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Hodge Classes in the Cohomology of Local Systems

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by
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Dedicated to My Mother.

ABSTRACT OF THE DISSERTATION

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Let S be an arithmetic quotient of a Hermitian symmetric domain and X/S be a family of varieties over S. One interesting problem is to find the Hodge classes of X, and if possible, to prove the Hodge conjecture for X. Using techniques from automorphic forms, we studied the Hodge conjecture for certain families of varieties over arithmetic quotients of balls and the Siegel domain of degree two. As a byproduct, we derived formulas for Hodge numbers in terms of automorphic forms.

Chapter 1

Introduction

1.1 The Hodge Conjecture

The Hodge conjecture is one of the most important open problems in mathematics.

Conjecture (Hodge conjecture)**.** Let X be a non-singular complex projective manifold. Then every Hodge class on X is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of X.

Here X is nonsingular projective, therefore is a ka\'hler manifold. The decomposition of complex forms into holomorphic and antiholomorphic forms induces a decomposition of $H^*(X, \mathbb{C})$:

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X).$$

The Hodge (p, p)-classes are defined as $H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$. Given a subvariety Y of codimension p, integration over Y defines a cohomology class, which turns out to be a Hodge (p, p)-class. It is known that the integral Hodge conjecture is not true.

The only case known in general is when $p = 1$, the Lefschetz theorem on $(1, 1)$-classes (and its dual, for $p = n - 1$). However, there are interesting *strategies* for attacking the Hodge conjecture in general. Inspired by Lefschetz's original approach, one idea features normal functions arising from fibering a Hodge class out over a base [BFNP, GG, KP]. Philosophically, it is also natural to break the Hodge Conjecture into a question about the absoluteness of Hodge classes (or field of definition of Hodge loci) and the Hodge conjecture for varieties defined over $\bar{\mathbb{Q}}$ [Vo].

1.2 Some Known Results

The Hodge conjecture is still open by now, but mathematicians has proved it for various types of varieties with totally different methods.

Abelian Varieties

Abelian varieties are group objects in the category of algebraic varieties. It is well known that the category of abelian varieties is equivalent to the category of weight one polarized Hodge structures. The study of Hodge conjecture for abelian varieties inspired the study of Mumford-Tate groups and domains. There are lots of results concerning the Hodge conjecture for abelian varieties.

Teh Hodge conjecture holds for a general abelian variety (Mattuck). This is because for a general abelian variety, the graded ring of Hodge classes is generated by Hodge $(1,1)$-classes. Using the same argument, the Hodge conjecture holds for an abelian avriety X which is isogeneous to a product of elliptic curves. Using Mumford-Tate groups, Tankeev showed that for a simple abelian variety X whose dimension is a prime number has no exceptional Hodge classes (Hodge classes not generated by Hodge $(1,1)$-classes.), there the Hodge cconjecture holds true for such abelian varieties. However, Mumford first constructed a simple four dimensional abelian varieties with exeptional Hodge classes.

An abelian variety of Weil-type of dimension $2n$ is a pair (X,K) with X a $2n$-dimensional abelian variety and $K \to \mathrm{End}(X)\otimes\mathbb{Q}$ is an imaginary quadratic field such that for all $x \in K$, its action on $T_0(X)$ has n eigenvalues x and n eigenvalues $\bar{x}$. A general $2n$-dimensional abelian variety has exceptional Hodge classes. The abelian variety of Weil-type are the only simple abelian varieties of dimension four with exceptional Hpdge classes.

Schoen proved that the hodge $(2,2)$-conjecture is true for the general four dimensional abelian variety of Weil-type $(X, \mathbb{Q}(\sqrt{-3}))$, $(X, \mathbb{Q}(\sqrt{-1}))$ with $\det H = 1$. Here H is a Hermitian form on $H_1(X, \mathbb{Q}) \times H_1(X, \mathbb{Q}) \to K$ and $\det H \in \mathbb{Q}^*/N(K^*)$. His proof uses the theory of Prym varieties.

As pointed out above, the absoluteness of Hodge conjectures is an important step in studying the Hodge conjecture. For abelian varieties, we have:

Theorem 1.2.1 (Deligne)**.** *Let A be an abelian variety over an algeraically closed field k, then every Hodge class is an absolute Hodge class.*

Special Varieties

There are various results for very special varieties. The Hodge conjecture holds for:

- hypersurfaces of degree one or two,
- (Zucker) cubic fourfolds,
- (Murre) unirational fourfolds,
- intersections of low degree hypersurfaces,
- (Shioda) certain Fermat varieties.

Locally Symmetric Varieties

Bergeron, Milson, and Moeglin studied the hodge conjecture for arithmetic ball quotients, arithmetic manifolds of orthogonal type. They showed that the hodge conjecture holds for such varieties in low degrees. Their proof uses automorphic form theory. The key point was to show certain automorphic representations are theta lifting of anoth group. Later, Bergeron, Milson, Moeglin, and Zhiuyuan Li studied moduli space of quasi-polarized $K3$-surfaces.

1.2.1 Main Results

Universal families over locally symmetric spaces (connected Shimura varieties) provide a source of varieties defined over $\bar{\mathbb{Q}}$ which come endowed with natural fibrations. In this paper, we investigate what can be said about Hodge classes on such automorphic total spaces. The central point is that, under the Decomposition Theorem [Sa, CM, KL], these Hodge classes live in the intersection cohomology of automorphic local systems on the Shimura variety, which can be calculated using representation-theoretic tools.

Let $\mathcal{X} \xrightarrow{\pi} \mathcal{S}$ be a family of varieties over a locally symmetric base. We are particularly interested in cycles in $\mathcal{X}$. In this case, the decomposition theorem provides a decomposition of the cohomology groups $H^*(\mathcal{X})$ into cohomology groups over the base $\mathcal{S}$ with coefficients in various local systems. Since the base is locally symmetric, various methods and techniques from representation theory can be used to understand the cohomology groups.

Before our work, Arapura [Ar1] studied families of genus-two curves $C_2(\Gamma)$ over $\mathcal{A}_2(\Gamma)$. He proved that the Hodge conjecture holds for X where $X \to Y$ is a good model of $\overline{C_2(\Gamma)} \to \overline{\mathcal{A}_2(\Gamma)}$. The key was a vanishing theorem given by an explicit computation. Chai and Faltings [FC] studied cohomology of local systems over $\mathcal{A}_g(\Gamma)$ in terms of the BGG complex. This leads to a quick proof of the vanishing theorem in [Ar1]. The main aim of my work is to reprove Arapura's results by a different method and extend it to some more general cases.

Let $\mathbf{H} \to \mathcal{S} = \Gamma \backslash G_{\mathbb{R}}/K$ be a homogeneous VHS over a locally symmetric variety, arising from a Hodge representation H of G. Given a cycle $\mathcal{Z} \in \mathrm{CH}^p(\bar{\mathcal{X}})$ on some family $\mathcal{X} \xrightarrow{\pi} \mathcal{S}$ underlying $\mathbf{H}$ (with $R^{2p-k}\pi_*\mathbb{Q} \cong \mathbb{H}$), the perverse Leray decomposition of $[\mathcal{Z}]$ yields a Hodge class in $\mathrm{IH}^k(\bar{\mathcal{S}}, \mathbb{H})$, where $\bar{\mathcal{S}}$ is the Baily-Borel compactification. So to predict such cycles (or rule them out), one needs a way to compute these Hodge classes.

Now suppose **H** is the local system associated with a finite-dimensional G-representation V. As an initial step, the relevant Matsushima formula (together with the Zucker conjecture [**?**, SS] and a result of Borel-Casselman [**?**]) identifies

$$\mathrm{IH}^k(\bar{\mathcal{S}}, \mathbb{H}_{\mathbb{R}}) \cong H^k(\mathfrak{g}, K; \mathcal{A}^2(G,\Gamma) \otimes V_{\mathbb{R}}); \tag{1.2.1}$$

and the description of the Hodge decomposition is also straightforward. Vanishing results for $(\mathfrak{g}, K)$-cohomology [BW] annihilate this group for k below the real rank r_G of G (in particular, for the space IH^1 of normal functions, unless $\mathcal{S}$ is a ball quotient). For the cases that remain, our goal was to use techniques of [BW, VZ] to study the Hodge Conjecture for the total spaces of universal families over locally symmetric varieties.

In particular, Vogan-Zuckerman [VZ] classified the cohomological $G(\mathbb{R})$-representations in terms of θ-stable parabolic subalgebras, and explicitly calculated their cohomology groups. To calculate the Hodge numbers for intersection cohomology groups of local systems on the left-hand side of (2.5.1), we need to modify their results to get a Hodge number formula for the Lie algebra cohomology groups on the right-hand side of (2.5.1) in terms of unitary cohomological representations and their multiplicities in $\mathcal{A}^2(G,\Gamma)$.

From the Hodge number formula, we get several vanishing theorems for the Hodge classes by considering all cohomological representations with respect to a given V. We therefore proved the Hodge conjecture for certain varieties. In particular, we get an alternate proof of Arapura's result [**?**] on Siegel modular 4-folds (arising from the universal genus 2 curve), and a proof for Picard modular fourfolds (arising from universal genus 4 Picard curve).

Theorem A. Let X be a compactification of the universal genus four Picard curve over an arithmetic quotient of $\mathbb{B}^3$. Then the HC holds for X.

However, things are more interesting when the cohomological representations do produce Hodge classes and the multiplicities appear in the formula. It is a central problem in automorphic form theory to study the multiplicities of a representation in the space $\mathcal{A}^2(G,\Gamma)\otimes V_{\mathbb{R}}$. The Arthur-Selberg trace formula is a powerful method to classify automorphic representations and study their multiplicities. It expresses the multiplicity of automorphic representations in terms of geometric objects, say, weighted orbital integrals. A complete classification of automorphic representations has already been given for symplectic and orthogonal groups by Arthur [Art], for quasi-split unitary groups by Mok [Mok], and for inner forms of unitary groups by Kaletha et al [KMSW].

The classifications are written in terms of A-packets. It is hard to get an explicit description of these representations and to express the multiplicities in terms of more tractable objects. However, for simple groups — say, $Sp(4)$ — a more detailed and explicit classification is given. See [Art] or Ralf Schmidt's work. The automorphic representations are divided into six types, and are parameterized by simple parameters, like characters or cuspidal representations of GL_2. With such a classification, Dan Petersen [**?**] was able to study the cohomology of local systems over $\mathcal{A}_2$. He calculated $H_c^*(\mathcal{A}_2, \mathbb{V}_{a,b})$ where $\mathbb{V}_{a,b}$ are local systems over $\mathcal{A}_2$. But we want a general formula for certain local systems.

We are particularly interested in the Saito-Kurokawa type since certain cohomological representations for $Sp(4)$ are archimedean components of automorphic representations of this type. With the understanding of local components of automorphic representations (in particular, the fixed vector under congruence subgroups), we are able to express the multiplicities of the non-tempered cohomological representations as dimensions of certain elliptic modular forms.

Propositon B Let $\Gamma = \Gamma^{\mathrm{para}}(p)$ be a paramodular subgroup of prime level; then the multiplicity of the non-tempered representation σ_k in $\mathcal{A}^2(Sp(4,\mathbb{R}),\Gamma)$ is given in terms of classical spaces of cusp forms:

- $\dim S_{2k-2}(SL(2,\mathbb{Z})) + \dim S_{2k-2}(\Gamma_0(p))^{new,+}$ if k is odd;

- $\dim S_{2k-2}(\Gamma_0(p))^{new,-}$ if k is even.

Now we consider paramodular subgroups $\Gamma^{para}(n)$ of $Sp(4)$. For small levels, the dimensions of the corresponding elliptic modular forms are zero. We thus obtain a proof the Hodge conjecture for certain Siegel modular fivefolds arising from the universal abelian surface over $\Gamma^{para}(p)\backslash\mathfrak{H}^2$.

Theorem B. The HC holds for the self-fiber product of the universal genus 2 curve, as well as for the universal abelian surface (and any compactification thereof), over $\Gamma^{\mathrm{para}}(p)\backslash\mathfrak{H}_2$ when $p = 1, 2, 3, 5$.

The dimension of the Hodge classes depends on the level of p. For instance, when $p = 7$ there is exactly one (rational) Hodge class in H^4 of our fivefold not pulled back from the base. It is an interesting problem to interpret such classes in terms of geometric objects, say, cycles or motives. These cycles should depend on the levels, and therefore have an arithmetic significance.

* * *

Chapter 2

Cohomology Theories

2.1 Locally Symmetric Varieties

Let G be a semisimple algebraic group defined over $\mathbb{Q}$ of Hermitian type with a fixed maximal compact subgroup K. The associated Hermitian symmetric domain D is $G(\mathbb{R})/K$. The isomorphism classes of irreducible hermitian symmetric domains are classified by the special nodes on connected Dynkin diagrams.

A locally symmetric variety (or a connected Shimura variety) $X = \Gamma\backslash D$ is just the quotient space of a Hermitian symmetric domain D by an arithmetic group Γ. It turns out to be defined over $\bar{\mathbb{Q}}$, and usually has moduli interpretations. Some simple examples are quotients of the upper half-plane, parametrizing families of elliptic curves; and quotients of higher dimensional symmetric spaces, parametrizing families of abelian varieties, curves, K3 surfaces, and other varieties. Universal families over locally symmetric varieties provide a source of varieties defined over $\bar{\mathbb{Q}}$ which come endowed with natural fibrations.

It turns out that X is a quasi-projective variety with Baily-Borel compactification $\overline{X}$. The compactification $\overline{X}$ is the projective space associated with the graded ring of automorphic forms for the powers of the canonical automorphy factor. Denote by $i : X \to \overline{X}$ the natural inclusion map.

2.2 The Decomposition Theorem

The aim of this paper is to prove the Hodge conjecture for certain families of varieties. Before that, we first briefly recall the perverse Leray decomposition theorem, see [CM] for details.

Let X be a complex projective manifold and let $\mathcal{D}_X$ be the derived category of bounded constructible sheaves and $\mathcal{P}_X$ be the full subcategory of *perverse sheaves.*

To any pair (U, L) where $j : U \to X$ is a Zariski open subset of X and L a local system over U, the *intersection complex* $IC_U(L)$ is defined as the intermediate extension $j_{!*}(L)$ and is the unique perverse extension of L to X with neither subobjects nor quotients supported on $X \backslash U$. Intersection complexes are simple objects in the category of perverse sheaves, and perverse sheaves are iterated extensions of intersection complexes.

The decomposition theorem studies the topological properties of proper maps between algebraic varieties.

Theorem 2.2.1 (Decomposition theorem)**.** *Let $f : Y \to X$ be a proper map of complex algebraic varieties. There exists an isomorphism in the constructible bounded derived category $\mathcal{D}_X$:*

$$Rf_* IC_Y \cong \bigoplus_i {}^{\mathfrak{p}}\mathcal{H}^i(Rf_* IC_Y)[-i].$$

Furthermore, the perverse sheaves ${}^{\mathfrak{p}}\mathcal{H}^i(Rf_* IC_Y)$ are semisimple; i.e., there is a decomposition into finitely many disjoint locally closed and nonsingular subvarieties $X = \coprod X_\alpha$ and a canonical decomposition into a direct sum of intersection complexes of semisimple local systems

$${}^{\mathfrak{p}}\mathcal{H}^i(Rf_* IC_Y) \cong \bigoplus_\alpha IC_{\overline{X_\alpha}}(L_\alpha).$$

The *intersection cohomology groups* (with middle perversity) $IH^*(X, L)$ for the local system L over a Zariski open subset U are simply defined to be the hypercohomology groups $H^*(X, IC_U(L))$. Taking hypercohomology of the decomposition theorem, we get:

Theorem 2.2.2 (Decomposition theorem for intersection cohomology groups)**.** *Let $f : Y \to X$ be a proper map of varieties. There exists finitely many triples $(X_\alpha, L_\alpha, d_\alpha)$ made of locally closed, smooth and irreducible algebraic subvarieties $X_\alpha \subset X$, semisimple local systems L_α on X_α and integer numbers d_α, such that for every open set $U \subset X$ there is an isomorphism*

$$IH^r(f^{-1}U) \cong \bigoplus_\alpha IH^{r-d_\alpha}(U \cap \overline{X}_\alpha, L_\alpha).$$

Remark 2.2.1. The decomposition is not uniquely defined. But in the case when X is quasi-projective, one can make distinguished choices that realize the summands as mixed Hodge substructures of a canonical mixed Hodge structure on $IH^*(Y)$. In particular, if Y is smooth and X, Y, f are projective, then $IH^*(Y) = H^*(Y)$ is the usual cohomology, and the intersection cohomology groups in the sum are equipped with canonical pure polarizable Hodge structures.

2.3 Zucker's conjecture

Let (V, ρ) be a (rational) representation of G, it defines a local system $\mathbb{V}$ over X. We are interested in the intersection cohomology $IH^*(\overline{X}, \mathbb{V})$.

The Hermitian symmetric domain D is equipped with a canonical Riemannian metric induced from the Killing form of the Lie algebra $\mathfrak{g}$. This metric is Γ-invariant, thus descends to a Riemannian metric over $X = \Gamma\backslash D$. We also choose and fix a metric on the local system $\mathbb{V}$. The L^2-cohomology groups $H^*_{(2)}(X, \mathbb{V})$ are defined to be the cohomology groups of the complex $(C^\bullet, d)$, where C^k is the space of $\mathbb{V}$-valued smooth k-forms over X such that the form itself and its exterior derivative are both square-integrable; the differential map d is simply the restriction of the usual exterior differential. Zucker's conjecture, now a theorem, compares the intersection cohomology over $\overline{X}$ and L^2-cohomology over X. It was proved in different ways by Eduard Looijenga ([Lo]) and by Leslie Saper and Mark Stern ([SS]).

Theorem 2.3.1 (Zucker's conjecture). *As real vector spaces, the intersection cohomology* $IH^*(\overline{X}, \mathbb{V}_{\mathbb{R}})$ *is isomorphic to the* L^2*-cohomology* $H^*_{(2)}(X, \mathbb{V}_{\mathbb{R}})$.

Zucker's conjecture is just an isomorphism of $\mathbb{R}$-vector spaces. However, both sides carry natural $\mathbb{R}$-Hodge structures. The Hodge structure of the intersection cohomology comes from Saito's mixed Hodge module theory (and is defined over $\mathbb{Q}$) while the Hodge structure of L^2-cohomology comes from harmonic analysis. It is an open question whether the isomorphism is actually an isomorphism of Hodge structures.

Harris-Zucker Conjecture. The isomorphism in Theorem 2.3.1 is an isomorphism of $\mathbb{R}$-Hodge structures.

The Harris-Zucker conjecture is known in several cases:

- Γ cocompact: because $X = overline X$ (no boundary);
- Hilbert modular surface or complex n-balls (Zucker [Zu]), trivial coefficient $\mathbb{C}$;
- Hilbert modular varieties [MSYZ];
- in general, for $i \leq c :=$ codimension of singular locus in Y^*. In this case actually $\mathrm{IH}^i(\overline{X}, \mathcal{H}^{2p-i}) = W_{2p}H^i(X, \mathcal{H}^{2p-i}) = H^i_{(2)}(X^*, \mathcal{H}^{2p-i})$ [HZ, §5].
- another general fact is that the Hodge structures on $H^*_{(2)}$ and IH^* are equal if we replace Y^* by $\hat{Y}$, though this isn't really a case of Harris-Zucker conjecture. It is due to Cattani-Kaplan-Schmid, Kashiwara-Kawai, and Saito (cf. [PSa]).

2.4 Relative Lie Algebra Cohomology and Cohomological Representations

2.4.1 Lie Algebra Cohomology and Spectral Decomposition

Let G be a Lie groups and U be a $(\mathfrak{g}, K)$-module. The relative Lie algebra cohomology groups are the cohomology groups of the complex $(C^{\cdot}, d)$ where $C^q = \mathrm{Hom}_{\mathfrak{k}}(\wedge^q(\mathfrak{g}/\mathfrak{k}), U)$. See [BW] for more details and properties.

It is well-known (de Rham isomorphism) that we may compute the cohomology groups of local systems by relative Lie algebra cohomology:

$$H^*(X, \mathbb{V}) \cong H^*(\mathfrak{g}, K, C^\infty(\Gamma\backslash G(\mathbb{R})) \otimes V).$$

The L^2-cohomology groups have a similar description

$$H^*_{(2)}(X, \mathbb{V}) \cong H^*(\mathfrak{g}, K, L^2(\Gamma\backslash G(\mathbb{R}))^\infty \otimes V).$$

The inclusion $L^2(\Gamma\backslash G(\mathbb{R}))^\infty \to C^\infty(\Gamma\backslash G(\mathbb{R}))$ induces a map

$$H^*(\mathfrak{g}, K, L^2(\Gamma\backslash G(\mathbb{R}))^\infty \otimes V) \to H^*(\mathfrak{g}, K, C^\infty(\Gamma\backslash G(\mathbb{R})) \otimes V),$$

which recovers the natural map $H^*_{(2)}(X, \mathbb{V}) \to H^*(X, \mathbb{V})$ under the above isomorphisms.

The unitary representation $L^2(\Gamma\backslash G)$ is the direct sum of a discrete spectrum and a continuous spectrum. The discrete spectrum is a Hilbert direct sum of unitary representations U_π, each with finite multiplicity $m_\pi(\Gamma)$. The contribution from the continuous spectrum to the $(\mathfrak{g}, K)$-cohomology vanishes if G has a discrete series. This is always true in our thesis. Let $\mathcal{A}(G,\Gamma)$ be the space of automorphic forms with respect to Γ and $\mathcal{A}^2(G,\Gamma) = \mathcal{A}(G,\Gamma) \cap L^2(\Gamma\backslash G)$. Then the decomposition of $L^2_{\text{disc}}(\Gamma\backslash G)$ is the same as the decomposition of $\mathcal{A}^2(G;\Gamma)$ as $\mathcal{A}^2(G;\Gamma)$ is precisely the space of smooth K-finite, $Z(\mathfrak{g})$-finite vectors in the discrete spectrum $L^2_{\text{disc}}(\Gamma\backslash G)$. In summary, the L^2-cohomology groups remain unchanged if we replace $L^2(\Gamma\backslash G(\mathbb{R}))^\infty$ with $\mathcal{A}^2(G;\Gamma)$, the subspace of L^2-automorphic forms:

$$H^*_{(2)}(X,\mathbb{V}) = H^*(\mathfrak{g},K,\mathcal{A}^2(G;\Gamma)\otimes V).$$

Combined with Zucker's conjecture, the intersection cohomology can be interpreted as relative Lie algebra cohomology:

$$IH^*(\overline{X},\mathbb{V}) = H^*(\mathfrak{g},K,\mathcal{A}^2(G;\Gamma)\otimes V) = \bigoplus_{U_\pi\in\mathcal{A}^2(G;\Gamma)} m_\pi(\Gamma)H^*(\mathfrak{g},K,U_\pi\otimes V).$$

The discrete spectrum is the direct sum of the cuspidal and the residual spectrum. Correspondingly, the intersection cohomology is the direct sum of a cuspidal part $IH^*_{\text{cusp}}(\overline{X},\mathbb{V})$ and a residual part $IH^*_{\text{res}}(\overline{X},\mathbb{V})$. The natural map $IH^*(\overline{X},,\mathbb{V}) \to H^*(X,\mathbb{V})$ is injective on the cuspidal part, but the restriction to the residual part is usually neither injective nor surjective.

2.4.2 The Vogan-Zuckerman Classification

Let G be a semisimple Lie group. Fix a maximal compact subgroup K. This is equivalent to choosing a Cartan involution θ or the Cartan decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$. Here we use $\mathfrak{g}_0$ (resp. $\mathfrak{k}_0$) to represent the *real* Lie algebra of G (resp. K), and $\mathfrak{g}$ (resp. $\mathfrak{k}$) to represent their complexifications. We always assume that G has a maximal compact torus T, which is equivalent to that G has discrete series representations. Let $\mathfrak{t}_0$ be the Lie algebra of T, it is a Cartan subalgebra of both K and G.

Let (π, U) be a G-representation. We will not distinguish the representation U and its associated $(\mathfrak{g}, K)$-module. The *relative Lie algebra cohomology groups* $H^*(\mathfrak{g}, K, U)$ are defined to be the cohomology groups of the complex $(C^*(\mathfrak{g}, K, U), d)$ where $C^*(\mathfrak{g}, K, U) = \mathrm{Hom}_K(\wedge^*(\mathfrak{g}/\mathfrak{k}), U)$ ([BW]).

Let (π, U) be an irreducible unitary $(\mathfrak{g}, K)$-module and V be a finite-dimensional irreducible representation of G. We say that (π, U) is *cohomological (w.r.t V)* if the $(\mathfrak{g}, K)$-cohomology groups $H^*(\mathfrak{g}, K; U \otimes V) \neq 0$. First, we have:

Lemma 2.4.1. *If $H^*(\mathfrak{g}, K, U \otimes V) \neq 0$, then the infinitesimal character of U is the same as the infinitesimal character of V^*, and the differential map d in the complex $(C^*(\mathfrak{g}, K, U \otimes V), d)$ is automatically zero.*

Cohomological $(\mathfrak{g}, K)$-modules are classified by Vogan and Zuckerman ([VZ]) in terms of θ-stable parabolic subalgebras. A θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{q}(X) \subset \mathfrak{g}$ is associated to an element $X \in i\mathfrak{t}_0$. It is defined as the direct sum

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u},$$

of the centralizer $\mathfrak{l}$ of X and the sum $\mathfrak{u}$ of the positive eigenspaces of $\mathrm{ad}(X)$ ($\mathfrak{t} \subset \mathfrak{l}$ by construction). Since $\theta(X) = X$, the subspace $\mathfrak{q}, \mathfrak{l}$ and $\mathfrak{u}$ are all invariant under θ, so is

$$\mathfrak{q} = \mathfrak{q} \cap \mathfrak{k} \oplus \mathfrak{q} \cap \mathfrak{p},$$

and so on.

The Lie algebra $\mathfrak{l}$ is the complexification of $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$. Let L be the connected subgroup of G with Lie algebra $\mathfrak{l}_0$. Fix a positive system $\Delta^+(\mathfrak{l})$ of the roots of $\mathfrak{t}$ in $\mathfrak{l}$. Then $\Delta^+(\mathfrak{g}, \mathfrak{t}) = \Delta^+(\mathfrak{l}) \cup \Delta(\mathfrak{u})$ is a positive system of the roots of $\mathfrak{t}$ in $\mathfrak{g}$. Let ρ be half the sum of the roots in $\Delta^+(\mathfrak{g},\mathfrak{t})$ and $\rho(\mathfrak{u}\cap\mathfrak{p})$ half the sum of roots in $\mathfrak{u}\cap\mathfrak{p}$.

A one-dimensional representation $\lambda : \mathfrak{l} \to \mathbb{C}$ is called *admissible* if it satisfies the following conditions:

(1) λ is the differential of a unitary character of L,

(2) if $\alpha \in \Delta(\mathfrak{u})$, then $\langle \alpha, \lambda|_{\mathfrak{t}} \rangle \geq 0$.

Given $\mathfrak{q}$ and an admissible λ, let $\mu(\mathfrak{q},\lambda)$ be the representation of K of highest weight $\lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u}\cap\mathfrak{p})$[1].

Proposition 2.4.2. *There exists a unique irreducible unitary $(\mathfrak{g},K)$-module $A_{\mathfrak{q}}(\lambda)$ such that*

(1) *$A_{\mathfrak{q}}(\lambda)$ contains the K-type $\mu(\mathfrak{q},\lambda)$.*

(2) *$A_{\mathfrak{q}}(\lambda)$ has infinitesimal character $\lambda|_{\mathfrak{t}} + \rho$.*

Remark 2.4.1. The K-representation $\mu(\mathfrak{q},\lambda)$ is minimal in the sense that all the K-types of $A_{\mathfrak{q}}(\lambda)$ are of the form

$$\delta = \lambda|_{\mathfrak{t}} + 2\rho(\mathfrak{u}\cap\mathfrak{p}) + \sum_{\beta\in\Delta(\mathfrak{u}\cap\mathfrak{p})} n_\beta \beta$$

with n_β non-negative integers.

Remark 2.4.2. We use the simplified symbols $\mu(\mathfrak{q})$, $A_{\mathfrak{q}}$ to denote $\mu(\mathfrak{q},0)$ and $A_{\mathfrak{q}}(0)$. It turns out that $\mu(\mathfrak{q})$ are actually K-representations inside the natural representation $\wedge^*\mathfrak{p}$ ([VZ]).

Remark 2.4.3. Given a finite-dimensional representation V, the Zuckerman translation functor is a functor twisting the infinitesimal character of represntations. The representations $A_{\mathfrak{q}}(\lambda)$ are exactly the Zuckerman translations of the representations $A_{\mathfrak{q}}$.

Proposition 2.4.3. *Let (π,U) be an irreducible unitary $(\mathfrak{g},K)$-module and V be a finite dimensional irreducible representation of G. Suppose $H^*(\mathfrak{g},K;U\otimes V)\neq 0$. Then there is a θ-stable parabolic subalgebra $\mathfrak{q}=\mathfrak{l}\oplus\mathfrak{u}$ of $\mathfrak{g}$, such that*

[1]The restriction of $\Delta^+(\mathfrak{g},\mathfrak{t})$ to $\mathfrak{k}$ is a positive system. This weight is dominant with respect to this positive system.

(1) *$V/\mathfrak{u}V$ is a one-dimensional unitary representation of L; write $-\lambda : \mathfrak{l} \to \mathbb{C}$ for its differential.*

(2) $\pi \cong A_{\mathfrak{q}}(\lambda)$. *Moreover, letting* $R = \dim(\mathfrak{u} \cap \mathfrak{p})$, *we have*

$$H^*(\mathfrak{g}, K; U \otimes V) = H^{*-R}(\mathfrak{l}, L \cap K; \mathbb{C}) = \mathrm{Hom}_{\mathfrak{l}\cap\mathfrak{k}}(\wedge^{*-R}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}).$$

2.5 The Hodge Conjecture

Now we turn to the discussion of the Hodge conjecture. Write $\Gamma(H) := \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H)$ for Hodge $(0,0)$-classes in a mixed Hodge structure H.

Conjecture. Let $Y/\mathbb{C}$ be a quasi-projective algebraic variety (possibly neither smooth nor complete) of dimension d. Then the cycle class map

$$\mathrm{CH}^p(Y) \to \Gamma(H^{\mathrm{BM}}_{2(d-p)}(Y, \mathbb{Q}(p-d))) \ \big(\cong \ \Gamma(H^{2p}(Y, \mathbb{Q}(p))) \text{ for } Y \text{ smooth}\big)$$

is surjective.[2] We abbreviate this conjecture by $\mathrm{HC}^p(X)$.

Now fix a smooth quasi-projective X, a possibly singular compactification $\bar{X}$ and a resolution $\hat{X} \overset{\rho}{\twoheadrightarrow} \bar{X}$. Write $Z = \bar{X} \setminus X$ and $\hat{Z} = \hat{X} \setminus X$ for the complements; assume that $\hat{Z}$ is a normal-crossing divisor. Statements about the relationship between X and $\bar{X}$ will obviously apply to $\hat{X}$ too, since $\bar{X}$ is more general.

Proposition 2.5.1. ***(i)*** $\mathrm{HC}^p(\hat{X}) \implies \mathrm{HC}^p(\bar{X})$.

(ii) $\mathrm{HC}^p(\bar{X}) \implies \mathrm{HC}^p(X)$.

(iii) $\mathrm{HC}^p(X) + \mathrm{HC}^p(Z) \implies \mathrm{HC}^p(\bar{X})$.

[2]Here $H^{\mathrm{BM}}_k(Y)$ denotes Borel-Moore homology [PSt], which carries a natural MHS. It is equal to the relative homology group $H_k(\bar{X}, \bar{X} \setminus X)$ for any compactification $\bar{X}$, but is independent of the choice of $\bar{X}$.

Sketch. Writing $q = d - p$, (i) is essentially because

$$W_0 H_{2q}^{\mathrm{BM}}(\bar{X}, \mathbb{Q}(q)) = \mathrm{gr}_0^W H^{2p}(\bar{X}, \mathbb{Q}(p)) \hookrightarrow H^{2p}(\hat{X}, \mathbb{Q}(p)),$$

which gives injectivity on Hodge classes, whereas cycles push down under ρ. For (ii) and (iii), the main idea (as in [Ar2]) is the diagram

$$\begin{array}{ccccccc} \mathrm{Hg}^p(Z) & \longrightarrow & \mathrm{Hg}^p(\bar{X}) & \longrightarrow & \mathrm{Hg}^p(X) & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathrm{CH}^p(Z) & \longrightarrow & \mathrm{CH}^p(\bar{X}) & \longrightarrow & \mathrm{CH}^p(X) & \longrightarrow & 0. \end{array}$$

with exact rows. □

The conclusion here is that one can focus on the Hodge conjecture for X without further discussion, because (a) it has a well-defined meaning and (b) the boundary in a compactification should be thought of as a separate issue (or a non-issue if $\dim(X) \leq 4$).

Now consider $X \overset{\pi}{\twoheadrightarrow} Y$ a smooth projective morphism, with X and Y smooth quasi-projective. We also have compactifications $\bar{\pi}\colon \bar{X} \to \bar{Y}$, $\hat{\pi}\colon \hat{X} \to \hat{Y}$ to projective morphisms, with assumptions as above ($\bar{Y}, \bar{X}$ possibly singular; $\hat{Y}, \hat{X}$ smooth with NC boundary). Writing $\mathcal{H}^r := R^r \pi_* \mathbb{Q}_X$, there are cycle maps from $\mathrm{CH}^p(X)$ to $\Gamma(H^i(Y, \mathcal{H}^{2p-i}(p)))$ by splitting the cycle-class into Leray graded pieces. Let $\mathrm{HC}_i^p(\pi)$ stand for surjectivity of these maps.

Moreover, by the Decomposition Theorem for $\hat{\pi}$ (e.g. see [KL] for a convenient presentation), we get cycle maps $\mathrm{CH}^p(\hat{X}) \to \Gamma(\mathrm{IH}^i(\hat{Y}, \mathcal{H}^{2p-i}(p)))$. We can also apply the DT to the composition $\bar{\pi} \circ \rho$ to get maps $\mathrm{CH}^p(\hat{X}) \to \Gamma(\mathrm{IH}^i(\bar{Y}, \mathcal{H}^{2p-i}(p)))$. Write $\mathrm{HC}_i^p(\hat{\pi})$, $\mathrm{HC}_i^p(\bar{\pi})$ for surjectivity of these maps.

Proposition 2.5.2. ***(i)*** $\mathrm{HC}_i^p(\pi)$ $(\forall i) \implies \mathrm{HC}^p(X)$.

(ii) $\mathrm{HC}_i^p(\pi)$ $(\forall (p,i) \in Q) \implies \mathrm{HC}^p(X)$ $(\forall p)$, *where Q is the parallelogram defined by $0 \leq i \leq d_Y$ and $\frac{i}{2} \leq p \leq \frac{i+(d_X-d_Y)}{2}$.*

Discussion. See [Ar2] for the proof. One key point is that $\mathrm{IH}^i(\hat{Y}, \mathcal{H}^{2p-i}) \twoheadrightarrow W_{2p}H^i(Y, \mathcal{H}^{2p-i})$, which means that Hodge classes surject and $\mathrm{HC}^p_i(\hat{\pi}) \implies \mathrm{HC}^p_i(\pi)$. (But this works for $\bar{\pi}$ too as we're about to see.) More precisely: a Hodge class in $W_{2p}H^i(Y, \mathcal{H}^{2p-i})$ lifts (nonuniquely) to a Hodge class in $\mathrm{IH}^i(\hat{Y}, \mathcal{H}^{2p-i})$ (or $\bar{Y}$); if that is given by a cycle on $\hat{X}$, then obviously the original class is given by its restriction to X. □

Proposition 2.5.3. ***(i)*** $\mathrm{HC}^p_i(\hat{\pi}) \implies \mathrm{HC}^p_i(\bar{\pi})$.

(ii) $\mathrm{HC}^p_i(\bar{\pi}) \implies \mathrm{HC}^p_i(\pi)$.

(iii) $\mathrm{HC}^p_i(\bar{\pi})\ (\forall (p,i) \in Q) \implies \mathrm{HC}^p(X)\ (\forall p)$.

Proof. By the decomposition theorem, $\mathrm{IH}^i(\bar{Y}, \mathcal{H}) \subseteq \mathrm{IH}^i(\hat{Y}, \mathcal{H})$ is a sub-HS; so classes lift and cycles push forward under ρ. This gives (i).

(ii) is the most subtle. Clearly, it is enough to show

$$\mathrm{IH}^i(\bar{Y}, \mathcal{H}^{2p-i}) \twoheadrightarrow W_{2p}H^i(Y, \mathcal{H}^{2p-i}) \tag{2.5.1}$$

so that Hodge classes lift. Essentially the same thing is proved in [PSa] (take the algebraic case of their main theorem). Here is a brief recap.

If we write H^{2p-i} for the IC-Hodge-module on $\bar{Y}$ associated to $\mathcal{H}^{2p-i}$, shifted so that the restriction to Y is $\mathcal{H}^{2p-i}$ (not $\mathcal{H}^{2p-i}[d_Y]$), then we have the localization sequence

$$\to \mathbb{H}^i(\bar{Y}, \mathsf{H}^{2p-i}) \to \mathbb{H}^i(\bar{Y}, R\jmath_*\jmath^*\mathsf{H}^{2p-i}) \to \mathcal{H}^{i+1}(\bar{Y}, \imath_*\imath^!\mathsf{H}^{2p-i}) \to$$

whose first 2 terms are $\mathrm{IH}^i(\bar{Y}, \mathcal{H}^{2p-i})$ and $H^i(Y, \mathcal{H}^{2p-i})$. Since $\imath^!$ and $\imath_*$ do not decrease weights (cf. [PSt, §14.1.1]), W_{2p} of the right-hand term is zero. Taking W_{2p} of the sequence then gives the desired result.

(iii) is just putting (ii) together with Proposition 2.5.2(ii). □

Consider the case where Y is a locally symmetric variety, $\bar{Y} = Y^*$ is the Baily-Borel compactification, and (say) $\hat{Y}$ is a toroidal one; $\hat{X}$ is a smooth compactification of a family X giving one of the Hermitian VHSs over Y. (I'll continue to write $\bar{\pi}$ rather than π^* which looks like a pullback.) The objects you are computing in your paper are the L^2-cohomology groups $H^i_{(2)}(Y^*, \mathcal{H}^{2p-i})$, which carry $\mathbb{R}$-HSs coming from (equivalently) $(\mathfrak{g}, \mathfrak{k})$-cohomology or harmonic forms. Everything stated so far is about the IH-groups because they are the ones the DT relates to cohomologies of the total space hence the Hodge conjecture.

Proposition 2.5.4. ***(i)*** *$H^i_{(2)}(Y^*, \mathcal{H}^{2p-i}) \cong \mathrm{IH}^i(Y^*, \mathcal{H}^{2p-i})$ as real vector spaces, and both have $\mathbb{R}$-MHS morphisms to $H^i(Y, \mathcal{H}^{2p-i})$ with the same image, namely $W_{2p}H^i(Y, \mathcal{H}^{2p-i})$.*
(ii) *If $(H^i_{(2)}(Y^*, \mathcal{H}^{2p-i}))^{p,p} = \{0\}$, then $\mathrm{HC}^p_i(\pi)$ holds.*
(iii) *If $(H^i_{(2)}(Y^*, \mathcal{H}^{2p-i}))^{p,p} = \{0\}$ for all $(p, i) \in Q$, then $\mathrm{HC}^p(X)$ holds.*

Sketch. (i) is Theorem 5.4 of [HZ, §5]. It says we can write (as $\mathbb{R}$-HSs)

$$H^i_{(2)}(Y^*, \mathcal{H}^{2p-i}) \cong W_{2p}H^i(Y, \mathcal{H}^{2p-i}) \oplus M'$$
$$\mathrm{IH}^i(Y^*, \mathcal{H}^{2p-i}) \cong W_{2p}H^i(Y, \mathcal{H}^{2p-i}) \oplus M''$$

for some weight-$2p$ $\mathbb{R}$-Hodge structures M' and M''.

To show (ii), note that the hypothesis implies $\Gamma(H^i(Y, \mathcal{H}^{2p-i}(p))) = \{0\}$. So $\mathrm{HC}^p_i(\pi)$ is true *vacuously*.

Clearly (iii) is a corollary of (ii) together with Proposition 2(ii). □

So (ii)-(iii) make precise *what* one is proving when checking there are no $\mathbb{R}$-Hodge classes in the relative Lie algebra cohomologies below: namely, the Hodge conjecture for X. This conclusion does *not* depend on knowledge of the Harris-Zucker conjecture.

Chapter 3

Computations of Hodge Numbers

3.1 Hodge Representations

Mumford-Tate groups and domains

Fix an integer n. Let (V, Q) be a pair consisting of a finite-dimensional $\mathbb{Q}$-vector space V and a non-degenerate bilinear form Q with $Q(u,v) = (-1)^n Q(v,u)$. A (polarized) Hodge structure of weight n on (V, Q) is a decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ satisfying the following conditions:

- $\overline{V^{p,q}} = V^{q,p}$;
- $Q(u,v) = 0$ for $u \in V^{p,q}, v \in H^{p',q'}, p \neq p'$;
- $i^{p-q} Q(u, \bar{u}) > 0$ for $0 \neq u \in V^{p,q}$.

The Hodge numbers $h^{p,q}$ are just the dimensions of $V^{p,q}$. Equivalently, a Hodge structure can be considered as a real representation $\varphi : S^1 \to \mathrm{Aut}(V_{\mathbb{C}}, Q)$ by setting $z \cdot v = z^{p-q} v$ for $z \in S^1, v \in V^{p,q}$ and extending by complex linearity.

Let $\mathcal{D} = \mathcal{D}_{\mathbf{h}} = \mathcal{G}_{\mathbb{R}}/\mathcal{G}_{\mathbb{R}}^0$ be the period domain parameterizing Q-polarized Hodge structures on V with Hodge numbers $\mathbf{h} = (h^{n,0}, \cdots, h^{0,n})$. Here $\mathcal{G}_{\mathbb{R}} = \mathrm{Aut}(V_{\mathbb{R}}, Q)$ is either an orthogonal group $O(a, 2b)$ (if n is even) or a symplectic group $Sp(2r, \mathbb{R})$ (if n is odd), and $\mathcal{G}_{\mathbb{R}}^0$ is the compact stabilizer of a fixed $\varphi \in \mathcal{D}$. To each Hodge structure $\varphi \in \mathcal{D}$ is associated a $\mathbb{Q}$-algebraic Hodge group $\mathbf{G}_{\varphi} \subset \mathrm{Aut}(V, Q)$, and a Mumford-Tate domain $D = D_{\varphi} = G_{\varphi} \cdot \varphi \subset \mathcal{D}$, where $G_\varphi = \mathbf{G}_\varphi(\mathbb{R})$. The Hodge group $\mathbf{G}_\varphi$ is defined to be the $\mathbb{Q}$-algebraic closure of $\varphi(S^1)$. It may also be defined as the stabilizer of the Hodge tensors of φ.

Example 3.1.1. We consider a pair (V, Q) where V is a $\mathbb{Q}$-vector space of dimension six and Q a nondegenerate alternating form. We assume there is an embedding $\mathbb{F}(:= \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\omega)) \hookrightarrow \mathrm{End}_{\mathbb{Q}}(V)$. Then we have a decomposition $V_{\mathbb{F}} = V_+ \oplus V_-$ into the $\pm$ eigenspaces of the action of $\mathbb{F}$. We assume further that $V_\pm$ are isotropic with respect to Q and the Hermitian form on $V_{+,\mathbb{C}}$ $H(u,v) = \pm iQ(u, \overline{v})$ has signature $(2,1)$.

Let φ be a weight one Hodge structure over (V, Q) commuting with the $\mathbb{F}$-action. So the Hodge decomposition $V_{\mathbb{C}} = V^{p,q}$ is compactible with the decomposition $V_{\mathbb{F}} = V_+ \oplus V_-$. We assume that $\dim V_+^{1,0} = V_-^{0,1} = 1$, $\dim V_-^{1,0} = V_+^{0,1} = 2$. The Hodge structures of this type are precisely the Hodge structure associated with Picard curves.

Let $G = \mathcal{U}(2,1) := \mathrm{Aut}_{\mathbb{Q}}(V, Q) \cap \mathrm{Res}_{\mathbb{F}/\mathbb{Q}} GL_{\mathbb{F}}(V)$. This is a $\mathbb{Q}$-algebraic group, and is a $\mathbb{Q}$-form of the real Lie group $U(2,1)$ of automorphisms of $\mathbb{C}^3$ preserving the Hermitian form H. By the construction, the Hodge group of such Hodge structures are subgroups of $U(2,1)$. And the equality holds for generic Hodge structures. The Mumford-Tate domain of a generic Hodge structure is the two-dimensional ball $\mathbb{B}$, the Hermitian symmetric domain associated with the group $U(2,1)$.

Hodge representations

Hodge representations were introduced by Green-Griffiths-Kerr [GGK1] to classify the Hodge groups of polarized Hodge structures and the corresponding Mumford-Tate subdomains of a period domain.

Definition 3.1.2. *Let G be a reductive $\mathbb{Q}$-algebraic group. A Hodge representation (G, ρ, φ) is given by a $\mathbb{Q}$-representation $\rho : M \to \mathrm{Aut}(V, Q)$ and a non-constant homomorphism of Lie groups $\varphi : \mathbb{S}^1 \to G(\mathbb{R})$ such that $(V, Q, \rho \circ \varphi)$ is a polarized Hodge structure.*

Green-Griffiths-Kerr showed that the Hodge groups $G = G_\varphi$ and Mumford-Tate domains $D = D_\varphi \subset \mathcal{D}_{\mathbf{h}}$ are in bijection with Hodge representations with Hodge numbers $\mathbf{h}_\varphi \leq \mathbf{h}$. The induced (real Lie algebra) Hodge representations

$$\mathbb{R} \to \mathfrak{g}_\mathbb{R} \to \mathrm{End}(V_\mathbb{R}, Q)$$

are enumerated by tuples $(\mathfrak{g}^{ss}_\mathbb{C}, E^{ss}, \mu, c)$ consisting of:

- a semisimple complex Lie algebra $\mathfrak{g}^{ss}_\mathbb{C} = [\mathfrak{g}_\mathbb{C}, \mathfrak{g}_\mathbb{C}]$,
- an element $E^{ss} \in \mathfrak{g}^{ss}_\mathbb{C}$ with the property that ad E^{ss} acts on $\mathfrak{g}^{ss}_\mathbb{C}$ diagonalizably with integer eigenvalues,
- a highest weight μ of $\mathfrak{g}^{ss}_\mathbb{C}$, and
- a constant $c \in \mathbb{Q}$ satisfying $\mu(E^{ss}) + c \in \frac{1}{2}\mathbb{Z}$.

Let's briefly recall the classification theorem. Given the Hodge decomposition $V_\mathbb{C} = \oplus V^{p,q}_\varphi$, the associated grading element (or infinitesimal Hodge structure) $E_\varphi \in i\mathfrak{g}_\mathbb{R}$ is defined by $E_\varphi(v) = \frac{1}{2}(p-q)v$ for $v \in V^{p,q}_\varphi$. In general, given a complex reductive Lie algebra $\mathfrak{g}_\mathbb{C}$, a grading element is any element $E \in \mathfrak{g}_\mathbb{C}$ with the property that $\mathrm{ad}(E) \in \mathrm{End}(\mathfrak{g}_\mathbb{C})$ acts diagonalizably on $\mathfrak{g}_\mathbb{C}$ with integer eigenvalues. That is,

$$\mathfrak{g}_\mathbb{C} = \bigoplus_{\ell \in \mathbb{Z}} \mathfrak{g}^{\ell,-\ell}, \qquad \text{with} \quad \mathfrak{g}^{\ell,-\ell} = \{\xi \in \mathfrak{g}_\mathbb{C} | [E, \xi] = \ell\xi\}.$$

Given the data $(\mathfrak{g}_\mathbb{C}, E)$, there is a unique real form $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}_\mathbb{C}$ such that the above decomposition is a weight zero Hodge structure on $\mathfrak{g}_\mathbb{R}$ that is polarized by $-\kappa$. The properties of E mean exactly that (G, Ad, E) is a Hodge representation. The data $(\mathfrak{g}_\mathbb{C}, E)$ determines the homogeneous space D_E and its complex dual $\check{D}_E$. More precisely, let $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}^{ss}$ denote the decomposition of $\mathfrak{g}$ into its center $\mathfrak{z}$ and semisimple factor $\mathfrak{g}^{ss}$. Let $E = E' + E^{ss}$ be the corresponding decomposition. The Hodge domain D_E is determined by $\mathfrak{g}^{ss}$ and E^{ss}.

The grading element E acts on any representation $G(\mathbb{R}) \to \mathrm{Aut}(V_{\mathbb{R}})$ by rational eigenvalues. The E-eigenspace decomposition $V_{\mathbb{C}} = \oplus_{k\in\mathbb{Q}} V_k$ is a Hodge decomposition if and only if those eigenvalues lie in $\frac{1}{2}\mathbb{Z}$. More precisely, if they lie in $\mathbb{Z}$ (resp. $\mathbb{Z}+\frac{1}{2}$), we can then choose any even (resp. odd) weight and "put" the Hodge structure in that weight. (The point is that E only knows $p-q$, not $p+q$.) Of course, Hodge classes have $p-q=0$ regardless of weight.

To classify the real Hodge representations, we will first need to study finite-dimensional real representations. Let $V_{\mathbb{R}}$ be an irreducible real representation of G. By Schur's lemma, $\mathrm{End}_G(V_{\mathbb{R}})$ is a division algebra over $\mathbb{R}$. We say that $V_{\mathbb{R}}$ is of real (resp. complex, quaternionic) type if $\mathrm{End}_G(V_{\mathbb{R}})$ is isomorphic to $\mathbb{R}$ (resp. $\mathbb{C}$. $\mathbb{H}$). If $V_{\mathbb{R}}$ is of real type, $V_{\mathbb{C}}$ is an irreducible self-dual representation V_+. If $V_{\mathbb{R}}$ is of complex or quaternionic type, $V_{\mathbb{C}}$ is the direct sum of an irreducible representation V_+ and its dual V_-, and $V_{\mathbb{R}} = \mathrm{Res}\ V_+ = \mathrm{Res}\ V_-$. Let μ, μ^* be the highest weight of V_+ and V_- respectively. Then we could read off the types of the representation $V_{\mathbb{R}}$ from the highest weight μ. See [GGK1] for more details.

Furthermore, the triple $(\mathfrak{g}_{\mathbb{C}}, E, \mu)$ is equivalent to a tuple $(\mathfrak{g}_{\mathbb{C}}^{ss}, E^{ss}, \mu^{ss}, c)$ by setting $c = \mu(E') \in \mathbb{Q}$ and μ^{ss} the restriction of μ to $\mathfrak{g}^{ss}$.

Example 3.1.3. We consider the group $SU(2,1)$. Let λ_i $(i=1,2)$ be its two fundamental weights. Let $\lambda = a\lambda_1 + b\lambda_2$ $(a, b \geq 0)$ be a dominant weight and V^λ be the irreducible representation (over $\mathbb{C}$) of highest weight λ. V^λ is self-dual if and only $a = b$, and V^λ is the complexification of an irreducible real representation $V_{\mathbb{R}}^\lambda$ of real type. Otherwise, the dual of V^λ is $V^{\lambda'}$ where $\lambda' = b\lambda_1 + a\lambda_2$. The direct sum $V^\lambda \oplus V^{\lambda'}$ is the complexification of an irreducible real representation $\tilde{V}_{\mathbb{R}}^\lambda \cong \mathrm{Res}_{\mathbb{C}/\mathbb{R}} V^\lambda$ of complex type. For example, the standard representation of $SU(2,1)$ on is of complex type, with complexification the direct sum of the complex standard representation and its dual.

The case for $G = SU(3,1)$ is similar. Let λ_i $(i = 1, 2, 3)$ be its three fundamental weights. Let $\lambda = \sum n_i\lambda_i$ be a dominant weight and $\lambda' = n_3\lambda_1 + n_2\lambda_2 + n_1\lambda_3$ its dual. Then V^λ is of real type if and only if $n_1 = n_3$. Otherwise $V^\lambda \oplus V^{\lambda'}$ has a natural real structure.

Example 3.1.4. If $G = Sp_4(\mathbb{R})$, all V^λ are of real type. The basechange map $V_{\mathbb{R}} \to V_{\mathbb{R}} \otimes \mathbb{C}$ establishes a bijection from the set of irreducible real representations to the set of irreducible complex representations.

3.1.1 Rational Hodge twists

Let $\mathbb{F}$ be an imaginary quadratic field $\mathbb{F}$ and (V, φ) a real Hodge structure with $\mathbb{F}$ acting as Hodge structure morphisms. Let $\iota : \mathbb{F} \to \mathbb{C}$ be a fixed embedding. Then we define V_+ (resp. $V_+^{p,q}$) to be the ι-eigenspace of V_+ (resp. $V^{p,q}$), and V_- (resp. $V_-^{p,q}$) to be the $\bar{\iota}$-eigenspace of V_- (resp. $V^{p,q}$). Then we get a decomposition of two conjugate pairs of complex Hodge structures V_+ and V_- since $\overline{V_+^{p,q}} = V_-^{q,p}$.

Now let $V = V^\lambda$ be an irreducible complex representation of G. The grading element acts on V with rational eigenvalues which differ by integers. If these eigenvalues are not half-integral, we can twist the action to make them half-integral. More precisely, as in [KK] we enlarge the MT group G to $U(1) \cdot G$, with complex Lie algebra $\mathbb{C} \oplus \mathfrak{g}$, and replace the grading element by $\tilde{E} = (1, E)$. We then define the twist $V\{c\}$ for $c \in \mathbb{Q}$ by

$$(V\{c\})_k := V_{k+c};$$

here c is simply the eigenvalue through which the "1" acts on V.

In the event that $\tilde{V}_{\mathbb{R}}$ is of complex type, with complexification $V_+ \oplus V_-$, we define $\tilde{V}\{c\}_{\mathbb{R}}$ to be the real irrep underlying $V_+\{c\} \oplus V_-\{-c\}$.

In [GGK1], the terminiology *half-twist* is used for the application of $\{-\frac{1}{2}\}$ to a pre-existing Hodge structure of type $V_+ \oplus V_-$. This changes the parity of the weight, and is usually thought of as adding 1 to the weight.

Example 3.1.5. We consider the real standard representation $\tilde{V}_{\mathbb{R}}^{\lambda_1}$ of $U(2,1)$. Its complexification is the direct sum of the standard representation $V_+ := V^{\lambda_1}$ and its dual $V_- := V^{\lambda_2}$. For V_+, E acts as $2/3$ on a one-dimensional subspace, and $-1/3$ on a two-dimensional subspace. For V_-, E acts as $1/3$ on a two-dimensional subspace, and $-2/3$ on a one-dimensional subspace.

Taking $c = 1/6$ makes the Hodge indexes half-integral as desired (E acts as $\pm 1/2$). We get a level-1 Hodge structure on $V^{\lambda_1}\{\frac{1}{6}\}_{\mathbb{R}}$, which we can take to have weight 1. In particular, $\dim V_+^{1,0} = \dim V_-^{0,1} = 1, \dim V_+^{0,1} = \dim V_-^{1,0} = 2$. This is exactly the Hodge structure of a Picard curve.

Taking a half-twist of the above Hodge structure, we get a Hodge structure of weight 2 with Hodge numbers $(1, 4, 1)$. It represents the transcendental cohomology of a family of $K3$ surfaces with Picard rank $\rho = 16$.

Taking half-twist again, we get a Hodge structure of weight three with Hodge numbers $(1, 2, 2, 1)$. It represents the Hodge structure of Rohde's family of Calabi-Yau threefolds.

3.2 A Hodge Number Formula

When G/K is Hermitian symmetric, the cohomology of a representation acquires a bigrading as follows. Write $\mathfrak{p}^+ \subset \mathfrak{p}$ (resp. $\mathfrak{p}^- \subset \mathfrak{p}$) for the holomorphic (resp. anti-holomorphic) tangent space of G/K at the origin. Then the decomposition

$$\wedge^* \mathfrak{p} = \oplus (\wedge^* \mathfrak{p}^+) \otimes (\wedge^* \mathfrak{p}^-)$$

induces a bigrading on the complex $\mathrm{Hom}_{\mathfrak{p}}(\wedge^* \mathfrak{p}, U \otimes V)$. For a cohomological representation, the differential map d is zero, so this bigrading descends to a bigrading on the cohomological spaces:

$$H^i(\mathfrak{g}, K; U \otimes V) = \bigoplus_{p+q=i} H^i(\mathfrak{g}, K; U \otimes V)^{p,q}.$$

This decomposition corresponds to the Hodge structure on the cohomology of locally symmetric spaces, and the computation is given in [VZ]. (That is, we are treating V as a trivial HS of type $(0,0)$; this will be fixed in a moment.) The Hodge types are calculated in terms of the decomposition of the θ-stable parabolic subalgebra $\mathfrak{q}$. We denote the Hodge type given in [VZ] by (p_U, q_U).

Now if V is a representation of real type of the group G, and the associated local system $\mathbb{V}$ is a VHS of weight w, then we need to shift the Hodge numbers to make $H^i(\mathfrak{g}, K; U \otimes V)$ a HS of weight $i + w$, viz.

$$H^i(\mathfrak{g}, K; U \otimes V) = \bigoplus_{p+q=i+w} H^i(\mathfrak{g}, K; U \otimes V)^{p,q}.$$

A class on the left-hand side can be written as $\sum_j \omega_j \otimes u_j \otimes v_j$ where $\omega_j \in (\wedge^* \mathfrak{p})^\vee$ is a "differential form" of type (p_U, q_U), while $v_j \in V$ and $u_j \in U$. From the construction of the representations $A_\mathfrak{q}$ and the computation of their cohomology, all the vectors v_j are in the same K-sub-representation of V, hence have the same Hodge type (p'_U, q'_U). So the Hodge type (p,q) for such a class is given by $(p_U + p'_U, q_U + q'_U)$.

Taking the sum of all possible representations, we get the following Hodge number formula for local systems.

Proposition 3.2.1. *Set* $h_i^{p,q}(\pi, V) := \dim(H^i_{(2)}(X^*, \mathbb{V})^{p,q})$, *then*

$$h_i^{p,q}(\pi, V) = \sum mult.(\Gamma, U_\pi) \dim(H^i(\mathfrak{g}, K; U_\pi \otimes V))$$

where U_π *sums over cohomological representations with nonzero* H^i *and such that* $(p_U + p'_U, q_U + q'_U) = (p,q)$.

Now if V is a representation of complex or quaternionic type of the group G, and the associated local system $\mathbb{V}$ is a VHS. Then $V_{\mathbb{C}} = V_{\mu} \oplus V_{\mu^*}$, and the computation of the Hodge types is the same as in the real case, but it is the direct sum of two representations.

Proposition 3.2.2. *Set* $h_i^{p,q}(\pi, V) := \dim(H^i_{(2)}(X^*, \mathbb{V})^{p,q})$*; then*

$$\begin{aligned} h_i^{p,q}(\pi, V) = &\sum \textit{mult.}(\Gamma, U_\pi) \dim(H^i(\mathfrak{g}, K; U_\pi \otimes V_\mu)) \\ &+ \sum \textit{mult.}(\Gamma, U_\pi^*) \dim(H^i(\mathfrak{g}, K; U_\pi^* \otimes V_{\mu^*})) \end{aligned}$$

where U_π *(resp.* U_π^**) sums over cohomological representations with respect to* V *(resp.* V^**) with nonzero* H^i *and such that* $(p_U + p'_U, q_U + q'_U) = (p, q)$ *(resp.* $(p_{U^*} + p'_{U^*}, q_{U^*} + q'_{U^*}) = (p, q)$*).*

If $\mathbb{V}$ is not a VHS, we could still make a rational half-twist to get a VHS. Let c be the shifting constant. We could pretend that the vector has a rational Hodge type (p'_U, q'_U), and then add the twisting constant c.

Proposition 3.2.3. *Set* $h_i^{p,q}(\pi, V) := \dim(H^i_{(2)}(X^*, \mathbb{V})^{p,q})$*; then*

$$\begin{aligned} h_i^{p,q}(\pi, V) = &\sum \textit{mult.}(\Gamma, U_\pi) \dim(H^i(\mathfrak{g}, K; U_\pi \otimes V_\mu)) \\ &+ \sum \textit{mult.}(\Gamma, U_\pi^*) \dim(H^i(\mathfrak{g}, K; U_\pi^* \otimes V_{\mu^*})) \end{aligned}$$

where U_π *(resp.* U_π^**) sums over cohomological representations with respect to* V *(resp.* V^**) with nonzero* H^i *and such that* $(p_U + p'_U + c, q_U + q'_U + c) = (p, q)$ *(resp.* $(p_{U^*} + p'_{U^*} - c, q_{U^*} + q'_{U^*} - c) = (p, q)$*).*

Remark 3.2.1. In practice, the group G could be reductive. Since the cohomology of local systems is not changed if we study the semisimple subgroup. We could calculate everything in the semisimple setting. Just note that, to calculate the Hodge type of the vector, v should be considered as a representation of G.

Remark 3.2.2. It seems that it is better to study packets of representations if representations in the same packet occurs with the same multiplicity, and have 'conjugate' Hodge numbers.

3.3 Some explicit computations I: Unitary Groups

$SU(2,1)$

As a Lie group, $SU(n,1)$ is the group of matrices M preserving the sesquilinear form defined by $\mathrm{diag}(I_n,-1)$: $M\mathrm{diag}(I_n,-1)\overline{M}^t = \mathrm{diag}(I_n,-1)$. The maximal torus of $SU(n,1)$ is the two-dimensional torus consisting of diagonal matrices $\{\mathrm{diag}(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, \cdots, e^{2\pi i\theta_n}), \sum\theta_i \equiv 0 \mod 2\pi\}$. The maximal compact subgroup is isomorphic to $U(n)$, consisting of matrices $\mathrm{diag}(A, \det(A)^{-1})$ where A is in $U(n)$.

The natural K-representation $\wedge^*\mathfrak{p}$ has a simple description [BW]. Let τ be the standard representation of $U(n)$ on $\mathbb{C}^n$. Set $\tau_1 = (\det)\otimes\tau$. Then as a K-representation, $\mathfrak{p} \cong \tau_1 \oplus \tau_1^*$. Wedge powers of τ_1 and τ_1^* are irreducible. And

$$\wedge^q\mathfrak{p} = \bigoplus_{r+s=q} \Lambda^{r,s}$$

with K acting on $\Lambda^{r,s}$ by $\wedge^r\tau_1 \otimes \wedge^s\tau_1^*$. Note that $\Lambda^{r,s}$ are not necessarily irreducible.

Irreducible representations of $U(2)$ are finite-dimensional. We have a surjective product map from $SU(2)\times U(1)$ to $U(2)$ with kernel $\{\pm 1\}$. The tensor product of k-th symmetric power ($k \in \mathbb{N}$) of the standard representation of $SU(2)$ and l-th character ($l \in \mathbb{Z}$) of $U(1)$ descends to an irreducible representation of $U(2)$ if and only if $k \equiv l \mod 2$. Therefore, the irreducible representations of $U(2)$ are parameterized by the pairs (k,l) with $k \in \mathbb{N}, l \in \mathbb{Z}$ and $k \equiv l \mod 2$. The representation τ_1 is $(1,1)$, and τ_1^* is $(1,-1)$.

The complexified Lie algebra $\mathfrak{su}(2,1)$ is isomorphic to $\mathfrak{sl}(3)$. Following notations in [GGK2], its simple roots are $\alpha_1 = e_2^* - e_1^*$ and $\alpha_2 = e_3^* - e_2^*$. The root α_1 is a compact root. From Weyl's unitary trick, finite-dimensional representations of $SU(2,1)$ are the same as finite-dimensional representations of $\mathfrak{su}(2,1) \cong \mathfrak{sl}(3)$. There are two fundamental representations corresponding to the two dominant weights: $\lambda_1 = \frac{1}{3}(\alpha_1 + 2\alpha_2)$ corresponds to the standard representation W, and $\lambda_2 = \frac{1}{3}(2\alpha_1 + \alpha_2)$ corresponds to the dual $W^* \cong \wedge^2 W$. Other representations are subrepresentations of tensor powers of W and W^*.

Now we list cohomological representations. Let V be a finite-dimensional representation of $SU(2,1)$.

We first consider the case when V is the trivial representation. We have three types of cohomological representations ([BW]):

- The three discrete series representations with the same infinitesimal characters as the trivial representation. One is the holomorphic discrete series D^+, with minimal K-type $(0,2)$; one is the anti-holomorphic discrete series D^-, with minimal K-type $(0,-2)$; and one is the non-holomorphic discrete series D^0; with minimal K-type $(2,0)$. The only nonvanishing cohomology is $H^2(\mathfrak{g}, K, U) = \mathbb{C}$.

- The two non-tempered representations $J_{1,0}$ and $J_{0,1}$. The minimal K-type of $J_{1,0}$ is $(1,1)$, and the minimal K-type of $J_{0,1}$ is $(1,-1)$. The nontrivial cohomology is $H^1(\mathfrak{g}, K, U) = H^3(\mathfrak{g}, K, U) = \mathbb{C}$.

- The trivial representation. The minimal K-type is $(0,0)$. The cohomology groups are those coming from the base varieties.

If V is regular, the only cohomological representations are discrete series representations.

If $V = V^\lambda$ is singular, then $\lambda = n\lambda_1$ or $n\lambda_2$. In this case, besides discrete series we have some nontempered cohomological representations. If $\lambda = n\lambda_1$ (resp. $n\lambda_2$), a twist of $J_{1,0}$ (resp. $J_{0,1}$) is also cohomological. We call it $J^n_{1,0}$ (resp. $J^n_{0,1}$). In order to list the corresponding Hodge types for $V = V^{n\lambda_1} \subseteq (V^{\lambda_1})^{\otimes n}$, we have to fix the Hodge structure on

V^{λ_1} as in Example 3.1.5. First, we take the Hodge type of V^{λ_1} to be twisted so that $h^{1,0} = 1$ and $h^{0,1} = 2$, so that $\mathcal{H}^1$ of the universal Picard curve takes the form $V^{\lambda_1} \oplus V^{\lambda_2}$ (where $V^{\lambda_2} = \overline{V^{\lambda_1}}$). Then the following Hodge structures appear:

- U = discrete series. The only non-vanishing cohomology is $H^2(\mathfrak{g}, K; U \otimes V^{n\lambda_1})$. The Hodge types are $(p, q) = (n + 2, 0)$,$(n + 1, 1)$, and $(n, 2)$.

- U = the representation $J^n_{1,0}$. The non-vanishing cohomologies are H^1 and H^3. Their Hodge types are $(n + 1, 0)$, and $(n + 2, 1)$.

If V^{λ_1} is twisted as for the local systems associated with $K3$ surfaces (resp. Calabi-Yau manifolds), just add $(1, 0)$ (resp. $(2, 0)$) to the Hodge numbers. Finally, in *all* cases, viewing $V^{n\lambda_2}$ as $\overline{V^{n\lambda_1}}$ as a Hodge representation, apply complex conjugation $(a, b) \mapsto (b, a)$ to the Hodge types to get the correct ones for $V^{n\lambda_2}$. In summary:

Proposition 3.3.1. *In these three geometric cases, we do not get Hodge classes in the cohomology of local systems of Picard curves or Calabi-Yau threefolds. In the case of $K3$ surfaces, we get Hodge classes of type $(2, 2)$ in $H^2(\mathfrak{g}, K; U \otimes V^{\lambda_1})$; similarly, for the universal abelian surfaces (=Jacobians of Picard curves) over the 2-ball, we get Hodge classes of type $(2, 2)$ in $H^2(\mathfrak{g}, K; U \otimes V^{2\lambda_1})$.*

$SU(3, 1)$

The representations of $U(3)$ are similar to the case of $U(2)$. We consider the surjection $SU(3) \times U(1) \to U(3)$ with kernel $A = \langle\omega\rangle \cong \mathbb{Z}/3\mathbb{Z}$ where ω is a primitive third root of unity. Any irreducible representation (π, V) of $SU(3)$ defines, by restriction, a character χ_π of A, which must be $\omega \mapsto \omega^{i(\pi)} (i(\pi) = 0, 1, 2)$. A pair (π, n) with $n \in \mathbb{Z}$ and $n \equiv i(\pi) \mod 3$ defines an irreducible representation of $U(3)$. And all irreducible representations of $U(3)$ are of this form. The representation τ_1 is $(\mathrm{st}, 1)$ and τ_1^* is $(\mathrm{st}^*, -1)$.

We have three fundamental representations. The standard representation $V^{\lambda_1} = W$, the wedge product $V^{\lambda_2} = \wedge^2 W$, and $V^{\lambda_3} = \wedge^3 W = W^*$. All other finite-dimensional representations are subrepresentations of tensor products of fundamental representations.

When V is the trivial representation, the cohomological representations U are:

- four discrete series D_i. The only non-vanishing cohomology is $H^3(\mathfrak{g}, K, U) = \mathbb{C}$. Their Hodge types are $(3,0)$, $(2,1)$, $(1,2)$, and $(0,3)$.

- non-tempered representations.

  - $J_{0,0} = \mathbb{C}$,

  - $J_{1,0}$ and $J_{0,1}$, whose only nonvanishing cohomology groups are $H^1(\mathfrak{g}, K, U) = H^3(\mathfrak{g}, K, U) = H^5(\mathfrak{g}, K, U) = \mathbb{C}$. Their Hodge types are $(1,0)$ ($(2,1)$, and $(3,2)$) and $(0,1)$ ($(1,2)$, and $(2,3)$).

  - $J_{2,0}$, $J_{1,1}$, $J_{0,2}$. The nonvanishing cohomology groups are $H^2(\mathfrak{g}, K, U) = H^4(\mathfrak{g}, K, U) = \mathbb{C}$. Their Hodge types are $(2,0)$, $(1,1)$, $(0,2)$ in degree two and $(3,1)$, $(2,2)$, $(1,3)$ in degree four.

For regular representations, only discrete series are cohomological. For singular representations, the twists of non-tempered representations are again cohomological, but we need to compute their minimal K-type to determine which representations are cohomological with respect to a finite-dimensional representation. The next proposition treats the most singular cases:

- If $\lambda = n\lambda_1$, a twist $J^n_{1,0}$ or $J^n_{2,0}$ is also cohomological;

- If $\lambda = n\lambda_2$, a twist of $J^n_{1,1}$ is also cohomological;

- If $\lambda = n\lambda_3$, a twist of $J^n_{0,1}$ or $J^n_{0,2}$ is also cohomological.

Proposition 3.3.2. *For these choices of λ, with Hodge types as in tensor powers of $\mathcal{H}^1$ of Picard curves, the only case to get Hodge classes in $H^*(\mathfrak{g}, K, U \otimes V^\lambda)$ is when $\lambda = n\lambda_2$ and U is a twisting of $J_{1,1}$.*

3.4 Some explicit Computations II: Symplectic Groups

As a Lie group, the symplectic group $Sp_4(\mathbb{R})$ is defined to be the group

$$\{X \in M_{4\times 4} | X^t J X = J\}$$

where J is the standard symplectic form $\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. The maximal torus of Sp_4 is the diagonal matrices $\mathrm{diag}(x, y, x^{-1}, y^{-1})$. The maximal compact subgroup is $K = U(2)$. Let $X' = A + iB \in U(2)$ where A, B are real matrices, the associated element in Sp_4 is $X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$.

There are two fundamental weights λ_1 and λ_2. The corresponding representation $V = V^{\lambda_1}$ is the standard representation W. V^{λ_2} is the kernel of the natural contraction map $\wedge^2 W \to \mathbb{C}$ induced by the symplectic form over W. So we have $\wedge^2 V^{\lambda_1} = V^{\lambda_2} \oplus \mathbb{C}$. We change the indexing by writing V^{λ_1} as $V_{1,0}$, V^{λ_2} as $V_{1,1}$, so that (more generally) the irreducible representations $V_{a,b} = V^{(a-b)\lambda_1 + b\lambda_2}$ are parameterized by pairs of integers $\{(a, b) | a \geq b \geq 0\}$.

Now set

$$\Xi_1 = \{(l_1, l_2) \in \mathbb{Z}^2 | l_1 > l_2 > 0\}, \qquad \Xi_2 = \{(l_1, l_2) \in \mathbb{Z}^2 | l_1 > -l_2 > 0\},$$

$$\Xi_3 = \{(l_1, l_2) \in \mathbb{Z}^2 | -l_2 > l_1 > 0\}, \qquad \Xi_4 = \{(l_1, l_2) \in \mathbb{Z}^2 | -l_2 > -l_1 > 0\}.$$

Then there is a one-to-one correspondence between the union of the four sets and the set of unitary equivalence classes of discrete series representations of $Sp_4(\mathbb{R})$. The integral point (l_1, l_2) is called the *Harish-Chandra parameter*.

1. $D(l_1, l_2)$ denotes the holomorphic discrete series representation with the minimal K-type $\det^{l_2+2} \otimes \mathrm{Sym}^{l_1-l_2-1}$ if $(l_1, l_2) \in \Xi_1$.

2. $D(l_1, l_2)$ denotes the large discrete series representation with the minimal K-type $\det^{l_2} \otimes \mathrm{Sym}^{l_1-l_2+1}$ if $(l_1, l_2) \in \Xi_2$.

3. $D(l_1, l_2)$ denotes the large discrete series representation with the minimal K-type $\det^{l_2-1} \otimes \mathrm{Sym}^{l_1-l_2+1}$ if $(l_1, l_2) \in \Xi_3$.

4. $D(l_1, l_2)$ denotes the anti-holomorphic discrete series representation with the minimal K-type $\det^{l_2-1} \otimes \mathrm{Sym}^{l_1-l_2-1}$ if $(l_1, l_2) \in \Xi_4$.

If $(l_1, l_2) \in \Xi_1$, the four discrete series representations

$$\{D(l_1, l_2), D(l_1, -l_2), D(l_2, -l_1), D(-l_2, -l_1)\}$$

form a single L-packet, and they have the same infinitesimal character. $D(l_1, l_2)$ is *holomorphic*, $D(-l_2, -l_1)$ is *anti-holomorphic*, and the other two discrete series representations are called *large discrete series representations*.

Next consider the parabolic subgroups

$$P_1(\mathbb{R}) = \left(\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \in Sp_4(\mathbb{R}) \right) \text{ and } P_2(\mathbb{R}) = \left(\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & * & * & * \end{pmatrix} \in Sp_4(\mathbb{R}) \right),$$

and let $M_k(\mathbb{R})A_k(\mathbb{R})N_k(\mathbb{R})$ be the Langlands decomposition of $P_k(\mathbb{R})$ for $k = 1$ or 2. Hence, we have $M_1(\mathbb{R}) \cong SL_2^{\pm}(\mathbb{R})$, $A_1(\mathbb{R}) \cong \mathbb{R}^+$, $N_1(\mathbb{R}) \cong \mathbb{R}^3$, $M_2(\mathbb{R}) \cong SL_2(\mathbb{R}) \times (\mathbb{Z}/2\mathbb{Z})$, $A_2(\mathbb{R}) \cong \mathbb{R}^+$, and $N_2(\mathbb{R}) \cong \mathbb{R} \ltimes \mathbb{R}^2$.

Let D_k^+ (resp. D_k^-) denote the holomorphic (resp. anti-holomorphic) discrete series of $SL_2(\mathbb{R})$ which has the minimal K-type $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mapsto e^{i(k+1)\theta}$ (resp. $e^{-i(k+1)\theta}$). The representation $D_k := \mathrm{Ind}_{SL_2(\mathbb{R})}^{SL_2^{\pm}(\mathbb{R})} D_k^+ \cong \mathrm{Ind}_{SL_2(\mathbb{R})}^{SL_2^{\pm}(\mathbb{R})} D_k^-$ is irreducible, and its restriction to $SL_2(\mathbb{R})$ is $D_k^+ \oplus D_k^-$. The quasi-character ν_1 on $\mathbb{R}^+$ is defined by $\nu_1(a) = a$. Let sgn denote the non-trivial character on $\mathbb{Z}/2\mathbb{Z}$.

For each integer $k > 2$, we denote by σ_k the Langlands quotient of the induced representation

$$\mathrm{Ind}_{M_1(\mathbb{R})A_1(\mathbb{R})N_1(\mathbb{R})}^{Sp_4(\mathbb{R})}(D_{2k-1} \otimes \nu_1 \otimes 1).$$

Similarly, for each integer $l > 1$, let $\omega_l^{\pm}$ denote the Langlands quotient of the induced representation

$$\mathrm{Ind}_{M_2(\mathbb{R})A_2(\mathbb{R})N_2(\mathbb{R})}^{Sp_4(\mathbb{R})}((D_l^{\pm} \otimes \mathrm{sgn}) \otimes \nu_1 \otimes 1).$$

The representations σ_k and $\omega_l^{\pm}$ are unitarizable and cohomological, but non-tempered.

Now let $V = V_{a,b}$ be an irreducible finite-dimensional representation. If V is regular ($a > b > 0$), only the discrete series representations are cohomological. If $a > b = 0$, $\omega_{a+2}^{\pm}$ is also cohomological. If $a = b > 0$, σ_{a+3} is also cohomological. If $a = b = 0$, all cohomological representations occur.

So when $V = V_{0,0}$ is the trivial representation, we have the following cohomological representations and Hodge types:

- four discrete series representations, with $H^3(\mathfrak{g}, K, U) = \mathbb{C}$. Their Hodge types are $(3, 0)$, $(2, 1)$, $(1, 2)$, and $(0, 3)$.

- non-tempered representations ω_2^+, σ_3, ω_2^-: $H^2(\mathfrak{g}, K, U) = H^4(\mathfrak{g}, K, U) = \mathbb{C}$. Their Hodge types for H^2 are $(2,0)$, $(1,1)$, and $(0,2)$ (for H^4, add $(1,1)$).

More generally, we arrive at the following

Proposition 3.4.1. *The only case to get Hodge classes in* $H^*(\mathfrak{g}, K, U \otimes V)$ *is when* V *is* $V_{a,a}$ *and* U *is the non-tempered representation* σ_{a+3}.

Chapter 4

Automorphic Forms and Multiplicity Formulas

4.1 Automorphic Forms and Automorphic representations

4.1.1 Automorphic forms

Let G be a reductive algebraic group over a number field F.

Definition 4.1.1. *A function f on $G(F)\backslash G(\mathbb{A}_F)$ is called an automorphic form if*

- *f is smooth. This means that the nonarchimedean component f_∞ is smooth, and locally constant in $G(\mathbb{A}_{F,f})$.*
- *f is right K-finite. Here $K = (K_p)$ is in the definition of $G(\mathbb{A}_F)$, and the condition means that the right K-translations of f span a finite dimensional vector space. Equivelently, f is K_∞-finite and is right invariant under an open compact subgroup of $G(\mathbb{A}_{F,f})$.*
- *f is of moderate growth. This means that $f(g)$ can be controlled by some powers of the matrix elements of g and $i(g)$ (after choose an embedding $G \to GL_N$).*
- *f is $Z(\mathfrak{g})$-finite. Here $Z(\mathfrak{g})$ is the center of the universal enveloping algebra. The $Z(\mathfrak{g})$-finiteness is equivalent to a system of differential equations for f_∞.*

We let $\mathcal{A}(G)$ denote the space of automorphic forms on G.

Definition 4.1.2. *An automorphic form f on G is called a cusp form if, for any parabolic F-subgroup $P = MN$ of G, the constant term*

$$f_N(g) = \int_{N(F)\backslash N(\mathbb{A}_F)} f(ng)dn$$

is zero as function on $G(\mathbb{A}_F)$.

Remark 4.1.1. In some literature, the automorphic form is defined with a character $\omega : Z(\mathbb{A}_F) \to \mathbb{C}$, so the function f are functions on $G(F)\backslash G(A)$ such that $f(zg) = \phi(z)f(g)$. Our space is just the sum over all characters.

We let $\mathcal{A}_0(G)$ be the space of cusp forms on G.

4.1.2 Automorphic Representations

Recall that $\mathcal{A}(G)$ denotes the space of automorphic forms on G. As usual , we define right translation r by $r(g)f(h) = f(hg^{-1})$. But we cannot regard $\mathcal{A}(G)$ as a $G(\mathbb{A}_F)$-module. At archimedean places, the K-finite property is not preserved by all $G(F_\infty)$-translations, so we can only get a $(\mathfrak{g}_\infty, K_\infty)$-module at archimedean places. The space $\mathcal{A}(G)$ is a $G(\mathbb{A}_{F,f}) \times (\mathfrak{g}_\infty, K_\infty)$-module. By abuse of language, we still say $\mathcal{A}(G)$ is a $G(\mathbb{A}_F)$-representation. An irreducible $G(\mathbb{A}_F)$-representation is called an *automorphic representation* if it is isomorphic to a subquotient of $\mathcal{A}(G)$.

Flath's theorem

We start by defining a *restricted tensor product of vector spaces.* Let Ξ be a finite subset, and Ξ_0 be a finite subset. Let $\{W_v\}_{v\in\Xi}$ be a family of $\mathbb{C}$-vector spaces and choose ϕ_{0v}[1] of local representations. for each $v \in \Xi - \Xi_0$. For all sets $\Xi_0 \subset S \subset \Xi$ of finite cardinality set

[1]Unlike the direct limit of a family of abelian groups, we have to specify nonzero vectors for almost all indices. This is because we cannot tensor with the "canonical element" $0 \in W_v$.

$W_S := \prod_{v\in S} W_v$. If $S \subset S'$, there is a map $W_S \to W_{S'}$ defined by:

$$\otimes_{v\in S} w_v \mapsto \otimes(\otimes_{v\in S'-S}\phi_{0v}).$$

The vector space

$$W := \otimes' \otimes W_v := \varinjlim_{S} W_S$$

is the restricted tensor product of the W_v with respect to the ϕ_{0v}. Thus W is the set of sequences $(W_v)_{v\in\Xi} \subset \otimes_v W_v$ such that $w_v = \phi_{0v}$ for all but finitely many $v \in \Xi$.

Remark 4.1.2. The isomorphism classes of W in general depend on the choice of ϕ_{0v}. However, if we replace $\phi_{0,v}$ by nonzero scalar multiples we obtain isomorphism vector spaces.

Example 4.1.3. One has

$$C_c^\infty(G(\mathbb{A}_F^\infty)) \cong \otimes' C_c^\infty(G((F_v)))$$

with respect to the idempotents $e_{K_v} := \frac{1}{\mathrm{vol}(K_v)} 1_{K_v}$ where K_v is a hyperspecial subgroup.

Theorem 4.1.4 (Flath's theorem)**.** *Every admissible irreducible representation W of $C_c^\infty(G(\mathbb{A}_F))$ can be written as*

$$W \cong \otimes_v' W_v$$

where the restricted tensor product is with respect to elements $\phi_{v_0} \in W_v^{K_v}$, $\dim W_v^{K_v} = 1$, and the isomorphism interwins the action of $C_c^\infty(\mathbb{A}_F)$ with the action of $\otimes_v'(G(F_v)$, the restricted tensor product being with respect to the idempotents e_{K_v}.

In particular, automorphic representations are restricted tensor products.

Let v be a nonarchimedean place, W_v be an irreducible smooth admissible representation of $G(F_v)$, then $\dim W_v^{K_v} \leq 1$. If $\dim V^K = 1$, the representation is called a *spherical* representation. One implication of Flath's theorem is that for almost all nonarchimedean places v, the representation W_v is spherical. The classification of spherical representations is well known by the Sataka isomorphism theorem.

The representations at the archimedean places are the admissible Harish-Chandra modules. Their classifications are well known by the langlands classification. Roughly speaking, all admissible representations are Laglands quotients of a parabolically induced representation.

4.1.3 Classical modular forms

We first consider elliptic modular forms with respect to the full modular group $SL_2(\mathbb{Z})$.

A modular form of weight k is just a holomorphic function $f(z)$ on the upper half plane $\mathbb{H}$ satisfies the functional equations

$$f(\gamma z) = \frac{1}{(cz+d)^k} f\left(\frac{az+b}{cz+d}\right), \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$f(z)$ then has a Fourier expansion $f(z) = a_n q^n$ where $q = e^{2\pi i z}$. $f(z)$ is called a cusp form if $a_0 = 0$. Let S_k be the space of cusp form s of weight k. There is a family of linear operators T_n, called the Hecke operators, on S_k. They can be defined in terms of their action on the Fourier coefficients of cuap forms. An eigenform f is a cusp form that is simitaneously eigenvaluea for all T_n.

Let f be an eigenform. We may associate an automorphic representation of π_f on $PGL_2(\mathbb{Q})\backslash PGL_2(\mathbb{A})$ in two steps. First, we have the isomorphism $PGL_2(\mathbb{R})/PGO_2 = \mathbb{H}$ where g is mapped to gi. f could be considered as a function on $PGL_2(\mathbb{R})$ that is right PGO_2-invariant but has a good transition formula with respect to the left $PGL_2(\mathbb{Z})$. We define a function ϕ_f on $PGL_2(\mathbb{Z})\backslash PGL_2(\mathbb{R})$ by the formula

$$\phi_f(g) = f(gi).$$

Then ϕ_f is left $PGL_2(\mathbb{Z})$-invariant but is an eigenfunction with eigenvalue $e^{2\pi ik}$ under the right translation of the circle $SO(2)$. From the isomorphism $PGL_2(\mathbb{Z})\backslash PGL_2(\mathbb{R}) = GL_2(\mathbb{Q})\backslash PGL_2(\mathbb{A})/U$ where U is a compact subgroup of $PGL_2(\mathbb{A}_f)$, we get a function on $GL_2(\mathbb{Q})\backslash PGL_2(\mathbb{A})$. The subspace generated by this vector is the automorphic representation π_f.

The archimedean component of π_f is the holomorphic discrete series D^+_{k-1}. The nonarchimedean components are all spherical representations. Let p be a prime, the Hecke eigenvalue of f is exactly the action of an element in the local spherical Hecke algebra.

The multiplicity one theorem says that the association $f \to \pi_f$ is injective. Note that only the archimedean component is not enough to distinguish eigenforms with the same weight.

Automorphic representations associated with elliptic modular forms

Let f be a cuspidal new form for $\Gamma_0(N)$ with weight $k \geq 2$, we associate an automorphic representation π_f of $GL_2(\mathbb{A})$ as in [Bu]. The representation π_f is irreducible if f is a new form, and π_f is a restricted tensor product of irreducible representations $\pi_{f,v}$ over local groups $GL_2(\mathbb{Q}_v)$. Given such an f, An explicit algorithm for computing local components is given in [LW].

If $v = \infty$, the representations of $GL_2(\mathbb{R})$ are well-known. In particular, $\pi_{f,\infty}$ is the unique discrete series subrepresentation of the representation constructed via unitary induction from the character

$$\begin{pmatrix} t_1 & * \\ & t_2 \end{pmatrix} \mapsto \left|\frac{t_1}{t_2}\right|^{k/2} \operatorname{sgn}(t_1)^k$$

of the Borel subgroup of $GL_2(\mathbb{R})$.

Now let p be a finite prime, the irreducible infinite-dimensional representations of $GL_2(\mathbb{Q}_p)$ fall into three classes: the principal series representations $\pi(\chi_1, \chi_2)$, the special representations St $\otimes \chi_1$, and the supercuspidal representations $\mathrm{Ind}_K^G \tau$ (See [BH]). Here, χ_1 and χ_2 are characters of $\mathbb{Q}_p^*$. The representation denoted by $\pi(\chi_1, \chi_2)$ is the principal series attached to the characters χ_1 and χ_2, defined whenever $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$. The representation St is the Steinberg representation. The supercuspidal representations of G are induced from certain finite-dimensional characters τ of compact-mod-center subgroups $K \subset G$.

4.2 The Trace Formula

The trace formula is one of the most powerful methods so far in studying the Langlands program. It describes the character of the representation of $G(\mathbb{A}_F)$ on the discrete part $L_0^2(G(F)\backslash G(\mathbb{A}_F))$ of $L^2(G(F)\backslash G(\mathbb{A}_F))$ in terms of geometric data, where G is a reductive algebraic group defined over a global field F and $\mathbb{A}_F$ is the ring of adeles of F. We usually calculate the geometric side to get information on the mysterious spectral side. Applications of the trace formula include the functoriality principle and classification of automorphic representations.

4.2.1 compact case

In this subsection, we assume that G is defined over $\mathbb{Q}$. $G(\mathbb{R})$ is then a Lie group. We use G to stand for this Lie group to simplify notations. Fix a Haar measure dg of G. Let Γ be a cocompact arithmetic subgroup of G. The Haar measure descends to a Haar measure dx on $\Gamma\backslash G$ by the left-invariance of dg. We consider the regular representation R of G on $L^2(\Gamma\backslash G)$:

$$[R(g)]\phi(x) = \phi(xg), g \in G, x \in \Gamma\backslash G.$$

This extends to a representation of the convolution algebra $C_c^\infty(G)$ [2] :

$$[R(f)]\phi(x) = \int_G f(g)\phi(xg)dg = \int_G f(x^{-1}g)\phi(g)dg$$

The operator $R(f)$ is of trace class, and we want to compute $\mathrm{tr}\,(R(f))$. There are two methods to compute it:

1. Geometric method. If $f \in C_c^\infty(G)$, the operator $R(f)$ has a kernel $K_f(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)$:

$$[R(f)]\phi(x) = \int_G f(x^{-1}g)\phi(g)dg = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\phi(y)dy.$$

   $\mathrm{tr}\,(R(f))$ is just the integral of the kernel over the diagonal (the trace of an "infinite dimensional" matrix is the "sum" over the diagonal):

$$\mathrm{tr}\,(R(f)) = \int_{\Gamma\backslash G} K_f(x,x)dx = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)dx.$$

   We can break the sum over γ into conjugacy classes of Γ. The conjugacy class

$$[\gamma] = \{\delta^{-1}\gamma\delta : \delta \in \Gamma_\gamma\backslash\Gamma\}$$

   where Γ_γ is the centralizer of γ in Γ, contributes

$$\int_{\Gamma\backslash G} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)dx = \int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)dx = \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)I(\gamma, f),$$

[2] Formally, $R(f) = \int_G R[g]f(g)dg$. It is the weighted integration of the operator $R(g)$ with respect to the measure $f(g)dg$. We can also define $R(d\mu) = \int_G R[g]d\mu(g)$ for a bounded measure $d\mu$.

where $I(\gamma, f)$ is the *orbital integral*

$$I(\gamma, f) = \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)dx.$$

In summary,

$$\text{tr}\,(R(f)) = \sum_{[\gamma]} \text{vol}(\Gamma_\gamma \backslash G_\gamma) I(\gamma, f)^3. \tag{4.2.1}$$

2. Spectral method. First recall that we have a very simple description of the regular representation in the compact case:

Proposition 4.2.1 (Gelfand-Graev-Piatetski-Shapiro)**.** *$L^2(\Gamma\backslash G)$ decomposes discretely into a direct sum of irreducible representations of G, each occuring with finite multiplicity* [4]*.*

Let $\hat{G}$ be the unitary dual of G, from the representation decomposition

$$R = \sum_{\pi \in \hat{G}} m(\pi)\pi,$$

we get another computation of the trace

$$\text{tr}\,(R(f)) = \sum_{\pi \in \hat{G}} m(\pi) \text{tr}\ \pi(f). \tag{4.2.2}$$

Then the *Selberg trace formula* says that the geometric trace (4.2.1) and the spectral

[3] Clearly, the volume and the orbital integral are all defined on conjugacy classes.
[4] If Γ is the trivial group, this is (part of) the Peter-Weyl theorem.

trace (4.2.2) are equal:

$$\sum_{[\gamma]} \text{vol}(\Gamma_\gamma \backslash G_\gamma) I(\gamma, f) = \sum_{\pi \in \hat{G}} m(\pi) \text{tr } \pi(f).$$

In particular, if we set $G = \mathbb{R} = \mathbb{G}_a(\mathbb{R})$ and $\Gamma = \mathbb{Z}$, we recover the Poisson summation formula:

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n), f \in C_c^\infty(\mathbb{R}). \tag{4.2.3}$$

4.2.2 Arthur's trace formula

Let G be an algebraic group defined over $\mathbb{Q}$. We are mainly interested in $L^2(\Gamma \backslash X)$ where Γ is a discrete subgroup of $G(\mathbb{R})$, or $\Gamma \backslash X = G(F) \backslash G(\mathbb{A})$. As a general principle in representation theory, the study of representations is essentially the study of characters, or, the traces of operators. The Selberg-Arthur trace formula describes the character of the representation of $G(\mathbb{A})$ on the discrete part $L_0^2(G(F) \backslash G(\mathbb{A}))$ of $L^2(G(F) \backslash G(\mathbb{A}))$ in terms of geometric data. In most cases of Selberg-Arthur trace formula, the quotient $G(F) \backslash G(\mathbb{A})$ is not compact, which causes the following problems:

(a). *Spectral side:* The representation on $L^2(G(F) \backslash G(\mathbb{A}))$ contains not only discrete components but also continuous components. So we need a good description of the spectrum decomposition.

(b). *Geometric side:* The kernel is no longer integrable over the diagonal, and the operators $R(f)$ are no longer of trace class. We have to modify divergent integrals to make them converge. Note that traces should be interpreted as distributions, not simply functions.

Let f be a smooth function of compact support, now the kernel operator is not of trace class. The divergence comes from parabolic subgroups. Fix a minimal parabolic subgroup P_0. We truncate the kernel function K_f by an alternating sum of functions parameterized by standard parabolic subgroups. The truncation depends on a parameter $T \in i\mathfrak{a}_P^*$ and is defined for sufficiently regular T. After the truncation, K_f^T is "supported in a large compact subset". So we may compute the geometric and spectral side of this kernel function. We, therefore, get a family of equations, depending on T.

But this is not what we really want. We need something intrinsic, that is, independent of our truncation parameter T. It turns out that both sides of the equations are polynomials of T. Therefore, the "constant term" could be an intrinsic formula. This is the trace formula we get:

Theorem 4.2.2 (Arthur's trace formula)**.**

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(f) = \sum_{\chi\in\mathcal{X}} J_{\chi}(f), \qquad f \in C_c^\infty(G(\mathbb{A})) \tag{4.2.4}$$

The left hand side is the geometric side of the trace formula, and is a sum over equivalence classes in the group of rational points $G(F)$ of G, while the right hand side is the spectral side of the trace formula and is a sum over certain representations of subgroups of $G(\mathbb{A})$. For unramified conjugacy classes or automorphic data, the terms are simply weighted orbital integral or weighted characters. The singular terms are much more complicated.

4.2.3 Refinements of the trace formula

The original trace formula is the equality of two expressions of the distribution $J(f)$. As we have seen, J is not invariant under conjugation, so is easy to use in practice. We always assume that we want to study spectral decompositions and therefore only conjugation-invariant objects are interested. So we need a refined version of the trace formula. We need to modify the distribution J to get a conjugation-invariant distribution $I(f)$. And the *stable trace formula* is an equality of two expressions of the new distribution $I(f)$. Another advantage of the stable trace formula is that we have explicit formulas to compute the local contributions over arbitrary conjugacy classes and characters, they are weighted sums of weighted orbital integrals or weighted characters.

Teo conjugacy classes are called stably conjugate if they are conjugate over $G(\overline{F})$. We need to stabilize the trace formula for two reasons.

- *Geometric side: transfer conjugacy classes.* Conjugacy class for GL_N can be expressed in terms of their characteristic polynomials. We can define the transfer of a conjugacy by determining its characteristic polynomial. However, characteristic polynomials only distinguish stable conjugacy.

- *Spectral side: L-packets.* Lnaglands correspondence associates L-packets to L-parameters. The functoriality is only a correspondence of L-packets. Therefore, we need a formula that studies the trace of representations in L-packets. The characters over L-packets are expected to be stable characters.

  Here is an example. Recall that a semisimple Lie group $G(\mathbb{R})$ has a discrete series if and only if it has a compact Cartan subgroup. More generally, a reductive Lie group $G(\mathbb{R})$ has a discrete series if and only if G-has an elliptic torus T_G over $\mathbb{R}$. any strongly regular elements elliptic conjugacy class for $G(\mathbb{R})$ intersects $T_{G,reg}$. Two elements in $T_{G,reg}$ are $G(\mathbb{R})$-conjugate if and only if they lie in the same $W_{\mathbb{R}}$-orbit.

  Let μ be a character of $Z(\mathfrak{g})$, there are exactly $|W_{\mathbb{C}}/W_{\mathbb{R}}|$-discrete series representations with infinitesimal character μ. Their characters can be written as a distribution on $T_{\mathbb{R}}$ (in fact a function on $T_{G,res}$ by Harish-Chardra's theorem). The explicit formula is given in [Sch1], they are sums over $W_{\mathbb{R}}$, so is (only) $G(\mathbb{R})$-invariant. However, the sums of characters in an L-packets of discrete series is a sum over $W_{\mathbb{C}}$-orbits, so is a stable character.

4.3 Automorphic Forms for $GSp(4)$

We first consider representations of $Sp(4,\mathbb{R})$. Let $P(\mathbb{R})$ be the Siegel parabolic subgroup, and $P(\mathbb{R}) = M(\mathbb{R})A(\mathbb{R})N(\mathbb{R})$ the Langlands decomposition of P. Hence we have $M(\mathbb{R}) \cong SL_2^{\pm}(\mathbb{R})$, $A(\mathbb{R}) \cong \mathbb{R}^+$, $N(\mathbb{R}) \cong \mathbb{R}^3$.

Let D_k^+ (resp. D_k^-) denote the holomorphic (resp. anti-holomorphic) discrete series of $SL_2(\mathbb{R})$ which has the minimal K-type $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mapsto e^{i(k+1)\theta}$ (resp. $e^{-i(k+1)\theta}$). The representation $D_k := \mathrm{Ind}_{SL_2(\mathbb{R})}^{SL_2^{\pm}(\mathbb{R})} D_k^+ \cong \mathrm{Ind}_{SL_2(\mathbb{R})}^{SL_2^{\pm}(\mathbb{R})} D_k^-$ is irreducible, and its restriction to $SL_2(\mathbb{R})$ is $D_k^+ \oplus D_k^-$. The quasi-character ν_1 on $\mathbb{R}^+$ is defined by $\nu_1(a) = a$. Let $k \geq 2$ be an integer. We denote by σ_k the Langlands quotient of the induced representation

$$\mathrm{Ind}_{P(\mathbb{R})}^{Sp(4,\mathbb{R})}(D_{2k-1} \otimes \nu_1 \otimes 1).$$

The representations σ_k are unitarizable and cohomological. The infinitesimal character is $(k-1, 2-k)$ and the minimal K-type is $k-1, 1-k$. Let σ_k^- be the cohomological representation of $PGSp(4,\mathbb{R})$ given in [Sch1]. It is of course the same as a representation of $GSp(4,\mathbb{R})$ with trivial central character. The restriction of σ_k^- to $Sp(4,\mathbb{R})$ is exactly σ_k.

Consider the natural maps $i : Sp(4,\mathbb{R}) \to GSp(4,\mathbb{R})$ and $p : GSp(4,\mathbb{R}) \to PGSp(4,\mathbb{R})$. The kernel of the composition $\pi \circ i$ is $\pm I_4$. Now let Γ be a congruence subgroup of $Sp(4,\mathbb{R})$ such that $-1 \in \Gamma$. We are interested in the multiplicity of σ_k inside $\mathcal{A}^2(\Gamma\backslash Sp(4,\mathbb{R}))$. Note that any paramodular subgroup of $Sp(4,\mathbb{R})$ satisfies the above condition. The congruence subgroup Γ can be identified with a congruence subgroup of $PGSp(4,\mathbb{R})$, we also denote it by Γ.

Lemma 4.3.1. *The multiplicity of σ_k in $\mathcal{A}^2(\Gamma\backslash Sp(4,\mathbb{R}))$ is equal to the multiplicity of σ_k^- in $\mathcal{A}^2(\Gamma\backslash PGSp_4(4,\mathbb{R}))$.*

Proof. The two spaces $\Gamma\backslash Sp(4,\mathbb{R})$ and $\Gamma\backslash PGSp(4,\mathbb{R})$ are actually the same. □

4.3.1 The Classification of Automorphic Representations for $GSp(4)$

The multiplicity of σ_k^- in $\mathcal{A}^2(\Gamma\backslash PGSp(4,\mathbb{R}))$ can be calculated in terms of automorphic forms. Let Γ' be the corresponding open subgroup of $GSp(4,\mathbb{A}_f)$. Let π be an automorphic representation of $GSp(4,\mathbb{A})$ and $\pi = \pi_f \otimes \pi_\infty$ be its decomposition into nonarchimedean and archimedean components. Then

$$\dim \mathrm{Hom}(\sigma_k^-, \mathcal{A}^2(\Gamma\backslash PGSp(4,\mathbb{R}))) = \sum_{\pi_\infty = \sigma_k^-} \dim \pi_f^{\Gamma'} \mathrm{mult}(\pi, \mathcal{A}^2(GSp(4,\mathbb{A}))).$$

Arthur [Art] classified the discrete spectrum of the group $GSp(4)$. They come in finite or infinite packets, of which there are six types. The general type consists of those representations that lift to cusp forms on $GL(4)$. The Yoshida type can be characterized as representations whose L-functions are of the form $L(s,\pi_1)L(s,\pi_2)$ with distinct cuspidal automorphic representations on $GL(2)$. At least conjecturally these two types consist of everywhere-tempered representations. Then there are three non-tempered types (Howe-Piatetski-Shapiro type, Saito-Kurakowa type, and Soudry type), associated with the three conjugacy classes of parabolic subgroups. Finally, there is a type consisting of one-dimensional representations. The automorphic representations of each type are parameterized by Arthur parameters, and their multiplicities are given explicitly in terms of Arthur parameters. See [Sch3] for a more detailed description of the six types.

Now we want to study the multiplicity of σ_k^- in $L^2_{\mathrm{disc}}(\Gamma\backslash PGSp_4(\mathbb{R}))$. Except for minimal K-type $(1,-1)$, the σ_k^- can only appear in packets of Saito-Kurokawa (SK) type. They cannot appear in packets of general type or Yoshida type, since they are non-tempered. And they cannot appear in packets of Howe-Piatetski-Shapiro type or Soudry type, since these can be explicitly calculated (See [Sch3]). Packets of SK-type are parametrized by pairs

(μ,σ), where μ is a cuspidal representation of $GL(2,\mathbb{A})$ with trivial central character, and σ is a quadratic Hecke character. All representations of SK-type are obtained via lifting from $GL(2)$ by Arthur's classification.

4.3.2 A Multiplicity Formula

Now let p be a prime and we consider $\Gamma = \Gamma^{\mathrm{para}}(p)$. Let Γ' be the corresponding open subgroup of $GSp(4,\mathbb{A}_f)$, then $\Gamma'_p = K(p)$, the local paramodular subgroup , and $\Gamma'_{p'} = GSp(4,\mathcal{O}_{p'})$ for $p' \neq p$. Therefore, we are looking for (discrete) automorphic representations π with trivial central character, with archimedean component $\pi_\infty = \sigma_k^-$, with p-component π_p such that π_p admits non-zero $K(p)$-invariant vectors, and the other non-archimedean components unramified. The archimedean condition forces π to be of SK-type.

Now we look at Table 2 in [Sch2]. We see that μ_∞ must be a discrete series representation of weight $2k-2$ so that μ corresponds to a cuspidal newform f of this weight. Looking at the possibilities for π_p, we see that it must be of type IIb or Vb or VIc, as the others do not admit $K(p)$-invariant vectors. We also see that σ must be unramified at every place, or there is at least one place where the representation does not have paramodular vectors. (Table A.12 in [RS]) Thus σ is in fact trivial, and μ has trivial central character. It follows that $f \in S_{2k-2}(\Gamma_0(N))$ for some N. In fact, to produce IIb at the place p we must have $N=1$, and to produce Vb or VIc at p we must have $N=p$. The next step is to determine which local new forms lift via Arthur's multiplicity formula. (The determination of the levels is because the local components of a new form can be explicitly computed.)

IIb. Arthur's multiplicity formula says that in order for μ to lift, we must have $(-1)^n = \epsilon(1/2,\mu)$, where n is the number of places where we do not have the base point in the local Arthur packet. Again from Table 2 we see $n=0$, so that we must have $\epsilon(1/2,\mu)=1$. The archimedean contribution to the epsilon-factor is $(-1)^{k-1}$. So we must have $\epsilon(1/2, \mu_p) = 1$ if k is odd, and $\epsilon(1/2, \mu_p) = -1$ if k is even. We see the following: If k is even, then there are no lifts producing IIb (at the place p), and if k is odd, then the number of lifts with IIb is $\dim S_{2k-2}(SL(2,\mathbb{Z}))$.

Vb. Now consider lifts with a Vb component at p. From Table 2, μ_p must be a non-trivial twist of the Steinberg representation. But it must also be an unramified twist, or we won't have $K(p)$-invariant vectors. Thus $\mu_p = \xi \mathrm{St}_{GL(2)}$, where ξ is the non-trivial, quadratic, unramified character of $\mathbb{Q}_p$. The local sign of such μ_p is $+1$. Thus, to satisfy Arthur's multiplicity formula, we must have k odd. Hence, for k odd, exactly the newforms in $S_{2k-2}(\Gamma_0(p))^+$ lift, where "+" indicates that the Atkin-Lehner eigenvalue at p is $+1$. The number of automorphic representations as above with Vb at the place p is therefore $\dim S_{2k-2}(\Gamma_0(p))^{new,+}$ for odd k. For even k there are no such representations.

VIc. Similarly, the number of automorphic representations as above with VIc at the place p is $\dim S_{2k-2}(\Gamma_0(p))^{new,-}$ for even k (while for odd k there are no such representations).

In summary, we have proved the following result.

Proposition 4.3.2. *Let $\Gamma = \Gamma^{\mathrm{para}}(p)$ be a paramodular subgroup of prime level, then the multiplicity of σ_k in $\mathcal{A}^2(\Gamma\backslash Sp(4,\mathbb{R}))$ is:*

- $\dim S_{2k-2}(SL(2,\mathbb{Z})) + \dim S_{2k-2}(\Gamma_0(p))^{new,+}$ *if k is odd;*
- $\dim S_{2k-2}(\Gamma_0(p))^{new,-}$ *if k is even.*

The dimension of elliptic modular forms can be found on `http://www.lmfdb.org/ModularForm/GL2/Q/holomorphic/`. This will be used in the next section.

Chapter 5

Geometric Applications

Finally we turn to Hodge classes and instances of the Hodge conjecture for universal families over the locally symmetric varieties, first in the case of an arithmetic quotient of Siegel upper half space, then for ball quotients of dimensions 2 and 3.

5.1 The Abelian Surfaces over Siegel Domain

5.1.1 Local systems and their cohomology

Let $\mathfrak{H}_2$ be the Siegel upper half space of degree two. The group $Sp(4,\mathbb{R})$ acts naturally on $\mathfrak{H}_2$ via Möbius transformation. The quotient $X(N) := \Gamma^{\text{para}}(N)\backslash\mathfrak{H}_2$ is a moduli space for complex abelian surfaces with polarization type $(1, N)$. More generally, we can take Γ to be any arithmetic subgroup and write $X(\Gamma)$. Let $\pi\colon \mathcal{A} \to X(\Gamma)$ be the universal family. The local system $R^1\pi_*\mathbb{C}$ is exactly the local system $\mathbb{V}_{1,0}$. All other local systems are sub-local systems of the higher direct images of the fiber product of the universal family. There is also a universal family of genus 2 curves $\mathcal{C} \to X(\Gamma)$; let $\bar{\mathcal{C}}$ denote a smooth compactification.

By Prop. 3.4.1, the only real Hodge classes in cohomologies of fiber powers $\mathcal{A}^k$ and $\mathcal{C}^k$ of the total spaces come from

$$H^2(\mathfrak{sp}_4, U(2); U \otimes V_{\mathbb{R}}) \cong \mathbb{R} \cong H^4(\mathfrak{sp}_4, U(2); U \otimes V_{\mathbb{R}})$$

when V is $V_{a,a}$ (for some $a \leq k$ resp. $\frac{k}{2}$) and U is the non-tempered representation σ_{a+3}. Note that when $a = 0$, these are just pullbacks of Hodge classes from $X(N)$, and this is all that occurs for the fourfold $\mathcal{C}$; for $\mathcal{A}$, $\mathbb{V}_{1,1}$ occurs in the relative $\mathcal{H}^2$.

Lemma 5.1.1. *These real Hodge classes are actually rational Hodge classes.*

Proof. Writing $X(N)^*$ for the Baily-Borel compactification, Props. 3.2.1 and 3.4.1 give

$$IH^2(X(N)^*, \mathbb{V}_{a,a}) \cong H^2(\mathfrak{sp}_4, U(2); \sigma_{a+3} \otimes V_{a,a;\mathbb{R}})^{\oplus \mathrm{mult.}(\Gamma^{\mathrm{para}}(N), \sigma_{a+3})}$$

which is pure of type $(a+1, a+1)$ (with the LHS obviously defined over $\mathbb{Q}$). The point is that the other cohomological representations for $\mathbb{V}_{a,a}$ (which are discrete series) only contribute to H^3. The H^4 case follows by duality. □

5.1.2 The universal curve

Let $\mathbb{V}$ be an arbitrary local system. It turns out that the only case for which $H^3(\mathfrak{sp}_4, U(2), U \otimes V) \neq 0$ is when U is a discrete series. However, these cohomology groups do not have real Hodge classes. Therefore, $IH^3(\mathcal{A}_2, \mathbb{V})^{2,2} = 0$ for any local system $\mathbb{V}$. This leads to another proof of Arapura's result in [Ar1].

Theorem 5.1.2 (Arapura)**.** *The Hodge conjecture for the universal curve $\mathcal{C}$ over $X(\Gamma)$ (and thus for $\overline{\mathcal{C}}$) is true.*

Proof. Let $\mathbb{V} = \mathbb{V}_{1,0} = R^1 f_* \mathbb{C}$, then $H^3(U, \mathbb{V}_{1,0})^{2,2} = 0$ since the intersection cohomology surjects onto the weight-4 part of the usual cohomology. (This is exactly the key vanishing theorem in Arapura's proof.) Since $\overline{\mathcal{C}} \backslash \mathcal{C}$ is a 3-fold, we get the HC for $\overline{\mathcal{C}}$ by Prop. 2.5.1(iii). □

5.1.3 Fiber products of the universal curve

Let $\mathcal{C}^n$ be the n-fold fiber power of the universal family. All local systems $\mathbb{V}_{a,a}$ occur in the higher direct images of $\mathcal{C}^n$ for some n. If $\Gamma = \Gamma^{\mathrm{para}}(p)$, we need to find $\dim S_{2a+4}(SL(2,\mathbb{Z})) + \dim S_{2a+4}(\Gamma_0(p))^{new,+}$ if a is even, and $\dim S_{2a+4}(\Gamma_0(p))^{new,-}$ if a is odd.

Theorem 5.1.3. *For $p = 1, 2, 3, 5$, the Hodge conjecture holds for $\mathcal{C}^2$ over $X(p)$.*

Proof. The only interesting higher direct image $\mathcal{H}^2_{\mathcal{C}^2/X(p)}$ has three components: the trivial local system, the adjoint local system $\mathbb{V}_{2,0}$, and the local system $\mathbb{V}_{1,1}$. Now by Prop. 3.4.1, $\mathbb{V}_{1,1}$ is the only possible non-trivial local system in $\mathcal{C}^2$ that could have Hodge classes. (The trivial ones lead to Hodge classes pulled back from the base; since this has dimension 3, those Hodge classes are algebraic.) If $\Gamma = Sp(4,\mathbb{Z})$, then there are no Hodge classes.

For $p = 2, 3, 5$, just note that $\dim S_6(\Gamma_0(p))^{new,-} = 0$ for $p = 2, 3, 5$. □

Remark 5.1.1. By the same calculation, we get a real Hodge class for $\mathcal{C}^2$ over $X(7)$ since $\dim S_6(\Gamma_0(p))^{new,-} = 1$. It would be an interesting problem to determine if this is a rational Hodge class, and, if so, find a cycle representing it.

5.1.4 The universal abelian surface

This also has the feature that the only local system supporting Hodge classes is the copy of $\mathbb{V}_{1,1}$ in $\mathcal{H}^2_{\mathcal{A}/X(p)}$.

Corollary 5.1.4. *The HC holds for any smooth compactification of the universal abelian surface $\mathcal{A}$ over $X(p)$, for $p = 1, 2, 3, 5$.*

5.2 Ball Quotients

5.2.1 Local systems and their cohomology

Let $\mathbb{B}^2 = SU(2,1)/U(2)$ be the two-dimensional ball, the Hermitian symmetric domain associated with the group $SU(2,1)$. Let $X = \Gamma\backslash\mathbb{B}^2$ be an arithmetic quotient, and $\pi : \mathcal{C} \to X$ be the universal family of Picard curves.[1] Denoting the representation $V^{a\lambda_1+b\lambda_2}$ by $V_{a,b}$, the

[1] The maximal choice of Γ would be of the form $SU(2,1;\mathcal{O}_K)$, with $K = \mathbb{Q}(\omega)$, $\omega = e^{2\pi i/3}$. The curves here are of the form $y^3 = P(x)$, with P of degree 4, see for example [GK].

local system $R^1\pi_*\mathbb{C}$ is equal to $\mathbb{V}_{1,0} \oplus \mathbb{V}_{0,1}$. All other local systems are sub-local systems of the higher direct images of the fiber product of the universal family.

The only real Hodge classes are in $H^2(\mathfrak{g}, K; U \otimes V)$ when $V = V_{a,a}$ and $U = U_{nh}$ is the non-holomorphic discrete series.

5.2.2 K3-surfaces

For $0 \leq k \leq 6$, there exist $K3$ surfaces S with an automorphism α_S of order three such that:

$$H^2(S, \mathbb{Q}) = T_S \oplus N_S, \quad N_S := H^2(S, \mathbb{Q})^{\alpha_S} \cong \mathbb{Q}^{8+2k}, \quad H^{2,0} \subset T_S \otimes \mathbb{C}.$$

The action of α_S^* defines a structure of $\mathbb{Q}(\omega)$-vector space on T_S. The moduli space of such $K3$ surfaces is a quotient of the q-ball (where $q = 6 - k$). Let $k = 4$; then we have:

Theorem 5.2.1. *The Hodge conjecture holds for the family of $K3$ surfaces over $\Gamma \backslash \mathbb{B}^2$.*

Proof. We do not get Hodge classes when V is the (half-twisted) standard representation, so $H^2(X, \mathcal{H}^2)$ has no Hodge classes. But the K3 surfaces have no H^1 or H^3. □

Since the universal $K3$ surface is a fourfold, the Hodge conjecture holds for any smooth compactification as well.

5.2.3 Rohde's Calabi-Yau threefolds

J.C. Rohde [Roh] constructed families of Calabi-Yau threefolds with $q = h^{2,1} = 6 - k$, for $0 \leq k \leq 6$, which are parameterized by a q-dimensional ball quotients. The construction is as follows. Let $\omega \in \mathbb{C}$ be a primitive cube root of unity and consider the elliptic curve

$$E = \mathbb{C}/\mathbb{Z} + \omega\mathbb{Z}.$$

Then E has a natural automorphism of order three α_E defined by $z \mapsto \omega z$. The automorphism α_E gives a decomposition of $H^1(E,\mathbb{C})$ into eigenspaces with eigenvalues ω and $\bar{\omega}$:

$$H^1(E,\mathbb{C}) = H^{1,0}(E)_\omega \oplus H^{0,1}(E)_{\bar{\omega}}.$$

Now we take S to be the $K3$ surface in the last subsection (so $k = 4$, $q = 2$). The weight three polarized rational Hodge structure of $\alpha = \alpha_S \otimes \alpha_E$-invariants in the tensor product $H^2(S,\mathbb{Q}) \otimes H^1(E,\mathbb{Q})$ is then of CY-type. Rohde shows that it is isomorphic to the third cohomology group of a CY threefold X_S which is a desingularization of the singular quotient variety $(S \times E)/\alpha$. Clearly, the moduli space of such X_S is the same as the moduli space of S, a quotient of the 2-ball.

The CY-threefold X_S still has an automorphism α_{X_S} of order three which is induced by $\alpha_S \times \alpha_E$. We have

$$H^3(X_S,\mathbb{C}) \cong (T_{S,\bar{\omega}} \otimes H^{1,0}(E)_\omega) \oplus (T_{S,\omega} \otimes H^{0,1}(E)_{\bar{\omega}}),$$

and

$$H^{2,1}(X_S) \cong T^{2,0}_{S,\bar{\omega}} \otimes H^{1,0}(E)_\xi, \qquad \dim H^{2,1}(X_S) = q.$$

Consider the universal family $\mathcal{X}$ over the ball quotient $X = \Gamma\backslash\mathbb{B}_2$. The H^1 of CY-threefolds vanishes automatically. The middle cohomology H^3 has Hodge numbers $(1,2,2,1)$. As Hodge structures, they are just the double half-twists we considered in this paper. From the previous calculations, we know $H^k(X,\mathbb{V}^3)$ has no Hodge classes. Since the fiberwise $\mathcal{H}^{2j}$'s are copies of the trivial local system, any Hodge classes "come from the base" hence are algebraic.

Proposition 5.2.2. *The Hodge Conjecture holds for the (open) total spaces of these families of CY 3-folds.*

5.2.4 Curves over three-dimensional ball quotients

We just consider the universal curve $\mathcal{C}$. The local system is associated with the standard representation. By Prop. 3.3.2, there are no Hodge classes in $H^*(X, \tilde{\mathbb{V}}_{\mathbb{R}}^{\lambda_1}) \otimes \mathbb{C} = H^*(X, \mathbb{V}^{\lambda_1} \oplus \mathbb{V}^{\lambda_3})$, so all Hodge classes come from the base. Since the base has dimension three, these classes are known to be algebraic. This proves

Theorem 5.2.3. *Let $\mathcal{C}$ be the universal curve over an arithmetic quotient of $\mathbb{B}_3$. Then the Hodge conjecture holds for $\mathcal{C}$, hence for any compactification $\overline{\mathcal{C}}$.*

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