that for every n-point subset $\{\mu_1, \ldots, \mu_n\} \subset \{\lambda_i\}$ we have

$$d = \text{dist}\big(\hat{\ell}_{\mu_1},\ \text{span}\{\hat{\ell}_{\mu_2}, \ldots, \hat{\ell}_{\mu_n}\}\big) \geq \epsilon.$$

Now, fix n points $\{\mu_1, \ldots, \mu_n\} \subset \{\lambda_i\}$ and let $m \in \{2, 3, \ldots, n\}$. n-weak separation implies that the vectors $\{\hat{\ell}_{\mu_1}, \hat{\ell}_{\mu_2}, \ldots, \hat{\ell}_{\mu_n}\}$ are linearly independent. In view of Lemma 1.3.10, we can write

$$\frac{\det\big[\langle \hat{\ell}_{\mu_i}, \hat{\ell}_{\mu_j}\rangle\big]_{1\leq i,j\leq m}}{\det\big[\langle \hat{\ell}_{\mu_i}, \hat{\ell}_{\mu_j}\rangle\big]_{2\leq i,j\leq m}} = \Big[\text{dist}\big(\hat{\ell}_{\mu_1},\ \text{span}\{\hat{\ell}_{\mu_2}, \ldots, \hat{\ell}_{\mu_m}\}\big)\Big]^2 \geq d^2 \geq \epsilon^2$$

$$\Rightarrow \det\big[(1 - w_j\overline{w}_i)\langle \hat{\ell}_{\mu_i}, \hat{\ell}_{\mu_j}\rangle\big]_{1\leq i,j\leq m} > 0,$$

where $w_1 = \epsilon/2$ and $w_2 = w_3 = \cdots = w_n = 0$. Since this is true for arbitrary $m \in \{2, 3, \dots, n\}$, Sylvester's criterion tells us that the matrix

$$\left[(1 - w_j\overline{w}_i)\langle \hat{\ell}_{\mu_i}, \hat{\ell}_{\mu_j}\rangle\right]_{1\leq i,j\leq n}$$

is positive semi-definite. Multiplying the previous matrix by the dyad $\left[||\ell_{\mu_i}|| \cdot ||\ell_{\mu_j}||\right]$, we obtain the positivity of

$$\left[(1 - w_j\overline{w}_i)\ell(\mu_j, \mu_i)\right]_{1\leq i,j\leq n},$$

which can be rewritten as

$$\left[\ell(\mu_j, \mu_i) - v_j\overline{v}_i s(\mu_j, \mu_i)\right]_{1\leq i,j\leq n},$$

where $v_1 = \frac{\epsilon}{2}\frac{||\ell_{\mu_1}||}{||s_{\mu_1}||}$ and $v_2 = v_3 = \cdots = v_n = 0$. But (s, ℓ) has the Pick property, so we can deduce the existence of a multiplier $\phi \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_\ell)$ of norm at most 1 such that $\phi(\mu_1) = \frac{\epsilon}{2}\frac{||\ell_{\mu_1}||}{||s_{\mu_1}||}$ and $\phi(\mu_j) = 0$, for $j = 2, 3, \dots, n$.

We have proved one implication from part (a). For the converse, simply reverse the steps in the previous proof (the Pick property of (s, ℓ) is no longer necessary).

The proof of (b) is essentially identical to that of (a). One point worth mentioning is that the inequalities $\det\left[(1 - w_j\overline{w}_i)\langle \hat{\ell}_{\mu_i}, \hat{\ell}_{\mu_j}\rangle\right]_{1\leq i,j\leq m} > 0$, for all $m \geq 2$, allow us to deduce (through Sylvester's criterion and standard approximation arguments) the positivity of the infinite matrix $\left[(1 - w_j\overline{w}_i)\langle \hat{\ell}_{\mu_i}, \hat{\ell}_{\mu_j}\rangle\right]_{i,j}$. The rest of the proof carries over without change. □

Next, we use the Column-Row property for spaces with a complete Pick factor ([79, Theorem 3.18]) to characterize $\text{Mult}(\mathcal{H}_s, \mathcal{H}_\ell)$-interpolating sequences in terms of the $\mathcal{H}_s$-Carleson measure condition and strong separation by ℓ. We state the result here for the reader's convenience.

Theorem 1.3.12 (Hartz, [79])**.** *Suppose s and ℓ are as above and let $\Phi \in Mult(\mathcal{H}_s, \mathcal{H}_\ell \otimes \ell^2)$ be a column multiplication operator. Then, the row multiplication operator Φ^T is bounded and*

$$||\Phi^T|| \leq_{Mult(\mathcal{H}_s\otimes\ell^2, \mathcal{H}_\ell)} \leq ||\Phi||_{Mult(\mathcal{H}_s, \mathcal{H}_\ell\otimes\ell^2)}.$$

We now give our characterization. Our argument is motivated by the proof of Theorem 4.4 in [79].

Theorem 1.3.13. *Suppose s and ℓ are as above and let $\{\lambda_i\} \subset X$. Then, $\{\lambda_i\}$ is interpolating for Mult($\mathcal{H}_s, \mathcal{H}_\ell$) if and only if it satisfies the Carleson measure condition for $\mathcal{H}_s$ and is strongly separated by ℓ. In this case, there exists a bounded linear operator of interpolation associated with $\{\lambda_i\}$.*

Proof. First, suppose that $\{\lambda_i\}$ is interpolating for Mult$(\mathcal{H}_s, \mathcal{H}_\ell)$. By Theorem 1.3.7(a), we obtain that $\{\lambda_i\}$ satisfies (CM) with respect to $\mathcal{H}_s$.
Recall also (see (1.6)) that every $\phi \in$ Mult$(\mathcal{H}_s, \mathcal{H}_\ell)$ satisfies

$$|\phi(z)| \leq ||\phi||_{\text{Mult}(\mathcal{H}_s,\mathcal{H}_\ell)} \cdot \frac{||\ell_z||}{||s_z||},$$

for every $z \in X$. We can thus define

$$S : \text{Mult}(\mathcal{H}_s, \mathcal{H}_\ell) \to \ell^\infty$$

$$\phi \mapsto \left\{\phi(\lambda_i) \cdot \frac{||s_{\lambda_i}||}{||\ell_{\lambda_i}||}\right\}_{i\geq 1}.$$

S is well-defined, linear and bounded by $||\phi||_{\text{Mult}(\mathcal{H}_s,\mathcal{H}_\ell)}$. Since $\{\lambda_i\}$ is interpolating, S is also onto ℓ^∞. A standard application of the Open Mapping Theorem then allows us to deduce the existence of a constant $C > 0$ *(the constant of interpolation)*, such that for every $\{w_i\} \in \ell^\infty$ we can find $\phi \in$ Mult$(\mathcal{H}_s, \mathcal{H}_\ell)$ with the property that $\phi(\lambda_i) = w_i||\ell_{\lambda_i}||/||s_{\lambda_i}||$ and $||\phi||_{\text{Mult}(\mathcal{H}_s,\mathcal{H}_\ell)} \leq C\cdot||\{w_i\}||_\infty$. In view of Lemma 1.3.11(b), this implies that $\{\lambda_i\}$ is strongly separated by ℓ.

For the converse, suppose that $\{\lambda_i\}$ satisfies (CM) with respect to s and is strongly separated by ℓ.
By Lemma 1.3.11(b), there exist multipliers $\{\phi_i\} \subset$ Mult$(\mathcal{H}_s.\mathcal{H}_\ell)$ and $M > 0$ such that

(i) $\phi_i(\lambda_j) = \delta_{ij}\frac{||\ell_{\lambda_i}||}{||s_{\lambda_i}||}$, for every i, j;
(ii) $||\phi_i||_{\text{Mult}(\mathcal{H}_s,\mathcal{H}_\ell)} \leq M$, for every i.

Also, by Theorem 1.3.9, there exists a multiplier $\Psi \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_s \otimes \ell^2)$ such that

$$\Psi(\lambda_i) = \begin{pmatrix} * \\ \vdots \\ * \\ 1 \\ * \\ \vdots \end{pmatrix},$$

for every i.

Consider now the bounded diagonal operator $\text{diag}\{\phi_1, \phi_2, \phi_3, \ldots\} \in \text{Mult}(\mathcal{H}_s \otimes \ell^2, \mathcal{H}_\ell \otimes \ell^2)$ and define

$$\Phi := \text{diag}\{\phi_1, \phi_2, \phi_3, \ldots\} \cdot \Psi \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_\ell \otimes \ell^2).$$

Notice that

$$\Phi(\lambda_i) = \frac{||\ell_{\lambda_i}||}{||s_{\lambda_i}||} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix},$$

for every i. Theorem 1.3.12 now tells us that $\Phi^T \in \text{Mult}(\mathcal{H}_s \otimes \ell^2, \mathcal{H}_\ell)$ and thus, letting $\Delta : \ell^\infty \to \text{Mult}(\mathcal{H}_s \otimes \ell^2)$ denote the embedding via diagonal operators, we can define

$$T : \ell^\infty \to \text{Mult}(\mathcal{H}_s, \mathcal{H}_\ell)$$

$$\{w_i\} \mapsto \Phi^T \cdot \Delta(\{w_i\}) \cdot \Psi.$$

T is well-defined, bounded and also satisfies

$$[T(\{w_i\})](\lambda_j) = w_j \frac{||\ell_{\lambda_j}||}{||s_{\lambda_j}||},$$

for every j. Thus, T is the linear operator of interpolation with respect to $\{\lambda_i\}$ and our proof is complete. □

We are now ready to prove Theorem 1.3.4. Of critical importance will be the observation (see [8, Proposition 9.11] for a proof) that every sequence $\{\lambda_i\} \subset X$ satisfying (CM) with respect to $\mathcal{H}_s$ can be written as a union of n sequences that are weakly separated by s, where n is a finite integer. In this setting, it turns out that n-weak separation by ℓ (for the n that is given as a consequence of [8, Proposition 9.11]) is *precisely* what is missing for $\{\lambda_i\}$ to be Mult$(\mathcal{H}_s, \mathcal{H}_\ell)$-interpolating.

Proof of Theorem 1.3.4. By Theorem 1.3.13, every Mult$(\mathcal{H}_s, \mathcal{H}_\ell)$-interpolating sequence satisfies (CM) with respect to s and is strongly separated by ℓ, hence also n-weakly separated by ℓ for every $n \geq 2$.

For the converse, suppose $\{\lambda_i\} \subset X$ satisfies (CM) with respect to s and is n-weakly separated by ℓ, for every $n \geq 2$. If $\{\lambda_i\}$ also happens to be weakly separated by s, then it must be interpolating for Mult$(\mathcal{H}_s)$ (and hence interpolating for Mult$(\mathcal{H}_s, \mathcal{H}_\ell)$ by Theorem 1.3.7(b)). If not, then there exists $n \geq 2$ such that $\{\lambda_i\}$ can be written as a union of n sequences that are weakly separated by s. In other words, there exist disjoint sequences $\{p_i^1\}, \{p_i^2\}, \dots, \{p_i^n\} \subset X$ and a number $0 < c < 1$ such that $\{\lambda_i\} = \cup_{k=1}\{p_j^k\}$ and also for every $i \neq j$,

$$|\langle \hat{s}_{p_i^1}, \hat{s}_{p_j^1}\rangle|^2, \ |\langle \hat{s}_{p_i^2}, \hat{s}_{p_j^2}\rangle|^2, \ \dots \ , \ |\langle \hat{s}_{p_i^n}, \hat{s}_{p_j^n}\rangle|^2 \leq c. \tag{1.9}$$

Now, choose an arbitrary point from $\{\lambda_i\}$. Without loss of generality, we may choose a point $p_{m_1}^1$ from $\{p_i^1\}$. Consider the pseudometric

$$d_s(\lambda_1, \lambda_2) = \sqrt{1 - |\langle \hat{s}_{\lambda_1}, \hat{s}_{\lambda_2}\rangle|^2}$$

associated with $\mathcal{H}_s$. By (1.9), we obtain that $d_s(p_i^k, p_j^k) \geq \sqrt{1-c}$, for every $i \neq j$ and every $k \in \{1, 2, \dots, n\}$. Now, for any fixed $k \in \{2, 3, \dots, n\}$ and $i \neq j$, the fact that d_s is a pseudometric implies

$$\sqrt{1-c} \leq d_s(p_i^k, p_j^k) \leq d_s(p_i^k, p_{m_1}^1) + d_s(p_{m_1}^1, p_j^k).$$

Consequently, for every $k \in \{2, 3, \dots, n\}$, there exists at most one point $p^k_{m_k}$ such that $d_s(p^1_{m_1}, p^k_{m_k}) < \frac{\sqrt{1-c}}{2}$ (if such a point does not exist, pick an arbitrary point of $\{p^k_i\}$ to be $p^k_{m_k}$). Hence, there exists $c' > 0$ (depending only on c) such that for every $k \in \{1, 2, \dots, n\}$ and every $j \neq m_k$, we have

$$|\langle \hat{s}_{p^1_{m_1}}, \hat{s}_{p^k_j} \rangle|^2 \leq c' < 1. \tag{1.10}$$

Also, for every $k \in \{1, 2, \dots, n\}$ and every $j \neq m_k$, the Pick property of s allows us to find a contractive multiplier $\phi^k_j \in \text{Mult}(\mathcal{H}_s)$ such that $\phi^k_j(p^k_j) = 0$ and

$$\phi^k_j(p^1_{m_1}) = \sqrt{1 - |\langle \hat{s}_{p^1_{m_1}}, \hat{s}_{p^k_j} \rangle|^2}. \tag{1.11}$$

Consider now the product

$$\prod_{k=1}^{n} \prod_{j \neq m_k} \phi^k_j. \tag{1.12}$$

More precisely, we take any weak-star cluster point of the partial products. We thus obtain a contractive multiplier $\Phi \in \text{Mult}(\mathcal{H}_s)$ such that $\Phi(p^k_j) = 0$, for every $j \neq m_k$.

Now, since $\{\lambda_i\}$ is n-weakly separated by ℓ, Lemma 1.3.11(a) tells us that there exists a contractive multiplier $\Psi_{m_1, m_2, \dots, m_n} \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_\ell)$ such that $\Psi_{m_1, m_2, \dots, m_n}(p^k_{m_k}) = 0$ for every $k \geq 2$ and also $\Psi_{m_1, m_2, \dots, m_n}(p^1_{m_1}) = \epsilon \frac{||\ell_{p^1_{m_1}}||}{||s_{p^1_{m_1}}||}$, where the constant $\epsilon > 0$ does not depend on the choice of points $p^1_{m_1}, \dots, p^n_{m_n}$.

Finally, we put

$$\tilde{\Phi} = \Psi_{m_1, m_2, \dots, m_n} \cdot \Phi \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_\ell).$$

This is a contractive multiplier that is zero at every point of the sequence $\{\lambda_i\}$ except $p^1_{m_1}$. We also have (by (1.11))

$$\begin{aligned} \tilde{\Phi}(p^1_{m_1}) &= \Psi_{m_1, m_2, \dots, m_n}(p^1_{m_1}) \cdot \Phi(p^1_{m_1}) \\ &= \epsilon \frac{||\ell_{p^1_{m_1}}||}{||s_{p^1_{m_1}}||} \prod_{k=1}^{n} \prod_{j \neq m_k} \sqrt{1 - |\langle \hat{s}_{p^1_{m_1}}, \hat{s}_{p^k_j} \rangle|^2}. \end{aligned} \tag{1.13}$$

Now, the fact that $\{\lambda_i\}$ satisfies (CM) with respect to s implies the existence of a constant $C > 0$ such that for every $f \in \mathcal{H}_s$ the inequality

$$\sum \frac{|f(\lambda_i)|^2}{||s_{\lambda_i}||^2} \leq C||f||^2_{\mathcal{H}_s}$$

holds true. Choosing $f = \hat{s}_{p^1_{m_1}}$, we obtain

$$\sum_{k=1}^{n} \sum_{j} \left|\langle \hat{s}_{p^1_{m_1}}, \hat{s}_{p^k_j} \rangle\right|^2 \leq C. \tag{1.14}$$

Combining (1.10), (1.13) and (1.14), we can conclude that

$$\frac{||s_{p^1_{m_1}}||}{||\ell_{p^1_{m_1}}||} \tilde{\Phi}(p^1_{m_1})$$

is bounded below by a positive number that only depends on the constants ϵ, c and C and not on the specific point $p^1_{m_1}$ we started with. Thus, $\{\lambda_i\}$ is strongly separated by ℓ. By Theorem 1.3.13, $\{\lambda_i\}$ must be Mult($\mathcal{H}_s, \mathcal{H}_\ell$)-interpolating. □

Remark 1.3.14. In the setting of Theorem 1.3.4, suppose $\{\lambda_i\} \subset X$ is a union of n disjoint sequences (where $n \geq 2$) that are interpolating for Mult($\mathcal{H}_s$). The previous proof tells us that $\{\lambda_i\}$ is Mult($\mathcal{H}_s, \mathcal{H}_\ell$)-interpolating if and only if it is n-weakly separated by ℓ.

1.3.4 A Counterexample to Question 1.3.1

Now, suppose we have a sequence $\{\lambda_i\}$ satisfying (CM) with respect to s and suppose also that $n \geq 3$ is the smallest integer such that $\{\lambda_i\}$ can be written as a union of n disjoint sequences that are weakly separated by s. Then, not even $(n-1)$-weak (let alone weak) separation by ℓ can, in general, guarantee that $\{\lambda_i\}$ is Mult($\mathcal{H}_s, \mathcal{H}_\ell$)-interpolating. This is essentially the content of Theorem 1.3.5, which we now prove.

Proof of Theorem 1.3.5. Let s be the Szegő kernel on the unit disk $\mathbb{D}$. For every $n \geq 3$, choose n disjoint sequences $\{\lambda_i^{n,1}\}_i, \{\lambda_i^{n,2}\}_i, \dots, \{\lambda_i^{n,n}\}_i \subset \mathbb{D}$ that are interpolating for Mult($\mathcal{H}_s$) and such that their i^{th} terms satisfy

$$\lim_i d_s(\lambda_i^{n,k}, \lambda_i^{n,m}) = 0, \tag{1.15}$$

for every $k, m \in \{1, 2, \dots, n\}$. We also require the existence of $c > 0$ (which could depend on n) such that

$$d_s(\lambda_i^{n,k}, \lambda_j^{n,m}) \geq c, \tag{1.16}$$

for all n, i, j, k, m such that $i \neq j$. Finally, choose these sequences in such a way that $\lambda_i^{n,k} = \lambda_j^{\nu,m}$ is equivalent to $i = j, n = \nu$ and $k = m$, i.e. there are no common points between them. Here is one way of constructing such sequences: let $\{a_1^1\}$ be an arbitrary point in $\mathbb{D}$ and suppose that, for $m \geq 1$, the m-point set $\{a_1^m, \dots, a_m^m\}$ has been determined. Then, choose $m+1$ points $a_1^{m+1}, \dots, a_{m+1}^{m+1}$ such that

(i) $\frac{1-|a_i^{m+1}|}{1-|a_j^k|} \leq \rho < 1, \quad \forall k \in \{1, \dots, m\}, i \in \{1, \dots, m+1\}, j \in \{1, \dots, k\}$.

(ii) $d_s\big(a_i^{m+1}, a_j^{m+1}\big) \leq \frac{1}{m+1}, \quad$ for all i, j.

Essentially, our sequences come in m-point packets (where m is increasing) that are well-separated from one another but such that the points in each packet are increasingly close to each other. Now, put $\{\lambda_i^{n,k}\}_{i\geq 1} = \{a_{c(n,k)}^i\}_{i \geq d(n)}$, where $c(n,k) = k + \sum_{j=1}^{n-1} j$ and $d(n) = \sum_{j=1}^{n} j$. Then, no two sequences will have any points in common. Also, by item (i), each $\{\lambda_i^{n,k}\}_{i\geq 1}$ converges to $\partial\mathbb{D}$ exponentially and so must be interpolating for $\text{Mult}(\mathcal{H}_s)$. The same condition guarantees that (1.16) must be valid for some constant $c > 0$. Finally, item (ii) implies that (1.15) is also satisfied.

We now turn to the construction of the kernel ℓ. For every $n \geq 3$, choose n linearly dependent vectors $\{v_{n,1}, v_{n,2}, \dots, v_{n,n}\}$ in ℓ^2 with the following property: any choice of $n-1$ vectors among $\{v_{n,1}, v_{n,2}, \dots, v_{n,n}\}$ produces a linearly independent set. Also, define $u : \mathbb{D} \to \ell^2$ by

$$u(\lambda) = \begin{cases} v_{n,k}, & \text{if } \lambda = \lambda_i^{n,k} \text{ for some } i, n, k \\ \\ e_1, & \text{for every other point } \lambda \in \mathbb{D}. \end{cases}$$

(The definition of u on points other than $\lambda_i^{n,k}$ is not important.) Since $\lambda_i^{n,k} = \lambda_j^{\nu,m}$ is equivalent to $i = j, n = \nu$ and $k = m$, the function u is well-defined. Finally, consider the positive semi-definite kernel $\mathrm{g} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ given by $\mathrm{g}(\lambda, \mu) = \langle u(\lambda), u(\mu)\rangle_{\ell^2}$ and put $\ell := s \cdot \mathrm{g}$.

Now, fix $n \geq 3$ (this will remain fixed for the rest of the proof). Since any $(n-1)$-point subset of $\{v_{n,1}, v_{n,2}, \dots, v_{n,n}\}$ is linearly independent, there exists $\epsilon > 0$ such that each vector $v_{n,j}$ always has distance greater than ϵ from the span of any $(n-2)$-point subset of the remaining $n-1$ vectors.

Thus, we can deduce (in a manner identical to the proof of Lemma 1.3.11) the existence of $\epsilon_1 > 0$ such that for any $(n-1)$-point subset $\{\nu_1, \nu_2, \dots, \nu_{n-1}\}$ of $\{1, 2, \dots, n\}$ the matrix

$$\left[(1 - w_j \overline{w}_i)\langle v_{n,\nu_j}, v_{n,\nu_i}\rangle\right]_{1\leq i,j\leq n-1},$$

is positive semi-definite, where $w_1 = \epsilon_1$ and $w_2 = \cdots = w_{n-1} = 0$. By definition of the kernel g, we obtain that for every such subset $\{\nu_1, \nu_2, \dots, \nu_{n-1}\}$ and every choice of $n-1$ (not necessarily distinct) integers $m_1, m_2, \dots, m_{n-1}$, the matrix

$$\left[(1 - w_j \overline{w}_i) g(\lambda_{m_j}^{n,\nu_j}, \lambda_{m_i}^{n,\nu_i})\right]_{1\leq i,j\leq n-1},$$

is positive semi-definite. Now, multiply with the corresponding Grammian associated to s and apply the Schur product theorem to deduce positivity of the matrix

$$\left[(1 - w_j \overline{w}_i) \ell(\lambda_{m_j}^{n,\nu_j}, \lambda_{m_i}^{n,\nu_i})\right]_{1\leq i,j\leq n-1}.$$

Since (s, ℓ) has the Pick property, this last positivity condition implies that for any $(n-1)$-point subset $\{\nu_1, \nu_2, \dots, \nu_{n-1}\}$ of $\{1, 2, \dots, n\}$ and every $m_1, m_2, \dots, m_{n-1} \geq 1$, there exists a contractive multiplier $\Phi \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_\ell)$ such that

$$\Phi(\lambda_{m_1}^{n,\nu_1}) = \epsilon_1 \frac{||\ell_{\lambda_{m_1}^{n,\nu_1}}||}{||s_{\lambda_{m_1}^{n,\nu_1}}||} \quad \text{and} \quad \Phi(\lambda_{m_i}^{n,\nu_i}) = 0, \tag{1.17}$$

for all $i \in \{2, \dots, n-1\}$.

Now, recall that each sequence $\{\lambda_i^{n,k}\}_i$ is interpolating for $\text{Mult}(\mathcal{H}_s)$ and hence their union $\cup_{k=1}^n \{\lambda_i^{n,k}\}$ must satisfy the Carleson measure condition for $\mathcal{H}_s$ (this is because of the elementary fact that the sum of two Carleson measures is a Carleson measure). In view of the separation condition (1.16) and the Pick property of s, we can mimic the construction of the Blaschke-type multiplier (1.12) from the proof of Theorem 1.3.4 to deduce the existence of an $\epsilon_2 > 0$ with the property that, for every point $\lambda_i^{n,k}$, there exists a contractive multiplier $\Psi \in \text{Mult}(\mathcal{H}_s)$ such that

$$\Psi(\lambda_i^{n,k}) = \epsilon_2 \quad \text{and} \quad \Psi(\lambda_j^{n,m}) = 0, \tag{1.18}$$

for all $m \in \{1, 2, \dots, n\}$ and every integer $j \neq i$.

Combining (1.17) and (1.18) (and also using the basic fact that the product of a function in $\text{Mult}(\mathcal{H}_s)$ with a function in $\text{Mult}(\mathcal{H}_s, \mathcal{H}_\ell)$ yields a multiplier in $\text{Mult}(\mathcal{H}_s, \mathcal{H}_\ell)$), we deduce that for every $(n-1)$-point subset $\{\mu_1, \mu_2, \dots, \mu_{n-1}\}$ of the union $\cup_k \{\lambda_i^{n,k}\}$ there exists a multiplier $\phi \in \text{Mult}(\mathcal{H}_s, \mathcal{H}_\ell)$ of norm at most 1 such that $\phi(\mu_1) = \epsilon_1 \epsilon_2 \frac{||\ell_{\mu_1}||}{||s_{\mu_1}||}$ and $\phi(\mu_j) = 0$, for $j = 2, 3, \dots, n-1$. By Lemma 1.3.11(a), we conclude that $\cup_k \{\lambda_i^{n,k}\}$ must be $(n-1)$-weakly separated by ℓ.

We now show that $\cup_k \{\lambda_i^{n,k}\}$ is not n-weakly separated by ℓ. We proceed by contradiction; suppose instead that there exists $\epsilon > 0$ such that for every n-point subset $\{\mu_1, \mu_2, \dots, \mu_n\}$ of $\cup_k \{\lambda_i^{n,k}\}$ the matrix

$$\left[(1 - w_j \overline{w}_i) \ell(\mu_i, \mu_j)\right]_{1 \leq i,j \leq n}$$

is positive semi-definite, where $w_1 = \epsilon$ and $w_2 = \dots = w_n = 0$. Choosing $\mu_k = \lambda_m^{n,k}$ gives us the positivity of

$$\left[(1 - w_j \overline{w}_i) \ell(\lambda_m^{n,i}, \lambda_m^{n,j})\right]_{1 \leq i,j \leq n},$$

for every $m \geq 1$. Next, multiply the previous matrix by the transpose of $[s(\lambda_m^{n,i}, \lambda_m^{n,j})]$ (which must be positive semi-definite as well) and the dyad $[s(\lambda_m^{n,i}, \lambda_m^{n,i})^{-1} s(\lambda_m^{n,j}, \lambda_m^{n,j})^{-1}]$. The Schur product theorem then allows us to obtain

$$\det \left[(1 - w_j \overline{w}_i) g(\lambda_m^{n,i}, \lambda_m^{n,j}) \frac{|s(\lambda_m^{n,i}, \lambda_m^{n,j})|^2}{s(\lambda_m^{n,i}, \lambda_m^{n,i}) s(\lambda_m^{n,j}, \lambda_m^{n,j})}\right]_{1 \leq i,j \leq n} \geq 0. \tag{1.19}$$

However, condition (1.15) implies that $\lim_m \frac{|s(\lambda_m^{n,i}, \lambda_m^{n,j})|^2}{s(\lambda_m^{n,i}, \lambda_m^{n,i}) s(\lambda_m^{n,j}, \lambda_m^{n,j})} = 1$, for every $i, j \in \{1, \dots, n\}$. Thus, letting $m \to \infty$ in (1.19) and using the definition of g, we obtain

$$\det \left[(1 - w_j \overline{w}_i) \langle v_{n,i}, v_{n,j} \rangle\right]_{1 \leq i,j \leq n} \geq 0,$$

which implies

$$\det \left[\langle v_{n,i}, v_{n,j} \rangle\right]_{1 \leq i,j \leq n} \geq \epsilon^2 ||v_{n,1}||^2 \det \left[\langle v_{n,i}, v_{n,j} \rangle\right]_{2 \leq i,j \leq n} > 0,$$

a contradiction, as the vectors $\{v_{n,1}, \dots, v_{n,n}\}$ are linearly dependent.

To sum up, we have showed that, for every $n \geq 3$, the sequence $\cup_{k=1}^{n}\{\lambda_i^{n,k}\}$ satisfies (CM) with respect to s and is $(n-1)$-weakly separated, but not n-weakly separated by ℓ. The proof is complete. □

Remark 1.3.15. Notice that the choice of the Szegő kernel s in our counterexample is not really important; all that was required for the proof to go through was a complete Pick kernel s containing, for every $n \geq 3$, disjoint sequences $\{\lambda_i^{n,1}\}_i, \{\lambda_i^{n,2}\}_i, \dots, \{\lambda_i^{n,n}\}_i$ in the underlying set that satisfy (1.15) and (1.16).

Remark 1.3.16. Here is a simpler counterexample (which only works for *fixed* n). Choose $n \geq 3$ and consider the kernel $s(\lambda, \mu) = \frac{1}{1 - z^n \overline{w}^n}$ defined on $\mathbb{D} \times \mathbb{D}$. Also, let $\omega_1, \dots, \omega_n$ denote the n^{th} roots of unity and choose n vectors $v_1, \dots, v_n$ in $\mathbb{C}^{n-1}$ with the property that every choice of $n-1$ vectors among them produces a linearly independent set. Put

$$u(\lambda) = \begin{cases} v_k, & \text{if } \lambda = \frac{1}{2}\omega_k \\ \\ v_1, & \text{for every other point } \lambda \in \mathbb{D}, \end{cases}$$

and set $g(\lambda, \mu) = \langle u(\lambda), u(\mu) \rangle$, $\ell := s \cdot g$ and $z_k = \frac{1}{2}\omega_k$. By assumption, there exists $\epsilon > 0$ such that the matrix

$$\left[(1 - w_j \overline{w}_i) g(z_{\mu_i}, z_{\mu_j})\right]_{1 \leq i,j \leq n-1}$$

is positive semi-definite for every $(n-1)$-point subset $\{\mu_1, \dots, \mu_{n-1}\}$ of $\{1, \dots, n\}$, where $w_1 = \epsilon$ and $w_2 = \dots = w_{n-1} = 0$. But since $s(z_i, z_j) = 4/3$ for every i, j, we immediately obtain the positivity of the matrix

$$\left[(1 - w_j \overline{w}_i) \ell(z_{\mu_i}, z_{\mu_j})\right]_{1 \leq i,j \leq n-1}.$$

Thus, the sequence $\{z_1, \dots, z_n\}$ is $(n-1)$-weakly separated by ℓ and also (trivially) satisfies the Carleson measure condition for $\mathcal{H}_s$. However, the kernel functions $\ell_{z_1}, \dots, \ell_{z_n}$ are linearly dependent, which implies that $\{z_1, \dots, z_n\}$ is not n-weakly separated.

Remark 1.3.17. Given the artificial nature of the kernel g, one might wonder whether more natural counterexamples to Question 1.3.1 can be found. Subsection 1.3.8 contains a “nicer” counterexample involving holomorphic kernels on the bidisk.

1.3.5 When is (CM)+(WS) Sufficient?

Once more, suppose that s, ℓ are two kernels on X such that $\ell/s >> 0$ and s is a normalized complete Pick kernel. The proof of Theorem 1.3.5 shows that a potential reason for the failure of the $\mathcal{H}_s$-Carleson measure condition and weak separation by ℓ to always be sufficient conditions for Mult($\mathcal{H}_s.\mathcal{H}_\ell$)-interpolation is that X might contain weakly separated (by ℓ) sequences that are not n-weakly separated by ℓ for some $n \geq 3$. This motivates the following definition.

Definition 1.3.18. Let ℓ be a kernel on a set X. ℓ will be said to have the *automatic separation property* if any sequence $\{\lambda_i\} \subset X$ that is weakly separated by ℓ must always be n-weakly separated by ℓ for every $n \geq 3$. Such kernels will also be called *AS kernels.*

Remark 1.3.19. The AS property is not very intuitive from a geometric viewpoint. Indeed, if ℓ has the automatic separation property, then, for every $n \geq 2$, a normalized kernel function $\hat{\ell}_\mu$ can be “close” to the span of n other normalized kernel functions if and only if it is actually “close” to one of them (see [65, Section 6] for a direct proof of the fact that the Szegő kernel on $\mathbb{D}$ has this property). Nevertheless, as we shall see in subsections 1.3.6-1.3.7, it turns out that a surprisingly large number of well-known function spaces possess AS kernels.

As an immediate consequence of Theorem 1.3.4, we obtain that Question 1.3.1 has a positive answer for every pair (s, ℓ) such that ℓ has the automatic separation property.

Corollary 1.3.20. *Let s, ℓ be two kernels on a set X such that ℓ is an AS kernel and s is a normalized complete Pick factor of ℓ. Then, a sequence $\{\lambda_i\} \subset X$ is interpolating for Mult($\mathcal{H}_s, \mathcal{H}_\ell$) if and only if it satisfies the Carleson measure condition for $\mathcal{H}_s$ and is weakly separated by ℓ.*

We now show that, under some additional weak hypotheses on the kernels s and ℓ, Question 1.3.1 having a positive answer for the pair (s, ℓ) is actually *equivalent* to ℓ being an AS kernel.

Theorem 1.3.21. *Suppose X is a topological space, ℓ a kernel on X with a normalized complete Pick factor s and the following properties are satisfied:*

(Q1) $\ell : X \times X \to \mathbb{C}$ *is continuous;*

(Q2) If $\{\lambda_i\} \subset X$ *satisfies* $||\ell_{\lambda_i}|| \to \infty$*, then* $||\ell_{\lambda_i}||^{-1}\ell(\lambda_i, \mu) \to 0$ *for every* $\mu \in X$*;*

(Q3) Let $\{\lambda_i\} \subset X$*. Then, either* $||\ell_{\lambda_i}|| \to \infty$ *or* $\{\lambda_i\}$ *contains a subsequence converging to a point inside X;*

(Q4) If $||\ell_{\lambda_i}|| \to \infty$*, then* $||s_{\lambda_i}|| \to \infty$*.*

Then, the following assertions are equivalent:

(i) For every $\{\lambda_i\} \subset X$*,* $\{\lambda_i\}$ *is interpolating for* $Mult(\mathcal{H}_s.\mathcal{H}_\ell)$ *if and only if it satisfies the Carleson measure condition for* $\mathcal{H}_s$ *and is weakly separated by* ℓ*.*

(ii) ℓ *has the automatic separation property.*

Proof. Let ℓ be a kernel with a complete Pick factor s defined on a topological space X such that properties (Q1)-(Q4) are all satisfied.

If (ii) holds, then (i) must also hold by Corollary 1.3.20.

Now, suppose that (ii) fails. Thus, there exists a sequence $\{\lambda_i\}$ and $n \geq 2$ such that $\{\lambda_i\}$ is n-weakly separated but not $(n+1)$-weakly separated by ℓ. This implies the existence of $n+1$ subsequences $\{\lambda_m^1\}_m, \{\lambda_m^2\}_m, \dots, \{\lambda_m^{n+1}\}_m \subset \{\lambda_i\}$ (which may contain repeated points) such that the points $\lambda_m^1, \lambda_m^2, \dots, \lambda_m^{n+1}$ are distinct from one another, for every m, and also

$$\lim_m \operatorname{dist}\big(\hat{\ell}_{\lambda_m^1},\ \operatorname{span}\{\hat{\ell}_{\lambda_m^2}, \dots, \hat{\ell}_{\lambda_m^{n+1}}\}\big) = 0. \tag{1.20}$$

Since

$$\Big[\operatorname{dist}\big(\hat{\ell}_{\lambda_m^1},\ \operatorname{span}\{\hat{\ell}_{\lambda_m^2}, \dots, \hat{\ell}_{\lambda_m^{n+1}}\}\big)\Big]^2 = \frac{\det\big[\langle \hat{\ell}_{\lambda_m^i}, \hat{\ell}_{\lambda_m^j}\rangle\big]_{1\leq i,j\leq n+1}}{\det\big[\langle \hat{\ell}_{\lambda_m^i}, \hat{\ell}_{\lambda_m^j}\rangle\big]_{2\leq i,j\leq n+1}},$$

condition (1.20) (and the fact that all determinants $\det\left[\langle \hat{\ell}_{\lambda_m^i}, \hat{\ell}_{\lambda_m^j}\rangle\right]_{2\leq i,j\leq n+1}$ are uniformly bounded with respect to m, given that their entries cannot exceed 1 in modulus) implies that

$$\lim_m \det\left[\langle \hat{\ell}_{\lambda_m^i}, \hat{\ell}_{\lambda_m^j}\rangle\right]_{1\leq i,j\leq n+1} = 0. \tag{1.21}$$

By n-weak separation, we also obtain the existence of $\epsilon > 0$ such that for every n-point subset $\{\mu_1, \dots, \mu_n\}$ of $\{\lambda_i\}$ we have

$$\operatorname{dist}\left(\hat{\ell}_{\mu_1},\ \operatorname{span}\{\hat{\ell}_{\mu_2}, \dots, \hat{\ell}_{\mu_n}\}\right) > \epsilon. \tag{1.22}$$

We now use (Q1) and (Q2) to deduce a useful lemma about the behavior of the pseudometric d_ℓ.

Lemma 1.3.22. *Let ℓ be a kernel on X satisfying (Q1) and (Q2). If $\{w_i\}$ and $\{z_i\}$ are two sequences in X such that $w_i \to w$ for some $w \in X$ and $||\ell_{z_i}|| \to \infty$, then*

$$d_\ell(w_i, z_i) \to 1.$$

Proof of Lemma 1.3.22. Since $w_i \to w$, continuity of ℓ (property (Q1)) implies that $d_\ell(w_i, w) \to 0$. Also, by (Q2), we obtain that $\langle \hat{\ell}_{z_i}, f\rangle \to 0$ whenever f is a finite linear combination of kernel functions. But since linear combinations of kernel functions are dense in $\mathcal{H}_\ell$, it must be true that $\hat{\ell}_{z_i} \to 0$ weakly in $\mathcal{H}_\ell$. Finally, d_ℓ is a pseudometric and so we can write

$$-d_\ell(w, w_i) + d_\ell(w, z_i) \leq d_\ell(w_i, z_i) \leq 1,$$

where $d_\ell(w, w_i) \to 0$ and $d_\ell(w, z_i) = \sqrt{1 - |\langle \hat{\ell}_w, \hat{\ell}_{z_i}\rangle|^2} \to 1$, as $\hat{\ell}_{z_i} \to 0$ weakly. Thus, $d_\ell(w_i, z_i) \to 1$ and the proof is complete. □

We can also assume, without loss of generality, that each subsequence $\{\lambda_m^k\}_m$ either satisfies $||s_{\lambda_m^k}|| \to \infty$ and $||\ell_{\lambda_m^k}|| \to \infty$ or converges to a point $p_k \in X$. This is possible because of (Q3) and (Q4) (we can keep extracting subsequences until we have the desired properties). There are now three separate cases to examine.

First, suppose that all of the sequences $\{\lambda_m^k\}_m$ converge to points p_k in X. Combining (1.21) with the continuity of ℓ (property (Q1)), we deduce that

$$\det\left[\langle \hat{\ell}_{p_i}, \hat{\ell}_{p_j}\rangle\right]_{1\le i,j\le n+1} = 0. \tag{1.23}$$

Hence, the kernel functions $\ell_{p_1}, \dots, \ell_{p_{n+1}}$ are linearly dependent. Consider the finite sequence $\{p_1, \dots, p_{n+1}\} \subset X$. This sequence trivially satisfies the Carleson measure condition for $\mathcal{H}_s$, is n-weakly separated (because of (1.22)), and hence weakly separated, but not $(n+1)$-weakly separated by ℓ (because of (1.23)). Thus, (i) fails in this case.

Suppose now that at least one sequence $\{\lambda_m^k\}_m$ satisfies $||s_{\lambda_m^k}|| \to \infty$ and $||\ell_{\lambda_m^k}|| \to \infty$. Suppose also that at least one sequence $\{\lambda_m^{k'}\}_m$ converges to a point $p_{k'} \in X$. We will arrive at a contradiction. Without loss of generality, we can reindex our sequences so that $\{\lambda_m^1\}, \dots, \{\lambda_m^r\}$ converge to points $p_1, \dots, p_r$ in X, while $\{\lambda_m^{r+1}\}, \dots, \{\lambda_m^{n+1}\}$ satisfy $||s_{\lambda_m^k}|| \to \infty$ and $||\ell_{\lambda_m^k}|| \to \infty$ (where $1 \le r \le n$). In view of (1.22), we can write

$$\det\left[\langle \hat{\ell}_{\lambda_m^i}, \hat{\ell}_{\lambda_m^j}\rangle\right]_{1\le i,j\le r} \ge \epsilon^2 \det\left[\langle \hat{\ell}_{\lambda_m^i}, \hat{\ell}_{\lambda_m^j}\rangle\right]_{2\le i,j\le r} \ge \dots \ge \epsilon^{2(r-1)}. \tag{1.24}$$

Similarly, we obtain

$$\det\left[\langle \hat{\ell}_{\lambda_m^i}, \hat{\ell}_{\lambda_m^j}\rangle\right]_{r+1\le i,j\le n+1} > \epsilon^{2(n-r)}. \tag{1.25}$$

Now, Lemma 1.3.22 tells us that (as $m \to \infty$)

$$\langle \hat{\ell}_{\lambda_m^i}, \hat{\ell}_{\lambda_m^j}\rangle \to 0, \ \ \forall i \in \{1, \dots, r\}, \ \forall j \in \{r+1, \dots, n+1\}. \tag{1.26}$$

We can thus write

$$\det\left[\langle \hat{\ell}_{\lambda_m^i}, \hat{\ell}_{\lambda_m^j}\rangle\right]_{1\le i,j\le n+1} =$$
$$= \Big(\det\left[\langle \hat{\ell}_{\lambda_m^i}, \hat{\ell}_{\lambda_m^j}\rangle\right]_{1\le i,j\le r}\Big)\Big(\det\left[\langle \hat{\ell}_{\lambda_m^i}, \hat{\ell}_{\lambda_m^j}\rangle\right]_{r+1\le i,j\le n+1}\Big) + e_m,$$

where $e_m \to 0$ as $m \to \infty$ because of (1.26). Hence, letting $m \to \infty$ in the previous equality and using (1.24) and (1.25) gives us

$$\liminf_m \det\left[\langle \hat{\ell}_{\lambda_m^i}, \hat{\ell}_{\lambda_m^j}\rangle\right]_{1\le i,j\le n+1} \ge \epsilon^{2(n-1)} > 0,$$

a contradiction.

Finally, suppose that all of our sequences $\{\lambda^k_m\}_m$ satisfy $||\ell_{\lambda^k_m}|| \to \infty$ and $||s_{\lambda^k_m}|| \to \infty$. By [17, Proposition 5.1], we can extract a subsequence $\{\lambda^k_{m_j}\}_j$ from each $\{\lambda^k_m\}_m$ that is interpolating for Mult($\mathcal{H}_s$). Hence, the union $\cup_{k=1}^{n+1}\{\lambda^k_{m_j}\}$ will satisfy the Carleson measure condition for $\mathcal{H}_s$. By assumption, it will also be n-weakly separated but not $(n+1)$-weakly separated by ℓ. Thus, (i) fails and our proof is complete. □

Remark 1.3.23. Property (Q2) is definitely satisfied whenever each kernel function ℓ_μ is bounded on *X*, however this is not necessary in general. For instance, the kernel $\ell(\lambda, \mu) = e^{\lambda\overline{\mu}}$ of the Bargmann-Fock space on $\mathbb{C}$ satisfies (Q2), even though not every ℓ_μ is a bounded function.

Before we proceed, we record the following useful lemma. It essentially says that factoring a kernel does not increase the distance between the normalized kernel functions.

Lemma 1.3.24. *Suppose* g, ℓ *are two reproducing kernels on X such that* ℓ/g *is positive semi-definite. Then,*

$$\mathit{dist}\big(\hat{g}_{\mu_1},\ \mathit{span}\{\hat{g}_{\mu_2}, \dots, \hat{g}_{\mu_n}\}\big) \leq \mathit{dist}\big(\hat{\ell}_{\mu_1},\ \mathit{span}\{\hat{\ell}_{\mu_2}, \dots, \hat{\ell}_{\mu_n}\}\big),$$

for any n-point subset $\{\mu_1, \mu_2, \dots, \mu_n\}$ *of X (where* $n \geq 2$*).*

Proof. Assume that there exists $\epsilon > 0$ such that

$$\mathrm{dist}\big(\hat{g}_{\mu_1},\ \mathrm{span}\{\hat{g}_{\mu_2}, \dots, \hat{g}_{\mu_n}\}\big) > \epsilon$$

(if no such ϵ exists, then there is nothing to prove). In view of Lemma 1.3.10, we obtain

$$\frac{\det[\langle \hat{g}_{\mu_i}, \hat{g}_{\mu_j}\rangle]_{1\leq i,j\leq m}}{\det[\langle \hat{g}_{\mu_i}, \hat{g}_{\mu_j}\rangle]_{2\leq i,j\leq m}} > \epsilon^2, \quad \text{for all } m \in \{2, \dots, n\}.$$

Hence, from Sylvester's Criterion, we can deduce the positivity of the matrix

$$[(1 - w_j\overline{w}_i)\langle \hat{g}_{\mu_i}, \hat{g}_{\mu_j}\rangle]_{1\leq i,j\leq n},$$

where $w_1 = \epsilon$ and $w_2 = \cdots = w_n = 0$. Taking the Schur product with the positive semi-definite matrix $[\langle \hat{\ell}_{\mu_i}, \hat{\ell}_{\mu_j}\rangle / \langle \hat{g}_{\mu_i}, \hat{g}_{\mu_j}\rangle]$ then gives us

$$[(1 - w_j\overline{w}_i)\langle \hat{\ell}_{\mu_i}, \hat{\ell}_{\mu_j}\rangle]_{1\le i,j\le n} >> 0,$$

which implies that

$$\text{dist}(\hat{\ell}_{\mu_1},\ \text{span}\{\hat{\ell}_{\mu_2}, \dots, \hat{\ell}_{\mu_n}\}) = \sqrt{\frac{\det[\langle \hat{\ell}_{\mu_i}, \hat{\ell}_{\mu_j}\rangle]_{1\le i,j\le n}}{\det[\langle \hat{\ell}_{\mu_i}, \hat{\ell}_{\mu_j}\rangle]_{2\le i,j\le n}}} \ge \epsilon.$$

This concludes the proof of the lemma. □

Let us now look at an example of a pair (s, ℓ) for which Question 1.3.1 has a positive answer but such that ℓ is not an AS kernel.

Example. Let $n \ge 2$. Consider the restricted Szegő kernel $s(z, w) = \frac{1}{1-z\overline{w}}$ on $\frac{1}{2}\mathbb{D}$ and choose $n + 1$ disjoint sequences $\{\lambda_m^1\}, \{\lambda_m^2\}, \dots, \{\lambda_m^{n+1}\} \subset \frac{1}{2}\mathbb{D}$ that converge to the boundary of $\frac{1}{2}\mathbb{D}$ and such that their m^{th} terms satisfy

$$\lim_m \langle \hat{s}_{\lambda_m^i}, \hat{s}_{\lambda_m^j}\rangle = 1, \tag{1.27}$$

for every i, j. Also, choose an orthonormal sequence $\cup_{1\le k\le n+1}\{e_m^k\}_m$ in ℓ^2 and define $u : \frac{1}{2}\mathbb{D} \to \ell^2$ by

$$u(\lambda) = \begin{cases} e_m^k, & \text{if } \lambda = \lambda_m^k \text{ for some } m \ge 1 \text{ and } k \in \{1, \dots, n\} \\ \sum_{i=1}^n e_m^i + \frac{1}{m} e_m^{n+1}, & \text{if } \lambda = \lambda_m^{n+1} \text{ for some } m \ge 1 \\ e_1^1, & \text{for every other point } \lambda. \end{cases}$$

(The definition of u on points different from λ_m^k is not important.) Finally, set $g : \frac{1}{2}\mathbb{D} \times \frac{1}{2}\mathbb{D} \to \mathbb{C}$ to be equal to $g(\lambda, \mu) = \langle u(\lambda), u(\mu)\rangle$ and define $\ell := s \cdot g$. Note that ℓ does not satisfy (Q3) from Theorem 1.3.21.

Consider now a sequence $\{\nu_i\} \subset \frac{1}{2}\mathbb{D}$ that satisfies the Carleson measure condition for $\mathcal{H}_s$. The only way this can happen is if $\{\nu_i\}$ is actually a finite sequence (this is because $||s_{\nu_i}||$ is uniformly bounded above). But then, $\{\nu_i\}$ will (trivially) be interpolating for Mult($\mathcal{H}_s$) and hence also for Mult($\mathcal{H}_s, \mathcal{H}_\ell$). Thus, Question 1.3.1 has a positive answer for the pair (s, ℓ).

We now show that ℓ is not an AS kernel. By definition of g, we obtain that $\operatorname{dist}(\hat{g}_{\mu_1},\ \operatorname{span}\{\hat{g}_{\mu_2}, \dots, \hat{g}_{\mu_n}\})$ is uniformly bounded below, where $\{\mu_1, \dots, \mu_n\}$ is any n-point subset of $\cup_k\{\lambda_m^k\}$. In view of Lemma 1.3.24, this implies that $\cup_k\{\lambda_m^k\}$ is n-weakly separated by ℓ. However, notice that

$$\frac{\det\left[\langle g_{\lambda_m^i}, g_{\lambda_m^j}\rangle\right]_{1\le i,j\le n+1}}{\det\left[\langle g_{\lambda_m^i}, g_{\lambda_m^j}\rangle\right]_{1\le i,j\le n}}$$

$$= \left[\operatorname{dist}\left(g_{\lambda_m^{n+1}},\ \operatorname{span}\{g_{\lambda_m^1}, \dots, g_{\lambda_m^n}\}\right)\right]^2$$

$$= \left[\operatorname{dist}\left(\left(\sum_{i=1}^n e_m^i + \frac{1}{m}e_m^{n+1}\right),\ \operatorname{span}\{e_m^1, \dots, e_m^n\}\right)\right]^2 \to 0,$$

as $m \to \infty$. Thus, $\det\left[\langle \hat{g}_{\lambda_m^i}, \hat{g}_{\lambda_m^j}\rangle\right]_{1\le i,j\le n+1} \to 0$ and so, in view of (1.27), we obtain

$$\det\left[\langle \hat{\ell}_{\lambda_m^i}, \hat{\ell}_{\lambda_m^j}\rangle\right]_{1\le i,j\le n+1} \to 0$$

as $m \to \infty$. But then,

$$\left[\operatorname{dist}\left(\hat{\ell}_{\lambda_m^{n+1}},\ \operatorname{span}\{\hat{\ell}_{\lambda_m^1}, \dots, \hat{\ell}_{\lambda_m^n}\}\right)\right]^2 = \frac{\det\left[\langle \hat{\ell}_{\lambda_m^i}, \hat{\ell}_{\lambda_m^j}\rangle\right]_{1\le i,j\le n+1}}{\det\left[\langle \hat{\ell}_{\lambda_m^i}, \hat{\ell}_{\lambda_m^j}\rangle\right]_{1\le i,j\le n}} \to 0,$$

as $m \to \infty$ (note that the determinant in the denominator is uniformly bounded below by n-weak separation). Hence, $\cup_k\{\lambda_m^k\}$ is not $(n+1)$-weakly separated by ℓ, which implies that ℓ does not have the automatic separation property.

1.3.6 Automatic Separation by Multipliers

In this subsection, we look at kernels enjoying a separation property which is stronger than the AS property. To be precise, we will be concerned with kernels ℓ defined on a set X such that for every $\{\lambda_i\} \subset X$, weak separation by ℓ implies weak separation by Mult($\mathcal{H}_\ell$).

Definition 1.3.25. Let ℓ be a kernel on a set X. We will say that ℓ has the *multiplier separation property* if, for every $\delta > 0$, there exists an $\epsilon > 0$ such that, for any two points $\lambda_i \neq \lambda_j$ in X satisfying $d_s(\lambda_i, \lambda_j) > \delta$, there exists $\phi_{ij} \in$ Mult($\mathcal{H}_\ell$) of norm at most 1 satisfying $\phi_{ij}(\lambda_i) = \epsilon$ and $\phi_{ij}(\lambda_j) = 0$.

Remark 1.3.26. For any kernel ℓ and any sequence $\{\lambda_i\} \subset X$, weak separation by $\text{Mult}(\mathcal{H}_\ell)$ always implies weak separation by ℓ. This is a consequence of the positivity of (1.7) (for $k = \ell$). Hence, for kernels satisfying the multiplier separation property, weak separation by the kernel always coincides with weak separation by the multiplier algebra.

First, we show that the multiplier separation property does indeed imply the AS property.

Proposition 1.3.27. *Let ℓ be a kernel on X with the multiplier separation property. Then, ℓ satisfies the automatic separation property.*

Proof. Let $n \geq 3$ and suppose $\{\lambda_i\} \subset X$ is weakly separated by ℓ. Thus, there exists $\delta > 0$ such that $d_s(\lambda_i, \lambda_j) > \delta$ for every $i \neq j$. By assumption, there exists $\epsilon > 0$ such that for every $i \neq j$, we can find $\phi_{ij} \in \text{Mult}(\mathcal{H}_\ell)$ of norm at most 1 such that $\phi_{ij}(\lambda_i) = \epsilon$ and $\phi_{ij}(\lambda_j) = 0$. Now, suppose $\{\mu_1, \dots, \mu_n\}$ is an arbitrary n-point subset of $\{\lambda_i\}$. Consider the contractive multiplier $\Phi := \prod_{i=2}^n \phi_{\mu_1\mu_i}$, which satisfies $\Phi(\mu_2) = \dots = \Phi(\mu_n) = 0$ and $\Phi(\mu_1) = \epsilon^{n-1}$. But then, we will have $||\Phi\hat{\ell}_{\mu_1}||_{\mathcal{H}_\ell} \leq 1$ and so we can write

$$
\begin{aligned}
&\frac{1}{\text{dist}(\hat{\ell}_{\mu_1},\ \text{span}\{\hat{\ell}_{\mu_2}, \dots, \hat{\ell}_{\mu_n}\})} \\
&= \inf\Big\{||f|| \,:\, f \in \big(\,\text{span}\{\hat{\ell}_{\mu_2}, \dots, \hat{\ell}_{\mu_n}\}\big)^{\perp},\ \langle f, \hat{\ell}_{\mu_1}\rangle = 1\Big\} \\
&= \inf\{||f|| \,:\, f \in \mathcal{H}_\ell,\ f(\mu_2) = \dots = f(\mu_n) = 0,\ f(\mu_1) = ||\ell_{\mu_1}||\} \\
&\leq \frac{||\Phi\hat{\ell}_{\mu_1}||}{\epsilon^{n-1}} \\
&\leq \frac{1}{\epsilon^{n-1}},
\end{aligned}
$$

hence $\{\lambda_i\}$ must be n-weakly separated by ℓ. This concludes the proof. □

Suppose now that we have a kernel (or a collection of kernels) with the multiplier separation property. Our next result shows that performing certain operations on these kernels allows us to construct new ones possessing the same property.

Proposition 1.3.28. *Suppose $\ell : X\times X \to \mathbb{C}$ and $k : S\times S \to \mathbb{C}$ are two kernels with the multiplier separation property and let $\phi : S \to X$ be a function and $a \geq 1$. Suppose also that ρ denotes any of the following kernels:*

(a) $\ell \otimes k$*;*

(b) ℓ^a *(here, we also assume that ℓ is non-vanishing);*

(c) $\ell \circ \phi$ *(defined by $\ell \circ \phi(\lambda, \mu) = \ell(\phi(\lambda), \phi(\mu))$);*

(d) a kernel $\rho : X \times X \to \mathbb{C}$ such that both ρ/ℓ and ℓ^a/ρ are positive semi-definite;

(e) a kernel $\rho : X \times X \to \mathbb{C}$ such that ℓ/ρ is positive semi-definite and $||\phi||_{Mult(\mathcal{H}_\rho)} \leq C \sup_{x\in X} |\phi(x)|$, for some constant $C \geq 1$, for every ϕ in $Mult(\mathcal{H}_\rho)$;

(f) a kernel $\rho : X \times X \to \mathbb{C}$ such that $\mathcal{H}_\rho$ and $\mathcal{H}_\ell$ have equivalent norms.

Then, ρ must also have the multiplier separation property.

Before we go into the proof, we require the following simple lemma.

Lemma 1.3.29. *Suppose g, ℓ are two reproducing kernels on X such that ℓ/g is positive semi-definite. Then, $Mult(\mathcal{H}_g) \subset Mult(\mathcal{H}_\ell)$ and $||\phi||_{Mult(\mathcal{H}_\ell)} \leq ||\phi||_{Mult(\mathcal{H}_g)}$, for every $\phi \in Mult(\mathcal{H}_g)$.*

Proof. We know that $||\phi||_{\text{Mult}(\mathcal{H}_g)} \leq M$ if and only if the matrix $\left[\left(M^2 - \phi(\lambda_i)\overline{\phi(\lambda_j)}\right)g(\lambda_i, \lambda_j)\right]$ is positive semi-definite for any choice of points λ_i in X. Taking the Schur product with $\left[(\ell/g)(\lambda_i, \lambda_j)\right]$ then yields the desired result. □

Proof of Proposition 1.3.28. For (a), let $\delta > 0$ and suppose $(\lambda_i, \mu_i), (\lambda_j, \mu_j)$ are two points in $X \times S$ satisfying $d_{\ell\otimes k}\big((\lambda_i, \mu_i), (\lambda_j, \mu_j)\big) > \delta$. Thus, we must have

$$\frac{|(\ell \otimes k)\big((\lambda_i, \mu_i), (\lambda_j, \mu_j)\big)|^2}{(\ell \otimes k)\big((\lambda_i, \mu_i), (\lambda_i, \mu_i)\big)(\ell \otimes k)\big((\lambda_j, \mu_j), (\lambda_j, \mu_j)\big)}$$
$$= \frac{|\ell(\lambda_i, \lambda_j)k(\mu_i, \mu_j)|^2}{\ell(\lambda_i, \lambda_i)\ell(\lambda_j, \lambda_j)k(\mu_i, \mu_i)k(\mu_j, \mu_j)}$$
$$\leq 1 - \delta^2,$$

which implies that either $|\langle \hat{\ell}_{\lambda_i}, \hat{\ell}_{\lambda_j}\rangle|^2 \le \sqrt{1-\delta^2}$ or $|\langle \hat{k}_{\mu_i}, \hat{k}_{\mu_j}\rangle|^2 \le \sqrt{1-\delta^2}$. Without loss of generality, assume that $|\langle \hat{\ell}_{\lambda_i}, \hat{\ell}_{\lambda_j}\rangle|^2 \le \sqrt{1-\delta^2}$, hence

$$d_\ell(\lambda_i, \lambda_j) \ge \sqrt{1-\sqrt{1-\delta^2}}.$$

The fact that ℓ has the multiplier separation property then implies the existence of $\epsilon > 0$ (depending only on δ) and $\phi_{ij} \in \mathrm{Mult}(\mathcal{H}_\ell)$ of norm at most 1 such that $\phi_{ij}(\lambda_i) = \epsilon$ and $\phi_{ij}(\lambda_j) = 0$. In view of (1.7), we obtain

$$(1 - \phi_{ij}(x)\overline{\phi_{ij}(y)})\ell(x, y) \ge 0, \tag{1.28}$$

for every choice of points $x, y \in X$. We now extend ϕ to $X \times S$ by putting $\phi(x, s) = \phi(x)$, for every $(x, s) \in X \times S$. Condition (1.28) then becomes $(1 - \phi_{ij}(x, s)\overline{\phi_{ij}(y, t)})\ell(x, y) \ge 0$, which, after taking the Schur product with $k(s, t)$, gives us

$$(1 - \phi_{ij}(x, s)\overline{\phi_{ij}(y, t)})\ell(x, y)k(s, t)$$

$$= (1 - \phi_{ij}(x, s)\overline{\phi_{ij}(y, t)})(\ell \otimes k)\big((x, s), (y, t)\big) \ge 0,$$

for every choice of points $(x, s), (y, t) \in X \times S$. Hence, ϕ_{ij} is a contractive multiplier of $\mathcal{H}_{\ell\otimes k}$, which concludes the proof of (a).

For (b), let $a \ge 1$. It is easy to see that a sequence is weakly separated by ℓ if and only if it is weakly separated by ℓ^a (actually, $a > 0$ suffices for this, as $\sqrt{1 - x^a} \sim \sqrt{1 - x}, 0 \le x \le 1$) . Thus, assuming $\{\lambda_i\} \subset X$ is weakly separated by ℓ^a, we can deduce the existence of $\epsilon > 0$ and $\{\phi_{ij}\} \subset \mathrm{Mult}(\mathcal{H}_\ell)$ such that $\phi_{ij}(\lambda_i) = \epsilon$, $\phi_{ij}(\lambda_j) = 0$ and ϕ_{ij} has norm at most 1, for all i, j. Clearly, each ϕ_{ij} also satisfies (1.28). Taking the Schur product with the positive-semidefinite kernel $\ell^{a-1}(x, y)$ then gives us that each $\phi_{ij} \in \mathrm{Mult}(\mathcal{H}_{\ell^a})$ (and has norm at most 1), which concludes the proof of (b).

For (c), let $\delta > 0$ and assume that s_i, s_j are two points in S satisfying $d_{\ell\circ\phi}(s_i, s_j) > \delta$. It is known (see [113, Theorem 5.7]) that there exists an isometry $\Gamma : \mathcal{H}_{\ell\circ\phi} \to \mathcal{H}_\ell$ satisfying $\Gamma((\ell\circ\phi)_s) = \ell_{\phi(s)}$, for every $s \in S$. This implies that $d_{\ell\circ\phi}(s, t) = d_\ell(\phi(s), \phi(t))$, for every $s, t \in S$. Thus, we obtain

$d_\ell(\phi(s_i), \phi(s_j)) > \delta$. But ℓ has the multiplier separation property, so we deduce the existence of $\epsilon > 0$ (depending only on δ) and $\psi_{ij} \in \mathrm{Mult}(\mathcal{H}_\ell)$ of norm at most 1 such that $\psi_{ij}(\phi(s_i)) = \epsilon$ and $\psi_{ij}(\phi(s_j)) = 0$. Since (again by [113, Theorem 5.7]) $||f||_{\mathcal{H}_{\ell\circ\phi}} = \inf\{||F||_{\mathcal{H}_\ell} : f = F \circ \phi\}$, for every $f \in \mathcal{H}_{\ell\circ\phi}$, we obtain $||\psi_{ij} \circ \phi||_{\mathrm{Mult}(\mathcal{H}_{\ell\circ\phi})} \leq ||\psi_{ij}||_{\mathrm{Mult}(\mathcal{H}_\ell)} \leq 1$ and so $\psi_{ij} \circ \phi$ is a separating multiplier with the desired properties.

For (d), suppose that ρ is a kernel on X satisfying the given hypotheses. Let $\delta > 0$ and assume that λ_i, λ_j are two points in X with $d_\rho(\lambda_i, \lambda_j) > \delta$. By Sylvester's criterion, we obtain the positivity of the 2×2 matrix

$$\begin{bmatrix} (1-\delta^2)\rho(\lambda_i, \lambda_i) & \rho(\lambda_i, \lambda_j) \\ \rho(\lambda_j, \lambda_i) & \rho(\lambda_j, \lambda_j). \end{bmatrix}$$

Taking the Schur product with the positive 2×2 matrix $\left[(\ell^a/\rho)(\lambda_i, \lambda_j)\right]$ then yields the positivity of

$$\begin{bmatrix} (1-\delta^2)\ell^a(\lambda_i, \lambda_i) & \ell^a(\lambda_i, \lambda_j) \\ \ell^a(\lambda_j, \lambda_i) & \ell^a(\lambda_j, \lambda_j) \end{bmatrix},$$

which implies that $d_{\ell^a}(\lambda_i, \lambda_j) > \delta$. Thus, we obtain $d_\ell(\lambda_i, \lambda_j) > \sqrt{1 - \sqrt[a]{1-\delta^2}}$. But ℓ has the multiplier separation property, so we deduce the existence of $\epsilon > 0$ (depending only on δ and a) and $\phi_{ij} \in \mathrm{Mult}(\mathcal{H}_\ell)$ of norm at most 1 such that $\phi_{ij}(\lambda_i) = \epsilon$ and $\phi_{ij}(\lambda_j) = 0$. By Lemma 1.3.29, ϕ_{ij} must also be a contractive multiplier of $\mathcal{H}_\rho$, which concludes the proof.

For (e), again suppose that ρ is a kernel on X satisfying the given hypotheses and let λ_i, λ_j be two points in X satisfying $d_\rho(\lambda_i, \lambda_j) > \delta$. Reasoning as in the proof of (d), we obtain the existence of $\epsilon > 0$ (depending only on δ) and $\phi_{ij} \in \mathrm{Mult}(\mathcal{H}_\ell)$ of norm at most 1 such that $\phi_{ij}(\lambda_i) = \epsilon$ and $\phi_{ij}(\lambda_j) = 0$. But $\mathcal{H}_\ell$ is a Hilbert function space, so, in view of our assumptions, we can write

$$||\phi_{ij}||_{\mathrm{Mult}(\mathcal{H}_\rho)} \leq C \sup_{x \in X} |\phi_{ij}(x)| \leq C||\phi_{ij}||_{\mathrm{Mult}(\mathcal{H}_\ell)} \leq C,$$

which implies that ϕ_{ij} is a separating multiplier with the desired properties.

Finally, suppose that $\mathcal{H}_\ell$ and $\mathcal{H}_\rho$ have equivalent norms. This implies that the evaluation functionals $T_x : \mathcal{H}_\ell \to \mathbb{C}$, $T_x(f) = f(x)$ and $S_x : \mathcal{H}_\rho \to \mathbb{C}$, $S_x(f) = f(x)$ also have equivalent norms (with constants independent of $x \in X$). Hence, there exist $C_1, C_2 > 0$ such that

$$C_1 ||\ell_x||_{\mathcal{H}_\ell} \leq ||\rho_x||_{\mathcal{H}_\rho} \leq C_2 ||\ell_x||_{\mathcal{H}_\ell}, \tag{1.29}$$

for every $x \in X$. But since we know that, for any kernel k and points $x, y \in X$,

$$\begin{aligned}\frac{1}{d_k(x,y)} &= \frac{1}{\operatorname{dist}\big(\hat{k}_x,\ \operatorname{span}\{\hat{k}_y\}\big)} \\ &= \inf\Big\{||f||_{\mathcal{H}_k} : f \in \big(\operatorname{span}\{\hat{k}_y\}\big)^{\perp},\ \langle f, \hat{k}_x\rangle = 1\Big\} \\ &= \inf\big\{||f||_{\mathcal{H}_k} : f \in \mathcal{H}_k,\ f(y) = 0,\ f(x) = ||k_x||_{\mathcal{H}_k}\big\},\end{aligned} \tag{1.30}$$

(1.29) and the equivalence of $|| \cdot ||_{\mathcal{H}_\ell}$ and $|| \cdot ||_{\mathcal{H}_\rho}$ allow us to deduce the existence of constants $C_1', C_2' > 0$ such that

$$C_1' d_\ell(x,y) \leq d_\rho(x,y) \leq C_2' d_\ell(x,y),$$

for all $x, y \in X$. This double inequality, combined with the fact that the associated multiplier norms $|| \cdot ||_{\mathrm{Mult}(\mathcal{H}_\ell)}$ and $|| \cdot ||_{\mathrm{Mult}(\mathcal{H}_\rho)}$ must also be equivalent, finishes off the proof. □

Remark 1.3.30. Let ℓ, k be two kernels on X with the multiplier separation property. In view of Proposition 1.3.28, the product $\ell \cdot k$ (which is the restriction of $\ell \otimes k$ along the diagonal) must also have the multiplier separation property.

Remark 1.3.31. Assume that ℓ, ρ, k are kernels on X such that ℓ and k have the multiplier separation property and ρ/ℓ and k/ρ are both positive semi-definite. Then, ρ needn't even be an AS kernel. Indeed, define

$$\rho(\lambda, \mu) = \frac{1}{1 - \lambda^2 \overline{\mu}^2} + \frac{1}{1 - \lambda^3 \overline{\mu}^3}, \quad \ell(\lambda, \mu) = \frac{1}{1 - \lambda^6 \overline{\mu}^6},$$

and

$$k(\lambda, \mu) = \frac{1}{(1 - \lambda^2 \overline{\mu}^2)(1 - \lambda^3 \overline{\mu}^3)},$$

for $\lambda, \mu \in \mathbb{D}$. It is not hard to see that ℓ is a factor of both $\frac{1}{1-\lambda^2\overline{\mu}^2}$ and $\frac{1}{1-\lambda^3\overline{\mu}^3}$, hence ρ/ℓ is positive semi-definite. Also, since ℓ is a CP kernel, it evidently possesses the multiplier separation property.

On the other hand, note that

$$\frac{k(\lambda,\mu)}{\rho(\lambda,\mu)} = \frac{1/2}{1-\langle \frac{1}{\sqrt{2}}(\lambda^2,\lambda^3), \frac{1}{\sqrt{2}}(\mu^2,\mu^3)\rangle}, \quad \lambda,\mu \in \mathbb{D},$$

and so k/ρ is positive semi-definite (actually a CP kernel). Being the product of two CP kernels, k must satisfy the multiplier separation property. However, note that (letting $\omega = e^{2\pi i/3}$)

$$\rho_z - \rho_{\omega z} + \rho_{-\omega z} - \rho_{-z} = 0,$$

for all $z \in \mathbb{D}$. Also, any two-vector subset of $\{\rho_z, \rho_{\omega z}, \rho_{-\omega z}, \rho_{-z}\}$ is linearly independent if $z \neq 0$. This implies that ρ does not have the AS property.

We now present examples of kernels satisfying the multiplier separation property.

Example (*Products of powers of* 2*-point Pick kernels*).

Let $k = k_1^{t_1} \otimes k_2^{t_2} \otimes \cdots \otimes k_n^{t_n}$, where $n \geq 1$, $t_i \geq 1$ and each k_i is an irreducible kernel on X_i with the 2-point scalar Pick property (note that, in view of [8, Lemma 7.2], every such kernel must be nonzero on $X_i \times X_i$). If $x, y \in X_i$ satisfy $d_{k_i}(x,y) = \delta$, then the 2-point Pick property implies the existence of a contractive multiplier $\phi_{xy} \in \mathrm{Mult}(\mathcal{H}_{k_i})$ such that $\phi_{xy}(x) = \delta$ and $\phi_{xy}(y) = 0$. Thus, each k_i has the multiplier separation property and we also deduce, in view of Proposition 1.3.28, that the same must be true for k. Examples of such kernels (which are actually products of powers of complete Pick kernels) include those of the form

$$k((\lambda_1, \dots, \lambda_m), (\mu_1, \dots, \mu_m)) = \prod_{i=1}^{m} \frac{1}{(1 - \langle b_i(\lambda_i), b_i(\mu_i)\rangle)^{t_i}},$$

where $t_i \geq 1$, $b_i : X_i \to \mathbb{B}_d$ and $(\lambda_1, \dots, \lambda_m), (\mu_1, \dots, \mu_m)$ lie in the polydomain $X_1 \times X_2 \times \cdots \times X_m$.

If each k_i is defined on the same set X, Proposition 1.3.28(c) and the previous result imply that $\tilde{k} := k_1^{t_1} k_2^{t_2} \cdots k_n^{t_n}$ must also have the automatic separation property whenever every k_i is an irreducible 2-point Pick kernel.

Example (*Hardy spaces on planar domains*).

Suppose that $\Omega \subset \mathbb{C}$ is a domain with boundary $\partial\Omega$ consisting of a finite collection of smooth curves. Let $d\sigma$ be arclength measure on $\partial\Omega$ and define $H^2(\Omega)$ to be the closure in $L^2(\partial\Omega, d\sigma)$ of the subspace consisting of restrictions to $\partial\Omega$ of functions holomorphic on $\overline{\Omega}$ (see [5] and [68] for the basic theory of these spaces). While the choice of the measure $d\sigma$ is not canonical, all the standard choices lead to the same space of holomorphic functions on Ω with equivalent norms. A fascinating result due to Arcozzi, Rochberg and Sawyer (see [20, Corollary 13]) then tells us that $H^2(\Omega)$ admits an equivalent norm with the property that with the new norm the space is a reproducing kernel Hilbert space with a complete Pick kernel. In view of Proposition 1.3.28(f), we obtain that the kernel of $H^2(\Omega)$ has the multiplier separation property.

Examples 1.3.6-1.3.6 serve as manifestations of a general observation: the multiplier separation property will always be present in kernels obtained by performing any of the operations from Proposition 1.3.28 to one or more 2-point Pick kernels. We state this as a corollary.

Corollary 1.3.32. *Let ℓ be a kernel on X defined by performing a finite number of any of the operations from Proposition 1.3.28 to one or more kernels having the 2-point Pick property. Then, ℓ has the multiplier separation property.*

1.3.7 Automatic Separation for General Spaces

First, we prove a result (which mirrors Proposition 1.3.28) showing that certain operations on kernels preserve the AS property.

Proposition 1.3.33. *Suppose $\ell : X \times X \to \mathbb{C}$ and $k : S \times S \to \mathbb{C}$ are two AS kernels and let $\phi : S \to X$ be a function and $a \geq 1$. Suppose also that ρ denotes any of the following kernels:*

(a) $\ell \otimes k$;

(b) ℓ^a (here, we also assume that ℓ is non-vanishing);

(c) $\ell \circ \phi$ (defined by $\ell \circ \phi(\lambda, \mu) = \ell(\phi(\lambda), \phi(\mu))$);

(d) a kernel $\rho : X \times X \to \mathbb{C}$ such that both ρ/ℓ and ℓ^a/ρ are positive semi-definite;

(e) a kernel $\rho : X \times X \to \mathbb{C}$ such that $\mathcal{H}_\rho$ and $\mathcal{H}_\ell$ have equivalent norms.

Then, ρ must also have the AS property.

Proof. The ideas here are very similar to those used in the proof of Proposition 1.3.28 (one difference being that we have to use Lemma 1.3.24 in place of Lemma 1.3.29), so we only prove (c)-(e). For (c), let $\{s_i\} \subset S$ be weakly separated by $\ell \circ \phi$, hence we can find $\epsilon > 0$ such that any two points $s_i \neq s_j$ satisfy $d_{\ell\circ\phi}(s_i, s_j) > \epsilon$. Recall (as in the proof of Proposition 1.3.28(c)) that

$$\langle (\ell \circ \phi)_s, (\ell \circ \phi)_t \rangle_{\mathcal{H}_{\ell\circ\phi}} = \langle \ell_{\phi(s)}, \ell_{\phi(t)} \rangle_{\mathcal{H}_\ell}, \tag{1.31}$$

for every $s, t \in S$. This implies that $d_\ell(\phi(s_i), \phi(s_j)) = d_{\ell\circ\phi}(s_i, s_j) > \epsilon$, for every $i \neq j$, hence the sequence $\{\phi(s_i)\} \subset X$ is weakly separated by ℓ. But ℓ has the AS property, so $\{\phi(s_i)\}$ must also be n-weakly separated by ℓ, for every $n \geq 3$. Thus, we can find positive constants $\epsilon_n > 0$ such that for every n-point subset $\{\phi(\mu_1), \dots, \phi(\mu_n)\}$ of $\{\phi(s_i)\}$ and for $w_1 = \epsilon_n$, $w_2 = \dots = w_n = 0$, the matrix

$$[(1 - w_j\overline{w}_i)\langle \hat{\ell}_{\phi(\mu_i)}, \hat{\ell}_{\phi(\mu_j)} \rangle_{\mathcal{H}_\ell}]_{1\leq i,j\leq n}$$

is positive semi-definite. In view of (1.31), the same must be true for the matrix

$$[(1 - w_j\overline{w}_i)\langle \widehat{(\ell \circ \phi)}_{\mu_i}, \widehat{(\ell \circ \phi)}_{\mu_j} \rangle_{\mathcal{H}_{\ell\circ\phi}}]_{1\leq i,j\leq n}.$$

Lemma 1.3.10 then implies that $\{s_i\}$ is n-weakly separated by $\ell \circ \phi$, for every $n \geq 3$, so the proof of (c) is complete.

For (d), suppose that ρ is a kernel on X satisfying the given hypotheses. Let $\{\lambda_i\} \subset X$ be a sequence satisfying $d_\rho(\lambda_i, \lambda_j) > \epsilon > 0$, for every $i \neq j$. As in the proof of Proposition 1.3.28(d), we obtain $d_\ell(\lambda_i, \lambda_j) > \sqrt{1 - \sqrt[q]{1 - \epsilon^2}}$, for every $i \neq j$. This implies that $\{\lambda_i\}$ is weakly, and hence n-weakly, separated by ℓ, for every $n \geq 3$. Lemma 1.3.24 then allows us to deduce that $\{\lambda_i\}$ must also be n-weakly separated by ρ, for every $n \geq 3$, which concludes the proof.

Finally, suppose that $\mathcal{H}_\ell$ and $\mathcal{H}_\rho$ have equivalent norms. As in the proof of Proposition 1.3.28 (f), there exist constants $C_1, C_2 > 0$ such that (1.29) is satisfied. But then, we also know that for any kernel $k : X \times X \to \mathbb{C}$ and any n-point set $\{\mu_1, \dots, \mu_n\} \subset X$ we can write

$$\frac{1}{\mathrm{dist}(\hat{k}_{\mu_1},\ \mathrm{span}\{\hat{k}_{\mu_2}, \dots, \hat{k}_{\mu_n}\})}$$

$$= \inf\{||f|| : f \in \mathcal{H}_k,\ f(\mu_2) = \cdots = f(\mu_n) = 0,\ f(\mu_1) = ||k_{\mu_1}||\}.$$

This equality, combined with (1.29) and the equivalence of norms for $\mathcal{H}_\ell, \mathcal{H}_\rho$, implies that a sequence in X is n-weakly separated by ℓ if and only if it is n-weakly separated by ρ, for any $n \geq 2$, and so, since ℓ is an AS kernel, we are done. □

Remark 1.3.34. Let ℓ, k be two AS kernels on X. In view of Proposition 1.3.33, the product $\ell \cdot k$ must have the AS property as well.

Next, we establish a general criterion for the AS property, one that is closely related to the nature of interpolating sequences for $\mathcal{H}_\ell$. In particular, we will show that, under some mild additional assumptions, the kernel ℓ satisfies the AS property if and only if any weakly separated finite union of "sufficiently sparse" sequences in X forms an $\mathcal{H}_\ell$-interpolating sequence.

Theorem 1.3.35. *Suppose X is a topological space, ℓ is a kernel on X and the following properties are satisfied:*

- *(Q0) No finite collection of kernel functions $\ell_{\lambda_1}, \dots, \ell_{\lambda_m}$ can form a linearly dependent set if $|\{\lambda_1, \dots, \lambda_m\}| = m$;*
- *(Q1) $\ell : X \times X \to \mathbb{C}$ is continuous;*
- *(Q2) If $\{\lambda_i\} \subset X$ satisfies $||\ell_{\lambda_i}|| \to \infty$, then $||\ell_{\lambda_i}||^{-1}\ell(\lambda_i, \mu) \to 0$ for every $\mu \in X$;*
- *(Q3) Let $\{\lambda_i\} \subset X$. Then, either $||\ell_{\lambda_i}|| \to \infty$ or $\{\lambda_i\}$ contains a subsequence converging to a point inside X.*

Then, the following assertions are equivalent:

- *(i) ℓ has the automatic separation property.*
- *(ii) Suppose $\{\lambda_i\}$ is any weakly separated by ℓ sequence that satisfies $||\ell_{\lambda_i}|| \to \infty$ as $i \to \infty$. Then, for any $n \geq 3$ and any decomposition $\{\lambda_i\} = \cup_{j=1}^{\infty} \cup_{k=1}^{n} \{\mu_j^k\}$ where $||\ell_{\mu_j^k}|| \to \infty$ as $j \to \infty$, for every $1 \leq k \leq n$, we can always find a subsequence $\{m_i\}$ such that $\cup_{j=1}^{\infty} \cup_{k=1}^{n} \{\mu_{m_j}^k\}$ is interpolating for $\mathcal{H}_\ell$.*

Note that the extra hypotheses imposed on ℓ include properties (Q1)-(Q3) from the statement of Theorem 1.3.21, as well as a new condition (Q0) ensuring that no finite collection of kernel functions can be linearly dependent. This is to avoid the somewhat trivial situation where the failure of the AS property would be caused by the existence of linearly dependent kernel functions $\ell_{\lambda_1}, \dots, \ell_{\lambda_m}$ (where $m \geq 3$) such that no two kernels $\ell_{\lambda_i}, \ell_{\lambda_j}$ are linearly dependent if $i \neq j$.

Proof. First, suppose that ℓ satisfies the hypotheses of (ii). Working towards a contradiction, assume that there exists $\{\lambda_i\} \subset X$ and $n \geq 3$ such that $\{\lambda_i\}$ is $(n-1)$-weakly separated but not n-weakly separated by ℓ. Thus, we can find a subsequence $\cup_{j=1}^{\infty} \cup_{k=1}^{n} \{\mu_j^k\}$ of $\{\lambda_i\}$ where $|\{\mu_j^1, \dots, \mu_j^n\}| = n$ for every j (but we may have $\mu_j^k = \mu_i^r$ if $k \neq r$) and

$$\lim_j \operatorname{dist}\big(\hat{\ell}_{\mu_j^1},\ \operatorname{span}\{\hat{\ell}_{\mu_j^2}, \dots, \hat{\ell}_{\mu_j^n}\}\big) = 0. \tag{1.32}$$

In view of Q(0), $\cup_{j=1}^{\infty} \cup_{k=1}^{n} \{\mu_j^k\}$ must contain infinitely many points. We can also assume, without loss of generality, that each subsequence $\{\mu_j^k\}_j$ either satisfies $||\ell_{\mu_j^k}|| \to \infty$ or converges to a point in X. This is possible because of (Q3) (we can keep extracting subsequences until we have the desired properties).

Now, if every $\{\mu_j^k\}_j$ converges to a point p_k in X, we can show (as in the proof of Theorem 1.3.21) that the kernel functions $\ell_{p_1}, \dots, \ell_{p_n}$ are linearly dependent, which contradicts (Q0). On the other hand, if at least one subsequence $\{\mu_j^{k'}\}_j$ satisfies $||\ell_{\mu_j^{k'}}|| \to \infty$ and at least one subsequence $\{\mu_j^k\}_j$ converges to a point in X, we arrive at a contradiction (using properties (Q1)-(Q3)) in precisely the same manner as in the proof of Theorem 1.3.21. Finally, if $||\ell_{\mu_j^k}|| \to \infty$ as $j \to \infty$, for every k, the sequence $\cup_{j=1}^{\infty} \cup_{k=1}^{n} \{\mu_j^k\}$ will satisfy the hypotheses of (ii). Thus, we can extract a subsequence $\cup_{j=1}^{\infty} \cup_{k=1}^{n} \{\mu_{m_j}^k\}$ of $\cup_{j=1}^{\infty} \cup_{k=1}^{n} \{\mu_j^k\}$ that is interpolating for $\mathcal{H}_\ell$. In particular, $\cup_{j=1}^{\infty} \cup_{k=1}^{n} \{\mu_{m_j}^k\}$ must be n-weakly separated by ℓ, which contradicts (1.32). Hence, ℓ must be an AS kernel.

For the converse, assume that ℓ possesses the automatic separation property. Let $n \geq 3$ and suppose $\{\lambda_i\}$ is a weakly separated by ℓ sequence that can be written as $\{\lambda_i\} = \cup_{j=1}^{\infty} \cup_{k=1}^{n} \{\mu_j^k\}$, where $||\ell_{\mu_j^k}|| \to \infty$ for every $1 \leq k \leq n$. Assume also, for convenience, that $|\{\mu_j^1, \dots, \mu_j^n\}| = n$ for every j.

The AS property tells us that $\{\lambda_i\}$ has to be n-weakly separated by ℓ, for every $n \geq 3$. Arguing as in the proof of Lemma 1.3.11, we can show that there exists $\epsilon > 0$ such that the matrices

$$\left[(1 - w_{m,k}\overline{w}_{m.r})\langle \hat{\ell}_{\mu_j^r}, \hat{\ell}_{\mu_j^k}\rangle\right]_{1\leq k,r\leq n}$$

are positive semi-definite for every $m \in \{1, \dots, n\}$ and $j \geq 1$, where $w_{k,k} = \epsilon$ and $w_{m,k} = 0$ if $m \neq k$ (without the assumption that $|\{\mu_j^1, \dots, \mu_j^n\}| = n$, we would need to consider matrix blocks of non-constant size $|\{\mu_j^1, \dots, \mu_j^n\}| \leq n$, but that wouldn't affect the proof in any way). Adding together the matrices corresponding to $m = 1, 2, \dots, n$, we obtain the positivity condition

$$\sum_{k,r=1}^{n} a^k\overline{a^r}\langle \hat{\ell}_{\mu_j^r}, \hat{\ell}_{\mu_j^k}\rangle \geq c\sum_{k=1}^{n} |a^k|^2, \tag{1.33}$$

for every $j \geq 1$ and every choice of scalars a^k, a^r (c can be taken to be ϵ^2/n).

Our next step will be to extract a subsequence $\cup_{j=1}^{\infty} \cup_{k=1}^{n} \{\mu_{m_j}^k\}$ that has a bounded below Grammian. To achieve this, it suffices to find a subsequence $\{m_j\}$ such that for every $\rho \geq 1$,

$$\sum_{i,j=1}^{\rho}\sum_{k,r=1}^{n} a_i^k\overline{a_j^r}\left\langle \hat{\ell}_{\mu_{m_j}^r}, \hat{\ell}_{\mu_{m_i}^k}\right\rangle > \frac{c}{2}\left[2 - \sum_{i=1}^{\rho} 2^{-i}\right]\sum_{i=1}^{\rho}\sum_{k=1}^{n} |a_i^k|^2, \tag{1.34}$$

for every choice of scalars a_i^k, a_j^r.

We will construct $\{m_j\}$ inductively. First, choose $m_1 = 1$. Then, (1.34) will be satisfied (for $\rho = 1$) because of (1.33). Now, suppose that ρ integers $\{m_1, \dots, m_\rho\}$ have been chosen so that (1.34) is satisfied. Without loss of generality, we can take our scalars a_i^k to satisfy $\sum_{i=1}^{\rho+1}\sum_{k=1}^{n} |a_i^k|^2 = 1$. By assumption, we know that $||\ell_{\mu_j^k}|| \to \infty$ for every k and hence, by Lemma (1.3.22), there exists an integer $m_{\rho+1}$ such that

$$|\langle \hat{\ell}_{\mu_{m_{\rho+1}}^r}, \hat{\ell}_{\mu_{m_i}^k}\rangle| \leq \frac{2^{-(\rho+2)}}{2n^2\rho}, \tag{1.35}$$

for every $k, r \in \{1, \dots, n\}$ and $i \in \{1, \dots, \rho\}$. This implies that

$$2\left|\sum_{i=1}^{\rho}\sum_{k,r=1}^{n} a_i^k\overline{a_{\rho+1}^r}\left\langle \hat{\ell}_{\mu_{m_{\rho+1}}^r}, \hat{\ell}_{\mu_{m_i}^k}\right\rangle\right| \leq 2^{-(\rho+2)}c. \tag{1.36}$$

Now, we can combine our inductive hypothesis with (1.33) and (1.36) to obtain

$$\sum_{i,j=1}^{\rho+1}\sum_{k,r=1}^{n} a_i^k\overline{a_j^r}\left\langle \hat{\ell}_{\mu_{m_j}^r}, \hat{\ell}_{\mu_{m_i}^k}\right\rangle$$

$$= \sum_{i,j=1}^{\rho} \sum_{k,r=1}^{n} a_i^k \overline{a_j^r} \left\langle \hat{\ell}_{\mu_{m_j}^r}, \hat{\ell}_{\mu_{m_i}^k} \right\rangle + 2\Re\left[\sum_{i=1}^{\rho} \sum_{k,r=1}^{n} a_i^k \overline{a_{\rho+1}^r} \left\langle \hat{\ell}_{\mu_{m_{\rho+1}}^r}, \hat{\ell}_{\mu_{m_i}^k} \right\rangle \right]$$
$$+ \sum_{k,r=1}^{n} a_{\rho+1}^k \overline{a_{\rho+1}^r} \left\langle \hat{\ell}_{\mu_{m_{\rho+1}}^r}, \hat{\ell}_{\mu_{m_{\rho+1}}^k} \right\rangle$$
$$> \frac{c}{2}\left[2 - \sum_{i=1}^{\rho} 2^{-i}\right] \sum_{i=1}^{\rho} \sum_{k=1}^{n} |a_i^k|^2 - 2^{-(\rho+2)}c + c\sum_{k=1}^{n} |a_{\rho+1}^k|^2 >$$
$$> \frac{c}{2}\left[2 - \sum_{i=1}^{\rho} 2^{-i}\right] \sum_{i=1}^{\rho+1} \sum_{k=1}^{n} |a_i^k|^2 - 2^{-(\rho+2)}c$$
$$= \frac{c}{2}\left[2 - \sum_{i=1}^{\rho+1} 2^{-i}\right],$$

as desired.

We have thus found a subsequence $\cup_{j=1}^{\infty} \cup_{k=1}^{n} \{\mu_{m_j}^k\}$ that has a bounded below Grammian. Now, since (by Lemma 1.3.22) $\hat{\ell}_{\mu_{m_j}^k}$ converges to 0 weakly for every $k \in \{1, \dots, n\}$, we easily see that it must have a subsequence (which we denote by $\{\hat{\ell}_{\nu_j^k}\}$) that is a Riesz sequence. This implies that $\cup_{j=1}^{\infty} \cup_{k=1}^{n} \{\nu_j^k\}$ must have a bounded Grammian and so it has to be interpolating for $\mathcal{H}_\ell$ by Lemma 1.3.8. This concludes the proof. □

An apparent limitation in applying Theorem 1.3.35 is that one first needs to know that there exists a sufficient condition for $\mathcal{H}_\ell$-interpolation having the form {(WS) by ℓ + (D)}, with the property that condition (D) is always satisfied by any finite union of "sufficiently sparse" sequences. Of course, establishing the sufficiency of such a condition in the first place is usually a highly non-trivial problem! Nevertheless, as there already exists a large literature on the subject of interpolating sequences in different function spaces, Theorem 1.3.35 will be of use in discovering new examples of AS kernels.

First, we prove a lemma which will aid us in situations where a different metric or distance function is used instead of the one induced by the kernel.

Lemma 1.3.36. *Let ℓ be a kernel on the topological space X satisfying conditions (Q1)-(Q3) from the statement of Theorem 1.3.35 and also let $\rho : X \times X \to [0, \infty)$ be a continuous function such* *that $\rho(\lambda,\mu)=0$ if and only if $\lambda=\mu$. Assume that if $\{\mu_i\}\subset X$ converges to a point in X and $\{\nu_i\}\subset X$ satisfies $||\ell_{\nu_i}||\to\infty$, then there exists $\epsilon>0$ and $i_0\in\mathbb{N}$ such that*

$$\rho(\mu_i,\nu_i)>\epsilon,$$

for all $i\geq i_0$. Finally, suppose that there exists a condition (C) such that a sequence $\{\lambda_i\}\subset X$ is interpolating for $\mathcal{H}_\ell$ if and only if it satisfies (C) and also there exists $\delta>0$ such that $\rho(\lambda_i,\lambda_j)>\delta$, for every $i\neq j$. Then, a sequence in X is interpolating for $\mathcal{H}_\ell$ if and only if it satisfies (C) and is weakly separated by ℓ.

Proof. One direction is obvious; since any sequence $\{\lambda_i\}$ satisfying $\rho(\lambda_i,\lambda_j)>\delta$, for $i\neq j$, and (C) is interpolating for $\mathcal{H}_\ell$, it must also be weakly separated by ℓ.

For the converse, suppose that $\{\lambda_i\}$ satisfies (C) and is weakly separated by ℓ. Aiming towards a contradiction, assume that there exist subsequences $\{\mu_i\},\{\nu_i\}\subset\{\lambda_i\}$ such that $\mu_i\neq\nu_i$ for every i and also

$$\lim_i \rho(\mu_i,\nu_i)=0. \tag{1.37}$$

We can additionally assume (in view of (Q3)) that $\{\mu_i\}$ either converges to a point in X or satisfies $||\ell_{\mu_i}||\to\infty$ and an analogous assumption can be made for $\{\nu_i\}$. Now, if both of them converge to points $p_1,p_2\in X$, the assumptions that ℓ is continuous and $\{\lambda_i\}$ is weakly separated imply that $d_\ell(p_1,p_2)\neq 0$, hence $p_1\neq p_2$. But then, continuity of ρ implies that

$$0=\lim_i \rho(\mu_i,\nu_i)=\rho(p_1,p_2)\neq 0,$$

a contradiction. On the other hand, if $\{\mu_i\}$ converges to a point in X and $\{\nu_i\}$ satisfies $||\ell_{\nu_i}||\to\infty$, we can find (in view of our assumptions) $\epsilon>0$ and $i_0\in\mathbb{N}$ such that $\rho(\mu_i,\nu_i)>\epsilon$ for all $i\geq i_0$, which again contradicts (1.37). Finally, assume that $\{\mu_i\},\{\nu_i\}$ both satisfy $||\ell_{\mu_i}||\to\infty$ and $||\ell_{\nu_i}||\to\infty$. Mimicking the proof of "(i)⇒(ii)" from Theorem 1.3.35, we can extract a subsequence $\{m_i\}$ such that the union $\{\mu_{m_i}\}\cup\{\nu_{m_i}\}$ is interpolating for $\mathcal{H}_\ell$. This implies that $\{\mu_{m_i}\}\cup\{\nu_{m_i}\}$ must also satisfy $\rho(\mu_{m_i},\nu_{m_i})>\delta>0$, for all i, which contradicts (1.37). Our proof is complete. □

Remark 1.3.37. In the setting of Lemma 1.3.36, assume that condition (C) possesses the following additional property: every sequence in X that can be written as $\{\lambda_i\} = \cup_{j=1}^{\infty} \cup_{k=1}^{n} \{\mu_j^k\}$, where $||\ell_{\mu_j^k}|| \to \infty$ as $j \to \infty$, for every $1 \leq k \leq n$, contains a subsequence of the form $\cup_{j=1}^{\infty} \cup_{k=1}^{n} \{\mu_{m_j}^k\}$ that satisfies (C). If we also have that no two functions ℓ_z, ℓ_w are linearly dependent if $z \neq w$, the conclusion of Lemma 1.3.36 can be strengthened to give: $\{\lambda_i\} \subset X$ is weakly separated by ℓ if and only if there exists $\delta > 0$ such that $\rho(\lambda_i, \lambda_j) > \delta$, for every $i \neq j$. We omit the proof of this claim, as it is very similar to that of the previous Lemma.

Now, recall that, as seen in subsection 1.3.6, every kernel possessing the multiplier separation property must also be an AS kernel. In particular, Corollary 1.3.32 tells us that, in a certain sense, "proximity" to powers of products of 2-point Pick kernels guarantees that the kernel will have the AS property. Well-studied examples of spaces associated with such kernels include (apart, of course, from complete Pick spaces) Bergman spaces with standard weights on the n-dimensional unit ball, where the reproducing kernel is equal to

$$\ell_\alpha(\mathbf{z}, \mathbf{w}) = \frac{1}{(1 - \langle \mathbf{z}, \mathbf{w} \rangle)^{n+1+\alpha}}, \quad \mathbf{z}, \mathbf{w} \in \mathbb{B}_n, \ \alpha > -1.$$

The norm of $\mathcal{H}_{\ell_\alpha}$ is given by integration against the weighted Lebesgue measure dv_α defined as $dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z)$, where c_α ensures that dv_α is a probability measure. See [149] for more details on these spaces.

What happens then when we move beyond such weights? It would be natural to look at so-called "large" Bergman spaces, where the weights are rapidly decreasing. One major difficulty when studying such spaces arises from the lack of an explicit expression for the reproducing kernels (see [83] and the references therein for more information). For a particular example, let $\mathcal{H}_k$ denote the Bergman space on $\mathbb{D}$ with weight $\exp\left(-\frac{1}{1-|z|^2}\right)$ and let

$$z_j = 1 - 2^{-j}, \ \ w_j = z_j + i2^{-5j/4}, \ j \geq 0.$$

As explained in [17, Example 4.13], $\{z_j\} \cup \{w_j\}$ forms a sequence in $\mathbb{D}$ that is $\mathcal{H}_k$-interpolating (in particular, $\{z_j\} \cup \{w_j\}$ must be weakly separated by k), but not weakly separated by $H^\infty = \text{Mult}(\mathcal{H}_k)$. Thus, k does not have the multiplier separation property and Proposition 1.3.27 does not apply.

Still, it turns out that k, as well as the kernels of many other large Bergman spaces on $\mathbb{D}$, is an AS kernel.

Example (*Large Bergman spaces on* $\mathbb{D}$).

Given an increasing function $h : [0,1) \to [0,\infty)$ such that $h(0) = 0$ and $\lim_{r\to 1} h(r) = +\infty$, we extend it by $h(z) = h(|z|), z \in \mathbb{D}$. We also assume that $h \in C^2(\mathbb{D})$ and $\Delta h(z) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) h(x+iy) \geq 1$. Define the weighted Bergman space

$$A_h^2(\mathbb{D}) = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : ||f||_h^2 = \int_{\mathbb{D}} |f(z)|^2 e^{-2h(z)} dm(z) < \infty \right\},$$

where dm is area measure.

$A_h^2(\mathbb{D})$-interpolating sequences for weights of polynomial growth (i.e. $h(z) = -\alpha \log(1 - |z|)$, $\alpha > 0$) were characterized by Seip in [126] (see also [128] for a thorough account). Later on, Borichev, Dhuez and Kellay [38] tackled the case of radial weights of arbitrary (more than polynomial) growth. To state their results, let

$$\rho(r) = \left[(\Delta h)(r)\right]^{-1/2}, \quad 0 \leq r < 1.$$

Certain natural growth restrictions are imposed on ρ in the setting of [38]. Examples of admissible h include

$$h(r) = \log\log\frac{1}{1-r} \cdot \log\frac{1}{1-r}, \quad h(r) = \frac{1}{1-r} \quad \text{and} \quad h(r) = \exp\frac{1}{1-r}.$$

Now, let $\mathcal{D}(z,r)$ denote the disc of radius r centered at z and define

$$d_\rho(z,w) = \frac{|z-w|}{\min\{\rho(z),\rho(w)\}}, \quad z,w \in \mathbb{D}.$$

A subset $\Gamma \subset \mathbb{D}$ will be called d_ρ*-separated* if there exists $c > 0$ such that $d_\rho(z,w) > c$, for all $z,w \in \Gamma$ such that $z \neq w$. Also, define the *upper* d_ρ*-density* of Γ to be

$$D_\rho^+(\Gamma) = \limsup_{R\to\infty} \limsup_{|z|\to 1, z\in\mathbb{D}} \frac{\mathrm{Card}(\Gamma \cap \mathcal{D}(z, R\rho(z)))}{R^2}.$$

By [38, Theorem 2.4], a sequence $\{\lambda_i\}$ is interpolating for $A_h^2(\mathbb{D})$ if and only if it is d_ρ-separated and satisfies $D_\rho^+(\Gamma) < \frac{1}{2}$.

Can Theorem 1.3.35 be applied here? Let ℓ_h denote the associated reproducing kernel. First, note that $A^2_h(\mathbb{D})$ satisfies condition (Q0), as it contains all polynomials (thus, if $\mu_1, \dots, \mu_m \in \mathbb{D}$ are m distinct points, we can always find $f \in A^2_h(\mathbb{D})$ such that $f(\mu_1) = \cdots = f(\mu_{m-1}) = 0$ and $f(\mu_m) \neq 0$). It also satisfies (Q1)-(Q3) and $||(\ell_h)_{\lambda_i}|| \to \infty$ if and only if $|\lambda_i| \to 1$ (see Theorems 3.2-3.3 in [83]). Next, we observe that the conditions of Lemma 1.3.36 are satisfied as well. Indeed, if $\{\mu_i\} \subset \mathbb{D}$ converges to a point in $\mathbb{D}$ and $\{\nu_i\} \subset \mathbb{D}$ satisfies $||(\ell_h)_{\nu_i}|| \to \infty$, then the fact that ρ decreases to 0 near the point 1 (one of the additional restrictions imposed on ρ) implies that we can find $\epsilon > 0$ such that $d_\rho(\mu_i, \nu_i) > \epsilon$ for all i. Thus, we can deduce that $\{\lambda_i\}$ is interpolating for $A^2_h(\mathbb{D})$ if and only if it is weakly separated by ℓ_h and satisfies $D^+_\rho(\Gamma) < \frac{1}{2}$ (actually, in view of Remark 1.3.37, one can easily show that d_ρ-separation is equivalent to weak separation by ℓ_h). Finally, suppose that $\{\lambda_i\}$ is a weakly separated by ℓ_h sequence that can be written as $\{\lambda_i\} = \cup_{j=1}^\infty \cup_{k=1}^n \{\mu_j^k\}$, where $||(\ell_h)_{\mu_j^k}|| \to \infty$ as $j \to \infty$, for every $1 \leq k \leq n$. Then, we can always find a subsequence $\{m_j\}$ such that $S = \cup_{j=1}^\infty \cup_{k=1}^n \{\mu_{m_j}^k\}$ satisfies $D^+_\rho(S) = 0$. Hence, S must be interpolating for $A^2_h(\mathbb{D})$ and Theorem 1.3.35 allows us to conclude that ℓ_h has the AS property.

For our next example, we turn to spaces of Bargmann-Fock type. Since these are spaces of entire functions, their multiplier algebras will consist solely of constants (as every multiplier must be bounded). This implies that the associated reproducing kernels will not contain any (non-trivial) complete Pick factors, thus Bargmann-Fock spaces are not really relevant in the context of Question 1.3.1. Still, it is worth noting that their kernels satisfy, in general, the AS property.

Example (*Bargmann-Fock spaces on $\mathbb{C}^n$*). Given $n \geq 1$ and $\alpha > 0$, the Bargmann-Fock space over $\mathbb{C}^n$ is defined as

$$F^2_\alpha = \left\{ f \in \mathrm{Hol}(\mathbb{C}^n) : ||f||^2_\alpha = \int_{\mathbb{C}^n} |f(z)|^2 e^{-\alpha|z|^2} dm(z) < \infty \right\},$$

where dm denotes Lebesgue measure on $\mathbb{C}^n$.

Interpolating sequences for F^2_α in one variable have been completely characterized by Seip [127] and Seip-Wallstén [129]. For the general case, a sufficient condition for F^2_α-interpolation was given by Massaneda and Thomas in [94] (the authors also gave a necessary condition, although, as they admit, the gap between the two is rather large). Given $\{\lambda_i\} \subset \mathbb{C}^n$, we say that $\{\lambda_i\}$ is *separated* if there exists $\delta > 0$ such that $|\lambda_i - \lambda_j| > \delta$, for all $i \neq j$. Also, let $B(z,r)$ denote the ball of center $z \in \mathbb{C}^n$ and radius r. Given $\Gamma \subset \mathbb{C}^n$, the *upper density* of Γ is defined as

$$D^+(\Gamma) = \limsup_{r\to\infty} \sup_{z\in\mathbb{C}^n} \frac{\mathrm{Card}(\Gamma \cap B(z,r))}{r^2}.$$

[94, Theorem 5.1] states that any $\{\lambda_i\} \subset \mathbb{C}^n$ that is separated and satisfies $D^+(\{\lambda_i\}) < \alpha/n$ must be interpolating for F^2_α.

Now, let $\ell(z,w) = e^{\alpha z\overline{w}}$ denote the reproducing kernel of F^2_α. Since $|\langle \hat{\ell}_z, \hat{\ell}_w\rangle|^2 = e^{-\alpha|z-w|^2}$, a sequence $\{\lambda_i\}$ is weakly separated by ℓ if and only if it is separated. Also, it can be easily verified that ℓ satisfies conditions (Q0)-(Q3) from Theorem 1.3.35. Next, suppose that $\{\lambda_i\}$ is a weakly separated by ℓ sequence that can be written as $\{\lambda_i\} = \cup_{j=1}^\infty \cup_{k=1}^n \{\mu_j^k\}$, where $||\ell_{\mu_j^k}|| \to \infty$ as $j \to \infty$, for every $1 \leq k \leq n$. This implies that $|\mu_j^k| \to \infty$, for every $1 \leq k \leq n$, hence we can always find a subsequence $\{m_j\}$ such that $S = \cup_{j=1}^\infty \cup_{k=1}^n \{\mu_{m_j}^k\}$ satisfies $D^+(S) = 0$. In view of [94, Theorem 5.1], S must be interpolating for F^2_α. By Theorem 1.3.35, we can conclude that ℓ has the AS property.

Remark 1.3.38. Let $\phi : \mathbb{C} \to \mathbb{R}$ be a subharmonic function and consider the weighted Bargmann-Fock space defined on $\mathbb{D}$ by

$$F^2_\phi = \left\{ f \in \mathrm{Hol}(\mathbb{C}) : ||f||^2_\phi = \int_{\mathbb{C}} |f(z)|^2 e^{-2\phi(z)} dm(z) < \infty \right\}.$$

In [34] and [110], the results of Seip [127] and Seip-Wallstén [129] were extended to F^2_ϕ with $\Delta\phi \simeq 1$. Later on, Marco, Massaneda and Ortega-Cerdà [91] described interpolating sequences for F^2_ϕ for a wide class of ϕ such that $\Delta\phi$ is a doubling measure. A further extension was achieved by Borichev, Dhuez and Kellay in [38], where a class of radial ϕ having more than polynomial growth was considered. Somewhat more recently, a sufficient condition for interpolation in "small" Bargmann-Fock spaces (where $\phi(z) = \alpha(\log^+|z|)^2$) was given by Baranov, Dumont, Hartmann and Kellay in [29, Theorem 1.6]. A common characteristic shared by all conditions that appear in the previously mentioned results (regardless of whether they are both necessary and sufficient or merely sufficient for F^2_ϕ-interpolation) is that they have the form: {{separation by d} + (D)}, where $d : \mathbb{C} \times \mathbb{C} \to [0, \infty)$ plays the role of a distance function and (D) is a density condition that is, roughly, always satisfied by any finite union of "sufficiently sparse" sequences. Thus, letting ℓ_ϕ denote the kernel of F^2_ϕ, Theorem 1.3.35 tells us that, for any ϕ corresponding to one of the previous cases and such that ℓ_ϕ satisfies (Q0)-(Q3) and d satisfies the hypotheses of Lemma 1.3.36, ℓ_ϕ will have the AS property.

We end this subsection by giving a general class of pairs (s, ℓ) for which Question 1.3.1 has a positive answer.

Theorem 1.3.39. *Let ℓ be a kernel on X defined by performing a finite number of any of the operations from Proposition 1.3.33 to one or more kernels having the 2-point Pick property and/or to one or more kernels from Example 1.3.7. Suppose also that s is a complete Pick factor of ℓ. Then, a sequence $\{\lambda_i\} \subset X$ is interpolating for Mult$(\mathcal{H}_s, \mathcal{H}_\ell)$ if and only if it satisfies the Carleson measure condition for $\mathcal{H}_s$ and is weakly separated by ℓ.*

Proof. This is a consequence of Corollary 1.3.20 and of Propositions 1.3.27 and 1.3.33, since 2-point Pick kernels and the kernels from Example 1.3.7 are all AS kernels. □

1.3.8 A Holomorphic Counterexample

In this subsection, we construct a holomorphic pair (s, ℓ) on $\mathbb{D}^2$, where s is a CP factor of ℓ and ℓ satisfies properties (Q0)-(Q3) from Theorem 1.3.35, such that there exists an (infinite) sequence $\{\lambda_i\} \subset \mathbb{D}^2$ satisfying the Carleson measure condition for $\mathcal{H}_s$ and being weakly separated by ℓ, but not Mult$(\mathcal{H}_s, \mathcal{H}_\ell)$-interpolating. In particular, we will construct a sequence that is weakly but not 4-weakly separated by ℓ.

Example. Let $\mathcal{H}_k$ denote the weighted Bergman space on $\mathbb{D}$ with weight $e^{-\frac{1}{1-|z|^2}}$ and define the kernels

$$\ell((\lambda_1, \lambda_2), (\mu_1, \mu_2)) = \frac{k(\lambda_1, \mu_1) + k(\lambda_2, \mu_2)}{(1 - \lambda_1\overline{\mu}_1)(1 - \lambda_2\overline{\mu}_2)},$$

and

$$s((\lambda_1,\lambda_2),(\mu_1,\mu_2)) = \frac{1}{2-\lambda_1\overline{\mu}_1 - \lambda_2\overline{\mu}_2} = \frac{\frac{1}{2}}{1-\left\langle \frac{1}{\sqrt{2}}(\lambda_1,\lambda_2), \frac{1}{\sqrt{2}}(\mu_1,\mu_2)\right\rangle},$$

where $(\lambda_1,\lambda_2),(\mu_1,\mu_2) \in \mathbb{D}\times\mathbb{D}$.

Evidently, s is a complete Pick kernel. Also, letting $H^2_{\mathbb{D}^2}$ denote the Hardy space on $\mathbb{D}^2$, we observe that the vector-valued function $\phi(\lambda_1,\lambda_2) = \begin{bmatrix}\lambda_1/\sqrt{2} & \lambda_2/\sqrt{2}\end{bmatrix} \in \text{Mult}(H^2_{\mathbb{D}^2}\otimes\mathbb{C}^2, H^2_{\mathbb{D}^2})$ is a contractive multiplier. This implies that s is a complete Pick factor of $1/(1-\lambda_1\overline{\mu}_1)(1-\lambda_2\overline{\mu}_2)$ and hence also of ℓ. Regarding ℓ, it is easily verified that $\ell((\lambda_1,\lambda_2),(\mu_1,\mu_2))$ is of the form $\sum_{n,m=0}^{\infty} a_{n,m}(\lambda_1\overline{\mu_1})^n(\lambda_2\overline{\mu_2})^m$, where every $a_{n,m}$ is nonzero. We deduce that the monomials $\lambda_1^n\lambda_2^m$ form a complete orthogonal set for $\mathcal{H}_\ell$. Thus, ℓ satisfies (Q0) (and the same must be true for s). In view of [83, Theorems 3.2-3.3], ℓ must also satisfy (Q1)-(Q3) (this is because k already satisfies these properties). Finally, it is worth noting that the existence of the factor $1/(1-\lambda_1\overline{\mu}_1)(1-\lambda_2\overline{\mu}_2)$ implies that $\text{Mult}(\mathcal{H}_\ell) = H^\infty(\mathbb{D}^2)$.

Now, define the sequence

$$\lambda_{4j+n} = (z_{4j+n}, w_{4j+n}) = \begin{cases} (1-2^{-j}, 1-2^{-j}), & \text{if } n=0; \\ (1-2^{-j}, 1-2^{-j}+i2^{-5j/4}), & \text{if } n=1; \\ (1-2^{-j}+i2^{-5j/4}, 1-2^{-j}), & \text{if } n=2; \\ (1-2^{-j}+i2^{-5j/4}, 1-2^{-j}+i2^{-5j/4}), & \text{if } n=3, \end{cases}$$

where $j \geq 1$. Letting

$$K((\lambda_1,\lambda_2),(\mu_1,\mu_2)) = k(\lambda_1,\mu_1) + k(\lambda_2,\mu_2),$$

it can be easily verified that

$$K_{\lambda_{4j}} - K_{\lambda_{4j+1}} - K_{\lambda_{4j+2}} + K_{\lambda_{4j+3}} = 0,$$

for all $j \geq 1$. Thus,

$$\det[\langle \hat{K}_{\lambda_{4j+n}}, \hat{K}_{\lambda_{4j+m}} \rangle]_{0 \leq n,m \leq 3} = 0, \tag{1.38}$$

for all $j \geq 1$. Also, let r and t denote the Szegő kernels on $\mathbb{D}$ and $\mathbb{D}^2$, respectively. A short calculation reveals that

$$\lim_j \frac{r(1-2^{-j}, 1-2^{-j}+i2^{-5j/4})}{\sqrt{r(1-2^{-j},1-2^{-j})r(1-2^{-j}+i2^{-5j/4}, 1-2^{-j}+i2^{-5j/4})}} = 1.$$

Thus, we obtain

$$\lim_j \langle \hat{t}_{\lambda_{4j+n}}, \hat{t}_{\lambda_{4j+m}} \rangle = \lim_j \left[\langle \hat{r}_{z_{4j+n}}, \hat{r}_{z_{4j+m}} \rangle \cdot \langle \hat{r}_{w_{4j+n}}, \hat{r}_{w_{4j+m}} \rangle \right] = 1,$$

for all $n, m \in \{0, 1, 2, 3\}$. In view of (1.38) and the previous limit, we can write

$$\det[\langle \hat{\ell}_{\lambda_{4j+n}}, \hat{\ell}_{\lambda_{4j+m}} \rangle]_{0 \leq n,m \leq 3}$$

$$= \det[\langle \hat{K}_{\lambda_{4j+n}}, \hat{K}_{\lambda_{4j+m}} \rangle \cdot \langle \hat{t}_{\lambda_{4j+n}}, \hat{t}_{\lambda_{4j+m}} \rangle]_{0 \leq n,m \leq 3} \to 0, \tag{1.39}$$

as $j \to \infty$.

Next, we prove the existence of $\epsilon > 0$ such that

$$d_\ell(\lambda_{4j+n}, \lambda_{4j+m}) > \epsilon, \tag{1.40}$$

for all $j \geq 1$ and $n, m \in \{0, 1, 2, 3\}, n \neq m$. First, note that the discussion preceding Example 1.3.7 implies the existence of $\delta > 0$ such that

$$d_k(1-2^{-j}, 1-2^{-j}+i2^{-5j/4}) > \delta, \quad \forall j \geq 1. \tag{1.41}$$

Now, let $j \geq 1$ and $n, m \in \{0, 1, 2, 3\}$, with $n \neq m$. Without loss of generality, we may assume that $z_{4j+n} = 1 - 2^{-j}$ and $z_{4j+m} = 1 - 2^{-j} + i2^{-5j/4}$ (if $z_{4j+n} = z_{4j+m}$, we would work with w_{4j+n} and w_{4j+m} instead). Note that $|w_{4j+n}|, |w_{4j+m}|, |z_{4j+n}| \leq |z_{4j+m}|$, hence $||k_{w_{4j+n}}||, ||k_{w_{4j+m}}||, ||k_{z_{4j+n}}|| \leq ||k_{z_{4j+m}}||$ (k is rotationally invariant). Also, in view of (1.30) and (1.41), we can find $f \in \mathcal{H}_k$ such that $||f||_{\mathcal{H}_k} < 1/\delta$, $f(z_{4j+n}) = 0$ and $f(z_{4j+m}) = ||k_{z_{4j+m}}||$. Define

$$F(\lambda_1, \lambda_2) = \frac{\sqrt{||k_{z_{4j+m}}||^2 + ||k_{w_{4j+m}}||^2}}{||k_{z_{4j+m}}||} f(\lambda_1), \quad (\lambda_1, \lambda_2) \in \mathbb{D}^2.$$

[113, Theorem 5.4] implies that $||f||_{\mathcal{H}_K} \leq ||f||_{\mathcal{H}_k}$. Thus,

$$||F||_{\mathcal{H}_K} \leq \frac{\sqrt{||k_{z_{4j+m}}||^2 + ||k_{z_{4j+m}}||^2}}{||k_{z_{4j+m}}||} ||f||_{\mathcal{H}_k} < \sqrt{2}/\delta.$$

Also, observe that $F(\lambda_{4j+n}) = 0$ and

$$F(\lambda_{4j+m}) = \frac{\sqrt{||k_{z_{4j+m}}||^2 + ||k_{w_{4j+m}}||^2}}{||k_{z_{4j+m}}||} f(z_{4j+m}) = ||K_{\lambda_{4j+m}}||.$$

In view of (1.30), we obtain $d_K(\lambda_{4j+n}, \lambda_{4j+m}) > \delta/\sqrt{2}$. An application of Lemma 1.3.24 then gives us (1.40).

Next, note that both $||\ell_{\lambda_{4j+n}}||, ||s_{\lambda_{4j+n}}|| \to \infty$ as $j \to \infty$, for all $n \in \{0, 1, 2, 3\}$. Thus, there exists a subsequence $\{m_j\}$ such that $\cup_{n=0}^{3}\{\lambda_{4m_j+n}\}$ satisfies the Carleson measure condition for $\mathcal{H}_s$. Also, after some calculations, we can deduce the existence of $\epsilon' > 0$ with the property

$$d_t(\lambda_{4j+n}, \lambda_{4i+m}) > \epsilon', \tag{1.42}$$

for all $i \neq j$ and all $n, m \in \{0, 1, 2, 3\}$. Lemma 1.3.24 then implies that

$$d_\ell(\lambda_{4j+n}, \lambda_{4i+m}) > \epsilon', \tag{1.43}$$

for all $i \neq j$ and all $n, m \in \{0, 1, 2, 3\}$. (1.40) combined with (1.43) tell us that $\cup_{n=0}^{3}\{\lambda_{4j+n}\}$ (and hence $\cup_{n=0}^{3}\{\lambda_{4m_j+n}\}$ as well) is weakly separated by ℓ. However, Lemma 1.3.10 and (1.39) imply that $\cup_{n=0}^{3}\{\lambda_{4j+n}\}$ (and hence $\cup_{n=0}^{3}\{\lambda_{4m_j+n}\}$ as well) is not 4-weakly separated by ℓ. We deduce that the pair (s, ℓ) constitutes a counterexample to Question 1.3.1.

Remark 1.3.40. The specific choice of s in the previous example is not important; all that was required for the proof to go through was a CP factor s of ℓ such that $||s_{\lambda_{4j+n}}|| \to \infty$ as $j \to \infty$, for all $n \in \{0, 1, 2, 3\}$.

Chapter 2

Operator Inequalities on the Annulus

The material contained in this chapter originates in the following three papers:

Paper III G. Tsikalas. "A note on a spectral constant associated with an annulus". In: *Oper. Matrices* 16.1 (2022), pp. 95–99

Paper IV G. Tsikalas. "A von Neumann type inequality for an annulus". In: *J. Math. Anal. Appl.* 506.2 (2022), Paper No. 125714, 12

Paper V M. T. Jury and G. Tsikalas. "Positivity conditions on the annulus via the double-layer potential kernel". 2023, submitted. arXiv: 2307.13387

One exception is Section 2.4, which is unpublished.

2.1 Introduction

Let X be a closed set in the complex plane and let $\mathcal{R}(X)$ denote the algebra of complex-valued bounded rational functions on X, equipped with the supremum norm

$$||f||_X = \sup\{|f(x)| : x \in X\}.$$

Suppose that T is a bounded linear operator acting on the (complex) Hilbert space H. Suppose also that the spectrum $\sigma(T)$ of T is contained in the closed set X. Let $f = p/q \in \mathcal{R}(X)$. As the poles of the rational function f are outside of X, the operator $f(T)$ is naturally defined as $f(T) = p(T)q(T)^{-1}$ or, equivalently, by the Riesz-Dunford functional calculus (see e.g. [55] for a treatment of this topic). Given a fixed constant $K > 0$, we will say that the set X is a *K-spectral set* for T if $\sigma(T) \subseteq X$ and the inequality

$$||f(T)|| \leq K||f||_X$$

holds for every $f \in \mathcal{R}(X)$. The set X is a *spectral set* for T if it is a K-spectral set with $K = 1$.

Spectral sets were introduced and studied by von Neumann in [105], where he proved the celebrated result that an operator T is a contraction if and only if the closed unit disk is a spectral set for T. They have applications to the approximate computation of norms of matrix functions, an essential task in many fields of pure and applied mathematics, including numerical and functional analysis (see e.g. [76, 82, 111]). Inequalities of von Neumann-type have also found deep applications in complex geometry, such as an alternative proof of Lempert's theorem on the Carathéodory metric [6] and the solution of the Carathéodory extremal problem for the symmetrized bidisc [10]. We refer the reader to the book [111] and the survey [28] for more detailed presentations and more information on K-spectral sets.

2.1.1 The Hardy Space and von Neumann's Inequality

von Neumann's original proof of his inequality consisted in first showing that it holds for the Möbius transformations of the disk, and then reducing the case of any general analutic function to this special case. Since then, his result has been proved in many different ways (in [111] alone, there exist five different proofs). A particularly illuminating proof was given by Sz.-Nagy in [136] as an application of his famous dilation theorem, which asserts that every contraction operator can be dilated to a unitary operator. We now present another well-known proof of this inequality that is closely tied to the Hardy space H^2.

Recall that H^2 is the Hilbert space of functions f analytic in the unit disc $\mathbb{D} = \{|z| < 1\}$ and satisfying

$$||f||_2^2 := \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

Its multiplier algebra can be isometrically identified with H^∞, the algebra of all bounded holomorphic functions on $\mathbb{D}$ equipped with the norm $||\phi||_\infty := \sup_{z\in\mathbb{D}} |\phi(z)|$ (see also Chapter 1). Now, a well-known characterization tells us that a function $\phi : \mathbb{D} \to \mathbb{C}$ is a multiplier of H^2 with norm less than or equal to 1 if and only if there exists a positive semi-definite function $k : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ such that

$$1 - \phi(\lambda)\overline{\phi(\mu)} = k(\lambda, \mu)(1 - \lambda\bar{\mu}) \quad \text{on} \quad \mathbb{D} \times \mathbb{D}.$$

Since k is positive semi-definite on $\mathbb{D} \times \mathbb{D}$, it can be written as a (possibly infinite) sum of dyads (i.e. positive semi-definite functions of the form $f(\lambda)\overline{f(\mu)}$). Taking ϕ to be a polynomial, we can then apply both sides of the last equality to an arbitrary contraction T by means of the *hereditary functional calculus* (see Section 2.8.3 in [14] for the details) to obtain

$$I - \phi(T)\phi(T)^* = \sum_{i\geq 0} f_i(T)(I - TT^*)f_i(T)^*.$$

Since $||T|| \leq 1$ is equivalent to $I - TT^* \geq 0$, standard positivity arguments can be used to conclude that $I - \phi(T)\phi(T)^* \geq 0$, as desired. In Section 2.3, we will deduce a certain von Neumann-type inequality for an operator class on the annulus by applying analogous positivity arguments to model formulas in an appropriate function space setting.

2.1.2 Crouzeix's Conjecture and the Double-Layer Potential

Given a Hilbert space H and $T \in \mathcal{B}(H)$, we let

$$W(T) = \{\langle Tx, x\rangle : x \in H, ||x|| = 1\}$$

denote the *numerical range* of T. The following conjecture first appeared in 2004 [62] and has since continued to generate intense research activity.

Crouzeix's conjecture. $\overline{W(T)}$ *is a* 2*-spectral set for* T, *for every* $T \in \mathcal{B}(H)$.

This conjecture is already known to hold if $W(T)$ is disk (a result originally due to Okubo and Ando [108] via dilation theory), if T is 2×2 [62] and for a few other special classes of matrices, see [75], [58], [50], [77]. The basic technique used in the proof of those special cases is to consider a conformal mapping $\phi : W(T)^{\circ} \to \mathbb{D}$ and then show that $\phi(T)$ must be similar to a contraction via a similarity transformation with condition number at most 2.

In [61], Crouzeix and Palencia, improving earlier results of Delyon-Delyon [66] and Crouzeix [57], showed that $\overline{W(T)}$ is always a $(1+\sqrt{2})$-spectral set for T, and this remains the best known general estimate to date. The key ingredient in the Crouzeix-Palencia proof is the use of an integral representation formula for operators based on the Cauchy transform and the so-called double layer potential kernel. More precisely, given any smoothly bounded, open $\Omega \subset \mathbb{C}$, let $\mathcal{A}(\Omega)$ denote the uniform algebra of continuous functions f on $\overline{\Omega}$ that are holomorphic on Ω. Assuming $f \in \mathcal{A}(\Omega)$ and $T \in \mathcal{B}(H)$ is such that $\sigma(T) \subset \Omega$, we may consider the Cauchy transforms of f and $\overline{f}$

$$f(T) = (Cf)(T) = \frac{1}{2\pi i}\int_{\partial\Omega} f(\sigma)(\sigma - T)^{-1} d\sigma,$$

$$(C\overline{f})(T) = \frac{1}{2\pi i}\int_{\partial\Omega} \overline{f(\sigma)}(\sigma - T)^{-1} d\sigma.$$

We also define the transform of f by the double-layer potential kernel

$$S(f,T) = \int_{\partial\Omega} \mu(\sigma(s),T) f(\sigma(s)) ds,$$

where s denotes the arc length of $\sigma = \sigma(s)$ on the (counter-clockwise) oriented boundary $\partial\Omega$ and $\mu(\sigma(s),T)$ is the self-adjoint operator defined (for $\sigma(s) \notin \sigma(T)$) as

$$\mu(\sigma(s),T) = \frac{1}{2\pi i}\big(\sigma'(s)(\sigma(s) - T)^{-1} - \overline{\sigma'(s)}(\overline{\sigma(s)} - T^*)^{-1}\big).$$

Note that $S(f,T) = f(T) + (C\overline{f})(T)^*$ and thus $\int_{\partial\Omega} \mu(\sigma,T) ds = 2I$.

The double-layer potential kernel and the numerical range can then be related as follows: we have $\overline{W(T)} \subset \Omega$ if and only if $\mu(\sigma,T)$ is positive for every $\zeta \in \partial\Omega$. This leads to the estimate

$$||f(T) + (C\bar{f})(T)^*|| \leq 2 \sup_{z\in\Omega} |f(z)|,$$

for every such T and $f \in \mathcal{A}(\Omega)$. An abstract functional analysis lemma (see [121][Lemma 1.1], further developments in [53]) can then used, in conjuction with the fact that $f \mapsto C\overline{f}$ is contractive for convex Ω, to estimate $||f(T)||$ directly and conclude the Crouzeix-Palencia proof.

A special class of functions that have played a vital role in recent investigations of the Crouzeix conjecture, are the so-called extremal functions. Given $A \in \mathbb{C}^{n\times n}, \sigma(A) \subset \Omega$ convex, a function $f \in H^\infty(\Omega)$ is *extremal* for (A, Ω) if it maximizes $||f(A)||/||f||_\Omega$ amongst all functions in $H^\infty(\Omega)$. Extremal functions always exist and, by Nevanlinna-Pick theory, must have the form $f = B \circ \phi$, where $\phi : \Omega \to \mathbb{D}$ is a conformal mapping and B is a finite Blaschke product of degree at most $n-1$. The structure and properties of these functions and associated operators $f(A)$ (e.g. the uniqueness and possible degrees of B) have been studied in [120] and in [35], however many questions remain. A very important result in this direction is the following orthogonality property: in the previous setting, if we also have $||f(A)|| > 1$ and $||f(A)x|| = ||f(A)||$ for the unit vector x, then

$$\langle f(A)x, x\rangle = 0.$$

This characterizes extremal pairs (f, A) in the 2×2 case (but not in the 3×3 case) and leads to a very simple proof of Crouzeix's conjecture in the case that $W(A)$ is a disk (see [43]).

We mention one more application of the double-layer potential kernel, which will motivate our investigations in Section 2.5. Let $\rho > 0$. Denote by $\mathcal{C}_\rho$ the class of all bounded Hilbert space operators $T \in \mathcal{B}(H)$ that have a unitary ρ-dilation, i.e. there exists a Hilbert space $K \supset H$ and a unitary $U \in \mathcal{B}(K)$ such that

$$T^n = \rho P_H U^n|_H, \quad n = 1, 2, \dots,$$

where P_H is the orthogonal projection of K onto H. The elements of $\mathcal{C}_\rho$ are referred to as ρ-contractions. $\mathcal{C}_\rho$ was introduced by Sz.-Nagy and Foiaş [138] (see also [137, Chapter 1]) and has subsequently been investigated by many authors (see e.g. [46] and the references therein). It is known that $\mathcal{C}_1$ is precisely the set of contractions on H [136], while $\mathcal{C}_2$ is the set of operators whose numerical range is contained in $\overline{\mathbb{D}}$ (see [31], [33]). An alternate characterization of $\mathcal{C}_\rho$ (see [47, p. 315]) then states that T is a ρ-contraction if and only if the operator

$$\begin{aligned} S_{\mathbb{D},\rho} &: \mathcal{A}(\mathbb{D}) \to \mathcal{B}(H) \\ f &\mapsto \frac{1}{\rho}\Big[f(T) + (C\overline{f})(T)^* + \frac{(\rho-2)}{2\pi i}\int_{\partial\mathbb{D}} f(\zeta)d\zeta\Big] \end{aligned} \tag{2.1}$$

$$= \frac{1}{\rho}[f(T) + (\rho - 1)f(0)]$$

is contractive, where $C\overline{f}$ is taken over $\mathbb{D}$. The mapping $S_{\mathbb{D},\rho}$, a translate of the double-layer potential integral operator, has proved itself a valuable tool for better understanding $\mathcal{C}_\rho$. As a recent example, Clouâtre, Ostermann and Ransford [53], building on ideas from [43], used the contractivity of $S_{\mathbb{D},\rho}$ as a stepping stone for an alternative, simple proof of the fact that $\overline{\mathbb{D}}$ is a ρ-spectral set for T whenever T is a ρ-contraction (with $\rho \geq 1$), a result originally proved by Okubo and Ando [108].

2.1.3 The Annulus as a *K*-spectral Set

Let $X = \overline{A_R} := \{1/R \leq |z| \leq R\}$ $(R > 1)$ denote the closed annulus, the intersection of the two closed disks $D_1 = \{|z| \leq R\}$ and $D_2 = \{|z| \geq 1/R\}$. The intersection of two spectral sets is not necessarily a spectral set; counterexamples for the annulus were presented in [104], [112] and [143]. However, the same question for *K*-spectral sets remains open (the counterexamples for the spectral set case show that we will not, in general, be able to use the same constant for the intersection). Regarding the annulus in particular, Shields proved that, given an invertible operator $T \in \mathcal{B}(H)$ with $||T|| \leq R$ and $||T^{-1}|| \leq R$, $\overline{A_R}$ is a *K*-spectral set for T with $K = 2 + \sqrt{(R^2 + 1)/(R^2 - 1)}$, see [133, Proposition 23] . This bound is large if R is close to 1. In this context, Shields raised the question of finding the smallest constant $K = K(R)$ such that $\overline{A_R}$ is $K(R)$-spectral, see [133, Question 7] . In particular, he asked whether this optimal constant $K(R)$ would remain bounded.

Motivated by Shields' results, we define the *quantum annulus* $\mathbb{QA}_R$ as the class of all operators $T \in \mathcal{B}(H)$ with $||T|| \leq R$ and $||T^{-1}|| \leq R$. Thus, $K(R)$ can be equivalently defined as the smallest constant K such that $\overline{A_R}$ is a K-spectral set for every $T \in \mathbb{QA}_R$. The question whether $K(R)$ is uniformly bounded was answered positively by Badea, Beckermann and Crouzeix in [27, Theorem 1.2], where they obtained that (for every $R > 1$)

$$\frac{4}{3} < \gamma(R) := 2(1 - R^{-2}) \prod_{n=1}^{\infty} \left(\frac{1 - R^{-8n}}{1 - R^{4-8n}}\right)^2 \leq K(R) \leq 2 + \frac{R+1}{\sqrt{R^2 + R + 1}} \leq 2 + \frac{2}{\sqrt{3}}.$$

The quantity $\gamma(R)$ was numerically shown to be greater than or equal to $\pi/2$ (leading to the universal lower bound $\pi/2$ for $K(R)$) and, in addition, tends to 2 as R tends to infinity.

Two subsequent improvements were made to the upper bound for $K(R)$: the first one in [59, Lemma 2.1] by Crouzeix and the most recent one in [63, p. 7] by Crouzeix and Greenbaum, where it was proved that

$$K(R) \leq 1 + \sqrt{2}, \quad \forall R > 1.$$

As for the lower bound, Badea obtained in [26, p. 242] the statement

$$\frac{3}{2} < 2\frac{1 + R^2 + R}{1 + R^2 + 2R} \leq K(R), \quad \forall R > 1,$$

where the quantity $2(1 + R^2 + R)/(1 + R^2 + 2R)$ again tends to 2 as R tends to infinity. In Section 2.2, we will improve the aforementioned estimates by showing that 2 is actually a universal lower bound for $K(R)$, for every $R > 1$.

What happens if we restrict T to the smaller class

$$\mathcal{F}_R := \{T \in \mathcal{B}(H) : R^2 T^{-1}(T^{-1})^* + R^2 T T^* \leq R^4 + 1\}? \tag{2.2}$$

$$= \{T \in \mathcal{B}(H) : q(T, T^*) \geq 0\},$$

where q is the hereditary Laurent polynomial $q(z, w) = (R^2 - z\overline{w})(R^2 - z^{-1}\overline{w}^{-1})$? In Section 2.3, we will show that $\overline{A_R}$ is a $\sqrt{2}$-spectral set for any $T \in \mathcal{F}_R$, the constant $\sqrt{2}$ being optimal.

In [30], Bello and Yakubovich asked whether a certain refinement of $\mathcal{F}_R$ is equal to the class of all operators that have $\overline{A_R}$ as a spectral set. In Section 2.4, we will show that the answer to their question is negative.

Let us now briefly return to the above-mentioned CrouzGreen [63] upper bound for $K(R)$. The CrouzGreen argument rests on an extension of the Crouzeix-Palencia proof to more general (not necessarily convex) planar domains Ω, including the annulus. In particular, the key observation in their proof is that the double-layer potential kernel $\mu(\sigma, T)$ is positive on ∂A_R whenever T lies in the quantum annulus. Thus, if one wants to gain a better understanding of K-spectral estimates over A_R, the following question emerges naturally: for which operators T will $\mu(\sigma, T)$ be positive over ∂A_R? In Section 2.5, which is joint work with Michael Jury, we will study the scale of operator classes $\mathbb{DLA}_R(c)$, where $\mathbb{DLA}_R(0)$ is defined precisely as the set of all operators T with spectrum in the annulus that satisfy this positivity condition. We will obtain relevant K-spectral estimates which will allow us to unify and generalize existing results from the literature on the annulus. For certain 2×2 matrices, we will also give sharp K-spectral constants that motivate the following conjecture:

Conjecture. *Let $c \geq 0$. Then A_R is a $(2+c)$-spectral set for every $T \in \mathbb{DLA}_R(c)$.*

Note that a positive resolution of the above conjecture would imply that $K(R) = 2$ for every $R > 1$, which is supported by numerical data.

2.2 Estimates for the Quantum Annulus

Recall that

$$\mathbb{QA}_R = \{T \in \mathcal{B}(H) : ||T|| \leq R, ||T^{-1}|| \leq R\},$$

for every $R > 1$. Our objective is to show:

Theorem 2.2.1. *Let $K(R)$ denote the smallest positive constant such that $\overline{A_R}$ is a $K(R)$-spectral set for the bounded linear operator T whenever $T \in \mathbb{QA}_R$. Then,*

$$K(R) \geq 2, \quad \forall R > 1.$$

Proof. Fix $R > 1$. For every $n \geq 2$, define

$$g_n(z) = \frac{1}{R^n}\left(\frac{1}{z^n} + z^n\right) \in \mathcal{R}(A_R).$$

It is easy to see that

$$||g_n||_{A_R} = g_n(R) = 1 + \frac{1}{R^{2n}}. \tag{2.3}$$

To achieve the stated improvement, we will apply g_n to a bilateral shift operator S acting on a particular weighted sequence space $L^2(\beta)$. First, define the sequence $\{\beta(k)\}_{k\in\mathbb{Z}}$ of positive numbers (weights) as follows:

$$\beta(2ln + q) = R^q, \ \forall q \in \{0, 1, \dots, n\}, \forall l \in \mathbb{Z};$$

$$\beta((2l+1)n+q) = R^{n-q}, \ \ \forall q \in \{0,1,\dots,n\}, \forall l \in \mathbb{Z}.$$

Consider now the space of sequences $f = \{\hat{f}(k)\}_{k\in\mathbb{Z}}$ such that

$$||f||_\beta^2 := \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2[\beta(k)]^2 < \infty.$$

We shall use the notation $f(z) = \sum_{k\in\mathbb{Z}} \hat{f}(k)z^k$ (formal Laurent series), whether or not the series converges for any (complex) values of z. Our weighted sequence space will be denoted by

$$L^2(\beta) := \{f = \{\hat{f}(k)\}_{k\in\mathbb{Z}} : ||f||_\beta^2 < \infty\}.$$

This is a Hilbert space with the inner product

$$\langle f, g\rangle_\beta := \sum_{k\in\mathbb{Z}} \hat{f}(k)\overline{\hat{g}(k)}[\beta(k)]^2.$$

Consider also the linear transformation (bilateral shift) S of multiplication by z on $L^2(\beta)$:

$$(Sf)(z) = \sum_{k\in\mathbb{Z}} \hat{f}(k)z^{k+1}.$$

In other words, we have

$$\widehat{(Sf)}(k) = \hat{f}(k-1), \ \ \forall k \in \mathbb{Z}.$$

Observe that

$$||S|| = \sup_{k\in\mathbb{Z}} \frac{\beta(k+1)}{\beta(k)} = R$$

and

$$||S^{-1}|| = \sup_{k\in\mathbb{Z}} \frac{\beta(k)}{\beta(k+1)} = R.$$

Now, let $m \geq 3$ and define $h = \{\hat{h}(k)\}_{k\in\mathbb{Z}} \in L^2(\beta)$ by putting:

$$\hat{h}(2ln) = \frac{1}{m}, \ \ \forall l \in \{0,1,2\dots,m^2\};$$

$$\hat{h}(k) = 0, \ \ \text{in all other cases.}$$

We calculate

$$||h||_\beta^2 = \sum_{l=0}^{m^2} \frac{1}{m^2}[\beta(2ln)]^2 = \sum_{l=0}^{m^2} \frac{1}{m^2}\cdot 1^2 = \frac{m^2+1}{m^2},$$

hence

$$||h||_\beta = \frac{\sqrt{m^2+1}}{m}. \tag{2.4}$$

Also, put $f = (S^{-n} + S^n)h$ and notice that

$$\begin{aligned} ||(S^{-n} + S^n)h||_\beta^2 &= ||f||_\beta^2 \\ &\geq \sum_{l=1}^{m^2} |\hat{f}((2l-1)n)|^2[\beta((2l-1)n)]^2 \\ &= \sum_{l=1}^{m^2} \left(\frac{2}{m}\right)^2 R^{2n} \\ &= 4R^{2n}. \end{aligned}$$

Thus,

$$||(S^{-n} + S^n)h||_\beta \geq 2R^n. \tag{2.5}$$

Using (2.3), (2.4) and (2.5), we can now write

$$\begin{aligned} K(R) &\geq \frac{||g_n(S)||}{||g_n||_{A_R}} \\ &= \frac{1}{R^n} \cdot \frac{||S^{-n} + S^n||}{1 + R^{-2n}} \\ &\geq \frac{1}{R^n + R^{-n}} \cdot \frac{||(S^{-n} + S^n)h||_\beta}{||h||_\beta} \\ &\geq \frac{1}{R^n + R^{-n}} \cdot \frac{2R^n}{\frac{\sqrt{m^2+1}}{m}}. \end{aligned}$$

Letting $m \to \infty$, we obtain

$$K(R) \geq \frac{1}{R^n + R^{-n}} \cdot \frac{2R^n}{1} = \frac{2R^n}{R^n + R^{-n}} \xrightarrow{n\to\infty} 2, \quad \text{as } R > 1.$$

The proof is complete.

□

2.3 Estimates for $\mathcal{F}_R$

This section revolves around the spectral constant associated with the operator class $\mathcal{F}_R$ as defined above (2.2). However, to more easily communicate with the setting of the paper where these results are contained, we will instead work with the rescaled annulus $A_r = \{r < |z| < 1\}$ (where $0 < r < 1$) and replace $\mathcal{F}_R$ by the updated class

$$\mathcal{F}_r := \{T \in \mathcal{B}(H) : r^2 T^{-1}(T^{-1})^* + TT^* \leq r^2 + 1, \, \sigma(T) \subset A_r\}.$$

We note that the spectral constant remains unchanged if we drop the assumption $\sigma(T) \subset A_r$; we choose to include it in order not to have to add aproximation arguments to our proof.

To arrive at our estimates, we will apply standard positivity arguments to model formulas in the setting of the holomorphic function space $\mathcal{H}^2(A_r)$ induced on A_r by the kernel

$$k_{A_r}(\lambda, \mu) := \frac{1 - r^2}{(1 - \lambda\bar{\mu})(1 - r^2/\lambda\bar{\mu})}, \quad \forall \lambda, \mu \in A_r.$$

Our main result is the following theorem, the proof of which is contained in subsection 2.3.3.

Theorem 2.3.1. *For every $\phi \in H^\infty(A_r)$,*

$$\sup_{T \in \mathcal{F}_r} ||\phi(T)|| = ||\phi||_{\mathrm{Mult}(\mathcal{H}^2(A_r))} \leq \sqrt{2}||\phi||_\infty,$$

where the constant $\sqrt{2}$ is the best possible.

The fact that k_{A_r} is a complete Pick kernel also allows us to show the following extension result.

Theorem 2.3.2. *Let $0 < r < 1$. For every $\phi \in H^\infty(A_r)$, the quantity*

$$\min\{||\psi||_{Mult(H^2_2)} : \psi \in \mathrm{Mult}(H^2_2) \text{ and } \psi\Big(\frac{z}{\sqrt{r^2+1}}, \frac{r}{\sqrt{r^2+1}}\frac{1}{z}\Big) = \phi(z), \, \forall z \in A_r\}$$

lies in the interval $[||\phi||_\infty, \sqrt{2}||\phi||_\infty]$. Moreover, the constant $\sqrt{2}$ is the best possible.

Here, H^2_2 denotes the 2-dimensional Drury-Arveson space on the open unit ball $\mathbb{B}_2 \subseteq \mathbb{C}^2$. We refer the reader to Chapter 1 for background on reproducing kernel Hilbert spaces and complete Pick kernels.

2.3.1 Preliminaries

First, we record the following well-known result, which will be needed in the sequel. Suppose

$$k(y,x) = \frac{1}{1 - \langle b(y), b(x)\rangle}$$

is a complete Pick kernel on a set X embedding into the Drury-Arveson space H^2_d, where $b : X \to \mathbb{B}_d$. Then, for every $\phi \in \mathrm{Mult}(\mathcal{H}_k)$ we obtain that

$$||\phi||_{\mathrm{Mult}(\mathcal{H}_k)} = min\{||\psi||_{\mathrm{Mult}(H^2_d)} : \psi \in \mathrm{Mult}(H^2_d) \text{ and } \psi(b(x)) = \phi(x),\ \forall x \in X\}. \tag{2.6}$$

For the proof of Theorem 2.3.1, we will be making use of the *Riesz-Dunford functional calculus* in the setting of the annulus A_r. Instead of employing the standard Cauchy integral formula, we will adopt the equivalent Laurent series definition. Let $T \in \mathcal{B}(H)$ and suppose that the spectrum $\sigma(T)$ of T is contained in A_r. If $f = \sum_{n\in\mathbb{Z}} a_n z^n$ is any function holomorphic on A_r, then $f(T) \in \mathcal{B}(H)$ is defined as

$$f(T) = \sum_{n\in\mathbb{Z}} a_n T^n.$$

Observe that since $\sigma(T) \subseteq A_r$, the convergence of the above Laurent series is guaranteed.

We now set up the *hereditary functional calculus* on A_r. We say that h is a hereditary function on A_r if h is a mapping from $A_r \times A_r$ to $\mathbb{C}$ and has the property that

$$(\lambda, \mu) \mapsto h(\lambda, \bar{\mu}) \in \mathbb{C}$$

is a holomorphic function on $A_r \times A_r$. The set $\mathrm{Her}(A_r)$ of hereditary functions on A_r forms a complete metrizable locally convex topological vector space when equipped with the topology of uniform convergence on compact subsets of $A_r \times A_r$.

If $T \in \mathcal{B}(H)$ with $\sigma(T) \subseteq A_r$ and h is a hereditary function on A_r, then we may define $h(T) \in \mathcal{B}(H)$ by the following procedure. Expand h into a double Laurent series

$$h(\lambda, \mu) = \sum_{m,n\in\mathbb{Z}} c_{mn} \lambda^m \bar{\mu}^n \quad \text{for all } \lambda, \mu \in A_r,$$

and then define $h(T)$ by substituting T for λ and T^* for $\bar{\mu}$:

$$h(T) = \sum_{m,n\in\mathbb{Z}} c_{mn} T^m (T^*)^n.$$

There is a natural involution $h \mapsto h^*$ on $\mathrm{Her}(A_r)$, defined by

$$h^*(\lambda, \mu) = \overline{h(\mu, \lambda)}, \quad \text{for all } \lambda, \mu \in A_r.$$

It is easy to see that $h^*(T) = h(T^*)$.

Finally, we record the following fundamental lemma (the counterpart of Theorem 2.88 in [14] for the annulus) which is essentially the holomorphic version of Moore's theorem on the factorisation of positive semi-definite kernels. It will allow us to decompose positive semi-definite hereditary functions as sums of dyads.

Lemma 2.3.3. *Suppose U is a positive semi-definite hereditary function on A_r, then there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions holomorphic on A_r such that*

$$U(\lambda, \mu) = \sum_{n=1}^{\infty} f_n(\lambda)\overline{f_n(\mu)}, \ \ \forall \lambda, \mu \in A_r,$$

the series converging uniformly on compact subsets of $A_r \times A_r$.

2.3.2 The Space $\mathcal{H}^2(A_r)$

Fix $r < 1$ and let $A_r = \{r < |z| < 1\}$. Denote by $H^2(A_r)$ the classical Hardy space on an annulus. This is the Hilbert function space

$$H^2(A_r) = \{f \in \mathrm{Hol}(A_r) : \sup_{r<\rho<1} \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{it})|^2 \, dt < \infty\}$$

equipped with the norm (for $f = \sum_{n\in\mathbb{Z}} a_n z^n$)

$$||f||^2_{H^2(A_r)} = \sum_{-\infty}^{\infty} (r^{2n} + 1)|a_n|^2.$$

An important observation is that the multiplier algebra $\mathrm{Mult}(H^2(A_r))$ is isometrically isomorphic to the algebra $H^\infty(A_r)$ of bounded holomorphic functions on A_r.

Now, we define the space $\mathcal{H}^2(A_r)$ by equipping $H^2(A_r)$ with a different norm:

$$||f||^2_{\mathcal{H}^2(A_r)} = \sum_{-\infty}^{-1} r^{2n}|a_n|^2 + \sum_{0}^{\infty} |a_n|^2.$$

These two norms are equivalent, as

$$||f||^2_{\mathcal{H}^2(A_r)} \leq ||f||^2_{H^2(A_r)} = \sum_{-\infty}^{\infty}(r^{2n}+1)|a_n|^2$$

$$\leq 2\sum_{-\infty}^{-1} r^{2n}|a_n|^2 + 2\sum_{0}^{\infty} |a_n|^2 = 2||f||^2_{\mathcal{H}^2(A_r)}.$$

Hence,

$$||f||_{\mathcal{H}^2(A_r)} \leq ||f||_{H^2(A_r)} \leq \sqrt{2}||f||_{\mathcal{H}^2(A_r)}, \tag{2.7}$$

for every $f \in \mathcal{H}^2(A_r)$. Notice also that the set

$$\left\{\frac{z^n}{r^n}\right\}_{n\leq -1} \cup \left\{z^n\right\}_{n\geq 0}$$

is an orthonormal basis for $\mathcal{H}^2(A_r)$. Applying Parseval's identity, we can then calculate the kernel function for $\mathcal{H}^2(A_r)$ as follows

$$k_{A_r}(\lambda,\mu) = \langle k_{A_r}(\cdot,\mu), k_{A_r}(\cdot,\lambda)\rangle$$

$$= \sum_{-\infty}^{-1}\langle k_{A_r}(\cdot,\mu), z^n/r^n\rangle\langle z^n/r^n, k_{A_r}(\cdot,\lambda)\rangle + \sum_{0}^{\infty}\langle k_{A_r}(\cdot,\mu), z^n\rangle\langle z^n, k_{A_r}(\cdot,\lambda)\rangle$$

$$= \sum_{-\infty}^{-1} \frac{\lambda^n}{r^n}\frac{\bar{\mu}^n}{r^n} + \sum_{0}^{\infty} \lambda^n\bar{\mu}^n$$

$$= (1-r^2)\frac{1}{\left(1-\frac{r^2}{\lambda\bar{\mu}}\right)\left(1-\lambda\bar{\mu}\right)}, \quad \forall \lambda,\mu \in A_r.$$

There are a few interesting observations we can make here. Recall that $a_2(\lambda,\mu)$ (where $\lambda = (\lambda_1,\lambda_2)$ and $\mu = (\mu_1,\mu_2)$) denotes the reproducing kernel of the Drury-Arveson space H^2_2 defined on the

2-dimensional complex unit ball $\mathbb{B}_2 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 < 1\}$. Denote also by $s_2(\lambda, \mu)$ the kernel of the classical Hardy space $H^2(\mathbb{D}^2)$ defined on the bidisk $\mathbb{D}^2 = \{(z_1, z_2) : |z_1|, |z_2| < 1\}$,

$$s_2(\lambda, \mu) = \frac{1}{(1-\lambda_1\overline{\mu_1})(1-\lambda_2\overline{\mu_2})}.$$

A short calculation then leads us to the equalities

$$k_{A_r}(\lambda, \mu) =$$

$$= \Big(\frac{1-r^2}{1+r^2}\Big) a_2\Big(\Big(\frac{\lambda}{\sqrt{r^2+1}}, \frac{r}{\sqrt{r^2+1}}\frac{1}{\lambda}\Big), \Big(\frac{\mu}{\sqrt{r^2+1}}, \frac{r}{\sqrt{r^2+1}}\frac{1}{\mu}\Big)\Big) \tag{2.8}$$

$$= (1-r^2) s_2\Big(\Big(\lambda, \frac{r}{\lambda}\Big), \Big(\mu, \frac{r}{\mu}\Big)\Big), \tag{2.9}$$

for every λ and μ in A_r. We can now apply the pull-back theorem for reproducing kernels (see e.g. Theorem 5.7 in [113]) to obtain two new descriptions of the norm of $\mathcal{H}^2(A_r)$. By (2.8), we obtain that for every $f \in \mathcal{H}^2(A_r)$:

$$||f||_{\mathcal{H}^2(A_r)} = \sqrt{\frac{1+r^2}{1-r^2}} \min\{||g||_{H^2_2} : g \in H^2_2 \text{ and } g\Big(\frac{z}{\sqrt{1+r^2}}, \frac{r}{\sqrt{1+r^2}}\frac{1}{z}\Big) = f(z), \forall z \in A_r\},$$

while (2.9) gives us:

$$||f||_{\mathcal{H}^2(A_r)} = \sqrt{\frac{1}{1-r^2}} \min\{||g||_{H^2(\mathbb{D}^2)} : g \in H^2(\mathbb{D}^2) \text{ and } g\Big(z, \frac{r}{z}\Big) = f(z), \forall z \in A_r\}.$$

We will now use (2.7) to compare the norm of the multipliers of $\mathcal{H}^2(A_r)$ with the supremum norm on $H^\infty(A_r)$.

Proposition 2.3.4. *For every $\phi \in H^\infty(A_r)$,*

$$||\phi||_\infty \le ||\phi||_{\text{Mult}(\mathcal{H}^2(A_r))} \le \sqrt{2}||\phi||_\infty.$$

Moreover, the constant $\sqrt{2}$ is the best possible.

Proof. Since $\mathcal{H}^2(A_r)$ is a reproducing kernel Hilbert space, the inequality $||\phi||_\infty \le ||\phi||_{\text{Mult}(\mathcal{H}^2(A_r))}$ is automatic, for all ϕ in $\text{Mult}(\mathcal{H}^2(A_r))$.

Now, fix $\phi \in H^\infty(A_r)$. (2.7) allows us to write

$$||\phi f||_{\mathcal{H}^2(A_r)} \le ||\phi f||_{H^2(A_r)} \le ||\phi||_\infty ||f||_{H^2(A_r)} \le \sqrt{2}||\phi||_\infty ||f||_{\mathcal{H}^2(A_r)},$$

for every $f \in \mathcal{H}^2(A_r)$. Hence, $||\phi||_{\text{Mult}(\mathcal{H}^2(A_r))} \leq \sqrt{2}||\phi||_\infty$, as desired.

Now, to prove that the constant $\sqrt{2}$ is the best possible, we define

$$g_n(z) = \frac{r^n}{z^n} + z^n, \ \ \forall z \in A_r, \ \ \forall n \geq 1.$$

Then, for every $z \in A_r$:

$$|g_n(z)| \leq \frac{r^n}{|z|^n} + |z|^n \leq 1 + r^n.$$

Hence, $||g_n||_\infty = 1 + r^n$ for all $n \geq 1$. Notice also that

$$\begin{aligned}\frac{||g_n||_{\text{Mult}(\mathcal{H}^2(A_r))}}{||g_n||_\infty} &= \frac{||g_n||_{\text{Mult}(\mathcal{H}^2(A_r))}}{1+r^n} \\ &\geq \frac{||g_n \cdot 1||_{\mathcal{H}^2(A_r)}}{1+r^n} \\ &= \frac{\sqrt{2}}{1+r^n} \xrightarrow{n\to\infty} \sqrt{2}.\end{aligned}$$

This concludes our proof. □

We return to equality (2.8). This can be written equivalently as

$$k_{A_r}(\lambda,\mu) = \frac{1-r^2}{1+r^2}\frac{1}{1-\langle b_r(\lambda), b_r(\mu)\rangle_{\mathbb{C}^2}}, \quad \forall \lambda,\mu \in A_r,$$

where

$$b_r(\lambda) := \left(\frac{\lambda}{\sqrt{r^2+1}}, \frac{r}{\sqrt{r^2+1}}\frac{1}{\lambda}\right)$$

and $|b_r(\lambda)| < 1$ in A_r.

Hence, k_{A_r} is a complete Pick kernel. This allows us to draw an interesting connection between the supremum norm of $H^\infty(A_r)$ and the multiplier norm of $\text{Mult}(H^2_2)$, formulated as an extension result of holomorphic functions off a subvariety of $\mathbb{B}_2$. It is the content of Theorem 2.3.2, which we now prove.

Proof of Theorem 2.3.2. Since $\frac{1+r^2}{1-r^2}k_{A_r}$ is a (normalized) complete Pick kernel, we can use (2.6) to deduce that for every $\phi \in \text{Mult}(\mathcal{H}^2(A_r)) = H^\infty(A_r)$,

$$||\phi||_{\text{Mult}(\mathcal{H}^2(A_r))} = min\{||\psi||_{\text{Mult}(H^2_2)} : \psi \in \text{Mult}(H^2_2) \text{ and } \psi(b_r(z)) = \phi(z), \ \forall z \in A_r\}.$$

Applying Proposition 2.3.4 then concludes the proof. □

Remark 2.3.5. A rescaled version of the space $\mathcal{H}^2(A_r)$ was also considered by Arcozzi, Rochberg, Sawyer in [20]. There, the authors proved the much more general fact that every Hardy space over a finitely connected domain with smooth boundary curves admits an equivalent norm originating from a complete Pick kernel.

2.3.3 Proof of Theorem 2.3.1

Proof of Theorem 2.3.1. Since we have already established Proposition 2.3.4, it remains to show that

$$\sup_{T\in\mathcal{F}_r} ||\phi(T)|| = ||\phi||_{\mathrm{Mult}(\mathcal{H}^2(A_r))},$$

for every $\phi \in H^\infty(A_r)$.

First, suppose $||\phi||_{\mathrm{Mult}(\mathcal{H}^2(A_r))} \leq 1$. We then obtain the existence of a positive-semidefinite kernel $U : A_r \times A_r \to \mathbb{C}$ such that

$$(1 - \phi(\lambda)\overline{\phi(\mu)})k_{A_r}(\lambda, \mu) = U(\lambda, \mu),$$

hence we have the model formula

$$1 - \phi(\lambda)\overline{\phi(\mu)} = \frac{U(\lambda, \mu)}{1 - r^2}(1 + r^2 - r^2/\lambda\bar{\mu} - \lambda\bar{\mu}).$$

Evidently, U is a positive semi-definite element of $\mathrm{Her}(A_r)$. Applying Lemma 2.3.3, we obtain the existence of a sequence $\{f_n\}_{n\in\mathbb{N}}$ of elements of $\mathrm{Hol}(A_r)$ such that

$$1 - \phi(\lambda)\overline{\phi(\mu)} = \frac{1}{1 - r^2}\sum_{n=1}^{\infty} f_n(\lambda)(1 + r^2 - r^2/\lambda\bar{\mu} - \lambda\bar{\mu})\overline{f_n(\mu)},$$

with uniform convergence on compact subsets of $A_r \times A_r$.

Now, let $T \in \mathcal{F}_r$. We can view each side of our previous equality as a hereditary function on A_r and substitute T into both sides using our hereditary functional calculus on A_r (since $\sigma(T) \subseteq A_r$). This results in the equality

$$1 - \phi(T)\phi(T)^* = \frac{1}{1 - r^2}\sum_{n=1}^{\infty} f_n(T)(1 + r^2 - r^2T^{-1}(T^{-1})^* - TT^*)f_n(T)^*. \tag{2.10}$$

However, observe that since $T \in \mathcal{F}_r$ we can write

$$1 + r^2 - r^2 T^{-1}(T^{-1})^* - TT^* \geq 0$$

$$\Rightarrow f_n(T)(1 + r^2 - r^2 T^{-1}(T^{-1})^* - TT^*)f_n(T)^* \geq 0$$

$$\Rightarrow \sum_{n=1}^{\infty} f_n(T)(1 + r^2 - r^2 T^{-1}(T^{-1})^* - TT^*)f_n(T)^* \geq 0.$$

By (2.10), we can then conlude that

$$1 - \phi(T)\phi(T)^* \geq 0$$

$$\Rightarrow ||\phi(T)|| \leq 1.$$

We have showed that

$$\sup_{T \in \mathcal{F}_r} ||\phi(T)|| \leq ||\phi||_{\text{Mult}(\mathcal{H}^2(A_r))}.$$

We now show the reverse inequality. First, we prove a lemma.

Lemma 2.3.6. *Every* $T \in \mathcal{B}(H)$ *such that* $r^2 T^{-1}(T^{-1})^* + TT^* \leq r^2 + 1$ *satisfies*

$$r^2 \leq TT^* \leq 1.$$

Thus, we also have that $\sigma(T) \subset \overline{A_r}$ *for every such operator.*

Proof of Lemma 2.3.6. Suppose instead that $||T^*|| = ||T|| = 1 + \delta > 1$. Thus, for any $\epsilon \in (0, \delta)$ there exists $y \in H$ with $||y|| = 1$ such that $||T^* y|| > 1 + \delta - \epsilon$. Since $||T^* x|| \leq (1 + \delta)||x||$, we also obtain $\frac{1}{1+\delta}||x|| \leq ||(T^{-1})^* x||$, for every $x \in H$. We can now write

$$\frac{r^2}{(1+\delta)^2} + (1 + \delta - \epsilon)^2 < r^2||(T^{-1})^* y||^2 + ||T^* y||^2$$

$$= \langle (r^2 T^{-1}(T^{-1})^* + TT^*)y, y \rangle \leq r^2 + 1.$$

Letting $\epsilon \to 0$, we obtain

$$\frac{r^2}{(1+\delta)^2} + (1+\delta)^2 \leq r^2 + 1,$$

a contradiction. Hence, $||T|| \leq 1$ and an analogous argument shows that $||rT^{-1}|| \leq 1$ as well. □

Our next step will be to prove the following special case.

Lemma 2.3.7. *Let ϕ be holomorphic in a neighborhood of $\overline{A_r}$ with the property that $||\phi(T)|| \leq 1$ for all $T \in \mathcal{B}(H)$ such that $r^2T^{-1}(T^{-1})^* + TT^* \leq r^2 + 1$. Then, $||\phi||_{\text{Mult}(\mathcal{H}^2(A_r))} \leq 1$.*

Proof of Lemma 2.3.7. If $T \in \mathcal{B}(H)$ is such that $r^2T^{-1}(T^{-1})^* + TT^* \leq r^2 + 1$, then by Lemma 2.3.6 T also satisfies $r^2 \leq TT^* \leq 1$. In particular, $\sigma(T)$ has to lie in $\overline{A_r}$ and so the operator $\phi(T)$ is indeed well-defined whenever $\phi \in \text{Hol}(\overline{A_r})$.

Now, suppose that ϕ satisfies the given hypotheses and consider the bilateral shift operator $(Sf)(z) = zf(z)$ defined on $\mathcal{H}^2(A_r)$. A standard computation shows that $S^*k_{A_r}(\cdot, \lambda) = \bar{\lambda}k_{A_r}(\cdot, \lambda)$, for every $\lambda \in A_r$. Notice also that for every λ, μ in A_r,

$$\langle (r^2 + 1 - r^2S^{-1}(S^{-1})^* - SS^*)k_{A_r}(\cdot, \mu), k_{A_r}(\cdot, \lambda)\rangle$$

$$= (r^2+1)\langle k_{A_r}(\cdot,\mu), k_{A_r}(\cdot,\lambda)\rangle - r^2\langle (S^{-1})^*k_{A_r}(\cdot,\mu), (S^{-1})^*k_{A_r}(\cdot,\lambda)\rangle - \langle S^*k_{A_r}(\cdot,\mu), S^*k_{A_r}(\cdot,\lambda)\rangle$$

$$= (r^2+1)k_{A_r}(\lambda,\mu) - \frac{r^2}{\lambda\bar{\mu}}k_{A_r}(\lambda,\mu) - \lambda\bar{\mu}k_{A_r}(\lambda,\mu)$$

$$= 1 - r^2,$$

a (trivial) positive semi-definite kernel on $A_r \times A_r$.. Since linear combinations of kernel functions are dense in $\mathcal{H}^2(A_r)$, our previous equality implies that

$$r^2 + 1 - r^2S^{-1}(S^{-1})^* - SS^* \geq 0.$$

But our hypotheses on ϕ then allow us to deduce that

$$||\phi||_{\text{Mult}(\mathcal{H}^2(A_r))} = ||\phi(S)|| \leq \sup_{T \in \mathcal{F}_r} ||\phi(T)|| \leq 1,$$

which concludes the proof of the lemma. □

To complete our main proof, we will apply an approximation argument to extend the previous special case to every multiplier of $\mathcal{H}^2(A_r)$.

Suppose $\phi \in \text{Hol}(A_r)$ is such that $\sup_{T\in\mathcal{F}_r} ||\phi(T)|| \leq 1$. For $n > 2/(1-r)$, define $A_{r,n} :=$

$\{r + 1/n < |z| < 1 - 1/n\}$. We will be needing the following lemma, the proof of which is just a simple calculation.

Lemma 2.3.8. *The following two inequalities hold for all $n > 2/(1-r)$:*

$$\frac{\left(r+\frac{1}{n}\right)^2}{1+\left(\frac{r+\frac{1}{n}}{1-\frac{1}{n}}\right)^2} \geq \frac{r^2}{1+r^2},$$

$$\frac{1}{\left(1-\frac{1}{n}\right)^2+\left(r+\frac{1}{n}\right)^2} \geq \frac{1}{1+r^2}.$$

Now, define the classes of operators

$$\mathcal{F}_{r,n} = \{T \in \mathcal{B}(H) : [r+(1/n)]^2 T^{-1}(T^{-1})^* + [1/(1-n)]^2 TT^* \leq 1 + [(r+(1/n))/(1-(1/n))]^2\}$$

and also the family of kernels

$$k_{r,n}(\lambda,\mu) := \frac{1}{\left(1-\frac{(r+1/n)^2}{\bar{\mu}\lambda}\right)\left(1-\frac{\bar{\mu}\lambda}{(1-1/n)^2}\right)}$$

$$= \frac{1}{\left(1+\frac{(r+1/n)^2}{(1-1/n)^2}-\frac{(r+1/n)^2}{\bar{\mu}\lambda}-\frac{\bar{\mu}\lambda}{(1-1/n)^2}\right)}.$$

Each $k_{r,n}$ is a positive-semidefinite kernel on $A_{r,n} \times A_{r,n}$, simply a rescaled version of k_{A_r}. Denote by $\mathcal{H}^2(A_{r,n})$ the corresponding Hilbert space of holomorphic functions on $A_{r,n}$.

Now, let $T \in \mathcal{F}_{r,n}$. By the appropriately rescaled version of Lemma 2.3.6, we obtain $\sigma(T) \subseteq \overline{A_{r,n}} \subseteq A_r$. Observe also that

$$[r+(1/n)]^2 T^{-1}(T^{-1})^* + [1/(1-n)]^2 TT^* \leq 1 + [(r+(1/n))/(1-(1/n))]^2$$

$$\iff \frac{\left(r+\frac{1}{n}\right)^2}{1+\left(\frac{r+\frac{1}{n}}{1-\frac{1}{n}}\right)^2}||(T^{-1})^*x||^2 + \frac{1}{\left(1-\frac{1}{n}\right)^2+\left(r+\frac{1}{n}\right)^2}||T^*x||^2 \leq ||x||^2,$$

for every $x \in H$. Using our inequalities from Lemma 2.3.8, we obtain

$$\frac{r^2}{r^2+1}||(T^{-1})^*x||^2 + \frac{1}{r^2+1}||T^*x||^2$$
$$\leq \frac{\left(r+\frac{1}{n}\right)^2}{1+\left(\frac{r+\frac{1}{n}}{1-\frac{1}{n}}\right)^2}||(T^{-1})^*x||^2 + \frac{1}{\left(1-\frac{1}{n}\right)^2+\left(r+\frac{1}{n}\right)^2}||T^*x||^2$$
$$\leq ||x||^2, \quad \forall x \in H.$$

Thus, $r^2T^{-1}(T^{-1})^* + TT^* \leq r^2+1$, which means that $T \in \mathcal{F}_r$. By our assumptions on ϕ, we then obtain that $||\phi(T)|| \leq 1$.

To sum up, we have proved (for every $n > 2/(1-r)$) that ϕ, a function holomorphic on a neighborhood of $\overline{A_{r,n}}$, satisfies $||\phi(T)|| \leq 1$ for all $T \in \mathcal{B}(H)$ such that

$$[r+(1/n)]^2(T^{-1})^*T^{-1} + [1/(1-n)]^2T^*T \leq 1 + [(r+(1/n))/(1-(1/n))]^2.$$

The appropriately rescaled version of Lemma 2.3.7 now allows us to conclude that $||\phi||_{\mathrm{Mult}(\mathcal{H}^2(A_{r,n}))} \leq 1$ and so there exists a positive semi-definite hereditary function $h_n : A_{r,n} \times A_{r,n} \to \mathbb{C}$ such that

$$1-\phi(\lambda)\overline{\phi(\mu)} = (1/k_{r,n}(\lambda,\mu))h_n(\lambda,\mu)$$
$$= \left(1+\frac{(r+1/n)^2}{(1-1/n)^2} - \frac{\left(r+1/n\right)^2}{\bar{\mu}\lambda} - \frac{\bar{\mu}\lambda}{\left(1-1/n\right)^2}\right)h_n(\lambda,\mu), \tag{2.11}$$

for all $\lambda, \mu \in A_{r,n}$ and for every $n > 2/(1-r)$.

Let $K \subseteq A_r \times A_r$ be compact and fix $N > 2/(1-r)$ large enough so that $K \subseteq A_{r,n} \times A_{r,n}$ for every $n \geq N$. Then, for every such n and for every $\lambda \in K$ we have

$$|h_n(\lambda,\lambda)| = h_n(\lambda,\lambda) = (1-|\phi(\lambda)|^2)k_{r,n}(\lambda,\lambda)$$

$$\leq \sup_{z\in K}\left[(1-|\phi(z)|^2)\frac{1}{\big(1-(r+1/n)^2/|z|^2\big)\big(1-|z|^2/(1-1/n)^2\big)}\right]$$

$$\leq \sup_{z\in K}\left[(1-|\phi(z)|^2)\frac{1}{\big(1-(r+1/N)^2/|z|^2\big)\big(1-|z|^2/(1-1/N)^2\big)}\right]$$

$$= M < \infty, \tag{2.12}$$

where M is independent of n. Notice now that by Lemma 2.3.3 there exists (for every $n \in \mathbb{N}$) a function $u_n : A_{r,n} \to l^2$ with the property that

$$h_n(\lambda,\mu) = \langle u_n(\lambda), u_n(\mu)\rangle_{l^2}, \quad \text{in } A_{r,n}\times A_{r,n}.$$

Hence, using the Cauchy-Schwarz inequality and the bound (2.12) we can write

$$|h_n(\lambda,\mu)|^2 \leq |h_n(\lambda,\lambda)||h_n(\mu,\mu)| \leq M^2,$$

for every $\lambda,\mu \in K$ and for every $n \geq N$. In other words, the sequence of holomorphic functions $\{(\lambda,\mu) \mapsto h_n(\lambda,\bar{\mu})\}_{n\geq N}$ is uniformly bounded on K. By Montel's theorem and the completeness of Her(A_r), we can then deduce the existence of an element $h \in$ Her(A_r) with the property that $h_{n_k} \to h$ uniformly on compact subsets of $A_r \times A_r$ for some subsequence $\{n_k\}$. Since every h_{n_k} is positive semi-definite, the same must be true for h as well. Now, equality (2.11) combined with the convergence $h_{n_k} \to h$ gives

$$1-\phi(\lambda)\overline{\phi(\mu)} = (1+r^2-r^2/\lambda\bar{\mu}-\lambda\bar{\mu})h(\lambda,\mu) \quad \text{on } A_r\times A_r,$$

and so

$$(1-\phi(\lambda)\overline{\phi(\mu)})k_{A_r}(\lambda,\mu) \geq 0 \quad \text{on } A_r\times A_r.$$

Thus, $||\phi||_{\mathrm{Mult}(\mathcal{H}^2(A_r))} \leq 1$ and our proof is complete. □

Remark 2.3.9. The class $\mathcal{F}_r$ was also considered in work of Bello, Yakubovich[30], where the authors obtained, with an alternative approach, that A_r is a complete $\sqrt{2}$-spectral set for every $T \in \mathcal{F}_r$.

Remark 2.3.10. By Lemma 2.3.6, every $T \in \mathcal{B}(H)$ such that $r^2T^{-1}(T^{-1})^* + TT^* \leq r^2 + 1$ also satisfies $r^2 \leq TT^* \leq 1$. The converse assertion is not true, even if we restrict ourselves to 2×2 matrices. Indeed, define $A \in \mathcal{B}(\mathbb{C}^2)$ by

$$A = \begin{bmatrix} \sqrt{r} & 1-r \\ 0 & \sqrt{r} \end{bmatrix}.$$

A short computations shows that $||A|| = ||rA^{-1}|| = 1$. However, notice that

$$\left\langle \left(r^2 + 1 - r^2A^{-1}(A^{-1})^* - AA^* \right) \begin{bmatrix} 1 \\ \sqrt{r} \end{bmatrix}, \begin{bmatrix} 1 \\ \sqrt{r} \end{bmatrix} \right\rangle$$

$$= r^2(r + 1 - 1/r - (2 - 1/r)^2),$$

which is negative for all $r \in (0, 1)$.

2.4 A Counterexample to a Question of Bello-Yakubovich

Define

$$\mathrm{Sp}(A_R) := \{T \in \mathcal{B}(H) : T \text{ has } \overline{A_R} \text{ as a spectral set}\},$$

$$\mathcal{F}_R := \{T \in \mathcal{B}(H) : TT^* + T^{-1}(T^{-1})^* \leq R^2 + R^{-2}\}.$$

In [30], Bello and Yakubovich showed the inclusion

$$\mathrm{Sp}(A_R) \subseteq \mathcal{F}_R \cap \mathcal{F}_R^* ,$$

where $\mathcal{F}_R^* := \{T \in \mathcal{B}(H) : T^* \in \mathcal{F}_R\}$. They then asked whether:

$$\mathrm{Sp}(A_R) = \mathcal{F}_R \cap \mathcal{F}_R^* \ ?$$

We show that the inclusion is strict, for every $R > 1$.

Theorem 2.4.1. *For $R > 1$, $Sp(A_R) \subsetneq \mathcal{F}_R \cap \mathcal{F}_R^*$.*

Proof. Put $g(z) = \frac{1}{R}(\frac{1}{z} + z)$, then $||g||_\infty = 1 + \frac{1}{R^2}$. Notice that for every $R > 1$:

$$\frac{2}{R^2 + R^{-2}} < \left(\frac{R + R^{-1}}{2}\right)^2 < \frac{R^2 + R^{-2}}{2}. \tag{*}$$

Proof of (*). Put $x = R^2 + R^{-2} > 2$, since $R > 1$. The above set of inequalities then becomes

$$\frac{2}{x} < \frac{x+2}{4} < \frac{x}{2},$$

which holds for every $x > 2$. □

We will now introduce a slight variation of the weighted shift presented in Section 2.2. First, define the sequence $\{\beta(k)\}_{k\in\mathbb{Z}}$ of positive numbers (weights) as follows:

$$\beta(2l + q) = P(R)^q, \quad \forall q \in \{0, 1\}, \forall l \in \mathbb{Z},$$

where P is any function of $R > 1$ with the property that $\left(\frac{R+R^{-1}}{2}\right)^2 < (P(R))^2 < \frac{R^2+R^{-2}}{2}$. Consider now the space of sequences $f = \{\hat{f}(k)\}_{k\in\mathbb{Z}}$ such that

$$||f||_\beta^2 := \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2[\beta(k)]^2 < \infty.$$

We shall use the notation $f(z) = \sum_{k\in\mathbb{Z}} \hat{f}(k)z^k$ (formal Laurent series), whether or not the series converges for any (complex) values of z. Our weighted sequence space will be denoted by

$$L^2(\beta) := \{f = \{\hat{f}(k)\}_{k\in\mathbb{Z}} : ||f||_\beta^2 < \infty\}.$$

This is a Hilbert space with the inner product

$$\langle f, g\rangle_\beta := \sum_{k\in\mathbb{Z}} \hat{f}(k)\overline{\hat{g}(k)}[\beta(k)]^2.$$

Consider also the linear transformation S of multiplication by z on $L^2(\beta)$:

$$(Sf)(z) = \sum_{k\in\mathbb{Z}} \hat{f}(n)z^{n+1}.$$

S is a weighted bilateral shift. In particular, putting $\{e_n\}_n := \{z^n/||z^n||_\beta\}_n$, we have that $Se_n = w_n e_{n+1}$, where

$$w_{2l} = P(R);$$

$$w_{2l+1} = \frac{1}{P(R)}.$$

It is true that $S \in \mathcal{F}_R \cap \mathcal{F}_{\mathcal{R}}^*$ if and only if:

$$\begin{cases} |w_k|^2 + |\frac{1}{w_{k-1}}|^2 \le R^2 + \frac{1}{R^2}, \ \forall k \in \mathbb{Z}; \\ |w_k|^2 + |\frac{1}{w_{k+1}}|^2 \le R^2 + \frac{1}{R^2}, \ \forall k \in \mathbb{Z}. \end{cases}$$

These are equivalent to the two inequalities

$$\begin{cases} 2(P(R))^2 \le R^2 + \frac{1}{R^2}, \\ \frac{2}{(P(R))^2} \le R^2 + \frac{1}{R^2}, \end{cases}$$

both of which hold because of (*) and the definition of P. Hence, $S \in \mathcal{F}_R \cap \mathcal{F}_{\mathcal{R}}^*$ for every $R > 1$.

Now, let $m \ge 3$ and define $h = \{\hat{h}(k)\}_{k\in\mathbb{Z}} \in L^2(\beta)$ by putting:

$$\hat{h}(2l) = \frac{1}{m}, \ \ \forall l \in \{0, 1, 2 \ldots, m^2\};$$

$$\hat{h}(k) = 0, \ \ \text{in all other cases.}$$

We calculate

$$||h||_\beta^2 = \sum_{l=0}^{m^2} \frac{1}{m^2}[\beta(2l)]^2 = \sum_{l=0}^{m^2} \frac{1}{m^2} \cdot 1^2 = \frac{m^2+1}{m^2},$$

hence

$$||h||_\beta = \frac{\sqrt{m^2+1}}{m}. \tag{2.13}$$

Also, put $f = (S^{-1} + S)h$ and notice that

$$\begin{aligned} ||(S^{-1} + S)h||_\beta^2 &= ||f||_\beta^2 \\ &\ge \sum_{l=1}^{m^2} |\hat{f}(2l-1)|^2[\beta(2l-1)]^2 \\ &= \sum_{l=1}^{m^2} \left(\frac{2}{m}\right)^2 (P(R))^2 \\ &= 4(P(R))^2. \end{aligned}$$

Thus,

$$||(S^{-1}+S)h||_\beta \geq 2P(R). \tag{2.14}$$

Using (2.13) and (2.14), we can now write

$$\frac{||g(S)||}{||g||_\infty}$$

$$= \frac{1}{R} \cdot \frac{||S^{-1}+S||}{1+R^{-2}}$$

$$\geq \frac{1}{R+R^{-1}} \cdot \frac{||(S^{-1}+S^{1})h||_\beta}{||h||_\beta}$$

$$\geq \frac{1}{R+R^{-1}} \cdot \frac{2P(R)}{\frac{\sqrt{m^2+1}}{m}}.$$

Letting $m \to \infty$ and using the definition of P, we obtain

$$\frac{||g(S)||}{||g||_\infty} \geq \frac{2P(R)}{R+R^{-1}} > 1.$$

Hence, for $R > 1$ we have $S \in (\mathcal{F}_R \cap \mathcal{F}_R^*) \setminus \mathrm{Sp}(A_R)$. □

2.5 Positivity Conditions on the Annulus via the Double-Layer Potential Kernel

We now revisit our discussion that took place in the end of subsection 2.1.3. Let $A_R = \{\frac{1}{R} < |z| < R\}$, with $R > 1$. In [63, Section 5] it was shown that the mapping

$$S_{R,0} : \mathcal{A}(A_R) \to \mathcal{B}(H)$$

$$f \mapsto \frac{1}{2}\left[f(T) + (C\overline{f})(T)^*\right],$$

where $(C\overline{f})(z) = \frac{1}{2\pi i}\int_{A_R} \frac{\overline{f(\zeta)}}{\zeta - z} d\zeta$, will always be contractive if $T \in \mathbb{QA}_R$. This observation, combined with the contractivity of $f \mapsto C\overline{f}$ over A_R and an abstract functional analysis lemma (see

[63, Theorem 2] or [121, Lemma 1.1]), suffices to establish that, given any $T \in \mathbb{QA}_R$, we must have

$$||f(T)|| \leq (1+\sqrt{2}) \sup_{z \in \overline{A_R}} |f(z)|, \quad \forall f \in \mathcal{A}(A_R),$$

i.e. $\overline{A_R}$ will be a $(1+\sqrt{2})$-spectral set for T whenever $T \in \mathbb{QA}_R$.

Now, observe that in the previous argument, the class $\mathbb{QA}_R$ enters the picture only through the contractivity of $S_{R,0}$. Thus, if one wants to gain a better understanding of K-spectral estimates over A_R, the following question emerges naturally: for which operators T will $S_{R,0}$ be contractive? This line of inquiry, together with the form of the mapping (2.1), is what motivates our definition of the operator class $\mathbb{DLA}_R(c)$ (where $c > -2$); an operator $T \in \mathcal{B}(H)$ with $\sigma(T) \subset A_R$ will belong to $\mathbb{DLA}_R(c)$ if and only if the mapping

$$S_{R,c} : \mathcal{A}(A_R) \to \mathcal{B}(H)$$

$$f = \sum_{n=-\infty}^{\infty} a_n z^n \mapsto \frac{1}{2+c}\left[f(T) + (C\overline{f})(T)^* + ca_0\right]$$

is contractive (see subsection 2.5.2 for the general definition and Theorem 2.5.17 for the equivalence between the two when $\sigma(T) \subset A_R$). Our goal will be to study the operator class $\mathbb{DLA}_R(c)$ and its completely contractive analogue $\mathbb{CDLA}_R(c)$ (see subsection 2.5.5). Note that the study of completely bounded maps and dilations in the setting of the double-layer potential kernel and Crouzeix's conjecture has already been successfully initiated in the papers [118] and [53, Section 6]. Also, we point out that, while the inclusion $\mathbb{CDLA}_R(c) \subseteq \mathbb{DLA}_R(c)$ follows immediately from the definitions, it is not known to us whether it is strict or not (see Question 2.5.25).

To state our first main result, a characterization of $\mathbb{CDLA}_R(c)$, we require the following generalization of the $\mathcal{C}_\rho$ classes, introduced by Langer [137, p. 53] (see also [135]).

Definition 2.5.1. Assume $A \in \mathcal{B}(H)$ is a bounded, positive operator that is also bounded below. The class $\mathcal{C}_A$ contains all operators $T \in \mathcal{B}(H)$ with the property that there exists a Hilbert space $K \supset H$ and a unitary $U \in \mathcal{B}(K)$ such that

$$A^{-1/2} T^n A^{-1/2} = P_H U^n|_H, \quad n = 1, 2, \dots .$$

Our characterization then proceeds as follows. Note that the inequality $T \geq 0$ implies that the Hilbert space operator T is positive, while $T > 0$ implies that it is strictly positive.

Theorem 2.5.2. *Let $T \in \mathcal{B}(H)$ and $c > -2$, with $\sigma(T) \subset A_R$. Then, $T \in \mathbb{CDLA}_R(c)$ if and only if $T/R \in \mathcal{C}_{2+c-A}$ and $T^{-1}/R \in \mathcal{C}_A$ for some $A \in \mathcal{B}(H)$ such that $A > 0$ and $2 + c - A > 0$.*

Dropping the assumption $\sigma(T) \subset A_R$ leads us to Theorem 2.5.22.

As remarked previously, the only way the class $\mathbb{QA}_R$ enters in the proof of the $K = 1 + \sqrt{2}$ spectral estimate in [63] is through the inclusion $\mathbb{QA}_R \subset \mathbb{CDLA}_R(0)$. This suggests that the spectral constant for $\mathbb{QA}_R$ may coincide with the one for $\mathbb{CDLA}_R(0)$. Utilizing the solution of an extremal problem over A_R due to McCullough and Shen [100], we are able to prove a partial result in the setting of 2×2 matrices that supports this idea.

Theorem 2.5.3. *Let $c \geq 0$ and assume $T \in \mathbb{DLA}_R(c)$ is a 2×2 matrix with a single eigenvalue. Then, $\overline{A_R}$ will be a $K(R)$-spectral set for T, where*

$$K(R) = 2 + c\frac{R^2 - 1}{R^2 + 1} \leq 2 + c.$$

Note that one can also take advantage of the machinery established in [43] and [63] to prove general K-spectral estimates for $\mathbb{DLA}_R(c)$ and $\mathbb{CDLA}_R(c)$; see Theorem 2.5.28 (see also Remark 2.5.30). In fact, our approach yields sharper estimates for certain operator classes; see Remark 2.5.29.

The rest of Section 2.5 is organized as follows: subsection 2.5.1 contains a few preliminary lemmata on the $\mathcal{C}_A$ classes. In subsections 2.5.2, 2.5.3.2.5.4, we explore basic properties of $\mathbb{DLA}_R(c)$ and demonstrate its connection with the double-layer potential. In subsection 2.5.5, we characterize $\mathbb{CDLA}_R(c)$ through Theorems 2.5.2 and 2.5.22. Finally, subsection 2.5.6 contains the proofs of Theorems 2.5.3 and 2.5.28.

2.5.1 Preliminaries

We first record an alternate characterization of $\mathcal{C}_A$ that is usually easier to work with. We have included proofs for the convenience of the reader.

Lemma 2.5.4. *Assume $A \in \mathcal{B}(H)$ is a bounded, strictly positive operator and let $T \in \mathcal{B}(H)$. Then, $T \in \mathcal{C}_A$ if and only if $\sigma(T) \subset \overline{\mathbb{D}}$ and*

$$2\Re(1 - zT)^{-1} + A - 2 \geq 0, \quad \forall z \in \mathbb{D}. \tag{2.15}$$

Proof. Assume first that $T \in \mathcal{C}_A$. Since $\mathcal{C}_A \subset \mathcal{C}_{||A||}$, we must have $\sigma(T) \subset \overline{\mathbb{D}}$ (see e.g. [137, p. 43]). Also, in view of [137, p. 53], $T \in \mathcal{C}_A$ is equivalent to

$$\langle Ah, h\rangle - 2\Re\langle z(A - I)Th, h\rangle + |z|^2\langle (A - 2I)Th, Th\rangle \geq 0,$$

for all $h \in H, |z| \leq 1$. This inequality can be rewritten as

$$\langle A(I - zT)h, (I - zT)h\rangle - 2\langle (I - zT)h, (I - zT)h\rangle + 2\Re\langle (I - zT)h, h\rangle \geq 0$$

for all $h \in H, |z| \leq 1$. Setting $h = (I - zT)^{-1}h$, we obtain

$$\langle (A - 2 + (I - zT)^{-1} + (1 - \bar{z}T^*)^{-1})h, h\rangle \geq 0$$

for all $h \in H, |z| \leq 1$, which is the desired conclusion.

For the converse, simply roll back the previous steps. □

Note that one could define $\mathcal{C}_A$ more generally, for A bounded and self-adjoint, through (2.15). Using this definition, it is easy to see that $\mathcal{C}_A$ is non-empty if and only if $A \geq 0$ (in which case it will contain the zero operator).

Lemma 2.5.5. *Assume $A \in \mathcal{B}(H)$ is a bounded, strictly positive operator and let $T \in \mathcal{B}(H)$ be such that $\sigma(T) \subset \mathbb{D}$. Then, $T \in \mathcal{C}_A$ if and only if*

$$(1 - e^{i\theta}T)^{-1} + (1 - e^{-i\theta}T^*)^{-1} + A - 2 \geq 0, \quad \forall \theta \in [0, 2\pi). \tag{2.16}$$

Proof. Fix an arbitrary $h \in H$ and define

$$\Phi_h : \mathbb{D} \to \mathbb{C}$$

$$z \mapsto \langle (2\Re(1 - zT)^{-1} + A - 2)h, h \rangle.$$

Since $\sigma(T) \subset \mathbb{D}$, Φ_h will be a harmonic function on $\mathbb{D}$ that extends continuously to $\overline{\mathbb{D}}$. By the minimum principle for harmonic functions, we then obtain that $\Phi_h(z) \geq 0$, for all $z \in \mathbb{D}$, if and only if

$$\Phi_h(e^{i\theta}) = \langle (2\Re(1 - e^{i\theta}T)^{-1} + A - 2)h, h \rangle \geq 0, \quad \forall \theta \in [0.2\pi).$$

Since h was arbitrary, we are done. □

Next, we record a few basic facts concerning completely bounded maps. Let $A \subset \mathcal{B}(H)$ denote an *operator algebra*, i.e. a unital subalgebra of the C^*-algebra of bounded linear operators on some Hilbert space H. Given a natural number $n \geq 1$, we denote by $M_n(A)$ the algebra of $n \times n$ matrices with entries from A, which we view as a subalgebra of bounded linear operators acting on $H^{(n)} = H \oplus H \oplus \cdots \oplus H$. In particular, $M_n(A)$ is endowed with a norm under this identification. Given a map $\Phi : A \to \mathcal{B}(H)$, for each $n \geq 1$, we may define the coordinate-wise map $\Phi^{(n)} : M_n(A) \to \mathcal{B}(K^{(n)})$ as

$$\Phi^{(n)}([a_{ij}]) = [\Phi(a_{ij})], \quad [a_{ij}] \in M_n(A).$$

If Φ is linear (or anti-linear), we say that Φ is completely bounded if the quantity

$$||\Phi||_{cb} = \sup_n ||\Phi^{(n)}||$$

is finite. We say that Φ is completely contractive if $||\Phi||_{cb} \leq 1$. Furthermore, Φ will be called positive if it maps positive elements of A to positive operators in $\mathcal{B}(K)$. Φ will be said to be completely positive if $\Phi^{(n)}$ is positive for every $n \geq 1$. For more details on these concepts, see [111].

2.5.2 $\mathcal{C}_{s,t}(R)$

Before we dive into the study of $\mathbb{DLA}_R(c)$, we will establish some basic properties of $\mathcal{C}_{s,t}(R)$. These are operator classes that generate special examples of $\mathbb{DLA}_R(c)$ operators.

Definition 2.5.6. Let $R > 1$ and $s, t > 0$. Define

$$\mathcal{C}_{s,t}(R) = \{T \in \mathcal{B}(H) \mid T/R \in \mathcal{C}_s \text{ and } T^{-1}/R \in \mathcal{C}_t\}.$$

First, we record $\mathcal{C}_{s,t}(R)$-membership criteria concerning normal matrices. These are easy consequences of known results about $\mathcal{C}_\rho$.

Lemma 2.5.7. *Let $s > 0$ and assume $T \in \mathcal{B}(H)$ is in $\mathcal{C}_s$.*

(i) If $s \geq 1$, then $\sigma(T) \subset \{|z| \leq 1\}$.

(ii) If $s \leq 1$, then $\sigma(T) \subset \{|z| \leq s/(2-s)\}$.

Proof. This is Lemma 5 in [32]. □

Proposition 2.5.8. *Fix $R > 1$ and assume $s, t > 0$. Also, let $N \in \mathcal{B}(H)$ be normal.*

(i) If $s, t \geq 1$, then $\mathcal{C}_{s,t}(R)$ will always be non-empty. Moreover, $N \in \mathcal{C}_{s,t}(R)$ if and only if $\frac{1}{R^2} \leq N^*N \leq R^2$.

(ii) If $s \geq 1$ and $t \leq 1$, then $\mathcal{C}_{s,t}(R)$ will be non-empty if and only if

$$\frac{2}{R^2+1} \leq t.$$

Moreover, $N \in \mathcal{C}_{s,t}(R)$ if and only if $\frac{(2-t)^2}{R^2t^2} \leq N^*N \leq R^2$.

(iii) If $s \leq 1$ and $t \geq 1$, then $\mathcal{C}_{s,t}(R)$ will be non-empty if and only if

$$\frac{2}{R^2+1} \leq s.$$

Moreover, $N \in \mathcal{C}_{s,t}(R)$ if and only if $\frac{1}{R^2} \leq N^*N \leq R^2\frac{s^2}{(2-s)^2}$.

(iv) If $s \leq 1$ and $t \leq 1$, then $\mathcal{C}_{s,t}(R)$ will be non-empty if and only if

$$\frac{1}{R}\frac{2-t}{t} \leq R\frac{s}{2-s}.$$

Moreover, $N \in \mathcal{C}_{s,t}(R)$ if and only if $\frac{1}{R^2}\frac{(2-t)^2}{t^2} \leq N^*N \leq R^2\frac{s^2}{(2-s)^2}$.

Proof. We only consider the case $s \geq 1$, $t \leq 1$ (the remaining three cases can be proved in essentially identical ways). The assertion about N is an immediate consequence of [32, Theorem 6]. Note also that, if $\frac{2}{R^2+1} \leq t$, there exists a scalar $a \in \mathbb{C}$ satisfying $\frac{2-t}{Rt} \leq |a| \leq R$. This scalar lies in $\mathcal{C}_{s,t}(R)$, hence the class will be non-empty. Conversely, assume that $\frac{2}{R^2+1} > t$ and that there exists $T \in \mathcal{C}_{s,t}(R)$. Since $T/R \in \mathcal{C}_s$, Lemma 2.5.7 tells us that $\sigma(T/R) \subset \{|z| \leq 1\}$, hence $\sigma(T) \subset \{|z| \leq R\}$. Also, the fact that $T^{-1}/R \in \mathcal{C}_t$ implies (in view of the same lemma) that $\sigma((2-t)t^{-1}T^{-1}/R) \subset \{|z| \leq 1\}$. Thus, we can conclude

$$\sigma(T) \subset \{R^{-1}(2-t)t^{-1} \leq |z| \leq R\} = \emptyset,$$

as $\frac{2}{R^2+1} > t$, a contradiction. Thus, $\mathcal{C}_{s,t}(R)$ must be empty. □

We now show that the $\mathcal{C}_{s,t}(R)$ classes are, in a certain sense, "rigid" with respect to the parameters s, t. To do this, we require $\mathcal{C}_{s,t}(R)$-membership conditions for 2×2 matrices with a single eigenvalue.

Lemma 2.5.9. *Fix $R > 1$. Also, let $s, t > 0$ and $T = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, where $a, b \in \mathbb{C}$.*

(i) Assume $s, t \geq 1$. Then, $T \in \mathcal{C}_{s,t}(R)$ if and only if $\frac{1}{R} \leq |a| \leq R$ and

$$R|b| - sR^2 + (2-s)|a|^2 \leq 2(1-s)R|a|$$

and

$$R|b| - t|a|^2R^2 + (2-t) \leq 2(1-t)R|a|.$$

(ii) Assume $s \geq 1$ and $t \leq 1$. Then, $T \in \mathcal{C}_{s,t}(R)$ if and only if $\frac{1}{R}\frac{2-t}{t} \leq |a| \leq R$ and

$$R|b| - sR^2 + (2-s)|a|^2 \leq 2(1-s)R|a|$$

and

$$R|b| - t|a|^2R^2 + (2-t) \le 2(t-1)R|a|.$$

(iii) Assume $s \le 1$ *and* $t \ge 1$*. Then,* $T \in \mathcal{C}_{s,t}(R)$ *if and only if* $\frac{1}{R} \le |a| \le R\frac{s}{2-s}$ *and*

$$R|b| - sR^2 + (2-s)|a|^2 \le 2(s-1)R|a|$$

and

$$R|b| - t|a|^2R^2 + (2-t) \le 2(1-t)R|a|.$$

(iv) Assume $s \le 1$ *and* $t \le 1$*. Then,* $T \in \mathcal{C}_{s,t}(R)$ *if and only if* $\frac{1}{R}\frac{2-t}{t} \le |a| \le R\frac{s}{2-s}$ *and*

$$R|b| - sR^2 + (2-s)|a|^2 \le 2(s-1)R|a|$$

and

$$R|b| - t|a|^2R^2 + (2-t) \le 2(t-1)R|a|.$$

Proof. The lemma is a consequence of the following observation: if $s \ge 1$, then $\begin{pmatrix} a/R & b/R \\ 0 & a/R \end{pmatrix} \in \mathcal{C}_s$ if and only if $|a| \le R$ and

$$R|b| - sR^2 + (2-s)|a|^2 \le 2(1-s)R|a|, \tag{2.17}$$

while if $0 < s < 1$ we have $\begin{pmatrix} a/R & b/R \\ 0 & a/R \end{pmatrix} \in \mathcal{C}_s$ if and only if $|a| \le R\frac{s}{2-s}$ and

$$R|b| - sR^2 + (2-s)|a|^2 \le 2(s-1)R|a|. \tag{2.18}$$

So, assume first that $s \ge 1$. In view of [109, Theorem 3.1], we have that $\begin{pmatrix} a/R & b/R \\ 0 & a/R \end{pmatrix} \in \mathcal{C}_s$ if and only if $|a| \le R$ and

$$|b|^2/R^2 \le \left|s + (s-2)|a|^2/R^2 - 2(s-1)|a|/R\right|^2. \tag{2.19}$$

Observe that

$$s + (s-2)|a|^2/R^2 - 2(s-1)|a|/R$$

$$= (s-1)(|a|-R)^2/R^2 + 1 - |a|^2/R^2,$$

which is non-negative because $|a| \le R$. We thus obtain, after taking square roots, that (2.19) is equivalent to (2.17), as desired.

Assume now that $0 < s < 1$. Using [109, Theorem 3.1] again, we obtain that $\begin{pmatrix} a/R & b/R \\ 0 & a/R \end{pmatrix} \in \mathcal{C}_s$ if and only if $|a| \le R\frac{s}{2-s}$ and

$$|b|^2/R^2 \le \left|s + (s-2)|a|^2/R^2 - 2(1-s)|a|/R\right|^2. \tag{2.20}$$

Observe that

$$s + (s-2)|a|^2/R^2 - 2(1-s)|a|/R$$

$$= (s-1)(|a|+R)^2/R^2 + 1 - |a|^2/R^2$$

$$= (|a|/R+1)\big((s-1)(|a|/R+1) + 1 - |a|/R\big)$$

$$= (|a|/R+1)\big((s-2)|a|/R + s\big)$$

which is non-negative because $|a| \le R\frac{s}{2-s}$. Hence, (2.20) is equivalent to (2.18) and the proof is complete. □

Theorem 2.5.10. *Assume $s, t > 0$ and $R > 1$ are such that $\mathcal{C}_{s,t}(R)$ is non-empty. Then, there exists $T \in \mathcal{C}_{s,t}(R)$ such that for every $\epsilon > 0$ and every $s', t' > 0$ we have*

$$T \notin \mathcal{C}_{s',t-\epsilon}(R) \quad \textit{and} \quad T \notin \mathcal{C}_{s-\epsilon,t'}(R).$$

Proof. We only deal with the case $s \ge 1, t \le 1$, as the computations required for the remaining three cases are very similar in nature.

So, let $s \ge 1, t \le 1$ and assume also that $(2-t)/(Rt) \le R$, hence $\mathcal{C}_{s,t}(R)$ is non-empty. Note that if $(2-t)/(Rt) = R$, then any class either of the form $\mathcal{C}_{s',t-\epsilon}(R)$ or of the form $\mathcal{C}_{s-\epsilon,t'}(R)$ will be empty. Obviously, the conclusion of the theorem will hold in this case. Thus, we may actually assume that $(2-t)/(Rt) < R$. We are looking for an operator that satisfies the conditions in the statement of the theorem and has the form $T = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, where $a, b \in \mathbb{C}$. In view of Lemma (2.5.9), we know that such a T will belong to $\mathcal{C}_{s,t}(R)$ if and only if $\frac{1}{R}\frac{2-t}{t} \leq |a| \leq R$ and

$$R|b| - sR^2 + (2-s)|a|^2 \leq 2(1-s)R|a| \tag{2.21}$$

and

$$R|b| - t|a|^2R^2 + (2-t) \leq 2(t-1)R|a|. \tag{2.22}$$

Our goal is to find a pair (a_0, b_0) such that $(2-t)/(Rt) < |a_0| < R$ and both (2.21) and (2.22) become equalities. Indeed, assuming such a_0 and b_0 exist, let T_0 denote the corresponding operator and consider any $\epsilon > 0$. We will then have that $T_0 \in \mathcal{C}_{s,t}(R)$ and also that

$$R|b_0| - (s-\epsilon)R^2 + (2-(s-\epsilon))|a_0|^2 + 2((s-\epsilon)-1)R|a_0|$$

$$= R|b_0| - sR^2 + (2-s)|a_0|^2 + 2(s-1)R|a_0| + \epsilon(R^2 + |a_0|^2 - 2R|a_0|)$$

$$= \epsilon(R - |a_0|)^2 > 0,$$

which shows that $T_0 \notin \mathcal{C}_{s-\epsilon,t'}(R)$, for any $t' > 0$. An analogous argument involving the second inequality also shows that $T_0 \notin \mathcal{C}_{s',t-\epsilon}(R)$ for any $s' > 0$, as desired.

Now, assume that we have equality in both (2.21) and (2.22) (without any extra restrictions). We can then extract a quadratic equation involving $|a|$ only:

$$(s - tR^2 - 2)|a|^2 + 2(2-s-t)R|a| + 2 - t + sR^2 = 0 \tag{2.23}$$

Notice that, if we are able to find a solution $|a_0|$ of the above equation that also satisfies $(2-t)/(Rt) < |a_0| < R$, our proof will be complete. Indeed, $|b_0|$ will then be uniquely determined as

$$|b_0| = sR + \frac{(s-2)|a_0|^2}{R} + 2(1-s)|a_0| = t|a_0|^2R + \frac{t-2}{R} + 2(t-1)|a_0|$$

and the associated T_0 will have the desired properties.

Assume first that $s = tR^2 + 2$. (2.23) then turns into the linear equation

$$2(2-s-t)R|a| + 2 - t + sR^2 = 0,$$

where $s = tR^2 + 2$. Let a_0 be any number satisfying the previous equation, i.e.

$$|a_0| = \frac{2(R^2+1) + t(R^4-1)}{2R(R^2+1)t}.$$

One can then verify that $|a_0| < R$ is equivalent to the inequality $\frac{2}{R^2+1} < t$, which is equivalent to $(2-t)/(Rt) < R$, hence it must be true. An analogous argument applies for the inequality $|a_0| > (2-t)/(Rt)$.

Assume now that $s \neq tR^2 + 2$. After some calculations, we obtain (for equation (2.23)) the (non-negative) discriminant

$$\varDelta = 4[(R^2+1)^2 st - 2(R^2+1)(s+t) + 4(R^2+1)].$$

Choose a_0 to be any number satisfying

$$|a_0| = \frac{2R(s+t-2) - \sqrt{\varDelta}}{2(s - tR^2 - 2)}.$$

Suppose first that $s > tR^2 + 2$. The inequality $|a_0| < R$ is equivalent to

$$tR^2 + 2 = \frac{R^2(R^2+1)^2t^2 + 2(R^2+1)t - 4(R^2+1)}{(R^2+1)((R^2+1)t-2)} < s,$$

hence it must be true. The proof in the case that $s < tR^2 + 2$ is entirely analogous.

We now prove that $|a_0| > (2-t)/(tR)$. The computations here are somewhat more unpleasant. First, we assume that $s > tR^2 + 2$, in which case $|a_0| > (2-t)/(tR)$ is equivalent to the inequality

$$\frac{[(R^2+1)t-2]^2}{R^2}s^2 + \Big(-\frac{16}{R^2} + \frac{8}{R^2}(R^2+2)t + 2(R^2+1)[1-2/R^2]t^2 - (R^2+1)^2t^3\Big)s$$
$$+\frac{16}{R^2} - \frac{16}{R^2}t + 4(1/R^2 - R^2 - 1)t^2 + 2(R^2+1)t^3 > 0.$$

But this last inequality can be rewritten as

$$\frac{1}{R^2}((R^2+1)t-2)(s-tR^2-2)([(R^2+1)t-2]s + 2(2-t)) \geq 0,$$

which holds for $s > tR^2 + 2$ and $(R^2+1) > (R^2+1)t > 2$, as desired. The proof in the case that $s < tR^2 + 2$ is entirely analogous. □

2.5.3 $\mathbb{DLA}_R(c)$

We now define a new operator class attached to the annulus A_R.

Definition 2.5.11. Let $c \in \mathbb{R}$ and $R > 1$. $\mathbb{DLA}_R(c)$ denotes the class of all operators $T \in \mathcal{B}(H)$ such that

(i) $\sigma(T) \subset \overline{A_R}$ and

(ii) $2\Re\left[(1 - zT/R)^{-1} + (1 - wT^{-1}/R)^{-1}\right] - 2 + c \geq 0, \quad \forall z, w \in \mathbb{D}$.

First, we prove a few elementary properties of $\mathbb{DLA}_R(c)$, including membership criteria for normal operators.

Lemma 2.5.12. *Let $c > -2$, $R > 1$ and assume $A \in \mathcal{B}(H)$ is a self-adjoint operator such that $A > 0$ and $2 + c - A > 0$. If $T \in \mathcal{B}(H)$ is such that $T/R \in \mathcal{C}_{2+c-A}$ and $T^{-1}/R \in \mathcal{C}_A$, then $T \in \mathbb{DLA}_R(c)$.*

Proof. Assume $T \in \mathcal{B}(H)$ satisfies our hypotheses. Lemma 2.5.4 allows us to deduce that $\sigma(T/R),\ \sigma(T^{-1}/R) \subset \overline{\mathbb{D}}$, hence $\sigma(T) \subset \overline{A_R}$. Also, (2.15) gives us

$$2\Re(1 - zT/R)^{-1} + 2 + c - A - 2 \geq 0$$

and

$$2\Re(1 - wT^{-1}/R)^{-1} + A - 2 \geq 0,$$

for all $z, w \in \mathbb{D}$. Adding these two inequalities concludes the proof. □

Proposition 2.5.13. *Let $c \in \mathbb{R}$, $R > 1$ and assume $N \in \mathcal{B}(H)$ is normal.*

(i) If $c \geq 0$, then $\mathbb{DLA}_R(c)$ will be non-empty for every $R > 1$.

(ii) If $-2 < c < 0$, then $\mathbb{DLA}_R(c)$ will be non-empty if and only if there exists $s \in (0, 2 + c)$ such that

$$\max\left\{1, \frac{2 - s}{s}\right\} \leq R^2 \min\left\{1, \frac{2 + c - s}{s - c}\right\}.$$

In fact, $\mathbb{DLA}_R(c)$ is non-empty if and only if it contains a scalar $a \in \mathbb{C}$.

(iii) *If* $c \leq -2$, *then* $\mathbb{DLA}_R(c)$ *will be empty for every* $R > 1$.

(iv) $\mathbb{DLA}_R(c) \subset \mathcal{C}_{2-c,2-c}(R)$, *for every* $c > -2$.

(v) *Assume* $c > -2$. *Then,* $N \in \mathbb{DLA}_R(c)$ *if and only if*

$$\sigma(N) \subset \bigcup_{0<s<2+c} \mathcal{C}_{2+c-s,s}(R).$$

(vi) *If, in addition, we assume* $c \geq 0$, *then* $N \in \mathbb{DLA}_R(c)$ *is equivalent to* $R^{-2} \leq N^*N \leq R^2$.

Proof. First, we will prove that if $c > -2$, then $\mathbb{DLA}_R(c)$ is non-empty if and only if there exists $s \in (0, 2+c)$ such that $\mathcal{C}_{2+c-s,s}(R)$ is non-empty. Once this has been shown, the last assertion in (ii) will follow immediately from Lemma 2.5.12 and the fact that $\mathcal{C}_{2+c-s,s}(R)$ is non-empty if and only if it contains a scalar (see Proposition 2.5.8).

First, observe that Lemma 2.5.12 implies $\mathcal{C}_{2+c-s,s}(R) \subset \mathbb{DLA}_R(c)$ for every $s \in (0, 2+c)$, hence one direction is obvious. For the converse, assume $\mathbb{DLA}_R(c)$ is non-empty. Thus, there exists $T \in \mathcal{B}(H)$ such that

$$\left\langle \left(2\Re\left[(1 - zT/R)^{-1} + (1 - wT^{-1}/R)^{-1}\right] - 2 + c \right) h, h \right\rangle \geq 0, \tag{2.24}$$

for all $z, w \in \mathbb{D}$ and $x \in H$. Now, let $\lambda \in \mathbb{C}^*$ be in the approximate point spectrum of T (this is always non-empty, as it contains the topological boundary of $\sigma(T)$; see [78, Problem 63]). Thus, there exists a sequence $\{h_n\} \subset H$ such that $||h_n|| = 1$ ($n = 1, 2, \dots$) and $(\lambda - T)h_n \to 0$ as $n \to \infty$. From this last limit we easily obtain

$$\lim_n \left(\lambda^k - T^k\right) h_n = 0,$$

for all $k \in \mathbb{Z}$. Thus, we can write $\langle T^k h_n, h_n \rangle \to \lambda^k$, for all $k \in \mathbb{Z}$, and so

$$\left\langle \sum_{k=0}^{m} (zT/R)^k h_n, h_n \right\rangle \to \sum_{k=0}^{m} (z\lambda/R)^k \tag{2.25}$$

and

$$\left\langle \sum_{k=0}^{m} (wT^{-1}/R)^k h_n, h_n \right\rangle \to \sum_{k=0}^{m} (w/(R\lambda))^k, \tag{2.26}$$

as $n \to \infty$, for all $m \geq 0$ and $z, w \in \mathbb{D}$.

Now, fix $z, w \in \mathbb{D}$. We have convergence $\sum_{k=0}^{m}(zT/R)^k \to (1-zT/R)^{-1}$ and $\sum_{k=0}^{m}(wT^{-1}/R)^k \to (1-wT^{-1}/R)^{-1}$ as $m \to \infty$ in the operator norm, since $\sigma(zT/R), \sigma(wT^{-1}/R) \subset \mathbb{D}$. Hence, in view of (2.25)-(2.26), we can conclude that

$$\lim_{n\to\infty} \langle (1-zT/R)^{-1}h_n, h_n\rangle = \lim_{m,n\to\infty} \left\langle \sum_{k=0}^{m}(zT/R)^k h_n, h_n \right\rangle$$
$$= (1-z\lambda/R)^{-1} \tag{2.27}$$

and

$$\lim_{n\to\infty} \langle (1-wT^{-1}/R)^{-1}h_n, h_n\rangle = \lim_{m,n\to\infty} \left\langle \sum_{k=0}^{m}(wT^{-1}/R)^k h_n, h_n \right\rangle$$
$$= (1-w/(R\lambda))^{-1}. \tag{2.28}$$

We now set $h = h_n$ in (2.24) and let $n \to \infty$. In view of the real-part versions of (2.27)-(2.28), we obtain

$$2\Re\left[(1-z\lambda/R)^{-1} + (1-w/(\lambda R))^{-1}\right] - 2 + c \geq 0,$$

for all $z, w \in \mathbb{D}$. This last inequality can also be written as

$$\inf_{z\in\mathbb{D}}\left[2\Re(1-z\lambda/R)^{-1} + c\right] \geq \sup_{w\in\mathbb{D}}\left[-2\Re(1-w/(\lambda R))^{-1} + 2\right]$$

and thus we can deduce the existence of $s \in \mathbb{R}$ such that

$$2\Re(1-z\lambda/R)^{-1} + (1-w/(\lambda R))^{-1} + c \geq s \geq -2\Re(1-w/(\lambda R))^{-1} + 2,$$

for all $z, w \in \mathbb{D}$. In view of Lemma 2.5.4, we obtain $\lambda/R \in \mathcal{C}_{2+c-s}$ and $1/(\lambda R) \in \mathcal{C}_s$, which concludes the proof of our assertion in the beginning. If $c \geq 0$, then we can always choose $s = 1$, as $\mathcal{C}_{2+c-1,1}(R)$ will be non-empty by Lemma 2.5.8(i). This gives us (i) and if $-2 < c < 0$, we obtain (ii) by the same lemma.

For (iii), assume $c \leq -2$ and let $T \in \mathbb{DLA}_R(c)$. Set $w = 0$ in the definition of $\mathbb{DLA}_R(c)$ to obtain $T/R \in \mathcal{C}_{c+2}$, which implies that $c = -2$ (since $\mathcal{C}_\rho$ is empty for $\rho < 0$). But then, we obtain

$T/R \in \mathcal{C}_0$ and so $T \equiv 0$, which contradicts the inclusion $\sigma(T) \subset \overline{A_R}$. Thus, $\mathbb{DLA}_R(c)$ must be empty.

For (iv), assume $c > -2$ and let $T \in \mathbb{DLA}_R(c)$. Setting $w = 0$ in the definition of $\mathbb{DLA}_R(c)$ gives us $T/R \in \mathcal{C}_{2-c}$, while setting $z = 0$ (and letting $w \in \mathbb{D}$) gives us $T^{-1}/R \in \mathcal{C}_{2-c}$. Thus, $\mathbb{DLA}_R(c) \subset \mathcal{C}_{2-c,2-c}(R)$.

For (v), let $c > -2$. Since N is normal, it has a spectral decomposition (see [55, Chapter IX])

$$N = \int_{\sigma(N)} \lambda \, dE(\lambda).$$

Using this, we obtain that $N \in \mathbb{DLA}_R(c)$ if and only if its spectrum is contained in $\overline{A_R}$ and

$$\begin{aligned} &2\Re\big[(1 - zN/R)^{-1} + (1 - wN^{-1}/R)^{-1}\big] - 2 + c \\ &= \int_{\sigma(N)} \Big(2\Re\big[(1 - z\lambda/R)^{-1} + (1 - w/(\lambda R))^{-1}\big] - 2 + c\Big)\, dE(\lambda) \geq 0, \end{aligned}$$

for all $z, w \in \mathbb{D}$, which can be equivalently restated as

$$2\Re\big[(1 - z\lambda/R)^{-1} + (1 - w/(\lambda R))^{-1}\big] - 2 + c \geq 0,$$

for all $z, w \in \mathbb{D}$ and all $\lambda \in \sigma(N)$. Mimicking our argument from the proof of (i), we deduce, for every $\lambda \in \sigma(N)$, the existence of $s \in (0, 2 + c)$ (depending on λ) such that $\lambda \in \mathcal{C}_{2+c-s,s}(R)$. Hence, we can write

$$\sigma(N) \subset \bigcup_{0<s<2+c} \mathcal{C}_{2+c-s,s}(R),$$

which concludes the proof.

For (vi), assume $c \geq 0$. Since N is normal, the condition $R^{-2} \leq N^*N \leq R^2$ is equivalent to $\sigma(N) \subset A_R$, which is necessary for membership in $\mathbb{DLA}_R(c)$. Conversely, if $R^{-2} \leq N^*N \leq R^2$, then

$$\sigma(N) \subset \mathcal{C}_{1,1}(R) \subset \bigcup_{0<s<2+c} \mathcal{C}_{2+c-s,s}(R),$$

where the last inclusion holds because $c \geq 0$. Thus, we must have $N \in \mathbb{DLA}_R(c)$. □

Under the extra assumption $\sigma(T) \subset A_R$, we can restrict the parameters z, w in the definition of $\mathbb{DLA}_R(c)$ to the boundary of the disk (this is the $\mathbb{DLA}_R(c)$-version of Lemma 2.16).

Lemma 2.5.14. *Assume $c > -2$ and $T \in \mathcal{B}(H)$ is such that $\sigma(T) \subset A_R$. Then, $T \in \mathbb{DLA}_R(c)$ if and only if*

$$2\Re\big[(1-e^{i\theta}T/R)^{-1} + (1-e^{i\psi}T^{-1}/R)^{-1}\big] - 2 + c \geq 0, \quad \forall \theta, \psi \in [0, 2\pi).$$

Proof. First, assume $T \in \mathbb{DLA}_R(c)$. Fix an arbitrary $h \in H$ and $w \in \mathbb{D}$ and define

$$\Phi_{h,w} : \mathbb{D} \to \mathbb{C}$$

$$z \mapsto \big\langle \big(2\Re\big[(1-zT/R)^{-1} + (1-wT^{-1}/R)^{-1}\big] - 2 + c\big)h, h\big\rangle.$$

Since $\sigma(T) \subset A_R$, $\Phi_{h,w}$ will be a harmonic function on $\mathbb{D}$ that extends continuously to $\overline{\mathbb{D}}$. By the minimum principle for harmonic functions, we then obtain that

$$\Phi_{h,w}(z) = \big\langle \big(2\Re\big[(1-zT/R)^{-1} + (1-wT^{-1}/R)^{-1}\big] - 2 + c\big)h, h\big\rangle \geq 0,$$

for all $z \in \mathbb{D}$, if and only if

$$\Phi_{h,w}(e^{i\theta}) = \big\langle \big(2\Re\big[(1-e^{i\theta}T/R)^{-1} + (1-wT^{-1}/R)^{-1}\big] - 2 + c\big)h, h\big\rangle \geq 0,$$

for all $\theta \in [0, 2\pi)$. Since h and w were arbitrary, we conclude that

$$2\Re\big[(1-e^{i\theta}T/R)^{-1} + (1-wT^{-1}/R)^{-1}\big] - 2 + c \geq 0,$$

for all $\theta \in [0, 2\pi)$ and all $w \in \mathbb{D}$. We can now apply the minimum principle to the function

$$w \mapsto \big\langle \big(2\Re\big[(1-e^{i\theta}T/R)^{-1} + (1-wT^{-1}/R)^{-1}\big] - 2 + c\big)h, h\big\rangle$$

to conclude the proof.

For the converse, simply roll back the steps in the previous proof. □

Now, recall that the $\mathcal{C}_\rho$ classes are strictly monotone with respect to ρ, i.e. $\rho < \rho'$ implies $\mathcal{C}_\rho \subsetneq \mathcal{C}_{\rho'}$. We are going to prove an analogous monotonicity result for $\mathbb{DLA}_R(c)$.

Theorem 2.5.15. *If $-2 < c < c'$ and $\mathbb{DLA}_R(c')$ is non-empty, then*

$$\mathbb{DLA}_R(c) \subsetneq \mathbb{DLA}_R(c').$$

Proof. If $c < c'$, it is obvious by the definition of $\mathbb{DLA}_R(c)$ that $\mathbb{DLA}_R(c) \subseteq \mathbb{DLA}_R(c')$. To show that the inclusion is actually strict, we are going to divide the proof into two cases.

First, assume that there exist $s, t > 0$ such that $s \geq 1, s + t = 2 + c'$ and $\mathcal{C}_{s.t}(R)$ is non-empty. Note that (in view of Lemma 2.5.8 and the monotonicity of the $\mathcal{C}_\rho$ classes), if such s, t exist and $t \geq 1$, we can replace them by new parameters s', t' such that $s' \geq 1$, $t' < 1$, $s' + t' = 2 + c'$, $\frac{2}{R^2+1} < t$ and $\mathcal{C}_{s'.t'}(R)$ is non-empty. So, we may also assume that $t < 1$ and $\frac{2}{R^2+1} < t$.

Now, we will take advantage of the matrices we calculated in the proof of Theorem 2.5.10. Recall first that, in view of Lemma 2.5.9, a matrix

$$T = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

with $a, b > 0$ lies in $\mathcal{C}_{s,t}(R)$ if and only if $(2-t)/(Rt) \leq a \leq R$ and

$$2\Re\left[\frac{1}{1 - ae^{i\theta}/R}\right] + (s-2) \geq \frac{b}{R|1 - ae^{i\theta}/R|^2} \tag{2.29}$$

holds for $\theta = 0$ and

$$2\Re\left[\frac{1}{1 - e^{i\psi}/(aR)}\right] + (t-2) \geq \frac{b}{a^2} \frac{1}{R|1 - e^{i\psi}/(aR)|^2} \tag{2.30}$$

holds for $\psi = \pi$. Since $\frac{2}{R^2+1} < t$, the proof of Theorem 2.5.10 tells us that we can find a, b such that $(2-t)/(Rt) < a < R$ and we have equality in (2.29) for $\theta = 0$ and in (2.30) for $\psi = \pi$. Since, for these values of a, b, we have $T \in \mathcal{C}_{s,t}(R)$ and $s + t = 2 + c'$, Lemma 2.5.12 tells us that $T \in \mathbb{DLA}_R(c')$.

Now, we claim that T cannot lie in $\mathbb{DLA}_R(c)$. Indeed, assume $T \in \mathbb{DLA}_R(c)$. Taking determinants in the inequality

$$2\Re\left[(1 - zT/R)^{-1} + (1 - wT^{-1}/R)^{-1}\right] - 2 + c \geq 0$$

implies that

$$\begin{aligned} &2\Re\left[\frac{1}{1 - ae^{i\theta}/R} + \frac{1}{1 - e^{i\psi}/(aR)}\right] - 2 + c \\ &\geq \left|\frac{be^{i\theta}}{R(1 - ae^{i\theta}/R)^2} - \frac{b}{a^2}\frac{e^{i\psi}}{R(1 - e^{i\psi}/(aR))^2}\right| \end{aligned}$$

for all θ and ψ. In particular, we can choose $\theta_0 = 0$ and $\psi_0 = \pi$, which implies (since we have equality in both (2.29) and (2.30))

$$\begin{aligned}
&2\Re\left[\frac{1}{1-ae^{i\theta_0}/R}+\frac{1}{1-e^{i\psi_0}/(aR)}\right]-2+c\\
&\geq\left|\frac{be^{i\theta_0}}{R(1-ae^{i\theta_0}/R)^2}-\frac{b}{a^2}\frac{e^{i\psi_0}}{R(1-e^{i\psi_0}/(aR))^2}\right|\\
&=\left|\frac{be^{i\theta_0}}{R(1-ae^{i\theta_0}/R)^2}\right|+\left|\frac{b}{a^2}\frac{e^{i\psi_0}}{R(1-e^{i\psi_0}/(aR))^2}\right|\\
&=2\Re\left[\frac{1}{1-ae^{i\theta_0}/R}\right]+(s-2)+2\Re\left[\frac{1}{1-e^{i\psi_0}/(aR)}\right]+(t-2)\\
&=2\Re\left[\frac{1}{1-ae^{i\theta_0}/R}\right]+2\Re\left[\frac{1}{1-e^{i\psi_0}/(aR)}\right]+c'-2,
\end{aligned}$$

a contradiction, since $c < c'$. Thus, $T \notin \mathbb{DLA}_R(c')$.

Now, assume that there do not exist $s, t > 0$ such that $s \geq 1, s + t = 2 + c'$ and $\mathcal{C}_{s.t}(R)$ is non-empty. Since $T \in \mathcal{C}_{s,t}(R)$ if and only if $T^{-1} \in \mathcal{C}_{t,s}(R)$, we also deduce that there do not exist $s, t > 0$ such that $t \geq 1, s+t = 2+c'$ and $\mathcal{C}_{s.t}(R)$ is non-empty. Thus, $-2 < c' < 0$. Now, $\mathbb{DLA}_R(c')$ is non-empty, so Proposition 2.5.13(i) tells us that there exists $s \in (0, 2 + c')$ such that $\mathcal{C}_{s,2+c'-s}(R)$ is non-empty. In view of our previous remarks, we must have $s, 2 + c' - s < 1$. Lemma 2.5.8 then tells us that

$$\frac{1}{R}\frac{s-c'}{2+c'-s} \leq R\frac{s}{2-s}.$$

Note that the right-hand side tends to 0 as $s \to 0$, while the left-hand side remains bounded below by a strictly positive number. Thus, shrinking s, we may replace it by a positive number δ such that $\frac{1}{R}\frac{\delta-c'}{2+c'-\delta} = R\frac{\delta}{2-\delta}$. In view again of Lemma 2.5.8, we obtain that $\mathcal{C}_{\delta,2+c'-\delta}(R)$ is non-empty. Setting $a = R\frac{\delta}{2-\delta}$, we obtain, from the same Lemma, that $a \in \mathcal{C}_{\delta,2+c'-\delta}(R) \subset \mathbb{DLA}_R(c')$.

Now, we shall show that $a \notin \mathbb{DLA}_R(c)$. Assume instead that $a \in \mathbb{DLA}_R(c)$. Proposition 2.5.13 then implies that we can find $s_0 \in (0, 2 + c)$ such that $a \in \mathcal{C}_{s_0,2+c-s_0}(R)$. But $c < c'$, hence we must have either $s_0 < \delta$ or $2 + c - s_0 < 2 + c' - \delta$. Assume that $s_0 < \delta$, then

$$R\frac{s_0}{2-s_0} < R\frac{\delta}{2-\delta} = a,$$

which contradicts $a \in \mathcal{C}_{s_0,2+c-s_0}(R)$. Similarly, if $2+c-s_0 < 2+c'-\delta$ we can write

$$\frac{1}{R}\frac{2-(2+c-s_0)}{2+c-s_0} > \frac{1}{R}\frac{2-(2+c'-\delta)}{2+c'-\delta} = \frac{1}{R}\frac{\delta-c'}{2+c'-\delta} = a,$$

which again contradicts $a \in \mathcal{C}_{s_0,2+c-s_0}(R)$. Thus, $a \notin \mathbb{DLA}_R(c)$ and we are done. □

It is well-known (see e.g. [144]) that given any $T \in \mathcal{C}_\rho$ ($\rho > 0$) and any $f \in \mathcal{A}(\mathbb{D})$ such that $||f||_\infty \leq 1$ and $f(0) = 0$, we must have $f(T) \in \mathcal{C}_\rho$. We end this subsection with a proposition that is motivated by this result.

Proposition 2.5.16. *Assume* $c > -2$ *and* $T \in \mathbb{DLA}_R(c)$. *Then, for any* $f, g \in \mathcal{A}(\mathbb{D})$ *that are bounded by* 1 *and satisfy* $f(0) = g(0) = 0$, *we have*

$$2\Re\left[(1-zf(T/R))^{-1} + (1-wg(T^{-1}/R))^{-1}\right] - 2 + c \geq 0,$$

for all $z, w \in \mathbb{D}$.

Proof. By assumption, we know that

$$\Re\left[\sum_{n=1}^{\infty}\left(\frac{Tz}{R}\right)^n + 1 + \sum_{n=1}^{\infty}\left(\frac{T^{-1}w}{R}\right)^n\right] \geq \frac{-c}{2}, \tag{2.31}$$

for all $z, w \in \mathbb{D}$. Fix $w \in \mathbb{D}$, let $x \in H$ and choose a decreasing null sequence $\{\epsilon_k\}$. Set

$$S_k = S_k(w) = 1 + c/2 + \epsilon_k + \Re\sum_{n=1}^{\infty}\left(\frac{T^{-1}w}{R}\right)^n \in \mathcal{B}(H).$$

(2.31) now tells us that the holomorphic function

$$F_k : \mathbb{D} \to \mathbb{C}$$

$$z \mapsto \sum_{n=1}^{\infty}\frac{z^n}{R^n}\langle T^n x, x\rangle + \langle S_k x, x\rangle$$

has positive real part and satisfies $F_k(0) = \langle S_k x, x\rangle > 0$ (put $z = 0$ in (2.31)). Thus, Herglotz's theorem implies the existence of a positive measure $\mu_{x,w,k}$ on the unit circle such that

$$F_k(0) + \sum_{n=1}^{\infty}\frac{z^n}{R^n}\langle T^n x, x\rangle = \int \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}d\mu_{x,w,k}(\theta),$$

for all $z \in \mathbb{D}$. Expanding the integrand and equating coefficients, we obtain

$$\frac{1}{R^n}\langle T^n x, x\rangle = 2\int e^{-in\theta} d\mu_{x,w,k}(\theta),$$

for every $n \geq 1$. Thus, if p is any polynomial such that $p(0) = 0$, we can write

$$\langle p(T/R)x, x\rangle = 2\int p(e^{-i\theta}) d\mu_{x,w,k}(\theta).$$

Replacing p by p^n, we obtain

$$\langle p^n(T/R)x, x\rangle = 2\int p^n(e^{-i\theta}) d\mu_{x,w,k}(\theta),$$

for every $n \geq 1$. Thus, if we also assume that $|p| \leq 1$, we obtain that $p(T/R)$ has its spectrum inside $\mathbb{D}$ (since the same must be true for T/R) and we can write

$$\left\langle \left(S_k + \sum_{n=1}^{\infty} z^n p^n(T/R)\right)x, x\right\rangle$$

$$= F_k(0) + 2\sum_{n=1}^{\infty} z^n \int p(e^{-i\theta})^n d\mu_{x,w,k}(\theta) = \int \frac{1 + zp(e^{-i\theta})}{1 - zp(e^{-i\theta})} d\mu_{x,w,k}(\theta).$$

Now, if $f \in \mathcal{A}(\mathbb{D})$ is bounded by 1 and satisfies $f(0) = 0$, a standard approximation argument shows that

$$\left\langle \left(S_k + \sum_{n=1}^{\infty} z^n f^n(T/R)\right)x, x\right\rangle = \int \frac{1 + zf(e^{i\theta})}{1 - zf(e^{i\theta})} d\mu_{x,w,k}(\theta).$$

The integrand has positive real part for all z and θ, hence

$$\Re\left[\sum_{n=1}^{\infty} \left(zf(T/R)\right)^n + 1 + \epsilon_k + \sum_{n=1}^{\infty} \left(wT^{-1}/R\right)^n\right] \geq \frac{-c}{2},$$

for all k and for all $z, w \in \mathbb{D}$. Letting $k \to \infty$, we obtain

$$\Re\left[\sum_{n=1}^{\infty} \left(zf(T/R)\right)^n + 1 + \sum_{n=1}^{\infty} \left(wT^{-1}/R\right)^n\right] \geq \frac{-c}{2},$$

for all $z, w \in \mathbb{D}$. Using this last inequality, we may repeat the previous argument with the roles of z and w swapped, thus obtaining the desired result. □

2.5.4 The Double-Layer Potential Kernel

In this subsection, we finally exhibit the connection between $\mathbb{DLA}_R(c)$ and the double-layer potential kernel over the annulus. Given any smoothly bounded, open $\Omega \subset \mathbb{C}$ and any $T \in \mathcal{B}(H)$ such that $\sigma(T) \subset \Omega$, recall that the transform of f by the double-layer potential kernel is defined as

$$S(f,T) = \int_{\partial\Omega} \mu(\sigma(s),T) f(\sigma(s)) ds,$$

where s denotes the arc length of $\sigma = \sigma(s)$ on the (counter-clockwise) oriented boundary $\partial\Omega$ and $\mu(\sigma(s),T)$ is the self-adjoint operator defined (for $\sigma(s) \notin \sigma(T)$) as

$$\mu(\sigma(s),T) = \frac{1}{2\pi i}\big(\sigma'(s)(\sigma(s) - T)^{-1} - \overline{\sigma'(s)}(\overline{\sigma(s)} - T^*)^{-1}\big).$$

Note that $S(f,T) = f(T) + (C\overline{f})(T)^*$ and thus $\int_{\partial\Omega} \mu(\sigma,T) ds = 2I$.

The definition of $\mathbb{DLA}_R(c)$ can now be recast (for $\sigma(T) \subset A_R$) as follows.

Theorem 2.5.17. *Assume $R > 1, c > -2$ and $T \in \mathcal{B}(H)$ satisfies $\sigma(T) \subset A_R$. Then, $T \in \mathbb{DLA}_R(c)$ if and only if the mapping*

$$S_{R,c} : \mathcal{A}(A_R) \to \mathcal{B}(H)$$

$$f = \sum_{n\in\mathbb{Z}} a_n z^n \mapsto \frac{1}{2+c}\Big[\int_{\partial A_R} \mu(\sigma,T) f(\sigma) ds + c a_0\Big]$$

is contractive.

Proof. Write $\partial A_R = \Gamma_1 \cup \Gamma_{-1}$, where Γ_1 (the outer circle) is counter-clockwise oriented, while Γ_{-1} (the inner circle) is clockwise oriented. Also, in view of [111, Corollary 2.9] and [111, Proposition 2.12], the fact that $S_{R,c}$ is unital allows us to deduce that $S_{R,c}$ is contractive if and only if

$$S'_{R,c} : \mathcal{A}(A_R) + \mathcal{A}(A_R)^* \to \mathcal{B}(H) \tag{2.32}$$

$$f + \overline{g} = \sum_{n\in\mathbb{Z}} a_n z^n + \sum_{n\in\mathbb{Z}} \overline{b_n}\overline{z}^n \mapsto S_{R,c}(f) + S_{R,c}(g)^*$$

$$= \frac{1}{2+c}\Big[\int_{\partial A_R} \mu(\sigma,T)(f + \overline{g})(\sigma) ds + c(a_0 + \overline{b_0})\Big]$$

is positive. Let $C(\partial A_R)$ denote the algebra of continuous functions on ∂A_R and recall that the closure of $\mathcal{A}(A_R) + \mathcal{A}(A_R)^*$ in $C(\partial A_R)$ is the codimension one subspace (see [111, p. 80])

$$\mathcal{M}_R = \Big\{ f \in C(\partial A_R) : \frac{1}{2\pi R} \int_{\Gamma_1} f(\sigma) ds = \frac{R}{2\pi} \int_{\Gamma_{-1}} f(\sigma) ds \Big\}.$$

Thus, our goal will be to show that $T \in \mathbb{DLA}_R(c)$ if and only if $\tilde{S}_{R,c}$ is positive over $\mathcal{M}_R$, where

$$\tilde{S}_{R,c} : \mathcal{M}_R \to \mathcal{B}(H)$$
$$f \mapsto \frac{1}{2+c} \Big[\int_{\partial A_R} \mu(\sigma, T) f(\sigma) ds + \frac{c}{2\pi i} \int_{\Gamma_1} \frac{f}{\sigma} \sigma' ds \Big]$$

(note that $S'_{R,c} \equiv \tilde{S}_{R,c}$ over $\mathcal{A}(A_R) + \mathcal{A}(A_R)^*$). We require the following lemma.

Lemma 2.5.18. *In the setting of Theorem 2.5.17, $T \in \mathbb{DLA}_R(c)$ if and only if*

$$R\mu(\sigma_1, T) + R^{-1}\mu(\sigma_2, T) + \frac{c}{2\pi} \geq 0,$$

for all $\sigma_1 \in \Gamma_1$ and $\sigma_2 \in \Gamma_{-1}$.

Proof of Lemma 2.5.18. For $\sigma_1 \in \Gamma_1$, the arc-length parametrization gives $\sigma_1 = Re^{i\theta}, \sigma_1' = ie^{i\theta}$, while for $\sigma_2 \in \Gamma_{-1}$ we obtain $\sigma_2 = R^{-1}e^{-i\psi}, \sigma_2' = -ie^{-i\psi}$, where $\theta, \psi \in [0, 2\pi)$. Thus, we may write

$$2\pi\big[R\mu(\sigma_1, T) + R^{-1}\mu(\sigma_2, T)\big] + c$$
$$= 2\Re\big[Re^{i\theta}(Re^{i\theta} - T)^{-1}\big] + 2\Re\big[-R^{-1}e^{-i\psi}(R^{-1}e^{-i\psi} - T)^{-1}\big] + c$$
$$= 2\Re\big[(1 - e^{-i\theta}T/R)^{-1} + (1 - e^{-i\psi}T^{-1}/R)^{-1}\big] + c - 2,$$

which is positive for all $\theta, \psi \in [0, 2\pi)$ if and only if $T \in \mathbb{DLA}_R(c)$. □

Now, assume that $\tilde{S}_{R,c}$ is positive and that $T \notin \mathbb{DLA}_R(c)$. In view of Lemma 2.5.18, there exist $\eta_1 \in \Gamma_1, \eta_2 \in \Gamma_{-1}$ and a unit vector $\upsilon \in H$ such that

$$\Big\langle \Big(R\mu(\eta_1, T) + R^{-1}\mu(\eta_2, T) + \frac{c}{2\pi} \Big) \upsilon, \upsilon \Big\rangle = 2k < 0. \tag{2.33}$$

Since $\sigma(T) \subset A_R$, we know that both of the maps $\sigma_1 \mapsto \langle \mu(\sigma_1, T)v, v\rangle$ and $\sigma_2 \mapsto \langle \mu(\sigma_2, T)v, v\rangle$ are continuous. Thus, in view of (2.33), we can find small arcs $I_1 \subset \Gamma_1$ and $I_{-1} \subset \Gamma_{-1}$ of equal length and centered at η_1 and η_2 respectively such that

$$\left\langle\left(R\mu(\sigma_1, T) + R^{-1}\mu(\sigma_2, T) + \frac{c}{2\pi}\right)v, v\right\rangle \leq k < 0,$$

for all $\sigma_1 \in I_1$ and all $\sigma_2 \in I_{-1}$. From this, we easily deduce the existence of $t \in \mathbb{R}$ such that

$$\left\langle\left(R\mu(\sigma_1, T) + \frac{c}{2\pi}\right)v, v\right\rangle \leq k/2 + t \tag{2.34}$$

and

$$\left\langle R^{-1}\mu(\sigma_2, T)v, v\right\rangle \leq k/2 - t, \tag{2.35}$$

for all $\sigma_1 \in I_1$ and all $\sigma_2 \in I_{-1}$. Now, take $g : \partial A_R \to \mathbb{C}$ to be a continuous function such that $0 \leq g \leq 1$, $g(z) = 0$ for z outside $I_1 \cup I_{-1}$ and also $d = \frac{1}{2\pi R}\int_{\Gamma_1} g(\sigma)ds = \frac{R}{2\pi}\int_{\Gamma_{-1}} g(\sigma)ds > 0$. Then, $g \in \mathcal{M}_R$ and so $\tilde{S}_{R,c}(g)$ must be a positive operator. However, observe that by (2.34) and (2.35),

$$\begin{aligned}(2+c)\langle \tilde{S}_{R,c}(g)v, v\rangle &= \int_{\partial A_R} \langle \mu(\sigma, T)v, v\rangle g(\sigma)ds + \frac{c}{2\pi R}\int_{\Gamma_1} g(\sigma_1)ds \\ &= \int_{\Gamma_1} \langle R\mu(\sigma_1, T)v, v\rangle \frac{g(\sigma_1)}{R}ds + \int_{\Gamma_{-1}} \langle R^{-1}\mu(\sigma_2, T)v, v\rangle Rg(\sigma_2)ds + \frac{c}{2\pi R}\int_{\Gamma_1} g(\sigma_1)ds \\ &\leq (k/2 + t)\frac{1}{R}\int_{\Gamma_1} g(\sigma_1)ds + (k/2 - t)R\int_{\Gamma_{-1}} g(\sigma_2)ds \\ &= 2\pi dk < 0,\end{aligned}$$

a contradiction. Thus, we must have $T \in \mathbb{DLA}_R(c)$.

Conversely, assume $T \in \mathbb{DLA}_R(c)$. Fix $v \in H$. Lemma 2.5.18 tells us that we can find $t \in \mathbb{R}$ such that

$$\left\langle\left(R\mu(\sigma_1, T) + \frac{c}{2\pi}\right)v, v\right\rangle \geq t \tag{2.36}$$

and

$$\left\langle R^{-1}\mu(\sigma_2, T)v, v\right\rangle \geq -t, \tag{2.37}$$

for all $\sigma_1 \in \Gamma_1$ and all $\sigma_2 \in \Gamma_{-1}$. Now, let $f \in \mathcal{M}_R$ be positive. Since $\frac{1}{2\pi R}\int_{\Gamma_1} f(\sigma)ds = \frac{R}{2\pi}\int_{\Gamma_{-1}} f(\sigma)ds$, we may write (in view of (2.36)-(2.37))

$$\begin{aligned}
&(2+c)\langle \tilde{S}_{R,c}(f)v, v\rangle \\
&= \int_{\Gamma_1} \left\langle \left(R\mu(\sigma_1, T) + \frac{c}{2\pi}\right)v, v\right\rangle \frac{f(\sigma_1)}{R} ds + \int_{\Gamma_{-1}} \langle R^{-1}\mu(\sigma_2, T)v, v\rangle R f(\sigma_2) ds \\
&\geq \frac{t}{R}\int_{\Gamma_1} f(\sigma_1)ds - tR\int_{\Gamma_{-1}} f(\sigma_2)ds \\
&= 0.
\end{aligned}$$

Since $v \in H$ was arbitrary, $\tilde{S}_{R,c}(f)$ has to be a positive operator and we are done. □

2.5.5 A Completely Contractive Analogue of $\mathbb{DLA}_R(c)$

We now introduce and characterize $\mathbb{CDLA}_R(c)$, the "complete version" of $\mathbb{DLA}_R(c)$. Our main result is Theorem 2.5.2. One can also work with operators satisfying $\sigma(T) \subset \overline{A_R}$ to obtain the more general Theorem 2.5.22.

Definition 2.5.19. Assume $R > 1, c > -2$ and $T \in \mathcal{B}(H)$ satisfies $\sigma(T) \subset A_R$. Then, $T \in \mathbb{CDLA}_R(c)$ if the mapping

$$S_{R,c} : \mathcal{A}(A_R) \to \mathcal{B}(H)$$
$$f = \sum_{n\in\mathbb{Z}} a_n z^n \mapsto \frac{1}{2+c}\left[\int_{\partial A_R} \mu(\sigma, T)f(\sigma)ds + ca_0\right]$$

is completely contractive.

We first establish one direction of Theorem 2.5.2.

Lemma 2.5.20. *Assume $R > 1, c > -2$ and $T \in \mathcal{B}(H)$ satisfies $\sigma(T) \subset A_R$. If there exists $A \in \mathcal{B}(H)$ such that $A > 0$, $2 + c - A > 0$ and $T/R \in \mathcal{C}_{2+c-A}$ and $T^{-1}/R \in \mathcal{C}_A$, then $T \in \mathbb{CDLA}_R(c)$.*

Proof. Assume T and A satisfy the given hypotheses. We again write $\partial A_R = \Gamma_1 \cup \Gamma_{-1}$, where Γ_1 (the outer circle) is counter-clockwise oriented, while Γ_{-1} (the inner circle) is clockwise oriented.

Now, define the self-adjoint operator

$$\nu_A(\sigma, T) := \mu(\sigma, T) + \frac{\sigma'}{\sigma}\frac{(c - A)}{2\pi i}, \ \ \forall \sigma \in \partial\Gamma_1,$$

and

$$\nu_A(\sigma, T) := \mu(\sigma, T) - \frac{\sigma'}{\sigma}\frac{A}{2\pi i}, \ \ \forall \sigma \in \partial\Gamma_{-1}.$$

Note that, if $\sigma \in \Gamma_1$, we have $\sigma = Re^{i\theta}$ and $s = R\theta$, thus

$$\nu_A(\sigma, T) = \mu(Re^{i\theta}, T) + \frac{c - A}{2\pi R} \geq 0,$$

for all θ, as $T/R \in \mathcal{C}_{2+c-A}$. Also, if $\sigma \in \Gamma_{-1}$, we can write $\sigma = R^{-1}e^{-i\phi}$ and $s = R^{-1}\phi$, hence

$$\begin{aligned}\nu_A(\sigma, T) &= \mu(R^{-1}e^{-i\phi}, T) + \frac{R}{2\pi}A \\ &= \frac{1}{2\pi}\big(- e^{-i\phi}(R^{-1}e^{-i\phi} - T)^{-1} + e^{i\phi}(R^{-1}e^{i\phi} - T^*)^{-1}\big) + \frac{R}{2\pi}A \\ &= \frac{R}{2\pi}\Big(2\Re\big(R^{-1}T^{-1}e^{-i\phi}(I - T^{-1}e^{-i\phi}R^{-1})^{-1}\big) + A\Big) \\ &= \frac{R}{2\pi}\big(2\Re(I - T^{-1}e^{-i\phi}R^{-1})^{-1} + A - 2\big) \geq 0,\end{aligned}$$

for all ϕ, as $T^{-1}/R \in \mathcal{C}_A$.

Next, we consider the coordinate-wise map $S^{(m)}_{R,c} : M_m(\mathcal{A}(A_R)) \to \mathcal{B}(H^{(m)})$, for $m \geq 1$. Here, $M_m(\mathcal{A}(A_R))$ denotes the algebra of all matrix-valued $F : A_R \to \mathbb{C}^{m\times m}$ that are (coordinate-wise) analytic and admit a continuous extension to $\overline{A_R}$. For any such $F = \sum_{n\in\mathbb{Z}} A_n \otimes z^n$, we can write

$$\begin{aligned}(2 + c)S^{(m)}_{R,c}(F) &= \int_{\partial A_R} F(\sigma) \otimes \mu(\sigma, T)ds + cA_0 \otimes I \\ &= \int_{\partial A_R} F(\sigma) \otimes \mu(\sigma, T)\, ds + \frac{c}{2\pi i}\int_{\Gamma_1} \frac{\sigma'}{\sigma}F(\sigma) \otimes I\, ds \\ &= \int_{\partial A_R} F(\sigma) \otimes \mu(\sigma, T)\, ds + \frac{c}{2\pi i}\int_{\Gamma_1} \frac{\sigma'}{\sigma}F(\sigma) \otimes I\, ds - \frac{A}{2\pi i}\int_{\partial A_R} \frac{\sigma'}{\sigma}F(\sigma) \otimes I\, ds,\end{aligned}$$

$$= \int_{\partial A_R} F(\sigma) \otimes \nu_A(\sigma, T)\, ds.$$

But now, since $\nu_A(\sigma, T) \geq 0$ for every σ in ∂A_R and also $\int_{A_R} \nu_A(\sigma, T)\, ds = (2 + c)I$, one can show (see e.g. the proof of Lemma 2.2 in [60]) that $S_{R,c}^{(m)}$ is contractive, for every $m \geq 1$. This concludes the proof. $\square$

We now prove a lemma; the $\mathcal{C}_A$ classes do not contain any invertible operators if A is not invertible.

Lemma 2.5.21. *Let $A \in \mathcal{B}(H)$ be a positive operator that is not invertible. Assume also that $T \in \mathcal{B}(H)$ satisfies $\sigma(T) \subset \mathbb{D}$ and*

$$2\Re(1 - zT)^{-1} + A - 2 \geq 0, \quad \forall z \in \mathbb{D}.$$

Then, T is not invertible.

Proof. Arguing as in the proof of Theorem 2.5.17, one can show that T satisfies

$$2\Re(1 - zT)^{-1} + A - 2 \geq 0, \quad \forall z \in \mathbb{D}.$$

if and only if the mapping

$$S_A : \mathcal{A}(\mathbb{D}) \to \mathcal{B}(H)$$
$$f \mapsto \int_{\partial \mathbb{D}} \mu(\sigma, T) f(\sigma) ds + f(0)(A - 2)$$

is contractive. But then, we know (see e.g. [111, Chapters 2-3]) that S_A is contractive if and only if it has a completely positive extension to all of $C(\partial\mathbb{D})$. Let $\tilde{S}_A$ denote such an extension. The non-unital version of Stinespring's Theorem [111, Theorem 4.1] then implies the existence of a Hilbert space $K \supset H$, a unital $*$-homomorphism $\pi : C(\partial\mathbb{D}) \to \mathcal{B}(K)$ and a bounded operator $V : H \to K$ such that

$$\tilde{S}_A(f) = V^* \pi(f) V, \quad \forall f \in C(\partial\mathbb{D}). \tag{2.38}$$

Set $U = \pi(z)$. It is easy to see that U will then be a unitary operator. Setting $f \equiv 1$ in (2.38) gives us $A = V^*V$, thus the polar decomposition of V will be given by $YA^{1/2}$, where Y is some partial isometry. But observe also that putting $f = z$ in (2.38) gives us

$$T = A^{1/2} Y^* U Y A^{1/2}.$$

Since $A^{1/2}$ is not invertible, we conclude that T cannot be invertible. $\square$

Before we finish off the proof of Theorem 2.5.2, a few dilation-theoretic observations are in order. Assume $T \in \mathcal{B}(H)$, with $\sigma(T) \subset A_R$. It is then well-known that $T \in \mathbb{CDLA}_R(c)$ if and only if the mapping (2.32) is completely positive. Further, by Stinespring's Theorem, this is equivalent to the existence of a Hilbert space $K \supset H$ and a unital $*$-homomorphism $\pi : C(\partial A_R) \to \mathcal{B}(K)$ such that, for $f = \sum a_n z^n \in \mathcal{A}(A_R)$,

$$(2+c)S_{R,c}(f) = f(T) + (C\overline{f})(T)^* + ca_0 = (2+c)P_H \pi(f)|_H.$$

Set $\pi(z) = N$. Then, N will be a normal operator satisfying $\sigma(N) \subset \partial A_R$ and our previous equality becomes, for $f(z) = z^n$,

$$T^n + (C\overline{z^n})(T)^* = (2+c)P_H N^n|_H, \quad \forall n \neq 0.$$

After some computations, one verifies that

$$(C\overline{z^n})(\zeta) = \frac{1}{2\pi i}\int_{\partial A_R} \frac{\overline{z^n}}{z-\zeta}\, dz = R^{-2|n|}\zeta^{-n},$$

hence

$$T^n + R^{-2|n|} T^{-n*} = (2+c)P_H N^n|_H, \quad \forall n \neq 0. \tag{2.39}$$

On the other hand, Theorem 2.5.2 tells us that $T \in \mathbb{CDLA}_R(c)$ if and only if there exists $A \in \mathcal{B}(H)$ such that $A > 0,\ 2 + c - A > 0$ and $T/R \in \mathcal{C}_{2+c-A}$ and $T^{-1}/R \in \mathcal{C}_A$. In view of Definition 2.5.1, this is equivalent to the existence of a Hilbert space $K' \supset H$ and unitaries $U_1, U_{-1} \in \mathcal{B}(K')$ such that

$$R^{-n} T^n = (2+c-A)^{1/2} P_H U_1^n (2+c-A)^{1/2}|_H, \quad \forall n \geq 1, \tag{2.40}$$

and

$$R^{-n} T^{-n} = A^{1/2} P_H U_{-1}^n A^{1/2}|_H, \quad \forall n \geq 1. \tag{2.41}$$

Thus, the content of Theorem 2.5.2 is that 2.39 holds if and only if there exists $A \in \mathcal{B}(H)$ with $0 < A < 2 + c$ such that (2.40) and (2.41) hold. It would be of interest to find a direct proof of this assertion, using only the dilations U_1, U_{-1} and N.

Proof of Theorem 2.5.2. All that is left is to establish the converse of Lemma 2.5.20.

Accordingly, assume $T \in \mathbb{CDLA}_R(c)$, with $\sigma(T) \subset A_R$. By Arveson's Theorem, $S_{R,c}$ extends to a completely positive map $\Psi_{R,c} : C(\partial A_R) \to \mathcal{B}(H)$. Next, consider the mapping

$$\psi_{R,c} : C(\partial A_R) \to \mathcal{B}(H)$$

$$f \mapsto \frac{1}{2+c}\Big[\int_{\partial A_R} \mu(\sigma, T) f(\sigma) ds + \frac{c}{2\pi i} \int_{\Gamma_1} \frac{f}{\sigma} \sigma' ds \Big],$$

which is an alternate (not necessarily positive) extension of $S_{R,c}$. Since $||\mu(\sigma, T)||$ is uniformly bounded with respect to σ (because of the assumption $\sigma(T) \subset A_R$), we can estimate

$$\Big|\Big| \int_{\partial A_R} \mu(\sigma, T) f(\sigma) ds \Big|\Big| \leq ||f||_\infty \int_{\partial A_R} ||\mu(\sigma, T)|| ds \leq M ||f||_\infty, \quad \forall f \in C(\partial A_R).$$

Thus, $\psi_{R,c}$ is bounded. Observe also that both $\psi_{R,c}$ and $\Psi_{R,c}$ are actually extensions of the (completely positive) map $\tilde{S}_{R,c} : \mathcal{M}_R \to \mathbb{C}$ defined in the proof of Theorem 2.5.17. For $\psi_{R,c}$ this is obvious, while for $\Psi_{R,c}$ it holds because the completely contractive map $S_{R,c}$ has a unique completely positive extension to the closure of $\mathcal{A}(A_R) + \mathcal{A}(A_R)^*$.

Now, define $S : C(\partial A_R) \to \mathcal{B}(H)$ as $S = \psi_{R,c} - \Psi_{R,c}$. Fix an orthonormal basis $\{e_j\}$ of H and put

$$S_{ij} : C(\partial A_R) \to \mathbb{C}$$

$$f \mapsto \langle S(f) e_j, e_i \rangle.$$

S_{ij} will then be a bounded linear functional that vanishes on $\mathcal{M}_R$, which is a codimension one subspace of $C(\partial A_R)$. Hence, each S_{ij} lies in the one-dimensional annihilator $\mathrm{Ann}[\mathcal{M}_R] \subset \{L : C(\partial A_R) \to \mathbb{C} \text{ linear, bounded}\}$ of $\mathcal{M}_R$. But we also know that the (nonzero) map

$$\phi : C(\partial A_R) \to \mathbb{C}$$

$$f \mapsto \frac{1}{(2+c)} \frac{1}{2\pi i} \int_{\partial A_R} \frac{f(\zeta)}{\zeta} d\zeta$$

lies in Ann$[\mathcal{M}_R]$. Thus, for every i, j, there exists $k_{ij} \in \mathbb{C}$ such that $S_{ij} = k_{ij}\phi$. Define the (a priori unbounded) operator A acting on H by $\langle Ae_j, e_i \rangle = k_{ij}$, for all i, j, and let A_J denote its compression to $H_J := \text{span}\{e_j : j \in J\}$, where J is any finite subset of $\mathbb{N}$. Hence, we obtain

$$\frac{1}{2+c}\left[\frac{1}{2\pi i} \int_{\partial A_R} \frac{f(\zeta)}{\zeta} d\zeta\right] A_J = P_{H_J} S(f)|_{H_J}, \tag{2.42}$$

for every $f \in C(\partial A_R)$ and every J. Set $f = f_0$ in this last equality, where $f_0 \equiv 1$ on Γ_1 and $f_0 \equiv 0$ on Γ_{-1}. This gives us

$$A_J = (2+c) P_{H_J} S(f_0)|_{H_J},$$

for all J. Since $S(f_0) = \psi(f_0) - \Psi(f_0)$ is bounded and self-adjoint, we obtain that A is a bounded, self-adjoint operator. Also, in view of (2.42), we may deduce that

$$\Psi_{R,c}(f) = \psi_{R,c}(f) - S(f) = \int_{\partial A_R} \nu_{A,c}(\sigma, T) f(\sigma) ds$$

for every continuous f, where

$$\nu_{A,c}(\sigma, T) = \begin{cases} \mu(\sigma, T) + \frac{1}{2\pi i} \frac{\sigma'}{\sigma}(c - A), & \text{for } \sigma \in \Gamma_1, \\ \mu(\sigma, T) - \frac{1}{2\pi i} \frac{\sigma'}{\sigma}(A), & \text{for } \sigma \in \Gamma_{-1}. \end{cases}$$

Since $\Psi_{R,c}$ is (completely) positive on $C(\partial A_R)$, we easily obtain that
$\nu_{A,c}(\sigma, T) \geq 0$ for every $\sigma \in \partial A_R$, hence (as in the proof of Lemma 2.5.20) $T/R \in \mathcal{C}_{2+c-A}$ and $T^{-1}/R \in \mathcal{C}_A$. The fact that $\mathcal{C}_{2+c-A}$ and $\mathcal{C}_A$ are non-empty immediately implies $0 \leq A \leq 2 + c$ (see the remark after Lemma 2.15). But we also know that T is invertible, so Lemma 2.5.21 tells us that both A and $2 + c - A$ have to be invertible as well. This concludes the proof. □

We now drop the assumption $\sigma(T) \subset A_R$.

Theorem 2.5.22. *Assume $R > 1, c > -2$ and $T \in \mathcal{B}(H)$ satisfies $\sigma(T) \subset \overline{A_R}$. Then, there exists $A \in \mathcal{B}(H)$ such that $A > 0,\ 2 + c - A > 0$ and $T/R \in \mathcal{C}_{2+c-A}$ and $T^{-1}/R \in \mathcal{C}_A$ if and only if $T \in \mathbb{CDLA}_{R'}(c)$ for every $R' > R$.*

Proof. First, assume $T \in \mathbb{CDLA}_{R'}(c)$ for every $R' > R$ and fix a decreasing sequence $\epsilon_k \to 0$. Put $R_k = R + \epsilon_k$. Since $\sigma(T) \subset A_{R_k}$, Theorem 2.5.2 tells us that there exists a sequence $\{A_k\} \subset \mathcal{B}(H)$ such that $0 < A_k < 2 + c$ and $T/R_k \in \mathcal{C}_{2+c-A_k}$ and $T^{-1}/R_k \in \mathcal{C}_{A_k}$, for all $k \geq 1$. Now, from $T/R_k \in \mathcal{C}_{2+c-A_k}$ we get

$$\left\langle \left(2\Re(1 - zT/R_k)^{-1} + c - A_k\right)v, v\right\rangle \geq 0, \tag{2.43}$$

for all $z \in \mathbb{D}, v \in H$ and $k \geq 1$. Since $\{A_k\}$ is uniformly bounded, we may replace it, without loss of generality, by a WOT-convergent subsequence. Note also that $\sigma(zT/R_k) \subset \mathbb{D}$, for all z, k. Letting $k \to \infty$ in (2.43) (while keeping z and v fixed) is then easily seen to imply

$$\left\langle \left(2\Re(1 - zT/R)^{-1} + c - A\right)v, v\right\rangle \geq 0,$$

where A is the WOT limit of $\{A_k\}$ (notice that A has to be self-adjoint, being the WOT limit of self-adjoint operators). Since this last inequality holds for any $z \in \mathbb{D}$ and $v \in H$, we conclude that $T/R \in \mathcal{C}_{2+c-A}$, while an entirely analogous argument shows that $T^{-1}/R \in \mathcal{C}_A$. Finally, the fact that both $\mathcal{C}_{2+c-A}$ and $\mathcal{C}_A$ contain an invertible operator implies, as seen previously, that $0 < A < 2 + c$.

For the converse, observe that (in view of Lemma 2.15) having $T/R \in \mathcal{C}_{2+c-A}$ and $T^{-1}/R \in \mathcal{C}_A$ implies that $T/R' \in \mathcal{C}_{2+c-A}$ and $T^{-1}/R' \in \mathcal{C}_A$ for every $R' > R$. Since $\sigma(T) \subset A_{R'}$, Theorem 2.5.2 allows us to deduce that $T \in \mathbb{CDLA}_{R'}(c)$ for every $R' > R$. □

Now, we record the following analogue of Proposition 2.5.13 for $\mathbb{CDLA}_R(c)$. The proof is essentially an application of Proposition 2.5.13 combined with Lemma 2.5.20, so we omit the details.

Proposition 2.5.23. *Let $c \in \mathbb{R}$, $R > 1$ and assume $N \in \mathcal{B}(H)$ is normal.*

(i) If $c > -2$, then $\mathbb{CDLA}_R(c)$ will be non-empty if and only if there exists $s \in (0, 2+c)$ such that $\mathcal{C}_{2+c-s,s}(R)$ is non-empty.
In particular, if $c \geq 0$, then $\mathbb{CDLA}_R(c)$ will be non-empty for every $R > 1$.

(ii) If $c \leq -2$, then $\mathbb{CDLA}_R(c)$ will be empty for every $R > 1$.

(iii) $\mathbb{CDLA}_R(c) \subset \mathcal{C}_{2-c,2-c}(R)$, *for every* $c > -2$.

(iv) *If, in addition, we assume* $c \geq 0$, *then* $N \in \mathbb{CDLA}_R(c)$ *is equivalent to* $R^{-2} \leq N^*N \leq R^2$.

It is also worth noting that, like $\mathbb{DLA}_R(c)$, $\mathbb{CDLA}_R(c)$, is (eventually) strictly monotone with respect to c. This has essentially already been given to us by the proof of Theorem 2.5.15 plus Lemma 2.5.20.

Theorem 2.5.24. *If* $-2 < c < c'$ *and* $\mathbb{CDLA}_R(c')$ *is non-empty, then*

$$\mathbb{CDLA}_R(c) \subsetneq \mathbb{CDLA}_R(c').$$

Proof. If $c < c'$, it is obvious by the definition of $\mathbb{CDLA}_R(c)$ that $\mathbb{CDLA}_R(c) \subseteq \mathbb{CDLA}_R(c')$. To show that the inclusion is actually strict, we divide the proof into two cases, like with Theorem 2.5.15.

First, assume that there exist $s, t > 0$ such that $s \geq 1, s + t = 2 + c'$ and $\mathcal{C}_{s.t}(R)$ is non-empty. As in the proof of Theorem 2.5.15, we may assume that $t < 1$ and $\frac{2}{R^2+1} < t$. In this setting, we were able to construct $T \in \mathcal{C}_{s,t}(R) \subset \mathbb{CDLA}_R(c')$ such that $T \notin \mathbb{DLA}_R(c)$, hence also $T \notin \mathbb{CDLA}_R(c)$. We thus obtain strict inclusion.

On the other hand, assume that there do not exist $s, t > 0$ such that $s \geq 1, s+t = 2+c'$ and $\mathcal{C}_{s.t}(R)$ is non-empty. In this setting, we found $\delta > 0$ and $a \in \mathbb{C}$ such that $a \in \mathcal{C}_{\delta,2+c'-\delta}(R) \subset \mathbb{CDLA}_R(c')$, but $a \notin \mathbb{CDLA}_R(c)$, as desired. □

We end with a question. While the inclusion $\mathbb{CDLA}_R(c) \subseteq \mathbb{DLA}_R(c)$ is obvious, we have not been able to determine whether it is actually strict or not.

Question 2.5.25. Let $R > 1, c > -2$ and assume $\mathbb{CDLA}_R(c)$ is non-empty. Is it true that

$$\mathbb{CDLA}_R(c) \subsetneq \mathbb{DLA}_R(c)?$$

A negative answer to the above question would imply that, given $\sigma(T) \subset A_R$, having

$$2\Re\left[(1 - zT/R)^{-1} + (1 - wT^{-1}/R)^{-1}\right] - 2 + c \geq 0, \quad \forall z, w \in \mathbb{D},$$

is equivalent to the existence of $0 < A < 2 + c$ such that

$$2\Re(1 - zT/R)^{-1} + c - A \geq 0, \quad \forall z \in \mathbb{D},$$

and

$$2\Re(1 - wT^{-1}/R)^{-1} + A - 2 \geq 0, \quad \forall w \in \mathbb{D}.$$

While this seems unlikely to hold, the computational difficulty in verifying membership conditions of the form $T/R \in \mathcal{C}_{2+c-A}(R)$ and $T^{-1}/R \in \mathcal{C}_A(R)$, for arbitrary $0 < A < 2 + c$ and T non-normal, does not make it easy to come up with a counterexample.

2.5.6 *K*-spectral Estimates

In this subsection, we show how the methods established in [43] and [63] can be used to derive *K*-spectral estimates for $\mathbb{DLA}_R(c)$ and $\mathbb{CDLA}_R(c)$. We shall need a few preliminary lemmata, the scalar-valued versions of which are all contained in [63].

Lemma 2.5.26. *The map*

$$\alpha : \mathcal{A}(A_R) \to \mathcal{A}(A_R)$$
$$f \mapsto C\overline{f}$$

is completely contractive, for every $R > 1$.

Proof. By [63, Lemma 8], we know that $||\alpha(f)|| \leq ||f||$ whenever f is a scalar-valued rational function in A_R that is bounded by 1. A standard approximation argument shows that α must be contractive. The proof of [60, Lemma 2.1] then implies that α is completely contractive. □

Lemma 2.5.27. *Let $R > 1$ and $T \in \mathcal{B}(H)$ be such that $\sigma(T) \subset A_R$. Assume that there exists a bounded linear functional $\gamma : \mathcal{A}(A_R) \to \mathbb{C}$ and a constant $p > 0$ such that the mapping*

$$f \mapsto \frac{1}{2p}\big(f(T) + \alpha(f)(T)^* + \gamma(f)\big)$$

is completely contractive on $\mathcal{A}(A_R)$. Then, $\overline{A_R}$ is a complete K-spectral set for T with constant $K = p + \sqrt{1 + p^2 + ||\gamma||_{cb}}$.

Proof. In the setting of [63, Theorem 2], replace Ω by A_R, c_1 by 1 (this is possible because of Lemma 2.5.26), c_2 by p and $\hat{\gamma}$ by $||\gamma||_{\mathrm{cb}}$. While the proof of Theorem 2 in [63] was given in the scalar-valued setting, it can be repeated, mutatis mutandis, in the matrix-valued setting to give the exact same K-spectral estimate (see also Remark (i) after the proof of Theorem 1.1 in [53]), where

$$K = c_2 + \sqrt{c_2^2 + c_1 + \hat{\gamma}} = p + \sqrt{1 + p^2 + ||\gamma||_{\mathrm{cb}}}.$$

□

We are now in a position to show:

Theorem 2.5.28. *Let* $c > -2$ *and assume* $T \in \mathbb{DLA}_R(c)$ *(resp.* $T \in \mathbb{CDLA}_R(c)$*). Then,* $\overline{A_R}$ *will be a K-spectral (resp. complete K-spectral) set for T, where*

$$K = 1 + \frac{c}{2} + \sqrt{\left(1 + \frac{c}{2}\right)^2 + 1 + |c|}.$$

Proof. Let $T \in \mathbb{CDLA}_R(c)$ (the case $T \in \mathbb{DLA}_R(c)$ is essentially contained in [63, Theorem 2]). By assumption, the mapping

$$f \mapsto \frac{1}{2+c}\big(f(T) + \alpha(f)(T)^* + ca_0\big)$$

is completely contractive on $\mathcal{A}(A_R)$. If $\gamma : \mathcal{A}(A_R) \to \mathbb{C}$ is given by $\gamma(\sum a_n z^n) = ca_0$, it can be easily verified that $||\gamma||_{\mathrm{cb}} = |c|$. Thus, one can apply Lemma 2.5.27 with $p = 1 + c/2$ and $\gamma(f) = ca_0$ to deduce the desired result. □

Remark 2.5.29. Let $\mathbb{NA}_R$ denote the *numerical annulus*, i.e. the class of all $T \in \mathcal{B}(H)$ such that $w(T) \leq R$ and $w(T^{-1}) \leq R$. K-spectral estimates for $\mathbb{NA}_R$ have been studied in [59] and, more recently, in [63]. Since

$$\mathbb{NA}_R \equiv \mathcal{C}_{2,2}(R) \subset \mathbb{DLA}_R(2),$$

Theorem 2.5.28 tells us that $\overline{A_R}$ is a $(2 + \sqrt{7})$-spectral set for T whenever $T \in \mathbb{NA}_R$. This improves on the estimates from [63, Section 6].

Remark 2.5.30. The main result of [125] can be used to obtain sharper spectral estimates in certain cases. Indeed, let $T \in \mathbb{DLA}_R(c)$ be a matrix with $\sigma(T) \subset A_R$. In the setting of [125, Theorem 5], choose $A = \mathcal{A}(A_R)$ and set $\gamma(f) = f(T)$ and $\Phi(f) = C\overline{f}$. Since $c \geq 0$, we have $\mathbb{QA}_R \subset \mathbb{DLA}_R(c)$, which implies that $||\gamma|| \geq 2$ (see [141]). Now, assume, in addition, the existence of an *extremal pair* $(f_0, x_0) \in \mathcal{A}(A_R) \times H$ for γ (see [125, p. 2]). We have $||f_0|| = ||x_0|| = 1$ and $||\gamma|| = ||\gamma(f_0)x_0||$. Also, there exists an *extremal measure* associated with (f_0, x_0) (see [125, Proposition 3] and the discussion afterwards). This observation, combined with the contractivity of Φ (Lemma 2.5.26), allows us to deduce that $|\langle\gamma(\Phi(f_0)f_0)x_0, x_0\rangle| \leq 1$, see e.g. the proof of [125, Theorem 11]). Thus, if we define ω in the dual of $\mathcal{A}(A_R)$ as $\omega(\sum_n a_n z^n) = -ca_0$, [125, Theorem 5] implies that

$$\begin{aligned} ||\gamma|| &\leq \frac{1}{2}||\gamma_\Phi - \omega|| + \sqrt{\left(\frac{1}{2}||\gamma_\Phi - \omega||\right)^2 + |\langle\gamma(\Phi(f_0)f_0)x_0, x_0\rangle|} \\ &\leq \frac{2+c}{2}||S_{R,c}|| + \sqrt{\left(\frac{2+c}{2}||S_{R,c}||\right)^2 + 1} \\ &\leq 1 + \frac{c}{2} + \sqrt{\left(1 + \frac{c}{2}\right)^2 + 1}, \end{aligned}$$

which gives us the sharper constant $K' = 1 + \frac{c}{2} + \sqrt{\left(1 + \frac{c}{2}\right)^2 + 1}$. Note that if T has distinct eigenvalues, then the existence of an extremal function f_0 (that extends continuously to the boundary) can be obtained as in the proof of [62, Theorem 2.1] (see [98, Section 3] and [70, Exercise 5, p.162] for the structure of solutions to extremal Pick problems over the annulus). While the authors deem it very likely that the extremal function continues to enjoy boundary continuity even in mixed Carathéodory-Pick interpolation problems (corresponding to the general case where one might have repeated eigenvalues), they are not aware of any reference that describes the properties of extremal functions for such problems.

We now show Theorem 2.5.3 from the introduction, which offers improved K-spectral estimates for 2×2 matrices with a single eigenvalue. The key ingredient of the proof will be a function-theoretic result from [100]. To more easily connect with the setting of that paper, we will work with the annulus,

$$\mathcal{A}_q := \{q < |z| < 1\},$$

which is conformally equivalent to $A_{1/\sqrt{q}}$. The definition of $\mathbb{DLA}_R(c)$ can then be updated as follows:

Definition 2.5.31. Let $c \in \mathbb{R}$ and $0 < q < 1$. $\mathcal{DLA}_q(c)$ denotes the class of all operators $T \in \mathcal{B}(H)$ such that

(i) $\sigma(T) \subset \overline{\mathcal{A}_q}$ and

(ii) $2\Re\big[(1 - zT)^{-1} + (1 - wqT^{-1})^{-1}\big] - 2 + c \geq 0, \quad \forall z, w \in \mathbb{D}$.

Now, for $w \in \mathbb{D}$ and $a \in \mathcal{A}_q$, define $\psi_w(z) = \frac{z-w}{1-\overline{w}z}$ and

$$\mathcal{F}_{a,w} = \{f : \overline{\mathcal{A}_q} \to \mathbb{D} \mid f \text{ analytic and } f(a) = w\}.$$

Lemma 2.5.32. *Let $w \in \mathbb{D}$ and $a \in \mathcal{A}_q$. Then,*

$$\sup\{|f'(a)| \mid f \in \mathcal{F}_{a,w}\} \leq (1 - |w|^2)\Big(\frac{1}{1 - |a|^2} + \frac{q}{|a|^2 - q^2}\Big).$$

Proof. Put $k^q(a, a) = \sum_{n\in\mathbb{Z}} \frac{|a|^{2n}}{1+q^{2n+1}}$. The solution to the above extremal problem for $w = 0$ can be found in [100, p. 1119]. In particular, it is known that

$$\sup\{|f'(a)| \mid f \in \mathcal{F}_{a,0}\} = k^q(a, a).$$

Now, if $h \in \mathcal{F}_{a,w}$, it can be easily verified that $\psi_w \circ h \in \mathcal{F}_{a,0}$, hence

$$\frac{|h'(a)|}{1 - |w|^2} = |(\psi_w \circ h)'(a)| \leq k^q(a, a).$$

Since $h \in \mathcal{F}_{a,w}$ was arbitrary, we can deduce that

$$\begin{aligned}
\sup\{|f'(a)| \mid f \in \mathcal{F}_{a,w}\} &\leq (1 - |w|^2)k^q(a, a) \\
&= (1 - |w|^2)\Big(\sum_{n=0}^{\infty} \frac{|a|^{2n}}{1 + q^{2n+1}} + \sum_{n=-1}^{-\infty} \frac{|a|^{2n}}{1 + q^{2n+1}}\Big) \\
&\leq (1 - |w|^2)\Big(\sum_{n=0}^{\infty} |a|^{2n} + \frac{1}{q}\sum_{n=1}^{\infty} \frac{q^{2n}}{|a|^{2n}}\Big) \\
&= (1 - |w|^2)\Big(\frac{1}{1 - |a|^2} + \frac{q}{|a|^2 - q^2}\Big).
\end{aligned}$$

□

We also require the following computational lemmata.

Lemma 2.5.33. *Let $a, u \in \mathbb{C}$ and assume $T = \begin{pmatrix} a & u \\ 0 & a \end{pmatrix} \in \mathcal{DLA}_q(c)$ with $q < |a| < 1$. Then,*

$$\left|\frac{e^{i\theta}}{(1-ae^{i\theta})^2} - \frac{qe^{i\psi}}{a^2(1-qe^{i\psi}a^{-1})^2}\right||u| \leq 2\Re\left(\frac{1}{1-ae^{i\theta}} + \frac{1}{1-qe^{i\psi}a^{-1}}\right) + c - 2,$$

for all θ and ψ.

Proof. The proof of Lemma 2.5.14 carries over to the $\mathcal{DLA}_q(c)$ setting. Thus, the fact that $T \in \mathcal{DLA}_q(c)$ and $\sigma(T) = \{a\} \subset \mathcal{A}_q$ allows us to deduce

$$2\Re\left[(1-e^{i\theta}T)^{-1} + (1-e^{i\psi}qT^{-1})^{-1}\right] - 2 + c \geq 0, \quad \forall \theta, \psi \in [0, 2\pi).$$

Taking determinants then leads to the desired inequality. □

Lemma 2.5.34. *Let $C > 0$ and $w \in \mathbb{D}$. Then,*

$$\left\|\begin{pmatrix} w & C(1-|w|^2) \\ 0 & w \end{pmatrix}\right\| \leq \max\{1, C\}.$$

Proof. Given any matrix of the form $P = \begin{pmatrix} e & f \\ 0 & e \end{pmatrix}$, it is well-known that $||P|| \leq 1$ if and only if $|f| \leq 1 - |e|^2$. Thus, if $C \leq 1$, we immediately obtain that

$$\left\|\begin{pmatrix} w & C(1-|w|^2) \\ 0 & w \end{pmatrix}\right\| \leq 1.$$

Now, assume $C > 1$. Note that

$$C^2 - \begin{pmatrix} w & C(1-|w|^2) \\ 0 & w \end{pmatrix}\begin{pmatrix} w & C(1-|w|^2) \\ 0 & w \end{pmatrix}^*$$

$$= \begin{pmatrix} -|w|^2 + C^2(2-|w|^2)|w|^2 & -C\overline{w}(1-|w|^2) \\ -Cw(1-|w|^2) & C^2 - |w|^2 \end{pmatrix} \geq 0,$$

since the $(1,1)$-entry of this last matrix is clearly positive, while its determinant is equal to

$$|w|^2(C^2-|w|^2)(2C^2-C^2|w|^2-1)-C^2|w|^2(1-|w|^2)^2$$

$$=|w|^2(C^2-1)(2C^2-|w|^2(1+C^2))>0.$$

This concludes the proof. □

We are now prepared for the main result of this subsection.

Proof of Theorem 2.5.3. First, we will prove the theorem in the setting of $\mathcal{A}_q$.

Let $T \in \mathcal{DLA}_q(c)$ be a 2×2 matrix with a single eigenvalue. Since unitary equivalence respects K-spectral estimates (see e.g. [28, Example 4, p. 107-5]), we may assume that T is of the form $\begin{pmatrix} a & u \\ 0 & a \end{pmatrix}$. We may also take $a>0$ (as $\mathcal{A}_q$ is invariant under rotations). Finally, it suffices to work with $q<a<1$ (the general case follows by a standard approximation argument).

Now, let $f : \overline{\mathcal{A}_q} \to \mathbb{D}$ be analytic. We may write

$$||f(T)|| = \left|\left|\begin{pmatrix} f(a) & f'(a)u \\ 0 & f(a) \end{pmatrix}\right|\right|$$

$$= \left|\left|\begin{pmatrix} f(a) & |f'(a)u| \\ 0 & f(a) \end{pmatrix}\right|\right|$$

$$= \left|\left|\begin{pmatrix} f(a) & \frac{|f'(a)u|}{1-|f(a)|^2}(1-|f(a)|^2) \\ 0 & f(a) \end{pmatrix}\right|\right|.$$

If $\frac{|f'(a)u|}{1-|f(a)|^2} \le 1$, Lemma 2.5.34 gives us $||f(T)|| \le 1$, which is stronger than the desired estimate. On the other hand, if $\frac{|f'(a)u|}{1-|f(a)|^2} > 1$, Lemmata 2.5.32 and 2.5.34 imply that

$$||f(T)|| = \left|\left|\begin{pmatrix} f(a) & \frac{|f'(a)u|}{1-|f(a)|^2}(1-|f(a)|^2) \\ 0 & f(a) \end{pmatrix}\right|\right|$$

$$\le \frac{|f'(a)|}{1-|f(a)|^2}|u|$$

$$\leq \left(\frac{1}{1-a^2} + \frac{q}{a^2-q^2}\right)|u|.$$

Assume that $a \geq \sqrt{q}$. In Lemma 2.5.33, take $\theta = 0, \psi = \pi$. The resulting bound on $|u|$ then allows us to write:

$$||f(T)||$$
$$\leq \left(\frac{1}{1-a^2} + \frac{q}{a^2-q^2}\right)\left(\frac{1}{(1-a)^2} + \frac{q}{a^2(1+q/a)^2}\right)^{-1}\left(\frac{2}{1-a} + \frac{2}{1+q/a} + c - 2\right)$$
$$\leq 2 + c\frac{1-q}{1+q},$$

where the last inequality can be seen (after some computations) to be equivalent to $a \geq \sqrt{q}$. This concludes the proof in this case. If $a < \sqrt{q}$, one can choose $\theta = \pi$ and $\psi = 0$ in the above calculation and argue in an analogous manner. Thus, we have shown that $\overline{\mathcal{A}_q}$ is a K_q-spectral for $T \in \mathcal{DLA}_q(c)$ whenever T is a 2×2 matrix with a single eigenvalue, where $K_q = 2 + c\frac{1-q}{1+q}$.

We now convert this estimate to the A_R-setting. Assume $T \in \mathbb{DLA}_R(c)$ is a 2×2 matrix with a single eigenvalue and set $q = R^{-2}$. Then, $\tilde{T} := T/R \in \mathcal{DLA}_q(c)$ and is also, evidently, still a 2×2 matrix with a single eigenvalue. In view of our previous result, $\overline{\mathcal{A}_q}$ will be a K-spectral set for $\tilde{T}$, where

$$K = 2 + c\frac{1-q}{1+q} = 2 + c\frac{R^2-1}{R^2+1}.$$

This is easily seen to imply (see e.g. [28, Fact 2, p. 107-3]) that $\overline{A_R}$ is a K-spectral set for T, which concludes our proof. □

Chapter 3

Denjoy-Wolff Points on the Bidisk

The material contained in this chapter originates in the following paper:

Paper VI M. T. Jury and G. Tsikalas. "Denjoy-Wolff points on the bidisk via models". In: *Integral Equations Operator Theory* (to appear)

3.1 Introduction

Let $\mathbb{D}$ denote the open unit disk. Given a holomorphic map $f : \mathbb{D} \to \mathbb{D}$ without fixed points, a theorem of Wolff [148] states that there exists a boundary point $\tau \in \partial\mathbb{D}$ such that every closed disk internally tangent to $\mathbb{D}$ at τ (in other words, every horocycle containing τ) is invariant under f. From this, one can deduce the classical Denjoy-Wolff Theorem [67], [146], [147]: the sequence of iterates

$$f^n := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$$

converges to τ uniformly on compact subset of $\mathbb{D}$. In this setting, the (unique) point τ is termed the *Denjoy-Wolff point* of f. See [42] for a nice exposition of the details and many historical remarks.

A lot of work has been devoted to obtaining higher-dimensional generalizations of the Denjoy-Wolff Theorem. The first such result is due to Hervé [81], who proved an exact analogue of the Denjoy-Wolff Theorem for fixed-point-free self-maps of the unit ball $\mathbb{B}_n \subset \mathbb{C}^n$ (see also [90]). Later, Abate [1] (see also the excellent survey [2]) achieved a generalization of this result to all smoothly bounded strongly convex domains in $\mathbb{C}^n$, paving the way for further extensions to smoothly bounded pseudoconvex domains of both finite and infinite type (see [89] and the references therein). More recently, Budzyńska [40] (see also [39] and [41]) showed that the smoothness assumption can be dropped if one restricts to strictly convex domains.

Unfortunately, the situation becomes considerably more complicated in general bounded domains. The proofs of the above results utilize certain f-invariant domains (usually termed *horospheres*, as they generalize Wolff's horocycles) which may have too large intersections with the boundary of the domain in the general case, making it difficult to control the behavior of the iterates. Indeed, even though several different types of horospheres have been considered in the literature with varying degrees of generality (see e.g. [2], [4], [40], [51], [72], [102], where the focus is either on bounded convex or bounded symmetric domains), boundary smoothness or extra convexity assumptions (or a mixture of both) are generally required to control the size of the intersection with the boundary. This is true even in very simple finite-dimensional domains, such the unit polydisk $\mathbb{D}^n$, where the presence (for $n \geq 2$) of large "flat" boundary components prevents the iterates from converging. In such a case, one seeks to understand the cluster points of $\{f^n\}$. Although holomorphic dynamics on $\mathbb{D}^n$ (for general n) have been studied by a number of authors (see e.g. [3],[4], [25], [52], [72], [103]), progress on iteration-theoretic questions remains limited.

Somewhat stronger conclusions can be drawn if one restricts their attention to the bidisk. Let $F = (\phi, \psi) : \mathbb{D}^2 \to \mathbb{D}^2$ be holomorphic and without fixed points. The best known general results regarding the behavior of the iterates $\{F^n\}$ in this setting can be found in the classical paper [80] of Hervé (see also [72], [74], [107], [134] for more recent work concerning the bidisk). Hervé observed that all holomorphic maps $\phi : \mathbb{D}^2 \to \mathbb{D}$ (that are not coordinate projections) can be classified into two separate categories (see Definition 3.2.6) based on the location of the Denjoy-Wolff points of the slice functions $\phi_\mu : \mathbb{D} \to \mathbb{D}$, where $\phi_\mu(\lambda) = \phi(\lambda, \mu)$ for all $\lambda, \mu \in \mathbb{D}$. He then gave a description of the cluster points of $\{F^n\}$ by considering three distinct cases (see Theorem 3.2.7), depending on the categories that the coordinate functions ϕ and ψ belong to. [80] also contains numerous examples demonstrating that, from a certain viewpoint, these results are optimal.

In this chapter, motivated by the model-theoretic techniques of [12] and [124], we propose new definitions for Denjoy-Wolff-type points of holomorphic functions $\phi : \mathbb{D}^2 \to \mathbb{D}$ (see Definition 3.2.8). These will be boundary points where ϕ satisfies a mild regularity condition (termed B-points following [12], see Section 3.2 for definitions) and appropriate contractivity assumptions stated in terms of the model function. We prove several equivalent characterizations of our Denjoy-Wolff points, some of which are particularly easy to verify in practice and involve certain directional derivatives of ϕ at the points in question (see Theorems 3.2.9, 3.2.10 and 3.2.11). This constitutes a departure from the usual criteria for Denjoy-Wolff points used in the setting of $\mathbb{D}^2$, which depend on the existence of invariant horospheres. With these tools in our disposal, we are able to refine Hervé's theorem. Among several results, we show that if the coordinate functions ϕ and ψ of F possess certain Denjoy-Wolff points but don't have angular gradients there (i.e. the points in question are B- but not C-points), then one gains much tighter control over the behavior of the iterates $\{F^n\}$(see Theorems 3.2.12 and 3.2.13). Roughly, this is because the structure of the model function at Denjoy-Wolff points that are not C-points allows one to deduce many different (contractive) versions of Julia's inequality there, thus increasing the supply of invariant horospheres available (see Corollaries 3.4.5 and 3.4.9). We also provide examples to illustrate the different cases contained in our theorems.

Our work is arranged as follows. Section 3.2 contains the necessary background on the notions of a model of a function, B-points and C-points and the main result of [80]. It also presents our new definitions of Denjoy-Wolff points and the main results of this paper. In Section 3.3, we prove general results concerning the relation between the model function and certain directional derivatives at B-points, as well as a refined version of Julia's inequality for the bidisk (see Theorem 3.3.10). These will be much needed in the sequel but are also of independent interest. In Section 3.4, we prove several equivalent characterizations of our Denjoy-Wolff points (see Theorems 3.2.9, 3.2.10 and 3.2.11), uniqueness results (Propositions 3.4.4 and 3.4.8) and useful corollaries involving weighted Julia inequalities (Corollaries 3.4.5 and 3.4.9). Next, in Section 3.5, we revisit Hervé's Theorem and establish several partial refinements using our tools from the previous sections. These refinements include Theorems 3.2.12, 3.2.13 and 3.5.2. We also provide relevant examples (see Examples 3.5.2 and 3.5.3). Finally, in Section 3.6, we discuss Frosini's work on Denjoy-Wolff-type points on the bidisk and show how our main results can be used to recover a theorem from [72] on the classification of a certain type of these points.

3.2 Background and Main Results

3.2.1 Models

Let $\mathcal{S}$ and $\mathcal{S}_2$ denote the one- and two-variable *Schur classes*, i.e. the sets of analytic functions on $\mathbb{D}$ and $\mathbb{D}^2$ respectively that are bounded by 1 in modulus. We require the notion of a model of a Schur-class function, as seen in [12]. It is well known that every function in $\mathcal{S}_2$ possesses such a model, however this ceases to be the case in higher-dimensional polydisks (this is a consequence of the fact that von Neumann's inequality fails in more than two variables, see also [14, Section 9.7]).

Definition 3.2.1. Let $\phi \in \mathcal{S}_2$. We say that (M, u) is a *model* for ϕ if $M = M^1 \oplus M^2$ is an orthogonally decomposed separable Hilbert space and $u : \mathbb{D}^2 \to M$ is an analytic map such that, for all $\lambda = (\lambda^1, \lambda^2), \mu = (\mu^1, \mu^2) \in \mathbb{D}^2$,

$$1 - \phi(\lambda)\overline{\phi(\mu)} = (1 - \lambda^1\overline{\mu^1})\langle u^1_\lambda, u^1_\mu\rangle + (1 - \lambda^2\overline{\mu^2})\langle u^2_\lambda, u^2_\mu\rangle. \tag{3.1}$$

In equation (3.1) we have written u_λ for $u(\lambda), u^1(\lambda) = P_{M^1}u(\lambda)$, and $u^2(\lambda) = P_{M^2}u(\lambda)$. In general, given $v \in M$, we will write v^1 for $P_{M^1}v$ and v^2 for $P_{M^2}v$. Note that we may suppose, without loss of generality, that $\{u^j(\lambda) : \lambda \in \mathbb{D}^2\}$ spans a dense subspace of M^j , since otherwise we may replace M^j by this span. However, it needn't be true that $\{u(\lambda) : \lambda \in \mathbb{D}^2\}$ spans a dense subspace of M (these observations can be found in [12, Section 3]).

3.2.2 B-points and C-points

If $S \subset \mathbb{D}^2$ and $\tau \in \partial\mathbb{D}^2$, we say that S *approaches* τ *nontangentially* if $\tau \in \mathrm{cl}(S)$ (where $\mathrm{cl}(S)$ denotes the topological closure of S) and there exists a constant $c > 0$ such that

$$||\tau - \lambda|| \leq c(1 - ||\lambda||), \tag{3.2}$$

for all $\lambda \in S$, where $||(\lambda^1, \lambda^2)|| = \max\{|\lambda^1|, |\lambda^2|\}$.

Now, let $\phi \in \mathcal{S}_2$ and $\tau \in \partial\mathbb{D}^2$. τ is said to be a *B-point for* ϕ if the Carathéodory condition

$$\liminf_{\lambda \to \tau} \frac{1 - ||\phi(\lambda)||}{1 - ||\lambda||} < \infty \tag{3.3}$$

holds. The nontangential limit of ϕ at any such τ always exists [3] and will be denoted by $\phi(\tau)$.

While in one variable the Julia-Carathéodory Theorem [44] tells us that a function in S has an angular derivative at any B-point τ, a function $\phi \in \mathcal{S}_2$ does not necessarily have an angular gradient at all of its B-points. If ϕ does have an angular gradient at τ, we will say that τ is a *C-point for* ϕ. In any case, ϕ will always have a directional derivative at a B-point in any direction pointing into the bidisk. Moreover, as was shown in [12], the directional derivatives in question will vary holomorphically with respect to direction (actually, the derivatives can be described in terms of certain one-variable *Pick functions* [9], though we won't be needing this result here).

To state the relevant theorems, we need some notation. Let (M, u) be a model for $\phi \in \mathcal{S}_2$ and define the *nontangential cluster set* X_τ of the model at a B-point τ of ϕ to be the set of weak limits of weakly convergent sequences $\{u_{\lambda_n}\}$ over all sequences $\{\lambda_n\}$ that converge nontangentially to τ in $\mathbb{D}^2$. X_τ turns out to be a subset of the *cluster set* of (M, u) at τ, which is defined as the set of limits in M of the weakly convergent sequences $\{u_{\lambda_n}\}$ as $\{\lambda_n\}$ ranges over all sequences in $\mathbb{D}^2$ that tend to τ in such a way that

$$\frac{1 - |\phi(\lambda_n)|}{1 - ||\lambda_n||} \tag{3.4}$$

remains bounded. The cluster set at τ will be denoted by Y_τ. Also, let $\mathbb{H} = \{z \in \mathbb{C} : \Re z > 0\}$, $\mathbb{T} = \partial\mathbb{D}$

and define, for every $\tau \in \partial\mathbb{D}^2$,

$$\mathbb{H}(\tau) = \begin{cases} \tau^1\mathbb{H} \times \tau^2\mathbb{H} & \text{if } \tau \in \mathbb{T}^2, \\ \tau^1\mathbb{H} \times \mathbb{C} & \text{if } \tau \in \mathbb{T} \times \mathbb{D}, \\ \mathbb{C} \times \tau^2\mathbb{H} & \text{if } \tau \in \mathbb{D} \times \mathbb{T}. \end{cases}$$

For the remainder of this subsection, fix a function $\phi \in \mathcal{S}_2$ with model (M, u) and a B-point $\tau \in \partial\mathbb{D}^2$. The next lemma can be easily obtained from (3.1).

Lemma 3.2.2 (see [12], Proposition 4.2)**.** *We have $X_\tau \neq \emptyset$. Moreover, for all $x \in Y_\tau$ and $\lambda \in \mathbb{D}^2$,*

$$1 - \phi(\lambda)\overline{\phi(\tau)} = \sum_{|\tau^j|=1} (1 - \lambda^j\overline{\tau^j})\langle u_\lambda^j, x^j\rangle. \tag{3.5}$$

As a consequence, we obtain:

Lemma 3.2.3 (see [12], Lemma 8.10)**.** *If $|\tau^j| < 1$ for $j = 1$ or 2, then*

$$Y_\tau = \{u_\tau\}, \quad \textit{where } u_\tau^j = 0.$$

A consequence of the following theorem is that facial B-points are always C-points (see [13] for more results in that direction).

Theorem 3.2.4 (see [12], Corollary 8.11)**.** *τ is a C-point for ϕ if and only if X_τ is a singleton set.*

Now, since τ is a B-point for ϕ, we know that for every $\delta \in \mathbb{H}(\tau)$ the directional derivative

$$D_{-\delta}\phi(\tau) = \lim_{t \to 0+} \frac{\phi(\tau - t\delta) - \phi(\tau)}{t}$$

exists. Much more can be said.

Theorem 3.2.5 (see [12], Theorems 7.1, 7.8)**.** *For any $\delta \in \mathbb{H}(\tau)$, the nontangential limit (in the norm of M)*

$$x_\tau(\delta) = \lim_{\tau - z\delta \xrightarrow{nt} \tau} u_{\tau - z\delta}$$

exists in M. In addition,

(1) $x_\tau(\cdot)$ *is a holomorphic M-valued function on* $\mathbb{H}(\tau)$*;*

(2) $x_\tau(\delta) \in X_\tau$ *for all* $\delta \in \mathbb{H}(\tau)$*;*

(3) $x_\tau(z\delta) = x_\tau(\delta)$ *for all* $z \in \mathbb{C}$ *such that* $\delta, z\delta \in \mathbb{H}(\tau)$ *(i.e.* $x_\tau(\cdot)$ *is homogeneous of degree* 0 *in* δ*);*

(4) $D_{-\delta}\phi(\tau)$ *is analytic, homogeneous of degree* 1 *in* δ *and satisfies*

$$D_{-\delta}\phi(\tau) = -\phi(\tau) \sum_{|\tau^j|=1} \overline{\tau^j}\delta^j ||x_\tau^j(\delta)||^2.$$

3.2.3 Horocycles and Horospheres

The language of horospheres and horocycles will be required for our iteration-theoretic results. Recall that a *horocycle* in $\mathbb{D}$ is a set of the form $E(\tau, R)$ for some $\tau \in \mathrm{cl}(\mathbb{D})$ and $R > 0$, where

$$E(\tau, R) = \left\{\lambda \in \mathbb{D} : \frac{|\lambda - \tau|^2}{1 - |\lambda|^2} < R\right\}$$

for $\tau \in \mathbb{T}$, while $E(\tau, R) = \mathbb{D}$ otherwise. Letting $D(z, r)$ denote the Euclidean disk in $\mathbb{C}$ with centre z and radius $r > 0$, it is not hard to see that, given any $\tau \in \mathbb{T}$, we always have

$$E(\tau, R) = D\left(\frac{\tau}{R+1}, \frac{R}{R+1}\right).$$

Also, for $\tau = (\tau^1, \tau^2) \in \partial\mathbb{D}^2$ and $R_1, R_2 > 0$, we define the (weighted) *horosphere* $E(\tau, R_1, R_2)$ to be the set $E(\tau^1, R_1) \times E(\tau^2, R_2)$.

Now, given $\phi \in \mathcal{S}$ and a B-point $\tau \in \mathbb{T}$, it is known that

$$\alpha := \lim_{\substack{\lambda \xrightarrow{\text{nt}} \tau}} \frac{1 - |\phi(\lambda)|}{1 - |\lambda|} \geq 0$$

exists. Julia's inequality [44], [84] (see also the more modern [132]) then states that

$$\phi\big(E(\tau, R)\big) \subset E(\phi(\tau), \alpha R), \tag{3.6}$$

for all $R > 0$. Generalizations of this result to the bidisk are contained in [2] and [145] (see also [12, Section 4] for a model-theoretic proof). In particular, given $\phi \in \mathcal{S}_2$ and a B-point $\tau \in \partial\mathbb{D}^2$, it is known that, for any $\alpha \geq 0$, we have

$$\liminf_{\lambda \to \tau} \frac{1 - |\phi(\lambda)|}{1 - ||\lambda||} \leq \alpha$$

if and only if

$$\phi(E(\tau, R, R)) \subset E(\phi(\tau), \alpha R), \tag{3.7}$$

for all $R > 0$ (if $\alpha = 0$, then ϕ is constant). In Section 3.3, we use ideas from [12] to establish a refined version of the previous equivalence, one that is expressed in terms of weighted horospheres (see Theorem 3.3.10).

Lastly, we will occasionally be making use of the *horospheric topology* on $\mathrm{cl}(\mathbb{D}^2)$, which is the topology with base consisting of all open sets of $\mathbb{D}^2$ together with all sets of the form $\{\tau\} \cup E(\tau, R_1, R_2)$, where $\tau \in \partial\mathbb{D}^2$ and $R_1, R_2 > 0$ (see [12, Section 4] for more details). Note that (3.7) tells us that $\phi(\lambda) \to \phi(\tau)$ whenever τ is a B-point and $\lambda \to \tau$ horospherically.

3.2.4 Hervé's Result

For $i \in \{1, 2\}$, define the coordinate projections $\pi^i : \mathbb{D}^2 \to \mathbb{D}, \pi^i(\lambda) = \lambda^i$. Given $\phi \in \mathcal{S}_2$ and $\mu \in \mathbb{D}$, we will denote by $\phi_\mu \in \mathcal{S}$ the slice function

$$\phi_\mu(\lambda) = \phi(\lambda, \mu) \quad (\lambda \in \mathbb{D}).$$

Also, we let $\widetilde{\phi} \in \mathcal{S}_2$ denote the function $\widetilde{\phi}(\lambda) = \phi(\lambda^2, \lambda^1)$, for obtained from ϕ by interchanging the arguments.

Holomorphic functions $\phi : \mathbb{D}^2 \to \mathbb{D}$ can be classified according to the Denjoy-Wolff points of their slices.

Definition 3.2.6. Assume $\phi \in \mathcal{S}_2$. ϕ is said to be a:

(i) *left Type I* function if $\phi \neq \pi^1$ and there exists $\tau^1 \in \mathbb{T}$ such that τ^1 is the common Denjoy-Wolff point of the maps $\phi_\mu \in \mathcal{S}$, for all $\mu \in \mathbb{D}$;

(ii) *right Type I* function if $\widetilde{\phi}$ is a left Type I function;

(iii) *left Type II* function if $\phi \neq \pi^1$ and there exists a holomorphic map $\xi : \mathbb{D} \to \mathbb{D}$ such that, for all $\lambda, \mu \in \mathbb{D}$, we have $\phi_\mu(\lambda) = \lambda$ if and only if $\xi(\mu) = \lambda$;

(iv) *right Type II* function if $\widetilde{\phi}$ is a left Type II function.

Surprisingly, it turns out that any $\phi \in \mathcal{S}_2$ that is not a coordinate projection will either be a left Type I or a left Type II function (respectively, either a right Type I or a right Type II function), a result originally proved by Hervé in [80]. In Section 3.4, we give a new proof of this using purely model-theoretic methods (see Theorem 3.4.3).

Using the Type I/Type II terminology, the main result of [80] can be stated as follows.

Theorem 3.2.7 (Hervé)**.** *Let $F = (\phi, \psi) : \mathbb{D}^2 \to \mathbb{D}^2$ be a holomorphic self-map of the bidisk without fixed points. Then, one and only one of the following cases occurs:*

(i) if $\psi \equiv \pi^2$ (respectively, $\phi \equiv \pi^1$), then $\{F^n\}$ converges uniformly on compact sets to (τ^1, π^2), where $\tau^1 \in \mathbb{T}$ (respectively, to (π^1, τ^2), where $\tau^2 \in \mathbb{T}$);

(ii) if ϕ is a left Type I and ψ is right Type I function, then there exist $\tau^1, \tau^2 \in \mathbb{T}$ such that

(a) either every cluster point of $\{F^n\}$ has the form (τ^1, h), where h is either a holomorphic function $\mathbb{D}^2 \to \mathbb{D}$ or the constant τ^2,

(b) or every cluster point of $\{F^n\}$ has the form (g, τ^2), where g is either a holomorphic function $\mathbb{D}^2 \to \mathbb{D}$ or the constant τ^1;

(iii) if ϕ is a left Type I function and ψ is a right Type II function (respectively, ϕ is a left Type II function and ψ is a right Type I function), there exists $\tau^1 \in \mathbb{T}$ such that every cluster point of $\{F^n\}$ has the form (τ^1, h), where $h \in \mathcal{S}_2$ (respectively, there exists $\tau^2 \in \mathbb{T}$ such that every cluster point of $\{F^n\}$ has the form (g, τ^2), where $\mathrm{g} \in \mathcal{S}_2$);

(iv) if ϕ is a left Type II and ψ is a right Type II function, then there exist $\tau^1, \tau^2 \in \mathbb{T}$ such that $\{F_n\}$ converges uniformly on compact sets to (τ^1, τ^2).

3.2.5 Principal Results

We begin with our model-theoretic definitions of Denjoy-Wolff-type points.

Definition 3.2.8. Let $\phi \in \mathcal{S}_2$ with model (M, u). Assume first that $\phi \neq \pi^1$.

(i) A point $(\tau^1, \sigma) \in \mathbb{T} \times \text{cl}(\mathbb{D})$ will be called a *left Type I DW point* for ϕ if it is a B-point, $\phi(\tau^1, \sigma) = \tau^1$ and there exists $u_{(\tau^1,\sigma)} \in Y_{(\tau^1,\sigma)}$ such that $||u^1_{(\tau^1,\sigma)}|| \leq 1$ and $u^2_{(\tau^1,\sigma)} = 0$.

(ii) A point $\tau = (\tau^1, \tau^2) \in \mathbb{T}^2$ will be called a *left Type II DW point* for ϕ if it is a B-point, $\phi(\tau) = \tau^1$, there exists $u_\tau \in Y_\tau$ such that $||u^1_\tau|| < 1$ and τ is not a left Type I DW point for ϕ. In particular, if $K > 0$ is any constant such that

$$||u^1_\tau||^2 + K||u^2_\tau||^2 \leq 1,$$

we will say that τ is a *left Type II DW point with constant K*.

Now, assume instead that $\phi \neq \pi^2$.

(iii) A point $(\sigma, \tau^2) \in \text{cl}(\mathbb{D}) \times \mathbb{T}$ will be called a *right Type I DW point* for ϕ if (τ^2, σ) is a left Type I DW point for $\widetilde{\phi}$.

(iv) A point $\tau = (\tau^1, \tau^2) \in \mathbb{T}^2$ will be called a *right Type II DW point* for ϕ (with constant $K > 0$) if $\tilde{\tau} = (\tau^2, \tau^1)$ is a left Type II DW point for $\widetilde{\phi}$ (with constant $K > 0$).

An immediate consequence of Definition 3.2.8 is that every left (resp., right) Type II DW point is a left (resp., right) Type II DW point with constant K, for some $K > 0$.

The following characterizations are proved in Section 3.4 (notice that the property of being a Type I/Type II point turns out not to depend on the model of the function).

Theorem 3.2.9. *Let $\phi \in \mathcal{S}_2$ with model (M, u) and $\tau^1 \in \mathbb{T}$. Assume also that $\phi \neq \pi^1$. The following assertions are equivalent:*

(i) there exists $\sigma \in cl(\mathbb{D})$ such that (τ^1, σ) is a left Type I DW point for ϕ;

(ii) every point in $\{\tau^1\} \times cl(\mathbb{D})$ is a left Type I DW point for ϕ;

(iii) ϕ *is a left Type I function and the common Denjoy-Wolff point of all slice functions* $\phi_\mu \in \mathcal{S}$ *is* τ^1*;*

(iv) *there exists* $\sigma \in cl(\mathbb{D})$ *such that* (τ^1, σ) *is a B-point,* $\phi(\tau^1, \sigma) = \tau^1$ *and*

$$\frac{D_{-(\tau^1, \sigma M)}\phi(\tau^1, \sigma)}{-\tau^1} \leq 1, \quad \forall M > 0;$$

(v) *for every* $\sigma \in cl(\mathbb{D})$*,* (τ^1, σ) *is a B-point,* $\phi(\tau^1, \sigma) = \tau^1$ *and*

$$\frac{D_{-(\tau^1, \sigma M)}\phi(\tau^1, \sigma)}{-\tau^1} \leq 1, \quad \forall M > 0.$$

Moreover, assuming that any of the above statements holds and letting $\phi'_\mu(\tau^1)$ *denote the angular derivative of* ϕ_μ *at* τ^1*, we obtain*

$$\lim_{M \to \infty} D_{-(\tau^1, \sigma M)}\phi(\tau^1, \sigma) = -\tau^1 \phi'_\mu(\tau^1),$$

for all $\mu \in \mathbb{D}$ *and all* $|\sigma| \leq 1$*.*

There is an analogous statement for right Type I DW points (we need to assume that $\phi \neq \pi^2$*).*

Theorem 3.2.10. *Let* $\phi : \mathbb{D}^2 \to \mathbb{D}$ *be holomorphic with model* (M, u)*. Also, let* $\tau = (\tau^1, \tau^2) \in \mathbb{T}^2$*,* $K > 0$ *and assume that* $\phi \neq \pi^1$*. The following assertions are equivalent:*

(i) τ *is a left Type II DW point for* ϕ *with constant K;*

(ii) ϕ *is a left Type II function. Also, letting* $\xi : \mathbb{D} \to \mathbb{D}$ *denote the holomorphic function such that* $\phi(\xi(\mu), \mu) = \xi(\mu)$*, for all* $\mu \in \mathbb{D}$*, we have that* τ^2 *is a B-point for* ξ*,* $\xi(\tau^2) = \tau^1$ *and*

$$\liminf_{z \to \tau^2} \frac{1 - |\xi(z)|}{1 - |z|} \leq \frac{1}{K};$$

(iii) τ *is a B-point for* ϕ*,* $\phi(\tau) = \tau^1$*, the quantity* $D_{-(\tau^1, \tau^2 M)}\phi(\tau)$ *is not constant with respect to* $M > 0$ *and also there exists* $A \geq K$ *such that*

$$\frac{D_{-(\tau^1, \tau^2 A)}\phi(\tau)}{-\tau^1} = 1.$$

Moreover, assuming that any of the above statements holds,

$$A = \left[\liminf_{z \to \tau^2} \frac{1 - |\xi(z)|}{1 - |z|} \right]^{-1}$$

will be the maximum among all constants $K > 0$ such that τ is a left Type II DW point for ϕ with constant K. It will also be the unique positive number such that $D_{-(\tau^1,\tau^2 A)}\phi(\tau)/(-\tau^1) = 1$.

There is an analogous statement for right Type II DW points (we need to assume that $\phi \neq \pi^2$).

A consequence of Theorem 3.2.10 is that not all Type II functions have Type II DW points (just choose e.g. any left Type II function such that ξ has no B-points). However, Type II DW points do appear naturally when investigating iteration-theoretic questions. In particular, if $F = (\phi, \psi) : \mathbb{D}^2 \to \mathbb{D}^2$ has no fixed points, ϕ is left Type II and ψ is right Type II, then both ϕ and ψ will have Type II DW points (see Theorem 3.5.1 for details).

Theorems 3.2.9-3.2.10 allow us to give a simple, unified characterization of Type I/II DW points, one that is expressed in terms of directional derivatives and is easier to verify in practice than checking for invariant horospheres. To state it, set (for any function $\phi \in \mathcal{S}_2$ such that $\tau \in \partial\mathbb{D}^2$ is a B-point)

$$K_\tau(M) = \frac{D_{-(\tau^1,\tau^2 M)}\phi(\tau)}{-\phi(\tau)} \quad (M > 0).$$

It can be shown (see Proposition 3.3.5) that $K_\tau(M)$ is nonnegative and increasing with respect to M. This observation, combined with Theorems 3.2.9-3.2.10, leads to:

Theorem 3.2.11. *Let $\phi \in \mathcal{S}_2$ and assume $\tau = (\tau^1, \tau^2) \in \partial\mathbb{D}^2$ is a B-point for ϕ such that $\phi(\tau) = \tau^1$. Assume also that $\phi \neq \pi^1$.*

(a) If $|\tau^2| < 1$, then τ is a left Type I DW point that is also a C-point for ϕ if and only if

$$K_\tau(M) = \alpha \leq 1, \quad \forall M > 0.$$

In any other case, τ will be neither a left Type I nor a left Type II DW point.

(b) If $|\tau^2| = 1$, then τ is a:

(i) left Type I DW point that is also a C-point if and only if

$$K_\tau(M) = \alpha \leq 1, \quad \forall M > 0;$$

(ii) left Type I DW point that is not a C-point if and only if $\{K_\tau(M)\}_M$ *is non-constant and*

$$K_\tau(M) < 1, \quad \forall M > 0;$$

(iii) left Type II DW point if and only if $\{K_\tau(M)\}_M$ *is non-constant and there exists* $A > 0$ *such that*

$$K_\tau(A) = 1;$$

(iv) neither a left Type I nor a left Type II DW point if and only if

$$K_\tau(M) > 1, \quad \forall M > 0.$$

There is an analogous statement for right Type I/II DW points (we need to assume that $\phi \neq \pi^2$*).*

Using our work on DW points, we are able to offer the following refinements of Theorem 3.2.7.

Theorem 3.2.12. *Assume* $F = (\phi, \psi) : \mathbb{D}^2 \to \mathbb{D}^2$ *is holomorphic,* ϕ *is left Type I and* ψ *is right Type II. Let* τ^1 *denote the common Denjoy-Wolff point of all slice functions* ϕ_μ*. If there exists* $\sigma \in \mathbb{T}$ *such that* (τ^1, σ) *is a right Type II DW point for* ψ *but not a C-point for* ϕ*, then* $F^n \to (\tau^1, \sigma)$ *uniformly on compact subsets of* $\mathbb{D}^2$.

Theorem 3.2.13. *Assume* $F = (\phi, \psi) : \mathbb{D}^2 \to \mathbb{D}^2$ *is holomorphic,* ϕ *is left Type I and* ψ *is right Type I. Let* τ^1 *and* τ^2 *denote the common Denjoy-Wolff points of all slice functions* $\phi(\cdot, \mu)$ *and* $\psi(\lambda, \cdot)$*, respectively. If* $\tau = (\tau^1, \tau^2)$ *is not a C-point for* ϕ*, then every cluster point of* $\{F^n\}$ *will have the form* (τ^1, h)*, where* h *is either a holomorphic function* $\mathbb{D}^2 \to \mathbb{D}$ *or the constant* τ^2*. An analogous conclusion can be reached if* τ *is not a C-point for* ψ*.*

Applications are contained in Examples 3.5.2 and 3.5.3. A further refinement can be found in Theorem 3.5.2.

3.3 B-points and Directional Derivatives along $(\tau^1, \tau^2 M)$

This section contains several technical results that build upon the model theory of [12] and [13], the highlights being Theorems 3.3.3 and 3.3.9-3.3.10. These will be critical for our work in Sections 3.4, 3.5, but are also interesting in their own right.

Now, choose an arbitrary $\phi \in \mathcal{S}_2$ with model (M, u) and a B-point $\tau = (\tau^1, \tau^2) \in \partial\mathbb{D}^2$. These will be fixed for the remainder of this section. Recall that we can define

$$x_\tau(\delta) = \lim_{\tau - z\delta \xrightarrow{nt} \tau} u_{\tau - z\delta},$$

for any $\delta \in \mathbb{H}(\tau)$, where the limit is with respect to the norm of M. The following easy consequence of (3.5) will be used repeatedly throughout the paper.

Lemma 3.3.1. *Assume $\tau \in \mathbb{T}^2$. Then, for any $u_\tau \in Y_\tau$ we have*

$$\langle x_\tau^1(\delta), u_\tau^1\rangle + \frac{\overline{\tau^2}\delta^2}{\overline{\tau^1}\delta^1}\langle x_\tau^2(\delta), u_\tau^2\rangle = ||x_\tau^1(\delta)||^2 + \frac{\overline{\tau^2}\delta^2}{\overline{\tau^1}\delta^1}||x_\tau^2(\delta)||^2,$$

for all $\delta \in \mathbb{H}(\tau)$.

Proof. Applying (3.5) twice gives us

$$\langle u_\lambda^1, u_\tau^1\rangle + \frac{1 - \lambda^2\overline{\tau^2}}{1 - \lambda^1\overline{\tau^1}}\langle u_\lambda^2, u_\tau^2\rangle = \langle u_\lambda^1, x_\tau^1(\delta)\rangle + \frac{1 - \lambda^2\overline{\tau^2}}{1 - \lambda^1\overline{\tau^1}}\langle u_\lambda^2, x_\tau^2(\delta)\rangle,$$

for all $\lambda \in \mathbb{D}^2$ and $\delta \in \mathbb{H}(\tau)$. Setting $\lambda = \tau - r\delta$ and letting $r \to 0+$ then finishes off the proof. □

We also require the following lemma.

Lemma 3.3.2. *Assume $\tau \in \mathbb{T}^2$. If $u_\tau, v_\tau \in Y_\tau$ are such that $u_\tau^i = v_\tau^i = 0$ for some $i \in \{1, 2\}$, we must have $u_\tau = v_\tau$.*

Proof. Without loss of generality, assume $i = 2$. Applying (3.5) twice, we obtain

$$1 - \phi(\lambda)\overline{\phi(\tau)} = (1 - \lambda^1\overline{\tau^1})\langle u_\lambda^1, u_\tau^1\rangle,$$

$$= (1 - \lambda^1\overline{\tau^1})\langle u_\lambda^1, v_\tau^1\rangle,$$

for all $\lambda \in \mathbb{D}^2$. Thus, $\langle u^1_\lambda, u^1_\tau - v^1_\tau \rangle = 0$ for all λ. This equality, combined with the fact that both v^1_τ and v^1_τ are weak limits of vectors in the span of $\{u^1_\lambda : \lambda \in \mathbb{D}^2\}$ implies that

$$||u^1_\tau||^2 = ||v^1_\tau||^2 = \langle u^1_\tau, v^1_\tau \rangle.$$

Thus, $v^1_\tau = u^1_\tau$ and we are done. □

Our next result shows that the presence of vectors with null components in X_τ has a surprisingly strong impact on the boundary regularity of the function. We exclude facial B-points from our theorem, since they are automatically C-points.

Theorem 3.3.3. *Assume $\tau \in \mathbb{T}^2$ and also that there exists $x_\tau(\delta) \in X_\tau$ with $x^i_\tau(\delta) = 0$ for some $i \in \{1, 2\}$. Then, τ is a C-point for ϕ.*

Proof. Without loss of generality, assume that there exists $x_\tau(\delta_0) \in X_\tau$ with $x^2_\tau(\delta_0) = 0$. We may assume that $x^1_\tau(\delta_0) \neq 0$, else ϕ would be a unimodular constant. In view of Lemma 3.3.1, we obtain

$$\langle x^1_\tau(\delta), x^1_\tau(\delta_0) \rangle = ||x^1_\tau(\delta)||^2 + \frac{\overline{\tau^2}\delta^2}{\overline{\tau^1}\delta^1} ||x^2_\tau(\delta)||^2, \tag{3.8}$$

for all $\delta \in \mathbb{H}(\tau)$. Choose any open subset Ω of $\mathbb{H}(\tau)$ with the property that $\frac{\overline{\tau^2}\delta^2}{\overline{\tau^1}\delta^1}$ has positive real part for all $\delta \in \Omega$. (3.8) then implies that

$$||x^1_\tau(\delta)|| \leq ||x^1_\tau(\delta_0)||$$

for all $\delta \in \Omega$. Indeed, if this were not the case, we would be able to write

$$\begin{aligned} \Re\langle x^1_\tau(\delta), x^1_\tau(\delta_0) \rangle &\leq ||x^1_\tau(\delta)|| \cdot ||x^1_\tau(\delta_0)|| \\ &< ||x^1_\tau(\delta)||^2 \\ &\leq ||x^1_\tau(\delta)||^2 + \Re\left(\frac{\overline{\tau^2}\delta^2}{\overline{\tau^1}\delta^1}\right) ||x^2_\tau(\delta)||^2 \end{aligned}$$

whenever $\delta \in \Omega$, a contradiction.

Now, assume $||x_\tau^1(\delta)|| = ||x_\tau^1(\delta_0)||$ for all $\delta \in \Omega$. The previous chain of inequalities then implies that

$$\langle x_\tau^1(\delta), x_\tau^1(\delta_0)\rangle = ||x_\tau^1(\delta)||^2 = ||x_\tau^1(\delta_0)||^2,$$

for all $\delta \in \Omega$. This gives us $x_\tau^1(\delta) = x_\tau^1(\delta_0)$ on Ω, and hence also on $\mathbb{H}(\tau)$. In view of (3.8), we obtain that $x_\tau^2(\cdot)$ must be identically zero. Hence, $X_\tau = \{(x_\tau^1(\delta_0), 0)\}$ and we obtain (by Lemma 3.2.4) that τ is a C-point.

Assume, on the other hand, that we can find $\delta_1 \in \mathbb{H}(\tau)$ such that $||x_\tau^1(\delta_1)|| < ||x_\tau^1(\delta_0)||$. Applying (3.5) again, with $\delta = \delta_0$ and $u_\tau = x_\tau(\delta_1)$, we deduce that

$$\langle x_\tau^1(\delta_0), x_\tau^1(\delta_1)\rangle = ||x_\tau^1(\delta_0)||^2,$$

a contradiction. This concludes the proof. □

Remark 3.3.4. If we merely assume the existence of $u_\tau \in Y_\tau$ such that $u_\tau^i = 0$ for some $i \in \{1, 2\}$, τ will not necessarily be a C-point; see Example 3.5.2.

Next, we show that the directional derivatives of ϕ along $(\tau^1, \tau^2 M)$ can be naturally associated with an increasing (with respect to M) sequence of positive numbers. Indeed, put $\delta_M = (\tau^1, \tau^2 M)$ and define

$$K_\tau(M) := \frac{D_{-\delta_M}\phi(\tau)}{-\phi(\tau)} = ||x_\tau^1(\delta_M)||^2 + M||x_\tau^2(\delta_M)||^2,$$

for all $M > 0$.

Proposition 3.3.5. *For any $u_\tau \in Y_\tau$ we have*

$$K_\tau(M) \leq ||u_\tau^1||^2 + M||u_\tau^2||^2, \quad \forall M > 0,$$

with equality if and only if $x_\tau(\delta_M) = u_\tau$. In particular, $K_\tau(M)$ is increasing with respect to M. It will be strictly increasing if and only if $X_\tau \neq \{(x_\tau^1, 0)\}$.

Proof. First, assume τ is a facial B-point. If $|\tau^2| < 1$, then Lemma 3.2.3 tells us that $Y_\tau = X_\tau = \{(x_\tau^1, 0)\}$, hence $K_\tau(M)$ is constant and there is nothing to prove. If $|\tau^1| < 1$, then $Y_\tau = X_\tau = \{(0, x_\tau^2)\}$, $K_\tau(M)$ is strictly increasing with respect to M and the theorem obviously holds.

Now, assume $\tau \in \mathbb{T}^2$ and fix $u_\tau \in Y_\tau$, $M > 0$. In view of Lemma 3.3.1, we can apply Cauchy-Schwarz to obtain

$$\begin{aligned} K_\tau(M) &= \langle x_\tau^1(\delta_M), u_\tau^1 \rangle + M \langle x_\tau^2(\delta_M), u_\tau^2 \rangle \\ &\leq ||x_\tau^1(\delta_M)|| \cdot ||u_\tau^1|| + \big(\sqrt{M}||x_\tau^2(\delta_M)||\big)\big(\sqrt{M}||u_\tau^2||\big) \qquad (3.9) \\ &\leq \sqrt{K_\tau(M)}\sqrt{||u_\tau^1||^2 + M||u_\tau^2||^2}. \qquad (3.10) \end{aligned}$$

Thus, $K_\tau(M) \leq ||u_\tau^1||^2 + M||u_\tau^2||^2$.

When does equality hold? For (3.9) to hold as an equality, we must have $c^i \in \mathbb{R}^+ \cup \{0\}$ such that $c^i x_\tau^i(\delta_M) = u_\tau^i$, for $i \in \{1, 2\}$. For (3.10), we need

$$||x_\tau^1(\delta_M)|| \cdot ||u_\tau^2|| = ||x_\tau^2(\delta_M)|| \cdot ||u_\tau^1||. \qquad (3.11)$$

Now, assume that either $x_\tau^i(\delta_M) = 0$ or $u_\tau^i = 0$ for some i. For definiteness, let us assume $u_\tau^2 = 0$ (the other cases are proved in an identical way). In view of (3.11), we must have either $x_\tau^2(\delta_M) = 0$ or $u_\tau^1 = 0$. If the latter holds, we obtain $u_\tau = 0$, hence ϕ is a unimodular constant and there is nothing to prove. Thus, we may assume $x_\tau^2(\delta_M) = 0$. In this case, we may replace u_τ^1 by $c^1 x_\tau^1(\delta)$ in the equality

$$||x_\tau^1(\delta_M)||^2 = K_\tau(M) = \langle x_\tau^1(\delta_M), u_\tau^1 \rangle + M \langle x_\tau^2(\delta_M), u_\tau^2 \rangle = \langle x_\tau^1(\delta_M), u_\tau^1 \rangle$$

to obtain $||x_\tau^1(\delta_M)||^2 = c^1 ||x_\tau^1(\delta_M)||^2$. If $x_\tau^1(\delta_M) = 0$, we again obtain that ϕ is a unimodular constant, while $x_\tau^1(\delta_M) \neq 0$ implies $c^1 = 1$, hence $x_\tau(\delta_M) = u_\tau$.

On the other hand, assume $x_\tau^1(\delta_M), x_\tau^2(\delta_M), u_\tau^1, u_\tau^2$ are all nonzero. (3.11) then gives us $c^1 = c^2 = c$. Replacing u_τ^i by $c x_\tau^i(\delta_M)$ in the equality

$$K_\tau(M) = \langle x_\tau^1(\delta_M), u_\tau^1 \rangle + M \langle x_\tau^2(\delta_M), u_\tau^2 \rangle,$$

we obtain $c = 1$, hence $x_\tau(\delta_M) = u_\tau$.

Now, we show that $K_\tau(M)$ is increasing with respect to M. Indeed, let $N > M > 0$. Setting $u_\tau = x_\tau(\delta_N)$ in our previous result implies

$$K_\tau(M) \leq ||x_\tau^1(\delta_N)|| + M||x_\tau^2(\delta_N)|| \leq K_\tau(N),$$

as desired.

Now, if $X_\tau = \{(x_\tau^1, 0)\}$, it is evident that $K_\tau(M)$ will be constant (and equal to $||x_\tau^1||^2$ for all M). On the other hand, assume X_τ is not a singleton of the form $\{(x_\tau^1, 0)\}$ but that we can also find positive numbers $M < N$ such that $K_\tau(M) = K_\tau(N)$. As we have already seen, this implies that $x_\tau(\delta_M) = x_\tau(\delta_N)$, which, combined with $K_\tau(M) = K_\tau(N)$, allows us to deduce that $x_\tau^2(\delta_M) = x_\tau^2(\delta_N) = 0$. Theorem 3.3.3 then tells us that $X_\tau = \{(x_\tau^1(\delta_M), 0)\}$, a contradiction. □

We now explore some consequences of Proposition 3.3.5.

Corollary 3.3.6. *Given any $M > 0$ and $u_\tau \in Y_\tau$, we must either have $||x_\tau^1(\delta_M)|| \leq ||u_\tau^1||$ or $||x_\tau^2(\delta_M)|| \leq ||u_\tau^2||$. Moreover, if*

(i) $||x_\tau^1(\delta_M)|| = ||u_\tau^1||$ (resp., $||x_\tau^2(\delta_M)|| = ||u_\tau^2||$), then $||x_\tau^2(\delta_M)|| \leq ||u_\tau^2||$ (resp., $||x_\tau^1(\delta_M)|| \leq ||u_\tau^1||$);

(ii) $||x_\tau(\delta_M)|| = ||u_\tau||$, then $x_\tau(\delta_M) = u_\tau$.

Proof. Assume that $||x_\tau^i(\delta_M)|| > |u_\tau^i||$ for all $i \in \{1, 2\}$. Thus, $K_\tau(M) > ||u_\tau^1||^2 + M||u_\tau^2||^2$, which contradicts Proposition 3.3.5. The rest of the corollary is proved by applying Proposition 3.3.5 in an analogous manner. □

Corollary 3.3.7. *Given any $M, N > 0$, one and only one of the following cases can occur:*

(i) $||x_\tau^1(\delta_M)|| < ||x_\tau^1(\delta_N)||$ (resp., $||x_\tau^2(\delta_M)|| < ||x_\tau^2(\delta_N)||$) and $||x_\tau^2(\delta_M)|| > ||x_\tau^2(\delta_N)||$ (resp., $||x_\tau^1(\delta_M)|| > ||x_\tau^1(\delta_N)||$);

(ii) $x_\tau(\delta_M) = x_\tau(\delta_N)$.

Proof. Suppose first that $||x_\tau^1(\delta_M)|| < ||x_\tau^1(\delta_N)||$. If $||x_\tau^2(\delta_M)|| \leq ||x_\tau^2(\delta_N)||$, one obtains

$$K_\tau(N) > ||x_\tau^1(\delta_M)||^2 + N||x_\tau^2(\delta_M)||^2,$$

which contradicts Proposition 3.3.5. Thus, $||x_\tau^2(\delta_M)|| > ||x_\tau^2(\delta_N)||$. The proof in the case that $||x_\tau^2(\delta_M)|| < ||x_\tau^2(\delta_N)||$ proceeds in an entirely analogous manner.

Now, assume $||x_\tau^1(\delta_M)|| = ||x_\tau^1(\delta_N)||$. If $||x_\tau^2(\delta_M)|| < ||x_\tau^2(\delta_N)||$, then we again obtain $K_\tau(N) > ||x_\tau^1(\delta_M)||^2 + N||x_\tau^2(\delta_M)||^2$, a contradiction (the inequality $||x_\tau^2(\delta_M)|| > ||x_\tau^2(\delta_N)||$ can be ruled out in an analogous way). Thus, we must have $||x_\tau(\delta_M)|| = ||x_\tau(\delta_N)||$, which, by Proposition 3.3.5, gives us $x_\tau(\delta_M) = x_\tau(\delta_N)$. $\square$

Our next proposition (while fitting the theme of this section) will not be used in the sequel, so we record it without a proof (one can use Proposition 3.3.5 in combination with the previous two lemmas).

Proposition 3.3.8. *Assume $\tau \in \mathbb{T}^2$. Then,*

$$\lim_{M\to 0+} ||x_\tau^1(\delta_M)|| = \lim_{M\to 0+} \sqrt{K_\tau(M)} = \inf_{M>0} \{||x_\tau^1(\delta_M)||\}$$

(resp., $\lim_{M\to+\infty} ||x_\tau^2(\delta_M)|| = \lim_{M\to+\infty} \sqrt{K_\tau(M)} = \inf_{M>0} \{||x_\tau^2(\delta_M)||\}$*).*
Moreover, if there exist $u \in Y_\tau$ and a sequence $\{M_k\}$ of positive numbers such that $M_k \to 0$ (resp. $M_k \to \infty$) and $||u_\tau^1|| \le ||x_\tau^1(\delta_{M_k})||$ (resp., $||u_\tau^2|| \le ||x_\tau^2(\delta_{M_k})||$) for all k, then

$$\lim_k x_\tau^1(\delta_{M_k}) = u_\tau^1 \quad \textit{in norm}$$

(resp., $\lim_k x_\tau^2(\delta_{M_k}) = u_\tau^2$ *in norm).*

We now prove a theorem that describes those vectors in Y_τ with null components (recall that, by Lemma 3.3.2, these vectors, if they exist, must be unique).

Theorem 3.3.9. *Assume the B-point $\tau \in \mathbb{T}^2$ is such that there exists $u_\tau \in Y_\tau$ with $u_\tau^2 = 0$. Then,*

$$\lim_{M\to+\infty} K_\tau(M) = ||u_\tau^1||^2$$

and also

$$\lim_{M\to+\infty} x_\tau(\delta_M) = u_\tau$$

in norm.

Moreover, τ is a C-point for ϕ if and only if $X_\tau = \{u_\tau\}$. In this case, every $v_\tau \in Y_\tau$ such that $v_\tau \neq u_\tau$ must satisfy $||v_\tau^1|| > ||u_\tau^1||$ and $v_\tau^2 \neq 0$.

Proof. We prove the C-point portion of the theorem first. If $X_\tau = \{u_\tau\}$, Theorem 3.2.4 implies that τ is a C-point. Conversely, assume that τ is a C-point. Write $X_\tau = \{x_\tau\}$. Proposition 3.3.5 then implies that

$$||x^1_\tau||^2 + M||x^2_\tau||^2 \leq ||u^1_\tau||^2,$$

for all $M > 0$. Thus, $x^2_\tau = 0$. Lemma 3.3.2 then gives us $x_\tau = u_\tau$ and we can also write

$$1 - \phi(\lambda)\overline{\phi(\tau)} = (1 - \lambda^1\overline{\tau^1})\langle u^1_\lambda, x^1_\tau\rangle \quad (\lambda \in \mathbb{D}^2). \tag{3.12}$$

Assume $x^1_\tau \neq 0$ (else the result will be be trivial) and let $v_\tau \in Y_\tau$. Lemma 3.3.1 implies that

$$\langle x^1_\tau, v^1_\tau\rangle = ||x^1_\tau||^2.$$

Thus, either $||v^1_\tau|| > ||x^1_\tau||$, or $||v^1_\tau||^2 = ||x^1_\tau||^2 = \langle x^1_\tau, v^1_\tau\rangle$, which leads to $v^1_\tau = x^1_\tau$. But then, comparing

$$1 - \phi(\lambda)\overline{\phi(\tau)} = (1 - \lambda^1\overline{\tau^1})\langle u^1_\lambda, v^1_\tau\rangle + (1 - \lambda^2\overline{\tau^2})\langle u^2_\lambda, v^2_\tau\rangle$$

with (3.12) gives us $v^2_\tau = 0$, thus $v_\tau = x_\tau$ as desired. Finally, if $v^2_\tau = 0$, we can apply Lemma 3.3.2 to conclude that $v_\tau = x_\tau$.

Now, we prove the first part of the theorem. If τ is a C-point, the theorem follows by our previous result. So, assume that τ is not a C-point, in which case Proposition 3.3.5 implies that $K_\tau(M)$ is strictly increasing. The bound $K_\tau(M) \leq ||u^1_\tau||^2$, for every $M > 0$, implies that $\lim_{M\to+\infty} x^2_\tau(\delta_M) = 0$ in norm and also that $\{||x^1_\tau(\delta_M)||\}$ is bounded with respect to M.

Now, let $\{M_k\}$ be any sequence converging to $+\infty$ such that $x^1_\tau(\delta_{M_k})$ converges to some $x^1 \in M^1$ weakly. Also, fix a decreasing null sequence $\{\epsilon_k\}$. In view of Theorem 3.2.5, we can find $\{\lambda_k\} \subset \mathbb{D}^2$ that converges to τ and such that $|\phi(\lambda_k) - \phi(\tau)| < \epsilon_k$ and also $||x_\tau(\delta_{M_k}) - u_{\lambda_k}|| < \epsilon_k$, for all k. Thus, $\lim_k \phi(\lambda_k) = \phi(\tau)$, $u^1_{\lambda_k}$ converges weakly to x^1 and $u^2_{\lambda_k}$ converges to 0 in norm. Now, the model formula (3.1) implies that

$$1 - \phi(\lambda)\overline{\phi(\lambda_k)} = (1 - \lambda^1\overline{\lambda^1_k})\langle u^1_\lambda, u^1_{\lambda_k}\rangle + (1 - \lambda^2\overline{\lambda^2_k})\langle u^2_\lambda, u^2_{\lambda_k}\rangle,$$

for all k and $\lambda \in \mathbb{D}^2$. Letting $k \to \infty$ then gives us,

$$1 - \phi(\lambda)\overline{\phi(\tau)} = (1 - \lambda^1\overline{\tau^1})\langle u_\lambda^1, x^1\rangle, \tag{3.13}$$

for all λ. However, since $u_\tau \in Y_\tau$ we can also write (in view of (3.5))

$$1 - \phi(\lambda)\overline{\phi(\tau)} = (1 - \lambda^1\overline{\tau^1})\langle u_\lambda^1, u_\tau^1\rangle,$$

for all λ. Comparing this equality with (3.13) then gives us $\langle u_\lambda^1, u_\tau^1\rangle = \langle u_\lambda^1, x^1\rangle$ for all λ. Since both vectors u_τ^1, x^1 are weak limits of elements from $\{u_\lambda^1 : \lambda \in \mathbb{D}^2\}$, we may conclude that $u_\tau^1 = x^1$. But then, observe that (by a standard property of weak limits)

$$\begin{aligned} ||u_\tau^1||^2 &= ||x^1||^2 \\ &\leq \liminf_k ||x_\tau^1(\delta_{M_k})||^2 \\ &\leq \limsup_k ||x_\tau^1(\delta_{M_k})||^2 \\ &\leq \limsup_k K_\tau(M_k) \\ &\leq ||u_\tau^1||^2, \end{aligned}$$

which implies that $\lim_k x_\tau^1(\delta_{M_k}) = x^1$ in norm and also that $\lim_k K_\tau(M_k) = ||u_\tau^1||^2$. We conclude that $x_\tau^1(\delta_M)$ converges to u_τ^1 in norm and also that $\lim_{M\to+\infty} K_\tau(M) = ||u_\tau^1||^2$, as desired. □

We end this section with a weighted version of Julia's inequality for the bidisk, which will be of critical importance in Section 3.5. Our methods are motivated by the proof of [12, Theorem 4.9].

Theorem 3.3.10. *Let $\phi \in \mathcal{S}_2$ and $\tau = (\tau^1, \tau^2) \in \mathbb{T}^2$. Assume also that α and M are positive numbers. The following assertions are equivalent:*

(i) τ is a B-point for ϕ and

$$\frac{D_{-(\tau^1,\tau^2 M)}\phi(\tau)}{-\phi(\tau)} \leq \alpha;$$

(ii) There exists a sequence $\{\lambda_n\} \subset \mathbb{D}^2$ *such that* $\lambda_n \to \tau$, $\lim_n \frac{1-|\lambda_n^2|}{1-|\lambda_n^1|} = M$ *and*

$$\lim_n \frac{1-|\phi(\lambda_n)|}{1-||\lambda_n||} \leq \begin{cases} \alpha & \text{if } M \geq 1; \\ \frac{\alpha}{M} & \text{if } M < 1; \end{cases}$$

(iii) There exists $\omega \in \mathbb{T}$ *such that*

$$\phi(E(\tau, R_1, R_2)) \subset E(\omega, \max\{aR_1, aR_2/M\}), \quad \forall R_1, R_2 > 0.$$

If (iii) holds, ω *will necessarily be equal to* $\phi(\tau)$.

Proof. Let (M, u) be a model for ϕ.

First, we show that (i) implies (ii). Assuming (i) holds, set $\delta_M = (\tau^1, \tau^2 M)$ and fix a decreasing null sequence $\{r_n\}$. Since τ is a B-point, Theorem 3.2.5 allows us to deduce that

$$||x_\tau^1(\delta_M)||^2 + M||x_\tau^2(\delta_M)||^2 \leq \alpha, \tag{3.14}$$

and also $\lim_n u_{\tau - r_n\delta_M} = x_\tau(\delta_M)$ (in norm). Now, assume $M \geq 1$ and put $\lambda_n = \tau - r_n\delta_M$. (3.1) allows us to write:

$$\begin{aligned} \lim_n \frac{1-|\phi(\lambda_n)|}{1-||\lambda_n||} &= \lim_n \frac{1-|\phi(\lambda_n)|^2}{1-||\lambda_n||^2} \\ &= \lim_n \frac{1-|\phi(\lambda_n)|^2}{1-|\lambda_n^1|^2} \\ &= \lim_n \left(||u_{\lambda_n}^1||^2 + \frac{1-|\lambda_n^2|^2}{1-|\lambda_n^1|^2} ||u_{\lambda_n}^2||^2 \right) \\ &= ||x_\tau^1(\delta_M)||^2 + M||x_\tau^2(\delta_M)||^2, \end{aligned}$$

which, combined with (3.14), gives us (ii). The proof for $M < 1$ is entirely analogous and is omitted.

Next, we show that (ii) implies (iii). The assumptions in (ii) clearly imply that τ is a B-point for ϕ. But then, we can argue as above to deduce that (3.14) holds. Thus, we can use (3.5) to obtain

$$|1 - \phi(\lambda)\overline{\phi(\tau)}| \leq |1 - \lambda^1\overline{\tau^1}| \cdot ||x^1(\delta_M)|| \cdot ||u_\lambda^1|| + |1 - \lambda^2\overline{\tau^2}| \cdot ||x^2(\delta_M)|| \cdot ||u_\lambda^2||,$$

for all $\lambda \in \mathbb{D}^2$. Setting $R_j = \frac{|\tau^j - \lambda^j|^2}{1-|\lambda^j|^2}$ $(j = 1, 2)$ and applying Cauchy-Schwarz then gives us

$$
\begin{aligned}
|\phi(\tau) - \phi(\lambda)|^2 &\leq \left(||x^1(\delta_M)||^2 + M||x^2(\delta_M)||^2\right)\left(|\tau^1 - \lambda^1|^2||u_\lambda^1||^2 + \frac{|\tau^2 - \lambda^2|^2}{M}||u_\lambda^2||^2\right) \\
&\leq \alpha \max\{R_1, R_2/M\}\left((1 - |\lambda^1|^2)||u_\lambda^1||^2 + (1 - |\lambda^2|^2)||u_\lambda^2||^2\right) \\
&= \max\{\alpha R_1, \alpha R_2/M\}\left(1 - |\phi(\lambda)|^2\right),
\end{aligned}
$$

which implies

$$
\frac{|\phi(\tau) - \phi(\lambda)|^2}{1 - |\phi(\lambda)|^2} \leq \max\{\alpha R_1, \alpha R_2/M\} \quad (\lambda \in \mathbb{D}^2)
$$

and our proof is complete.

Lastly, we show that (iii) implies (i). Set $\lambda_n = \tau - r_n\delta_M$, where $\{r_n\}$ is a decreasing null sequence. Assuming (iii) holds, we obtain

$$
\begin{aligned}
\frac{|\omega - \phi(\lambda_n)|^2}{1 - |\phi(\lambda_n)|^2} &\leq \alpha \max\left\{\frac{|\tau^1 - \lambda_n^1|^2}{1 - |\lambda_n^1|^2}, \frac{1}{M}\frac{|\tau^2 - \lambda_n^2|^2}{1 - |\lambda_n^2|^2}\right\} \\
&= \alpha \max\left\{\frac{r_n}{2 - r_n}, \frac{r_n}{2 - Mr_n}\right\} \\
&= R_n.
\end{aligned}
$$

Thus, we can write

$$
\phi(\lambda_n) \in \mathrm{cl}\left(E(\omega, R_n)\right) = \mathrm{cl}\left(D\left(\frac{\omega}{R_n + 1}, \frac{R_n}{R_n + 1}\right)\right),
$$

for all $n \geq 1$.

Now, assume $M \geq 1$ (the proof in the case where $M < 1$ will be entirely analogous). Then, $R_n = \frac{\alpha r_n}{2-Mr_n}$, $||\lambda_n|| = 1 - r_n$ and we can compute

$$
\begin{aligned}
\frac{1 - |\phi(\lambda_n)|}{1 - ||\lambda_n||} &= \frac{1 - |\phi(\lambda_n)|}{r_n} \\
&\leq \frac{|\phi(\lambda_n) - \omega|}{r_n} \\
&\leq \frac{2}{r_n}\frac{R_n}{R_n + 1} \\
&= \frac{2\alpha}{2 + (\alpha - M)r_n} \to \alpha,
\end{aligned}
$$

as $n \to \infty$. This implies that τ is a B-point for ϕ and $\omega = \phi(\tau)$. Also, since

$$\lim_n \frac{1 - |\phi(\lambda_n)|}{1 - ||\lambda_n||} = ||x_\tau^1(\delta_M)||^2 + M||x_\tau^2(\delta_M)||^2,$$

we obtain that (3.14) holds. Theorem 3.2.5 then finishes off the proof. □

Remark 3.3.11. In this theorem, we only considered points τ in the distinguished boundary. For facial B-points, the situation is more straightforward; see [13, Theorem 3.2].

Remark 3.3.12. Observe that if we assume

$$\lim_{r \to 1-} \frac{1 - |\phi(r\tau)|}{1 - ||r\tau||} = \liminf_{\lambda \to \tau} \frac{1 - |\phi(\lambda)|}{1 - ||\lambda||} \leq \alpha,$$

we obtain that (ii) holds with $M = 1$. Hence,

$$\phi(E(\tau, R, R)) \subset E(\phi(\tau), \alpha \max\{R, R\}) = E(\phi(\tau), \alpha R),$$

for all $R > 0$, which is the usual statement of Julia's inequality over the bidisk.

Remark 3.3.13. Julia-type inequalities like the one in Theorem 3.3.10(iii) were also considered by Frosini in [72], where she used Busemann sublevel sets to obtain analogous statements. Specifically, her Julia-type lemma [72, Theorem 1] depends on the behavior of ϕ along chosen complex geodesics that approach the boundary point τ. Theorem 3.3.10 can then be viewed as a refinement of that result, as it essentially says that every "weighted" version of Julia's inequality is equivalent to an inequality involving certain directional derivatives of ϕ at the corresponding boundary B-point.

3.4 Criteria for Denjoy-Wolff points

We will now use our work from Section 3.3 to study Type I/II DW points, as defined in subsection 3.2.5.

We start with two lemmas.

Lemma 3.4.1. *Let $\phi \in \mathcal{S}_2$ with model (M, u) and assume $\xi(\mu), \mu$ are points in $\mathbb{D}$ such that $\phi(\xi(\mu), \mu) = \xi(\mu)$. Then, $||u^1_{(\xi(\mu),\mu)}|| \leq 1$. Also, $||u^1_{(\xi(\mu),\mu)}|| = 1$ if and only if $u^2_{(\xi(\mu),\mu)} = 0$.*

Proof. In (3.1), set $(\lambda^1, \lambda^2) = (\mu^1, \mu^2) = (\xi(\mu), \mu)$ to obtain

$$1 - |\xi(\mu)|^2 = 1 - |\phi(\xi(\mu), \mu)|^2 = (1 - |\xi(\mu)|^2)||u^1_{(\xi(\mu),\mu)}||^2 + (1 - |\mu|^2)||u^2_{(\xi(\mu),\mu)}||^2.$$

Since $|\xi(\mu)|, |\mu| < 1$, the conclusions of the lemma follow easily. □

The next lemma is well-known (e.g. it appears as Theorem 2 in [80]). We include a proof for the sake of completeness.

Lemma 3.4.2. *Assume $\phi \in \mathcal{S}_2$ and that there exists $\mu_0 \in \mathbb{D}$ such that the slice function ϕ_{μ_0} is the identity on $\mathbb{D}$. Then, $\phi \equiv \pi_1$.*

Proof. Let (M, u) be a model for ϕ. We can use (3.1) to obtain

$$1 = \frac{1 - |\phi(\lambda, \mu_0)|^2}{1 - |\lambda|^2} = ||u^1_{(\lambda,\mu_0)}||^2 + \frac{1 - |\mu_0|^2}{1 - |\lambda^2|}||u^2_{(\lambda,\mu_0)}||^2, \tag{3.15}$$

for all $\lambda \in \mathbb{D}$. Thus, $||u^1_{(\lambda,\mu_0)}|| \leq 1$, for all $\lambda \in \mathbb{D}$, with equality if and only if $u^2_{(\lambda,\mu_0)} = 0$.

Now, fix $\tau^1 \in \mathbb{T}$ and let $\lambda \to \tau^1$ in (3.15) to obtain that (τ^1, μ_0) is a B-point for ϕ, $\phi(\tau^1, \mu_0) = \tau^1$ and also that there exists $u_{(\tau^1,\mu_0)} \in Y_{(\tau^1,\mu_0)}$ such that $||u^1_{(\tau^1,\mu_0)}|| \leq 1$ and $u^2_{(\tau^1,\mu_0)} = 0$. (3.5) then implies that

$$1 - \phi(\lambda, \mu)\overline{\tau^1} = (1 - \lambda\overline{\tau^1})\langle u^1_{(\lambda,\mu)}, u^1_{(\tau^1,\mu_0)}\rangle, \tag{3.16}$$

for all $\lambda, \mu \in \mathbb{D}$. Setting $\mu = \mu_0$ then gives us

$$1 - \lambda\overline{\tau^1} = (1 - \lambda\overline{\tau^1})\langle u^1_{(\lambda,\mu_0)}, u^1_{(\tau^1,\mu_0)}\rangle,$$

hence

$$\langle u^1_{(\lambda,\mu_0)}, u^1_{(\tau^1,\mu_0)}\rangle = 1 \geq ||u^1_{(\lambda,\mu_0)}||^2, ||u^1_{(\tau^1,\mu_0)}||^2.$$

This implies that $u^1_{(\lambda,\mu_0)} = u^1_{(\tau^1,\mu_0)}$ (and both have to be unit vectors) and also $u^2_{(\lambda,\mu_0)} = 0$, for all $\lambda \in \mathbb{D}$. Hence,

$$1 - \phi(\lambda, \mu)\overline{\lambda'} = 1 - \phi(\lambda, \mu)\overline{\phi(\lambda', \mu_0)}$$

$$= (1 - \lambda\overline{\lambda'})\langle u^1_{(\lambda,\mu)}, u^1_{(\lambda',\mu_0)}\rangle =$$

$$= (1 - \lambda\overline{\lambda'})\langle u^1_{(\lambda,\mu)}, u^1_{(\tau^1,\mu_0)}\rangle,$$

for all $\lambda, \mu, \lambda' \in \mathbb{D}$. Since both sides are affine functions of λ', we obtain $\phi(\lambda, \mu) = \lambda$, for all $\lambda, \mu \in \mathbb{D}$, as desired. □

Now, we use model theory to give a new proof of the fact that every $\phi \in \mathcal{S}_2$ (that is not a coordinate projection) must either be a Type I or a Type II function, a result originally due to Hervé (see [80, Theorem 1]).

Theorem 3.4.3. *Every $\phi \in \mathcal{S}_2$ such that $\phi \neq \pi^1$ (resp., $\phi \neq \pi^2$) is either a left Type I (resp. right Type I) or a left Type II (resp. right Type II) function.*

Proof. First, we prove the left Type I/II version of the theorem. Note that, since $\phi \neq \pi^1$, Lemma 3.4.2 implies that ϕ_μ is not the identity on $\mathbb{D}$, for any $\mu \in \mathbb{D}$. Thus, every such slice function will have a unique Denjoy-Wolff point (either on the interior of the disk or on the boundary).

To begin, assume that there exists some $\mu_0 \in \mathbb{D}$ such that the slice ϕ_{μ_0} has its Denjoy-Wolff point τ^1 on $\mathbb{T}$. Let $\lambda_n = \rho_n \tau^1$, where $\{\rho_n\}$ is an increasing sequence of positive numbers tending to 1. By the single-variable theory of Denjoy-Wolff points, we have $\lim_n \phi_{\mu_0}(\lambda_n) = \tau^1$ and

$$\frac{1 - |\phi(\lambda_n, \mu_0)|^2}{1 - ||(\lambda_n, \mu_0)||^2} = \frac{1 - |\phi_{\mu_0}(\lambda_n)|^2}{1 - |\lambda_n|^2} \to \alpha_{\mu_0} \leq 1,$$

as $n \to \infty$. Thus, (τ^1, μ_0) is a B-point for ϕ. Using the model formula for ϕ, we also see that

$$||u^1_{(\lambda_n,\mu_0)}||^2 + \frac{1 - |\mu_0|^2}{1 - |\lambda_n|^2}||u^2_{(\lambda_n,\mu_0)}||^2 = \frac{1 - |\phi_{\mu_0}(\lambda_n)|^2}{1 - |\lambda_n|^2}, \quad \forall n \geq 1.$$

Letting $n \to \infty$ and taking into account that $\frac{1-|\mu_0|^2}{1-|\lambda_n|^2} \to \infty$, we obtain the existence of $u_{(\tau^1,\mu_0)} \in Y_{(\tau^1,\mu_0)}$ satisfying $||u^1_{(\tau^1,\mu_0)}|| \leq \alpha_{\mu_0} \leq 1$ and $u^2_{(\tau^1,\mu_0)} = 0$. In view of (3.5), we can write

$$1 - \phi(\lambda, \mu)\overline{\tau^1} = (1 - \lambda\overline{\tau^1})\langle u^1_{(\lambda,\mu)}, u^1_{(\tau^1,\mu_0)}\rangle, \tag{3.17}$$

for all $\lambda, \mu \in \mathbb{D}$.

Now, assume there exists some slice ϕ_{μ_1} such that $\mu_1 \neq \mu_0$ and ϕ_{μ_1} has an interior fixed point $p \in \mathbb{D}$. Set $(\lambda, \mu) = (p, \mu_1)$ in (3.17) to obtain

$$1 - p\overline{\tau^1} = (1 - p\overline{\tau^1})\langle u^1_{(p,\mu_1)}, u^1_{(\tau^1,\mu_0)}\rangle,$$

hence $\langle u^1_{(p,\mu_1)}, u^1_{(\tau^1,\mu_0)}\rangle = 1$. Lemma 3.4.1 then implies that $u^1_{(p,\mu_1)} = u^1_{(\tau^1,\mu_0)}$ (and both will be unit vectors) and $u^2_{(p,\mu_1)} = 0$. Thus, we may substitute $(\mu^1, \mu^2) = (p, \mu_1)$ in (3.1) to obtain

$$1 - \phi(\lambda, \mu)\overline{p} = (1 - \lambda\overline{p})\langle u^1_{(\lambda,\mu)}, u^1_{(p,\mu_1)}\rangle = (1 - \lambda\overline{p})\langle u^1_{(\lambda,\mu)}, u^1_{(\tau^1,\mu_0)}\rangle, \tag{3.18}$$

for all $\lambda, \mu \in \mathbb{D}$. Comparing (3.18) with (3.17) then allows us to deduce that ϕ is equal to the identity, a contradiction.

So far, we have proved that every slice function ϕ_μ must have its Denjoy-Wolff point on the boundary of $\mathbb{D}$ (under the assumption that at least one of them does). We now show that τ^1 (the Denjoy-Wolff point of the slice ϕ_{μ_0} we started with) is actually the Denjoy-Wolff point of all slices ϕ_μ. Indeed, suppose we can find a slice ϕ_{μ_1} with a different Denjoy-Wolff point $\sigma^1 \in \mathbb{T}$. Arguing as in the beginning of the proof, we obtain that (σ^1, μ_1) is a B-point for ϕ, its value at (σ^1, μ_1) is σ^1 and also there exists $u_{(\sigma^1,\mu_1)} \in Y_{(\sigma^1,\mu_1)}$ satisfying $||u^1_{(\sigma^1,\mu_1)}|| \leq 1$ and $u^2_{(\sigma^1,\mu_1)} = 0$. (3.5) implies that

$$1 - \phi(\lambda, \mu)\overline{\sigma^1} = (1 - \lambda\overline{\sigma^1})\langle u^1_{(\lambda,\mu)}, u^1_{(\sigma^1,\mu_1)}\rangle, \quad \forall \lambda, \mu \in \mathbb{D}. \tag{3.19}$$

If in (3.19) we let $(\lambda, \mu) \to (\tau^1, \mu_0)$ in such a way that $u^1_{(\lambda,\mu)}$ converges weakly to $u^1_{(\tau^1,\mu_0)}$, we obtain (since $\phi(\tau^1, \mu_0) = \tau^1$ and $\sigma^1 \neq \tau^1$)

$$\langle u^1_{(\sigma^1,\mu_1)}, u^1_{(\tau^1,\mu_0)}\rangle = 1 \geq ||u^1_{(\sigma^1,\mu_1)}||^2, ||u^1_{(\tau^1,\mu_0)}||^2,$$

hence $u^1_{(\sigma^1,\mu_1)} = u^1_{(\tau^1,\mu_0)}$. Comparing (3.17) with (3.19) then gives us that ϕ is equal to the identity, a contradiction.

On the other hand, assume that every slice ϕ_μ has a (necessarily unique) interior fixed point $\xi(\mu)$. To show that ϕ is a left Type II function, it suffices to prove that $\xi : \mathbb{D} \to \mathbb{D}$ is actually a holomorphic function. First, note that putting $(\lambda^1, \lambda^2) = (\xi(\mu), \mu)$ and $(\mu^1, \mu^2) = (\xi(\mu'), \mu')$ in (3.1) gives us

$$1-\xi(\mu)\overline{\xi(\mu')}$$

$$=(1-\xi(\mu)\overline{\xi(\mu')})\langle u^1_{(\xi(\mu),\mu)},u^1_{(\xi(\mu'),\mu')}\rangle+(1-\mu\overline{\mu'})\langle u^2_{(\xi(\mu),\mu)},u^2_{(\xi(\mu'),\mu')}\rangle, \tag{3.20}$$

for all $\mu,\mu'\in\mathbb{D}$.

Now, if $||u^1_{(\xi(\mu'),\mu')}||=1$ for some $\mu'\in\mathbb{D}$, the model formula for ϕ yields (since $u^2_{(\xi(\mu'),\mu')}=0$ in view of Lemma 3.4.1)

$$1-\phi(\lambda,\mu)\overline{\xi(\mu')}=(1-\lambda\overline{\xi(\mu')})\langle u^1_{(\lambda,\mu)},u^1_{(\xi(\mu'),\mu')}\rangle, \tag{3.21}$$

for all λ,μ. Plugging in $(\lambda,\mu)=(\xi(\mu),\mu)$ gives us $\langle u^1_{(\xi(\mu),\mu)},u^1_{(\xi(\mu'),\mu')}\rangle=1$, for all μ, hence $u^1_{(\xi(\mu),\mu)}=u^1_\xi$ =constant (of norm 1) and $u^2_{(\xi(\mu),\mu)}=0$ for all μ. Thus, we obtain

$$1-\phi(\lambda,\mu)\overline{\xi(\sigma)}=(1-\lambda\overline{\xi(\sigma)})\langle u^1_{(\lambda,\mu)},u^1_\xi\rangle, \tag{3.22}$$

for all $\lambda,\mu,\sigma\in\mathbb{D}$.

There are now two separate cases to examine. Either $\phi(\lambda,\mu)=\lambda\langle u^1_{(\lambda,\mu)},u^1_\xi\rangle$ for all λ,μ, in which case (3.22) implies that $\langle u^1_{(\lambda,\mu)},u^1_\xi\rangle=1$ (for all λ,μ), hence $\phi=\pi^1$, a contradiction, or we can find $\lambda_0,\mu_0\in\mathbb{D}$ such that $\phi(\lambda_0,\mu_0)\neq\lambda_0\langle u^1_{(\lambda_0,\mu_0)},u^1_\xi\rangle$. Then, (3.22) implies that

$$\overline{\xi(\sigma)}=\frac{\langle u^1_{(\lambda_0,\mu_0)},u^1_\xi\rangle-1}{\lambda_0\langle u^1_{(\lambda_0,\mu_0)},u^1_\xi\rangle-\phi(\lambda_0,\mu_0)},$$

for all $\sigma\in\mathbb{D}$. Thus, ξ is constant (and trivialy holomorphic).

There is one more possibility to consider: suppose that $||u^1_{(\xi(\mu'),\mu')}||<1$ for all μ'. (3.20) then becomes

$$1-\xi(\mu)\overline{\xi(\mu')}=(1-\mu\overline{\mu'})\frac{\langle u^2_{(\xi(\mu),\mu)},u^2_{(\xi(\mu'),\mu')}\rangle}{1-\langle u^1_{(\xi(\mu),\mu)},u^1_{(\xi(\mu'),\mu')}\rangle}, \tag{3.23}$$

for all $\mu,\mu'\in\mathbb{D}$. In other words,

$$\frac{1-\xi(\mu)\overline{\xi(\mu')}}{1-\mu\overline{\mu'}}$$

is the Schur product of the positive-semidefinite kernels

$$\langle u^2_{(\xi(\mu),\mu)},u^2_{(\xi(\mu'),\mu')}\rangle$$

and

$$\frac{1}{1-\langle u^1_{(\xi(\mu),\mu)}, u^1_{(\xi(\mu'),\mu')}\rangle}$$

(the latter is actually a complete Pick kernel, see [8, Chapter 8]), hence it must be positive semi-definite as well. Automatic holomorphy of models (see [14, Proposition 2.32]) then implies that ξ is a holomorphic function on $\mathbb{D}$, concluding the proof.

The right Type I/II version of the theorem follows by applying the left Type I/II version to the function $\widetilde{\phi} : \mathbb{D}^2 \to \mathbb{D}$ defined by $\widetilde{\phi}(\lambda) = \phi(\lambda^2, \lambda^1)$, for all $\lambda \in \mathbb{D}^2$. □

Next, we provide criteria for Type I DW points, as stated in subsection 3.2.5. Recall that, given $\phi \in \mathcal{S}_2$ with model (M, u) and a B-point $\tau \in \mathbb{T}^2$, we have defined $\delta_M = (\tau^1, \tau^2 M)$ and

$$K_\tau(M) = ||x^1_\tau(\delta_M)||^2 + M||x^2_\tau(\delta_M)||^2,$$

for all $M > 0$.

Proof of Theorem 3.2.9. First, we show that (iii) implies (ii). Indeed, assume that τ^1 is the common Denjoy-Wolff point of all slice functions ϕ_μ and let $|\sigma| \leq 1$. We will show that (τ^1, σ) is a left Type I DW point for ϕ.

Fix a sequence $\{\mu_n\} \subset \mathbb{D}$ tending to σ. Now, since τ^1 is the Denjoy-Wolff point of ϕ_μ, we obtain that τ^1 is a B-point for ϕ_μ, $\phi_\mu(\tau^1) = \tau^1$ and also

$$\lim_{\lambda \xrightarrow{\text{nt}} \tau^1} \frac{1-|\phi_\mu(\lambda)|^2}{1-|\lambda|^2} \leq 1,$$

for all $\mu \in \mathbb{D}$. Thus, it is possible to choose a sequence $\{\lambda_n\} \subset \mathbb{D}$ converging to τ^1 nontangentially, and sufficiently fast, so that we obtain $\lim_n \phi_{\mu_n}(\lambda_n) = \tau^1$, $\lim_n \frac{1-|\lambda_n|^2}{1-|\mu_n|^2} = 0$ and also

$$\limsup_n \frac{1-|\phi(\lambda_n,\mu_n)|^2}{1-||(\lambda_n,\mu_n)||^2} = \limsup_n \frac{1-|\phi_{\mu_n}(\lambda_n)|^2}{1-|\lambda_n|^2} \leq 1, \tag{3.24}$$

which implies that (τ^1, σ) is a B-point for ϕ and also $\phi(\tau^1, \sigma) = \tau^1$. Moreover, the model formula for ϕ tells us

$$||u^1_{(\lambda_n,\mu_n)}||^2 + \frac{1-|\mu_n|^2}{1-|\lambda_n|^2}||u^2_{(\lambda_n,\mu_n)}||^2 = \frac{1-|\phi(\lambda_n,\mu_n)|^2}{1-|\lambda_n|^2},$$

for all n. Letting $n \to \infty$ and taking into account the limits $\lim_n \frac{1-|\lambda_n|^2}{1-|\mu_n|^2} = 0$ and (3.24), we can deduce the existence of $u_{(\tau^1,\sigma)} \in Y_{(\tau^1,\sigma)}$ such that $||u^1_{(\tau^1,\sigma)}|| \leq 1$ and $u^2_{(\tau^1,\sigma)} = 0$. This implies that (τ^1, σ) is a left Type I DW point for ϕ. Since σ was arbitrary, (ii) has been established.

That (ii) implies (i) is obvious.

Now, we prove that (i) implies (iii). So, assume that there exists $|\sigma| \leq 1$ such that (τ^1, σ) is a B-point for ϕ, $\phi(\tau^1, \sigma) = \tau^1$ and also there exists $u_{(\tau^1,\sigma)} \in Y_{(\tau^1,\sigma)}$ such that $||u^1_{(\tau^1,\sigma)}|| \leq 1$ and $u^2_{(\tau^1,\sigma)} = 0$. We obtain

$$1 - \phi_\mu(\lambda)\overline{\tau^1} = 1 - \phi(\lambda, \mu)\overline{\tau^1} = (1 - \lambda\overline{\tau^1})\langle u^1_{(\lambda,\mu)}, u^1_{(\tau^1,\sigma)}\rangle, \tag{3.25}$$

for all $\lambda, \mu \in \mathbb{D}$. If we fix μ, we may repeat the proof of "(ii) implies (iii)" from Theorem 3.3.10 to obtain

$$\frac{|\tau^1 - \phi_\mu(\lambda)|^2}{1 - |\phi_\mu(\lambda)|^2} = \frac{|\tau^1 - \phi(\lambda, \mu)|^2}{1 - |\phi(\lambda, \mu)|^2} \leq \alpha_\sigma \frac{|\tau^1 - \lambda|^2}{1 - |\lambda|^2},$$

for all $\lambda, \mu \in \mathbb{D}$, where $\alpha_\sigma = ||u^1_{(\tau^1,\sigma)}||^2$. Such an equality is then known to imply (see subsection 3.2.2) that τ^1 is a B-point for ϕ_μ, $\phi_\mu(\tau^1) = \tau^1$ and also that the angular derivative of ϕ_μ at τ^1 is equal to $\alpha_\sigma \leq 1$, for all $\mu \in \mathbb{D}$. Since we also know (in view of Lemma 3.4.2) that $\phi_\mu \neq \mathrm{Id}_{\mathbb{D}}$, for all $\mu \in \mathbb{D}$, we can conclude that τ^1 is the common Denjoy-Wolff point of every slice function, i.e. (iii) holds.

Before we proceed, a few important observations are in order. Our previous arguments show that, if at least one point in the closed face $\{\tau^1\} \times \mathrm{cl}(\mathbb{D})$ is a left Type I DW point for ϕ, then for every $|\sigma| \leq 1$ there exists $u_{(\tau^1,\sigma)} = (u^1_{(\tau^1,\sigma)}, 0) \in Y_{(\tau^1,\sigma)}$ such that $||u^1_{(\tau^1,\sigma)}|| \leq 1$ and also (3.25) holds, for all $\lambda, \mu \in \mathbb{D}$. Since σ was arbitrary, (3.25) implies that the vectors $u^1_{(\tau^1,\sigma)}$ do not actually depend on σ, thus $u_{(\tau^1,\sigma)} = u_{\tau^1} = (u^1_{\tau^1}, 0)$ for all $\sigma \in \mathrm{cl}(\mathbb{D})$. In particular, letting $\phi'_\mu(\tau^1)$ denote the angular derivative of ϕ_μ at τ^1, we obtain

$$\phi'_\mu(\tau^1) = ||u_{\tau^1}||^2 \leq 1, \tag{3.26}$$

for all $|\mu| < 1$. Also, notice that, in view of Lemma 3.3.2, u_{τ^1} will be the unique vector in $Y_{(\tau^1,\sigma)}$ with M^2-component equal to 0, for all $|\sigma| \leq 1$.

Next, we show that (iii) implies (v). Fix an arbitrary $\sigma \in \mathrm{cl}(\mathbb{D})$. By our previous results, (τ^1, σ) is a B-point for ϕ, $\phi(\tau^1, \sigma) = \tau^1$ and also there exists $u_{\tau^1} = (u^1_{\tau^1}, 0) \in Y_{(\tau^1,\sigma)}$ (not depending on σ) such that $||u_{\tau^1}|| \leq 1$. If we also assume $|\sigma| < 1$, then (τ^1, σ) is a facial B-point, so [13, Theorem 3.2] implies that $Y_{(\tau^1,\sigma)} = \{u_{\tau^1}\}$ and also

$$\frac{D_{-(\tau^1,\sigma M)}\phi(\tau^1,\sigma)}{-\tau^1} = \frac{D_{-(\tau^1,\sigma M)}\phi(\tau^1,\sigma)}{-\phi(\tau^1,\sigma)} = ||u_{\tau^1}||^2 \leq 1,$$

for all $M > 0$, as desired. On the other hand, assume that $|\sigma| = 1$. We may apply Theorems 3.2.5 and 3.3.9 to obtain that

$$\frac{D_{-(\tau^1,\sigma M)}\phi(\tau^1,\sigma)}{-\tau^1} = K_{(\tau^1,\sigma)}(M) \leq ||u_{\tau^1}||^2 \leq 1,$$

for all $M > 0$. Actually, one can deduce the even stronger statement

$$\lim_{M\to\infty} \frac{D_{-(\tau^1,\sigma M)}\phi(\tau^1,\sigma)}{-\tau^1} = \lim_{M\to\infty} K_{(\tau^1,\sigma)}(M) = ||u_{\tau^1}||^2 = \phi'_\mu(\tau^1),$$

for all $\mu \in \mathbb{D}$. Since σ was arbitrary, we have established (v).

That (v) implies (iv) is evident, so all that remains is to show that (iv) implies (iii). So, assume there exists $(\tau^1, \sigma) \in \mathbb{T} \times \mathrm{cl}(\mathbb{D})$ such that the assumptions of (iv) are satisfied. If $|\sigma| < 1$, then $Y_{(\tau^1,\sigma)} = \{(u^1_{(\tau^1,\sigma)}, 0)\}$ and for any $M > 0$ we have

$$||u_{(\tau^1,\sigma)}||^2 = \frac{D_{-(\tau^1,\sigma M)}\phi(\tau^1,\sigma)}{-\tau^1} \leq 1.$$

This shows that (τ^1, σ) is a left Type I DW point for ϕ, which gives us (i), hence (iii) holds. On the other hand, assume $|\sigma| = 1$. Fix an increasing sequence $\{M_k\}$ tending to ∞. Since, by assumption, we have

$$\frac{D_{-(\tau^1,\sigma M_k)}\phi(\tau^1,\sigma)}{-\tau^1} \leq 1,$$

for all k, Theorem 3.3.10 implies that

$$\phi\big(E((\tau^1,\sigma), R_1, R_2)\big) \subset E(\tau^1, \max\{R_1, R_2/M_k\}),$$

for all $k \geq 1$ and $R_1, R_2 > 0$. Letting $k \to \infty$ yields

$$\phi\big(E((\tau^1,\sigma), R_1, R_2)\big) \subset E(\tau^1, R_1\}),$$

for all $R_1, R_2 > 0$, which translates into the inequality

$$\frac{|\tau^1 - \phi_\mu(\lambda)|^2}{1 - |\phi_\mu(\lambda)|^2} \leq \frac{|\tau^1 - \lambda|^2}{1 - |\lambda|^2},$$

for all $\lambda, \mu \in \mathbb{D}$. As already mentioned during the proof of "(i) implies (iii)", this implies that τ^1 is the Denjoy-Wolff point of ϕ_μ, for all μ, hence (iii) holds.

Finally, to prove the right Type I-version of the theorem, notice that the function $\widetilde{\phi} : \mathbb{D}^2 \to \mathbb{D}$ defined by $\widetilde{\phi}(\lambda) = \phi(\lambda^2, \lambda^1)$ ($\lambda \in \mathbb{D}^2$) has $(\widetilde{M}, \tilde{u})$ as a model, where $\tilde{u} : \mathbb{D}^2 \to \widetilde{M} = M^2 \oplus M^1$ is defined as

$$\tilde{u}(\lambda) = \langle \tilde{u}^1_\lambda, \tilde{u}^2_\lambda, \rangle = \langle u^2_{(\lambda^2,\lambda^1)}, u^1_{(\lambda^2,\lambda^1)} \rangle,$$

for all $\lambda \in \mathbb{D}^2$. By definition, (σ, τ^2) is a right Type I DW point for ϕ if and only if (τ^2, σ) is a left Type I DW point for $\widetilde{\phi}$. Thus, to obtain the right Type I-version of Theorem 3.2.9, one simply has to apply the left Type I-version of that same theorem to $\widetilde{\phi}$. □

We also establish a uniqueness result for Type I DW points.

Proposition 3.4.4. *Let $\phi \in \mathcal{S}_2$ be a left Type I function with model (M, u) such that $\phi \neq \pi^1$ and $\tau^1 \in \mathbb{T}$ is the common Denjoy-Wolff point of all maps ϕ_μ. Then, there exists $u_{\tau^1} = (u^1_{\tau^1}, 0) \in M$ such that $||u_{\tau^1}||^2 = \phi'_\mu(\tau^1) \leq 1$, for all μ. Moreover, given any $\sigma = (\sigma^1, \sigma^2) \in \mathbb{T} \times cl(\mathbb{D})$, if*

(i) $\sigma^1 = \tau^1$ and $|\sigma^2| = 1$, we have $u_{\tau^1} \in Y_\sigma$. Also, given any $v_\sigma \in Y_\sigma$, we have $v^2_\sigma = 0$ if and only if $v_\sigma = u_{\tau^1}$. If, in addition, we assume that σ is a C-point, we obtain that every $v_\sigma \in Y_\sigma$ that is not equal to u_{τ^1} must satisfy $||v^1_\sigma|| > ||u_{\tau^1}||$ and $v^2_\sigma \neq 0$;

(ii) $\sigma^1 = \tau^1$ and $|\sigma^2| < 1$, we have $Y_\sigma = \{u_{\tau^1}\}$;

(iii) $\sigma^1 \neq \tau^1$, σ is a B-point for ϕ and $\phi(\sigma) = \sigma^1$, then every $v_\sigma \in Y_\sigma$ must satisfy either $||v^1_\sigma|| > 1$ or $||v^1_\sigma|| = 1$ and $v^2_\sigma \neq 0$.

Consequently, if $\sigma = (\sigma^1, \sigma^2) \in \mathbb{T} \times cl(\mathbb{D})$, then σ is a left Type I DW point for ϕ if and only if $\sigma^1 = \tau^1$. Also, no point in $\mathbb{T} \times cl(\mathbb{D})$ can be a left Type II DW point for ϕ.

There is an analogous statement for right Type I DW points (we need to assume that $\phi \neq \pi^2$).

Proof. Let ϕ be a left Type I function satisfying our assumptions and denote by $u_{\tau^1} \in Y_{(\tau^1,\sigma)}$ (for all $|\sigma| \leq 1$) the vector described after the "(i) implies (iii)" part of the proof of Theorem 3.2.9. Also, let $\sigma = (\sigma^1, \sigma^2) \in \mathbb{T} \times \mathrm{cl}(\mathbb{D})$.

First, assume $\sigma^1 = \tau^1$ and $|\sigma^2| = 1$. The conclusions of (i) then follow by invoking Lemma 3.3.2 and Theorem 3.3.9.

On the other hand, if $\sigma^1 = \tau^1$ and $|\sigma^2| < 1$, an application of Theorem 3.2.4 does the job.

Now, assume $\sigma^1 \neq \tau^1$, σ is a B-point for ϕ and $\phi(\sigma) = \sigma^1$. Let $v_\sigma \in Y_\sigma$ be such that $||v_\sigma^1|| \leq 1$ and choose $\{(\lambda_n, \mu_n)\} \subset \mathbb{D}^2$ that converges to σ and also satisfies $\lim_n \phi(\lambda_n, \mu_n) = \sigma^1$ and $u_{(\lambda_n,\mu_n)} \to v_\sigma$ weakly as $n \to \infty$. Setting $(\lambda, \mu) = (\lambda_n, \mu_n)$ in (3.25) and letting $n \to \infty$ then allows us to obtain

$$1 - \sigma^1\overline{\tau^1} = (1 - \sigma^1\overline{\tau^1})\langle v_\sigma^1, u_{\tau^1}^1\rangle.$$

Since $\sigma^1 \neq \tau^1$, we obtain

$$\langle v_\sigma^1, u_{\tau^1}^1\rangle = 1 \geq ||v_\sigma^1||^2, ||u_{\tau^1}^1||^2,$$

which implies that $v_\sigma^1 = u_{\tau^1}^1$ and both have to be unit vectors. However, if we also assume that $v_\sigma^2 = 0$, we obtain that σ is a left Type I DW point for ϕ. In view of Theorem 3.2.9, this implies that the common Denjoy-Wolff point of all maps ϕ_μ is $\sigma^1 \neq \tau^1$, a contradiction (since $\phi \neq \pi^1$). Thus, we must have $v_\sigma^2 \neq 0$ and the proof of (iii) is complete.

Finally, to prove the right Type I-version of the theorem, apply the left Type I-version to $\widetilde{\phi}$. □

Note also the following consequence of Theorem 3.2.9, which (especially the second part) will be instrumental in Section 3.5.

Corollary 3.4.5. *Let $\phi : \mathbb{D}^2 \to \mathbb{D}$, $\phi \neq \pi^1$, be holomorphic. Then, ϕ has a left Type I DW point of the form $(\tau^1, \sigma) \in \mathbb{T} \times cl(\mathbb{D})$ if and only if*

$$\frac{|\tau^1 - \phi(\lambda,\mu)|^2}{1 - |\phi(\lambda,\mu)|^2} \leq \frac{|\tau^1 - \lambda^2|^2}{1 - |\lambda|^2}, \quad \forall(\lambda,\mu) \in \mathbb{D}^2.$$

If, in addition, we assume that $\tau = (\tau^1, \tau^2)$ is not a C-point for some $\tau^2 \in \mathbb{T}$, then for any increasing sequence $\{M_k\} \subset \mathbb{R}^+$ tending to ∞ one can find a sequence $\{r_k\}$ such that $r_k > 1$, $r_k \to 1$

and

$$\frac{|\tau^1 - \phi(\lambda,\mu)|^2}{1-|\phi(\lambda,\mu)|^2} \leq \max\left\{\frac{1}{r_k}\frac{|\tau^1-\lambda|^2}{1-|\lambda|^2}, \frac{1}{M_k}\frac{|\tau^1-\lambda|^2}{1-|\lambda|^2}\right\},$$

for all $\lambda, \mu \in \mathbb{D}$ *and* $k \geq 1$.

There is an analogous statement for right Type I DW points.

Proof. We only prove the left Type I-version. Since τ^1 will be the Denjoy-Wolff point of every map ϕ_μ, to obtain the first part of the theorem it suffices (in view of Theorem 3.2.9) to apply the one-variable Julia's inequality to every ϕ_μ.

To prove the second part, assume that there exists $\tau^2 \in \mathbb{T}$ such that $\tau = (\tau^1, \tau^2)$ is not a C-point (it will necessarily be a B-point). In view of Proposition 3.3.5 and Theorem 3.3.9, $\{K_\tau(M_k)\}_k$ will be strictly increasing, hence

$$||x^1_\tau(\delta_{M_k})||^2 + M_k||x^2_\tau(\delta_{M_k})||^2 < 1,$$

for all $k \geq 1$. In particular, we can find $r_k > 1$ such that

$$K_\tau\left(\frac{M_k}{r_k}\right) \leq \frac{1}{r_k}$$

for all $k \geq 1$. Theorems 3.2.5 and 3.3.10 then allows us to deduce the desired inequality. □

Next, we turn to Type II DW points.

Proof of Theorem 3.2.10. Let (M, u) be a model for ϕ.

First, we show that (i) implies (ii). By assumption, τ is a B-point for ϕ that is not a left Type I DW point, $\phi(\tau) = \tau^1$ and also there exists $u_\tau \in Y_\tau$ such that $||u^1_\tau|| < 1$ and

$$||u^1_\tau||^2 + K||u^2_\tau||^2 \leq 1. \tag{3.27}$$

To begin, we show that ϕ has to be a left Type II function. Indeed, assume instead that ϕ is a left Type I function, $\sigma^1 \in \mathbb{T}$ being the common Denjoy-Wolff point of all maps ϕ_μ. We cannot have $\sigma^1 = \tau^1$, since then τ would be (in view of Theorem 3.2.9) a left Type I DW point, contradicting the definition of a left Type II DW point. On the other hand, if $\sigma^1 \neq \tau^1$, we obtain a contradiction in view of Proposition 3.4.4(iii). Thus, ϕ cannot be a left Type I function and we conclude (by Theorem 3.4.3) that ϕ is a left Type II function.

Now, let $\xi : \mathbb{D} \to \mathbb{D}$ denote the holomorphic function that keeps track of the unique (interior) fixed point of each slice ϕ_μ, i.e. we have $\phi(\xi(\mu), \mu) = \xi(\mu)$, for all $\mu \in \mathbb{D}$. Let $0 < K' < K$. Since (3.27) holds and $u_\tau^2 \neq 0$, we must have $r||u_\tau^1||^2 + K'||u_\tau^2||^2 \leq 1$ whenever $r > 1$ is sufficiently close to 1, hence

$$||u_\tau^1||^2 + \frac{K'}{r}||u_\tau^2||^2 \leq \frac{1}{r}.$$

Proposition 3.3.5 then implies that

$$\frac{D_{-(\tau^1, \tau^2 K'/r)}\phi(\tau)}{-\tau^1} = K_\tau(K'/r) \leq 1/r.$$

In view of Theorem 3.3.10, we obtain

$$\frac{|\tau^1 - \phi(\lambda, \mu)|^2}{1 - |\phi(\lambda, \mu)|^2} \leq \max\left\{\frac{1}{r}\frac{|\tau^1 - \lambda|^2}{1 - |\lambda|^2}, \frac{1}{K'}\frac{|\tau^2 - \mu|^2}{1 - |\mu|^2}\right\}, \tag{3.28}$$

for all $\lambda, \mu \in \mathbb{D}$. Plugging in $\lambda = \xi(\mu)$ in (3.28) then gives us

$$\frac{|\tau^1 - \xi(\mu)|^2}{1 - |\xi(\mu)|^2} \leq \max\left\{\frac{1}{r}\frac{|\tau^1 - \xi(\mu)|^2}{1 - |\xi(\mu)|^2}, \frac{1}{K'}\frac{|\tau^2 - \mu|^2}{1 - |\mu|^2}\right\},$$

for all $\mu \in \mathbb{D}$. Since $1/r < 1$, this last inequality implies

$$\frac{1}{r}\frac{|\tau^1 - \xi(\mu)|^2}{1 - |\xi(\mu)|^2} \leq \frac{1}{K'}\frac{|\tau^2 - \mu|^2}{1 - |\mu|^2}$$

whenever $r > 1$ is sufficiently close to 1. Letting $r \to 1$ first and $K' \to K$ afterwards yields

$$\frac{|\tau^1 - \xi(\mu)|^2}{1 - |\xi(\mu)|^2} \leq \frac{1}{K}\frac{|\tau^2 - \mu|^2}{1 - |\mu|^2},$$

for all $\mu \in \mathbb{D}$. The one-variable Julia's inequality (see Section 3.2) then allows us to deduce that τ^2 is a B-point for ξ, $\xi(\tau^2) = \tau^1$ and also

$$A := \left(\liminf_{\mu \to \tau^1} \frac{1 - |\xi(\mu)|}{1 - |\mu|}\right)^{-1} \geq K. \tag{3.29}$$

To show that (ii) implies (i), assume that ϕ is a left Type II function and ξ satisfies the given hypotheses. Substituting $\lambda = \xi(\mu)$ into the model formula

$$1 - |\phi(\lambda,\mu)|^2 = (1-|\lambda|^2)||u^1_{(\lambda,\mu)}||^2 + (1-|\mu|^2)||u^2_{(\lambda,\mu)}||^2$$

yields

$$\begin{aligned}\frac{1-|\phi(\xi(\mu),\mu)|^2}{1-|\mu|^2} &= \frac{1-|\xi(\mu)|^2}{1-|\mu|^2} \\ &= \frac{1-|\xi(\mu)|^2}{1-|\mu|^2}||u^1_{(\xi(\mu),\mu)}||^2 + ||u^2_{(\xi(\mu),\mu)}||^2, \end{aligned} \tag{3.30}$$

for all $\mu \in \mathbb{D}$. By assumption, we can find a (radial) sequence $\{\mu_n\} \subset \mathbb{D}$ such that $\lim_n \mu_n = \tau^2$, $\lim_n \xi(\mu_n) = \lim_n \phi(\xi(\mu_n),\mu_n) = \tau^1$ and

$$\lim_n \frac{1-|\xi(\mu_n)|}{1-|\mu_n|} = \lim_n \frac{1-|\xi(\mu_n)|^2}{1-|\mu_n|^2} \leq \frac{1}{K}.$$

Note also that $\lim_n \frac{1-|\xi(\mu_n)|}{1-|\mu_n|} > 0$, else the single-variable Julia's inequality would imply that ξ is a unimodular constant, a contradiction. Thus, plugging in $\mu = \mu_n$ in (3.30) and letting $n \to \infty$ allows us to conclude that τ is a B-point for ϕ, $\phi(\tau) = \tau^1$ and also there exists $u_\tau \in Y_\tau$ such that

$$||u^1_\tau||^2 + K||u^2_\tau||^2 \leq 1.$$

Moreover, since ϕ is a left Type II function, Theorem 3.2.9 implies that τ cannot be a left Type I DW point and $u^2_\tau \neq 0$, hence $||u^1_\tau|| < 1$ and we are done.

Note that the previous argument actually shows that A (as defined in (3.29)) is the maximum among all constants $K > 0$ such that τ is a left Type II DW point for ϕ with constant K.

Next, we show that (i) implies (iii). So, assume that all relevant assumptions are satisfied. Note that we cannot have

$$\frac{D_{-(\tau^1,\tau^2 M)}\phi(\tau)}{-\tau^1} = K_\tau(M) \leq 1$$

for all $M > 0$, as in such a case Theorem 3.2.9 would imply that τ is a left Type I DW point, a contradiction. Since $K_\tau(M)$ is continuous, increasing and $K_\tau(K) \leq 1$, there must exist $C \geq K$ such that $K_\tau(C) = 1$. Moreover, $K_\tau(M)$ cannot be constant (again by Theorem 3.2.9), hence (iii) holds.

We now prove the converse. Assume τ is a B-point for ϕ, $\phi(\tau) = \tau^1$, $K_\tau(M)$ is not constant with respect to M and also there exists $C \geq K$ such that $K_\tau(C) = 1$, hence

$$||x_\tau^1(\delta_C)||^2 + C||x_\tau^2(\delta_C)||^2 = 1.$$

We cannot have $x_\tau^2(\delta_C) = 0$ (else, Theorem 3.3.3 would imply that $K_\tau(M)$ is constant), thus $||x_\tau^1(\delta_C)|| < 1$. Moreover, τ cannot be a left Type I DW point, as, in view of Theorem 3.2.9 and the equality $K_\tau(C) = 1$, the only way for this to be possible would be having $K_\tau(M) = 1$, for all $M > 0$, a contradiction. Thus, τ is a left Type II DW point with constant $C \geq K$ and we are done.

We can say more about the constant C (which is uniquely determined, as $K_\tau(M)$ is strictly increasing). Indeed, our previous argument shows that τ is a left Type II DW point with constant C. Now, if $C' > C$, then

$$1 = K_\tau(C) < K_\tau(C'),$$

and thus, in view of "(i) implies (iii)", we obtain that τ cannot be a left Type II DW point with costant C'. This means that C is the largest constant with this property, hence $C = A$, as defined in (3.29).

Finally, as seen in the end of the proof of Theorem 3.2.9, to show the right Type II-version of the theorem we only need apply the left Type II-version to $\widetilde{\phi}$. □

Proof of Theorem 3.2.11. Combine Theorems 3.2.9-3.2.10 with Lemma 3.2.3, Theorem 3.3.3 and Proposition 3.3.5. □

Remark 3.4.6. Let $\xi : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then, one can always find $\phi \in \mathcal{S}_2$ (that will necessarily be a left Type II function) such that $\phi(\xi(\mu), \mu) = \xi(\mu)$ for all $\mu \in \mathbb{D}$. Indeed, it can be easily verified that the function

$$\phi(\lambda, \mu) := \frac{\lambda + \xi(\mu)}{2}$$

has the property in question.

Remark 3.4.7. As already mentioned in subsection 3.2.5, there exist left Type II functions that do not have left Type II DW points. Indeed, if e.g. ϕ is any left Type II function such that the map ξ

satisfies $\xi(\mathbb{D}) \subset r\mathbb{D}$ for some $r \in (0,1)$, then Theorem 3.2.10 implies that ϕ does not have any left Type II DW points (on account of ξ not having any B-points).

We can also prove certain uniqueness results for Type II DW points.

Proposition 3.4.8. *Let $\phi : \mathbb{D}^2 \to \mathbb{D}$ with model (M, u) be such that $\tau = (\tau^1, \tau^2) \in \mathbb{T}^2$ is a left Type II DW point, with $\xi : \mathbb{D} \to \mathbb{D}$ satisfying $\phi(\xi(\mu), \mu) = \xi(\mu)$, for all $\mu \in \mathbb{D}$, and $A > 0$ defined as in (3.29). Then, the following assertions all hold.*

(i) $x_\tau(\delta_A)$ is the unique vector $u_\tau \in Y_\tau$ such that

$$||u_\tau^1||^2 + A||u_\tau^2||^2 \leq 1. \tag{3.31}$$

(ii) No point in $\mathbb{T} \times cl(\mathbb{D})$ can be a left Type I DW point for ϕ.

(iii) If $\sigma \in \mathbb{T}$ and $\sigma \neq \tau^1$, then (σ, τ^2) is not a left Type II DW point for ϕ.

There is an analogous result for right Type II DW points.

Proof. First, we prove (i). Note that $x_\tau(\delta_A)$ certainly satifies

$$||x_\tau^1(\delta_A)||^2 + A||x_\tau^2(\delta_A)||^2 = 1,$$

as $K_\tau(A) = 1$. Also, if $u_\tau \in Y_\tau$ is such that (3.31) holds, Proposition 3.3.5 implies that $||u_\tau^1||^2 + A||u_\tau^2||^2 = 1$ and $x_\tau(\delta_A) = u_\tau$, as desired.

(ii) is an immediate consequence of Proposition 3.4.4.

Finally, (iii) is a simple application of Theorem 3.2.10, since ξ cannot have two distinct values (at least not in the sense of nontangential limits) at its B-point τ^2. □

The following Julia-type inequalities are obtained as a consequence of Theorem 3.2.10. The significance of parts (ii) and (iii) will be made apparent in Section 3.5.

Corollary 3.4.9. *Assume $\phi : \mathbb{D}^2 \to \mathbb{D}$ has a left Type II DW point $\tau = (\tau^1, \tau^2) \in \mathbb{T}^2$ and let $A > 0$ be defined as in (3.29). Also, fix $A_- < A$ and $r_1 < 1$.*

(i) For all $(\lambda,\mu) \in \mathbb{D}^2$*, we have*

$$\frac{|\tau^1 - \phi(\lambda,\mu)|^2}{1 - |\phi(\lambda,\mu)|^2} \leq \max\left\{\frac{|\tau^1 - \lambda|^2}{1 - |\lambda|^2}, \frac{1}{A}\frac{|\tau^2 - \mu|^2}{1 - |\mu|^2}\right\};$$

(ii) Moreover, if $r_2 > 1$ *is sufficiently close to 1, then*

$$\frac{|\tau^1 - \phi(\lambda,\mu)|^2}{1 - |\phi(\lambda,\mu)|^2} \leq \max\left\{\frac{1}{r_2}\frac{|\tau^1 - \lambda|^2}{1 - |\lambda|^2}, \frac{1}{A_-}\frac{|\tau^2 - \mu|^2}{1 - |\mu|^2}\right\},$$

for all $(\lambda,\mu) \in \mathbb{D}^2$*;*

(iii) Finally, if $x_\tau^1(\delta_A) \neq 0$ *and* $A < A_+$ *is sufficiently close to A, then*

$$\frac{|\tau^1 - \phi(\lambda,\mu)|^2}{1 - |\phi(\lambda,\mu)|^2} \leq \max\left\{\frac{1}{r_1}\frac{|\tau^1 - \lambda|^2}{1 - |\lambda|^2}, \frac{1}{A_+}\frac{|\tau^2 - \mu|^2}{1 - |\mu|^2}\right\},$$

for all $(\lambda,\mu) \in \mathbb{D}^2$.

There is an analogous result for right Type II DW points.

Proof. To prove (i), combine Theorems 3.3.10 and 3.2.10.

For (ii), note that, since $||x_\tau^1(\delta_A)||^2 + A||x_\tau^2(\delta_A)||^2 = 1$, $A_- < A$ and $x_\tau^2(\delta_A) \neq 0$ (by definition of a left Type II DW point), one obtains that

$$r_2||x_\tau^1(\delta_A)||^2 + A_-||x_\tau^2(\delta_A)||^2 \leq 1,$$

for all $r_2 > 1$ sufficiently close to 1, hence $K_\tau(A_-/r_2) \leq 1/r_2$. An application of Theorem 3.3.10 then finishes the job.

(iii) is proved in an analogous manner (note that we have to assume $x_\tau^1(\delta_A) \neq 0$, since not all left Type II DW points have this property). □

3.5 Refining Hervé's Theorem

Let $F = (\phi, \psi)$ denote a holomorphic self-map of $\mathbb{D}^2$ without interior fixed points. We use

$$F^n = (\phi_n, \psi_n) = \underbrace{F \circ F \circ \cdots \circ F}_{n \text{ times}}$$

to denote the sequence of iterates of F. Note that $\phi \circ F^n = \phi_{n+1}$ and $\psi \circ F^n = \psi_{n+1}$, for all $n \geq 1$.

Hervé analysed the behavior of $\{F^n\}$ by looking at three separate cases, depending on the Type of ϕ and ψ. In this section, we study the connection between Hervé's results from [80] and the DW points we defined in Section 3.4. In particular, we will show how the conclusions of Theorem 3.2.7 can be strengthened if one assumes that the DW points of ϕ and/or ψ are not C-points (i.e. the functions do not possess angular gradients there).

3.5.1 The (Type II, Type II) case

We begin with the case where ϕ and ψ are left Type II and right Type II functions, respectively. Even though not every Type II function will, in general, have Type II DW points (see Remark 3.4.7), F having no interior fixed points changes the situation dramatically, as seen in the following theorem. A proof of it (without the model terminology) is essentially contained in [73, Theorem 2] (see also [80, Section 16]). We give an alternative proof by using the results we have developed so far.

Theorem 3.5.1. *Assume $F = (\phi, \psi) : \mathbb{D}^2 \to \mathbb{D}^2$ is holomorphic and ϕ, ψ are left Type II and right Type II functions, respectively. Also, let $\xi, \eta : \mathbb{D} \to \mathbb{D}$ denote the (unique) functions such that $\phi(\xi(\mu), \mu) = \xi(\mu)$ and $\psi(\lambda, \eta(\lambda)) = \eta(\lambda)$, for all $\lambda, \mu \in \mathbb{D}$. Then, F has no interior fixed points if and only if*

(i) there exist $\tau \in \mathbb{T}^2$ and $K > 0$ such that τ is simultaneously a left Type II DW point for ϕ with constant K and a right Type II DW point for ψ with constant $1/K$ and also

(ii) $\phi \circ \eta \neq Id_{\mathbb{D}}$ and $\eta \circ \phi \neq Id_{\mathbb{D}}$.

Moreover, assuming F has no interior fixed points, the point $\tau = (\tau^1, \tau^2) \in \mathbb{T}^2$ above is uniquely determined: τ^1 is the Denjoy-Wolff point of $\xi \circ \eta$, while τ^2 is the Denjoy-Wolff point of $\eta \circ \xi$.

Proof. Let $(M, u), (N, v)$ be models for ϕ and ψ, respectively. Also, for $\tau \in \partial\mathbb{D}^2$, we will denote the corresponding cluster sets by Y_τ^ϕ and Y_τ^ψ.

First, assume F has no interior fixed points. Let $0 < r_n \uparrow 1$ and consider the functions $r_n \cdot F$. Since $\mathrm{cl}(r_n F(\mathbb{D}^2)) \subset \mathbb{D}^2$, for every n, the Earle-Hamilton Theorem [69] implies that each $r_n F$ has a fixed point $(\lambda_n, \mu_n) \in \mathbb{D}^2$. Since F has no fixed points in $\mathbb{D}^2$, we obtain that $(\lambda_n, \mu_n) \to \partial\mathbb{D}^2$. There are three possible cases to examine.

If $\lim_n \frac{1-|\lambda_n|^2}{1-|\mu_n|^2} = 0$, then $(\lambda_n, \mu_n) \to \tau = (\tau^1, \sigma) \in \mathbb{T} \times \mathrm{cl}(\mathbb{D})$. We can use the model formula for ϕ to write

$$1 - |\lambda_n|^2 \geq 1 - \frac{1}{r_n^2}|\lambda_n|^2 = 1 - |\phi(\lambda_n, \mu_n)|^2$$
$$= (1 - |\lambda_n|^2)||u^1_{(\lambda_n,\mu_n)}||^2 + (1 - |\mu_n|^2)||u^2_{(\lambda_n,\mu_n)}||^2.$$

Thus, for n large enough, we deduce

$$1 \geq \frac{1 - |\phi(\lambda_n, \mu_n)|^2}{1 - ||(\lambda_n, \mu_n)||^2} = \frac{1 - |\phi(\lambda_n, \mu_n)|^2}{1 - |\lambda_n|^2}$$
$$= ||u^1_{(\lambda_n,\mu_n)}||^2 + \frac{1 - |\mu_n|^2}{1 - |\lambda_n|^2}||u^2_{(\lambda_n,\mu_n)}||^2. \tag{3.32}$$

Letting $n \to \infty$, we obtain (in view of $\lim_n \frac{1-|\lambda_n|^2}{1-|\mu_n|^2} = 0$ and $\lim_n \phi(\lambda_n, \mu_n) = \tau^1$) that $\tau = (\tau^1, \sigma)$ is a B-point for ϕ, $\phi(\tau) = \tau^1$ and also there exists a weak limit $u_\tau \in Y_\tau^\phi$ such that $||u_\tau^1|| \leq 1, u_\tau^2 = 0$. This implies that τ is a left Type I DW point for ϕ, contradicting the fact that ϕ is a left Type II function.

If $\lim_n \frac{1-|\lambda_n|^2}{1-|\mu_n|^2} = \infty$, one can argue in a manner analogous to the previous case to deduce that ψ has a right Type I DW point, which is again a contradiction.

Finally, assume that $\lim_n \frac{1-|\lambda_n|^2}{1-|\mu_n|^2} = \frac{1}{K} \in (0, \infty)$. Hence, $(\lambda_n, \mu_n) \to \tau = (\tau^1, \tau^2) \in \mathbb{T}^2$. Letting $n \to \infty$ in (3.32) then yields that τ is a B-point for ϕ, $\phi(\tau) = \tau^1$ and also there exists $u_\tau \in Y_\tau^\phi$ such that $||u_\tau^1||^2 + K||u_\tau^2||^2 \leq 1$. Note that $u_\tau^2 \neq 0$, else τ would be a left Type I DW point. Thus, since ϕ is a left Type II function, τ must be a left Type II DW point for ϕ with constant K. Further, an analogous argument involving the model formula for ψ shows that τ is a B-point for ψ, $\phi(\tau) = \tau^2$ and also there exists $\upsilon_\tau \in Y_\tau^\psi$ such that $(1/K)||\upsilon_\tau^1||^2 + ||\upsilon_\tau^2||^2 \leq 1$. Also, $\upsilon_\tau^1 \neq 0$, since ψ is not a right Type I function. Thus, τ must be a right Type II DW point for ψ with constant $1/K$, which proves (i). To show that (ii) holds, note that if e.g. $\xi(\eta(\lambda)) = \lambda$ for some $\lambda \in \mathbb{D}$, then

$$F(\xi(\eta(\lambda)), \eta(\lambda)) = \big(\phi(\xi(\eta(\lambda)), \eta(\lambda)), \psi(\xi(\eta(\lambda)), \eta(\lambda))\big)$$

$$= (\xi(\eta(\lambda)), \eta(\lambda)),$$

a contradiction. In particular, we obtain the even stronger conclusion that neither $\xi \circ \eta$ nor $\eta \circ \xi$ can have interior fixed points.

Conversely, assume that (i) and (ii) both hold. In view of Theorem 3.2.10, (i) implies that τ^1 and τ^2 are B-points for η and ξ respectively, $\xi(\tau^2) = \tau^1$, $\eta(\tau^1) = \tau^2$ and also (by the single-variable Julia's inequality)

$$\xi(E(\tau^2, R)) \subset E(\tau^1, R/K) \quad \text{and} \quad \eta(E(\tau^1, R)) \subset E(\tau^2, KR),$$

for all $R > 0$. Thus, $(\xi \circ \eta)(E(\tau^1, R)) \subset \xi(E(\tau^2, KR)) \subset E(\tau^1, R)$, for all $R > 0$, which (combined with the fact that $\xi \circ \eta \neq \mathrm{Id}_{\mathbb{D}}$ must have a unique Denjoy-Wolff point) allows us to deduce that τ^1 is the Denjoy-Wolff point of $\xi \circ \eta$. An analogous argument shows that τ^2 is the Denjoy-Wolff point of $\eta \circ \xi$. Thus, the point τ is indeed uniquely determined. Also, notice that, in view of these observations, neither $\xi \circ \eta$ nor $\eta \circ \xi$ can have interior fixed points. Now, let (λ_0, μ_0) be an interior fixed point of F. We obtain

$$\phi(\lambda_0, \mu_0) = \lambda_0 \quad \text{and} \quad \psi(\lambda_0, \mu_0) = \mu_0.$$

Thus, $\xi(\mu_0) = \lambda_0$ and $\eta(\lambda_0) = \mu_0$, which implies that $\xi(\eta(\lambda_0)) = \lambda_0$, a contradiction. □

Now, let $F = (\phi, \psi) : \mathbb{D}^2 \to \mathbb{D}^2$, $\tau \in \mathbb{T}^2$ and $K > 0$ be as in Theorem 3.5.1, with F having no interior fixed points. Recall that, in this setting, one obtains a perfect analogue of the one-variable Denjoy-Wolff Theorem, i.e. the sequence of iterates $\{F^n\}$ converges uniformly on compact sets to τ (Theorem 3.2.7(iv)). A crucial ingredient for Hervé's proof of this fact is given by the invariant horospheres

$$F(E(\tau, R, KR)) \subset E(\tau, R, KR), \tag{3.33}$$

obtained as an application of Corollary 3.4.9.

So, we know that the entire sequence $\{F^n\}$ has to converge to τ, but can we use (3.33) to say more? Our main result in this subsection is a refinement of [80, Lemme 2], which concerns the location of the orbits $\{F^n(\lambda,\mu)\}_n$ with respect to the boundary of the invariant horospheres (3.33). To set up the statement, fix $(\lambda_0,\mu_0)\in\mathbb{D}^2$. For convenience, we will write $F^n=(\phi_n,\psi_n)$ in place of $F^n(\lambda_0,\mu_0)=(\phi_n(\lambda_0,\mu_0),\psi_n(\lambda_0,\mu_0))$. We also define:

$$A_n=\frac{|\tau^1-\phi_n|^2}{1-|\phi_n|^2}\quad\text{and}\quad B_n=\frac{|\tau^2-\psi_n|^2}{1-|\psi_n|^2}.$$

Theorem 3.5.2. *Let $F=(\phi,\psi):\mathbb{D}^2\to\mathbb{D}^2$, $\tau\in\mathbb{T}^2$ and $K>0$ be as in Theorem 3.5.1, with F having no interior fixed points. Then, either $F^n\to\tau$ in the horospheric topology or there exist $\rho_0,\rho_1\geq 0$ (depending on (λ_0,μ_0)) that are not both 0 such that*

$$A_{2n}\to\rho_0,\quad A_{2n+1}\to\rho_1,\quad B_{2n+1}\to K\rho_0,\quad B_{2n}\to K\rho_1.$$

Moreover, if τ is not a C-point for either ϕ or ψ, we can take $\rho_0=\rho_1$.

Proof. For every $n\geq 1$, let R_n denote the smallest radius such that $F_n\in E_n:=\mathrm{cl}\big(E(\tau,R_n,KR_n)\big)$. In view of (3.33), the sequence $\{R_n\}$ is non-increasing. $\{A_n\},\{B_n\}$ needn't also be non-increasing, however they have to satisfy (by definition of R_n) $\max\{KA_n,B_n\}=KR_n$, for all n.

Now, if $R_n\to 0$, then $A_n,B_n\to 0$ and we conclude that $F_n\to\tau$ in the horospheric topology. So, assume R_n converges to $\rho>0$.

First, consider the case where τ is not a C-point for either ϕ or ψ. Without loss of generality, we may suppose that τ is not a C-point for ϕ. Let u_τ denote any vector in Y_τ^ϕ such that $||u_\tau^1||^2+K||u_\tau^2||^2\leq 1$ (its existence is guaranteed by Theorem 3.2.10). In view of Theorem 3.3.3, it must be true that $u_\tau^1\neq 0$. We will show that $A_n\to\rho$ and $B_n\to K\rho$.

Indeed, aiming towards a contradiction, assume $B_n\not\to K\rho$ (the case where $A_n\not\to\rho$ can be treated in an analogous manner). In view of the equality $\max\{KA_n,B_n\}=KR_n$, there exists a subsequence $\{n_k\}$ and $r\in(0,\rho)$ such that $B_{n_k}\leq Kr$ for all k. This implies that $A_{n_k}=R_{n_k}$ for all k. Now, given $0 < K_- < K$ sufficiently close to K, we can choose $r_2 > 1$ sufficiently close to 1 such that $Kr/K_- < \rho/r_2$ and also, in view of Corollary 3.4.9(ii),

$$\frac{|\tau^1 - \phi(\lambda,\mu)|^2}{1 - |\phi(\lambda,\mu)|^2} \leq \max\left\{\frac{1}{r_2}\frac{|\tau^1 - \lambda|^2}{1 - |\lambda|^2}, \frac{1}{K_-}\frac{|\tau^2 - \mu|^2}{1 - |\mu|^2}\right\},$$

for all $\lambda, \mu \in \mathbb{D}$. In particular, we have

$$\begin{aligned} A_{n_k+1} &= \frac{|\tau^1 - \phi_{n_k+1}|^2}{1 - |\phi_{n_k+1}|^2} \\ &\leq \max\left\{\frac{1}{r_2}A_{n_k}, \frac{1}{K_-}B_{n_k}\right\} \\ &\leq \max\left\{\frac{1}{r_2}R_{n_k}, \frac{1}{K_-}Kr\right\} \\ &= \frac{R_{n_k}}{r_2}, \end{aligned} \tag{3.34}$$

as $\frac{Kr}{K_-} < \frac{\rho}{r_2} \leq \frac{R_{n_k}}{r_2}$, for all k. Now, let v_τ denote any vector in Y_τ^ψ such that $\tilde{K}||v_\tau^1||^2 + ||v_\tau^2||^2 \leq 1$, where $\tilde{K} = 1/K$ (as in the case of u_τ, we obtain the existence of this vector by Theorem 3.2.10). We look at two separate cases, depending on whether $v_\tau^2 \neq 0$.

So, assume $v_\tau^2 \neq 0$. In this case, given $r_1 < 1$ sufficiently close to 1, we can find $\tilde{K} < \tilde{K}_+$ sufficiently close to $\tilde{K}$ such that $\frac{\tilde{K}_+ r}{r_1} < \tilde{K}\rho$ and also, in view of the right Type II version of Corollary 3.4.9(iii),

$$\frac{|\tau^2 - \psi(\lambda,\mu)|^2}{1 - |\psi(\lambda,\mu)|^2} \leq \max\left\{\frac{1}{\tilde{K}_+}\frac{|\tau^1 - \lambda|^2}{1 - |\lambda|^2}, \frac{1}{r_1}\frac{|\tau^2 - \mu|^2}{1 - |\mu|^2}\right\},$$

for all $\lambda, \mu \in \mathbb{D}$. In particular, we have

$$\begin{aligned} B_{n_k+1} &\leq \max\left\{\frac{A_{n_k}}{\tilde{K}_+}, \frac{B_{n_k}}{r_1}\right\} \\ &\leq \max\left\{\frac{R_{n_k}}{\tilde{K}_+}, \frac{r}{\tilde{K}r_1}\right\} \\ &= \frac{R_{n_k}}{\tilde{K}_+}, \end{aligned} \tag{3.35}$$

as $\frac{r}{\tilde{K}r_1} < \frac{\rho}{\tilde{K}_+} \leq \frac{R_{n_k}}{\tilde{K}_+}$, for all k. Combining (3.34) with (3.35), we obtain

$$KR_{n_k+1} = \max\{KA_{n_k+1}, B_{n_k+1}\} < cKR_{n_k},$$

for some $c \in (0,1)$ and all k large enough. Letting $k \to \infty$ then leads to a contradiction.

Now, assume $v_\tau^2 = 0$. In view of (3.34), we can find $r' < \rho$ such that for all k large enough we have $A_{n_k+1} \leq r' < \rho$. Also, since $v_\tau^1 \neq 0$, we can mimic the proof of (3.34) (with ψ in place of ϕ) to obtain $B_{n_k+2} < c_1 K R_{n_k+1}$ for some $c_1 \in (0,1)$ and all k large enough. Similarly, since $u_\tau^1 \neq 0$, we can mimic the proof of (3.35) (with ϕ in place of ψ) to obtain the existence of $c_2 \in (0,1)$ such that $A_{n_k+2} \leq c_2 R_{n_k+1}$, for all k large enough. Thus, we arrive at the conclusion $KR_{n_k+2} = \max\{KA_{n_k+2}, B_{n_k+2}\} < \max\{c_1, c_2\} KR_{n_k+1,}$ for all k large enough, which yields a contradiction when we let $k \to \infty$.

The only case left to examine is when $R_n \to \rho > 0$ and $u_\tau^1 = v_\tau^2 = 0$. Mimicking the proof of "(ii) implies (iii)" from Theorem 3.3.10, we may conclude that

$$A_{n+1} \leq \frac{B_n}{K} \text{ and } B_{n+1} \leq KA_n,$$

for all $n \geq 1$. Thus,

$$A_{n+2} \leq A_n \text{ and } B_{n+2} \leq B_n,$$

which means that the sequences $\{A_{2n}\}, \{A_{2n+1}\}, \{B_{2n}\}$ and $\{B_{2n+1}\}$ are all non-increasing. Thus, there exist nonnegative numbers $\rho_0, \rho_1, \rho_0', \rho_1'$ such that $A_{2n} \to \rho_0, A_{2n+1} \to \rho_1$, $B_{2n+1} \to \rho_1'$ and $B_{2n} \to \rho_0'$. The inequalities $A_{2n+1} \leq \frac{B_{2n}}{K}$ and $B_{2n} \leq KA_{2n-1}$ give us $\rho_1 \leq \rho_0'/K$ and $\rho_0' \leq K\rho_1$, respectively. Thus, $\rho_0' = K\rho_1$ and an entirely analogous argument shows that $\rho_1' = K\rho_0$. We conclude that

$$A_{2n} \to \rho_0, \quad A_{2n+1} \to \rho_1, \quad B_{2n+1} \to K\rho_0, \quad B_{2n} \to K\rho_1,$$

where $\max\{\rho_0, \rho_1\} = \rho$ (by definition of ρ) and so ρ_0, ρ_1 cannot be zero at the same time. This concludes the proof. □

3.5.2 The (Type I,Type II) case

Assume now that ϕ and ψ are left Type I and right Type II functions, respectively. This immediately implies that $F = (\phi, \psi)$ does not have any interior fixed points. In this setting, Hervé proved that any cluster point of the sequence of iterates $\{F^n\}$ must be of the form (τ^1, h), where h is either a holomorphic function $\mathbb{D}^2 \to \mathbb{D}$ or a unimodular constant and τ^1 is the common Denjoy-Wolff point of all slices ϕ_μ (Theorem 3.2.7(iii)). Examples showing that this conclusion cannot, in general, be improved, are contained in [80, Section 11].

Now, if we, in addition, assume the existence of $\sigma \in \mathbb{T}$ such that (τ^1, σ) is a right Type II DW point for ψ, stronger conclusions can be drawn about the cluster set of $\{F^n\}$.

Proposition 3.5.3. *Assume $F = (\phi, \psi) : \mathbb{D}^2 \to \mathbb{D}^2$ is such that ϕ is a left Type I function (with τ^1 being the common Denjoy-Wolff point of all slices ϕ_μ) and ψ has a right Type II DW point of the form $\tau = (\tau^1, \sigma) \in \mathbb{T}^2$. Then, there exists $K > 0$ such that*

$$F(E(\tau, R, KR)) \subset E(\tau, R, KR),$$

for all $R > 0$. Thus, any cluster point of the sequence of iterates $\{F^n\}$ must be of the form (τ^1, h), where h is either a holomorphic function $\mathbb{D}^2 \to \mathbb{D}$ or the constant σ.

Proof. Assuming ψ has a right Type II DW point of the form $\tau = (\tau^1, \sigma)$, one can combine Corollary 3.4.5 with Corollary 3.4.9 to conclude that

$$F(E(\tau, AR, R)) \subset E(\tau, AR, R),$$

for all $R > 0$, where $A = \left(\liminf_{\lambda \to \tau^1} \frac{1-|\eta(\lambda)|}{1-|\lambda|} \right)^{-1} > 0$ and $\eta : \mathbb{D} \to \mathbb{D}$ is the holomorphic function satisfying $\psi(\lambda, \eta(\lambda)) = \eta(\lambda)$ for all $\lambda \in \mathbb{D}$.

To obtain the conclusion regarding the behavior of the iterates, combine the previous result with Theorem 3.2.7(iii) and the observation that, for any $R > 0$, $\mathrm{cl}(E(\tau, AR, R)) \cap \mathbb{T}^2 = \tau$. □

Remark 3.5.4. In the absence of a right Type II DW of the form (τ^1, σ) for ψ, the behavior of $\{F^n\}$ could be considerably more complicated. Indeed, it could even happen that infinitely many unimodular constants $\{\sigma(i) \mid i \in I\}$ exist such that the constant $(\tau^1, \sigma(i))$ is a cluster point of $\{F^n\}$, for every $i \in I$; see the 2nd example in [80, Section 11].

In the setting of Proposition 3.5.3, it is clear (in view of Theorem 3.2.9) that (τ^1, σ) will always be a left Type I DW point for ϕ, no matter the value of σ. Surprisingly, having (τ^1, σ) not be a C-point for ϕ will force the entire sequence $\{F^n\}$ to converge to (τ^1, σ). This is the content of Theorem 3.2.12, the proof of which does not make use of Hervé's results.

Proof of Theorem 3.2.12. Assume $\tau = (\tau^1, \sigma) \in \mathbb{T}^2$ satisfies the hypotheses of the theorem. Clearly, ϕ and ψ will be left Type I and right Type II functions, respectively, with the common Denjoy-Wolff point of all slices ϕ_μ being τ^1. By Proposition 3.5.3, there exists $K > 0$ such that

$$F(E(\tau, KR, R)) \subset E(\tau, KR, R), \tag{3.36}$$

for all $R > 0$. Now, fix $(\lambda_0, \mu_0) \in \mathbb{D}^2$. For convenience, we will write $F^n = (\phi_n, \psi_n)$ in place of $F^n(\lambda_0, \mu_0) = (\phi_n(\lambda_0, \mu_0), \psi_n(\lambda_0, \mu_0))$. We also define:

$$A_n = \frac{|\tau^1 - \phi_n|^2}{1 - |\phi_n|^2} \quad \text{and} \quad B_n = \frac{|\sigma - \psi_n|^2}{1 - |\psi_n|^2},$$

for all $n \geq 1$. Corollary 3.4.5 then yields that $\{A_n\}$ is non-increasing.

First, we show that $A_n \to 0$. Indeed, assume instead that $A_n \to \rho > 0$. (3.36) implies that there exists $B > 0$ such that $B_n < B$, for all $n \geq 1$. Also, let $\{M_k\} \subset \mathbb{R}^+$ be any increasing sequence tending to ∞. Corollary 3.4.5 implies that we can find a decreasing sequence $\{r_k\}$, $r_k \to 1$ such that

$$A_{n+1} \leq \max\left\{\frac{A_n}{r_k}, \frac{B_n}{M_k}\right\}, \tag{3.37}$$

for all $n, k \geq 1$. Let $\epsilon > 0$ and choose $k = k_0$ to be such that $B/M_{k_0} < \rho$. Also, since $r_{k_0} > 1$, we can find $N \geq 1$ such that $A_N/r_{k_0} < \rho$. Thus, (3.37) yields

$$A_{N+1} \leq \max\left\{\frac{A_N}{r_{k_0}}, \frac{B_N}{M_{k_0}}\right\} < \rho,$$

a contradiction. Hence, $A_n \to 0$. We will show that $B_n \to 0$ as well. Indeed, assume that $B_n \not\to 0$. (3.36) combined with the fact that $A_n \to 0$ implies that $\liminf_n B_n = s > 0$. Also, given $0 < K_- < K$, Corollary 3.4.9 yields that for any $t_2 > 1$ sufficiently close to 1 one obtains

$$B_{n+1} \leq \max\left\{\frac{A_n}{K_-}, \frac{B_n}{t_2}\right\}, \tag{3.38}$$

for all $n \geq 1$. Now, choose n_0 such that $A_{n_0}/K_- < s/2$ and also $B_{n_0}/t_2 < s$. In view of (3.38), we obtain

$$B_{n_0+1} \leq \max\left\{\frac{A_{n_0}}{K_-}, \frac{B_{n_0}}{t_2}\right\} < s,$$

a contradiction. We conclude that $A_n, B_n \to 0$, which gives us $F^n = F^n(\lambda_0, \mu_0) \to (\tau^1, \sigma)$. Since (λ_0, μ_0) was arbitrary, we are done. □

Remark 3.5.5. We have actually reached the even stronger conclusion that, in the setting of Theorem 3.2.12, the iterates $F^n(\lambda)$ converge to (τ^1, σ) in the horospheric topology, for any $\lambda \in \mathbb{D}^2$.

Example. Define $\phi, \psi : \mathbb{D}^2 \to \mathbb{D}$ by

$$\phi(\lambda) = \frac{1 - \lambda^1\lambda^2}{2 - \lambda^1 - \lambda^2}$$

and

$$\psi(\lambda) = \begin{cases} \frac{(\lambda^2-\lambda^1)-2(1-\lambda^1)(1-\lambda^2)\log\left(\frac{1+\lambda^2}{1-\lambda^2}\frac{1-\lambda^1}{1+\lambda^1}\right)}{(\lambda^2-\lambda^1)+2(1-\lambda^1)(1-\lambda^2)\log\left(\frac{1+\lambda^2}{1-\lambda^2}\frac{1-\lambda^1}{1+\lambda^1}\right)} & \text{if } \lambda^1 \neq \lambda^2, \\ \frac{-3+5\lambda^1}{5-3\lambda^1} & \text{if } \lambda^1 = \lambda^2, \end{cases}$$

for all $\lambda \in \mathbb{D}^2$ (ψ has been taken from [96]).

Since the slice function ϕ_0 has 1 as its Denjoy-Wolff point, Theorem 3.2.9 implies that the entire closed face $\{1\} \times \mathrm{cl}(\mathbb{D})$ consists of B-points for ϕ and also $\phi(1, \sigma) = 1$, for all $|\sigma| \leq 1$. Actually, it is easy to see that ϕ extends analytically across $(1, \sigma)$ whenever $\sigma \neq 1$. Now, for $\sigma = 1$, it can be verified that

$$\frac{D_{-(1,M)}\phi(1,1)}{-\phi(1,1)} = -D_{-(1,M)}\phi(1,1) = \frac{M}{M+1} < 1,$$

for all $M > 0$. Thus, $(1, 1)$ is not a C-point for ϕ and also, since $\lim_{M\to\infty} M/(M+1) = 1$, the angular derivative of every slice function ϕ_μ at its Denjoy-Wolff point 1 has to be equal to 1 (this can be also verified directly, as the slice functions are easy to compute in this case).

Now, we look at ψ. Since $\psi(0, 0) = 0$, ψ is clearly a left (also a right) Type II function. Also, as shown in [96], $(1, 1)$ is a B-point for ψ that is not a C-point and $\psi(1, 1) = 1$. We wish to determine whether $(1,1)$ is also a left Type II DW point for ψ. However, computing the function $\xi : \mathbb{D} \to \mathbb{D}$ such that $\psi(\xi(\mu), \mu) = \xi(\mu)$, for all $\mu \in \mathbb{D}$, seems impractical here. Instead, we will look at the directional derivatives of ψ at $(1,1)$ along $\delta_M = (1, M)$ and then use Theorem 3.2.11. Indeed, in [96, Section 4] it was determined that

$$\begin{aligned} K_{(1,1)}(M) &= \frac{D_{-(1,M)}\psi(1,1)}{-\psi(1,1)} = -D_{-(1,M)}\psi(1,1) \\ &= 4M \int_{-1}^{1} \frac{dt}{(1-t) + (1+t)M} \\ &= \begin{cases} 4\frac{M \ln M}{M-1} & \text{if } M \neq 1, \\ 4 & \text{if } M = 1. \end{cases} \end{aligned}$$

Since $K_{(1,1)}(1) > 1$ and $\lim_{M \to 0+} K_{(1,1)}(M) = 0$, there exists $C > 0$ such that $K_{(1,1)}(C) = 1$. Theorem 3.2.11 then implies that $(1,1)$ is a left Type II DW point for ψ. Also, since $\psi(\lambda^1, \lambda^2) = \psi(\lambda^2, \lambda^1)$, $(1,1)$ must also be a right Type II DW point for ψ.

Now, define $F = (\phi, \psi) : \mathbb{D}^2 \to \mathbb{D}^2$. In view of our previous observations, we have that $(1,1)$ is a left Type I DW point for ϕ that is not a C-point and it is also a right Type II DW point for ψ. Theorem 3.2.12 then allows us to conclude that $F^n \to (1,1)$ uniformly on compact subsets of $\mathbb{D}^2$.

Before ending this subsection, we remark that the (Type II, Type I) case can be treated in an entirely analogous way.

3.5.3 The (Type I, Type I) case

Finally, assume that ϕ and ψ are left Type I and right Type I functions, respectively, hence $F = (\phi, \psi)$ does not have any interior fixed points. The following characterization is an easy consequence of Theorem 3.2.9, so we omit the proof.

Proposition 3.5.6. *Let $F = (\phi, \psi) : \mathbb{D}^2 \to \mathbb{D}^2$ be holomorphic. Then, ϕ and ψ are left Type I and right Type I functions, respectively, if and only if there exists $\tau = (\tau^1, \tau^2) \in \mathbb{T}^2$ that is a left Type I DW point for ϕ and a right Type I DW point for ψ.*

Now, let τ^1 and τ^2 be as in Proposition 3.5.6. In this setting, Hervé proved that either every cluster point of $\{F^n\}$ will be of the form (τ^1, h), where h is either a holomorphic function $\mathbb{D}^2 \to \mathbb{D}$ or the constant τ^2, or every cluster point will be of the form (g, τ^2), where g is either a holomorphic function $\mathbb{D}^2 \to \mathbb{D}$ or the constant τ^1 (Theorem 3.2.7(ii)). Also, it is not hard to see that in e.g. the former case, there exists a (parabolic) fractional linear transformation T with Denjoy-Wolff point τ^2 such that, whenever both (τ^1, h_1) and (τ^1, h_2) appear as *non-constant* cluster points of $\{F^n\}$, it must be true that $h_1 = T \circ h_2$ (see the 2nd remark in [80, Section 14]). Examples showing that these conclusions cannot, in general, be improved are contained in [80, Section 15].

Unfortunately, the proof of Theorem 3.2.7(ii) (to be found in [80, Sections 12-13]) does not make it clear whether it is possible to determine "beforehand" which of the two constants (τ^1 or τ^2) will be the one that appears as a coordinate in every cluster point of $\{F^n\}$. We will show that, under the extra assumption of (τ^1, τ^2) not being a C-point for either ϕ or ψ, one can draw stronger conclusions. Our proof is independent of Hervé's result.

Proof of Theorem 3.2.13. Assume $\tau = (\tau^1, \tau^2) \in \mathbb{T}^2$ satisfies the hypotheses of the theorem. Clearly, ϕ and ψ will be left Type I and right Type I functions, respectively. Also, Corollary 3.4.9 tells us that

$$F(E(\tau, R_1, R_2)) \subset E(\tau, R_1, R_2), \tag{3.39}$$

for all $R_1, R_2 > 0$. For any fixed $(\lambda_0, \mu_0) \in \mathbb{D}^2$, define:

$$A_n = \frac{|\tau^1 - \phi_n(\lambda_0, \mu_0)|^2}{1 - |\phi_n(\lambda_0, \mu_0)|^2} \quad \text{and} \quad B_n = \frac{|\tau^2 - \psi_n(\lambda_0, \mu_0)|^2}{1 - |\psi_n(\lambda_0, \mu_0)|^2},$$

for all $n \geq 1$. (3.39) then implies that both $\{A_n\}$ and $\{B_n\}$ are non-increasing. We can then argue as in the proof of Theorem 3.2.12 to deduce that $A_n \to 0$ (assuming τ is not a C-point for ϕ). Thus, every cluster point of $\{F^n\}$ will be of the form (τ^1, h), where h is holomorphic on $\mathbb{D}^2$ and bounded by 1. Moreover, since $\{B_n\}$ is bounded, one can deduce that h will have to be either a holomorphic map $\mathbb{D}^2 \to \mathbb{D}$ or the constant τ^2. □

Remark 3.5.7. We have actually reached the even stronger conclusion that, in the setting of Theorem 3.2.13 with e.g. τ not being a C-point for ϕ, the points $\phi_n(\lambda)$ converge to τ^1 in the horospheric topology of the unit disk, for any $\lambda \in \mathbb{D}^2$.

Example. Define $\phi : \mathbb{D}^2 \to \mathbb{D}$ by

$$\phi(\lambda) = -\frac{3\lambda^1\lambda^2 - \lambda^1 - \lambda^2 - 1}{3 - \lambda^1 - \lambda^2 - \lambda^1\lambda^2},$$

for all $\lambda \in \mathbb{D}^2$ (this example appears in [134]). It can be easily verified that the Denjoy-Wolff point of the slice function $\phi_0(z) = (z+1)/(3-z)$ is equal to 1. Theorem 3.2.9 then implies that the closed face $\{1\} \times \mathrm{cl}(\mathbb{D})$ consists of B-points for ϕ and also $\phi(1,\sigma) = 1$, for all $|\sigma| \leq 1$. Moreover, we can compute

$$\frac{D_{-(1,M)}\phi(1,1)}{-\phi(1,1)} = -D_{-(1,M)}\phi(1,1) = \frac{M}{M+1},$$

for all $M > 0$. Thus, $(1,1)$ is not a C-point for ϕ (and also $\phi'_\mu(1) = \lim_{M\to\infty} M/(M+1) = 1$, for all $\mu \in \mathbb{D}$).

Now, let $F = (\phi, \psi) : \mathbb{D}^2 \to \mathbb{D}^2$, where ψ is any (holomorphic) right Type I function such that the Denjoy-Wolff point of all slice functions $\psi(\lambda, \cdot)$ is equal to 1. Theorem 3.2.13 then implies that every cluster point of $\{F^n\}$ will be of the form $(1, h)$, where h is either a holomorphic function $\mathbb{D}^2 \to \mathbb{D}$ or the constant 1. Now, if we take ψ to be e.g.

$$\psi(\lambda^1, \lambda^2) = \frac{1 - \lambda^1\lambda^2}{2 - \lambda^1 - \lambda^2},$$

our observations from Example 3.5.2 (and the fact that $\psi(\lambda^1,\lambda^2) = \psi(\lambda^2,\lambda^1)$) show that $(1,1)$ will be a right Type I DW point for ψ that is not a C-point. Applying Theorem 3.2.13 again then yields (for this particular choice of ψ) that $F^n \to (1,1)$ uniformly on compact subsets of $\mathbb{D}^2$.

3.6 Connection with Frosini's Work

Points of Denjoy-Wolff type for holomorphic maps $F : \mathbb{D}^2 \to \mathbb{D}^2$ have been investigated by Frosini in [72], [73], [74]. She defined Denjoy-Wolff points for F as those fixed boundary points where F-invariant horospheres are centered, with the exact definition depending on the kind of horospheres in question. In particular, motivated by the definition of "small" and "big" horospheres found in [3], she defined (see [74, Definitions 3.2-3.3]) *quasi-Wolff* and *Wolff* points for F as those fixed boundary points where small horospheres are mapped into big ones and small horospheres are mapped into small ones, respectively. Unfortunately, the existence of quasi-Wolff points is, in general, not very helpful for describing the behavior of $\{F^n\}$, as big horospheres offer very limited control over the iterates. On the other hand, while Wolff points do offer much more restrictive Julia-type inequalities, they do not always exist (see [74, Theorem 4.1] for a characterization of the set of Wolff points for any self-map F of $\mathbb{D}^2$). Finally, in [72, Section 8], Frosini considered *generalized Wolff points*, which motivate our next definition.

Definition 3.6.1. Let $F = (\phi, \psi) : \mathbb{D}^2 \to \mathbb{D}^2$ be holomorphic with $\tau \in \partial\mathbb{D}^2$. If there exists $M \in (0, \infty)$ such that

$$F(E(\tau, R, MR)) \subset E(\tau, R, MR),$$

for all $R > 0$, τ will be called a *generalized Denjoy-Wolff point* for F.

As a consequence of Julia's inequality for the bidisk, any generalized Denjoy-Wolff point $\tau \in \partial\mathbb{D}^2$ of F must be a B-point point for both ϕ and ψ such that $F(\tau) = \tau$. Notice also that, in contrast to [72, Definition 33], we do not assume the existence of any complex geodesics, instead relying only on the existence of F-invariant "weighted" horospheres (although the definitions turn out to be equivalent, see Remark 3.3.13).

Let $W(F)$ denote the set of all generalized Denjoy-Wolff points of F. Our next result is a slight refinement of [72, Theorem 39], obtained as a straightforward application of the results developed in this paper. Note that τ^1, τ^2 will always denote points in $\mathbb{T}$.

Theorem 3.6.2. *Let $F = (\phi, \psi) : \mathbb{D}^2 \to \mathbb{D}^2$ be holomorphic such that $\phi \neq \pi^1$, $\psi \neq \pi^2$ and without any interior fixed points. Then, one and only one of the following three cases is possible:*

(i) $W(F) = \{(\tau^1, \tau^2)\}$ *if and only if ϕ is left Type II and ψ is right Type II;*

(ii) $\{\tau^1\}\times\mathbb{D}\subset W(F)\subset\big(\{\tau^1\}\times\mathbb{D}\big)\cup\{(\tau^1,\tau^2)\}$ *(resp.,* $\mathbb{D}\times\{\tau^2\}\subset W(F)\subset\big(\mathbb{D}\times\{\tau^2\}\big)\cup\{(\tau^1,\tau^2)\}$*) if and only if ϕ is left Type I and ψ is right Type II (resp., ϕ is left Type II and ψ is right Type I);*

(iii) $W(F)=\big(\{\tau^1\}\times\mathbb{D}\big)\cup\{(\tau^1,\tau^2)\}\cup\big(\mathbb{D}\times\{\tau^2\}\big)$ *if and only if ϕ is left Type I and ψ is right Type I.*

Proof. Theorem 3.4.3 implies that (i)-(iii) contain all possible cases.

First, assume ϕ is left Type II and ψ is right Type II. Theorem 3.5.1 and Corollary 3.4.9 imply that $W(F)\supset\{(\tau^1,\tau^2)\}$ for some $\tau^1,\tau^2\in\mathbb{T}$, where (τ^1,τ^2) is simultaneously a left Type II DW point for ϕ with constant M and a right Type II DW point for ψ with constant $1/M$. Now, assume $(\sigma^1,\sigma^2)\in W(F)$. If either $\sigma^1\in\mathbb{D}$ or $\sigma^2\in\mathbb{D}$, Corollary 3.4.5 would imply that either ψ is right Type I or ϕ is left Type I, respectively, a contradiction. Thus, $(\sigma^1,\sigma^2)\in\mathbb{T}^2$. But then, Theorems 3.3.10 and 3.2.10 yield that (σ^1,σ^2) is simultaneously a left Type II DW point for ϕ with constant $M'>0$ and a right Type II DW point for ψ with constant $1/M'$. In view of Theorem 3.5.1, we obtain $(\sigma^1,\sigma^2)=(\tau^1,\tau^2)$, hence $W(F)=\{(\tau^1,\tau^2)\}$.

Conversely, if $W(F)=\{(\tau^1,\tau^2)\}$, Corollary 3.4.5 implies that ϕ cannot be a left Type I function and ψ cannot be a right Type I function (else, $W(F)$ would also have to contain facial boundary points). Theorem 3.4.3 then yields that ϕ is left Type II and ψ is right Type II.

Next, we prove (ii). We will only deal with the (Type I, Type II) version. First, assume that ϕ is left Type I and ψ is right Type II, with τ^1 being the common Denjoy-Wolff point of all functions ϕ_μ. Corollary 3.4.5 implies that $\{\tau^1\}\times\mathbb{D}\subset W(F)$. If $W(F)=\{\tau^1\}\times\mathbb{D}$, we are done. Otherwise, assume that we can find a different point $(\sigma^1,\sigma^2)\in W(F)$. We must have $\sigma^1\in\mathbb{T}$, else ψ would be a right Type I function. Also, we may assume $\sigma^2\in\mathbb{T}$ (else we would have $\sigma^1=\tau^1$, in view of Corollary 3.4.5). Now, Theorem 3.3.10 (specifically, the fact that (iii) implies (i)) yields that (σ^1,σ^2) must be either a left Type I or a left Type II DW point for ϕ. Proposition 3.4.4 then tells us that $\sigma^1=\tau^1$. Note that (τ^1,σ^2) will have to be (in view of Theorem 3.3.10) a right Type II DW point for ψ. Also, if $(t^1,t^2)\in W(F)$ is not contained in $\{\tau^1\}\times\mathbb{D}$, our previous arguments show that $t^1=\tau^1$ and (t^1,t^2) is, in addition, a right Type II DW point for ψ. Proposition 3.4.8 then implies

$\sigma^2 = \tau^2$. We conclude that $\{\tau^1\} \times \mathbb{D} \subset W(F) \subset \big(\{\tau^1\} \times \mathbb{D}\big) \cup \{(\tau^1, \sigma^2)\}$, where $\sigma^2 \in \mathbb{T}$.

Conversely, assume $\{\tau^1\} \times \mathbb{D} \subset W(F) \subset \big(\{\tau^1\} \times \mathbb{D}\big) \cup \{(\tau^1, \tau^2)\}$, where $\tau^1, \tau^2 \in \mathbb{T}$. Corollary 3.4.5 then implies that ϕ is a left Type I and ψ is a right Type II function (else, $W(F)$ would have to contain a face of the form $\mathbb{D} \times \{\sigma^2\}$), as desired.

Finally, the proof of (iii) rests on Corollary 3.4.5, Proposition 3.4.4 and Theorem 3.3.10; one can argue in a manner analogous to the proof of (ii). We omit the details. □

Remark 3.6.3. As seen in the previous proof, the point (τ^1, τ^2) in (ii) will belong to W(F) if and only if it is a right Type II DW point for ψ.

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