Winter 12-15-2022

Essays in Microeconomics

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Essays in Microeconomics

by
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A dissertation presented to
The Graduate School
of Washington University in
partial fulfillment of the
requirements for the degree
of Doctor of Philosophy

August 2022
St. Louis, Missouri

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Acknowledgments

I want to thank my advisor Brian for his generosity and insights that have dramatically improved this work. I am also deeply indebted to SangMok and Paulo for their continuous support during my years as a Ph.D. student. I would also like to thank Jonathan and John for their time advising me and discussing my research with me. This work has also benefited from discussions with other faculty members and peer students at Washington University in St. Louis.

Lintao Ye

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August 2022

ABSTRACT OF THE DISSERTATION

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Professor Brian Rogers, Chair

This dissertation has four chapters. The first chapter studies the testable implications of stable weighted (hedonic) coalitions. The second chapter explores an extension of Bayesian persuasion where the receiver can acquire additional information after receiving information from the sender. The third chapter studies observable implications when a decision-maker endogenously forms consideration sets. The last chapter examines weighted network formation where agents have social status concerns. Omitted proofs in each chapter are presented in the last section of each chapter.

In the first chapter, we study the testable implications of a stable profile of weighted coalitions. We then apply our result to a model of weighted network formation, which subsumes aggregate matchings and the fractional stable roommates' problem.

In the second chapter, we study persuasion in a setting where a sender cares about a receiver's action, and the receiver can acquire additional information after receiving information from the sender. Our main result indicates that the sender has considerable persuasion abilities. For binary actions, the sender always benefits from persuasion when there is a need for persuasion. For multiple actions, we give a sufficient condition for the sender to benefit from persuasion. We argue that this condition is frequently satisfied.

In the third chapter, we model a decision-maker that is unable to consider all of the available alternatives due to costly attention. The decision-maker will optimally choose a subset of given alternatives by maximizing the expected utility of having that set minus the cost of attention required for considering that set. In particular, we provide a representation theorem for random choice rules where subsets of menus, which are interpreted as consideration sets, are formed by maximizing an objective function, and the probability of choosing alternatives outside this set is equal to zero.

In the last chapter, we study environments where individuals allocate resources across relationships with others, creating a weighted, directed network. Value is achieved both through an exogenous factor and maintaining close connections to high-value individuals. We consider two cases corresponding to the direction benefits flow along links. In Model T (for "taking") agents receive benefits through the links they create, whereas in Model G (for "giving") the reverse is true: agents pass value along their links. Equilibrium and socially efficient networks are characterized. In Model G equilibrium networks do not necessarily maximize group welfare, but in Model T efficient networks and equilibrium networks coincide despite extensive network externalities.

Chapter 1

Testable Implications of Weighted Coalition Formation

1.1 Introduction

Coalition formation is important since individuals carry out activities as coalitions in many social and economic situations. An individual often decides which activities to participate in and how much time to spend on them. A weighted coalition profile consists of a collection of activities and their weights. We study the problem of constructing preferences to rationalize a weighted coalition profile. When such preferences exist, we say that the weighted coalition profile is a stable profile of weighted coalitions. Below, we define terms and present our main result.

Let $N = \{1, \cdots, n\}$ be a finite set of players. A coalition $S \in 2^N \setminus \emptyset$ is a non-empty subset of players. For $i \in N$, $N_i = \{S \in 2^N \setminus \emptyset : i \in S\}$ is the set of coalitions i is a member of, $q_i > 0$ is i's capacity constraint.

Definition 1 (A weighted coalition profile). *A weighted coalition profile is a function* $x : 2^N \setminus \emptyset \to \mathbb{R}_+$ *so that* $\Sigma_{S \in N_i} x(S) \leq q_i$ *for every* i.

Every i has preferences over coalitions that she belongs to. In particular, i's preferences are represented by a linear order $\succeq_i$ (a complete, reflexive, anti-symmetric, and transitive binary relation) over N_i.[1] Given a weighted coalition profile x, if there exists S such that for every $i \in S$, either 1) there exists $S' \in N_i$ so that $x(S') > 0$ and $S \succ_i S'$, or 2) $\Sigma_{S'' \in N_i} x(S'') < q_i$, we say that S blocks x.

Definition 2 (A stable profile of weighted coalitions). *A stable profile of weighted coalitions is a weighted coalition profile that is not blocked by any coalition.*

We study the testable implications of a stable profile of weighted coalitions.[2] A weighted coalition profile x is **rationalizable** if there exists $(\hat{q}_i, \hat{\succeq}_i)_i$ with which x is a stable profile of weighted coalitions. Rationalizability depends on the set of positive weighted coalitions but not on the exact values of those positive weights.

Theorem 1. *A weighted coalition profile is rationalizable if and only if, for any subset of players, the number of positive weighted coalitions that contains only those players is less than or equal to the number of players.*

[1]The setup is a weighted version of the hedonic coalition formation model. For references of hedonic coalition formation, please refer to Dreze and Greenberg (1980), Banerjee, Konishi, and Sönmez (2001), Cechlárová, Romero-Medina, et al. (2001), Bogomolnaia and Jackson (2002), and Aziz et al. (2019).

We can generalize the setup so that a coalition consists of a subset of players as well as an action. In this case, for each S, there is a finite set of actions, denoted by A_S, and i's preferences is a linear order over $\cup_{S \in N_i} A_S$.

[2]The existence of a stable profile of weighted coalitions for any $(q_i, \succeq_i)_i$ is guaranteed by Scarf's lemma.

1.2 Illustration: a model of weighted network formation

One of the essential questions in the economics of networks is to understand which networks are stable when self-interested individuals strategically choose which links to form. Jackson and Wolinsky (1996) proposed the pairwise stable solution concept, which treats links as binary quantities and later became popular in the formal modeling of strategic network formation. The importance of considering links with different strengths, however, has been acknowledged since the work of Granovetter (1973), Granovetter (1983). Yet the vast majority of the weighted network formation papers use Nash equilibrium as the solution concept, e.g., Rogers (2008), Bloch and Dutta (2009), and Baumann (2021).

1.2.1 Setup

Let N be the players involved in a network of relationships. A network is a non-negative real-valued $n \times n$ symmetry matrix g, where $g_{ij} = g_{ji} \geq 0$ represent the link strength between i, j. Therefore, feasible networks are $\{g | \Sigma_{j \in N} g_{ij} \leq q_i, \forall i\}$. In friendship networks, q_i can be interpreted as the time that i spends with her friends. Players possess distinctive qualities, captured by $(\succeq_i)_i$, that benefits others through direct links.

The setup is completed by formally describing how players rank feasible networks. For every i, a partial order R_i (with P_i being the corresponding strict order) over feasible $g_i = (g_{i1}, \cdots, g_{in})$ exists and respects $\succeq_i$ in a sense of first-order stochastic dominance. Therefore, $g_i R_i g_i'$ if $\Sigma_{j' \succeq_i j} g_{ij'} \geq \Sigma_{j' \succeq_i j} g'_{ij'}$ for all $j \neq i$, and $g_i P_i g_i'$ if at least one of the inequalities is strict.

The solution concept is pairwise stable and adapted to the present context. Increasing link strength needs mutual consent, while decreasing needs the approval of one player. Like pairwise stability, we allow for deviations by at most two players simultaneously. However, we do not restrict to deviations that involve at most one link. In particular, we allow for deviations in which two players increase their link strength while decreasing their link strengths with other players. We say that i and j **block** g if there exists g' that differs from g only in rows i and j in a way that $g'_{ij} > g_{ij}$, $g'_{ik} \leq g_{ik}, g'_{jk} \leq g_{jk}$ for all $k \neq i, j$, $g'_i P_i g_i$, and $g_j P_j g'_j$ is not true.

Definition 3. *A network is stable if it is not blocked by any pair of players.*

1.2.2 Results

The following result is an application of Scarf's lemma.

Proposition 1. *A stable network exists.*

Our network setup generalizes some well-studied models in the literature. If $q_i = 1$ for all i and players compare neighbor profiles if and only if one dominates another, the model reduces to the fractional stable roommates' problem.[3] The generalization, however, is essential for our network application. First, if $q = (q_1, \cdots, q_n)$ is a unit vector, stable network structures are limited.[4] In this sense, it is crucial to have heterogeneous capacities to observe a broad class of network structures. In addition, while q being a unit vector is reasonable for the roommates' problem, it is not the case for friendship or co-authorship networks. Second, if players can only compare neighbor profiles for which one dominates the other, then preferences are sparse. Partial orders $(R_i)_i$ complete the preferences while maintaining the flexibility that players maybe unable to compare some neighbor profiles. If players are separated into two groups, then the model reduces to the aggregate matching model studied by Echenique, Lee, et al. (2013).

[3] It is known that a fractional stable roommates exists, see Irving (1985), Tan (1991) and Teo and Sethuraman (1998).

[4] A player either forms a pair with another player or is in a cycle, where link strengths all equal half.

We then move on to characterize the testable implications of this model. We say a network is rationalizable if $(\hat{q}_i, \hat{\succeq}_i, \hat{R}_i)_i$ exists so that the network is stable. Since rationalizability does not depend on the exact values of positive links, we focus on the associated unweighted graphs.

Some definitions from graph theory are useful. An undirected and unweighted graph $G = (V, E)$ is a tuple consisting of a finite set V of vertices and a finite set E of edges where each edge is an unordered pair of vertices. A path $\{v_0, \cdots, v_n\}$ in G connecting v_0 and v_n are such that $\{v_i, v_{i+1}\} \in E$ for all $i \in \{0, \cdots, n-1\}$, and $v_i \in V$ are distinct. A cycle is a path $(v_0, \cdots, v_n)$ for which $v_0 = v_n$. A subgraph of a graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. A graph G is connected if there is a path between every pair of vertices in G. A component of G is a subgraph that is connected and is not contained in any other connected subgraph of G.

Proposition 2. *A network is rationalizable if and only if its associated unweighted graph has at most one cycle in each component.*

Proof. Proposition 2 follows from Theorem 1 and the following fact about trees. If G has n vertices, then any two of the following three statements imply the third and characterize a tree: 1) G is connected, 2) G contains no cycle, 3) G has $n-1$ edges. $\square$

Proposition 2 subsumes Theorem 1 of Echenique, Lee, et al. (2013). The two results have some differences too. To illustrate the first difference, we introduce **line graphs**. A line graph of $G = (V, E)$ is a graph $G' = (V', E')$ where $V' = E$, and for any distinct $e_1, e_2 \in E$, $\{e_1, e_2\} \in E'$ if and only if $e_1 \cap e_2 \neq \emptyset$. Echenique, Lee, et al. (2013) stated their result in terms of the line graphs of the matchings.[5] Second, they study two-sided matchings, so the underlying graph is bipartite, and the cycles must have an even number of nodes.

The model allows for heterogeneous preferences while excluding complementarities. Therefore, Proposition 2 is a testable implication of pure substitutable neighbors. Below, we also characterize networks that are rationalizable with homogenous preferences. We say players have homogenous preferences when for any distinct i, i', j, j', $j \succ_i j'$ if and only if $j \succ_{i'} j'$. A forest is a graph without any cycle.

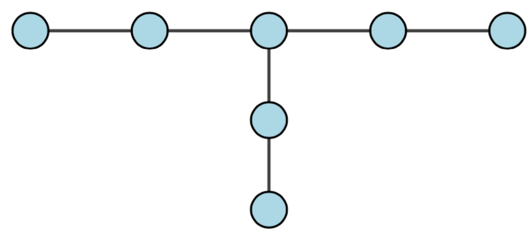

Figure 1.1: The network structure that is not rationalizable with homogenous preferences.

Proposition 3. *A network is rationalizable with homogenous preferences if and only if its associated unweighted graph is a forest without the structure in Figure 1.1 as a subgraph.*

1.3 Omitted proofs

1.3.1 Theorem 1

[5]Line graphs are usually much bigger than original graphs and have more cycles. Therefore, they have to define a concept that they call minimal cycle to illustrate the result. A path $(v_0, \cdots, v_n)$ is minimal if there is no proper subsequence if it that also connects v_0 and v_n. A cycle $(v_0, \cdots, v_n)$ is a **minimal cycle** if for any two vertices v_n and $v_{n'}$, the paths from v_n to $v_{n'}$ and from $v_{n'}$ to v_n are distinct and minimal.

Necessity

If there is a subset of players among which the number of positive weighted coalitions is more than the number of players. We look at these players and restrict their preferences to these positive weighted coalitions. Since the number of players is less than the number of coalitions, at least one coalition is not the worst for all its members. This coalition blocks the weighted coalition profile.

Sufficiency

For every i, we let $\hat{q}_i$ equal the sum of coalition weights i is a member of, and construct $(\hat{\succeq}_i)_i$ so that the weighted coalition profile is stable with respect to $(\hat{q}_i, \hat{\succeq}_i)_i$.

Step 1: If there is a player who appears in only one coalition, we let all other members of this coalition rank this coalition as the best. Excluding this player and this coalition, the number of players who appear in the remaining coalitions is larger than or equal to the number of remaining coalitions.

Step 2: Otherwise, we look for $m \in \{2, \cdots, n\}$ players who appear in the same m coalitions, and for any $1 \leq k < m$ players, they appear in at least $k+1$ coalitions. Pick a player and name a coalition it is a member of as the worst for this player. We eliminate this coalition and look at the remaining $m-1$ coalitions. Suppose a player appears in only one coalition; we name that coalition as the worst for that player.[6] Otherwise, we pick a player and name a coalition it is a member as the worst for this player. We eliminate another coalition and look at the remaining $m-2$ coalitions. Continue this process until we run out of all m players and coalitions. In this way, for each coalition, one player ranks it as the worst. For other

[6]It can not be the case that two or more players appear only in the same coalition as these players would appear in the same two coalitions. The argument holds when more coalitions are eliminated.

players who are also members of these m coalitions, we let them rank these m coalitions from the best in their preferences.

Step 3: Excluding those m players and coalitions, the number of players who appear in the remaining coalitions is larger than or equal to the number of remaining coalitions. Now we look at the remaining coalitions and players. If there is a player who appears in only one coalition, we let all other members of this coalition rank this coalition as the best of the remaining coalitions. Otherwise, we go to Step 2.

The algorithm guarantees that the number of players who appear in the remaining coalitions is larger than or equal to the number of remaining coalitions so that we do not run out of players eventually, and that for every coalition, there is a player who ranks it as the worst.

1.3.2 Proposition 1

We first state Scarf's lemma. Let A be an $n \times m$ nonnegative matrix with at least one nonzero entry in each row and each column. Remember $q = (q_1, \cdots, q_n)$ is a vector in $\mathbb{R}^n_+$. Associated with each row i of A is a linear order $\succeq_i$ over the set of columns j for which $A_{ij} > 0$. A vector $x \in \mathbb{R}^m_+$ satisfying $Ax \leq q$ dominates column j if there exists a row i such that $\Sigma_{j=1}^m A_{ij} x_j = q_i$ and $k \succeq_i j$ for all $k \in \{1, \cdots, m\}$ such that $A_{ik} > 0$, $x_k > 0$.

Lemma 1. *Scarf (1967): There exists a maximal vertex of $\{x \in \mathbb{R}^m_+ : Ax \leq q\}$ that dominates every column of A.*

To interpret Scarf's lemma, we consider A as a $0-1$ matrix, each row as a player, each column as a coalition, and $A_{ij} = 1$ if and only if row i is a member of coalition j. Therefore, a vector x satisfying $Ax \leq q$ is a weighted coalition profile. Scarf's lemma guarantees the existence of a stable profile of weighted coalitions by showing the presence of a dominating vertex in which, for any coalition, at least one member opposes increasing the weight for that coalition.

Now, let each column correspond to a two-player coalition. By Scarf's lemma, a dominating maximal vertex x^* exists. Increasing any coordinate of x^* alone is not feasible. If a pair blocks x^* and deviates to x. Let j be the column of this pair, by definition of a blocking pair, $x_j > x_j^*$, and $x_{j'} \leq x_{j'}^*$ for any other j'. Since x^* is a dominating vertex, one member of j, say i, satisfies that $\Sigma_{j=1}^m A_{ij} x^* = q_i$ and $j' \succeq_i j$ for all $j' \in \{1, \cdots, m\}$ such that $A_{ij'} > 0$ and $x_{j'}^* > 0$. Therefore, j' exists such that $j' \succ_i j$ and $x_{j'} < x_{j'}^*$. Consequently, $x^* P_i x$ since i's neighbor profile in x^* dominates its neighbor profile in x with respect to $\succeq_i$.

1.3.3 Proposition 3

A tree is a connected forest. A rooted tree is a tree in which one node, the root, is distinguished from the others. The root of a tree is usually drawn at the top. If a rooted tree is regarded as a directed graph in which each edge is directed from top to bottom, then every vertex v other than the root is connected by an edge from some other vertex v', called the parent of v. We also call v a child of vertex v'. A leaf is a vertex that has no children. An internal vertex is a vertex that has one or more children.[7]

Lemma 2. *A network is rationalizable with homogenous preferences if and only if its associated unweighted graph is a forest of rooted trees in which any vertex has at most one internal child.*

The intuition behind Lemma 2 is straightforward. When we construct homogenous preferences, one player will be the most preferred. Let that player be the root, then a player closer to it ranks higher than a farther player. In Figure 1.2, the two vertices connecting directly to the

[7] This definition is from Rahman et al. (2017).

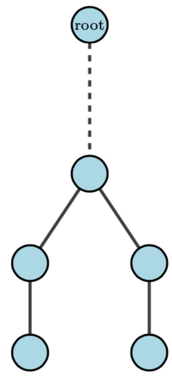

Figure 1.2: The type of rooted trees that is not rationalizable.

two leaves form a blocking pair. If a tree has the structure in Figure 1.1 as a subgraph, the situation in Figure 1.2 arises despite the root.

Chapter 2

Beneficial Persuasion

2.1 Introduction

In many situations, one person wishes to persuade another to take certain actions by providing information, while the other person also has other information sources available. The central question is, in such settings, to what extent can the former influence the latter's action by providing information. To study this question, we extend the Bayesian persuasion model of Kamenica and Gentzkow (2011) to assume that: 1) after receiving free information from the sender, the receiver can acquire additional information about states with a cost proportional to expected entropy reduction, and 2) the sender cares only about the receiver's action. The sender benefits from persuasion if sending information to the receiver can increase the receiver's probability of taking the sender's more preferred actions compared to not sending any information.

We first analyze the receiver's optimal information acquisition problem, and identify two facts about the receiver's optimal information acquisition that help derive our main result.[8] First, it is never optimal for the receiver to entirely rule out any initially possible state. Second, suppose the receiver chooses to acquire information at the prior belief and with some probability updates to a new belief. Then the receiver will not choose to acquire information if she initially has the new belief.

The sender has considerable persuasion abilities. For binary actions, the sender always benefits from persuasion when there is a need for persuasion.[9] Building on the two facts mentioned above, we show that the sender can design a Bayesian-consistent belief distribution for the receiver to strictly increase the receiver's probability of taking the sender's preferred action. For multiple actions, we consider the case where the receiver chooses to acquire information at the prior belief. A sufficient condition is given for the sender to benefit from persuasion. We argue that the sufficient condition is often satisfied. Roughly speaking, the argument is based on the fact that the receiver will not choose to obtain extreme beliefs.

We then discuss the model and the results. We first generalize the entropy-based cost function to a subclass of uniformly posterior-separable cost functions in a way that the two facts about the receiver's optimal information acquisition are still true.[10] Next, we argue that the tie-breaking rule, essential in Bayesian persuasion, is not nearly as important in this setting. Next, we discuss the case where the sender's preference is state-dependent. Finally, we discuss how the receiver's ability to acquire information changes the sender's persuasion behavior.

[8] The receiver's problem is well-studied in the literature. Examples include Matějka and McKay (2015), Caplin and Dean (2013), and Caplin, Dean, and Leahy (2019).

[9] There is a need for persuasion if the receiver does not choose the sender's preferred action with probability one at the prior belief.

[10] The definition of uniformly posterior-separable cost functions can be found in Caplin, Dean, and Leahy (2017).

2.1.1 Related literature

The paper closely relates to two strands of literature, the information design literature and the rational inattention literature.

Bloedel and Segal (2018) and Lipnowski, Mathevet, and Wei (2018) both study Bayesian persuasion with rationally inattentive receiver. They assume the receiver is rationally inattentive to the sender's information. When the sender's and the receiver's preferences are perfectly aligned, Lipnowski, Mathevet, and Wei (2018) show that full disclosure is optimal when the state set is binary. When there are multiple states, it is optimal for the sender to restrict some information to lead the receiver's attention to other aspects. Gentzkow and Kamenica (2014) show that the concavification approach extends to the setting where it is costly for the sender to send information if the sender's cost is proportional to the expected reduction of uncertainty.

Caplin and Dean (2013), Caplin, Dean, and Leahy (2017), and Caplin, Dean, and Leahy (2019) develop a posterior-based approach to solve the rational inattention model. Their approach provides a nice geometrical exposition of the receiver's optimal information acquisition.

The most closely related paper is Matyskova (2018).[11] In a similar framework, Matyskova (2018) shows that if the sender benefits from persuasion, we can find a sender-optimal persuasion strategy in the belief region where the receiver chooses not to learn. She then further shows that we can find a sender-optimal information strategy in the set consisting of extreme points of these non-learning regions. In contrast, we study the sender's persuasion abilities. We believe this is an important question because the whole point of persuasion is to benefit. Our main task is to compare the sender's well-being with and without persuasion.

[11] Matyskova (2018) is to be merged with another closely related paper by Alfonso Montes.

2.2 The model

2.2.1 Setup

There is a finite state set Ω with a typical element denoted by ω. We denote by $\Delta(\Omega)$ the set of all beliefs, $int(\Delta(\Omega))$ the interior of $\Delta(\Omega)$, and $\Delta(\Delta(\Omega))$ the set of all finite-support belief distributions. A typical belief is denoted by μ. The sender and the receiver share a prior belief $\mu_0 \in int(\Delta(\Omega))$.

First, the sender sends the receiver information about states. The sender's information is in the form of a persuasion environment $\tau \in \mathcal{T}_{\mu_0}$, where $\mathcal{T}_{\mu_0} \equiv \{\tau' \in \Delta(\Delta(\Omega)) | \Sigma_{\mu \in Supp(\tau')} \tau'(\mu)\mu = \mu_0\}$. The sender with probability $\tau(\mu)$ induces the receiver's belief to some $\mu \in Supp(\tau)$. There is no persuasion if $Supp(\tau) = \{\mu_0\}$.

Given any induced $\mu \in Supp(\tau)$, the receiver then acquires additional information by designing a costly signaling structure $\phi_u \in \Phi(\mu)$, where $\Phi(\mu) \equiv \{\phi' \in \Delta(\Delta(\Omega)) | \Sigma_{\mu' \in Supp(\phi')} \phi'(\mu')\mu' = \mu\}$. The receiver with probability $\phi_u(\mu')$ updates her belief to some $\mu' \in Supp(\phi_u)$.

Given any $\mu' \in Supp(\phi_u)$, the receiver finally chooses an action a from a finite action set A.

The sender cares about the receiver's action, while the receiver cares about the states. We denote by $\succ$ the sender's strict preference order over A, and $\succeq$ the corresponding weak order. The receiver's payoff from a in state ω is denoted by $u(a, \omega)$. We assume for any $a \in A$, there exists $\mu \in \Delta(\Omega)$ such that $\{a\} = \operatorname{argmax}_{a' \in A} \mathbb{E}_\mu[u(a', \omega)]$. The receiver has an entropy-based cost function for signaling structures:

$$c(\phi_u, \mu) = \lambda(-\Sigma_{\omega \in \Omega} \mu(\omega) \ln \mu(\omega) - (\Sigma_{\mu' \in Supp(\phi_u)} \phi_u(\mu')(-\Sigma_{\omega \in \Omega} \mu'(\omega) \ln \mu'(\omega)))),$$

where $\lambda \in (0, \infty)$ is the per unit cost of expected entropy reduction. The receiver's net payoff equals the payoffs from actions minus the cost for the signaling structure.

2.2.2 Beneficial persuasion

We begin our analysis by formalizing the receiver's problem. Given any μ, the receiver's problem is

$$\max_{\phi_u \in \Phi(\mu)} \Sigma_{\mu' \in Supp(\phi_u)} \phi_u(\mu') \max_{a \in A} \mathbb{E}_{\mu'}[u(a, \omega)] - c(\phi_u, \mu).$$

We assume the vectors $\{e^{\frac{u(a,\omega)}{\lambda}} \in \mathbb{R}^{\Omega} | a \in A\}$ are affinely independent. It is well-known that this condition guarantees a unique solution to the receiver's optimal information acquisition.[12] We denote by ϕ^*_μ the unique solution to the receiver's problem at μ, and call it an optimal signaling structure. The receiver chooses not to acquire information at μ if $Supp(\phi^*_\mu) = \{\mu\}$.

We use the following two lemmas to prove our main result.

Lemma 3. *If $\mu \in int(\Delta(\Omega))$, then $Supp(\phi^*_\mu) \subset int(\Delta(\Omega))$.*

The receiver never entirely rules out any ex ante possible state. This is because the marginal cost to entirely rule out any ex ante possible state is infinite.

Lemma 4. *If $\mu' \in Supp(\phi^*_\mu)$ for some μ, then $Supp(\phi^*_{\mu'}) = \{\mu'\}$.*

If a belief is in the support of some other belief's optimal signaling structure, then the receiver chooses not to acquire information given this belief.[13] This is because the entropy-based cost function is uniformly posterior-separable. Therefore, the receiver's problem can be reformed as a concavification problem.

[12] Please refer to Matějka and McKay (2015) online appendix Lemma 2, or Caplin and Dean (2013) Theorem 2 for a proof. This condition is sufficient but not necessary.

[13] Matyskova (2018) refers to this fact as the receiver engages in only one round of learning.

To this end, we summarize the receiver's response at μ as a choice distribution $P_\mu \in \Delta(A)$, where

$$P_\mu(a) = \begin{cases} \phi^*_\mu(\mu') & \text{if } \exists \mu' \in Supp(\phi^*_\mu) \text{ such that } \{a\} = \text{argmax}_{a' \in A} \mathbb{E}_{\mu'}[u(a', \omega)] \\ 0 & \text{otherwise} \end{cases}.^{14}$$

Definition 4 (**Beneficial persuasion**). *The sender benefits from persuasion if there exists $\tau \in \mathcal{T}_{\mu_0}$ such that for all $a \in A$,*

$$\Sigma_{\mu \in Supp(\tau)} \tau(\mu) \Sigma_{a' \succeq a} P_\mu(a') \geq \Sigma_{a' \succeq a} P_{\mu_0}(a'),$$

and for some $a \in A$, the inequality is strict.

If the sender benefits from persuasion, the sender strictly benefits from persuasion for any payoff function that respects $\succ$. This definition can be helpful when it is easy for the sender to rank the receiver's actions but challenging to assign actual numbers to them.

2.3 Result

2.3.1 Binary actions

In this subsection, we assume $A = \{a_1, a_2\}$, and $a_1 \succ a_2$. There is no need for persuasion if the receiver takes a_1 with probability one at μ_0. The following proposition states that excluding the above-mentioned uninteresting case, the sender always benefits from persuasion.

[14] This is well-defined because $\text{argmax}_{a' \in A} \mathbb{E}_{\mu'}[u(a', \omega)]$ is singleton for any μ' in $Supp(\phi^*_\mu)$ and different beliefs in $Supp(\phi^*_\mu)$ lead to different actions. The first fact is shown in Subsection 2.5.4. The second is because strictly more informative signaling structures are strictly more costly than less informative signaling structures.

Proposition 4. *The sender benefits from persuasion when there is a need for persuasion.*

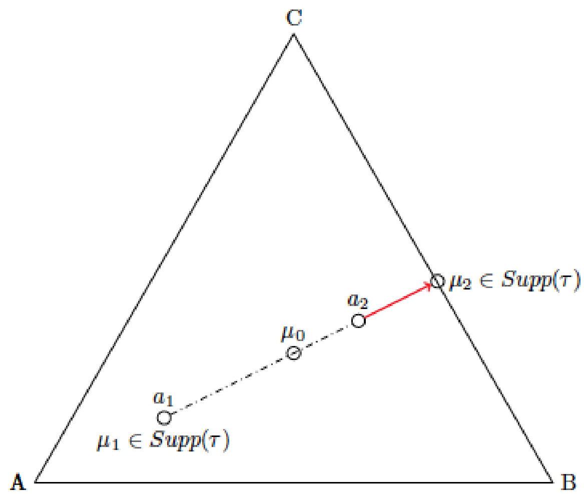

Figure 2.1: An illustration of the intuition behind Proposition 4.

We discuss the intuition of Proposition 4 when the receiver chooses to acquire information at μ_0. By Lemma 3, $\mu \in int(\Delta(\Omega))$ for any $\mu \in Supp(\phi^*_{\mu_0})$. Let $\mu_1 \in Supp(\phi^*_{\mu_0})$ lead to a_1, meaning $P_{\mu_0}(a_1) = \phi^*_{\mu_0}(\mu_1)$. Now we push the belief in $Supp(\phi^*_{\mu_0})$ that leads to a_2 away from μ_0 in the direction of $\overline{\mu_1\mu_0}$ until it hits the boundary of the probability simple. We denote this boundary belief as μ_2. Let the sender design a persuasion environment τ with $Supp(\tau) = \{\mu_1, \mu_2\}$. It is easy to see that $\tau(\mu_1) > \phi^*_{\mu_0}(\mu_1)$. By Lemma 4, if μ_1 is induced, the receiver will not choose to acquire information but instead just take a_1.

Proposition 4 combines the interior-posterior property of rational inattention with the perfect-bad-news logic from Bayesian persuasion.[15] The result also depends on the fact that the entropy-based cost function is uniformly posterior-separable because this implies Lemma 4.

[15]Notice that by pushing the belief away from μ_0 to the boundary of the probability simplex, the receiver's expected payoff from a_2 increases.

An immediate implication of Proposition 4 is that the sender prevents the receiver from acquiring information. If the receiver chooses to acquire information at some $\mu \in Supp(\tau)$, the sender can further decompose μ, with other beliefs in $Supp(\tau)$ unchanged, and increase the receiver's probability of taking the sender's preferred action. The sender's information benefits the receiver despite the fact that the sender uses the information to manipulate the receiver.

2.3.2 Multiple actions

In this subsection, we assume that the receiver chooses to acquire information at μ_0. We need some notations before we proceed. We denote by

$$M^a = \{\mu \in \Delta(\Omega) | P_\mu(a) = 1\}$$

the set of beliefs at which the receiver chooses a with probability one, and

$$B(\mu) = \{a | P_\mu(a) > 0, a \in A\}$$

the set of actions that the receiver will choose with positive probability given μ.[16]

Proposition 5. *The sender benefits from persuasion if $Supp(\phi^*_{\mu_0}) \subset int(conv(\cup_{a \in B(\mu_0)} M^a))$.*

We discuss the intuition of Proposition 5. Let $\mu_1 \in Supp(\phi^*_{\mu_0})$ lead to the sender's worst action in $B(\mu_0)$. We push μ_1 away from μ_0 in the direction of $\overline{\mu_0\mu_1}$ to the boundary of $conv(\cup_{a \in B(\mu_0)} M^a)$, and denote it as μ_2. Notice that we can write μ_0 as a convex combination of μ_2 and all other beliefs in $Supp(\phi^*_{\mu_0})$, and the weights of the beliefs in $Supp(\phi^*_{\mu_0})$ increase proportionally compared to their weights in $\phi^*_{\mu_0}$. Finally, we write μ_2 as a convex combination

[16] We also characterize M^a. For further references, please refer to Section 4 of Caplin, Dean, and Leahy (2019).

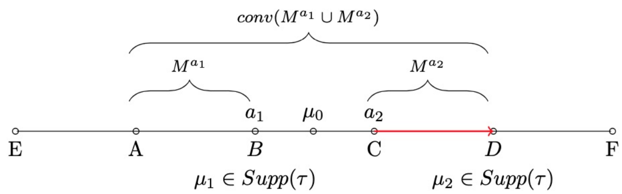

Figure 2.2: An illustration of the intuition behind Proposition 5.

of beliefs in $\cup_{a \in B(\mu_0)} M^a$ if $\mu_2 \notin \cup_{a \in B(\mu_0)} M^a$. By designing such a persuasion environment, the receiver's probability of choosing the sender's worst action in $B(\mu_0)$ is strictly decreased, while the receiver's probabilities of choosing other actions in $B(\mu_0)$ are all strictly increased.

The sufficient condition in Proposition 5 is often satisfied.[17] We let

$$M^{B(\mu_0)} = \{\mu \in \Delta(\Omega) | P_\mu(a) > 0 \text{ if and only if } a \in B(\mu_0)\}$$

be the set of beliefs at which the receiver chooses acquire information and subsequently takes one of the actions from $B(\mu_0)$. We know M^a and $M^{B(\mu_0)}$ are (convex) polytopes in the probability simplex.[18] The polytope of $M^{B(\mu_0)}$ is surrounded by the polytopes of $M^a, a \in B(\mu_0)$. The receiver's first-order condition from optimal information acquisition implies that beliefs in $Supp(\phi^*_{\mu_0})$ lie exactly in the intersection of hyperplanes that separate the surrounding regions of $M^a, a \in B(\mu_0)$ from the middle region of $M^{B(\mu_0)}$.[19] Therefore, it is easily true that beliefs in $Supp(\phi^*_{\mu_0})$ are interior points of $conv(\cup_{a \in B(\mu_0)} M^a)$.

If the state set is binary, then the sufficient condition is satisfied.

Corollary 1. *The sender benefits from persuasion if* $|\Omega| = 2$.

[17]We conjecture that the sufficient condition holds generically.

[18]In Subsection 2.5.4, we show that $M^a = \cap_{a' \in A} \{\mu \in \Delta(\Omega) | \Sigma_{\omega \in \Omega} \mu(\omega) e^{\frac{u(a',\omega) - u(a,\omega)}{\lambda}} \leq 1\}$. Notice that $\Sigma_{\omega \in \Omega} \mu(\omega) e^{\frac{u(a',\omega) - u(a,\omega)}{\lambda}} \leq 1$ is a half-plane in R^Ω.

[19]Caplin and Dean (2013), Caplin, Dean, and Leahy (2019) refer to this as the invariant likelihood ratio condition.

Proof. By assuming affine independence of the vectors $\{e^{\frac{u(a,\omega)}{\lambda}} \in \mathbb{R}^{\Omega} | a \in A\}$, we exclude non-uniqueness of the receiver's optimal information acquisition problem. Therefore, M^a are proper intervals. □

2.4 Discussion

2.4.1 Uniformly posterior-separable cost functions

In this subsection, we generalize the entropy-based cost function to a subclass of uniformly posterior-separable cost functions in a way that the receiver's optimal signaling structures still satisfy Lemma 3 and 4.

The receiver's cost function is uniformly posterior-separable if for any $\mu \in \Delta(\Omega)$ and any $\phi_u \in \Phi(\mu)$:

$$c(\phi_u, \mu) = f(\mu) - \Sigma_{\mu' \in Supp(\phi_u)} \phi_u(\mu') f(\mu'),$$

where $f : \Delta(\Omega) \to \mathbb{R}$ is strictly concave and C^1 on $int(\Delta(\Omega))$. In addition, we assume that for any $\mu \in \Delta(\Omega) \setminus int(\Delta(\Omega))$ and any $\mu' \in int(\Delta(\Omega))$:

$$\lim_{d(\mu,\mu') \to 0} \frac{f(\mu') - f(\mu)}{d(\mu, \mu')} \to \infty,$$

where d is the Euclidean distance. That is, from any direction to a boundary belief, the marginally decreased uncertainty measured by f is unbounded. The intuition is that it can be very costly for the receiver to entirely rule out any ex ante possible state.[20] Lemma 3 follows directly from this assumption because the benefit of a more informative signaling structure comes from two actions' payoff differences in two states, which is bounded.

[20] Mackowiak, Matejka, and Wiederholt (2018) argue that many micro-founded cost functions would share this property.

We follow Caplin, Dean, and Leahy (2017) to denote by $N^a(\mu') = \mathbb{E}_{\mu'}[u(a,\mu')] + f(\mu')$ the receiver's net payoff from a at μ'. The receiver's maximization problem can be rewritten as

$$\max_{\phi_u \in \Phi(\mu)} \Sigma_{\mu' \in Supp(\phi_u)} \phi_u(\mu') \max_{a \in A} N^a(\mu'),$$ [21]

meaning the receiver finds a signaling structure to maximize weighted net payoffs. Lemma 4 holds because if a belief is in the support of some other belief's optimal signaling structure, then the receiver's payoff at this belief is already on the concavified function.[22]

2.4.2 Tie-breaking rule? Not necessary

The Bayesian persuasion literature uses the solution concept of the sender-preferred subgame perfect equilibrium. Specifically, a tie-breaking rule requires the receiver to take the sender's preferred action when being indifferent.[23] The sender will design a belief that makes the receiver indifferent. This further implies that the sender can not make even the smallest mistake in designing beliefs, as this can lead to a complete failure of persuasion.

This tie-breaking rule plays a minimal role if the receiver can acquire information. This, of course, partly depends on the assumption that the vectors $\{e^{\frac{u(a,\omega)}{\lambda}} \in \mathbb{R}^{\Omega} | a \in A\}$ are affinely independent. In fact, this condition is not necessary.[24] When $|A| - |\Omega| \geq 2$, this condition will not satisfy. However, even in this case, it is still almost always true that the receiver's optimal signaling structure is unique.[25]

[21] We drop $f(\mu)$ since it does not affect the receiver's maximization problem.

[22] Caplin, Dean, and Leahy (2017) also define posterior-separable cost functions, which is a superclass of uniformly posterior-separable cost functions. For posterior-separable cost functions, Lemma 4 does not hold.

[23] Without this tie-breaking rule, an optimal persuasion environment is not well-defined.

[24] This condition does not use the fact that the receiver only picks actions with "good" payoffs to concavify.

[25] To see this, let $|A| = 3, |\Omega| = 2$, and the receiver's optimal signaling structure be unique. If we add another action, then the payoffs from this action have to be tangent to the concavified function for the receiver's optimal information acquisition problem to become non-unique.

Furthermore, the receiver's choice probabilities will be a continuous function of her beliefs.[26] Therefore, the consequence will be small if the sender makes a small mistake in designing beliefs. The sender still significantly benefits from persuasion as long as the mistake is small.

2.4.3 The sender's preference is state-dependent

This subsection shows that our main result does not generalize to the case where the sender's preference is state-dependent. We denote by $v(a, \omega)$ the sender's payoff from a at state ω.

Let $A = \Omega = \{-1, 1\}$, and $v(a, \omega) = -u(a, \omega) = a\omega$. It is easy to verify that the sender's expected payoff as a function of the receiver's belief is concave. Therefore, the sender can not benefit from persuasion for any prior belief.

The sender's persuasion abilities depend on how aligned are the sender's and the receiver's preferences. If their preferences are perfectly aligned, the sender will disclose all the information to the receiver. If their preferences are precisely the opposite, the sender will not send any information. Our main result shows that for the case where the sender cares only about the receiver's action, the sender has considerable persuasion abilities.

2.4.4 The change of the sender's persuasion behavior

The receiver's ability to acquire information changes the sender's persuasion behavior. We follow Caplin, Dean, and Leahy (2019) to refer to $B(\mu)$ as the receiver's consideration set at μ, and use $\mathcal{B} = \{B(\mu) | \mu \in \Delta(\Omega)\}$ to denote the receiver's all possible consideration sets. Lemma 4 implies that if $a \in B(\mu)$ for some μ, then $\{a\} \in \mathcal{B}$. Therefore, $\{a\} \in \mathcal{B}$ is a necessary condition for a to be feasible for persuasion.

Lemma 5. *Let* $\mathcal{C} = \{a | \{a\} \in \mathcal{B}\}$. *Then* $\mathcal{C}(\lambda_1) \subseteq \mathcal{C}(\lambda_2)$ *if* $\lambda_1 < \lambda_2$.

[26] We can prove this result by applying Berge's maximum theorem to Lemma 2 of Matějka and McKay (2015).

The set of actions that the receiver uses to concavify shrinks if the receiver's information cost decreases. When the receiver can not acquire information, the sender only needs to design a belief at which the desired action gives the receiver the highest expected payoff. When the receiver can acquire information, the sender needs to design a belief at which not only the desired action gives the receiver the highest expected payoff, but the receiver also chooses not to acquire information and subsequently takes other actions. This shows that the message of designing beliefs that make the receiver indifferent may be misleading. If someone is indifferent between two actions, this person is likely going to acquire additional information, which is the case for the entropy-based cost function.

2.5 Omitted proofs

2.5.1 Lemma 3

Let $\mu' \in Supp(\phi^*_\mu)$ be associated with a. That is, should μ' be induced, the receiver chooses a. It is well-known that conditionally on any ω, the receiver's probability of choosing a is positive. Since $\mu(\omega) > 0$ for any ω, $\mu'(\omega) > 0$ for any ω.

2.5.2 Lemma 4

Let $\mu' \in Supp(\phi^*_\mu)$, and $Supp(\phi^*_{\mu'}) \neq \{\mu'\}$. Then the receiver can construct ϕ'_μ at μ with $Supp(\phi'_\mu) = (Supp(\phi^*_\mu) \setminus \{\mu'\}) \cup Supp(\phi^*_{\mu'})$, and

$$\phi'_\mu(\mu'') = \begin{cases} \phi^*_\mu(\mu'') & \text{if } \mu'' \in (Supp(\phi^*_\mu) \setminus \{\mu'\}) \setminus Supp(\phi^*_{\mu'}) \\ \phi^*_\mu(\mu')\phi^*_{\mu'}(\mu'') & \text{if } \mu'' \in Supp(\phi^*_{\mu'}) \setminus (Supp(\phi^*_\mu) \setminus \{\mu'\}) \\ \phi^*_\mu(\mu'') + \phi^*_\mu(\mu')\phi^*_{\mu'}(\mu'') & \text{if } \mu'' \in (Supp(\phi^*_\mu) \setminus \{\mu'\}) \cap Supp(\phi^*_{\mu'}) \end{cases}.$$

We show that ϕ'_μ gives the receiver a higher expected payoff than ϕ^*_μ.

We denote by $N^a(\mu) = \mathbb{E}_\mu[u(a,\omega)] + (-\Sigma_{\omega\in\Omega}\mu(\omega)\ln\mu(\omega))$ the receiver's net payoff from a at μ. Therefore, the receiver's expected payoff from ϕ'_μ is

$$
\begin{aligned}
&\Sigma_{\mu''\in Supp(\phi'_u)}\phi'_u(\mu'')\max_{a\in A} N^a(\mu'') + \lambda\Sigma_{\omega\in\Omega}\mu(\omega)\ln\mu(\omega)\\
=&\Sigma_{\mu''\in Supp(\phi^*_\mu)\setminus\{\mu'\}}\phi^*_u(\mu'')\max_{a\in A} N^a(\mu'') + \phi^*_\mu(\mu')\Sigma_{\mu''\in Supp(\phi^*_{\mu'})}\phi^*_{u'}(\mu'')\max_{a\in A} N^a(\mu'') + \lambda\Sigma_{\omega\in\Omega}\mu(\omega)\ln\mu(\omega)\\
>&\Sigma_{\mu''\in Supp(\phi^*_\mu)\setminus\{\mu'\}}\phi^*_u(\mu'')\max_{a\in A} N^a(\mu'') + \phi^*_\mu(\mu')\max_{a\in A} N^a(\mu') + \lambda\Sigma_{\omega\in\Omega}\mu(\omega)\ln\mu(\omega).
\end{aligned}
$$

The inequality follows from the fact that $\{\mu'\} \neq Supp(\phi^*_{\mu'})$. The last expression is the receiver's expected payoff from ϕ^*_μ.Therefore, ϕ^*_μ is not optimal in the first place.

2.5.3 Proposition 4

If $P_{\mu_0}(a_1) = 0$, we show the sender can make the receiver choose a_1 with some positive probability. Since a_1 is not weakly dominated by a_2 for the receiver. There exists ω such that $u(a_1,\omega) > u(a_2,\omega)$. Let the sender design μ with $\mu(\omega) = 1$. If μ is induced, the receiver takes a_1.

If $P_{\mu_0}(a_1) > 0$, let $Supp(\phi^*_{\mu_0}) = \{\mu_1, \mu_2\}$ with $P_{\mu_0}(a_1) = \phi^*_{\mu_0}(\mu_1)$. By Lemma 3, $\mu_1, \mu_2 \in int(\Delta(\Omega))$, hence

$$\alpha = \min_{\omega\in\Omega}\frac{(1-P_{\mu_0}(a_1))\mu_2(\omega)}{\mu_1(\omega)} > 0.$$

Now we construct μ'_2 such that for any $\omega \in \Omega$,

$$\mu'_2(\omega) = \frac{\mu_0(\omega) - (\alpha + P_{\mu_0}(a_1))\mu_1(\omega)}{1-\alpha-P_{\mu_0}(a_1)}.$$

It is easy to verify that μ'_2 is a belief.

Let the sender design τ with $Supp(\tau) = \{\mu_1, \mu_2'\}$ and $\tau(\mu_1) = \alpha + P_{\mu_0}(a_1)$. By Lemma 4, $Supp(\phi^*_{\mu_1}) = \{\mu_1\}$. Therefore, by designing τ, the sender makes the receiver chooses a_1 with a probability that is at least $\alpha + P_{\mu_0}(a_1)$.

2.5.4 Derivation of M^a

In this subsection, we characterize M^a. We also show that $\operatorname{argmax}_{a \in A} \mathbb{E}_\mu[u(a', \omega)]$ is singleton for any μ in the support of an optimal signaling structure. The receiver's response P_μ can be equivalently found by solving

$$\max_{P \in \Delta(A)} \Sigma_{\omega \in \Omega} \lambda \mu(\omega) \ln(\Sigma_{a \in A} P(a) e^{\frac{u(a,\omega)}{\lambda}}),$$ [27]

where $P \in \Delta(A)$ is a probability distribution over actions. Notice that the objective function is strictly concave. Consider the partial derivatives of the objective function at $P(a) = 1$:

$$\frac{\partial}{\partial P(a)}|_{P(a)=1} = \lambda \Sigma_{\omega \in \Omega} \mu(\omega) \frac{e^{\frac{u(a,\omega)}{\lambda}}}{\Sigma_{a' \in A} P(a') e^{\frac{u(a',\omega)}{\lambda}}}|_{P(a)=1} = \lambda$$

for a, and

$$\frac{\partial}{\partial P(a')}|_{P(a)=1} = \lambda \Sigma_{\omega \in \Omega} \mu(\omega) \frac{e^{\frac{u(a',\omega)}{\lambda}}}{\Sigma_{a'' \in A} P(a'') e^{\frac{u(a'',\omega)}{\lambda}}}|_{P(a)=1} = \lambda \Sigma_{\omega \in \Omega} \mu(\omega) e^{\frac{u(a',\omega) - u(a,\omega)}{\lambda}}$$

for actions other than a. Therefore, the sufficient and necessary condition for the receiver to take a with probability one reduces to $\Sigma_{\omega \in \Omega} \mu(\omega) e^{\frac{u(a',\omega) - u(a,\omega)}{\lambda}} \leq 1$ for any $a' \in A$, meaning

$$M^a = \cap_{a' \in A} \{\mu \in \Delta(\Omega) | \Sigma_{\omega \in \Omega} \mu(\omega) e^{\frac{u(a',\omega) - u(a,\omega)}{\lambda}} \leq 1\}.$$

[27] For reference, see Lemma 2 of Matějka and McKay (2015).

Notice that M^a is a proper subset of the set of beliefs for which a gives the receiver the highest expected payoff. Notice also that M^a shrinks if λ decreases.

We assume a_1 and a_2 give the receiver the same highest expected payoff at μ. Then by Jensen's inequality

$$\Sigma_{\omega\in\Omega}\mu(\omega)e^{\frac{u(a_2,\omega)-u(a_1,\omega)}{\lambda}} = \mathbb{E}_\mu e^{\frac{u(a_2,\omega)-u(a_1,\omega)}{\lambda}} > e^{\frac{\mathbb{E}_\mu(u(a_2,\omega)-u(a_1,\omega))}{\lambda}} = 1,$$

which implies $\mu \notin M^{a_1}$, similarly $\mu \notin M^{a_2}$.[28] By Lemma 4, μ is not in the support of any optimal signaling structure.

2.5.5 Proposition 5

Let $\mu \in Supp(\phi^*_{\mu_0})$ lead to the sender's least preferred action in $B(\mu_0)$. We increase proportionally the weights assigned to other beliefs in $Supp(\phi^*_{\mu_0})$ by pushing μ to the boundary of $conv(\cup_{a\in B(\mu_0)}M^a)$. We denote this boundary belief by μ', and let

$$\mu_0 = \Sigma_{\mu''\in Supp(\phi^*_{\mu_0})\setminus\{\mu\}}\phi^*_{\mu_0}(\mu'')(1+\alpha)\mu'' + \beta\mu',$$

where $\alpha, \beta > 0$.

We know μ' can be written as a convex combination of at most $|B(\mu_0)|$ beliefs in $\cup_{a\in B(\mu_0)}M^a$. Let

$$\mu' = \Sigma_i\gamma_i\mu_i,$$

where $\mu_i \in \cup_{a\in B(\mu_0)}M^a$, $\gamma_i > 0$ for any i, and $\Sigma_i\gamma_i = 1$. If $\mu' \in \cup_{a\in B(\mu_0)}M^a$, then this step is not necessary.

[28] It is easy to see that $\mu \notin M^a$ for any other a.

Let the sender design τ with $Supp(\tau) = (Supp(\phi^*_{\mu_0}) \setminus \{\mu\}) \cup \{\mu_i\}_i$, and

$$\tau(\mu'') = \begin{cases} \phi^*_{\mu_0}(\mu'')(1+\alpha) & \text{if } \mu'' \in (Supp(\phi^*_{\mu_0}) \setminus \{\mu\}) \setminus \{\mu_i\}_i \\ \Sigma_i \mathbb{1}_{\{\mu_i\}}(\mu'')\beta\gamma_i & \text{if } \mu'' \in \{\mu_i\}_i \setminus (Supp(\phi^*_{\mu_0}) \setminus \{\mu\}) \\ \phi^*_{\mu_0}(\mu'')(1+\alpha) + \Sigma_i \mathbb{1}_{\{\mu_i\}}(\mu'')\beta\gamma_i & \text{if } \mu'' \in (Supp(\phi^*_{\mu_0}) \setminus \{\mu\}) \cap \{\mu_i\}_i \end{cases}.$$

By Lemma 4, if $\mu'' \in Supp(\phi^*_{\mu_0}) \setminus \{\mu\}$ is induced, the receiver chooses not to learn but takes the associated action in $B(\mu_0)$ with probability one. Therefore by designing τ, the receiver's probability of taking the sender's least preferred action in $B(\mu_0)$ is decreased, whereas the receiver's probability of taking other actions in $B(\mu_0)$ are all strictly increased.

Chapter 3

Endogenous Consideration Set Luce

This chapter is coauthored with Edward Honda.

3.1 Introduction

Suppose a decision-maker (DM) has to choose where to eat. Clearly, the DM will not consider every restaurant that she is aware of in the city of her residence since it would take too much time and effort to compare all of them. Instead, we would expect that the DM only considers familiar restaurants within a 10-minute drive. Many models incorporate this type of behavior with limited attention, and one way in which limited attention can be modeled is through zero probability choice in a random choice framework. Given a Random Choice Rule (RCR), when the RCR assigns a value of zero to an alternative in a given menu, we can interpret it as the agent not considering that alternative.

The Robert Duncan Luce (1959) model is one of the most widely used model of random choice, so an obvious strategy would be to introduce zero probability choice in this model.

However, it is a well known fact that the Axiom of Positivity, which precludes this zero probability choice is necessary for this model. To alleviate this problem, generalizations have been developed to drop the Axiom of Positivity. The most general of these has been proposed by Ahumada and Ülkü (2018) and Echenique and Saito (2019). One problem though, is that the considered subsets in the most general version of their model can lack structure or fail to be realistic since the function that maps a set of alternatives to a considered subset can be any arbitrary mapping. For instance, it can be the case that only x is considered when the DM is given x and y to choose from. However, when presented with x, y, z the DM may suddenly decide to consider only y. Such behavior in which the attractiveness of x and y seemingly reversed may be puzzling and make it difficult for an analyst to interpret how these consideration sets are formed by the DM.

With that being said, we want to focus on a subclass of random choice rules that allow for a clear understanding of the process of consideration set formation. Of course, we want the process to be realistic, and since it seems natural to believe that agents act to maximize their utility, we want the process to involve optimizing some sort of payoff. Therefore, our goal is to axiomatically characterize a class of random choice rules for which it looks as if the DM chooses consideration sets to maximize the expected utility that can be derived from the consideration set minus the cost of attention required for considering that set. Many special cases of the model of Ahumada and Ülkü (2018) and Echenique and Saito (2019) in the literature impose structure on this process, but they lack this type of endogenous consideration set formation in which the DM solves an optimization problem, and this may lead to counterintuitive behavior as we discuss in the related literature section.

We call our first general model the alternative-independent attention cost model, and our first result provides a representation theorem for this model. We show that only a pair of axioms that are very easy to understand and interpret in addition to the axiom of Ahumada and Ülkü (2018) and Echenique and Saito (2019) are necessary and sufficient for this model. The first of these axioms, Independence of Inclusion Choice (IIC), roughly speaking, says that when an alternative is revealed to be ”preferred to” another through the choice from binary menus, this relationship is preserved in other menus and is independent of the other alternatives. The second is a weakening of the Axiom of Positivity.

The alternative-independent model allows for a very general class of attention cost functions. In particular, the attention cost can be menu-dependent. However, we may want to impose some structure on the costs for the same reason that we want to impose structure on the process of formation. Our second result provides a representation theorem for such a model by taking a special case in which the attention cost of adding an alternative is constant and menu-independent. The additional axiom for this model, which we refer to as Dominance Consistency, ensures that the RCR complies with the additional structure by guaranteeing a common threshold for any alternative to be included.

In the alternative-independent model and its special case, the attention cost is independent of the particular alternatives in a consideration set. In reality, however, the attention cost required for a consideration set may depend on the particular alternatives in a consideration set. Therefore, in the end, we consider a complementary model with menu-independent attention costs. Our third result provides a representation theorem for this model. It requires the Strong Axiom of Revealed Preference (SARP) tailored to our setting and captures the idea that the DM has a strict preference order for consideration sets. In this sense, this model also uses a very straightforward axiom that is easy to interpret.

3.1.1 Related literature

As we previously mentioned, our goal is to provide some realistic structure to the way in which the consideration sets are formed. Several other models also impose structure on this formation that occurs prior to choosing an alternative with frequencies determined by the Luce formula, but none of them provide an endogenous formation that involves solving an optimization problem. Instead, they assume that there is some binary relation, which can be interpreted as a dominance relation, and an alternative is excluded from the consideration set whenever there is another alternative that dominates it. For the models of Cerreia-Vioglio et al. (2018), Doğan and Yıldız (2021), and Lindberg (2012), this binary relation is a weak order over the set of alternatives. The model of Horan (2021) uses a semi-order. McCausland (2009) uses a partial order as he uses $\mathbb{R}^n_+$ as the set of alternatives and any alternative, say x, is not considered when there is another alternative y that has one coordinate that is strictly greater than the corresponding coordinate of x and each of the other coordinates is greater than or equal to the corresponding coordinates of x. Echenique and Saito (2019) also consider special cases of their model that uses a strict partial order for this binary relation.

The use of this type of binary dominance relation, however, may have implications that are counterintuitive. It is that an alternative is in the consideration set regardless of the number of other alternatives as long as none of the others dominate it. So, for instance, suppose that there are 100 alternatives, none of which dominate each other pairwise. Then it is necessarily the case that all 100 alternatives will enter the consideration set. We believe that this is not very realistic. Even if none of the alternatives dominate each other, if attention is actually costly, then it seems natural that some alternatives will start to be omitted from the consideration set once the number of such alternatives becomes too large. Our model is able to avoid this as alternatives may be omitted from the consideration set even when the menu consists of alternatives for which both of two alternatives are chosen with strictly positive frequency when we only consider a single pair at a time.

There are other models within the random choice framework that study consideration set formation using a different approach from ours. Cattaneo et al. (2020), Manzini and Mariotti (2014), Aguiar et al. (2021), and Brady and Rehbeck (2016) provide models for which the consideration sets are formed randomly. An essential feature of those random attention models is the independence between the first stage consideration set formation and the second stage choices.

Instead of taking the RCR as the observable, Caplin, Dean, and Leahy (2019) assume that the model parameters like state dependent utilities of alternatives, prior beliefs about the state, and costs of learning are known and they characterize the consideration sets that are formed as a result of optimally choosing the random choice rule. This is obviously different from our approach in which we take the RCR as the primitive and rationalize the RCR. Also, the DM in our models optimally chooses the consideration set while the DM optimizes over the RCR in their model. Outside the consideration set formation framework, contributions to generalize the Luce model include Gul, Natenzon, and Pesendorfer (2014), Echenique, Saito, and Tserenjigmid (2018).

Finally, some works document empirical findings that are relevant to our work. In marketing literature, Roberts and Lattin (1991) discussed how consumers in their study only consider a subset of available brands due to a brand cost. Our study complements this literature by looking at all menus and providing testable implications regarding consideration set formation across menus. In a market share data for ice cream that Dubé, Hortaçsu, and Joo (2021) use to estimate their model, it is seen that some products have zero market share, thus supporting our claim that it is desirable to write a model without the Axiom of Positivity.

3.2 The model

3.2.1 Primitive

Let $X = \{x_1, \cdots, x_N\}$ with $N \geq 2$ be a set of alternatives. Let $K = \{A \in 2^X || A| \geq 2\}$ be the set of all non-trivial menus (subsets of X with at least two alternatives). The primitive of the model will be the random choice rule, defined as the following:

Definition 5. *A random choice rule(RCR) is a function $\rho : X \times K \to [0, 1]$ such that for all $A \in K$: 1) $\sum_{x_i \in A} \rho(x_i, A) = 1$, and 2) $\rho(x_i, A) = 0$ if $x_i \notin A$.*

The RCR we observe can be interpreted as the limiting frequency with which an alternative is chosen from a menu when a choice is made repeatedly.

3.2.2 Representation

We model a boundedly rational decision-maker (henceforth DM), who, given a menu, will only consider a subset of the menu due to costly attention and choose an alternative from that subset. We will refer to these subsets as consideration sets. Below, we give a formal description of our representation.

Each $x_i \in X$ has an additive utility of $\tilde{U}(x_i) = u(x_i) + \varepsilon_i$, where the ε_i's are i.i.d. $\forall\ 1 \leq i \leq N$ according to the standard Gumbel distribution, and u is an injective function from X to $\mathbb{R}$. We place the assumption that u is injective in order to rule out the unnecessary case that the DM is totally indifferent between any two alternatives.[29]

[29]We say that such a case is unnecessary because we can always approximate such a case arbitrarily well by taking one alternative to have $u(x)$ and the other to have $u(x) + \epsilon$ for an arbitrarily small $\epsilon > 0$.

Given a menu of such alternatives, the DM optimally forms a consideration set under the following environment. For each $A \in K$, the DM has an **alternative-independent** attention cost function $c_A(\cdot) : \{1, \cdots, |A|\} \to \mathbb{R}_+$, which weakly increases with the size of the consideration set. If $B \subseteq A$ is the set of alternatives being considered, then the attention cost is $c_A(|B|)$. The intuition is that the DM compares alternatives in a consideration set in order to choose the best, and this becomes more difficult as the size of the consideration set becomes larger.[30]

We let the attention cost depend on the menu so that the cost of comparing the same number of alternatives can be different in different environments. For example, the cost of comparing the same two cars can be different across dealerships, or the cost of comparing identical products can be different between two grocery stores. The menu-dependent assumption can also capture the idea that an extensive menu increases the cost of comparing its alternatives.[31]

Furthermore, the attention cost of comparing any two alternatives is zero.

Assumption 1. *For any $A \in K$ and any $B \subseteq A$, $c_A(|B|) = 0$ if $|B| = 2$.*

Hence, the DM will form a consideration set with at least two alternatives. The interpretation is that the DM always compares at least two alternatives when making a decision because such a comparison is easy enough.

Once the DM forms the consideration set and pays the attention cost of comparing the alternatives in it, she chooses the alternative in the consideration set with the highest realization of $\tilde{U}(x_i)$. Therefore, the DM's expected utility from consideration set B is given

[30] Therefore, the attention costs for singleton consideration sets are zeros.

[31] Many studies have found evidence that providing individuals with more options can be detrimental to choice. In the psychology literature, this phenomenon is known as "choice overload." In particular, Chernev, Böckenholt, and Goodman (2015) identify preference uncertainty as one of the four key factors that facilitate choice overload.

by

$$\mathbb{E}[\max_{x_i \in B} \tilde{U}(x_i)] = \ln(\Sigma_{x_i \in B} \exp(u(x_i))).$$

Therefore, adding an alternative into a consideration set always strictly increases the expected utility from that consideration set. In addition, if $u(x_i) > u(x_j)$, adding x_i increases the expected utility more than adding $u(x_j)$ irrelevant of the other alternatives in the consideration set.

The DM forms a consideration set to maximize the expected utility from this set minus the cost of considering the alternatives in it,

$$\max_{B \subseteq A} \ \ln(\Sigma_{x_i \in B} \exp(u(x_i))) - c_A(|B|). \tag{3.1}$$

Now, it is possible that the DM will be indifferent between adding some alternatives or leaving them out. In this case, we assume that the DM goes for more diversity and always adds the alternatives.

Definition 6. *A RCR has an alternative-independent attention cost representation if there exists 1) an injective $u : X \to \mathbb{R}$, and 2) a weakly increasing $c_A(\cdot) : \{1, \cdots, |A|\} \to \mathbb{R}_+$ with Assumption 1 for any $A \in K$ such that for any A and $x_i \in A$,*

$$\rho(x_i, A) = \begin{cases} \frac{\exp u(x_i)}{\sum_{x_j \in B^*} \exp u(x_j)} & \text{if } x_i \in B^* \\ 0 & \text{otherwise} \end{cases},$$

where $B^* = \underset{B \subseteq A}{argmax} \ \ln(\Sigma_{x_i \in B} \exp(u(x_i))) - c_A(|B|)$.

3.3 Characterization

3.3.1 Axioms

Here, we describe axioms on the RCR that we later prove to be necessary and sufficient for the alternative-independent attention cost representation. The first axiom is from Echenique and Saito (2019) and Ahumada and Ülkü (2018).

Axiom 1 (**Cyclical Independence**). *For sequences $(x_i)_{i=1}^s$ of X and $(A_i)_{i=1}^s$ of K, if $\rho(x_i, A_i) > 0, \rho(x_{i+1}, A_i) > 0$ (using Mod s addition) for $i = 1, \cdots, s$, then*

$$\frac{\rho(x_1, A_s)}{\rho(x_s, A_s)} = \frac{\rho(x_1, A_1)}{\rho(x_2, A_1)} \frac{\rho(x_2, A_2)}{\rho(x_3, A_2)} \cdots \frac{\rho(x_{s-1}, A_{s-1})}{\rho(x_s, A_{s-1})}.$$

Similar to Luce's IIA, Cyclical Independence captures the independence of the frequency ratio between two alternatives from the others. It is known that Cyclical Independence is the only axiom that is necessary and sufficient to rationalize a general Luce model without any restrictions on the mapping from menus to consideration sets.[32]

The second axiom regulates the RCR by relating binary choice frequencies to the inclusion choices of larger menus. An alternative that is chosen more frequently than another in their binary menu should always be included before the other for all larger menus that include them. For binary menus it is convenient to write $\rho(x_i, x_j)$ for $\rho(x_i, \{x_i, x_j\})$.

Axiom 2 (**Independence of Inclusion Choice (IIC)**). *For any $A \in K$, and $x_i, x_j \in A$, if $\rho(x_i, x_j) > \frac{1}{2}$ and $\rho(x_j, A) > 0$, then $\rho(x_i, A) > 0$.*

Notice that because u is injective, $\rho(x_i, x_j) \neq \frac{1}{2}$ for any $x_i, x_j \in X$. However, note that we can get rid of injection if the DM has a well-defined order of considering the alternatives with the same fixed utility. That is, if $\rho(x_i, x_j) = \frac{1}{2}$, then for any A, A' that include both

[32]Please refer to Echenique and Saito (2019) or Ahumada and Ülkü (2018) for details.

alternatives, $\rho(x_i, A) \geq \rho(x_j, A)$ if and only if $\rho(x_i, A') \geq \rho(x_j, A')$. This is like having IIC for alternatives with the same fixed utility.

The third axiom is a positivity requirement on the RCR.

Axiom 3 (**Weak Positivity**). *For any $A \in K$, there exists distinct $x_i, x_j \in A$, such that $\rho(x_i, A) > 0, \rho(x_j, A) > 0$.*

Weak Positivity is clearly weaker than the Axiom of Positivity, which seems to be too restrictive in reality. For example, as discussed earlier, Dubé, Hortaçsu, and Joo (2021) suggest that some ice cream brands seem to have zero market share. We also would expect the DM to drop some items that are very likely to be inferior to others especially when presented with a menu of a large number of items. Weak Positivity, however, seems more reasonable even as seen from experimental results. For example, Agranov and Ortoleva (2017), suggest that one may choose differently for the same decision problem even if a choice is made shortly after, and this would mean that there are at least two alternatives being chosen with positive probability. On the contrary, we are not aware of any experimental results that clearly support the Axiom of Positivity.

3.3.2 A representation theorem

We give our main result, a representation theorem for the alternative-independent model. We then move on to consider variations of the model including a special case and a model without Assumption 1.

Theorem 2. *A RCR satisfies Cyclical Independence, IIC, and Weak Positivity if and only if it has an alternative-independent attention cost representation.*

IIC characterizes consideration set formation. For the logit model, if the DM optimally chooses a subset of available alternatives, the primary mechanism for these inclusion choices, as illustrated by the expected utility formula, is that the DM includes alternatives with the highest fixed utilities. Therefore, the main contribution of this result is that we introduce a particular form of attention cost, which enables us to isolate this mechanism as an axiom. Weak Positivity is a technical simplification that forces the data to reveal enough information so that Cyclical Independence has enough bites.

Although we use the logit framework, IIC is straightforward enough that it is satisfied by many other models too. For instance, it will still be satisfied even if we generalize the alternative-independent model to any i.i.d. discrete choice model. It is also satisfied by the additive perturbed utility model of Fudenberg, Iijima, and Strzalecki (2015) without positivity. All those models share the characteristic that there are no complementarities between the alternatives. In addition, IIC is also simple, making it possible to bring it to data sets with limited consideration.

The consideration sets are uniquely identified for all menus. The exponential of the fixed utilities are identified up to a scaling parameter. That is, if u rationalizes the RCR, then u' rationalizes the RCR if and only if there is $\alpha > 0$ such that $\exp(u'(x_i)) = \alpha \exp(u(x_i))$ for any x_i. The alternative-independent model has general attention cost functions so that they are not identified.

A special case: constant marginal attention costs

We previously argued that the alternative-independent model allows an alternative to being omitted from a consideration set even when no single alternative in this menu dominates it. However, the alternative-independent model may be too general to allow the following example.

Example 1. *Let* $N = 4$, $\rho(x_1, x_3) = \rho(x_2, x_3) = 0.6, \rho(x_1, x_4) = \rho(x_2, x_4) = 0.9$, *and* $\rho(x_4, \{x_1, x_2, x_4\}) > \rho(x_3, \{x_1, x_2, x_3\}) = 0$.

To impose additional structures on the DM's consideration set formation and have some identification power for attention costs, we study a special case of the alternative-independent model in which the DM has constant marginal attention costs. Specifically, the marginal cost of adding additional alternatives into the consideration set is fixed at some $\bar{c} \in \mathbb{R}_{++}$, measured in utils, from the third alternative onwards. The DM forms a consideration set by

$$\max_{B \subseteq A, |B| \geq 2} \ln(\Sigma_{x_i \in B} \exp(u(x_i))) - \bar{c} \cdot (|B| - 2).$$

Before describing the next axiom, we introduce the following notation. For any $A \in K$, we let $f(A) = \{x \in A | \rho(x, A) > 0\}$ be the set of alternatives that are chosen with positive probabilities.

Axiom 4 (**Dominance Consistency**). *For any distinct* $A, A' \in K$, *and any* $x_i \in A$, $x_j \in A'$, *if* $\rho(x_i, A) = 0$, $\rho(x_j, A') > 0$, *and* $|f(A')| \geq 3$, *then*

$$\Sigma_{x' \in f(A)} \frac{\rho(x', x_i)}{1 - \rho(x', x_i)} > \Sigma_{y' \in f(A') \setminus \{x_j\}} \frac{\rho(y', x_j)}{1 - \rho(y', x_j)}. \tag{3.2}$$

The intuition behind Dominance Consistency is simple. The relation $\rho(x_i, x_j) > \frac{1}{2}$ is read "x_i is preferred to x_j", and the relation $\rho(y_i, y_j) > \rho(x_i, x_j) > \frac{1}{2}$ is further read "y_i is preferred to y_j more than x_i is preferred to x_j". Therefore, Dominance Consistency states that combining the alternatives in $f(A)$ is preferred to x_i more than combining the alternatives in $f(A') \setminus \{x_j\}$

is preferred to x_j. In the logit model, this means x_i will be chosen less often in $f(A) \cup \{x_i\}$ than x_j in $f(A')$. For the logit model, an alternative's choice frequency in a consideration set precisely reveals the marginal benefit of having that alternative in that consideration set.

Theorem 3. *A RCR satisfies Cyclical Independence, IIC, Weak Positivity, and Dominance Consistency if and only if it has a constant marginal attention cost representation.*

Theorem 3 shows that Dominance Consistency is the additional structure on the process of consideration set formation if we have constant marginal attention costs.[33]

Similar to the alternative-independent model, consideration sets are uniquely identified and the exponential of the fixed utilities are identified up to a scaling parameter. We can identify $\bar{c}$ if the observed data are rich enough. First, we assume that $\bar{c} \in (0, \ln(\frac{3}{2}))$ so that it is possible to observe menus that have consideration sets of three or more alternatives. Intuitively, the incentive for the DM to include x_i into a consideration set B depends on $\bar{c}$ and the ratio of the exponential utilities $\frac{\exp(u(x_i))}{\Sigma_{x_j \in B} \exp(u(x_j))}$; to uniquely identify $\bar{c}$, we need to be able to vary this exponential utility ratio freely in $(0, \frac{1}{3})$. We let X be infinite and maintain the assumption that menus are finite.[34]

Proposition 6. *If there is $x_i \in X$ such that all any $q \in [\frac{2}{5}, \frac{1}{2})$, there exists $x_j \in X$, $\rho(x_i, x_j) = q$, then $\bar{c}$ is uniquely identified.*

Without Assumption 1

We move on to consider the model without imposing Assumption 1 since some readers may find it unsatisfactory. In particular, one may think that if binary comparisons have zero attention costs, then the DM may conduct many binary comparisons and pick the best one.

[33]We also study a model in which the marginal cost of adding alternatives is not fixed. This requires a different version of Dominance Consistency. Please refer to the Appendix for details.

[34]Our characterization result holds in this case.

We can, of course, argue that the process of completing a series of binary comparisons is itself costly. Notwithstanding, we discuss the model without Assumption 1 so the DM may form singleton consideration sets.

We use the notation $\succ$, where $x_i \succ x_j$ if and only if $\rho(x_i, x_j) > \frac{1}{2}$. The following axiom, **Weak (Stochastic) Transitivity**, is necessary since it guarantees that $\succ$ will be a strict total order.[35]

Axiom 5 (Weak Transitivity)**.** *For any $x_i, x_j, x_k \in X$, if $\rho(x_i, x_j) > \frac{1}{2}$ and $\rho(x_j, x_k) > \frac{1}{2}$, then $\rho(x_i, x_k) > \frac{1}{2}$.*

We also need a slightly stronger version of IIC to rule out cases where $\rho(x_i, x_j) = 1$ and $\rho(x_j, A) > \rho(x_i, A) > 0$ for some A.

Axiom 6 (IIC$^+$)**.** *For any $A \in K$, and $x_i, x_j \in A$, if $\rho(x_i, x_j) > \frac{1}{2}$ and $\rho(x_j, A) > 0$, then $\rho(x_i, A) > \rho(x_j, A)$.*

The following example shows that Cyclical Independence, IIC$^+$, and Weak Transitivity are not sufficient to rationalize the alternative-independent model without Assumption 1.

Example 2. *Let $N = 4$ and $x_1 \succ x_2 \succ x_3 \succ x_4$. Let $\rho(x_1, x_4) = 2/3$, $\rho(x_2, x_3) = 3/4$, and all other menus have singleton consideration sets.*

The issue in Example 2 is that according to $\succ$, $\rho(x_1, x_4)$ has to be larger than $\rho(x_2, x_3)$. More generally, we partition X in a way that two alternatives are in the same partitioning set if there is a sequence as defined in Cyclical Independence between them. Then, alternative utilities in the same partitioning set are unique up to one shifting in the real line. However, alternative utilities from different partitioning sets also have to respect $\succ$.

[35] Marschak (1959) attributes the definition of weak and strong transitivity to Valavanis-Vail (1957). R. Duncan Luce and Suppes (1965) think that the term "stochastic" is inappropriate since the probabilities do not involve transitions over time.

We construct a binary choice data ρ^* that incorporates all the information revealed in ρ. For any different $x_i, x_j \in X$, if there exists $(y_i)_{i=1}^s$ of X, $(A_i)_{i=1}^{s-1}$ of K, $s \geq 2$ such that $x_i = y_1, x_j = y_s$, $\rho(y_i, A_i) > 0, \rho(y_{i+1}, A_i) > 0$ for $i = 1, \cdots, s-1$, then

$$\frac{\rho^*(x_i, x_j)}{\rho^*(x_j, x_i)} = \frac{\rho(y_1, A_1)}{\rho(y_2, A_1)} \cdots \frac{\rho(y_{s-1}, A_{s-1})}{\rho(y_s, A_{s-1})};$$

otherwise, $\rho^*(x_i, x_j) = \rho(x_i, x_j)$. Notice that ρ^* is well-defined because of Cyclical Independence. The data ρ^* generalizes the strict binary utility model in a way that allows each binary choice to reveal one of the two different levels of information.[36] If $\rho^*(x_i, x_j) \in (0, 1)$, it reveals how much one alternative is preferred to the other; if $\rho^*(x_i, x_j) = 0$ or 1, it only reveals which one is the preferred among the two.[37]

We impose an axiom on ρ^* to rule out Example 2.

Axiom 7 (**Quadruple Transitivity**). *For any x_i, x_j, y_i, y_j of X, if $\rho^*(x_i, x_j) \in (\frac{1}{2}, 1), \rho^*(y_i, x_i) = 1$, and $\rho^*(y_j, x_j) = 0$, then $\rho^*(y_i, y_j) > \rho^*(x_i, x_j)$.*

It is clear that Quadruple Transitivity is necessary for the model. To discuss its sufficiency, we introduce the following math lemma.

Lemma 6. *Let $a_1 < a_2 < \cdots < a_N$ be unknown real numbers. For any $a_n, n \in \{1, \cdots, N\}$, $a_n \in I_{-1}$ or $a_n \in I_1$, but not both. Element values in each $I_i, i = -1, 1$ are unique up to one shifting in the real line. We can find values for $a_n, n \in \{1, \cdots, N\}$ if and only if for any $a_{i1} > a_{i2}$ of I_i and $a_{j1} > a_{j2}$ of I_{-i}, $a_{i1} - a_{i2} > a_{j1} - a_{j2}$ whenever $i1 > j1$ and $i2 < j2$.*

[36] The definition of a strict binary utility model can be found in R. Duncan Luce and Suppes (1965). (see Definition 18).

[37] Those data can arise in a random utility model with attention costs when alternatives are divided into two partitioning sets. Pairs of alternatives within the same group are easy to compare. In contrast, pairs of alternatives from different sets are not.

With Lemma 6, it is not difficult to see that Cyclical Independence, IIC$^+$, Weak Transitivity, and Quadruple Transitivity are sufficient when X is divided into at most two partitioning sets according to ρ^*. That is, for any distinct x_i, x_j, x_k of X, at least one of the following three is true: $\rho^*(x_i, x_j) \in (0, 1), \rho^*(x_j, x_k) \in (0, 1)$, or $\rho^*(x_i, x_k) \in (0, 1)$.[38]

It is not straightforward to generalize the discussion here to the case where X is divided into more than two partitioning sets.

Example 3. *Let $N = 6$ and $x_1 \succ \cdots \succ x_6$. Let $\rho^*(x_1, x_6) = \rho^*(x_2, x_3) + \epsilon = \rho^*(x_4, x_5) + \epsilon = 2/3$, where $\epsilon > 0$ is very small, and all other menus have singleton consideration sets.*

Example 4. *Let $N = 6$ and $x_1 \succ \cdots \succ x_6$. Let $\rho^*(x_1, x_4) = \rho^*(x_3, x_6) = 1/2+\epsilon, \rho^*(x_2, x_5) = 1 - \epsilon$, where $\epsilon > 0$ is very small, and all other menus have singleton consideration sets.*

Axioms involving alternative sequences will be needed to rule out those types of examples entirely.

The discussion here shows that rationalizing the alternative-independent model without Assumption 1 is not easy, and the axioms will be difficult to interpret. Axioms have bites only if the date reveal enough information. It is difficult to impose axioms to rationalize a model if its data do not disclose enough information. Weak Positivity is a way to enforce the data to reveal enough information, and as far as we think, the cost of imposing Weak Positivity is low.

[38] Quadruple Transitivity can also help to characterize other binary choice models, e.g., the Fechnerian model and the simple scalability model, when their binary choices exhibit two levels of information. Debreu (1958) uses the quadruple condition of Davidson and Marschak (1959) and a continuous condition to give a representation theorem for the Fechnerian model. Fudenberg, Iijima, and Strzalecki (2015) show that their Acyclicity axiom also characterizes the Fechnerian model with positivity. Tversky and Russo (1969) rationalizes the simple scalability model using a version of strong transitivity.

Cyclical Independence and Luce's IIA

Cyclical Independence is difficult to check since it involves sequences of alternatives and menus. With Weak Positivity, however, we can relax Cyclical Independence in our representation theorem so that we have to consider only sequences of at most three alternatives. Cyclical Independence reduces to Luce's IIA for pairs of alternatives.[39] The following axiom is due to Robert Duncan Luce (1959).

Axiom 8 (**Product Rule**). *For any* $x_i, x_j, x_k \in X$*,*

$$\frac{\rho(x_i, x_k)}{\rho(x_k, x_i)} = \frac{\rho(x_i, x_j)}{\rho(x_j, x_i)} \frac{\rho(x_j, x_k)}{\rho(x_k, x_j)}.^{40}$$

Corollary 2. *A RCR satisfies Luce's IIA, IIC, Weak Positivity, and Product Rule if and only if it has an alternative-independent attention cost representation.*

We can get rid of Product Rule if we have a slightly stronger version of Weak Positivity. For example, if attention costs additionally satisfy that for any $|A| \geq 3$ and any $B \subseteq A$, $c_A(|B|) = 0$ if $|B| = 3$. Then the DM will compare at least three alternatives for menus with three or more alternatives. In this case, we will have Weak$^+$ Positivity, which additionally requires that for any $|A| \geq 3$, there exists distinct $x_i, x_j, x_k \in A$, such that $\rho(x_i, A) > 0, \rho(x_j, A) > 0, \rho(x_k, A) > 0$. Product Rule is implied by Luce's IIA and Weak$^+$ Positivity.

[39] When we say Luce's IIA, we mean Luce's IIA for pairs of alternatives and menus such that both alternatives are chosen with positive probabilities in both menus.

[40] Notice that this expression is well-defined because of Weak Positivity.

3.4 A complementary model: menu-independent attention costs

We now consider a **menu-independent** attention cost model in which attention costs depend on the alternatives in the consideration set, i.e., $c : 2^X \setminus \{\emptyset\} \to \mathbb{R}_+$.[41] We assume that $c(\cdot)$ is monotonic, $c(B) \leq c(B')$ if $B \subseteq B'$ so that adding an alternative into a consideration set increases the attention cost. The DM's consideration set formation in terms of net expected utility is thus represented by

$$\max_{B \subseteq A} \ \ln(\Sigma_{x_i \in B} \exp(u(x_i))) - c(B).$$

If two consideration sets give the same net expected utility, the DM's choice between the two is assumed to be the same for all menus that include them.[42] Thus, for any u and c, the DM's preferences over consideration sets is a strict total order. Given a strict order over consideration sets, we say u and c rationalize this order if, under u and c, the DM's preferences over consideration sets equal this order.

For such a model, we have the following lemma.

Lemma 7. *Fix u, for any strict total order over consideration sets, there exists a* ***monotonic*** *c so that u and c rationalize it.*

[41] One issue for this model is that singleton consideration sets may have positive attention costs. However, for more extensive consideration sets, this setup may describe attention costs better than the alternative-independent model as attention costs may easily depend on the particular alternatives being considered..

[42] It is not difficult to extend the analysis here to allow for non-uniqueness assuming we observe the set of consideration sets. However, in that case, the DM's choices have two sources of randomization. It is difficult to tell if a choice reversal is randomization between consideration sets or between alternatives within a consideration set.

Proof. First, pick any strict total order and rank consideration sets from the most preferred as $B_1, B_2, \cdots$. Second, for any B_j, let $c(B_j) = \ln(\Sigma_{x_i \in B_j} \exp(u(x_i))) + j\epsilon$, where $\epsilon > 0$. If ϵ is small enough, $c(B) \leq c(B')$ if $B \subseteq B'$. The constructed c, combined with u, rationalize the order since for any B_j, the net expected payoff from B_j equals $-j\epsilon$. $\square$

Therefore, all testable implications regarding consideration sets are that a strict order over consideration sets exists.

We follow the standard revealed preference theory to characterize this model.[43] The difference, though, is that the revealed preference relation will be defined over subsets of menus. For any distinct $B, B' \in 2^X \setminus \{\emptyset\}$, $B \succ_{cs} B'$ if there exists $A \in K$ such that $B = f(A)$ and $B' \subseteq A$. For any distinct $B, B' \in 2^X \setminus \{\emptyset\}$, $B \succ_{cs}^* B'$ if there exists $(B_i)_{i=1}^s, s \geq 2$ such that $B = B_1, B' = B_s$, and $B_1 \succ_{cs} \cdots \succ_{cs} B_s$.

Axiom 9 (**SARP-CS**). *For any distinct $B, B' \in 2^X \setminus \{\emptyset\}$, if $B \succ_{cs}^* B'$, then not $B' \succ_{cs} B$.*[44]

SARP-CS rules out preference cycles over consideration sets.

Theorem 4. *A RCR satisfies Cyclical Independence and SARP-CS if and only if it has a menu-independent attention cost representation.*

The proof is straightforward given Lemma 7. The construction of u follows Echenique and Saito (2019) and Ahumada and Ülkü (2018) using Cyclical Independence. The constructed strict total order is an extension of the indirectly-revealed preference $\succ_{cs}^*$. Notice that we can

[43] Classical references include Samuelson (1938), Houthakker (1950), Richter (1966), and Afriat (1967).

[44] SARP-CS implies the following intuitive condition: for any A, A', if $f(A') \subseteq A \subseteq A'$, then $f(A) = f(A')$. A consideration set for a larger menu will be the consideration set for a small menu if the small menu includes the consideration set. This is a restatement of Sen's property α adapted to the present context. Sen (1971), however, attributes it to Chernoff (1954). Property α is not enough since it does not rule out preference cycles over consideration sets. Property α is also close to the attention filter property of Masatlioglu, Nakajima, and Ozbay (2012).

not observe the DM's preferences over all the consideration sets. The extension, however, is trivial since the number of consideration sets is finite. Fixing u, we then can construct a monotonic c using the method in the proof of Lemma 7 so that u and c rationalize the order.

The menu-independent model is a complement to the alternative-independent model since SARP-CS uses no references of binary choice frequencies, whereas IIC requires the formation of consideration sets to depend entirely on binary choices. This is because alternatives can be complements in the menu-independent model. The source of complementarity, however, is from attention costs instead of payoffs. This is different from the rational inattention model in which complementarities are from the state-dependent payoffs. The menu-independent model is also different from the models mentioned in the literature that use binary relations over alternatives to regulate consideration sets.

Masatlioglu, Nakajima, and Ozbay (2012) and Lleras et al. (2017) impose exogenous structures like attention filter and competition filter on consideration sets that is independent of the underlying preferences.[45] In contrast, we introduce an attention cost to the logit model and show that a strict preference order over consideration sets exists. Experimental studies could compare these two approaches and their respective testable implications.[46]

Choice reversals, hence a violation of WARP, is one of the most consistent findings in experimental settings.[47] The analysis here suggests that instead of checking violations of WARP for alternatives, empirical researches could also check violations of WARP for the

[45]The menu-independent model satisfies the attention filter property when it comes to consideration set formation. As Masatlioglu, Nakajima, and Ozbay (2012) pointed out, the attention filter property is a minimal condition, and that this property should be satisfied when the DM chooses consideration sets by taking the cost of investigation and the expected benefit into account.

[46]Another difference is that Masatlioglu, Nakajima, and Ozbay (2012) and Lleras et al. (2017) have deterministic choice data, so the consideration sets are not readily identified.

[47]See, for example, Sippel (1997).

considered sets. In this sense, a violation of WARP is to observe two menus such that the DM's consideration sets for them are different and both are subsets of the other menu.

Finally, we discuss a special case in which the attention cost for x_i is fixed at c_i, and the attention cost for consideration set B is $\Sigma_{x_i \in B} c_i$.[48] We will assume Weak Positivity to make the discussion here easier. The main testable implication for this special case is captured in the following axiom, which imposes structure on the observables to comply with the fact that the cost of adding the same alternative is the same for all menus.

Axiom 10 (**Individual Dominance Consistency**). *For any distinct $A, A' \in K$, and $x_i \in A \cap A'$, if $\rho(x_i, A) = 0$, $\rho(x_i, A') > 0$, and $|f(A')| \geq 3$, then*

$$\Sigma_{x' \in f(A)} \frac{\rho(x', x_i)}{1 - \rho(x', x_i)} > \Sigma_{y' \in f(A') \setminus \{x_i\}} \frac{\rho(y', x_i)}{1 - \rho(y', x_i)}. \tag{3.3}$$

That is, an alternative will be included if and only if this alternative's probability of being chosen is above some threshold should it be included. Unlike Dominance Consistency, different alternatives will have different thresholds depending on their costs.

3.5 Omitted proofs

3.5.1 Theorem 2

[48] This special case also appears in Roberts and Lattin (1991).

Necessity

Cyclical Independence. For any $(x_i)_{i=1}^s$ and $(A_i)_{i=1}^s$ with $\rho(x_i, A_i) > 0, \rho(x_{i+1}, A_i) > 0$, $i = 1, \cdots, s$:

$$\begin{aligned}\frac{\rho(x_1, A_s)}{\rho(x_s, A_s)} &= \frac{exp(u(x_1))}{exp(u(x_s))}\\ &= \frac{exp(u(x_1))}{exp(u(x_2))}\frac{exp(u(x_2))}{exp(u(x_3))}\cdots\frac{exp(u(x_{s-1}))}{exp(u(x_s))}\\ &= \frac{\rho(x_1, A_1)}{\rho(x_2, A_1)}\frac{\rho(x_2, A_2)}{\rho(x_3, A_2)}\cdots\frac{\rho(x_{s-1}, A_{s-1})}{\rho(x_s, A_{s-1})}.\end{aligned} \tag{3.4}$$

IIC . $\rho(x_i, x_j) > \frac{1}{2}$ implies that $u(x_i) > u(x_j)$. $\rho(x_j, A) > 0$ means x_j is in the consideration set of A. Therefore, x_i must also be in the consideration set, or else the DM could replace x_j with x_i and increases the consideration set's expected utility without changing the attention cost. This means $\rho(x_i, A) > 0$.

Weak Positivity. This is because considering an additional alternative strictly increases the consideration set's expected payoff.

Sufficiency

Recall that $x_i \succ x_j$ if and only if $\rho(x_i, x_j) > \frac{1}{2}$.

Lemma 8. *Weak Positivity and Cyclical Independence imply that $\succ$ is transitive.*

Proof. Let $x_i \succ x_j$, $x_j \succ x_k$, we show $x_i \succ x_k$. Since $x_i \succ x_j, x_j \succ x_k$, $\rho(x_i, x_j) > \frac{1}{2}, \rho(x_j, x_k) > \frac{1}{2}$. By Weak Positivity, $\rho(x_i, x_j) < 1, \rho(x_j, x_k) < 1$. Therefore, by Cyclical Independence:

$$\frac{\rho(x_i, x_k)}{\rho(x_k, x_i)} = \frac{\rho(x_i, x_j)}{\rho(x_j, x_i)}\frac{\rho(x_j, x_k)}{\rho(x_k, x_j)} > 1 \implies x_i \succ x_k.$$

$\square$

We first pin down $\hat{u} : X \to R$. Let $\hat{u}(x_1) = 1$, and pick $\hat{u}(x_i)$, $i \in \{2, \cdots, N\}$ sequentially

$$\hat{u}(x_i) = \hat{u}(x_{i-1}) + \ln(\frac{\rho(x_i, x_{i-1})}{\rho(x_{i-1}, x_i)}).$$

We next pin down $\hat{c}_A(\cdot) : \{1, \cdots, |A|\} \to \mathbb{R}_+$ for any $A \in K$. Let $\hat{c}_A(|B|) = 0$ for any $|B| \leq |f(A)|$, and $\hat{c}_A(|B|) = \ln(\Sigma_{x_i \in A} \exp(u(x_i)))$ for any $|f(A)| < |B| \leq |A|$.[49]

We show for any $A \in K$ and any $x \in f(A)$, $\rho(x, A) = \frac{exp(\hat{u}(x))}{\Sigma_{x' \in f(A)} exp(\hat{u}(x'))}$. We first consider binary menus. For binary menus $\{x_i, x_{i-1}\}$, $i \in \{2, \cdots, N\}$, it is satisfied by the definition of $\hat{u}$. For other binary menus $\{x_i, x_{i+j}\}$, $j \geq 2$:

$$\begin{aligned} \frac{\rho(x_i, x_{i+j})}{\rho(x_{i+j}, x_i)} &= \frac{\rho(x_i, x_{i+1})}{\rho(x_{i+1}, x_i)} \cdots \frac{\rho(x_{i+j-1}, x_{i+j})}{\rho(x_{i+j}, x_{i+j-1},)} \\ &= \frac{exp(\hat{u}(x_i))}{exp(\hat{u}(x_{i+1}))} \cdots \frac{exp(\hat{u}(x_{i+j-1}))}{exp(\hat{u}(x_{i+j}))} = \frac{exp(\hat{u}(x_i))}{exp(\hat{u}(x_{i+j}))}. \end{aligned} \tag{3.5}$$

We then consider menus with three or more alternatives. By Cyclical Independence, $\frac{\rho(x_i, A)}{\rho(x_j, A)} = \frac{\rho(x_i, x_j)}{\rho(x_j, x_i)} = \frac{exp(\hat{u}(x_i))}{exp(\hat{u}(x_j))}$ for any $x_i, x_j \in f(A)$. We combine this with $\Sigma_{x \in f(A)} \rho(x, A) = 1$ to derive that

$$\rho(x, A) = \frac{exp(\hat{u}(x))}{\Sigma_{x' \in f(A)} exp(\hat{u}(x'))}.$$

Define $\hat{f} : K \to K$, where

$$\hat{f}(A) = \underset{B \subseteq A}{argmax} \; \ln(\Sigma_{x \in B} \exp(\hat{u}(x))) - \hat{c}_A(|B|).$$

[49]The ides is just to make attention costs sufficiently small for consideration sets with sizes less than or equal to $|f(A)|$, and sufficiently large for sizes beyond $|f(A)|$.

We want to show $\hat{f} = f$. We first show for any $A \in K$, $|\hat{f}(A)| = |f(A)|$. $|\hat{f}(A)| \geq |f(A)|$ since $\hat{c}_A(|B|) = 0$ for any $|B| \leq |f(A)|$. $|\hat{f}(A)| \leq |f(A)|$ since $\hat{c}_A(|B|) = \ln(\Sigma_{x_i \in A} \exp(u(x_i)))$ for any $|f(A)| < |B| \leq |A|$. We then show for any $A \in K$, if $x \in f(A)$, then $x \in \hat{f}(A)$. By Lemma 8, $\succ$ is a total order. By IIC, $f(A)$ is the first $|f(A)|$ alternatives in A with respect to $\succ$. If $x \in f(A)$, then x is among the first $|f(A)|$ alternatives in A with respect to $\succ$. Since $\frac{\rho(x_i,x_j)}{\rho(x_j,x_i)} = \frac{exp(\hat{u}(x_i))}{exp(\hat{u}(x_j))}$, $x_i \succ x_j$ if and only if $\hat{u}(x_i) > \hat{u}(x_j)$. Therefore, $\hat{u}(x)$ is among the first $|\hat{f}(A)|$ largest utilities in A. This means $x \in \hat{f}(A)$.

3.5.2 Theorem 3

For any $x_i \in A \in K$, $A^{\uparrow}_{x_i} = \{x_j \in A | \rho(x_j, x_i) > \frac{1}{2}\}$ is the set of alternatives in A that are chosen more frequently than x_i in their binary menus. IIC implies that alternatives in $A^{\uparrow}_{x_i}$ are included before x_i.

Necessity

We show that Dominance Consistency satisfies. Since $\rho(x_j, A') > 0$ and $|f(A')| \geq 3$,

$$\bar{c} \leq \ln(\Sigma_{x' \in f(A')} \exp(u(x'))) - \ln(\Sigma_{x' \in f(A') \setminus \{x_j\}} \exp(u(x'))). \tag{3.6}$$

Since $\rho(x_i, A) = 0$,

$$\bar{c} > \ln(\Sigma_{x' \in f(A)} \exp(u(x')) + \exp(u(x_i))) - \ln(\Sigma_{x' \in f(A)} \exp(u(x'))). \tag{3.7}$$

Therefore,

$$\frac{\exp(x_j)}{\Sigma_{y' \in f(A') \setminus \{x_j\}} \exp(u(y'))} > \frac{\exp(x_i)}{\Sigma_{x' \in f(A)} \exp(u(x'))} \implies \Sigma_{x' \in f(A)} \frac{\rho(x', x_i)}{\rho(x_i, x')} > \Sigma_{y' \in f(A') \setminus \{x_j\}} \frac{\rho(y', x_j)}{\rho(x_j, y')}.$$

Sufficiency

We first pin down $\hat{u} : X \to R$. Let $\hat{u}(x_1) = 1$, and pick $\hat{u}(x_i)$, $i \in \{2, \cdots, N\}$ sequentially

$$\hat{u}(x_i) = \hat{u}(x_{i-1}) + \ln(\frac{\rho(x_i, x_{i-1})}{\rho(x_{i-1}, x_i)}).$$

The proof that $\rho(x, A) = \frac{\exp(\hat{u}(x))}{\Sigma_{x' \in f(A)} \exp(\hat{u}(x'))}$ for any $x \in f(A)$ is the same with Theorem 2. We next pin down $\hat{c}$. Let $\hat{c} = \ln(\frac{1}{1-\underline{\rho}})$, where $\underline{\rho} = \min_{\rho(x_i, A) > 0, |f(A)| \geq 3} \rho(x_i, A)$.

Define $\hat{f} : K \to K$, where

$$\hat{f}(A) = \underset{B \subseteq A, |B| \geq 2}{argmax} \ \ln(\Sigma_{x \in B} \exp(\hat{u}(x))) - (|B| - 2)\hat{c}.$$

We want to show $\hat{f} = f$.

If $x \in f(A)$ and $|f(A)| \geq 3$, then $\rho(x, A) \geq \underline{\rho}$. Therefore, $\hat{c} \leq \ln(\frac{1}{1-\rho(x,A)}) \implies \hat{c} \leq \ln(\frac{\Sigma_{x' \in f(A)} \exp(\hat{u}(x'))}{\Sigma_{x' \in f(A)} \exp(\hat{u}(x')) - \exp(\hat{u}(x))})$. By IIC, $A_x^{\uparrow} \subset f(A)$. Therefore,

$$\hat{c} \leq \ln(\frac{\Sigma_{x' \in A_x^{\uparrow}} \exp(\hat{u}(x')) + \exp(\hat{u}(x))}{\Sigma_{x' \in A_x^{\uparrow}} \exp(\hat{u}(x'))}).$$

From the construction of $\hat{u}$, for any $x' \in A$, $\hat{u}(x') > \hat{u}(x)$ if and only if $x' \in A_x^{\uparrow}$. Therefore, $x \in \hat{f}(A)$.

If $x \in f(A)$ and $|f(A)| = 2$. By IIC, $A_x^{\uparrow}$ is empty or singleton. Since $\hat{u}(x') > \hat{u}(x)$ if and only if $x' \in A_x^{\uparrow}$. Therefore, $x \in \hat{f}(A)$.

If $x \notin f(A)$, by Dominance Consistency,

$$\Sigma_{x' \in f(A)} \frac{\rho(x', x)}{\rho(x, x')} > \Sigma_{y' \in f(A') \setminus \{x_j\}} \frac{\rho(y', x_j)}{\rho(x_j, y')},$$

where $\rho(x_j, A') = \underline{\rho}$. By IIC, $f(A) \subseteq A_x^{\uparrow}$. This implies $\ln(\frac{\Sigma_{x' \in A_x^{\uparrow}} exp(\hat{u}(x')) + exp(\hat{u}(x))}{\Sigma_{x' \in A_x^{\uparrow}} exp(\hat{u}(x'))}) < \hat{c}$. Since $\hat{u}(x') > \hat{u}(x)$ if and only if $x' \in A_x^{\uparrow}$. Therefore, $x \notin \hat{f}(A)$.

3.5.3 Lemma 6

Without loss of generality, we assume a_1 is in the first group, and a_2 is in the second group. For any $i \geq 3$, let $\mu(i) = j, j = 1, 2$ if a_i is in the jth group. Let

$$a_i = \begin{cases} a_1 + y_i & \text{if } \mu(i) = 1 \\ a_2 + y_i & \text{if } \mu(i) = 2 \end{cases}.$$

To find a_1 and a_2: $a_1 < a_2$, $a_1 + y_i < a_2 + y_{i+1}$ whenever $\mu(i) = 1$, $\mu(i+1) = 2$, and $a_2 + y_i < a_1 + y_{i+1}$ whenever $\mu(i) = 2$, $\mu(i+1) = 1$. To this end, collect all the inequalities:

$$\max_{i \geq 3, \mu(i)=2, \mu(i+1)=1} \{y_i - y_{i+1}\} < a_1 - a_2 < \min_{i' \geq 3, \mu(i')=1, \mu(i'+1)=2} \{0, y_{i'+1} - y_{i'}\}.$$

For any $i \geq 3$ with $\mu(i) = 2$ and $\mu(i+1) = 1$: $y_i - y_{i+1} < 0$. We show for any $i, i' \geq 3$ with $\mu(i+1) = \mu(i') = 1$ and $\mu(i) = \mu(i'+1) = 2$: $y_i - y_{i+1} < y_{i'+1} - y_{i'}$. If $i > i'$, $y_i - y_{i'+1} < y_{i+1} - y_{i'}$; if $i < i'$, $y_{i'} - y_{i+1} < y_{i'+1} - y_i$.

3.5.4 Non-constant marginal attention costs

We assume the marginal cost of adding alternatives is not fixed. In specific, there is a weakly increasing attention cost function $c(\cdot) : \{1, \cdots, |X|\} \to \mathbb{R}_+$ that satisfies Assumption 1. We will have a different version of Dominance Consistency to complete the representation theorem.

Axiom 11 (Dominancy Consistency)**.** *Let $A, A' \in K$ be such that $|f(A)| < |f(A')|$. Then for any $B \subset A \setminus f(A), B' \subset f(A')$ such that $|B| \leq |B'| = |f(A')| - |f(A)|$,*

$$\Sigma_{y' \in B'} \frac{1}{\Sigma_{y'' \in f(A') \setminus B'} \frac{\rho(y'', y')}{1 - \rho(y'', y')}} > \Sigma_{x' \in B} \frac{1}{\Sigma_{x'' \in f(A)} \frac{\rho(x'', x')}{1 - \rho(x'', x')}}. \tag{3.8}$$

The idea behind this axiom is still simple. If the consideration set for one menu is larger than another, then the marginal benefit of considering the additional alternatives in the first menu must be larger than the second should more alternatives are considered in the second menu.

Chapter 4

A Strategic Theory of Network Status

This chapter is coauthored with Brian Rogers.

4.1 Introduction

People devote much of their time to maintaining relationships with each other. In addition to having certain qualities independent of their social position, some aspects of what a person has to offer depend on who their acquaintances are. For instance, some people are more likely to pass on job offers because of their professional contacts, and others are likely to introduce two friends with common interests because they know many similar people. The process of network formation and its impact on the flow of information and services across individuals or organizations has broad implications. There is a vast array of contexts where the complex set of relationships among agents is an important, or even predominant, ingredient in determining consequences. The ease with which one finds information on the world wide web, the spread of infectious and electronic viruses, (un)employment dynamics, international trade, and co-authorship structures, to name just a few examples, have all been studied and shown to crucially depend on network structures.[50] From individuals' viewpoints, these relationships affect the range of available opportunities, the amount of information to which they have access, or their relative influence in the system. From a social perspective, the structure of relationships affects overall system performance, such as how quickly and widely information is transmitted, or how efficiently resources are allocated. Thus it is important to understand the incentives facing agents in these situations in order to gain insight into what kind of structures are likely to form, and how these relate to structures that are optimal for the group.

Much has been learned through the study of strategic network formation.[51] Yet the vast majority of the formal modeling of network formation treats links as binary quantities; that is, links are either present or absent.[52] While this adds tractability to the analysis, it is clearly a simplification of the kinds of links that are observed in many applications of interest. Relationships such as friendships, trade partnerships, and research collaborations are characterized not only by the presence or absence of pairwise interactions, but also by the intensity, frequency, or reliability with which they occur.

While binary links have proven to be a very useful approximation, the importance of considering links with different strengths, originating in the work of Granovetter (1973),

[50]See, for instance, Myers and Shultz (1951), Rees, Shultz, et al. (1970) for early studies of employment contacts, Calvo-Armengol and Jackson (2004), Calvó-Armengol and Jackson (2007), Galeotti and Merlino (2014) for models of employment dynamics, Adamic (1999) for analysis of world wide web data collected by James Pitkow that demonstrates short path lengths, Auerbach et al. (1984) for an analysis of contacts between homosexuals with AIDS, and Casella and Rauch (2002), Rauch and Casella (2003) for an analysis of international trade.

[51]Jackson and Wolinsky (1996) and Bala and Goyal (2000) are seminal contributions on which much of this literature is based. See Jackson (2005), Bloch and Dutta (2011) for thorough reviews.

[52]Several papers, including Bloch and Dutta (2009), Deroïan (2009), Baumann (2021), Salonen (2015), Brueckner (2006), Griffith (2020), and Ding (2019), explicitly consider network formation with continuous link decisions. Goyal (2005) discusses implications of allowing for more general link qualities. Goyal, Moraga-González, and Konovalov (2008) considers a special form of variable strength links in the context of joint R&D investments among firms, and derive a number of interesting predictions in that setting. Calvó-Armengol and Jackson (2007) model employment dynamics in a network setting that allows for very general information structures which can be interpreted as a weighted graph.

Granovetter (1983) has been acknowledged for some time. From an economic perspective, allowing for the possibility of variable-strength links adds two important ingredients: (i) the ability to study finer details of individual linking decisions and how they adjust to (small) changes in the underlying characteristics of others, and (ii) the possibility to analyze macro-level consequences of the tradeoffs involved in adjusting resources across an actor's maintained links, including what kinds of configurations are stable and efficient for the group in this richer environment.

In settings where network structures are important, some kind of service or information is transferred across actors. When two actors are connected, they each typically receive benefits from the other (though possibly in an asymmetric way). Thus an actor's investment in a link affects the direct flow of benefits not only to himself but also to the other actor involved in the relationship. In the analysis below, we separate these directions of benefits flow into "taking" and "giving". In Model T (for "taking"), agents receive benefits through the links they create, whereas in Model G (for "giving"), the direction of the benefit is the opposite.

The combination of values derived from the exogenous measure and the endogenous network pattern is the object of study here. To understand the way in which benefits accrue to agents through the network, consider three points. First, individuals allocate their budgets across links and realize diminishing marginal returns to investment in a particular link.[53] Second, the benefit that one individual receives from another is the product of the *value* of the other agent and the strength of the relevant link. Thus, more valuable agents and stronger links each confer a greater benefit to agents. Third, each agent's value is the sum of an intrinsic value and the benefits from all of the connections to other agents. Thus value is achieved both through intrinsic value and through strong connections to high-value agents.

[53]The constant marginal returns case is also handled.

Separating these effects into the taking and giving directions proves useful for analyzing equilibrium networks' efficiency properties. An important lesson is that the taking behavior is efficient, whereas the giving behavior need not be. In Model T, we derive the unusual result in the network formation literature that efficient networks and equilibrium networks must coincide. This result is true for both the diminishing marginal returns case and the constant marginal returns case. The coincidence of equilibrium networks and efficient networks fails for Model G. First, many inefficient equilibrium networks exist in Model G. Second, efficient networks and equilibrium networks are characterized by different first-order conditions with different interpretations. Efficient networks require agents to allocate their linking budgets across links according to other agents' *total* connectivities, whereas equilibrium choices require agents to allocate budgets across links according to other agents' *individual* connectivities. Finally, when there are only three agents, an efficient equilibrium network exists if and only if agents have identical linking budgets.

To motivate this paper's way of constructing utility, consider an attempt to quantify scholarly potential among academic economists. One measure of quality is represented by the CV, which contains, e.g., a publication record. This corresponds to the intrinsic quality of individuals. Another important consideration, though, is the content of recommendation letters for hiring or promotion decisions, which represent the choice variables in the model. Clearly, recommendations from different kinds of individuals should be treated differently. In particular, recommendations from people who are "highly regarded" should contribute more to an individual's status. In the model, being highly regarded comes, itself, from a combination of intrinsic value (i.e., good publications) and strong recommendations from other highly regarded individuals.

The benefit from a link depends on the value of the agent linked to, and the value of the agent depends on its set of links and the values of its neighbors, and so forth. Consequently, given linking choices, the equations that determine network values must be solved simultaneously in order to compute utilities. Importantly, this is true despite the fact that an agent's utility depends explicitly on only those links that directly involve the agent. The implicit dependence on the remaining links enters through the values of those to whom the agent is directly connected. One implication of this interdependence is the presence of "feedback effects," whereby the benefits associated with a particular link are counted many times. In the previous example, one can think of the feedback effects in the following terms: if individual i recommends individual j, then j's status increases and (s)he is more likely to get a good job, whereupon any recommendation or help that j provides to i (or any other agent) becomes more valuable.

Previous work has limited the extent to which link externalities are accounted for. Consider the "connections model" of Jackson and Wolinsky (1996), where the utility one agent derives from another is a function of the length of the shortest path between the agents. There is no added value of having multiple paths (of any length) between agents, even though redundancy may be an important consideration. In contrast, in the present model a small increase in the intensity of a link not only has a positive effect on the agents involved in the link, but also has an indirect positive effect on every other agent who is in some way connected to them. A second feature of the connections model is that the potential value of connecting to an agent is independent of the network structure. However, in some applications, one might desire, for instance, that this benefit depends on the degrees (number, or weight, of connections) of the nodes involved. Unlike the connections model, our model considers such factors in the utility calculation by assuming the recursive definition of values described above. In this sense our work is complementary to the literature that has stemmed from variations on the connections model.

Another contribution is that heterogeneity among agents is allowed along two dimensions. Agents may differ both in the measure of intrinsic quality or skill, and also in the amount of resources they can expend forming links.[54] Such heterogeneity is both natural and important for the model. Some individuals have more social interest or ability, and naturally form stronger links to others, while others are more valuable contacts for reasons independent of their social connections. These features of the model admit a more interesting analysis of the strategic considerations involved in link formation. For instance, when exogenous qualities are homogenous, the efficient networks are the same under giving and taking behavior, but the distribution of value can be very different. Second, in Model G with identical linking budgets, there always exists an efficient equilibrium network. Third, increasing either intrinsic value or the linking budget is beneficial to an agent. Whereas the exact tradeoff between them depends on the particular shape of the link investment function in the diminishing marginal returns case, we derive the exact tradeoff for the constant marginal returns case for Model T.

The paper proceeds as follows. The next section, Section 2, formally presents the framework for modeling the network game. The main body of results, characterizing which network structures are outcomes of Nash equilibria and which maximize aggregate utility, is contained in Section 3. Section 4 interprets the network values and connects them with various concepts in related work and discusses several applications. Finally, Section 5 concludes. All proofs, along with an analysis of the constant marginal returns case, are included in an Appendix.

4.2 A model of link formation

There is a finite set of agents $N = \{1, \cdots, n\}$, $n \geq 2$ endowed with publicly observed intrinsic values $\alpha = (\alpha_1, \cdots, \alpha_n)' \in R^n_+$ and budgets $\beta = (\beta_1, \cdots, \beta_n)$, with $0 < \beta_i < 1$ for each $i \in N$.

[54] In the binary link case, Galeotti (2006), extending the work of Bala and Goyal (2000), allows for similar kinds of heterogeneity. He finds that such heterogeneity is important in determining equilibrium network configurations.

The agents are players in a network formation game with strategies $\phi_i = (\phi_{i1}, \cdots, \phi_{in})$ that satisfy $\phi_{ii} = 0, \phi_{ij} \geq 0$ for all $i, j \in N$ and $\Sigma_j \phi_{ij} \leq \beta_i$, for all $i \in N$. A strategy profile is represented by a matrix $\Phi = [\phi_{ij}]_{i,j}$.

The intensity of the directed link ij is given by $f(\phi_{ij})$, where $f : [0, 1] \to R$ is continuous, continuously differentiable on $(0, 1)$, strictly increasing, and satisfies $f(0) = 0$. In addition to that, we make the following assumptions about f.

Assumption 2. *The function f is strictly concave,* $\lim_{x \to 0} f'(x) = \infty$*, and satisfies* $(n - 1)f(\frac{1}{n-1}) \leq 1$.

Assumption 2 reflects the notion of diminishing marginal returns to investment in a link. The first-order conditions characterizing optimal tradeoffs across links are very simple to write down under Assumption 2. The last condition in Assumption 2 means that the total link intensity from an agent is always strictly less than 1.[55] We also consider an alternative formulation in which the function f is the identity mapping. That formulation reflects the notion of constant marginal returns to investment in a link. In that case, the intensity of the directed link ij is simply ϕ_{ij}.

The bulk of the analysis hinges on which direction benefits flow through the network. In Model T (for "taking") agent i setting $\phi_{ij} > 0$ confers benefits to i from j. One can think of i placing queries to others in proportion to ϕ_{ij} asking for information. Given the linking choices and intrinsic values of others, agent i chooses the relative amount of benefits to receive from others by allocating her budget of resources. The opposite case is handled in Model G (for "giving"). There, agent i setting $\phi_{ij} > 0$ confers benefits to j from i. In this case, one can think of ϕ_{ij} as representing a nomination or recommendation of j by i, as in the case of

[55]It is a sufficient condition to guarantee that the utilities are well-defined when all paths of any length add value.

journal article citations. In this case, an agent chooses how to confer benefits, which will have indirect benefits to her given the dictated by the choices of others.

The description of the network formation game is completed by formally defining how network structures generate utilities. To this end, let $f(\Phi)$ denote the matrix with elements $f(\phi_{ij})$, i.e., $f(\Phi)$ is the adjacency matrix of the network structure generated by strategy profile Φ. Total utility is the sum of intrinsic values and the value derived through sharing others' values via the network structure. Let v_i denote the (endogenously determined) component of i's utility derived from her relations in the network $f(\Phi)$, so that i's (total) utility is $u_i = \alpha_i + v_i$. In Model T,

$$v_i = \sum_j (\alpha_j + v_j) f(\phi_{ij}),$$

whereas in Model G,

$$v_i = \sum_j (\alpha_j + v_j) f(\phi_{ji}).$$

Notice that network values are linear combinations of the total utilities of other players, weighted by the intensity of the corresponding link (which varies according to the taking or giving direction of the model). Thus, for instance, in Model T, $u_i = \alpha_i + v_i = \alpha_i + \sum_j u_j f(\phi_{ij})$. Collecting these equations in matrix notation, we obtain $u = \alpha + f(\Phi)u$, where $u = (u_1, \ldots, u_n)'$ is the column vector of utilities. Solving for u yields $u = (I - f(\Phi))^{-1}\alpha$, provided $I - f(\Phi)$ is invertible.[56] Letting $A = (I - f(\Phi))^{-1}$, we have $u = A\alpha$. A similar derivation for Model G produces $u = A'\alpha$. Section 4 contains a more thorough interpretation of the value computation and how it is related to various other models in the literature.

[56] $I - f(\Phi)$ is invertible since it is strictly diagonally dominant under Assumption 2.

Notice that under Assumption 2 the requirement on f implies that

$$A = \Sigma_{p=0}^{\infty} f(\Phi)^p.$$ [57]

That is, A can be written as the infinite sum of powers of the network structure matrix $f(\Phi)$. Each of these matrices, in turn, has a natural interpretation. $(f(\Phi)^p)_{ij}$ measures the total weight of all directed paths from i to j that have length p. Thus a typical element a_{ij} in A represents the total weight of *all* directed paths of all lengths from i to j. Because individual links take values strictly less than unity, longer paths have less weight.

4.3 Results

This section contains the main results of the analysis. In particular, we characterize Nash equilibrium networks and efficient networks in both models. Following the literature on strategic network formation, we restrict attention to pure strategy Nash equilibria. Formally, Φ is a Nash equilibrium profile if no player can strictly increase her utility by deviating to another strategy, given all other players' strategies. If Φ is a Nash equilibrium profile, then $f(\Phi)$ is a Nash network. $f(\Phi)$ is an efficient(utilitarian) network if and only if Φ maximizes the aggregate utility $U = \Sigma_{i \in N} u_i$ among all strategy profiles.

Most of the networks we focus on for the diminishing marginal returns case are interior networks, in which all links have positive value: $\phi_{ij} > 0$ for all distinct $i, j \in N$. One can think of these networks as a generalization of the complete network from the binary link setting. We proceed with two propositions for Model G, characterizing the Nash networks and the efficient networks, respectively.

[57] The last condition in Assumption 2 guarantees that the row sums of the non-negative matrix $f(\Phi)$ are all strictly less than 1 for any Φ. In this case, the spectral radius of $f(\Phi)$ is strictly less than 1. Therefore, $\lim_{p \to \infty} f(\Phi)^p = 0$.

Proposition 7. *In Model G, there exist multiple Nash networks. $f(\Phi)$ is an interior Nash network if and only if $\Sigma_j \phi_{ij} = \beta_i$ for all $i \in N$, and*

$$f'(\phi_{ij})a_{ji} = f'(\phi_{ij'})a_{j'i} \tag{4.1}$$

for all distinct $i, j, j' \in N$. Non interior Nash networks have the form that a partition of players exists such that there are no positive links between players in different partitions, and links within a partition constitute an interior Nash network restricted to that group. If f additionally satisfies that $\lim_{x\to 0} xf'(x) = \infty$, then a Nash network exists for any partition of players.

Proposition 8. *In Model G, an efficient network exists, is interior, and satisfies the conditions $\Sigma_j \phi_{ij} = \beta_j$ for all $i \in N$, and*

$$f'(\phi_{ij})\Sigma_k a_{jk} = f'(\phi_{ij'})\Sigma_k a_{j'k} \tag{4.2}$$

for all distinct $i, j, j' \in N$.

The characterizations in Propositions 7 and 8 shed light on the (possible) differences between Nash and efficient networks in Model G. Notice that there are many non-interior Nash networks in Model G, for instance, the empty network, which corresponds to a partition of singletons. The reason is related to the common misccordination problem in mutual link formation setting, that if an individual i has no paths pointing to her, i.e., if $\phi_{ji} = 0$ for all j, then i is indifferent over her entire strategy space. More generally, in a Nash network, $\phi_{ij} = 0$ implies that $a_{ji} = 0$, i.e., that there are no paths of any positive weight from j to i. All these non-interior Nash networks are inefficient.

The difference between interior Nash networks and efficient networks arises because for efficiency, linking choices are determined by total connectivities $\sum_k a_{jk}$, whereas the equilibrium choices depend on individual connectivities a_{ji}. This difference is somewhat alleviated with identical linking budgets. Efficiency requires the following regularity property to hold: the ratio of marginal linking choices to agents j and j', $\frac{f'(\phi_{ij})}{f'(\phi_{ij'})}$, is constant across individuals i. When budgets are homogeneous, it turns out that the *strongly regular* network, where $\phi_{ij} = \frac{\bar{\beta}}{n-1}$ for all $i, j \in N$, is the unique efficient network. But in this case, the strongly regular network is also a Nash network.[58]

The next result further demonstrates the difference between Nash and efficient networks in Model G with heterogeneous linking budgets.

Proposition 9. *Assume $N = 3$. In Model G, there exists an efficient Nash network if and only if $\beta_i = \bar{\beta}$ for all $i \in N$. In addition, the unique efficient Nash network in this case is the strongly regular network.*

The possibility of tension between equilibrium and efficiency, however, is not present in Model T. The next result shows that Nash and the efficient networks coincide for all parameters under Model T.

Proposition 10. *In Model T, Nash networks and the efficient networks coincide. The efficient Nash network is unique. It is interior, satisfies the conditions $\Sigma_j \phi_{ij} = \beta_i$ for all $i \in N$, and*

$$f'(\phi_{ij})u_j = f'(\phi_{ij'})u_{j'} \tag{4.3}$$

for all distinct $i, j, j' \in N$.

[58]Interestingly, there may also exist asymmetric interior Nash networks even in symmetric environments. Whether or not this occurs depends on the particular shape of f. If it is sufficiently concave, then only the strongly regular network is a Nash network.

The condition in Proposition 10 has a particularly simple structure. Individuals allocate their budgets so as to balance the marginal values of their *direct* links, treating the utilities of others as fixed. Model T produces the unusual result that Nash networks and efficient networks must coincide. The intuition for this fact can be summarized as follows. Each individual i makes her linking choices in order to maximize her derived network value. This means sending more effort to individuals with higher values, but sending some effort to everyone because $\lim_{x\to 0} f'(x) = \infty$. Now consider any other individual j linking to i. j also makes her decisions selfishly, and benefits from having high value individuals to link to. It turns out that i's optimal decision is also optimal from j's (and every other individual's) point of view. Put differently, if j were given the power to choose ϕ_i as well as ϕ_j, j's choices for i would be the choices that i finds optimal herself. This can be verified directly by considering the conditions $\frac{\partial u_j}{\partial \phi_{ik}} = \frac{\partial u_j}{\partial \phi_{ik'}}$, which generate the same characterization as i's optimization problem.

We also show that Nash networks and the efficient networks also coincide for the constant marginal returns case. In fact, an efficient network must be a Nash network in Model T as long as the utilities are well defined. To see this, notice that the argument in the previous paragraph indicates that the aggregate utility $U : \Phi \to R$ is an ordinal potential for the network formation game of Model T. It is known that if the ordinal potential admits a maximal value in the strategy profile set, then the corresponding game admits a pure-strategy equilibrium. Since the ordinal potential is the aggregate utility; efficient networks must be Nash networks. Our proof of the non-existence of inefficient Nash networks depends on the fact that the total link intensity from an agent is strictly less than one. It is not clear to us whether an inefficient Nash network exists without this assumption.

The coincidence of Nash and efficient networks, of course, fails for Model G. Under giving behavior, individuals control their utility only indirectly, in the sense that their linking choices confer direct benefits to others, rather than to themselves. Thus, to optimize, an individual i

sends more effort to those agents who have the strongest connectivity back to i. When other individuals make their linking choices to i, they care not what i's network value is, as in Model T, but instead on how strong the connectivities are from i back to themselves. Thus the linking choices of i are not in general optimal from any other agent's perspective, and this causes the tension between equilibrium and efficiency considerations. Another observation that this logic highlights is that the first-order conditions in Model G for equilibrium and efficiency do not depend on intrinsic values α, but only on the network, as measured by the entries in A. We have the following result.

Corollary 3. *In Model G, Nash and the efficient networks are independent of the intrinsic values α.*

The next result shows that aggregate utility at the efficient networks are the same under taking and giving when agents have homogeneous intrinsic values.

Corollary 4. *When $\alpha_i = \bar{\alpha}$ for all $i \in N$, the efficient network in Model T and Model G coincides.*

However, there is not a clear relationship between the distribution of utilities. Under the taking model, utilities are ranked the same way as budgets, but this need not be true under giving. As mentioned above, the reason for this is that individuals care only about the network structure when making their linking decisions, and not on the intrinsic values of the agents they link to. In Model T, however, changing α changes utilities both through the direct effect, and also through an equilibrium effect, whereby optimal decisions change in response to the exogenous change.

4.3.1 Empty and complete networks

This subsection presents some examples to further illustrate how the forces of the model operate. In particular, we are interested in identifying how particular configurations of modeling parameters map into stylized network architectures that have attracted attention in previous work. In a setting that incorporates variable link intensities, the set of networks is considerably more complex than in the binary link case.

Bala and Goyal (2000) consider a binary link model that is close to Model T. These authors call this "one-way flow," but do not consider the other direction of flow corresponding to Model G. Instead, they also consider "two-way flow," in which a directed link confers benefits to both agents involved.

In binary link models, when costs are sufficiently high, empty networks are both efficient and are an equilibrium outcome. The counterpart to high costs in this context is small linking budgets, in which case all feasible networks converge to an empty network as budgets vanish. In addition, the empty network is Nash in Model G for budgets of every size. The reason for this difference is that the benefit flow in previous work corresponds instead to the taking behavior of Model T.

At the other extreme, provided there is some decay (to allow for networks that are not minimally connected), complete networks emerge for sufficiently small costs. In the present context most Nash networks, and all efficient networks, are interior, which is one form of "completeness," in that every link has positive strength. However, a stronger analogue of the complete binary network is the regular network with "equal spread," in which $\phi_{ij} = \frac{\beta_i}{n-1}$ for all distinct $i, j \in N$. In this case, every link is given equal weight, so that all pairs of agents are directly connected and the network is symmetric in a rather strong sense.

This arises in Model T when both budgets and intrinsic values are homogeneous. The underlying symmetry of agents implies that the unique efficient Nash network is the strongly regular network. However, it is also possible for a regular network to occur in the presence of budget and quality differences across agents, provided that they balance each other in exactly the right way. That is, if an agent has a high intrinsic value, she must have a correspondingly low linking budget in order to induce the other agents to treat her the same as the others. For example, consider 3 agents with intrinsic values $\alpha = (3, 2, 2)$. When budgets are $\beta = (0.015, 0.1, 0.1)$, and the investment function is $f(x) = \sqrt{x}$, each agent splits her linking budget equally among the other two.[59]

Model G can also generate regular networks. In fact, it is in some sense biased in that direction because of the fact that the optimal network structures do not depend on intrinsic values. Thus, in contrast to Model T, differences in qualities do not translate into differences in linking choices. When agents have identical budgets, the strongly regular network is the unique efficient Nash network. But there can be inefficient asymmetric interior Nash networks even with identical budgets. For instance, when $f(x) = \delta x^\lambda$, and λ is sufficiently close to one, a 3-agent equilibrium exists in which two agents devote a vast majority of their budget to the third agent, who divides her own budget equally.

Another important network structure that has attracted attention in previous work is the star network. Since the networks identified in the diminishing marginal returns case are mostly interior, there is no direct correspondence. However, the star network turns out to be the predominant structure in Models T and G with constant marginal returns of link investment.

[59] However, it is possible that one agent's intrinsic value is sufficiently high that no matter how low her corresponding budget, agents always devote a majority of their effort to the high-value agent. This is the case in the above example if α_1 is increased to 4. The reason is that one agent can become arbitrarily more valuable than the others even on her own without any connections to the value of others.

4.4 Interpretation of network value and related literature

The equations describing the computation of values are simple, but some of the implications deserve comment. In this section, we first provide some interpretation of the model and relate it to a number of ideas proposed in other contexts. Second, we discuss applications where giving and taking, respectively, are predominant considerations. Finally, we discuss the literature on strategic network formation with weighted links.

4.4.1 Network centrality, interpersonal influence, and interdependent utilities

The idea that value is determined in part by the value of one's neighbors creates the necessity for values to be determined simultaneously, since each individual's value may depend, through paths in the networks, on the values of all other agents. Consider the following illustration. If an agent i connects to another agent j, i derives benefit proportional to j's utility from this link. Agent j's utility is determined in part by how well-connected she is. Thus if j strengthens a connection to a third agent k, j's value, and hence i's utility, increase as well. Furthermore, if agent k increases her value by strengthening one of her connections to yet another agent, this also has a positive externality on i's utility through an increase in agent j's value, since j's neighbor k became more valuable. Of course, this effect is in addition to effects on i through other paths. This logic shows that an increase in the quality of any link increases the utilities of all agents who are not completely disconnected from the augmented link, since an increase in any link quality brings the whole world closer together, benefiting everyone. In this sense the model may be said to take full account of network externalities, since a change in the quality of a link that does not involve a particular agent still has an affect on her utility, which is a feature that most strategic models studied in previous literature have failed to accommodate.

At the same time, this idea has motivated a literature in the study of social networks which attempts to measure the importance, status, influence, or centrality of a node in a network. The earliest notions of centrality in the social network literature worked on binary networks and did not take all these considerations into account. For instance, one simple measure of centrality is a node's degree, the number (or total weight) of incoming paths. While degree centrality is a useful concept, and certainly captures important features of a node's position, it ignores the origin of the incoming paths.[60] In an effort to allow for such a calculation, Katz (1953) developed the first notion that resembles the calculation of values in this model. If Φ is a binary network of relationships, the measure in Katz (1953) is a scaled version of $(I - \delta\Phi)^{-1}\alpha$, where α is a vector of ones, analogous to homogeneous intrinsic values, and $\delta < 1$ is an attenuation factor. Such an attenuation factor is necessary when dealing with binary networks rather than the weighted graphs studied here or else the contributions from longer paths cause the status measure to diverge to infinity. The fact that the links take fractional values in the present paper subsumes the role of the attenuation factor. The main innovation of Katz's approach to centrality is that it allows all paths in the network to be considered, rather than calculating the centrality score based on some subset of links, e.g., the set of links pointing to a given node in the case of degree centrality.

Building on Katz's work, Hubbell (1965) introduced the measure $(I - f(\Phi))^{-1}\alpha$, where again, α is assumed to be a vector of ones, but which operates directly on a weighted graph $f(\Phi)$.

[60] Similar comments can be made about many other notions of centrality, including closeness, betweenness and eccentricity, all of which do not take into account a node's importance as a function of its position in the network when computing its contribution to the centrality scores of others. Wasserman, Faust, et al. (1994) (Ch. 5) provide an excellent review of this literature, and Freeman (1978) discusses their differences in applicability.

As Hubbell notes, this formulation mathematically resembles the input-output model of interlinked sectors in an open economy due to Leontief (1951).[61] Hubbell's work introduces a measure of centrality that very closely resembles the formulation of utility here. While Katz's goal was to derive a measure of status, Hubbell aimed his work at clique identification by analyzing the spectral properties of $(I - f(\Phi))^{-1}$ to identify closely connected subgroups. While such an analysis is beyond the scope of this paper, an interesting question is what kind of clique structures form in the strategic setting studied here.

Related measures have gained popularity through the influential work of Bonacich (1972), Bonacich (1987), Bonacich and Lloyd (2001) and others. Bonacich's measure that is most relevant to this model, in that it operates on weighted and asymmetric graphs, is essentially the eigenvector corresponding to the largest eigenvalue of the network $f(\Phi)$. A number of similar indices have been proposed by various authors; Poulin, Boily, and Mâsse (2000) provide a useful review and comparison. Interestingly, many of the measures can be calculated in a number of ways, including solving a linear system of equations, computing an infinite sum of status contributions along paths of different lengths, and a factor analytic approach, thus providing various ways to interpret the centrality scores.

One way to understand our formulation of utility here is to view it as the outcome of a repeated process of interactions occurring along the links of the network. This is the approach taken by Friedkin (1991) Friedkin (2006) and Friedkin and Johnsen (1990) in studies of interpersonal influence. Under this approach, α is a vector of exogenous qualities, which are shared across links during an infinite sequence of time periods. At date t, the interaction is described by $u^t = \alpha + f(\Phi)u^{t-1}$, with $u^0 = \alpha$. At each date, individuals receive a contribution from their own exogenous quality, and contributions proportional to their links to others,

[61] Some of the tools useful for analyzing these models originated from the study of Leontief's model, and in general rely on results about linear systems, a good review of which is Gale (1989).

and others' acquired utility at the time.[62] Under our assumptions, the limit outcome of this process is $u^\infty = (I - f(\Phi))^{-1}\alpha$. As this is the expression for utility functions in the model, utility can be view as the long run result of repeated interactions through the network.

This formulation makes it obvious that the utility vector, or limit outcome of the dynamic process, dominates the vector of exogenous quality α. In the context of interpersonal influence, Friedkin (1991) refers to this effect as polarization, meaning the tendency of opinions to drift towards a more extreme position due to repeated social interactions (rather than to a separation of opinions within the group). The mathematical reason for this is that the value in α is counted multiple times in the computation of the utility vector. For our purposes, such a formulation captures synergies of interactions. For instance, research collaborations and joint R&D projects are often thought to create output beyond what the parties would generate independently. Katz (1953) rejected this kind of effect, and proposed instead to normalize the status measure explicitly, so that his measure has a more relative interpretation.

The fact that the model here produces Hubbell's status score as utilities provides an interesting re-interpretation of status. That is, in this model agents seek to maximize their status when forming their links, as measured in the way suggested by sociologists. Since the analysis here is strategic, one of the benefits of this paper is to understand what kinds of networks form when individuals are seeking explicitly to maximize their status in the group. The results on efficiency then highlight when such networks will and will not be efficient, in terms of maximizing the overall status of the group.

[62]In this intertemporal formulation, it is not clear that the Nash equilibrium of the game with utility functions u is the most compelling stability notion, since repeated game considerations may become important. While such an analysis is outside the scope of this paper, the game corresponding to utilities u is relevant if players are patient, the choices must be made once and for all, and interactions take place repeatedly thereafter.

This paper is not the first to make an economic connection to the centrality literature. A very different connection is contained in Ballester, Calvó-Armengol, and Zenou (2006), in which the equilibrium action is proportional to Bonacich centrality, taking the network as given. In addition, Martı Beltran (2005), studies a question that is complimentary to our analysis in that it asks what the optimal way is to distribute intrinsic value among the agents as a function of an exogenously given network.

It is also interesting to note that the model is mathematically quite similar to some of the work on interdependent utilities, as seen in Bergstrom (1989), Bergstrom (1999) and Bramoullé (2001).[63] Under that interpretation, the links in the network represent parameters in utility functions that describe how individuals' consumption utility affects the happiness of others. Again the analysis here is very different principally because that work always takes the network structure as given, and addresses interesting questions that do not arise in the present paper such as the effect of "negative links." On the other hand, this paper takes individual attributes as given, and asks which network structures form Nash equilibrium outcomes, and which structures maximize aggregate utility in the group.

4.4.2 A few applications

We briefly describe a few applications in which either giving or taking is a naturally dominant consideration to illustrate that, in addition to the benefit of separating these components for the evaluation of efficiency, the taking and giving models are themselves relevant in certain contexts.

Recall the example about scholarly potential from the Introduction. Here, links represent the giving of value, as it is the person receiving the recommendation who benefits, and so Model

[63] Other contributions in this field include Hori (1997) and Shinotsuka (2008).

G is most relevant. Constraints on letter writing capacity reflect the fact that individuals typically only have time enough for several letters in a given period and can only exert a limited amount of influence. Given the budget, individuals must choose how to allocate their recommendation influence across others.

Model G is also relevant in other contexts where the quantification of reputation is important. Examples include measuring the imputation of trustfulness in online P2P networks, the influence of scientific journals, and determining popularity in social friendship networks. In all these cases agents choose who to "nominate" or promote by their actions. Calculations similar to the framework presented here have indeed been utilized in some of these contexts.[64] In the case of P2P networks, which remain prevalent in online applications, an important issue is how to trust certain network nodes. A system of recommendations provides one answer to this problem. The model predicts what kinds of recommendation patterns are likely to emerge when nodes make their decisions in order to increase their own trustworthiness. The result that the strongly regular networks are efficient equilibrium networks when recommendation budgets are homogenous has implications for how to set up such a system.

Model T can be viewed as a strategic setting for the model of interpersonal influence proposed by Friedkin (1991), Friedkin (2006), and Friedkin and Johnsen (1990). Influence is often a desired quality, and so it is natural to think of a model of influence in a strategic setting. In this context, α is interpreted as a vector that represents the intrinsic value of each agent, in terms of, say, the quality of information they possess. Agents choose how to spend their effort gaining access to the value of others. Budget constraints reflect the fact that some people are more active, or better able, to get value from others. Individuals who have much

[64] Beth, Borcherding, and Klein (1994) and Kamvar, Schlosser, and Garcia-Molina (2003) examine the case of trust networks and have results that are suggestive of the model proposed here. Palacios-Huerta and Volij (2004) formally study the problem of journal rankings.

asked of them are central. Our analysis thus ties equilibrium centrality to the underlying quality and budget constraints of the individual.

4.4.3 Literature on strategic weighted network formation

We discuss the literature on strategic network formation with weighted links. The network utilities defined in these papers are very different from the model we study. This paper's utilities are defined interdependently with a centrality flavor, providing a connection between the strategic network formation literature and the centrality measure literature.

Bloch and Dutta (2009) study communication network formation and assume agents allocate a fixed amount of endowment among other agents to form weighted links. For most of their analysis, they assume a link technology that is additively separable and convex. Agents use the most reliable path, which is defined as the path that maximizes the product of link strength, to get benefits from others. Different from the utility definition in this paper, there is no added value of redundancy, or having multiple paths between agents. The utility that an agent obtains from a network is the sum of all these benefits. They characterize efficient and Nash stable networks, and show that the unique efficient network is a star. If the link technology is linear, the efficient network is the symmetric star where the hub invests equally among peripheral agents. If link technology is strictly convex, then the unique Nash stable network is a star where the hub invests all her endowment in a single link. Deroïan (2009) extends Bloch and Dutta (2009) to study a directed communication network formation model. They show that the complete wheel network is uniquely efficient and uniquely Nash stable in this context.

Baumann (2021) assumes agents have a limited resource to invest in forming weighted links with other agents or to use for private investment. The utility function from relations with others can be seen as a generalization of the Cobb-Douglas production function, whereas the utility function from self-investment is increasing and strictly concave. An agent's utility is the sum of her utilities from direct relations with others and self-investment. Therefore, agents do not derive any indirect benefits. Baumann (2021) shows that Nash equilibria are either "reciprocal" or "non-reciprocal." In a reciprocal equilibrium, any two agents invest equally in the link between them. In a non-reciprocal equilibrium, agents are partitioned into two groups, and they only invest in agents in the other group. The unweighted relationship graph of an equilibrium, in which two agents are linked if they both invest positively in each other, uniquely predicts the equilibrium values of each agent's network investment and utility value, as well as the ratio of any two agents' investment in each other. Salonen (2015), Brueckner (2006) also study weighted network formation, but with a focus on symmetric equilibria. Other recent contributions include Griffith (2020), Ding (2019).

Cabrales, Calvó-Armengol, and Zenou (2011) assume agents choose socialization efforts as well as productive investments. The socialization effort is a scalar decision. An agent chooses one socialization effort, affecting this agent's link intensities with all other agents. Their main result states that there are exactly two interior Nash equilibria, which can be Pareto-ranked, under certain parameter conditions. The socially efficient outcome lies between these two equilibria. This sort of approach to network formation also appears in Galeotti and Merlino (2014), which studies the job contact network formation. Canen, Jackson, and Trebbi (2019) generalize Cabrales, Calvó-Armengol, and Zenou (2011) to allow for homophily in link formation. Recently, Dasaratha (2020) studies a model in which agents decide both how much to invest in creating ideas and the level of openness, which then dictates link probabilities in a learning network. He shows that the equilibrium learning network is at a critical threshold between sparse and dense networks.

4.5 Conclusion

This paper studies strategic network formation with variable link intensities. In addition, the computation of network externalities is handled in a way that considers all possible paths in the network, resulting in a formulation that creates a bridge to the sociology literature on centrality in networks. A novel feature of this work is that both equilibrium and efficient networks display interesting heterogeneity in link strength, which other models of variable strength links do not predict. Moreover, variations in link strengths are related to underlying characteristics of individuals in intuitive ways. The model thus provides a better understanding of the relationship between the underlying heterogeneity of individuals and the kinds of network structures that are likely to form.

While in some contexts maintaining a link confers benefits to both parties, there are also situations where the flow of benefits is, at least primarily, in only one direction. By separating the flow of benefits into taking and giving, we are able to derive new insights into the question of efficiency regarding stable network structures. In particular, the source of inefficiency is identified with giving behavior, in which a link investment by an individual confers benefits to the individual who is linked to. There are a number of interesting directions for future work. First, binary link studies have analyzed one-way flow (corresponding to Model T) and two-way flow (where both parties benefit from the link), and shown that different networks arise at equilibrium in the two cases. Instead, we focus on two different kinds of one-way flow: taking and giving. The model here can be extended to allow for two-way flow.

Second, there are a number of models in the literature that predict different kinds of networks corresponding to different kinds of heterogeneity in the population. It will be important to relate the theoretical analysis to empirical studies, to begin to determine what kinds of network benefits are more appropriate for modeling. A good example of such work is Goyal, Van Der Leij, and Moraga-González (2006), who analyze a network of economists linked by joint research papers. Laboratory experiments provide another useful tool for empirical examination of models, and experiments incorporating variable link strengths promise to add insight into the analysis of strong and weak links.

4.6 Omitted proofs

This section contains all omitted proofs. We also present results of the constant marginal returns case.

4.6.1 Proposition 7

It is not hard to see that all paired networks, as well as the empty network, are Nash networks.[65]

We know that in Model G, $(I - f(\Phi)')u = \alpha$. Therefore, by Cramer's rule

$$u_i = \frac{\det((I - f(\Phi)')_i)}{\det(I - f(\Phi)')},$$

where $(I - f(\Phi)')_i$ is the matrix formed by replacing the ith column of $I - f(\Phi)'$ with α. Since ϕ_i is the ith column of $I - f(\Phi)'$; $\det((I - f(\Phi)')_i)$ is independent of ϕ_i. We use Laplace expansion along the ith column of $I - f(\Phi)'$ to compute $\det(I - f(\Phi)')$,

$$u_i = \frac{\det((I - f(\Phi)')_i)}{(-1)^{i+i}M_{ii} + \Sigma_{j\neq i}(-1)^{j+i}M_{ji}(-f(\phi_{ij}))},$$

[65] A network $f(\Phi)$ is said to be paired if for every i, $\phi_{ij} = \beta_i$ for some $j \neq i$, and for every i (except one agent in the case that n is odd), $\phi_{ij} = \beta_i$ implies $\phi_{ji} = \beta_j$.

where M_{ji} is the jth row and ith column minor of $I - f(\Phi)'$.[66] It can be shown that $\det(I - f(\Phi)')$ is positive for any $f(\Phi)$, $(-1)^{j+i}M_{ji}$ is non-negative for any $i, j \in N$, and $\det((I - f(\Phi)')_i)$ is positive.[67] Therefore, given Φ_{-i}, i's best response can be equivalently found by solving

$$\max_{\phi_i} \Sigma_{j \neq i}(-1)^{j+i}M_{ji}f(\phi_{ij})$$

such that $\Sigma_j \phi_{ij} \leq \beta_i$. Since f is strictly increasing, strictly concave, and $\lim_{x \to 0} f'(x) = \infty$. Therefore, i's unique best response ϕ_i satisfies that $\Sigma_j \phi_{ij} = \beta_i$, $\phi_{ij} > 0$ if and only if $(-1)^{j+i}M_{ji} > 0$, and

$$(-1)^{j+i}M_{ji}f'(\phi_{ij}) = (-1)^{j'+i}M_{j'i}f'(\phi_{ij'})$$

for any distinct j, j' such that both $(-1)^{j+i}M_{ji}$ and $(-1)^{j'+i}M_{j'i}$ are positive.[68]

It can be shown that $(-1)^{j+i}M_{ji} > 0$ for all $j \neq i$ if Φ_{-i} is interior.[69] Therefore, i's best response is interior and a continuous function of Φ_{-i} if Φ_{-i} is interior. Thus Φ is an interior equilibrium if and only if $\Sigma_j \phi_{ij} = \beta_i$ for all $i \in N$, and $(-1)^{j+i}M_{ji}f'(\phi_{ij}) = (-1)^{j'+i}M_{j'i}f'(\phi_{ij'})$ for all distinct $i, j, j' \in N$. Since the inverse of a matrix is the transpose of its cofactor matrix divided by its determinant, $a_{ji} = \frac{(-1)^{j+i}M_{ji}}{\det(I - f(\Phi)')}$. Therefore, Φ is an interior equilibrium if and only if $\Sigma_j \phi_{ij} = \beta_i$ for all $i \in N$, and $f'(\phi_{ij})a_{ji} = f'(\phi_{ij'})a_{j'i}$ for all distinct $i, j, j' \in N$.

For non-interior equilibria, notice that $a_{ji} > 0$ implies $\phi_{ij} > 0$. This implies that if $a_{ij} > 0$, then for any two agents k, l that have a path to or from i or j, $a_{kl}, a_{lk} > 0$. Consider the finest partition such that there is no positive link across partition elements. The argument

[66]The determinant of the submatrix obtained by removing the jth row and the ith column of $I - f(\Phi)'$.

[67]Since $I - f(\Phi)'$ is (column) strictly diagonally dominant and all its diagonal elements are positive, the real parts of all its eigenvalues are positive. Therefore, $\det(I - f(\Phi)')$ is positive. The inverse matrix $A' = (I - f(\Phi)')^{-1} = I + f(\Phi)' + (f(\Phi)')^2 + \cdots$ is non-negative. It equals the transpose of the cofactor matrix of $I - f(\Phi)'$ divided by $\det(I - f(\Phi)')$. Therefore, the cofactor matrix of $I - f(\Phi)'$ is non-negative.

[68]If there exists $j \neq i$ such that $(-1)^{j+i}M_{ji} > 0$. Otherwise, any strategy is a best response.

[69]Since a_{ji} is positive when Φ_{-i} is interior and $a_{ji} = \frac{(-1)^{j+i}M_{ji}}{\det(I - f(\Phi)')}$.

implies that within a partition element, links constitute an interior equilibrium restricted to that group. Finally, agents i, j in different elements of the partition, who have no paths between them ($a_{ij} = a_{ji} = 0$), do not have a unilateral profitable deviation of linking to one another.

Existence of an interior equilibrium is established as follows. Define an ϵ-strategy for i as one in which $\phi_{ij} \geq \epsilon$ for all $j \neq i$, and $\Sigma_j \phi_{ij} = \beta_i$. So the space of profiles of ϵ-strategies is compact and convex. We show that there exists an $\epsilon > 0$ such that all best responses to ϵ-strategies are ϵ-strategies. If Φ_{-i} are ϵ-strategies, for any ϕ_i, $a_{ji} \in [\epsilon, \frac{1}{1-\max_k (n-1) f(\frac{\beta_k}{n-1})}]$ for all $j \in N \setminus \{i\}$. Since i's best response satisfies that $f'(\phi_{ij})/f'(\phi_{ik}) = a_{ki}/a_{ji}$. Therefore, $f'(\phi_{ij})/f'(\phi_{ik}) \leq \frac{1}{\epsilon(1-\max_k (n-1) f(\frac{\beta_k}{n-1}))}$ for any $j, k \neq i$. Since $\lim_{x \to 0} f'(x)x \to \infty$, we pick ϵ small enough such that for all $x \in (0, \epsilon)$:

$$\frac{f'(x)}{f'(\frac{\min_i \beta_i - x}{n-2})} > \frac{1}{x(1 - \max_k (n-1) f(\frac{\beta_k}{n-1}))} > \frac{1}{\epsilon(1 - \max_k (n-1) f(\frac{\beta_k}{n-1}))}.$$

Therefore, all best responses to ϵ-strategies must be ϵ-strategies. Since best responses are continuous, an interior equilibrium exists.

4.6.2 Proposition 8

An efficient network exists since the strategy profile set is compact and $U = \Sigma_i u_i$ is continuous in Φ. Since U is strictly increasing in Φ, each budget constraint must bind. Also, since $\lim_{x \to 0} f'(x) \to \infty$, any optimum must be interior.

Interiority of the solution implies necessary first order conditions are $\frac{\partial U}{\partial \phi_{ij}} = \frac{\partial U}{\partial \phi_{ij'}}$ for all distinct $i, j, j' \in N$. Differentiating $A'(A')^{-1} = I$ yields $\frac{\partial A'}{\partial \phi_{ij}}(A')^{-1} + A'(\frac{\partial (A')^{-1}}{\partial \phi_{ij}}) = 0$. Right-multiplying by A' and substituting $I - f(\Phi)' = (A')^{-1}$ produce

$$\frac{\partial A'}{\partial \phi_{ij}} = A'(\frac{\partial f(\Phi)'}{\partial \phi_{ij}})A'.$$

Since $\frac{\partial u}{\partial \phi_{ij}} = \frac{\partial A'}{\partial \phi_{ij}}\alpha$, it follows that

$$\frac{\partial u}{\partial \phi_{ij}} = A'(\frac{\partial f(\Phi)'}{\partial \phi_{ij}})A'\alpha = A'(\frac{\partial f(\Phi)'}{\partial \phi_{ij}})u$$

after substituting $A'\alpha = u$. Therefore, if we differentiate U:

$$\frac{\partial U}{\partial \phi_{ij}} = \Sigma_k \frac{\partial u_k}{\partial \phi_{ij}} = f'(\phi_{ij})u_i \Sigma_k a_{jk}.$$

Substituting into the necessary first-order conditions produces the claimed expression.

4.6.3 Proposition 9

We first prove that if there exists an efficient Nash network, it must be the strongly regular network. If Φ is an efficient Nash equilibrium: $f'(\phi_{12})a_{21} = f'(\phi_{13})a_{31}, f'(\phi_{12})(a_{21} + a_{22} + a_{23}) = f'(\phi_{13})(a_{31} + a_{32} + a_{33})$. This implies

$$\frac{a_{21} + a_{22} + a_{23}}{a_{21}} = \frac{a_{31} + a_{32} + a_{33}}{a_{31}}.$$

Substituting into a_{ij} and rearranging produce

$$(f(\phi_{31}) - f(\phi_{21}))(1 - f(\phi_{31})f(\phi_{13}) - f(\phi_{21})f(\phi_{12}) - f(\phi_{23})f(\phi_{32}) - f(\phi_{32})f(\phi_{21})f(\phi_{13}) - f(\phi_{23})f(\phi_{31})f(\phi_{12})) = 0.$$

This implies

$$f(\phi_{21}) = f(\phi_{31}).$$ [70]

Similarly, $f(\phi_{12}) = f(\phi_{32})$ and $f(\phi_{13}) = f(\phi_{23})$. Because f is strictly increasing, $\phi_{21} = \phi_{31}, \phi_{12} = \phi_{32}$, and $\phi_{13} = \phi_{23}$. Substituting this back to $f'(\phi_{12})a_{21} = f'(\phi_{13})a_{31}$ produces

$$\frac{1 + f(\phi_{12})}{f'(\phi_{12})} = \frac{1 + f(\phi_{13})}{f'(\phi_{13})}.$$

Since f is strictly increasing and strictly concave, $\phi_{12} = \phi_{13}$. Therefore, $\phi_{12} = \phi_{13} = \phi_{21} = \phi_{23} = \phi_{31} = \phi_{32}$. and the efficient Nash network is strongly regular.

We then argue that the strongly regular network is indeed efficient when $\beta_i = \bar{\beta}$.[71] The maximum possible total link intensity from a player is $(n-1)f(\frac{\bar{\beta}}{n-1})$. Therefore the summation of the ith column of $f(\Phi)'$ is less than or equal to $(n-1)f(\frac{\bar{\beta}}{n-1})$, the summation of the ith column of $(f(\Phi)')^2$ is less than or equal to $((n-1)f(\frac{\bar{\beta}}{n-1}))^2$, and so on. The strongly regular network achieves upper bounds for all lengths and all players. Therefore the strongly regular network must be the unique efficient Nash equilibrium. Notice that this direction of the proof applies for all $N \geq 3$.

4.6.4 Proposition 10

We first show that efficient networks are Nash networks. Let $f(\Phi^*)$ be an efficient network, that is, $\Phi^* \in \operatorname{argmax}_\Phi \Sigma_i u_i(\Phi)$. This means given Φ^*_{-i},

$$\phi_i^* \in \underset{\phi_i}{\operatorname{argmax}} \, \Sigma_i u_i(\phi_i, \Phi^*_{-i}).$$

[70] It can be shown that $f(\phi_{12}) + f(\phi_{13}) < 1, f(\phi_{21}) + f(\phi_{23}) < 1$, and $f(\phi_{31}) + f(\phi_{32}) < 1$ imply $1 - f(\phi_{31})f(\phi_{13}) - f(\phi_{21})f(\phi_{12}) - f(\phi_{23})f(\phi_{32}) - f(\phi_{32})f(\phi_{21})f(\phi_{13}) - f(\phi_{23})f(\phi_{31})f(\phi_{12}) > 0$.

[71] It is obviously a Nash network.

If Φ^* is not a Nash equilibrium, then there exists $i \in N$, and $\hat{\phi}_i$ such that

$$u_i(\hat{\phi}_i, \Phi^*_{-i}) > u_i(\phi^*_i, \Phi^*_{-i}).$$

If $u_i(\hat{\phi}_i, \Phi^*_{-i}) > u_i(\phi^*_i, \Phi^*_{-i})$, then $u_j(\hat{\phi}_i, \Phi^*_{-i}) \geq u_j(\phi^*_i, \Phi^*_{-i})$, for all $j \neq i$.[72] This means $\Sigma_i u_i(\hat{\phi}_i, \Phi^*_{-i}) > \Sigma_i u_i(\phi^*_i, \Phi^*_{-i})$, contradicting the fact that $f(\Phi^*)$ is an efficient network.

We then show that an inefficient Nash network does not exist. If there exists an inefficient Nash equilibrium $\tilde{\Phi}$, then there exists i such that $u_i(\tilde{\Phi}) - u_i(\Phi^*) < 0$ and $u_i(\tilde{\Phi}) - u_i(\Phi^*) \leq u_j(\tilde{\Phi}) - u_j(\Phi^*)$ for all $j \in N$.[73] We show that $\tilde{\phi_i}$ is not a best response given $\tilde{\Phi}_{-i}$. By the definition of Model T: $u_i(\tilde{\Phi}) = \alpha_i + f(\tilde{\phi}_{i1})u_1(\tilde{\Phi}) + \cdots + f(\tilde{\phi}_{in})u_n(\tilde{\Phi})$. Since $\Sigma_j f(\phi^*_{ij}) < 1$, $u_i(\Phi^*) - u_i(\tilde{\Phi}) > \Sigma_j f(\phi^*_{ij})(u_j(\Phi^*) - u_j(\tilde{\Phi}))$. This implies

$$\begin{aligned}\alpha_i + f(\phi^*_{i1})u_1(\tilde{\Phi}) + \cdots + f(\phi^*_{in})u_n(\tilde{\Phi}) &> \alpha_i + f(\phi^*_{i1})u_1(\Phi^*) + \cdots + f(\phi^*_{in})u_n(\Phi^*) + u_i(\tilde{\Phi}) - u_i(\Phi^*) \\ &= u_i(\tilde{\Phi}).\end{aligned}$$

Fixing other players' utilities, i's utility increases by switching from $\tilde{\phi_i}$ to ϕ^*_i. Other players' utilities, of course, change if i changes her strategy.

We define a sequence of utility vectors $\{u^k\}_{k\geq 0}$, where

$$u^0 = (u_1(\tilde{\Phi}), \cdots, \alpha_i + f(\phi^*_{i1})u_1(\tilde{\Phi}) + \cdots + f(\phi^*_{in})u_n(\tilde{\Phi}), \cdots, u_n(\tilde{\Phi})),$$

[72]Given u_i and Φ^*_{-i}, u_{-i} can be equivalently found by solving

$$u_{-i} = (I - (\Phi^*_{-i})_{-i})^{-1} \begin{pmatrix} \alpha_1 + \phi^*_{1i}u_i \\ \vdots \\ \alpha_n + \phi^*_{ni}u_i \end{pmatrix},$$

where $(\Phi^*_{-i})_{-i}$ is defined as the square matrix derived by deleting the ith column of Φ^*_{-i}. It is easy to see that if $u_i(\hat{\phi}_i, \Phi^*_{-i}) > u_i(\phi^*_i, \Phi^*_{-i})$, then $u_j(\hat{\phi}_i, \Phi^*_{-i}) \geq u_j(\phi^*_i, \Phi^*_{-i})$, for all $j \neq i$.

[73]Φ^* is the efficient Nash equilibrium.

and

$$u^k = \Pi(u^{k-1}) = \begin{pmatrix} \alpha_1 + f(\tilde{\phi}_{12})u_2^{k-1} + \cdots + f(\tilde{\phi}_{1n})u_n^{k-1} \\ \vdots \\ \alpha_i + f(\phi_{i1}^*)u_1^{k-1} + \cdots + f(\phi_{in}^*)u_n^{k-1} \\ \vdots \\ \alpha_n + f(\tilde{\phi}_{n1})u_1^{k-1} + \cdots + f(\tilde{\phi}_{nn-1})u_{n-1}^{k-1} \end{pmatrix}$$

for $k = 1, 2, \cdots$. The sequence $\{u^k\}_{k\geq 0}$ increases monotonically and converges to a fixed point.[74] By the definition of Model T, $u(\phi_i^*, \tilde{\Phi}_{-i})$ equals this fixed point. Therefore, if i chooses ϕ_i^* instead of $\tilde{\phi}_i$,

$$u_i(\phi_i^*, \tilde{\Phi}_{-i}) > u_i(\tilde{\Phi}).$$

This also proves that the utility vector of the efficient Nash network is unique.

Existence of an efficient Nash network is clear since $U = \Sigma_i u_i$ is continuous in Φ and the strategy sets are compact. Since U is strictly increasing in each ϕ_{ij} and therefore all budget constraints must be bind in any efficient Nash network. The assumption that $\lim_{x\to 0} f'(x) = \infty$ implies that any efficient Nash network is interior. Therefore, maximizing U has necessary first-order conditions $\frac{\partial U}{\partial \phi_{ij}} = \frac{\partial U}{\partial \phi_{ij'}}$ for all distinct $i, j, j' \in N$. We have

$$\frac{\partial U}{\partial \phi_{ij}} = \Sigma_k \frac{\partial u_k}{\partial \phi_{ij}} = f'(\phi_{ij})u_j \Sigma_k a_{ki}.$$

Substituting into the first-order conditions then produces the claimed conditions. Any efficient Nash network has to satisfy the claimed conditions. Since the efficient Nash network utility vector is unique, the efficient Nash network is also unique.

[74]The sequence converges because the row sums of the matrix $(\phi_i^*, \tilde{\Phi}_{-i})$ are all strictly less than 1.

4.6.5 Corollary 4

Assume $\alpha_i = \bar{\alpha}$ for all $i \in N$. In both Models, aggregate utility is proportional to the sum of all elements in A by the constant $\bar{\alpha}$, and so is the same across models.

4.6.6 Constant marginal returns in link investment

In this subsection, we assume the function f is the identity mapping. Hence the link intensity of ij is simply ϕ_{ij}. We characterize Nash and the efficient networks for both Models G and T.

Proposition 11. *In Model G, there exist multiple Nash networks. $f(\Phi)$ is a Nash network if and only if for all $i \in N$ for which there exists j such that $\phi_{ji} > 0$ satisfies that $\Sigma_j \phi_{ij} = \beta_i$ and $\phi_{ij} > 0$ only if $j \in \text{argmax}_{j' \in N \setminus \{i\}} a_{j'i}$.*[75]

Proof. Similar to the proof of Proposition 7, i's best response given Φ_{-i} can be equivalently found by solving

$$\max_{\phi_i} \Sigma_{j \neq i} (-1)^{j+i} M_{ji} \phi_{ij}$$

such that $\Sigma_j \phi_{ij} \leq \beta_i$.[76] Therefore, for all $i \in N$ for which there exists $j \neq i$ such that $(-1)^{j+i} M_{ji} > 0$, ϕ_i is a best response if and only if $\Sigma_j \phi_{ij} = \beta_i$, and $\phi_{ij} > 0$ only if $j \in \text{argmax}_{j' \in N \setminus \{i\}} (-1)^{j'+i} M_{j'i}$. If $(-1)^{j+i} M_{ji} = 0$ for all $j \neq i$, then any strategy is a best response for i. There exists $j \neq i$ such that $(-1)^{j+i} M_{ji} > 0$ if and only if there exists $j \neq i$ such that $\phi_{ji} > 0$. $\square$

[75] If i has $\phi_{ji} = 0$ for all $j \in N$, any feasible strategy is a best response for i.
[76] Here $(-1)^{j+i} M_{ji}$ is the jth row and ith column minor of $I - \Phi'$.

Equilibrium choices require that players allocate their entire budgets to the player with the strongest individual connectivity. All paired networks are Nash networks. However, there are also other Nash networks.[77]

Proposition 12. *In Model G, there exist both efficient and inefficient Nash networks. If* $\beta_1 > \beta_2 > \cdots > \beta_n$*, then the unique efficient Nash network is the one for which* $\phi_{j1} = \beta_j$ *for* $j = 2, \cdots, n$*, and* $\phi_{12} = \beta_1$*. If* $\beta_i = \bar{\beta}$ *for all* $i \in N$*, then efficient networks are those for which* $\Sigma_j \phi_{ij} = \beta$ *for all* $i \in N$*.*

Proof. First, we show that if $\beta_1 > \cdots > \beta_n$, the unique efficient Nash network is the one for which $\phi_{12} = \beta_1$ and $\phi_{j1} = \beta_j$ for all $j = 2, \cdots, n$. In Model G, $u = A'\alpha = (I + \Phi' + (\Phi')^2 + \cdots)\alpha$. If we calculate $U = \Sigma_i u_i$, the parameter before α_i is the summation of the ith column of the matrix $I + \Phi' + (\Phi')^2 + \cdots$. The summation of the ith column of Φ' is less than or equal to β_i. The summation of the ith column of $(\Phi')^2$ is less than or equal to $\Sigma_j \phi_{ij} \beta_j$, which is less than or equal to $\beta_i \beta_1$ for $i = 2, \cdots, n$, and $\beta_1 \beta_2$ for $i = 1$. The summation of the ith column of $(\Phi')^3$ is less than or equal to $\Sigma_j \phi_{ij} (\Sigma_k \phi_{jk} \beta_k)$, which is less than or equal to $\beta_i \beta_1 \beta_2$ for all i. The same logic works for longer paths. The network for which $\phi_{12} = \beta_1$ and $\phi_{i1} = \beta_i$ for $i = 2, \cdots, n$ achieves upper bounds for all paths and all players. It is not hard to see that any deviation from this efficient network reduces U.

When $\beta_1 = \cdots = \beta_n$, all networks with $\Sigma_j \phi_{ij} = \beta_i$ for all i have the same U. Since U is constant on these networks, they must all be efficient. □

[77] Let $N = 3$, and $\beta = (0.4, 0.5, 0.6)$. The following strategy profile

$$\Phi = \begin{pmatrix} 0 & 0.35 & 0.05 \\ 0.4583 & 0 & 0.0417 \\ 0.3667 & 0.2333 & 0 \end{pmatrix}$$

also constitutes a Nash network.

For efficiency, players should confer their intrinsic values to the player with the largest linking budget to further confer these values. Generically, efficient networks are star networks in which the player with the largest linking budget is the center.[78]

Proposition 13. *In Model T, Nash networks and efficient networks coincide. If* $\frac{\alpha_1}{1-\beta_1} > \frac{\alpha_2}{1-\beta_2} > \cdots > \frac{\alpha_n}{1-\beta_n}$*, then efficient Nash networks are those for which* $\phi_{j1} = \beta_j$ *for* $j = 2, \cdots, n$*, and* ϕ_1 *satisfies that* $\Sigma_j \phi_{1j} = \beta_1$ *and* $\phi_{1j} > 0$ *only if* $j \in \operatorname{argmax}_{j' \in \{2, \cdots, n\}} \frac{\alpha_1 + \alpha_{j'}\beta_1}{1-\beta_{j'}\beta_1}$*. If* $\frac{\alpha_1}{1-\beta_1} = \frac{\alpha_2}{1-\beta_2} = \cdots = \frac{\alpha_n}{1-\beta_n}$*, then efficient Nash networks are those for which* $\Sigma_j \phi_{ij} = \beta_i$ *for all* $i \in N$*.*

Proof. The proof for Proposition 10 applies, so Nash networks and efficient networks coincide.

If $\frac{\alpha_1}{1-\beta_1} > \frac{\alpha_2}{1-\beta_2} > \frac{\alpha_3}{1-\beta_3}$, $\frac{\alpha_1+\alpha_2\beta_1}{1-\beta_1\beta_2} > \frac{\alpha_2+\alpha_1\beta_2}{1-\beta_1\beta_2}$ and $\alpha_1 + \frac{\alpha_2+\alpha_3\beta_2}{1-\beta_2\beta_3}\beta_1 > \frac{\alpha_2+\alpha_3\beta_2}{1-\beta_2\beta_3}$. The first inequality states that the player with the higher value of $\frac{\alpha_i}{1-\beta_i}$ gets a higher utility when two players point to each other. In addition, both $\frac{\alpha_1+\alpha_2\beta_1}{1-\beta_1\beta_2}$ and $\frac{\alpha_2+\alpha_1\beta_2}{1-\beta_1\beta_2}$ are smaller than $\frac{\alpha_1}{1-\beta_1}$ and larger than $\frac{\alpha_2}{1-\beta_2}$. The second inequality states that the player with the highest value of $\frac{\alpha_i}{1-\beta_i}$ gets the highest utility even if other players point to each other.

First, we consider the case where $\frac{\alpha_1}{1-\beta_1} > \cdots > \frac{\alpha_n}{1-\beta_n}$. Equilibrium choices require that players allocate all their linking budgets to the player with the highest utility. Therefore, in the efficient Nash networks $\phi_{j1} = \beta_j$ for all $j = 2, \cdots, n$, and ϕ_1 satisfies that $\Sigma_j \phi_{1j} = \beta_1$ and $\phi_{1j} > 0$ only if $j \in \operatorname{argmax}_{j' \in \{2, \cdots, n\}} \frac{\alpha_1 + \alpha_{j'}\beta_1}{1-\beta_{j'}\beta_1}$. Notice that player 2 is not necessarily the player with the second highest utility.

If $\frac{\alpha_1}{1-\beta_1} = \cdots = \frac{\alpha_n}{1-\beta_n}$, then all networks for which $\Sigma_j \phi_{ij} = \beta_i$ for all $i \in N$ generate the same utility vector. It can be verified by showing that the utility vector $(\frac{\alpha_1}{1-\beta_1}, \cdots, \frac{\alpha_n}{1-\beta_n})$ is the

[78] A network $f(\Phi)$ is said to be a star network if the associated undirected network $f(\tilde{\Phi})$, where $\tilde{\Phi}$ is derived by setting $\tilde{\phi}_{ij} = \tilde{\phi}_{ji} = \max\{\phi_{ij}, \phi_{ji}\}$ for any $i, j \in N$, is a star network.

solution to all these networks. Since all these networks generate the same utility vector, they must all be efficient Nash networks. □

Efficient Nash networks depend on budgets as well as intrinsic values. Generically, efficient Nash networks are star networks in which the player with the highest value of $\frac{\alpha_i}{1-\beta_i}$ is the center. The value of $\frac{\alpha_i}{1-\beta_i}$ can be seen as a player's composite character. Increasing either the intrinsic value or the linking budget is beneficial to a player. Having a low intrinsic value decreases the incentives for others to link to the player. But having a large linking budget can more than compensate for what is lacked in the intrinsic value. That is, a player with low intrinsic value can be the center of a star network provided it has a large enough linking budget. In this sense, investing in becoming "well-connected" can make up for a low value, and create enough value through paths to others to be a focal node in the network.

Star networks and heterogeneity

A well-known network architecture that occurs robustly in "two-way flow" models is the star network, in which one node, the center, is connected to each of the other nodes, and there are no other connections.[79] In the case of directed networks, stars can be either center-sponsored or periphery-sponsored (or a combination of the two), depending on which agents maintain the links. The star network is indicative of the role of "hubs" in real networks, where a few nodes have many connections. The prevalence of hubs has important implications for network performance, mainly through their role in decreasing the distances between nodes in the network.

[79] Jackson and Wolinsky (1996) and Bala and Goyal (2000) contain early theoretical treatments, and Goeree, Riedl, and Ule (2009) extend the analysis and report on experiments. They find that heterogeneity plays a major role in helping star networks to form empirically. Bloch and Dutta (2009) also identify star networks as the stable and efficient network architectures.

Interestingly, star networks are not prominent in previous work on "one-way flow" models, but it is easy to generate in Models T and G with a linear link investment function. In Model T with heterogeneous composite character values, the efficient Nash networks are periphery-sponsored star networks where the player with the highest composite character value serves as the hub. Periphery nodes allocate their entire linking budgets to the center since the center has the highest utility.

In Model G with heterogeneous linking budgets, efficient networks are periphery-sponsored star networks where the player with the highest linking budget serves as the hub. The efficient network is "unfair" in the sense that the two players with the first two highest linking budgets get very high utilities while all other players' utilities equal their intrinsic values. The efficient network is also a Nash network. However, this efficient Nash network is "unstable" in the sense that a small perturbation in a periphery node's strategy can destroy the equilibrium.[80]

[80] On the contrary, paired networks are "stable" in this sense.

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