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Weighted Estimates for the Bergman and Szegö Projections

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by

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Dedicated to my parents, Stephen and Pamela Wagner

ABSTRACT OF THE DISSERTATION

Weighted Estimates for the Bergman and Szegő Projections

by

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This thesis is a study of various weighted estimates for the Bergman and Szegő projections on domains in several complex variables. The starting point of our analysis is a bounded, pseudoconvex domain $D \subset \mathbb{C}^n$. The Bergman and Szegő projections are both orthogonal projections onto spaces of holomorphic functions associated with D. While it is immediate that both of these operators are bounded on L^2, it has been a topic of substantial interest to determine their mapping properties on L^p, where the boundary geometry of D plays a major role.

Given a linear operator T acting on measurable functions that is bounded on L^2 or L^p with respect to Lebesgue measure $d\mu$, it is of substantial interest in harmonic analysis to determine the absolutely continuous measures $\sigma d\mu$ such that T is also bounded on $L^p(\sigma d\mu)$. In the case that T is a Calderón-Zygmund operator, it is known that the correct sufficient condition for boundedness is the A_p condition. A closely related weight class, called the B_p class, is known to characterize the weighted L^p boundedness of the Bergman projection on the unit ball. The goal of this thesis is to establish weighted estimates for the two projection operators for weights in the A_p or (generalized) B_p classes in a much more general context. We especially focus on strongly pseudoconvex domains with minimal or near-minimal boundary smoothness.

In Chapter 1, we introduce the spaces of holomorphic functions and related projection operators, define the relevant weight classes, and discuss the history of unweighted and

weighted L^p estimates for the projection operators on various classes of domains. We also establish common notation and state the main results in the thesis.

In Chapter 2, we establish weighted L^p estimates for the Bergman projection when $1 < p < \infty$ and $\sigma \in B_p$ on several classes of smoothly bounded, pseudoconvex domains where explicit size and smoothness on the Bergman kernel are known. In the case of strongly pseudoconvex domains, the necessity of the B_p condition is also proved.

Next, in Chapter 3 we shift our focus to strongly pseudoconvex domains with a lower level of boundary regularity. For such domains, explicit estimates on the kernel functions are not known, so one must instead use an operator-theoretic technique pioneered by Kerzman and Stein. This technique involves relating the Bergman or Szegő projection to a non-orthogonal projection which has a non-canonical, yet explicitly constructed kernel. In this chapter, we specifically prove weighted L^p estimates for the Szegő projection for $1 < p < \infty$ and $\sigma \in A_p$ on strongly pseudoconvex domains with C^2 boundary.

In Chapter 4, we establish weighted L^p estimates for the Bergman projection for $1 < p < \infty$ and $\sigma \in B_p$ on strongly pseudoconvex domains with C^4 boundary. These estimates are obtained using very similar techniques to Chapter 3. At the end of the chapter, we prove weighted estimates in the minimally smooth (C^2) case for the Bergman projection for a special class of weights that are a power of the distance to the boundary. We also provide an application of the proof techniques to mapping properties of Toeplitz operators.

Finally, in Chapter 5 we establish some endpoint estimates for both projection operators. In particular, we prove that on a strongly pseudoconvex domain with C^4 boundary, if σ belongs to B_1, the Bergman projection maps L^1_σ to $L^{1,\infty}_\sigma$ (a weighted weak-type space). We also establish that on a strongly pseudoconvex domain with C^3 boundary, if σ belongs to A_1, the Szegő projection is also weighted weak-type $(1,1)$. Finally, we provide some other endpoint estimates, including weighted Kolmogorov and Zygmund inequalities, as well as an estimate for the Bergman projection for $p = \infty$ in terms of the Bloch norm.

Chapter 1

Introduction

1.1 The Bergman and Szegő Projections

Let $D \subset \mathbb{C}^n$ be a bounded domain. Let $\text{Hol}(D)$ denote the set of holomorphic functions on D. For $1 < p < \infty$, the *Bergman space* $A^p(D)$ is defined to be $\text{Hol}(D) \cap L^p(D)$, or alternately

$$A^p(D) := \left\{ f \in \text{Hol}(D) : \int_D |f|^p \, dV < \infty \right\}.$$

Here, dV represents Lebesgue measure on $\mathbb{C}^n$ canonically identified with $\mathbb{R}^{2n}$.

It is straightforward to show, using the mean value property for holomorphic functions and Hölder's Inequality, that if a sequence $\{f_n\} \subset A^p(D)$ converges in $L^p(D)$ to a function f, then $f_n \to f$ uniformly on compact sets. We conclude that f is holomorphic and hence $f \in A^p(D)$. It follows that $A^p(D)$ is a Banach space for $1 < p < \infty$ when equipped with the L^p norm.

In the special case $p = 2$, $A^2(D)$ is clearly a Hilbert space with the standard L^2 inner product

$$\langle f, g \rangle = \int_D f \overline{g} \, dV.$$

Since $A^2(D)$ is a closed subspace of $L^2(D)$, we know from elementary Hilbert space theory that there exists an orthogonal projection operator $\mathcal{B} : L^2(D) \to A^2(D)$, called the *Bergman projection.*

For any fixed $z \in D$, we can show that there exists a constant C_z so that $|f(z)| \leq$

$C_z\|f\|_{A^2(D)}$ for all $f \in A^2(D)$. This means that point evaluations are bounded linear functionals on $A^2(D)$. Accordingly, by the Riesz Representation Theorem, there exists a function $K_z \in A^2(D)$ (called the *reproducing kernel* at z) so that $\langle f, K_z\rangle = f(z)$ for all $f \in A^2(D)$. Let $K_D(z,w) = \overline{K_z(w)}$. It is easy to see $K_D(z,w) = \overline{K_D(w,z)}$. Then it is immediate $K_D(z,w)$ is holomorphic in the z variable and anti-holomorphic in the w variable and satisfies the reproducing property

$$f(z) = \int_D K_D(z,w) f(w)\, dV(w)$$

for all $f \in A^2(D)$.

We refer to the function K_D as the *Bergman kernel* for the domain D. Notice that integration against the Bergman kernel is exactly the Bergman projection operator: for any $f \in L^2(D)$, we have, using the self-adjointness of the Bergman projection:

$$\begin{aligned} \mathcal{B}f(z) &= \langle \mathcal{B}f, K_z\rangle \\ &= \langle f, \mathcal{B}K_z\rangle \\ &= \langle f, K_z\rangle \\ &= \int_D K_D(z,w) f(w)\, dV(w). \end{aligned}$$

The Bergman kernel can rarely be calculated explicitly. However, considerable information about the behavior of the kernel near the boundary has been obtained for several classes of domains, some of which we discuss later in this chapter. Let $\mathbb{B}_n = \{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 < 1\}$ denote the unit ball in $\mathbb{C}^n$. Then it is well-known that

$$K_{\mathbb{B}_n}(z,w) = \frac{n!}{\pi^n} \frac{1}{(1 - z\cdot\bar{w})^{n+1}}$$

where $z \cdot \bar{w} = \sum_{j=1}^n z_j \bar{w}_j$.

Now let D be a domain with C^2 boundary smoothness. This means that there is a C^2 function $\rho : \mathbb{C}^n \to \mathbb{R}$ that satisfies $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ and $\nabla\rho \neq 0$ on bD. The obvious analogue of this definition applies to define domains with C^k boundary, $k \geq 1$. We call ρ a *defining function* for the domain D. We now define the *Hardy space* $H^2(bD)$ as follows:

$$H^2(bD) := \overline{\{f \in L^2(bD) : f = F|_{bD}, F \in \mathrm{Hol}(D) \text{ and } F \in C(\overline{D})\}},$$

where the closure is taken in $L^2(bD)$. This definition coincides with more typical definitions in the case $D = \mathbb{D}$, for example. By its very definition, $H^2(bD)$ is a closed subspace of $L^2(bD)$ and hence a Hilbert space with the $L^2(bD)$ inner product. Thus, we know there exists an orthogonal projection $\mathcal{S} : L^2(bD) \to H^2(bD)$, which we call the *Cauchy-Szegő*, or simply *Szegő*, projection. We should note it is also possible to define $H^p(bD)$ spaces for $p \in (1, \infty)$ using an approach involving non-tangential maximal functions, see [39].

It is a fact that every element of $H^2(bD)$ can be associated with a unique holomorphic function $\tilde{f} \in \mathrm{Hol}(D)$ via its Poisson integral $\tilde{f} = Pf$. Moreover, it is a fact that for almost every $\zeta \in bD$, the following holds [33]:

$$\lim_{\varepsilon \to 0^+} \tilde{f}(\zeta - \varepsilon n_\zeta) = f(\zeta).$$

Here, n_ζ denotes the outward unit normal vector at ζ, which is well-defined in light of the C^2 boundary of D. Therefore, we say that functions in the Hardy space have radial boundary limits almost everywhere (actually they have non-tangential boundary limits in a precise sense [33, 39], but this is beyond the scope of the thesis), and it is meaningful to talk about functions in $H^2(bD)$ as boundary values of holomorphic functions .

For $z \in D$, define the linear functional E_z by $E_z(f) = Pf(z)$, $f \in H^2(bD)$. Then it is not difficult to show that $|Pf(z)| \leq C_z \|f\|_{H^2(bD)}$, so this evaluation functional is bounded. Accordingly, there is a unique function $K_z \in H^2(bD)$ satisfying $\langle f, K_z \rangle = Pf(z)$ for all

$f \in H^2(bD)$. Analogous to the Bergman case, for $z \in D$ and $w \in bD$, we will let $K_D(z,w) = \overline{K_z(w)}$ and refer to this function as the *Szegő* kernel. Although we use the same letter to denote the Bergman and Szegő kernels, the meaning will be clear from context.

The Szegő projection is given by integration against the Szegő kernel in the following sense:

$$\begin{aligned} P\mathcal{S}f(z) &= \langle \mathcal{S}f, K_z \rangle \\ &= \langle f, \mathcal{S}K_z \rangle \\ &= \langle f, K_z \rangle \\ &= \int_{bD} K_D(z,w) f(w) dS(w). \end{aligned}$$

Of course, we may recover the boundary values of $\mathcal{S}f$ by taking a radial limit in z of the integral. When $z \in bD$, this integration can be interpreted as a singular integral in certain cases.

As in the case of the Bergman kernel, the Szegő kernel can rarely be calculated explicitly, but relevant estimates have been obtained in a variety of cases. In the case of the unit ball, one has the following formula for the Szegő kernel:

$$K_{\mathbb{B}_n}(z,w) = \frac{(n-1)!}{2\pi^n} \frac{1}{(1 - z \cdot \bar{w})^n}.$$

1.2 Calderón-Zygmund Operators

Calderón-Zygmund operators have been extensively studied in harmonic analysis. These operators are singular integrals whose kernels satisfy certain size and smoothness conditions. In the past several decades, there has been ample interest in studying Calderón-Zygmund operators in a non-Euclidean setting. The most natural setting in which to study these operators is a *space of homogeneous type*. This concept was introduced by Coifman and Weiss in [11,12].

Definition 1.2.1. Let (X, μ) be a measure space. Suppose there is a function $d : X \times X \to \mathbb{R}$ that satisfies the following:

1. $d(x, y) \geq 0$ for all $x, y \in X$ and equals 0 if and only if $x = y$.

2. $d(x, y) = d(y, x)$ for all $x, y \in X$.

3. There exists a constant $c \geq 1$ so that for all $x, y, z \in X$ we have $d(x, y) \leq c(d(x, z) + d(z, y))$.

Then the function d is called a *quasi-metric* or *pseudometric*. We also assume that for every $x \in X$, the ball $B(x, r) = \{y \in X : d(x, y) < r\}$ is μ-measurable. Suppose also that there exists a constant $C > 0$ so that for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \leq C\mu B(x, r).$$

The triple (X, d, μ) is then called a *space of homogeneous type*.

Now, we can define Calderón-Zygmund operators (CZOs) on spaces of homogeneous type. We say a bounded operator T on $L^2(X, \mu)$ has an associated kernel $K : X \times X \setminus \{(x, x) : x \in X\} \to \mathbb{C}$ if for all $f \in L^2(X)$ and $x \notin \operatorname{supp}(f)$, we have

$$Tf(x) = \int_X K(x, y) f(y)\, d\mu(y).$$

Definition 1.2.2. We say a linear operator T is a *Calderón-Zygmund operator* on (X, d, μ) if T is bounded on $L^2(X, \mu)$ and has an associated kernel K that satisfies

1. There exists $C_1 > 0$ so for $x \neq y$, there holds

$$|K(x,y)| \leq \frac{C_1}{\mu(B(x,d(x,y)))}.$$

2. There exists $C_2 > 0$ and $\eta > 0$ so if $d(x,y) > C_2 d(x,x')$, then

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq \frac{C_1}{\mu(B(x,d(x,y)))} \left(\frac{d(x,x')}{d(x,y)} \right)^{\eta}.$$

In the Euclidean setting, the prototypical examples of CZOs are the Hilbert transform on $\mathbb{R}$ and the Riesz transforms on $\mathbb{R}^n$. Many mapping properties of CZOs are well-known. For example, CZOs are automatically bounded on L^p for $1 < p < \infty$. The relevance of CZOs to this thesis is that in many situations the Bergman and Szegő projections (or related operators) can be viewed as CZOs on a space of homogeneous type that reflects the geometry of the domain. It should be noted that the Bergman projection is fundamentally different from typical singular integrals because the Bergman kernel $K_D(z,w)$ has finite modulus for each pair $(z,w) \in D \times D$ (so in a sense the integral is not "singular" at all; however the kernel may blow up as z and w approach the boundary diagonal and this behavior is typical for many domains). In contrast, the Szegő projection can be viewed as a "true singular integral."

A recent theme in harmonic analysis has been to determine the boundedness properties of integral operators, particular CZOs, on weighted Lebesgue spaces. A weight σ is a locally integrable function that is positive almost everywhere. We write $L^p_\sigma(X)$ to denote the L^p space on X with absolutely continuous measure $\sigma d\mu$. The consideration of these problems goes back to the formulation of the A_p condition for the Hilbert transform by Hunt, Muckenhoupt, and Wheeden, see [23]. The following class of weights is fundamental to harmonic analysis:

Definition 1.2.3. Let (X,d,μ) be a space of homogeneous type. For $1 < p < \infty$, we say a weight $\sigma \in A_p$ (or belongs to the *Muckenhoupt class*) if

$$[\sigma]_{A_p} := \sup_{\substack{x\in X\\ r>0}} \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} \sigma \, d\mu \right) \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} \sigma^{-\frac{1}{p-1}} \, d\mu \right)^{p-1} < \infty.$$

The relevance of A_p weights to CZOs is clearly illustrated in the following theorem.

Theorem 1.2.1. Let (X, d, μ) be a space of homogeneous type, T a CZO, and $1 < p < \infty$. If $\sigma \in A_p$, then there exists a constant $C(T, p, \sigma)$ depending on T, p and σ so that for all $f \in L^p_\sigma(X)$,

$$\|Tf\|_{L^p_\sigma(X)} \leq C(T, p, \sigma)\|f\|_{L^p_\sigma(X)}.$$

There is also a relevant endpoint class called A_1 weights. These weights are contained in every A_p class for $1 < p < \infty$ and can be defined as follows:

Definition 1.2.4. Let (X, d, μ) be a space of homogeneous type. We say a weight $\sigma \in A_1$ if

$$[\sigma]_{A_1} := \sup_{\substack{x\in X\\ r>0}} \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} \sigma \, d\mu \right) \|\sigma^{-1}\|_{L^\infty(B(x,r))} < \infty.$$

1.3 Known L^p and Weak-Type Estimates

By their very definitions, it is clear that the Bergman and Szegő projections act boundedly on L^2 (in fact with norm 1). However, it is of interest to determine when the projection operators extend to bounded operators on $L^p(D)$ for $1 < p < \infty$. For example, the boundedness of the Bergman projection on $L^p(D)$ allows one to conclude that $(A^p(D))^* = A^q(D)$, where * denotes the dual space in the functional analysis sense and $\frac{1}{p} + \frac{1}{q} = 1$. These problems have a long history and are intimately connected to the geometry of the domain D. In the simplest case $D = \mathbb{D}$ (the unit disk, or unit ball with $n = 1$), it was shown by Zaharjuta and Judovič in [68] that the Bergman projection $\mathcal{B}$ extends to a bounded operator on $L^p(\mathbb{D})$. Rudin and Forelli extended this result to multiple dimensions by proving the L^p regularity of $\mathcal{B}$ on the unit ball $\mathbb{B}_n$ in [19]. Their method of proof uses Schur's Test for integral operators with positive kernels. In fact, their argument actually shows that the *positive Bergman operator*

$$\mathcal{B}^+ f(z) := \int_{\mathbb{B}_n} |K_{\mathbb{B}_n}(z, w)| f(w) \, dV(w)$$

is bounded on $L^p(\mathbb{B}_n)$ for $1 < p < \infty$. In terms of the Szegő projection, it is a classical result of M. Riesz that the Szegő projection on the unit circle $\mathbb{T} = b\mathbb{D}$ extends to a bounded operator on $L^p(\mathbb{T})$ for $1 < p < \infty$. In fact, the Szegő projection on the circle is closely connected to the Hilbert transform/conjugate operator on the circle (see [56]). The corresponding statement for the unit ball was in fact proven in [32].

Many researchers then studied mapping properties of the Bergman and Szegő projections on wider classes of domains in several complex variables. In what follows, we assume that $D \subset \mathbb{C}^n$ is a bounded domain with C^2 boundary smoothness. The following definitions are fundamental in several complex variables:

Definition 1.3.1. Let $D \subset \mathbb{C}^n$ be a bounded domain with C^2 defining function ρ. Then D is said to be *(strongly) pseudoconvex* if for all $\zeta \in bD$, the complex Hessian matrix

$$\left\{ \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(\zeta) \right\}_{j,k=1}^{n}$$

is positive semi-definite (positive definite) on the complex tangent space

$$T_\zeta(bD) = \left\{ w \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(\zeta) w_j = 0 \right\}.$$

One can check that this definition is independent of the defining function ρ. Pseudoconvexity can be thought of as an analog of convexity that is invariant under biholomorphic mappings. It is also a (highly non-obvious) fact that pseudoconvex domains are precisely the domains of holomorphy [33]. It therefore makes sense to restrict attention to pseudoconvex domains when studying mapping properties of the projection operators.

The prime example of a strongly pseudoconvex domain is the unit ball. Therefore, strongly pseudoconvex domains with C^∞ (smooth) boundary are a natural class of domains to consider next, as they have similar boundary geometry. In 1977, Phong and Stein proved the following [57]:

Theorem 1.3.1. Let D be smoothly bounded and strongly pseudoconvex. Then $\mathcal{B}$ and $\mathcal{S}$ extend to bounded operators on L^p for $1 < p < \infty$.

Their proof used explicit kernel estimates on the Bergman and Szegő kernels, Schur's Test (see [70, Theorem 2.9]) for the Bergman projection, and singular integral theory on the Heisenberg group for the Szegő projection. It is worth noting that these proofs are made possible by precise asymptotic expansions of the Bergman and Szegő kernel in this setting (the expansion for the Bergman kernel was first famously obtained by Fefferman near the boundary diagonal in [18], see also [7]).

Smoothly bounded, strongly pseudoconvex domains are an example of a broader class of domains known as finite type. Roughly speaking, this means each point on the boundary of the domain has finite order of contact with all one-dimensional analytic varieties. Let $\phi : \mathbb{D} \to \mathbb{C}$ be smooth. Fix a point $p \in \mathbb{D}$ and let k be the least positive integer so that there exists a (real) derivative D^β of order $|\beta| = k$ so that $D^\beta(\phi)(p)$ is non-vanishing. We call the integer k the multiplicity of ϕ at p and write $v_p(\phi) = k$. If instead ϕ maps into $\mathbb{C}^n$, we define $v_p(\phi)$ to be the minimum multiplicity of its entries. Below, we assume $p = 0$ and simply write $v(\phi)$. We provide the precise definition of finite type, which is originally due to D'Angelo in [14], see also [33, 50] (again, this definition is actually independent of the defining function).

Definition 1.3.2. Let D be a smoothly bounded domain with defining function ρ. We say a point $\zeta \in bD$ is *finite type* if there exists a positive constant $C < \infty$ so that

$$\sup_{\substack{\phi:\mathbb{D}\to\mathbb{C}^n \\ \phi \text{ holomorphic and non-constant} \\ \phi(0)=\zeta}} \frac{v(\rho \circ \phi)}{v(\phi)} \leq C.$$

The infimum of all such constants C is denoted $\Delta(bD, \zeta)$ and is referred to as the type of bD at ζ. We say that D is finite type if each point $\zeta \in bD$ is finite type.

The finite type condition is extremely important in several complex variables because it turns out to be a necessary and sufficient condition for local subelliptic estimates to hold for the $\bar{\partial}$-Neumann operator on smoothly bounded, pseudoconvex domains [8]. In addition, it is known that any biholomorphic mapping from Ω_1 to Ω_2 extends to a diffeomorphism of the closures if both domains are finite type [33].

For several classes of finite type domains, strong estimates on the singularities of the Bergman and/or Szegő kernels have been obtained. These classes of domains include convex domains of finite type, domains of finite type in $\mathbb{C}^2$, and decoupled domains. We will not formally define these domains, but see [45, 46, 48, 49, 53] for more details. In these situations, we can introduce a quasi-distance d on D so that, with respect to the space of homogeneous type (D, d, V), the Bergman projection is a Calderón-Zygmund operator in the sense of Definition 1.2.2. In [47, 50], McNeal developed a coherent framework for studying the behavior of the Bergman kernel on these domains, which he refers to as *simple domains* (we follow his terminology in the rest of this thesis). The following theorem in [47] is a consequence of these estimates:

Theorem 1.3.2. Let D be a simple domain. Then the Bergman projection $\mathcal{B}$ extends to a bounded operator on L^p for $1 < p < \infty$.

The most difficult part of this approach is obtaining the actual size and smoothness estimates on the Bergman kernel $K_D(z, w)$. Estimates for the Szegő kernel have also been obtained for these domains [7, 9, 44, 53], and the corresponding mapping properties have also been studied. The upshot of this approach is that the theory of Calderón-Zygmund operators can be brought to bear when such estimates are proven for the Bergman or Szegő kernel, so the L^p boundedness of the Bergman projection follows from standard methods. It is still an interesting open question whether a finite type assumption alone on D is sufficient for $L^p(D)$ boundedness of the Bergman projection in the reflexive range, see [69].

It should be noted that historically, estimates for the Bergman kernel on strongly pseudoconvex domains were obtained first by Fefferman and later refined by Boutet and Sjöstrand, using different methods (see [7, 18]). Strongly pseudoconvex domains were also not one of the types considered in [47], as the L^p mapping properties of the Bergman projection on these domains were already known (see [57]). However, in [50] McNeal demonstrates that strongly pseudoconvex domains fall into the same paradigm as the other domains considered. This means that one can use the exact same singular integral machinery as in [47] to prove the L^p regularity of the Bergman projection on strongly pseudoconvex domains, even though this was not originally how this result was obtained.

If the domain is C^k smooth rather than C^∞, a more indirect approach is needed because the methods used to obtain precise estimates on the Bergman or Szegő kernel depend critically on the C^∞ boundary hypothesis and break down if this smoothness is lost. Around the same time that decisive results were proved for strongly pseudoconvex, smoothly bounded domains, Kerzman and Stein developed a powerful idea that allowed them to relate the Szegő projection $\mathcal{S}$ to a "Cauchy" integral operator $\mathcal{C}$ via an operator equation (see [28, 29] for the one variable and several variable cases, respectively). In the case $n \geq 2$, there is no "universal" or canonical integral kernel that reproduces and produces holomorphic functions on all domains D, while in $\mathbb{C}$ the familiar Cauchy kernel satisfies these properties and does not depend on the domain (see [36, 60] for some excellent reference material on these holomorphic integral representations). Instead, the construction of the kernel of $\mathcal{C}$ must proceed through ad-hoc methods that depend on the domain D.

The construction of Kerzman and Stein in [29] is different but bears some similiarities to the earlier work of Henkin and Ramirez [22, 59]. The essential idea, exploited in [29] as well as numerous other papers in the literature, involves constructing an auxiliary operator $\mathbf{C}$ that also produces and reproduces holomorphic functions inside D from boundary data, and defining $\mathcal{C}$ to be a restriction of $\mathbf{C}$ to the boundary in an appropriate sense, so that $\mathcal{C}$ is a singular integral operator. This operator $\mathbf{C}$ is given as a sum, $\mathbf{C}_1 + \mathbf{C}_2$, where $\mathbf{C}_1$ is constructed using the theory of Cauchy-Fantappié integrals and $\mathbf{C}_2$ is a correction term obtained by solving a $\overline{\partial}$ problem on a strongly pseudoconvex, smoothly bounded domain that contains $\overline{D}$ (see, for example, [29, 40, 60]). Importantly, $\mathbf{C}_1$ has a completely explicit kernel. The operator $\mathcal{C}^* - \mathcal{C}$ then roughly measures the "error" introduced by considering $\mathcal{C}$ instead of $\mathcal{S}$. A similar trick can be employed for the Bergman projection [42].

If we are to restrict our attention to strongly pseudoconvex domains (which are the domains on which the above projection operators can be constructed), we must at minimum assume that the boundary of our domain D is C^2 (minimal smoothness). By passing through these auxiliary operators, it is possible to obtain boundedness properties for the operators $\mathcal{B}$ and $\mathcal{S}$ without relying on explicit bounds for the Bergman or Szegő kernels. This indirect approach was employed by Lanzani and Stein in [37, 40] to study these problems in the case that D has C^2 boundary. In particular, Lanzani and Stein proved the following, which is the best result possible on strongly pseudoconvex domains:

Theorem 1.3.3. Let D be strongly pseudoconvex with C^2 boundary. Then $\mathcal{B}$ and $\mathcal{S}$ extend to bounded operators on L^p for $1 < p < \infty$.

Motivated by weighted results for Calderón-Zygmund operators such as Theorem 1.2.1, it is natural to determine necessary and sufficient conditions on a weight σ so that $\mathcal{B}$ or $\mathcal{S}$ map L^p_σ to L^p_σ. The main results in the literature pertaining to the boundedness of the Bergman projection on weighted spaces are due to Békollè and Bonami and consider the underlying domain to be the unit ball $\mathbb{B}_n$ [3, 4], see also [58]. The correct condition for the weights is referred to as the Békollè-Bonami, or B_p, condition. We state this condition below as well as the corresponding theorem in [3].

Definition 1.3.3. We say a weight $\sigma \in B_p(\mathbb{B}_n)$ if

$$[\sigma]_{B_p} := \sup_{B(z,r);r>1-|z|} \left(\frac{1}{V(B(z,r))} \int_{B(z,r)} \sigma dV \right) \left(\frac{1}{V(B(z,r))} \int_{B(z,r)} \sigma^{-1/(p-1)} dV \right)^{p-1} < \infty,$$

where the quasi-balls B are taken in the quasi-metric defined by

$$d(z,w) = ||z| - |w|| + \left| 1 - \frac{\langle z, w \rangle}{|z||w|} \right|.$$

Theorem 1.3.4. Let $1 < p < \infty$ and σ be a weight. Then the Bergman projection $\mathcal{B}$ is bounded on $L^p_\sigma(\mathbb{B}_n)$ if and only if $\sigma \in B_p$.

Notice this weight class is defined using a Muckenhoupt-type condition, but it is slightly altered to reflect the fact that the behavior of the weight away from the boundary is not important. The relevant quasi-metric was mentioned as an example of a space of homogeneous type in [12].

Considering the Szegő projection, there appear to be few weighted results that appear explicitly in the literature. The work of Hunt, Muckenhoupt, and Wheeden in establishing the necessity and sufficiency of the A_p condition for the L^p_σ boundedness of the Hilbert transform on the unit circle (the conjugate operator) also implies the necessity and sufficiency of the A_p condition for the Szegő projection on the unit circle as well, see [23, 56]. Note that in this case, the "balls" in the definition of the A_p characteristic are simply intervals on $\mathbb{T}$, and the measure is simply arc length. We also mention that [52, 54] contain some related weighted results for the unit disk. From a heuristic point of view, since the Szegő projection involves integration on the boundary and is a true singular integral, the correct class of weights should be an adaptation of the A_p class in Euclidean harmonic analysis. Therefore, the correct weight condition for the Szegő projection to be bounded on $L^p_\sigma(bD)$ should be for σ to belong to an A_p class on the boundary, where the non-isotropic boundary "balls" reflect the boundary geometry of the domain. We remark that analogous weighted results for the unit ball are likely known to the experts. We will also consider weighted estimates for the Szegő projection on certain classes of pseudoconvex domains in this thesis.

It is typical behavior for Calderón-Zygmund operators to fail to map L^1 to L^1 boundedly. It is well-known that this is the case for the Hilbert and Riesz transforms, and it is not too difficult to see that this also must be the case for the Bergman and Szegő projections on the unit disk/circle. This failure can be seen as consequence of the Rudin-Forelli estimates in the case of the Bergman projection (see [70, Theorem 1.12]), and the fact that the Hilbert transform/conjugate operator is unbounded on L^1 in the case of the Szegő projection (see [56]). In fact, a recent result shows that the projections are *always* unbounded on L^1 for any smoothly bounded domain, even without a pseudoconvexity assumption [13].

However, there is a replacement inequality for the failed L^1 boundedness many of these cases. Recall that we say a measurable function f belongs to *weak* L^1 on D (D any domain), and write $f \in L^{1,\infty}(D)$, if

$$\|f\|_{L^{1,\infty}(D)} := \sup_{\lambda>0} \lambda V(\{z \in D : |f(z)| > \lambda\}) < \infty.$$

The quantity $\|\cdot\|_{L^{1,\infty}(D)}$ is actually a quasi-norm (the triangle inequality is satisfied with a constant), and it is also a fact that $L^{1,\infty}(D)$ is a quasi-Banach space. By Chebyshev's Inequality, $L^1(D) \subset L^{1,\infty}(D)$ with bounded inclusion. We say that a linear operator T acting on measurable functions is *weak-type (1,1)* if there exists a constant $C > 0$ so that

$$\sup_{\lambda>0} \lambda V(\{z \in D : |Tf(z)| > \lambda\}) \leq C\|f\|_{L^1(D)}.$$

The infimum of all such constants C so that the above inequality holds is denoted by $\|T\|_{L^1(D)\to L^{1,\infty}(D)}$ and is referred to as the weak-type norm of T.

It is well known that the Hilbert and Riesz tranforms are weak-type $(1,1)$ (in this case, $D=\mathbb{R}$ or $\mathbb{R}^n$). It is also known that the Szegő projection on the unit circle in weak-type $(1,1)$. In terms of the Bergman projection, a result of Békollè states that the Bergman projection on the unit ball is weak-type $(1,1)$ [3, 15]. In fact, Békollè actually obtained a weighted weak-type estimate when σ belongs to an endpoint class of B_p weights called B_1 weights. We give the precise definition of a B_1 weight on the unit ball $\mathbb{B}_n$ here:

Definition 1.3.4. We say a weight $\sigma \in B_1(\mathbb{B}_n)$ if

$$[\sigma]_{B_1} := \sup_{\substack{B(z,r)\\ r>1-|z|}} \left(\frac{1}{V(B(z,r))} \int_{B(z,r)} \sigma \, dV \right) \|\sigma^{-1}\|_{L^\infty(B(z,r))} < \infty.$$

where the quasi-balls B are taken in the quasi-metric defined by

$$d(z,w) = ||z|-|w|| + \left| 1 - \frac{\langle z, w\rangle}{|z||w|} \right|.$$

Békollè proved the following theorem:

Theorem 1.3.5. Let σ be a weight on $\mathbb{B}_n$. Then $\mathcal{B}$ is bounded from $L^1_\sigma(D)$ to $L^{1,\infty}(D)$, meaning there exists a constant C so that

$$\sup_{\lambda>0} \lambda\sigma(\{z\in\mathbb{B}_n : |\mathcal{B}f(z)|>\lambda\}) \le C\|f\|_{L^1_\sigma(\mathbb{B}_n)},$$

if and only if $\sigma \in B_1$.

Notably, even unweighted weak-type $(1,1)$ estimates for the Bergman and Szegő projections have not been previously obtained on general strongly pseuedoconvex domains to the best of our knowledge.

1.4 Common Notation

In this section, we fix some common notation that will hold for the rest of the thesis. The letters z,w, and ζ will be most commonly used to denote complex vectors in $\mathbb{C}^n$. Given a complex vector $z \in \mathbb{C}^n$, we will denote its complex components with respect to the canonical basis as $z_1, z_2, \ldots, z_n$, although we will sometimes designate components in an alternative coordinate system in the same way (this will be clear from context). In what follows, $D \subset \mathbb{C}^n$ will be a bounded pseudoconvex domain with boundary bD and defining function ρ that is at least class C^2. The Bergman projection will be denoted by $\mathcal{B}$ while the Szegő projection will be denoted by $\mathcal{S}$. We denote the Bergman (or Szegő) kernel for a domain D by $K_D(z, w)$. For a generic kernel of an integral operator on $\mathbb{C}^n$, we use the notation $K(z, w)$ or $k(z, w)$.

We will write dV to denote Lebesgue measure on D and dS to denote induced Lebesgue surface measure on bD. The notation $L^p_\sigma(D)$ is used to refer to the L^p space on D with measure σdV (and $L^p_\sigma(bD)$ has the analogous meaning). We will use the letter σ to represent a weight and use the notation $\sigma(E)$ to mean $\int_E \sigma\, dV$. We will use the notation $\langle f \rangle_{E,\mu}$ to denote the average $\frac{1}{\mu(E)} \int_E f\, d\mu$. When this measure μ is Lebesgue measure or induced Lebesgue surface measure, we will omit the μ subscript and write $\langle f \rangle_E$ to denote the average of f on E. We use the symbol * to denote the adjoint of an operator on L^2. Importantly, the adjoint is taken on the *unweighted* Lebesgue space. The letter d is typically reserved for a metric/quasi-metric that plays the relevant role in the space under consideration. We write a quasi-ball of center z_0 and radius r in this quasi-metric as $B(z_0, r)$. For $1 < p < \infty$, we will let $q = \frac{p}{p-1}$ denote the Hölder conjugate exponent to p.

The letter C is typically used (with some exceptions) to refer to a positive constant, which can change from line to line. If the constant is important/has a significant interpretation, we label it with a number in the chapter (i.e. beginning with C_1). We will typically use a lowercase c for the constant in the triangle inequality in Definition 1.2.1. Sometimes we will not explicitly write constants and instead use the notation $A \lesssim B$ to mean that there exists a positive constant C, independent of obvious parameters, so that $A \leq CB$. The constant C can depend on various quantities depending on the context, but these are typically quantities that are intrinsic to the domain D and not dependent on a choice of base point, for example. Similarly, we write $A \gtrsim B$ if there exists a positive constant c so $A \geq cB$, and write $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$.

1.5 Main Results in this Thesis

Our first major result in this thesis is a significant generalization of Békollè's result in [3] for simple domains. Recall these were the classes of domains considered by McNeal in [47, 50]. All of these domains admit a quasi-metric $d(z,w)$ that reflects their geometry and naturally occurs in the estimation of the Bergman kernel. This quasi-metric will be defined more precisely in Chapter 2. Here we define an appropriate class of weights that will be useful for the rest of thesis:

Definition 1.5.1. Let D be a simple domain in the sense of McNeal [50]. For $1 < p < \infty$, we say a weight σ belongs to the *Békollè-Bonami (B_p)* class associated to the quasi-metric d if the following quantity is finite:

$$[\sigma]_{B_p} := \sup_{\substack{B(z,r) \\ r > d(z,bD)}} \left(\frac{1}{V(B(z,r))} \int_{B(z,r)} \sigma \, dV \right) \left(\frac{1}{V(B(z,r))} \int_{B(z,r)} \sigma^{-1/(p-1)} dV \right)^{p-1}.$$

We can also define an endpoint B_1 class of weights on such a domain as follows:

Definition 1.5.2. We say a weight σ belongs to the B_1 class associated to the quasi-metric

d on a simple domain D if the following quantity is finite:

$$[\sigma]_{B_1} := \sup_{\substack{B(z,r)\\ r>d(z,bD)}} \left(\frac{1}{V(B(z,r))} \int_{B(z,r)} \sigma \, dV \right) \|\sigma^{-1}\|_{L^\infty(B(z,r))} < \infty.$$

Remark 1.5.1. Technically, the quasi-metric d will only be defined close to the boundary (see Chapter 2 for more details). However, the collection of balls in the supremum in both these definitions should be interpreted to also include the entire domain D.

Chapter 2 is concerned with proving the following theorem:

Theorem 1.5.1. Let D be a simple domain and $1 < p < \infty$. If $\sigma \in B_p$, then there exists $C > 0$ so that $\|\mathcal{B}f\|_{L^p_\sigma(D)} \leq C\|f\|_{L^p_\sigma(D)}$.

The main strategy employed in Chapter 2 is to use Calderón-Zygmund estimates obtained by McNeal and others combined with some of the techniques used in Bekolle's paper [3], suitably adapted to more general domains. We remark that a similar result to Theorem 2.1.1 for the special case of convex domains of finite type appears in [20] using different methods. In the special case that D is strongly pseudoconvex, this sufficient condition is also necessary:

Theorem 1.5.2. Let D be a smoothly bounded, strongly pseudoconvex domain, $1 < p < \infty$, and σ be a weight. Then $\mathcal{B}$ is bounded on $L^p_\sigma(D)$ if and only if $\sigma \in B_p$.

Next, we turn our attention to strongly pseudoconvex domains with minimal (C^2) or near-minimal smoothness. In Chapter 3, we prove that the familiar A_p condition from harmonic analysis is sufficient for the boundedness of the Szegő projection on strongly pseudoconvex domains with C^2 boundary. In particular, we have the following theorem:

Theorem 1.5.3. Let D be strongly pseudoconvex with C^2 boundary and $1 < p < \infty$. If $\sigma \in A_p$, then there exists $C > 0$ so that $\|\mathcal{S}f\|_{L^p_\sigma(bD)} \leq C\|f\|_{L^p_\sigma(bD)}$.

Of course, the relevant A_p class is defined using a quasi-metric on the boundary that appropriately reflects its geometry. The class is defined precisely in Chapter 3.

In Chapter 4, we consider a similar question for the Bergman projection, and obtain the following result:

Theorem 1.5.4. Let D be strongly pseudoconvex with C^4 boundary and $1 < p < \infty$. If $\sigma \in B_p$, then there exists $C > 0$ so that $\|\mathcal{B}f\|_{L^p_\sigma(D)} \leq C\|f\|_{L^p_\sigma(D)}$.

These results are significant because methods that involve direct kernel estimation are unavailable in the minimally smooth setting. The machinery used to prove these theorems goes back to theory developed by Kerzman, Stein, Ligocka, and Lanzani that relates the projection operator to a non-canonical, non-orthogonal projection via and operator equation. We discuss this more precisely in Chapter 3 Section 3.2. We also apply these techniques to obtain a specialized result for the Bergman projection on minimally smooth domains with weights that are a power of the distance to the boundary and deduce certain mapping properties of Toeplitz operators with power-distance symbols.

Finally, in Chapter 5 we discuss some endpoint results for the two projection operators on near-minimally smooth, strongly pseudoconvex domains. In particular, we obtain weighted weak-type estimates for the Bergman and Szegő projections for an appropriate endpoint class of weights:

Theorem 1.5.5. Let D be strongly pseudoconvex with C^4 boundary. If $\sigma \in B_1$, then there exists $C > 0$ so that $\|\mathcal{B}f\|_{L^{1,\infty}_\sigma(D)} \leq C\|f\|_{L^1_\sigma(D)}$.

Theorem 1.5.6. Let D be strongly pseudoconvex with C^3 boundary. If $\sigma \in A_1$, then there exists $C > 0$ so that $\|\mathcal{S}f\|_{L^{1,\infty}_\sigma(bD)} \leq C\|f\|_{L^1_\sigma(bD)}$.

These results extend the results of Chapter 3 and Chapter 4 to the $p = 1$ endpoint and also extend Theorem 1.3.5 to more general domains. The proofs again involve the use of the Kerzman-Stein equation, as well as a compactness criterion for integral operators on L^1 and a generalization of the classical Riesz-Kolmogorov theorem. We also obtain some additional weighted endpoint estimates, such as Kolmogorov and Zygmund inequalities, and an appropriate (unweighted) estimate at $p = \infty$.

Chapter 2

The Bergman Projection on Smooth Domains

2.1 Summary of Main Results

A natural question is whether Békollè's result in [3] can be generalized in a suitable sense to more general classes of pseudoconvex domains. In particular, in this chapter we establish analogous weighted estimates for several classes of smoothly bounded, pseudoconvex domains with "reasonable" geometry. This chapter comprises work which appears in [24].

Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain with C^∞ defining function ρ. We also assume that D is a simple domain in the sense of the Introduction (we restate the definition in the next section). The estimates on the Bergman kernel in these cases also facilitate the development of an appropriate B_p-type class of weights σ (see Definition 1.5.1) for which the Bergman projection $\mathcal{B}$ is bounded on $L^p_\sigma(D)$, which is the focus of this chapter. The important thing to keep in mind is that in every case we have a quasi-metric d that reflects the boundary geometry of the domain D. The following is our principal result in this chapter:

Theorem 2.1.1. Let D be a simple domain and $1 < p < \infty$. If $\sigma \in B_p$, then there exists $C > 0$ so that $\|\mathcal{B}f\|_{L^p_\sigma(D)} \leq C\|f\|_{L^p_\sigma(D)}$.

For strongly pseuedoconvex domains, the B_p condition is also necessary.

Theorem 2.1.2. Let D be a smoothly bounded, strongly pseudoconvex domain, $1 < p < \infty$, and σ be a weight. Then $\mathcal{B}$ is bounded on $L^p_\sigma(D)$ if and only if $\sigma \in B_p$.

This chapter is organized as follows. Section 2.2 explains how the quasi-metric d is constructed in all these cases and gives rise to a homogeneous structure, and also states some crucial kernel estimates. In Section 2.3, we prove Theorem 2.1.1. Finally, in Section 2.4, we establish the necessity of the B_p condition for strongly pseudoconvex domains and thus prove Theorem 2.1.2.

2.2 Preliminaries

All of the domains in this paper are pseudoconvex of finite type in the sense of D'Angelo (see Definition 1.3.2, [14, 33]). In what follows we assume that D is one of the following types of pseudoconvex domains:

1. strongly pseudoconvex;
2. convex of finite type;
3. finite type in $\mathbb{C}^2$;
4. decoupled finite type in $\mathbb{C}^n$.

Following McNeal in [50], we will refer to such a domain as a *simple domain*. In [50], McNeal shows that estimates for the Bergman kernel previously obtained in [45],[48], and [46] actually fall into a unified framework using a scaling approach which we describe below. This scaling approach enables one to leverage subelliptic estimates for the $\bar{\partial}$-Neumann operator to obtain estimates on the Bergman kernel. Results on the L^p regularity of the Bergman projection on smooth pseudoconvex domains of finite type have actually been obtained in a more general context (see [31]), but in this chapter we focus on these *simple domains* since the quasi-metric in each of these cases leads to a space of homogeneous type.

We describe, first in qualitative terms, the scaling approach used by McNeal to obtain kernel estimates on all of these domains. See [50] for a detailed explanation in the strongly pseudoconvex case and [45, 46, 48, 49] for the other cases, as well as [55] for a correction to a statement in McNeal's original construction in the convex case. Let ρ be a fixed smooth defining function for D. Let U be a small neighborhood of a point $p \in bD$ and fix a point $q \in U$. A holomorphic coordinate change $z = (z_1, \cdots, z_n) = \Phi(w)$ with $\Phi(q) = 0$ is employed so that z_1 is essentially in the complex normal direction (i.e the complex direction in the orthogonal complement of $T_{\pi(q)}$, where π denotes the orthogonal projection to the boundary). In particular, the coordinates can be chosen so $\frac{\partial \rho}{\partial z_1}$ is non-vanishing on U. The coordinates $z_2, z_3, \ldots, z_n$ are basically the complex tangential directions. The geometric properties of the domain dictate the following: how far can one move in each of the complex directions $z_1, z_2, \ldots, z_n$ if one does not want to perturb the defining function $\rho(z)$ by more than δ (more precisely, a universal constant times δ)? Clearly, one can move no more than some constant multiple of δ in the radial direction, but it is not at all clear for an arbitrary domain what the answer is for the tangential directions. In fact, roughly speaking, the finite type property of the domain is precisely what ensures that the domain is not "too flat" and that the amount we can move in the tangential directions (and intermediate directions) is somehow appropriately controlled. We make this notion precise in the following proposition, which can be found in [50]:

Proposition 2.2.1. Let D be a simple domain. Fix a point $p \in bD$. Then there exists a small neighborhood U such that for sufficiently small $\delta > 0$ and any point $q \in U \cap D$, there exist holomorphic coordinates $z = (z_1, z_2, \ldots, z_n)$ centered at q and defined on U and quantities $\tau_1(q, \delta), \tau_2(q, \delta), \ldots, \tau_n(q, \delta)$ with $\tau_1(q, \delta) = \delta$ so that if we consider the polydisc centered at q:

$$P(q, \delta) = \{z : |z_j| < \tau_j(q, \delta), 1 \leq j \leq n\},$$

one has the property that if $z \in P(q, \delta) \cap D$, then $|\rho(z) - \rho(q)| \lesssim \delta$, where the implicit constant is independent of q and δ. Moreover, $\left|\dfrac{\partial \rho}{\partial z_1}\right| > C$ for some $C > 0$ on $U \cap D$, where C is independent of q and δ. In particular, $\dfrac{\partial \rho}{\partial \mathrm{Re}\, z_1} > C$ on $U \cap D$.

The coordinates $(z_1, z_2, \ldots, z_n)$ can depend on δ, for example in the convex finite type case (see [48]), but z_1 is always essentially the radial direction. Crucially, the polydiscs also satisfy a kind of doubling property:

Proposition 2.2.2 ([50],[47])**.** There exist independent constants C_1, C_2 so the following hold for the polydiscs:

1. If $P(q_1, \delta) \cap P(q_2, \delta) \neq \emptyset$, then $P(q_1, \delta) \subset C_1 P(q_2, \delta)$ and $P(q_2, \delta) \subset C_1 P(q_1, \delta)$.
2. There holds $P(q_1, 2\delta) \subset C_2 P(q_1, \delta)$.

One can now introduce a local quasi-metric M on $U \cap D$ (see [47]):

Definition 2.2.1. Define the following function on $U \cap D \times U \cap D$:

$$M(z, w) = \inf_{\varepsilon > 0} \{\varepsilon : w \in P(z, \varepsilon)\}.$$

Then M defines a quasi-metric on $U \cap D$.

Note that the volume of a polydisc $P(q, \delta)$ is comparable to $\delta^2 \prod_{j=2}^{n} (\tau_j(q, \delta))^2$. The constants, of course, are dependent on the Jacobian of the biholomorphism, but can be seen to be independent of the base point q and δ. Moreover, this polydisc is comparable in measure to a non-isotropic ball of radius δ centered at q in the local quasi-metric. Let $U_1, \cdots, U_N$ be a finite covering of bD by Euclidean balls with radius ε. We can suppose each set $3U_j$ is a neighborhood where a local quasi-metric M_j can be constructed. To extend this quasi-metric M_j to a global quasi-metric d defined on a tubular neighborhood of the boundary, one can just patch the local metrics defined on $3U_j \cap D$ together in an appropriate way. Let $N = \bigcup_{j=}^{n} (3U_j \cap D)$ denote this relative neighborhood of bD. In particular, let $\phi \in C_0^\infty(3U_j)$

be a bump function satisfying $\phi_i \equiv 1$ on $2U_j$. For $z, w \in N$, we define

$$d(z,w) = \begin{cases} \sum_{j=1}^{n} \phi_j(z)\phi_j(w)M_j(z,w) & |z-w| < \varepsilon \\ |z-w| & |z-w| \geq \varepsilon. \end{cases}$$

Here, $M_j(z,w)$ should be interpreted to be 0 if either $z \notin 3U_j$ or $w \notin 3U_j$. The resulting quasi-metric is not continuous, but it can be checked that satisfies all the relevant properties. The balls in this quasi-metric still have volume comparable to a polydisc if they have small radius.

We remark that the construction in [47] is not quite correct in the claim that this construction defines a metric on $D \times D$. This metric is evidently only defined on $N \times N$. However, the next lemma shows that this does not present us with any difficulties in bounding the Bergman projection.

Lemma 2.2.1. Let $\mathcal{B}|_N$ denote the Bergman projection restricted to N; that is, for $f \in L^2(D)$ and $z \in D$,

$$\mathcal{B}(f)(z) := \chi_N(z) \int\limits_N K_D(z,w)f(w)dV(w),$$

where $K_D(z,w)$ denotes the Bergman kernel for D and χ denotes characteristic function.

Then, if $\mathcal{B}|_N$ is bounded on $L^p_\sigma(D)$ and $\sigma, \sigma' = \sigma^{-1/(p-1)}$ are integrable on D, then $\mathcal{B}$ is bounded on $L^p_\sigma(D)$.

Proof. Take $f \in L^p_\sigma(D)$ and write $f = f_1 + f_2$, where $f_1 := f\chi_N$ and $f_2 := f\chi_{D\setminus N}$. Then write

$$\begin{aligned}
\|\mathcal{B}f\|_{L^p_\sigma(D)} &\leq \|\mathcal{B}f_1\|_{L^p_\sigma(D)} + \|\mathcal{B}f_2\|_{L^p_\sigma(D)} \\
&\leq \|\mathcal{B}f_1\|_{L^p_\sigma(N)} + \|\mathcal{B}f_2\|_{L^p_\sigma(N)} + \|\mathcal{B}f_1\|_{L^p_\sigma(D\setminus N)} + \|\mathcal{B}f_2\|_{L^p_\sigma(D\setminus N)} \\
&= \|\mathcal{B}|_N f\|_{L^p_\sigma(D)} + \|\mathcal{B}f_2\|_{L^p_\sigma(N)} + \|\mathcal{B}f_1\|_{L^p_\sigma(D\setminus N)} + \|\mathcal{B}f_2\|_{L^p_\sigma(D\setminus N)} \\
&\lesssim \|f\|_{L^p_\sigma(D)} + \|\mathcal{B}f_2\|_{L^p_\sigma(N)} + \|\mathcal{B}f_1\|_{L^p_\sigma(D\setminus N)} + \|\mathcal{B}f_2\|_{L^p_\sigma(D\setminus N)}
\end{aligned}$$

where in the last line we used the hypothesis on $\mathcal{B}|_N$. Thus, if we can control the last three terms, we are done. Recall by a result of Kerzman and Boas, the Bergman kernel for simple domains extends to a C^∞ function on $\overline{D} \times \overline{D} \setminus \triangle(bD \times bD)$, where $\triangle(bD \times bD)$ denotes the boundary diagonal $\{(z, z) : z \in bD\}$(see [30], [5]). Thus, in particular $K_D(z, w)$ is bounded on compact subsets of $\overline{D} \times \overline{D}$ that do not intersect the boundary diagonal. We show how this is applied to the term $\|\mathcal{B}f_2\|_{L^p_\sigma(N)}$, as the other terms can be handled similarly. Then, using this fact about $K_D(z, w)$, Hölder's inequality, and the hypotheses on σ,

$$
\begin{aligned}
\|\mathcal{B}f_2\|^p_{L^p_\sigma(N)} &= \int\limits_N \left| \int\limits_{D\setminus N} K_D(z,w)f(w)dV(w)\right|^p \sigma(z)dV(z) \\
&\leq \int\limits_N \left(\int\limits_{D\setminus N} |K_D(z,w)||f(w)|dV(w)\right)^p \sigma(z)dV(z) \\
&\lesssim \int\limits_D \left(\int\limits_D |f(w)|dV(w)\right)^p \sigma(z)dV(z) \\
&= \sigma(D) \left(\int\limits_D |f(w)|\sigma(w)^{1/p}\sigma(w)^{-1/p}dV(w)\right)^p \\
&\leq \sigma(D) \left(\int\limits_D |f(w)|^p\sigma(w)dV(w)\right) \left(\int\limits_D \sigma'(w)dV(w)\right)^{p/q} \\
&= \|f\|^p_{L^p_\sigma(D)}\sigma(D)\left(\sigma'(D)\right)^{p/q} \\
&\lesssim \|f\|^p_{L^p_\sigma(D)}
\end{aligned}
$$

which establishes the result.

□

This lemma shows that we can reduce to considering N in place of D and $\mathcal{B}|_N$ in place of $\mathcal{B}$. Therefore, going forward, we will abuse notation by writing D when we really mean the neighborhood N.

It is proven in [47] that the triple (D, d, V) constitutes a space of homogeneous type with triangle inequality constant c. Note that the measure V is doubling on the non-isotropic balls essentially because of Proposition 2.2.2. Note if d is not symmetric, we can symmetrize it by taking $d(z,w) + d(w,z)$ as an equivalent quasi-metric. We denote a ball in the quasi-metric

d of center z_0 and radius r by

$$B(z_0, r) = \{z \in D : d(z, z_0) < r\}.$$

Since ρ can be taken to be defined on $\mathbb{C}^n$, this quasi-metric actually extends to $\overline{D} \times \overline{D}$ because a polydisc can be centered at $q \in bD$. Thus, for $z \in D$, define $d(z, bD)$ as follows:

$$d(z, bD) := \inf_{w \in bD} d(z, w).$$

It is trivial to verify that for $z, z' \in D$,

$$d(z, bD) \lesssim d(z', bD) + d(z, z').$$

One can actually show that the distance to the boundary in this quasi-metric is comparable to the Euclidean distance. We have the following lemma.

Lemma 2.2.2. Let $\operatorname{dist}(z, bD)$ denote the Euclidean distance of z to the boundary of D. Then we have

$$d(z, bD) \approx \operatorname{dist}(z, bD).$$

Proof. We can assume that z is sufficiently close to the boundary. Let $\pi(z)$ be the normal projection of z to the boundary, which is uniquely defined in a tubular neighborhood of the boundary [2]. Then $d(z, \pi(z)) \lesssim \operatorname{dist}(z, \pi(z)) = \operatorname{dist}(z, bD)$ by the structure of the quasi-metric (note that the first coordinate of the polydisc corresponds to the radial direction). This shows the bound $d(z, bD) \lesssim \operatorname{dist}(z, bD)$.

For the other bound, we only need consider the distance of z to points on the boundary in a local neighborhood U where the local quasi-metric is defined (because otherwise the distances will reduce to Euclidean distance). Let $\varepsilon = \operatorname{dist}(z, bD) \approx |\rho(z)|$. It is clear there is a universal constant $c > 0$ so that the shrunken polydisc $P(z, c\varepsilon)$ is strictly contained in D. This implies that $d(z, bD) \gtrsim \text{dist}(z, bD)$, as desired.

$\square$

The following estimates for the Bergman kernel were obtained by McNeal (see [45, 46, 48, 50] and also [53] for a slightly different approach due to Nagel, Stein, Rosay, and Wainger):

Theorem 2.2.1. Let D be a simple domain and $K_D(z, w)$ denote the Bergman kernel for D. Then near any $p \in bD$, there exists a coordinate system centered at $z = (z_1, z_2, \ldots, z_n)$ so that if α, β are multi-indices and $\mathcal{D}^\alpha$, $\overline{\mathcal{D}}^\beta$ denote holomorphic and anti-holomorphic derivatives, respectively, taken in these coordinate directions, we have the following:

$$|\mathcal{D}_z^\alpha \overline{\mathcal{D}}_w^\beta K_D(z, w)| \leq C_{\alpha,\beta} \delta^{-(2+\alpha_1+\beta_1)} \prod_{k=2}^{n} \tau_k(z, \delta)^{-(2+\alpha_k+\beta_k)}$$

where $\delta = |\rho(z)| + |\rho(w)| + M(z, w)$.

By using the global quasi-metric d, one can obtain global estimates on the Bergman kernel. The following was proven in [47]:

Theorem 2.2.2. Let D be a simple domain. Then the following hold:

1. *(Size)* There exists a constant C_3 so that for all $z, w \in D$:

$$|K_D(z, w)| \leq \frac{C_3}{V(B(z, d(z, w)))}.$$

2. *(Smoothness)* There exists a constant C_4 and $\nu > 0$ so that we have, provided $d(z, w) \geq C_4 d(z, z')$:

$$|K_D(z, w) - K_D(z', w)| \leq C_3 \left(\frac{d(z, z')}{d(z, w)}\right)^\nu \frac{1}{V(B(z, d(z, w)))}.$$

We actually get another size estimate for free, which will help us in the course of the proof. This lemma can actually be deduced directly from Theorem 2.2.1, but we provide another proof here (which actually shows any domain, not necessarily simple, whose Bergman kernel satisfies the estimates in Theorem 2.2.2 will necessarily satisfy an additional estimate).

Lemma 2.2.3. Suppose $K_D(z,w)$ is the Bergman kernel for D and K_D satisfies the size estimate above. Then there exists a constant C_5 so uniformly for all $z, w \in D$

$$|K_D(z,w)| \leq C_5 \min\left\{\frac{1}{V(B(z,d(z,bD)))}, \frac{1}{V(B(w,d(w,bD)))}\right\}.$$

Proof. Fix $z \in D$. We first claim that given $\varepsilon > 0$, there exists a $w' \in D$ so $|K_D(z,w)| \leq |K_D(z,w')|$ and $\operatorname{dist}(w', bD) \leq \varepsilon$. The claim follows immediately by applying the Maximum Principle to the closed domain $D_\varepsilon = \{w \in D : |\rho(w)| \geq \varepsilon\}$ and function $K_D(w,z) = \overline{K_D(z,w)}$, which is analytic in w.

Now choose $w' \in bD_\varepsilon$ satisfying the above conditions. Then we have, using Lemma 2.2.2:

$$d(z,bD) \leq cd(z,w') + cd(w',bD) \leq c'd(z,w') + c'\varepsilon$$

so we obtain the estimate

$$d(z,w') \geq \frac{1}{c'}d(z,bD) - \varepsilon.$$

Thus, applying the known size estimate, we get

$$|K_D(z,w)| \leq |K_D(z,w')| \leq \frac{C_3}{V(B(z,d(z,w')))} \leq \frac{C_3}{V(B(z,\frac{1}{c'}d(z,bD) - \varepsilon))}.$$

Since the inequality above holds for all $\varepsilon > 0$ and Lebesgue measure is doubling on these quasi-balls, we obtain

$$|K_D(z,w)| \leq \frac{C_5}{V(B(z,d(z,bD)))}$$

as desired. Note C_5 is independent of z. The other inequality follows by symmetry.

□

Remark 2.2.1. As a clear example of this property, consider the unit ball $\mathbb{B}_n$ where the Bergman kernel is given by $K_{\mathbb{B}_n}(z,w) = \frac{n!}{\pi^n}\frac{1}{(1-z\cdot\overline{w})^{n+1}}$. Then $|K_{\mathbb{B}_n}(z,w)| \lesssim \frac{1}{(1-|z|)^{n+1}}$.

It is a well-known fact in harmonic analysis (for example, see [16]) that if $B(z,r)$ is a ball of radius r, center z in a space of homogeneous type with measure μ, then there exists uniform constants C_6, m so that if $\lambda \geq 1$, we have

$$\mu(B(z,\lambda r)) \leq C_6\lambda^m \mu(B(z,r)). \tag{2.2.1}$$

Here the parameter m can be thought of as roughly corresponding to the "dimension" of the space. We will use this fact, referred to as the *strong homogeneity property*, in a crucial point in the proof of the main theorem.

To continue with the analysis, we need to define an appropriate maximal function with respect to the quasi-metric. In analogy with Békollè's result, we will also only consider balls that touch the boundary of D. We make the following definition:

Definition 2.2.2. For $z \in D$ and $f \in L^1(D)$, define the following maximal function:

$$\mathcal{M}f(z) := \sup_{\substack{B(w,r)\ni z \\ r>d(w,bD)}} \langle |f| \rangle_{B(w,r)}.$$

Proving Theorem 2.1.1 can be broken down into the task of proving the following two results (mimicking the approach taken by Békollè in [3]):

Theorem 2.2.3. Let $1 < p < \infty$ and suppose $\sigma \in B_p$. Then there exists a constant $C > 0$ so $\|\mathcal{M}f\|_{L^p_\sigma(D)} \leq C\|f\|_{L^p_\sigma(D)}$.

Theorem 2.2.4. Let $\mathcal{B}^+$ be the positive operator defined $\mathcal{B}^+ f(z) = \int_D |K_D(z,w)|f(w)\,dV(w)$. Let $1 < p < \infty$ and $\sigma \in B_p$. Then there exists a constant $C > 0$ so $\|\mathcal{B}^+ f\|_{L^p_\sigma(D)} \leq C\|\mathcal{M}f\|_{L^p_\sigma(D)}$.

We will prove these two theorems in the following section. It is worth pointing out that Theorem 2.2.3 in conjunction with Theorem 2.2.4 shows that Theorem 2.1.1 actually holds when $\mathcal{B}$ is replaced with $\mathcal{B}^+$, as is typical for Bergman-type operators.

2.3 The Sufficiency of the B_p Condition

We begin by proving Theorem 2.2.3. In what follows, we follow the general outline of the approach taken in [3]. To begin with, we define a regularizing operator R_k for $k \in (0,1)$:

$$R_k(f)(z) := \langle |f| \rangle_{B_k(z)}$$

where $B_k(z) = \{w \in D : d(w,z) < kd(z,bD)\}$.

Intuitively, this regularizing operator spreads out the mass of the weight. We will ultimately show it turns B_p weights into A_p weights. In what follows, we say a ball $B(z,r)$ *touches* bD if $r > d(z,bD)$. We begin with a simple proposition.

Proposition 2.3.1. There exists a constant $C_d > 1$ (depending on the quasi-metric d) so that if $k \in (0, \frac{1}{2C_d})$, then $z' \in B_k(z)$ implies $z \in B_{k'}(z')$, where $k' = \dfrac{C_d k}{1 - C_d k}$.

Proof. This is a trivial consequence of the (quasi)-triangle inequality. In fact, we can take $C_d = c > 1$, where c is the implicit constant in the triangle inequality. □

It is also routine to verify that the radius of $B_{k'}(z')$ is at most a fixed multiple of the radius of $B_k(z)$ and the quasi-balls have comparable Lebesgue measure, where the implicit constants are independent of $k \in (0, \frac{1}{2c})$. We need another simple proposition to furnish the next lemma.

Proposition 2.3.2. Let B be a quasi-ball of radius r, center z_0, that touches the boundary of D (i.e $r > d(z_0, bD)$). Let $k \in (0,1)$ be fixed. Then there exists an (absolute, independent of k) constant C so the dilated quasi-ball $\tilde{B}$ with radius Cr and center z_0 satisfies $\tilde{B} \supset B_k(w)$ for all $w \in B$.

Proof. Again, the proof is routine. This is also a simple consequence of the triangle inequality. □

We are now ready to prove the following significant lemma.

Lemma 2.3.1. There exists $C > 0$ so that for each $k \in (0, \frac{1}{2c})$ and $z_0 \in D$, we have $\mathcal{M}f(z_0) \leq C\mathcal{M}(R_k(f(z_0)))$.

Proof. Fix k and let B be an arbitrary ball touching bD and centered at z_0, $\tilde{B}$ an inflation of B with radius chosen as in the previous proposition so that $\tilde{B} \supset B_{k''}(w)$ for all $w \in B$, where $k'' = \frac{k}{c(k+1)}$. Then we have the following:

$$
\begin{aligned}
\mathcal{M}(R_k(f(z_0))) &\geq \frac{1}{V(\tilde{B})} \int\limits_{\tilde{B}} \frac{1}{V(B_k(z))} \int\limits_{B_k(z)} |f(w)| dV(w) dV(z) \\
&= \frac{1}{V(\tilde{B})} \int\limits_{D} \int\limits_{\tilde{B}} \frac{1}{V(B_k(z))} \chi_{B_k(z)}(w) |f(w)| dV(z) dV(w) \\
&\gtrsim \frac{1}{V(\tilde{B})} \int\limits_{B} \int\limits_{\tilde{B}} \frac{1}{V(B_{k''}(w))} \chi_{B_{k''}(w)}(z) |f(w)| dV(z) dV(w) \\
&= \frac{1}{V(\tilde{B})} \int\limits_{B} |f(w)| dV(w) \\
&\approx \frac{1}{V(B)} \int\limits_{B} |f(w)| dV(w)
\end{aligned}
$$

where we used both propositions and the fact that Lebesgue measure V is doubling on quasi-balls. Since the following estimate is true for all balls B centered at z_0, the conclusion follows.

□

We have the additional following lemma which is a straightforward application of Proposition 2.3.1.

Lemma 2.3.2. Let f, g be positive, locally integrable functions. Then there exists $C > 0$ so for each $k \in (0, \frac{1}{2c})$, we have the inequality:

$$\int_D f R_k(g) dV \leq C \int_D R_{k'}(f) g dV,$$

where $k' = \dfrac{ck}{1 - ck}$.

Proof. We have:

$$\begin{aligned}
\int_D f R_k(g) dV &= \int_D f(z) \frac{1}{V(B_k(z))} \int_{B_k(z)} g(w) dV(w) dV(z) \\
&= \int_D \int_D \frac{f(z)}{V(B_k(z))} \chi_{B_k(z)}(w) g(w) dV(z) dV(w) \\
&\lesssim \int_D \int_D \frac{f(z)}{V(B_{k'}(w))} \chi_{B_{k'}(w)}(z) g(w) dV(z) dV(w) \\
&= \int_D g(w) \frac{1}{V(B_{k'}(w))} \int_{B_{k'}(w)} f(z) dV(z) dV(w) \\
&= \int_D R_{k'}(f) g dV,
\end{aligned}$$

as desired. □

The next lemma is fairly straightforward, but does require some care.

Lemma 2.3.3. Fix $k \in (0, \frac{1}{2c})$. Then for any positive, locally integrable function g there holds

$$R_k(\mathcal{M}(g))(z) \approx \mathcal{M}(g)(z),$$

where the implicit constant is independent of k.

Proof. It suffices to prove that for any fixed $z \in D$, there holds for $w \in B_k(z)$

$$\mathcal{M}(g)(w) \lesssim \inf_{z' \in B_k(z)} \mathcal{M}(g)(z') \leq \mathcal{M}(g)(w),$$

where the implicit constant is absolute. Assuming the claim, then

$$\begin{aligned} R_k(\mathcal{M}(g))(z) &= \frac{1}{V(B_k(z))} \int_{B_k(z)} \mathcal{M}(g)(w) dV(w) \\ &\approx \frac{1}{V(B_k(z))} \int_{B_k(z)} \inf_{z' \in B_k(z)} \mathcal{M}(g)(z') dV(w) \\ &= \inf_{z' \in B_k(z)} \mathcal{M}(g)(z') \\ &\approx \mathcal{M}(g)(z). \end{aligned}$$

Now we prove the claim. The upper bound is trivial. Fix $z \in D$. It is clearly sufficient to show that for any quasi-ball B centered at $w \in B_k(z)$ touching the boundary with radius r, given any $z' \in B_k(z)$, there is a quasi-ball $\tilde{B}$ centered at z' with radius Cr so $\tilde{B} \supset B$. First, note that if B touches bD we must have $r \geq \frac{1}{4c} d(z, bD)$, otherwise

$$d(z, bD) \leq cd(z, w) + cd(w, bD) \leq ck[d(z, bD)] + cr \leq \frac{1}{2} d(z, bD) + \frac{1}{4} d(z, bD) < d(z, bD),$$

which is absurd. Therefore we may conclude $d(z, bD) \leq 4cr$.

Thus, if $w' \in B$, we have

$$\begin{aligned} d(w', z') &\leq c^2[d(w', w) + d(w, z) + d(z, z')] \\ &\leq c^2[r + 2kd(z, bD)] \\ &\leq 9c^3 r \end{aligned}$$

so the claim is established by taking $C = 9c^3$.

□

We will need the following proposition concerning a kind of doubling property for B_p weights, which appears to be well-known insofar as it is used implicitly in Békollè's original paper. The proof is largely the same as the proof for the doubling of A_p weights (see, for example, [21, Proposition 7.1.5]), so we omit it.

Proposition 2.3.3. Suppose $\sigma \in B_p$. Let B be a quasi-ball (not necessarily touching bD) such that λB touches bD, where $\lambda > 1$. Then for any $\lambda' > 1$, we have

$$\sigma(\lambda' B) \lesssim \sigma(B),$$

where the implicit constant depends only on $\max\{\lambda, \lambda'\}$.

We now proceed to the proof of Theorem 2.2.3.

Proof of Theorem 2.2.3. Using the results previously proven, we can make the following progress to proving the theorem, fixing $k \in (0, \frac{1}{2c})$ (some of the following implicit constants can depend on k, but k is fixed):

$$
\begin{aligned}
\int_D [\mathcal{M}(f)(z)]^p \sigma \, dV &\lesssim \int_D [R_k(\mathcal{M}(R_k(|f|)))]^p \sigma dV \\
&\leq \int_D R_k[[\mathcal{M}(R_k(|f|))]^p] \sigma dV \\
&\lesssim \int_D [\mathcal{M}(R_k(|f|))]^p R_{k'}(\sigma) dV \\
&\lesssim \int_D [\mathcal{M}(R_k(|f|))]^p R_k(\sigma) dV,
\end{aligned}
$$

where in the first inequality we use Lemmas 2.3.1 and 2.3.3, the second inequality is Hölder, the penultimate inequality is Lemma 2.3.2, and the last inequality is given by the doubling property of σ given in Proposition 2.3.3.

Now, if we can prove that the weight $R_k(\sigma)$ belongs to A_p, by ordinary weighted theory the last quantity will be controlled by a positive constant depending on p, σ and D times

$$
\int_D [R_k(|f|)]^p R_k(\sigma) dV.
$$

Assuming this, then we have

$$
\begin{aligned}
\int_D [R_k(|f|)]^p R_k(\sigma) dV &\leq \int_D R_k(|f|^p \sigma)[R_k(\sigma^{-1/(p-1)})]^{p-1} R_k(\sigma) dV \\
&\lesssim [\sigma]_{B_p} \int_D R_k(|f|^p \sigma) dV \\
&\lesssim [\sigma]_{B_p} \int_D |f|^p \sigma dV
\end{aligned}
$$

where in the first inequality we use Hölder, the second inequality comes from the fact that $[R_k(\sigma^{-1/(p-1)})]^{p-1} R_k(\sigma) \lesssim [\sigma]_{B_p}$ (to see this, inflate the quasi-balls $B_k(z)$ by at most a fixed amount so they touch the boundary), and for the last step use Lemma 2.3.2.

Thus, it remains to prove that $R_k(\sigma) \in A_p$. To see this we need to consider two cases for the quasi-ball $B(z_0, r)$ over which we take averages: the case where $d(z_0, bD) < 2cr$ (we can inflate the quasi-ball so it touches the boundary), and the case where $d(z_0, bD) \geq 2cr$. For the first case, we proceed as follows:

$$\frac{1}{V(B)} \int_B R_k(\sigma) dV \lesssim \frac{1}{V(B)} \int_B \sigma \, dV$$

using Lemma 2.3.2, while the other factor is controlled as follows:

$$\begin{aligned}
\left(\frac{1}{V(B)} \int_B [R_k(\sigma)]^{-1/(p-1)} dV \right)^{p-1} &= \left(\frac{1}{V(B)} \int_B \left(\frac{1}{B_k(z)} \int_{B_k(z)} \sigma(w) dV(w) \right)^{-1/(p-1)} dV(z) \right)^{p-1} \\
&\leq \left(\frac{1}{V(B)} \int_B \frac{1}{B_k(z)} \int_{B_k(z)} \sigma(w)^{-1/(p-1)} dV(w) \; dV(z) \right)^{p-1} \\
&\lesssim \left(\frac{1}{V(B)} \int_B \sigma^{-1/(p-1)} dV \right)^{p-1}
\end{aligned}$$

where for the first inequality we used Hölder and the second inequality we used Lemma 2.3.2. Thus, we clearly have:

$$\left(\frac{1}{V(B)} \int_B R_k(\sigma) dV \right) \left(\frac{1}{V(B)} \int_B [R_k(\sigma)]^{-1/(p-1)} dV \right)^{p-1} \lesssim [\sigma]_{B_p},$$

inflating the quasi-balls by a fixed amount so they touch the boundary if necessary.

For the other case, observe $d(z_0, bD) \geq 2cr$, so $r \leq \frac{1}{2c} d(z_0, bD)$. One can verify that given $w \in B$, the quasi-balls $B_k(z_0)$ and $B_k(w)$ have comparable radii. From this it is simple to deduce that if $C_B = R_k(\sigma)(z_0)$, then the following bounds hold for $z \in B$:

$$C_B \lesssim R_k(\sigma)(z) \lesssim C_B$$

where the implicit constants are absolute. It easily follows that $R_k(\sigma) \in A_p$.

□

We now state a couple of technical lemmas that will assist us in the proof of Theorem 2.2.4. In particular, they mitigate some difficulties that occur when passing from the proof for the unit ball to the more general cases we consider.

Lemma 2.3.4. Fix constants $\gamma, \alpha_1., \alpha_2$ Let $B_0 = B(z_0, r_0)$ be a quasi-ball with the property that if $z \in B_0$, then $d(z, bD) \leq \alpha_1 r_0$ and $B_0 \subset B(z, \alpha_1 r_0)$. Define

$$F = \{z \in B_0 : V(B(z, d(z, bD))) \leq \alpha_2 \gamma V(B_0)\}.$$

Then $F \subset \tilde{F}$, where

$$\tilde{F} = \{z \in B_0 : d(z, bD) \leq \alpha' \gamma^{\frac{1}{m}} r_0\}$$

and $\alpha' = \alpha_1 (C_6 \alpha_2)^{\frac{1}{m}}$. Here C_6 and m are the constants in the strong homogeneity property (2.2.1).

Proof. Using the strong homogeneity property,

$$V(B_0) \leq V(B(z, \alpha_1 r_0)) \leq C_6 \left(\frac{\alpha_1 r_0}{d(z, bD)} \right)^m V(B(z, d(z, bD))).$$

If we assume $z \in F$, by the definition of the set F, we get an upper bound on $V(B(z, d(z, bD)))$, and arrive at the inequality

$$V(B_0) \leq C_6 \left(\frac{\alpha_1 r_0}{d(z, bD)} \right)^m \alpha_2 \gamma V(B_0).$$

Dividing both sides by $V(B_0)$, we obtain

$$C_6 \left(\frac{\alpha_1 r_0}{d(z, bD)} \right)^m \alpha_2 \gamma \geq 1.$$

Rearranging this expression, we obtain

$$d(z, bD) \leq \alpha' \gamma^{\frac{1}{m}} r_0,$$

where $\alpha' = \alpha_1 (C_6 \alpha_2)^{\frac{1}{m}}$, as required.

$\square$

Lemma 2.3.5. Let α' be a fixed constant, $\gamma > 0$ a constant to be chosen later. Let $B_0 = B(z_0, r_0)$ be a quasi-ball that touches the boundary and $F = \{z \in B_0 : d(z, bD) \leq \alpha' \gamma^{1/m} r_0\}$. Then, if γ is sufficiently small,

$$V(F) \lesssim \gamma^{\frac{1}{m}} V(B_0).$$

Proof. We need to consider two cases: when r_0 is large and when r_0 is small. We first consider the case when $r_0 < R_D$ is small, where R_D is some appropriately chosen absolute constant that depends only on D. We may assume that B_0 lies completely in one of the neighborhoods U where the local quasi-metric was constructed. To obtain a favorable estimate on the measure of F in this case, it is easiest to consider the local coordinates constructed by McNeal.

Recall the quasi-metric d is constructed by patching together these local metrics, so it suffices to work with the local coordinates on a local level. Recalling z_0 denotes the center of B_0, we work with coordinates $z = (z_1, z_2, \ldots, z_n)$ centered at z_0 and with parameter $\delta = r_0$. Note that B can be taken to be $P(z_0, r_0)$, or at least some multiple that will not affect the argument. Let $z_j = x_j + \mathrm{i} y_j$, $1 \leq j \leq n$. For $z \in P(z_0, r_0)$, write $z = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$ and write $z' = (y_1, x_2, y_2, \ldots, x_n, y_n) \in \mathbb{R}^{2n-1}$. For $z \in$

$P(z_0, r_0)$, we define the function $R(z') = \sup\{x_1 : (x_1, z') \in D\}$. We need to do this because the polydisc may "extend" past the domain, but we are only considering the measure of the portion that lies in D.

One can show using geometric arguments that if $z \in F$ then one has the bounds $R(z') - C\alpha'\gamma^{\frac{1}{m}} \le x_1 \le R(z')$, where C is some absolute constant. The upper bound is clear by definition. The lower bound follows from the fact that $\frac{\partial \rho}{\partial x_1} > 0$ on V. Denote by $\sigma_1(z, bD)$ the distance from a point z to bD along the (real) line in the direction of (positive) x_1. We show $d(z, bD) \gtrsim \sigma_1(z, bD)$ for all z in this neighborhood. Note if we fix $z \in P(z_0, r_0)$, freezing all the variables except x_1, we can select w with $w = (x_1', y_1, x_2, y_2, \ldots, x_n, y_n)$ by increasing x_1 so $w \in bD$. Then by the Mean Value Theorem (in one real variable), there is a point ζ in the neighborhood $U \cap D$ so that

$$\begin{aligned}
d(z, bD) &\approx |\rho(z)| \\
&= |\rho(z) - \rho(w)| \\
&= \left|\frac{\partial \rho(\zeta)}{\partial x_1}\right| |x_1' - x_1| \\
&\approx x_1' - x_1 \\
&= \sigma_1(z, bD)
\end{aligned}$$

where the implicit constant is independent of z. Crucially we use the fact that $\frac{\partial \rho}{\partial x_1}$ is bounded away from zero by the coordinate construction. This shows that $\sigma_1(z, bD) \lesssim d(z, bD)$ and establishes the claim.

Thus, we can gain control on the measure of F by integrating in these coordinates, using

Fubini and noting the function $R(z')$ will vanish after the first variable is integrated:

$$\begin{aligned} V(F) &\lesssim \int\limits_{|z_n|\le\tau_n(z_0,r_0)} \int\limits_{|z_{n-1}|\le\tau_{n-1}(z_0,r_0)} \cdots \int\limits_{|y_1|\le r_0} \int\limits_{R(z')-C\alpha'\gamma^{1/m}r_0\le|x_1|\le R(z')} dx_1\,dy_1\ldots dy_{n-1}\,dy_n \\ &\approx \gamma^{\frac{1}{m}} r_0^2 \prod_{j=2}^{n} \left(\tau_j(z,r_0)\right)^2 \\ &\approx \gamma^{\frac{1}{m}} V(B_0) \end{aligned}$$

which yields the required estimate.

Now suppose that $r_0 \ge R_D$. Since we are assuming γ is small, we can cover F and bD with finitely many small (Euclidean balls) so that in each ball, the normal projection to the boundary is well-defined. Then, in each of these balls with center z_c we can introduce a smooth change of coordinates $z = (z_1, \ldots, z_n)$ centered at $\pi(z_c)$, where π denotes the normal projection to the boundary, so we have x_1 is in the real normal direction at $\pi(z_c)$ and the coordinates $y_1, x_2, y_2, \ldots, x_n, y_n$ lie in the real tangent plane at $\pi(z_c)$. A similar type of coordinate system is employed in [37, Lemma 4.1]. In each ball, we can perform an integration very similar to the one above in these coordinates and obtain $V(F) \le C_D \gamma^{\frac{1}{m}} r_0$, where C_D is a constant depending only on the ambient domain. Since r_0 is uniformly bounded above and below by assumption, we also have $V(B_0)$ is bounded above and below by a universal constant for the domain. Thus, we can deduce that $V(F) \lesssim \gamma^{\frac{1}{m}} V(B_0)$, as desired. $\square$

Next, we proceed to prove Theorem 2.2.4. In what follows, we consider the positive Bergman operator

$$\mathcal{B}^+ f(z) = \int\limits_D |K_D(z,w)| f(w) dV(w).$$

It is known for the strongly pseudoconvex and convex finite type cases that the positive operator $\mathcal{B}^+$ is bounded on $L^p(D)$, $1 < p < \infty$ (see [43], [57]). We remark that our proof obtains the same result for the other cases in addition to the weighted estimates (just take $\sigma = 1$).

Proof of Theorem 2.2.4. We proceed by proving a good-λ inequality as in classical singular integral theory and Békollè's paper. In particular, we will show that there exist positive constants C and δ so that given any $f \in L^1(D)$ and $\lambda, \gamma > 0$ we have

$$\sigma\left(\{\mathcal{B}^+ f > 2\lambda \text{ and } \mathcal{M}f \leq \gamma\lambda\}\right) \leq C\gamma^\delta \sigma\left(\{\mathcal{B}^+ f > \lambda\}\right).$$

By the regularity of σ, it suffices to prove

$$\sigma\left(\{z \in \mathcal{O} : \mathcal{B}^+ f > 2\lambda \text{ and } \mathcal{M}f \leq \gamma\lambda\}\right) \leq C\gamma^\delta \sigma\left(\mathcal{O}\right).$$

for any open set $\mathcal{O}$ containing $\{z \in D : \mathcal{B}^+ f > \lambda\}$. Applying a Whitney decomposition to $\mathcal{O}$, consider a fixed ball B_0 in the Whitney decomposition with center z_0 and radius r_0 (see [12, Theorem 3.2]) for the existence of Whitney decompositions in a space of homogeneous type). It suffices to show

$$\sigma\left(\{z \in B_0 : \mathcal{B}^+ f > 2\lambda \text{ and } \mathcal{M}f \leq \gamma\lambda\}\right) \leq C\gamma^\delta \sigma\left(B_0\right).$$

We may assume that there exists a $\zeta_0 \in B_0$ so that $\mathcal{M}f(\zeta_0) \leq \gamma\lambda$, otherwise the inequality is trivial. Also note we are free to take γ sufficiently small as the inequality is trivial for large γ. By properties of the Whitney decomposition, we know that for some inflation constant $c_1 > 1$, the ball $\tilde{B}_0$ with radius $c_1 r_0$ contains a point z' so that $\mathcal{B}^+ f(z') \leq \lambda$. Finally, let c_2 be chosen large enough so that the ball centered at z' with radius $c_2 r_0$ contains B_0 and let B_1 be the ball centered at z' with radius equal to $c_3 = \max\{d(z', bD), c_2 r_0\}$.

Write $f = f_1 + f_2$ where $f_1 = f\chi_{B_1}$ and $f_2 = f\chi_{D\setminus B_1}$. Without loss of generality, we may assume f is positive. We first show there exists an absolute constant A so that for $z \in B_0$,

$\mathcal{B}^+ f_2(z) \leq \lambda + A\gamma\lambda$.

We have, for $z \in B_0$,

$$\begin{aligned}
\mathcal{B}^+ f_2(z) &= \int\limits_{D\setminus B_1} |K_D(z,w)| f(w)\, dV(w) \\
&\leq \int\limits_{D} |K_D(z',w)| f(w)\, dV(w) + \int\limits_{D\setminus B_1} |K_D(z,w) - K_D(z',w)||f(w)|\, dV(w).
\end{aligned}$$

Obviously, for the first term we have

$$\int\limits_{D} |K_D(z',w)| f(w) dV(w) = \mathcal{B}^+ f(z') < \lambda.$$

The second term is handled as follows. First notice that if $w \in D \setminus B_1$, we have $d(z,w) \geq C_4 d(z,z')$, where C_4 is the smoothness constant in Theorem 2.2.2, provided c_2 is taken appropriately large. Also, it can be shown $d(z,w) \gtrsim c_3$. For $0 \leq k < \infty$, let

$$A_k = \{w \in D : 2^k c_4 \leq d(z,w) \leq 2^{k+1} c_4\}$$

where $c_4 = \inf_{w \in D\setminus B_1} d(z,w) \approx \max\{C_4 d(z,z'), c_3\}$. Then we estimate:

$$\begin{aligned}
\int\limits_{D\setminus B_1} |K_D(z,w) - K_D(z',w)||f(w)|dV(w) &\leq \int\limits_{D\setminus B_1} \left(\frac{d(z,z')}{d(z,w)}\right)^{\nu} \frac{|f(w)|}{V(B(z,d(z,w)))} dV(w) \\
&\leq \sum_{k=0}^{\infty} \int\limits_{A_k} \left(\frac{d(z,z')}{d(z,w)}\right)^{\nu} \frac{|f(w)|}{V(B(z,d(z,w)))} dV(w) \\
&\lesssim \sum_{k=0}^{\infty} \int\limits_{A_k} 2^{-k\nu} \frac{|f(w)|}{V(B(z,2^k c_4))} dV(w) \\
&= \sum_{k=0}^{\infty} \frac{2^{-k\nu}}{V(B(z,2^{k+1}c_4))} \int\limits_{A_k} \frac{|f(w)|V(B(z,2^{k+1}c_4))}{V(B(z,2^k c_4))} dV(w) \\
&\lesssim \mathcal{M}f(\zeta_0) \\
&\leq \gamma\lambda.
\end{aligned}$$

Now we must consider some cases. First consider the case when $d(z', bD) \geq c_2 r_0$. We then have the easy estimate:

$$\begin{aligned}
\mathcal{B}^+ f_1(z) &= \int\limits_{B_1} |K_D(z,w)||f(w)|dV(w) \\
&\leq \frac{1}{V(B(z,d(z,bD)))} \int\limits_{B_1} |f(w)|\, dV(w) \\
&\lesssim \langle |f| \rangle_{B_1} \\
&\lesssim \mathcal{M}f(\zeta_0) \\
&\leq \gamma\lambda.
\end{aligned}$$

By choosing γ sufficiently small, it is clear we can make the left hand side of the good-λ inequality equal to 0, so the inequality is trivial in this case.

Now for the other case suppose that $d(z', bD) < c_2 r_0$. Note that if $\mathcal{B}^+ f(z) > 2\lambda$, then by what we have shown above $\mathcal{B}^+ f_1(z) > b\lambda$ where $b = 2 - (1 + A\gamma)$. We estimate:

$$\begin{aligned} b\lambda &< \mathcal{B}^+ f_1(z) \\ &\leq \int\limits_{B_1} |K_D(z,w)||f(w)|\, dV(w) \\ &\leq \frac{1}{V(B(z, d(z, bD)))} \int\limits_{B_1} |f(w)|\, dV(w) \\ &= \frac{V(B(z', c_2 r_0))}{V(B(z, d(z, bD)))} \langle |f| \rangle_{B_1} \\ &\lesssim \frac{V(B(z', c_2 r_0))}{V(B(z, d(z, bD)))} \mathcal{M} f(\zeta_0) \\ &\leq \frac{V(B(z', c_2 r_0))}{V(B(z, d(z, bD)))} \gamma\lambda. \end{aligned}$$

This implies the following:

$$V(B(z, d(z, bD))) \lesssim \gamma V(B(z', c_2 r_0)) \lesssim \gamma V(B(z, c_2 r_0)).$$

Let

$$F = \{z \in B_0 : V(B(z, d(z, bD))) \leq \alpha\gamma V(B(z, c_2 r_0))\}$$

where α is the implicit constant above. By renaming α, we can replace $V(B(z, c_2 r_0))$ by $V(B_0)$, using the doubling property. Note that by the above we have proven that in this case

$$\{z \in B_0 : \mathcal{B}^+ f(z) > 2\lambda \text{ and } \mathcal{M} f(z) \leq \gamma\lambda\} \subset F.$$

We need to prove that we have good control over the measure of the set F. In particular, we claim $V(F) \lesssim \gamma^{\frac{1}{m}} V(B_0)$ where we recall m is the exponent, characteristic of the domain, that appears in (2.2.1) By Lemma 2.3.4, we can replace F with $\tilde{F} = \{z \in B_0 : d(z, bD) \leq \alpha' \gamma^{\frac{1}{m}} r_0\}$. By inflating B_0 if necessary, we can assume without loss of generality $r_0 > d(z_0, bD)$

so that B_0 touches bD. Then Lemma 2.3.5 establishes the claim.

Now we prove that B_p weights satisfy a kind of "fairness" property that is characteristic of A_∞ weights. As in the previous proofs, define a regularized weight as follows:

$$\sigma'(z) = R(\sigma)(z) = \langle \sigma \rangle_{B(z)},$$

where $B(z) = \{w \in D : d(w,z) < k_0 d(z, bD)\}$ for some appropriately chosen constant k_0.

Recall that by previous work, $\sigma' \in A_p$. First we show $\sigma'(B_0) \lesssim \sigma(B_0)$. Using basically the arguments of Lemma 2.3.2, we can show that

$$\sigma'(B_0) \lesssim \int_D \sigma(\zeta) \left(\frac{1}{V(B'(\zeta))} \int_{B_0 \cap B'(\zeta)} dV(z) \right) dV(\zeta),$$

where $B'(\zeta)$ is some fixed inflation of $B(\zeta)$. We claim that we can inflate B_0 by a fixed amount to a ball $\widehat{B_0}$ so that $\zeta \notin \widehat{B_0}$ implies $B'(\zeta) \cap B_0 = \emptyset$.

Then

$$\sigma'(B_0) \lesssim \sigma(\widehat{B_0}) \lesssim \sigma(B_0)$$

using the doubling property of σ. We can use a similar argument to verify that $\sigma(F) \lesssim \sigma'(F)$. In particular, one can check that

$$\sigma(F) \lesssim \int_D \frac{1}{V(B(\zeta))} \int_{B(\zeta) \cap F} \sigma(z) dV(z)\, dV(\zeta)\,.$$

One can check there exists a constant β so that if we define the set $\widehat{F}$

$$\widehat{F} = \{z \in \widehat{B_0} : d(z, bD) \leq \beta \gamma^{\frac{1}{m}} r_0\},$$

then $\widehat{F}$ has the property that if $\zeta \notin \widehat{F}$, then $B(\zeta) \cap F = \emptyset$. Then $\sigma(F) \lesssim \sigma'(\widehat{F})$. Note that by the reasoning leading to the computation of the Lebesgue measure of F, $V(\widehat{F}) \lesssim \gamma^{\frac{1}{m}} V(B_0)$. Then notice we obtain, by the fairness property of A_p weights there exists $\delta > 0$ so that (see [63, Chapter V])

$$\sigma(F) \lesssim \sigma'(\widehat{F}) \lesssim [V(\widehat{F})/V(\widehat{B_0})]^{\delta}\sigma'(B_0') \lesssim [V(\widehat{F})/V(B_0)]^{\delta}\sigma(B_0),$$

and $[V(\widehat{F})/V(B_0)] \lesssim \gamma^{\frac{1}{m}}$. Thus, the good-$\lambda$ inequality is demonstrated, renaming δ as $\frac{\delta}{m}$. The rest of the proof follows from standard relative distribution estimates.

□

Remark 2.3.1. In principle one could track constants in the proof of sufficiency and obtain an upper quantitative estimate for the norm of $\mathcal{B}$ or $\mathcal{B}^+$ on $L^p_\sigma(D)$ in terms of $[\sigma]_{B_p}$. However, such an estimate would almost certainly not be sharp. We resolve this issue in [25] using modern techniques of dyadic harmonic analysis as in [58].

2.4 The Necessity of the B_p Condition

We would now like to consider whether the condition $\sigma \in B_p$ is necessary for $\mathcal{B}$ to be bounded on $L^p_\sigma(D)$. In what follows we obtain a partial answer to this question, valid for any simple domain D. In the special case that D is strongly pseudoconvex, we will prove that the B_p condition is necessary. In general, we require additional hypotheses, in particular a lower bound on the kernel and the integrability of σ and its dual, for our proof technique. We first prove a lemma which is valid for any simple domain where the Bergman kernel satisfies an appropriate lower estimate. This lemma is an analogue of [3, Lemma 5] and essentially the same argument is given.

Lemma 2.4.1. Suppose the Bergman kernel $K_D(z, w)$ on a simple domain D satisfies the following property: there exists a constant $\varepsilon_0 > 0$ so that if

$$\max\{d(z, bD), d(w, bD)\} \leq 2cd(z, w)$$

and $d(z, w) \leq \varepsilon_0$, then we have

$$|K_D(z, w)| \gtrsim \frac{1}{V(B(w, d(z, w)))},$$

where the implicit constant is independent of z and w.

Now, let $B_1(\zeta_1, \varepsilon_0')$ be a ball of small radius $\varepsilon_0' = \frac{\varepsilon_0}{c(2+C_4)}$ touching bD. Then there exists a ball $B_2(\zeta_2, \varepsilon_0')$ also touching bD with $d(\zeta_1, \zeta_2) \approx \varepsilon_0'$ so that if $f \geq 0$ is a function supported in B_i and $z \in B_j$, with $i \neq j$ and $i, j \in \{1, 2\}$, then we have

$$|\mathcal{B}f(z)| \gtrsim \frac{1}{V(B_i)} \int\limits_{B_i} f(w) dV(w).$$

Proof. Without loss of generality, suppose $i = 1$. Choose B_2 also touching bD with $d(\zeta_1, \zeta_2) = c(C_4 + 1)\varepsilon_0'$, so that if $z \in B_2$ and $\zeta \in B_1$, we have the estimate $d(\zeta_1, z) \geq C_4 d(\zeta_1, \zeta)$, where C_4 is the constant that appears in the smoothness estimate. Also, note that our choice of ε_0' implies that $d(z, \zeta_1) \leq \varepsilon_0$. Moreover, calculations show that $\max\{d(z, bD), d(\zeta_1, bD)\} \leq 2cd(z, \zeta_1)$ (assuming without loss of generality that $C_4 \geq 1$). Therefore, we are in a position to apply the lower bound on the kernel $K_D(z, \zeta_1)$.

Then, estimate as follows (assuming C_4 is appropriately large relative to C_3):

$$
\begin{aligned}
|\mathcal{B}f(z)| &= \left|\int\limits_{B_1} K_D(z,\zeta)f(\zeta)\,dV(\zeta)\right| \\
&\geq \left|\int\limits_{B_1} K_D(z,\zeta_1)f(\zeta)\,dV(\zeta)\right| - \int\limits_{B_1} |K_D(z,\zeta_1)-K_D(z,\zeta)|f(\zeta)\,dV(\zeta) \\
&\geq |K_D(z,\zeta_1)|\int\limits_{B_1} f(\zeta)\,dV(\zeta) - C_3\int\limits_{B_1}\left(\frac{d(\zeta_1,\zeta)}{d(\zeta_1,z)}\right)^{\nu}\frac{1}{V(B(\zeta_1,d(\zeta_1,z)))}f(\zeta)\,dV(\zeta) \\
&\geq |K_D(z,\zeta_1)|\int\limits_{B_1} f(\zeta)\,dV(\zeta) - \frac{C_3}{C_4^{\nu}V(B(\zeta_1,d(\zeta_1,z)))}\int\limits_{B_1} f(\zeta)\,dV(\zeta) \\
&\gtrsim \frac{1}{V(B(\zeta_1,d(\zeta_1,z)))}\int\limits_{B_1} f(\zeta)\,dV(\zeta) \\
&\approx \frac{1}{V(B_1)}\int\limits_{B_1} f(w)dV(w).
\end{aligned}
$$

Note in the penultimate estimate we use the hypothesis of the lower bound on the kernel. □

Using this lemma we obtain the following theorem, which grants the necessity of the B_p condition under certain conditions.

Theorem 2.4.1. Suppose the Bergman kernel $K_D(z,w)$ on a simple domain D satisfies the lower bound in Lemma 2.4.1. Then if $\mathcal{B}$ maps $L^p_\sigma(D)$ to $L^p_\sigma(D)$ and additionally σ and $\sigma^{-\frac{1}{p-1}}$ are integrable, we must have $\sigma \in B_p$.

Proof. We follow closely a standard argument in harmonic analysis that is used in proving the necessity of the A_p condition for the Hilbert/Riesz transforms (see, for example, the proof of [21, Theorem 7.47])).

First, we note that the assumption that σ and its dual are integrable allows us to consider only small balls as in the proof of Lemma 2.4.1 when we compute the B_p characteristic. Let B_1 and B_2 be two small balls as considered in the lemma, and f a positive function supported on B_1. Note that Lemma 2.4.1 implies:

$$B_2 \subseteq \{\mathcal{B}(f)(z) \geq C\langle f\rangle_{B_1}\}$$

where C is the implicit constant in Lemma 2.4.1. Let $\mathcal{A} = \|\mathcal{B}\|_{L^p_\sigma(D)}$. Using the fact that $\mathcal{B}$ is bounded on $L^p_\sigma(D)$, we obtain:

$$\sigma(B_2) \lesssim \frac{\mathcal{A}^p}{(\langle f\rangle_{B_1})^p} \int_D |f|^p \sigma dV. \tag{2.4.1}$$

Note we may interchange the roles of B_1 and B_2 to obtain

$$\sigma(B_1) \lesssim \frac{\mathcal{A}^p}{(\langle f\rangle_{B_2})^p} \int_D |f|^p \sigma dV. \tag{2.4.2}$$

Now take $f = \chi_{B_2}$ to obtain $\sigma(B_1) \lesssim \mathcal{A}^p\sigma(B_2)$. Then substitute this into (2.4.1) to obtain

$$\sigma(B_1) \lesssim \frac{\mathcal{A}^{2p}}{(\langle f\rangle_{B_1})^p} \int_D |f|^p \sigma dV. \tag{2.4.3}$$

Finally, take $f = \sigma^{-\frac{1}{p-1}}\chi_{B_1}$ and substitute into (2.4.3) to obtain

$$\langle \sigma\rangle_{B_1} \left(\langle \sigma^{-\frac{1}{p-1}}\rangle_{B_1}\right)^{p-1} \lesssim \mathcal{A}^{2p}$$

which completes the proof since B_1 was an arbitrary small ball.

□

We next show that if D is strongly pseudoconvex and $\mathcal{B}$ is bounded on $L^p_\sigma(D)$, then it follows that $\sigma, \sigma^{-1/p-1}$ are integrable on D, making this additional assumption unnecessary.

Lemma 2.4.2. Let D be strongly pseudoconvex with smooth boundary. Suppose $\mathcal{B}$ is bounded on $L^p_\sigma(D)$. Then $\sigma, \sigma^{-1/p-1} \in L^1(D)$.

Proof. It suffices to prove $\sigma^{-\frac{1}{p-1}} \in L^1(D)$. Then the integrability of σ follows by a duality argument. Indeed, if $\mathcal{B}$ is bounded on $L^p_\sigma(D)$, then since the Bergman projection is self-adjoint $\mathcal{B}$ is also bounded on $L^q_{\sigma'}(D)$, where q is the dual exponent to p and $\sigma' = \sigma^{-\frac{1}{p-1}}$. The same arguments then imply that $(\sigma^{-\frac{1}{p-1}})^{-\frac{1}{q-1}} = \sigma$ is integrable.

We first claim that there exists an $\varepsilon > 0$ so that for any $w \in D$, there exists a point $z_0 \in D$ (depending on w) so that for all z in a small neighborhood of z_0 (call it N_{z_0}) and $w' \in B_E(w, \varepsilon)$, we have $|K_D(z, w')| \approx 1$ and for any $z_1, z_2 \in N_{z_0}$ and $w' \in B_E(w, \varepsilon)$, $\arg\{K_D(z_1, w'), K_D(z_2, w')\} \in [-\frac{1}{3}, \frac{1}{3}]$. Here $B_E(w, \varepsilon)$ denotes the Euclidean ball of radius ε.

To see this, note that if z_0 is chosen so $\operatorname{dist}(z_0, bD) > C$, for some fixed $C > 0$, then $|K_D(z, w')| \lesssim 1$ by Kerzman's result that the Bergman kernel extends to a C^∞ function off the boundary diagonal. So it remains to show that there exists an $\varepsilon > 0$ so $|K_D(z, w')| \gtrsim 1$ for z, w' as above, and that the argument condition is satisfied. The argument condition again follows from Kerzman's theorem, perhaps by shrinking N_{z_0} sufficiently small. Suppose the remainder of the claim is not true. Then there is a sequence of points w_n so for each z satisfying $\operatorname{dist}(z, bD) > C$ and n, there is a point $w_n' \in B_E(w_n, \frac{1}{n})$ so that $|K_D(z, w_n')| < \varepsilon_n$, where ε_n is a sequence that tends to 0. Passing to a subsequence, we have that $w_n' \to w'' \in \overline{D}$ with $K_D(z, w'') = 0$ for all z with $\operatorname{dist}(z, bD) > C$ (note that w_n' depends on z but the limit point w'' does not). First consider the case when $w'' \in D$. Then we immediately get a contradiction, since $\{z : \operatorname{dist}(z, bD) > C\}$ is open in $\mathbb{C}^n$, while the zero set of $K_D(\cdot, w'')$ is a complex variety of complex codimension one (note $K_D(\cdot, w'')$ is not identically zero).

Note that in fact we can repeat this procedure for each n taking z_0 so $\operatorname{dist}(z_0, bD) > \frac{1}{n}$. Then in fact we will obtain a sequence of limit points w_n''. Then passing to a subsequence if necessary, we can assume that $w_n'' \to w^* \in \overline{D}$. By the argument above, we may assume $w^* \in bD$. For each n, we can select a z_n so $\operatorname{dist}(w^*, z_n) \leq \frac{2}{n}$ and $\operatorname{dist}(z_n, bD) > \frac{1}{n}$. Then clearly $z_n \to w^*$ and also $K_D(z_n, w_n'') = 0$ for all n. Looking at the asymptotic expansion for the Bergman kernel in the strongly pseudoconvex case obtained in [7], we see that this is impossible. In particular, the asymptotic expansion takes the following form:

$$K_D(z,w) = a(z,w)\psi(z,w)^{-n-1},$$

where a is continuous on $\overline{D}\times\overline{D}$ and is non-vanishing on $\triangle(bD\times bD)$, and ψ is C^∞ on $\overline{D}\times\overline{D}$ with certain additional properties. In particular, ψ vanishes on the boundary diagonal. Thus, clearly we must have $a(w^*,w^*)=0$. But this is impossible as a does not vanish on the boundary diagonal. This establishes the claim.

We now show that the claim implies the integrability of $\sigma^{-\frac{1}{p-1}}$. First, let $f\in L^p_\sigma(D)$ be a positive function. We claim $f\in L^1(D)$. Fix $w\in D$ and let ε and z_0 be as in the above claim. Then the function $F(w') := K(z_0,w')^{-1}f(w')\chi_{B_E(w,\varepsilon)}(w')\in L^p_\sigma(D)$ by the claim. Notice

$$\mathcal{B}(F)(z) = \int\limits_{D\cap B_E(w,\varepsilon)} \frac{K(z,w')}{K(z_0,w')} f(w')\,dV(w')$$

is in $L^p_\sigma(D)$ by hypothesis and hence is finite almost everywhere. Thus in particular there exists a z' in N_{z_0} so $|\mathcal{B}(f)(z')|<\infty$. But then this implies, using the argument condition,

$$\left|\int\limits_{D\cap B_E(w,\varepsilon)} f(w')\,dV(w')\right| < \infty.$$

It is then possible to choose a finite covering $B_E(w_1,\varepsilon),\ldots B_E(w_n,\varepsilon)$ of D, which thus implies $f\in L^1(D)$.

Now, suppose to the contrary that $\sigma^{-1/p-1}$ is not integrable. Then there exists a positive function $g\in L^p(D)$ so that $\int_D g\sigma^{-1/p}=\infty$. But then taking $f=g\sigma^{-1/p}$, we see $f\in L^p_\sigma(D)$. This implies $f\in L^1(D)$, a contradiction since we know $f\notin L^1(D)$. □

Finally, we show that strongly pseudoconvex domains also satisfy the necessary lower bound on the Bergman kernel, so the B_p condition is both necessary and sufficient in this case.

Theorem 2.4.2. Let D be a smoothly bounded, strongly pseudoconvex domain, $1 < p < \infty$, and σ be a weight. Then $\mathcal{B}$ is bounded on $L^p_\sigma(D)$ if and only if $\sigma \in B_p$.

Proof. One direction is given by Theorem 2.1.1, so it suffices to establish that if $\mathcal{B}$ is bounded on $L^p_\sigma(D)$, then $\sigma \in B_p$. Throughout the proof, we assume that $d(z,w) \leq \varepsilon_0$ and $\max\{d(z,bD), d(w,bD)\} \leq 2cd(z,w)$ as in the statement of Lemma 2.4.1. As above, by a result of Boutet and Sjöstrand ([7]), we have $K(z,w) = a(z,w)\psi(z,w)^{-n-1}$, where a is C^∞ on $\overline{D} \times \overline{D} \setminus \triangle(bD \times bD)$ and continuous on $\overline{D} \times \overline{D}$, a does not vanish on the diagonal sufficiently close to the boundary, and ψ is a C^∞ function with $\psi(z,z) = -\rho(z)$, and the additional condition that $\partial_w \psi$, $\overline{\partial}_z \psi$ are vanishing of infinite order on the diagonal $w = z$. We claim that if we choose $d(z,w)$ small enough then we have $|\psi(z,w)| \lesssim d(z,w)$. To see this, note that Taylor's theorem together with the conditions on ψ imply

$$|\psi(z,w)| \leq |\rho(w)| + \left| \sum_{j=1}^{n} \frac{\partial \rho(w)}{\partial z_j}(z_j - w_j) + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho(w)}{\partial z_j \partial z_k}(z_j - w_j)(z_k - w_k) \right| + \mathcal{O}(|z-w|^2).$$

On the other hand, the quasi-metric can be explicitly written down (locally) using a biholomorphic change of coordinates centered at w (see [50]). First, we may by a unitary rotation plus normalization and translation assume $\partial \rho(w) = dz_1$ and $w = 0$. Then in these coordinates, $\sum_{j=1}^{n} \frac{\partial \rho(w)}{\partial z_j}(z_j - w_j) = z_1$. Then, define holomorphic coordinates $\zeta = (\zeta_1, \dots, \zeta_n)$ as follows:

$$\zeta_1 = z_1 + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho(w)}{\partial z_j \partial z_k} z_j z_k, \;\; \zeta_j = z_j, j = 2, \dots, n.$$

In particular,

$$d(z,w) \approx |z_1 - w_1| + \sum_{j=2}^{n} |z_j - w_j|^2,$$

where the components of z and w are computed in ζ coordinates. Since $|\rho(w)| \lesssim d(z,w)$ by hypothesis, then it is clear, applying the change of variables, that $|\psi(z,w)| \lesssim d(z,w)$.

Finally, it is easy to verify that in the strongly pseudoconvex case, $V(B(z,r)) \approx r^{n+1}$, because $\tau_j(z,\delta) = \delta^{1/2}$ for $j = 2, \ldots, n$ (see [50] for instance). Therefore, if $d(z,w)$ is chosen appropriately small we can obtain the following lower bounds:

$$
\begin{aligned}
|K_D(z,w)| &\gtrsim |\psi(z,w)|^{-n-1} \\
&\gtrsim (d(z,w))^{-n-1} \\
&\approx \frac{1}{(B(w,d(z,w)))}.
\end{aligned}
$$

This concludes the proof. □

Chapter 3

The Szegő Projection on Minimally Smooth Domains

3.1 Summary of Main Results

This chapter mainly contains material from the manuscript [67]. The main result of this chapter is a sufficient condition on a weight σ for the L^p_σ boundedness of the Szegő projection on domains with minimal smoothness. This condition on the weights is precisely the A_p condition in the setting of spaces of homogeneous type with the appropriate quasi-metric on bD. We will precisely define these metric quantities in Sections 3.3.1. We can state the main theorem as follows:

Theorem 3.1.1. Let D be strongly pseudoconvex with C^2 boundary and $1 < p < \infty$. If $\sigma \in A_p$, then there exists $C > 0$ so that $\|\mathcal{S}f\|_{L^p_\sigma(bD)} \leq C\|f\|_{L^p_\sigma(bD)}$.

We remark that in the case that D has C^3 boundary, our results for the Szegő projection can be considerably sharpened. In fact, in this case it is possible to explicitly relate the extension of the Szegő projection on the weighted space to the auxiliary operator $\mathcal{C}$ using an operator equation. See Theorem 3.3.1 in the beginning of Section 3.3 for more details.

Notably, our methods are only suited to proving the sufficiency of the A_p condition, not the necessity. To obtain any results concerning the necessity of the A_p condition, it seems likely one would instead have to study the operator $\mathcal{S}$ directly and obtain novel estimates on the kernel function, or else use special properties of the operators in an ingenious way.

3.2 An Outline of the Proof

In this section, we provide a broad strokes outline of the method of proof so the reader has an idea of how the various pieces will fit together. Much of the same strategy will apply in Chapter 4 as well. Recall that via an idea of Kerzman and Stein, the Szegő projection $\mathcal{S}$ can be related to a "Cauchy integral" $\mathcal{C}$. It can be shown that the operator $\mathcal{C}$ is a (non-orthogonal) projection from $L^2(bD)$ to $H^2(bD)$. Thus, we obtain the following two operator identities relating $\mathcal{S}$ and $\mathcal{C}$ on $L^2(bD)$:

$$\mathcal{S}\mathcal{C} = \mathcal{C}, \quad \mathcal{C}\mathcal{S} = \mathcal{S}.$$

Taking adjoints of the second identity, subtracting from the first and some further manipulation yields the following operator identity:

$$\mathcal{S}(I - (\mathcal{C}^* - \mathcal{C})) = \mathcal{C}. \tag{3.2.1}$$

We will subsequently refer to (3.2.1) as the *Kerzman-Stein equation.* Note that if $(I - (\mathcal{C}^* - \mathcal{C}))$ is invertible on $L^2(bD)$ (this is true and easy to see in the case D is C^∞, see [29]), we arrive at an explicit formula for $\mathcal{S}$ in terms of $\mathcal{C}$:

$$\mathcal{S} = \mathcal{C}(I - (\mathcal{C}^* - \mathcal{C}))^{-1}.$$

Now perhaps the reader can see the utility of such an approach in proving L^p estimates. To prove that the Szegő projection extends to a bounded operator on L^p, one must prove the following two facts concerning $\mathcal{C}$:

1. The operator $\mathcal{C}$ is bounded on L^p;

2. The operator $(I - (\mathcal{C}^* - \mathcal{C}))$ is invertible on L^p.

The regularity of the domain is crucial in assessing whether the operator $(I - (\mathcal{C}^* - \mathcal{C}))$ is invertible on L^p. If this operator is to be invertible, the "error" $\mathcal{C}^* - \mathcal{C}$ must be small in some appropriate sense (for example, compact, smoothing, and/or with norm less than 1). In particular, for the Szegő projection, we will require the domain to be C^3 for this method of inversion.

As mentioned previously, in [37, 40], Lanzani and Stein considered the situation of minimal regularity and proved that the Szegő and Bergman projections are bounded on L^p for $1 < p < \infty$. In the case of the Szegő projection they transfer the question of boundedness to real-variable singular integral theory via the theory of spaces of homogeneous type. Lanzani and Stein show that the kernel of the operator $\mathcal{C}$ satisfies the appropriate size and smoothness estimates with respect to this quasi-metric (more precisely, they consider the kernel of the "main part" of the operator, $\mathcal{C}^\sharp$; there is an error term they also must handle). The celebrated $T(1)$ Theorem in harmonic analysis is then invoked to establish that the operator $\mathcal{C}$ is bounded on $L^2(bD)$. This result together with the kernel estimates of course implies that $\mathcal{C}$ is bounded on $L^p(bD)$ for $1 < p < \infty$. With appropriate control on the "error term" $\mathcal{C}^* - \mathcal{C}$, Lanzani and Stein establish that $\mathcal{S}$ is bounded on $L^p(bD)$ for $1 < p < \infty$.

We follow the general program of Lanzani and Stein in the weighted setting in the next section. In particular, we use the same construction of the auxiliary operator that goes back to Kerzman and Stein in [29], and we obtain (3.2.1). We obtain weighted L^p bounds on the auxiliary operator $\mathcal{C}$ using the same real-variable singular integral approach in [40]. The weights belong to an A_p class induced by the quasi-metric on the boundary of D.

In both cases, to show that the operator $(I - (\mathcal{C}^* - \mathcal{C}))$ is invertible on L^p_σ when $\sigma \in A_p$, we prove that $\mathcal{C}^* - \mathcal{C}$ is compact on L^p_σ for $\sigma \in A_p$ and also "improves" L^p spaces. Using equation (3.2.1), this grants the boundedness of $\mathcal{S}$ on L^p_σ.

Because Lanzani and Stein assume less regularity, our approach entails an application of their arguments in a simpler setting, so some technical obstructions in their paper can be ignored. In particular, Lanzani and Stein consider an entire family of non-orthogonal projections $\mathcal{C}_\varepsilon$ with parameter ε, while we only need to consider a single auxiliary operator $\mathcal{C}$ (this can be viewed as a special case of the operators in [40] with $\varepsilon = 0$, see also [36]). A major technical obstruction in their papers is that the operator $(I - (\mathcal{C}_\varepsilon^* - \mathcal{C}_\varepsilon))$ is no longer invertible on L^p, so they must split it appropriately. In fact, the proof of our result in the unweighted case $\sigma = 1$ can be viewed as a simplification of the arguments leading to the main result in [40] in the case that D has C^3 boundary.

However, as mentioned previously, we are actually able to obtain the same "top level" result for the Szegő projection in the case of minimal (C^2) smoothness. We only focus on $p = 2$ for simplicity; the general result may be obtained via extrapolation (see [62]; extrapolation still holds in spaces of homogeneous type). Here we follow the approach in [37] of "partially inverting" $(I - (\mathcal{C}_\varepsilon^* - \mathcal{C}_\varepsilon))$ by writing

$$\mathcal{C}_\varepsilon^* - \mathcal{C}_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{D}_\varepsilon,$$

where $\mathcal{A}_\varepsilon$ has small norm for sufficiently small ε so $I - \mathcal{A}_\varepsilon$ is invertible on $L^2_\sigma(bD)$ using a Neumann series. The operator $\mathcal{D}_\varepsilon$ may in general be unbounded in norm as $\varepsilon \to 0$, but it does map $L^2_\sigma(bD)$ to $L^\infty(bD)$, which turns out to be enough. The reverse Hölder property of A_p weights is the only key property we use in the proof.

This chapter is organized as follows. Section 3.3 focuses on the case where D is C^3 and sharper results can be obtained, while Section 3.4 focuses on the minimal smoothness case and proves the full strength of Theorem 1.5.3. At the beginning of each section, the first subsection introduces the background material and the construction of the relevant integral operators. The latter subsections deal with the proofs.

3.3 The Szegő Projection on C^3 domains

In this section, we assume that D is a strongly pseudoconvex domain of class C^3. This implies that there exists a strictly plurisubharmonic defining function ρ, which is of class C^3 and will be fixed throughout. We aim to prove the following theorem, which corresponds to a special case of Theorem 3.1.1 but also provides more detailed information about the connection between the main and auxiliary operators that is unavailable in the minimal smoothness case.

Theorem 3.3.1. Let D be strongly pseudoconvex with C^3 boundary. Then for $1 < p < \infty$ and $\sigma \in A_p$, the following hold:

1. The operator $\mathcal{C}^* - \mathcal{C}$ is compact on $L^p_\sigma(bD)$.

2. The operator $I - (\mathcal{C}^* - \mathcal{C})$ is invertible on $L^p_\sigma(bD)$.

3. The Szegő projection $\mathcal{S}$ extends to a bounded operator on $L^p_\sigma(bD)$ and satisfies

$$\mathcal{S} = \mathcal{C}(I - (\mathcal{C}^* - \mathcal{C}))^{-1}.$$

3.3.1 Preliminaries for C^3 Domains

The first step, following the approach of Lanzani and Stein as well as many other authors, is to construct an integral operator that reproduces and produces holomorphic functions from integration of their boundary values. To begin with, define the Levi polynomial at $w \in bD$:

$$P_w(z) := \sum_{j=1}^{n} \frac{\partial \rho}{\partial w_j}(w)(z_j - w_j) + \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial w_j \partial w_k}(w)(z_j - w_j)(z_k - w_k).$$

Using the strict pseudoconvexity of D, it is possible to choose a C^∞ cutoff function χ

and a constant c so that $\chi \equiv 1$ when $|z-w| \leq c/2$ and $\chi \equiv 0$ when $|z-w| \geq c$ so that the function

$$g(z,w) := \chi(-P_w(z)) + (1-\chi)|w-z|^2$$

satisfies

$$\mathrm{Re}(g(z,w)) \gtrsim -\rho(z) + |w-z|^2 \tag{3.3.1}$$

for $z \in D$ (see [37]).

Recall that a *generating form* $\eta(z,w)$ is a form of type $(1,0)$ in w with C^1 coefficient functions such that $\langle \eta(z,w), w-z \rangle = 1$ for all $z \in D$ and w in a neighborhood of bD [38]. Here $\langle \cdot, \cdot \rangle$ denotes the action of a 1-form on a vector in $\mathbb{C}^n$. The importance of generating forms lies in the construction of Cauchy-Fantappié integrals. The upshot of (3.3.1) is that we can construct a generating form as follows: define the following $(1,0)$ form in w

$$G(z,w) := \chi\left(\sum_{j=1}^{n} \frac{\partial \rho}{\partial w_j}(w)\, dw_j - \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial w_j \partial w_k}(w)(w_k - z_k)\, dw_j\right) + (1-\chi)\sum_{j=1}^{n} (\overline{w}_j - \overline{z}_j)\, dw_j\,.$$

Then define for $w \in bD$, $z \in D$

$$\eta(z,w) := \frac{G(z,w)}{\langle G(z,w), w-z \rangle} = \frac{G(z,w)}{g(z,w)}.$$

Then it is immediate that η is a generating form. As in [40, 60], define the associated Cauchy-Fantappié integral operator

$$\mathbf{C}_1(f)(z) := \frac{1}{(2\pi i)^n} \int\limits_{w \in bD} f(w) j^*(\eta \wedge (\overline{\partial}\eta)^{n-1})(z,w) = \frac{1}{(2\pi i)^n} \int\limits_{w \in bD} \frac{f(w) j^*\left(G \wedge (\overline{\partial} G)^{n-1}(z,w)\right)}{(g(z,w))^n},$$

where $j : bD \hookrightarrow \mathbb{C}^n$ is the inclusion map and j^* denotes the pullback of j. The point is that this operator reproduces holomorphic functions that are continuous up the boundary, as made precise in the following proposition (see [40]):

Proposition 3.3.1. Let F be holomorphic on D and continuous on $\overline{D}$, and let $f = F|_{bD}$. Then there holds for $z \in D$

$$\mathbf{C}_1(f)(z) = F(z).$$

The problem now is that $\mathbf{C}_1$ does not necessarily produce holomorphic functions, as the form η is not necessarily holomorphic in z. This difficulty can be overcome by solving a $\overline{\partial}$ problem on a strongly pseudoconvex, smooth domain Ω that contains D (see [40], or for more details [60]). One has the following:

Proposition 3.3.2. There exists an $(n, n-1)$ form (in w) $C_2(z, w)$ that is C^1 in w and depends smoothly on the parameter $z \in \overline{D}$ so that the following hold for the operator $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2$:

(i) $\mathbf{C}(f)(z) = F(z)$ for F holomorphic on D and continuous on $\overline{D}$, where $f = F|_{bD}$;

(ii) $\mathbf{C}(f)(z)$ is holomorphic for $f \in L^1(bD)$.

Here,

$$\mathbf{C}_2(f)(z) = \int_{w \in bD} f(w) C_2(z, w).$$

Note that importantly

$$\sup_{z \in \overline{D}, w \in bD} |C_2(z, w)| < \infty. \tag{3.3.2}$$

Thus, $\mathbf{C}$ is an operator that produces and reproduces holomorphic functions from boundary data.

Next, we proceed to define the relevant quasi-metric on the boundary of D for our analysis in this chapter. Let $d(z, w) = |g(z, w)|^{1/2}$. Then $d(z, w)$ satisfies all the properties of a quasi-metric or quasi-distance (see [40]). Denote a ball in bD in the quasi-metric with center z

and radius δ by $B(z,\delta)$. It is a fact that

$$S(B(z,\delta)) \approx \delta^{2n}. \tag{3.3.3}$$

where S denotes induced Lebesgue surface measure on bD.

We also have the important estimates in [40]:

$$|z-w| \lesssim d(z,w) \lesssim |z-w|^{1/2}. \tag{3.3.4}$$

We now introduce the Leray-Levi measure λ on bD. This measure is defined

$$d\lambda(w) = j^*(\partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1})/(2\pi i)^n.$$

The use of this measure is crucial in Lanzani and Stein's paper in the computation of an adjoint operator that plays a crucial role in the application of the $T(1)$ theorem, but it turns out to be equivalent to Lebesgue measure in a certain strong sense. In particular, we have

$$d\lambda(w) = \Lambda(w)\, dS(w), \tag{3.3.5}$$

where $\Lambda(w)$ is a real-valued function bounded above and below for all $w \in bD$. More explicitly, the function Λ is given by

$$\Lambda(w) = (n-1)!(4\pi)^{-n}|\det\rho(w)||\nabla\rho(w)|$$

where $\det\rho(w)$ is the determinant of the $(n-1)\times(n-1)$ matrix of second derivatives taken in a rotated coordinate system evaluated at w (see [60] or [40] for details):

$$\left\{\frac{\partial^2\rho}{\partial z_j \partial \overline{z}_k}(w)\right\}_{j,k=1}^{n-1}.$$

Crucially, note that Λ is Lipschitz (in fact C^1), since ρ is of class C^3. The importance of this fact will become clear in the proof of Lemma 3.3.1.

The following proposition is immediate.

Proposition 3.3.3. The triple $(bD, d, d\lambda)$ forms a space of homogeneous type in the sense of Definition 1.2.1.

Note we could replace the Leray-Levi measure by induced Lebesgue measure and the above result would still be true, since the function $\Lambda(w)$ is bounded above and below uniformly. Below, for a measurable set $A \subset bD$, when we write $S(A)$, we refer to its Lebesgue surface measure, but in every case we could replace it by the Leray-Levi measure and the result would still be true.

We now want to essentially consider the restriction of the operator $\mathbf{C}$ to the boundary bD and obtain a singular integral operator $\mathcal{C}$ that maps $L^p(bD)$ to $L^p(bD)$. Explicitly, Lanzani and Stein define

$$\mathcal{C}(f)(z) = \mathbf{C}(f)(z)|_{bD}$$

when f satisfies a type of Hölder continuity, namely

$$|f(w_1) - f(w_2)| \lesssim d(w_1, w_2)^\alpha$$

for some α with $0 < \alpha \leq 1$. In this case one can show $\mathbf{C}(f)$ extends to a continuous function on $\overline{D}$, so the above definition makes sense. The operator $\mathcal{C}$, while initially defined only on certain functions, actually extends to a bounded linear operator on $L^p(bD)$ (this is proven in [40] using the $T(1)$ theorem).

Now, it is useful to break the operator $\mathbf{C}$ into a main term and an error term as follows:

$$\mathbf{C} = \mathbf{C}^\sharp + \mathbf{R},$$

where

$$\mathbf{C}^\sharp(f)(z) = \int\limits_{bD} \frac{f(w)}{g(z,w)^n}\, d\lambda(w)$$

and $\mathbf{R}$ absorbs the error from replacing the numerator of the Cauchy-Fantappié integral with the Leray-Levi measure as well as the error from the operator $\mathbf{C}_2$, which in fact has a bounded kernel by (3.3.1). If we let $R(z,w)$ denote the kernel of the operator $\mathbf{R}$, we can obtain the crucial estimate (see [40] again):

$$|R(z,w)| \lesssim d(z,w)^{-2n+1}. \tag{3.3.6}$$

Note that it is immediately obvious that the kernel of $\mathbf{C}^\sharp$ is bounded above by a multiple of $d(z,w)^{-2n}$, so we see that the operator $\mathbf{R}$ is "less singular" in a sense than the operator $\mathbf{C}^\sharp$.

As before, for functions that satisfy the Hölder continuity condition as above, we can define

$$\mathcal{C}^\sharp(f) = \mathbf{C}^\sharp(f)|_{bD}$$

and thus obtain the decomposition for the operator $\mathcal{C}$

$$\mathcal{C} = \mathcal{C}^\sharp + \mathcal{R}.$$

In what follows, we write $\sigma \in A_p$ to mean that σ is an A_p weight in the sense of Definition 1.2.3 with respect to the space of homogeneous type (bD, d, S). Additionally, we can define a standard maximal function with respect to this quasi-metric on bD:

Definition 3.3.1. The *Hardy-Littlewood Maximal Function* is defined, for $f \in L^1(bD)$

$$\mathcal{M}(f)(z) = \sup_{B(w,r)\ni z} \frac{1}{S(B(w,r))} \int\limits_{B(w,r)} |f(w)|\, dS(w)$$

We also define A_1 weights with respect to the same quasi-metric (the following definition is obviously equivalent to Definition 1.2.4 and is slightly easier to work with for our purposes):

Definition 3.3.2. A function $\sigma \in L^1(bD)$ that is positive almost everywhere is said to belong to the class A_1 if the following estimate holds for almost every $z \in bD$:

$$\mathcal{M}(\sigma)(z) \lesssim \sigma(z).$$

We have now set up all the machinery we need to prove Theorem 3.3.1.

3.3.2 The Main Term

We proceed to analyze the "main term" $\mathcal{C}^\sharp$. It should be noted in what follows that in the C^2 case considered in [40], certain implicit constants depend on ε and can even blow up as $\varepsilon \to 0$. This is not the case in the C^3 case, as there is only one ε, namely $\varepsilon = 0$, for which there is no analog in the C^2 case. We have the following size and smoothness estimates for the kernel of $\mathcal{C}^\sharp$ given in [40]:

Proposition 3.3.4. Let $K(z,w) = g(z,w)^{-n}$ denote the kernel of $\mathcal{C}^\sharp$ with respect to the Leray-Levi measure. Then there exist constants C_1, C_2 so the following holds:

(i) $|K(z,w)| \leq C_1 d(z,w)^{-2n}$;

(ii) $|K(z,w) - K(z,w')| \leq C_1 \frac{d(w,w')}{d(z,w)^{2n+1}}$ for $d(z,w) \geq C_2 d(w,w')$;

(iii) $|K(z,w) - K(z',w)| \leq C_1 \frac{d(z,z')}{d(z,w)^{2n+1}}$ for $d(z,w) \geq C_2 d(z,z')$.

Lanzani and Stein also prove the following result by invoking the $T(1)$ theorem:

Theorem 3.3.2. The operator $\mathcal{C}^\sharp$ is bounded on $L^2(bD)$.

Theorem 3.3.2 and Proposition 3.3.4 demonstrate that the operator $\mathcal{C}^\sharp$ is Calderón-Zygmund in the sense of Definition 1.2.2, and consequently the weighted theory of real-variable harmonic analysis applies to this case. Thus, we have the following result using Theorem 1.2.1:

Theorem 3.3.3. Let $1 < p < \infty$. Then if $\sigma \in A_p$, there exists $C > 0$ so $\|\mathcal{C}^\sharp f\|_{L^p_\sigma(bD)} \leq C\|f\|_{L^p_\sigma(bD)}$.

Proof. This is an easy consequence of Theorem 1.2.1. The only remark that needs to be made is that the equivalence of the Leray-Levi measure and Lebesgue measure in (3.3.5) must be invoked because the kernel above is with respect to Leray-Levi measure, not Lebesgue measure. In particular, if $\sigma \in A_p$ as we have defined it, then σ is in A_p with respect to the Leray-Levi measure. By Calderón-Zygmund theory on spaces of homogeneous type, the operator $\mathcal{C}^\sharp$ is bounded on $L^p_{\sigma\Lambda}(bD)$, and hence bounded on $L^p_\sigma(bD)$ by the equivalence of the measures.

□

3.3.3 The Error Terms

Let $\mathcal{C}^*$ denote the adjoint of $\mathcal{C}$ with respect to Lebesgue measure. We now proceed to deal with the error terms $\mathcal{R}$ as well as $\mathcal{C}^* - \mathcal{C}$. Both of these terms will play a role in the proof of the main theorem in the subsequent section. We know from (3.3.6) that the kernel of the "remainder operator" $\mathcal{R}$ is "less singular" than the main operator $\mathcal{C}^\sharp$. We proceed to show that this is also true for the kernel of the "difference operator" $\mathcal{C}^* - \mathcal{C}$. First we need a preliminary proposition, which is similar to an argument that can be found in [60]:

Proposition 3.3.5. The following estimate holds for $w, z \in bD$:

$$|g(z,w) - \overline{g(w,z)}| \lesssim |w-z|^3.$$

Proof. It suffices to prove the estimate when $|w-z| \leq c/2$, so we can assume $g(z,w) = -P_w(z)$ and $\overline{g(w,z)} = -\overline{P_z(w)}$. To avoid cumbersome notation, we use the shorthand $\frac{\partial \rho}{\partial w_j}(w) = \rho_j(w)$ and $\frac{\partial^2 \rho}{\partial w_j \partial w_k}(w) = \rho_{j,k}(w)$. Recall the Levi polynomial at w is defined as

$$P_w(z) = \sum_{j=1}^{n} \rho_j(w)(z_j - w_j) + \frac{1}{2}\sum_{j,k=1}^{n} \rho_{j,k}(w)(z_j - w_j)(z_k - w_k).$$

We also define the Levi form

$$L_w(z) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial w_j \partial \overline{w}_k}(w)(z_j - w_j)(\overline{z}_k - \overline{w}_k).$$

The Taylor expansion (in w) of $\rho_j(w)$ about $w = z$ is

$$\rho_j(w) = \rho_j(z) + \sum_{k=1}^{n} \rho_{j,k}(z)(w_k - z_k) + \sum_{k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}}(z)(\overline{w_k} - \overline{z_k}) + \mathcal{O}(|w-z|^2)$$

while the Taylor expansion of $\rho_{j,k}(w)$ gives

$$\rho_{j,k}(w) = \rho_{j,k}(z) + \mathcal{O}(|w-z|).$$

Substituting these Taylor expansions into $P_w(z)$, we obtain

$$\begin{aligned} P_w(z) = &\sum_{j=1}^{n} \rho_j(z)(z_j - w_j) - \frac{1}{2}\sum_{j,k=1}^{n} \rho_{j,k}(z)(w_j - z_j)(w_k - z_k) \\ &- \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}}(z)(w_j - z_j)(\overline{w_k} - \overline{z_k}) + \mathcal{O}(|w-z|^3). \end{aligned}$$

On the other hand, we have

$$\overline{P_z(w)} = \sum_{j=1}^{n} \overline{\rho_j(z)}(\overline{w_j} - \overline{z_j}) + \frac{1}{2}\sum_{j,k=1}^{n} \overline{\rho_{j,k}(z)}(\overline{w_j} - \overline{z_j})(\overline{w_k} - \overline{z_k}).$$

A computation shows

$$\overline{P_z(w)} - P_w(z) = 2\mathrm{Re}P_z(w) + L_z(w) + \mathcal{O}(|w-z|^3).$$

Then just use the well-known fact that

$$\rho(w) = \rho(z) + 2\mathrm{Re}P_z(w) + L_z(w) + \mathcal{O}(|w-z|^3),$$

together with the fact that $\rho(z) = \rho(w) = 0$ as $w, z \in bD$.

□

This proposition will allow us to prove the following lemma. Again, the argument is essentially from [60].

Lemma 3.3.1. Let $K(z,w)$ denote the kernel of $(\mathcal{C}^\sharp)^* - \mathcal{C}^\sharp$ with respect to induced Lebesgue measure dS. Then the following estimate holds:

$$|K(z,w)| \lesssim d(z,w)^{-2n+1}.$$

Proof. Here we need to come to grips with the distinction between the Leray-Levi measure $d\lambda$ and the induced Lebesgue measure dS. Let $(\mathcal{C}^\sharp)^\dagger$ denote the adjoint of $\mathcal{C}^\sharp$ taken *with respect to the Leray-Levi measure.*. Let $K_L(z,w)$ denote the kernel, with respect to $d\lambda$, of the operator $(\mathcal{C}^\sharp)^\dagger - \mathcal{C}^\sharp$. It is immediate that $K_L(z,w) = \overline{g(w,z)}^{-n} - g(z,w)^{-n}$. Compute to see

$$\begin{aligned}
|K(z,w)| &= \left|\Lambda(z)\overline{g(w,z)}^{-n} - g(z,w)^{-n}\Lambda(w)\right| \\
&\leq |\Lambda(z)-\Lambda(w)||\overline{g(w,z)}|^{-n} + |\Lambda(w)||K_L(w,z)| \\
&\lesssim |z-w|d(z,w)^{-2n} + |K_L(z,w)| \\
&\lesssim d(z,w)^{-2n+1} + |K_L(z,w)|.
\end{aligned}$$

Here we use the fact that Λ is Lipschitz. Then, compute to see:

$$\begin{aligned}
|K_L(z,w)| &= \left|\overline{g(w,z)}^{-n} - g(z,w)^{-n}\right| \\
&= \left|\frac{g(z,w)^n - \overline{g(w,z)}^n}{g(z,w)^n\overline{g(w,z)}^n}\right| \\
&= \left|\frac{\left(g(z,w)-\overline{g(w,z)}\right)\left(\sum_{t=0}^{n-1}(g(z,w))^t(\overline{g(w,z)})^{n-1-t}\right)}{g(z,w)^n\overline{g(w,z)}^n}\right| \\
&\lesssim \frac{|g(z,w)-\overline{g(w,z)}|d(z,w)^{2n-2}}{d(z,w)^{4n}} \\
&\lesssim d(z,w)^{-2n+1}
\end{aligned}$$

where in the last estimation we used Proposition 3.3.5. □

One can show using a special coordinate system that

$$\sup_{z\in bD}\int_{bD} d(z,w)^{-2n+1}\,dS(w) < \infty \tag{3.3.7}$$

(see [40] or [60]). This result can also be obtained by integrating over dyadic “annuli” as we will later see. Thus, we see that $\mathcal{R}$ and $(\mathcal{C}^\sharp)^* - \mathcal{C}^\sharp$ have integrable kernels, while $\mathcal{C}^\sharp$ does not.

Now we show the these kernel estimates are not only enough to guarantee boundedness on weighted L^p spaces; they are actually enough to guarantee compactness which is much better. The following proposition allows for good control of the integration of an A_1 weight σ against a kernel $K(z,w)$ which satisfies the size estimate above.

Proposition 3.3.6. Let $K(z,w)$ be a kernel measurable on $bD \times bD$ that satisfies the size estimate $|K(z,w)| \lesssim d(w,z)^{-2n+1}$, and let $\sigma \in A_1$. Then the following estimates hold for all $z, w \in bD$ and $\delta > 0$:

$$\int\limits_{B(z,\delta)} |K(z,w)|\sigma(w)\, dS(w) \lesssim \delta\sigma(z)$$

$$\int\limits_{B(w,\delta)} |K(z,w)|\sigma(z)\, dS(z) \lesssim \delta\sigma(w).$$

Proof. Break the region of integration up into dyadic annuli and estimate the integral as follows, using the A_1 condition on σ in the last step:

$$
\begin{aligned}
&\int\limits_{B(z,\delta)} |K(z,w)|\sigma(w)\,dS(w) \\
&\lesssim \int\limits_{B(z,\delta)} d(z,w)^{-2n+1}\sigma(w)\,dS(w) \\
&= \sum_{i=0}^{\infty} \int\limits_{B(z,2^{-i}\delta)\setminus B(z,2^{-(i+1)}\delta)} d(z,w)^{-2n+1}\sigma(w)\,dS(w) \\
&\leq \sum_{i=0}^{\infty} \int\limits_{B(z,2^{-i}\delta)\setminus B(z,2^{-(i+1)}\delta)} 2^{(-(i+1)(-2n+1))}\delta^{(-2n+1)}\sigma(w)\,dS(w) \\
&\leq \sum_{i=0}^{\infty} 2^{(-(i+1)(-2n+1))}\delta^{(-2n+1)}S(B(z,2^{-i}\delta))\frac{1}{S(B(z,2^{-i}\delta))}\int\limits_{B(z,2^{-i}\delta)} \sigma(w)\,dS(w) \\
&\leq \sum_{i=0}^{\infty} 2^{2n-1}2^{-i}\delta\mathcal{M}(\sigma)(z) \\
&\lesssim \delta\mathcal{M}(\sigma)(z) \\
&\lesssim \delta\sigma(z).
\end{aligned}
$$

Note all implicit equivalences are independent of w and z. The proof of the other statement is completely analogous.

□

Note if $K(z,w)$ is the kernel of an integral operator satisfying the size estimate of the previous proposition, then K is “integrable” in the sense that

$$
\sup_{z\in bD}\int\limits_{bD} |K(z,w)|\,dS(w) < \infty, \tag{3.3.8}
$$

and obviously (3.3.8) still holds if the roles of z and w are interchanged. This can be seen by taking $\sigma = 1$ and δ sufficiently large in Proposition 3.3.6. But in fact, we can say something slightly better. The proof of the following proposition is essentially a reprise of Proposition 3.3.6 taking $\sigma = 1$ with obvious modifications.

Proposition 3.3.7. Let $K(z,w)$ be a kernel measurable on $bD \times bD$ that satisfies the size estimate $|K(z,w)| \lesssim d(z,w)^{-2n+1}$, and let $\varepsilon \in [0, \frac{1}{2n-1})$. Then the following hold:

$$\sup_{z\in bD} \int_{bD} |K(z,w)|^{1+\varepsilon}\, dS(w) < \infty$$

$$\sup_{w\in bD} \int_{bD} |K(z,w)|^{1+\varepsilon}\, dS(z) < \infty.$$

As a consequence of this proposition, we can prove that an integral operator $\mathcal{K}$ that has a kernel with the above size estimate "improves" L^p spaces. This was noted before in [29] using a slightly different approach.

Proposition 3.3.8. Let $\mathcal{K}$ be an integral operator on $L^p(bD)$ with a kernel $K(z,w)$ that satisfies the size estimate $|K(z,w)| \lesssim d(z,w)^{-2n+1}$. Then $\mathcal{K}$ maps $L^p(bD)$ to $L^{p+\varepsilon}(bD)$ boundedly for $p \geq 1$ and $\varepsilon \in [0, \frac{1}{2n-1})$.

Proof. We first demonstrate the result for $p = 1$ and then show how this implies the result for $p > 1$. Take $f \in L^1(bD)$ and $\varepsilon \in [0, \frac{1}{2n-1})$. Then compute, using Minkowski's integral inequality and Proposition 3.3.7:

$$\begin{aligned}
\left(\int_{bD} \left| \int_{bD} K(z,w) f(w)\, dS(w) \right|^{1+\varepsilon} dS(z) \right)^{\frac{1}{1+\varepsilon}} &\leq \left(\int_{bD} \left(\int_{bD} |K(z,w)||f(w)|\, dS(w) \right)^{1+\varepsilon} dS(z) \right)^{\frac{1}{1+\varepsilon}} \\
&\leq \int_{bD} \left(\int_{bD} |K(z,w)|^{1+\varepsilon}\, dS(z) \right)^{\frac{1}{1+\varepsilon}} |f(w)|\, dS(w) \\
&\lesssim \|f\|_{L^1(bD)}.
\end{aligned}$$

To obtain the result for $p > 1$, proceed as follows, using Hölder's inequality with exponents p and q:

$$
\begin{aligned}
&\left(\int\limits_{bD}\left|\int\limits_{bD} K(z,w)f(w)\,dS(w)\right|^{p+\varepsilon} dS(z)\right)^{\frac{1}{p+\varepsilon}} \\
&\leq \left(\int\limits_{bD}\left(\int\limits_{bD} |K(z,w)|^{1/p}|K(z,w)|^{1/q}|f(w)|\,dS(w)\right)^{p+\varepsilon} dS(z)\right)^{\frac{1}{p+\varepsilon}} \\
&\leq \left(\int\limits_{bD}\left(\int\limits_{bD} |K(z,w)|\,dS(w)\right)^{\frac{p+\varepsilon}{q}} \left(\int\limits_{bD} |K(z,w)||f(w)|^p\,dS(w)\right)^{\frac{p+\varepsilon}{p}} dS(z)\right)^{\frac{1}{p+\varepsilon}} \\
&\lesssim \left(\int\limits_{bD}\left(\int\limits_{bD} |K(z,w)||f(w)|^p\,dS(w)\right)^{\frac{p+\varepsilon}{p}} dS(z)\right)^{\frac{1}{p+\varepsilon}} \\
&= \left(\left(\int\limits_{bD}\left(\int\limits_{bD} |K(z,w)||f(w)|^p\,dS(w)\right)^{\frac{p+\varepsilon}{p}} dS(z)\right)^{\frac{p}{p+\varepsilon}}\right)^{\frac{1}{p}} \\
&\leq \left(\int\limits_{bD}\left(\int\limits_{bD} |K(z,w)|^{1+\frac{\varepsilon}{p}}\,dS(z)\right)^{\frac{p}{p+\varepsilon}} |f(w)|^p\,dS(w)\right)^{\frac{1}{p}} \\
&\lesssim \|f\|_{L^p(bD)}.
\end{aligned}
$$

In the penultimate line, notice we apply Minkowski's integral inequality with exponent $\frac{p+\varepsilon}{p} = 1 + \frac{\varepsilon}{p}$ and with respect to measures $|f(w)|^p\,dS(w)$ and $dS(z)$.

□

Thus, we obtain the following important corollary:

Corollary 3.3.1. The operators $\mathcal{R}$, $\mathcal{R}^*$, and $(\mathcal{C}^\sharp)^* - \mathcal{C}^\sharp$ map $L^p(bD)$ to $L^{p+\varepsilon}(bD)$ for $p \geq 1$ and $\varepsilon \in [0, \frac{1}{2n-1})$.

We now turn to a proof of the major lemma concerning the error terms. This lemma adapts an argument that can be found in [60] to the weighted setting. It also should be noted that components of this proof are analogous to a "weighted Schur test."

Lemma 3.3.2. Let $K(z,w)$ be a measurable function on $bD \times bD$ satisfying the following estimates for all $z, w \in bD$ and all weights $\sigma \in A_1$:

(i) $\int_{B(z,\delta)} |K(z,w)|\sigma(w)\, dS(w) \lesssim C(\delta)\sigma(z)$;

(ii) $\int_{B(w,\delta)} |K(z,w)|\sigma(z)\, dS(z) \lesssim C(\delta)\sigma(w)$;

(iii) For any fixed $\delta > 0$, the kernel $K(z,w)$ is bounded on

$$(bD \times bD) \setminus \{(z,w) : d(z,w) < \delta\}$$

(with a bound that depends on δ).

Furthermore, suppose $C(\delta)$ tends to 0 as $\delta \to 0$. Then, for each $p \in (1,\infty)$, the integral operator $\mathcal{K}$ defined by $\mathcal{K}(f)(z) = \int_{bD} K(z,w) f(w)\, dS(w)$ is compact on $L^p_\sigma(bD)$ for all $\sigma \in A_p$.

Proof. First, consider the case when K is bounded on $bD \times bD$, say $\|K\|_{L^\infty(bD\times bD)} \leq M$. Let $\sigma \in A_p$. Then note that the kernel of the operator $\mathcal{K}$ with respect to the weighted measure $d\sigma = \sigma dS$ is $\tilde{K}(z,w) = K(z,w)\sigma^{-1}(w)$. To prove compactness on $L^p_\sigma(bD)$, it suffices to show the following double-norm is finite (it is well-known the finiteness of this double-norm implies compactness, for example see [17]):

$$\int\limits_{bD} \left(\int\limits_{bD} |\tilde{K}(z,w)|^q \, d\sigma(w) \right)^{p/q} d\sigma(z),$$

where q denotes the Hölder exponent conjugate to p. Then we have

$$\int\limits_{bD}\left(\int\limits_{bD}|\tilde{K}(z,w)|^q\,d\sigma(w)\right)^{p/q}d\sigma(z) = \int\limits_{bD}\left(\int\limits_{bD}|K(z,w)|^q\sigma^{-\frac{1}{p-1}}\,dS(w)\right)^{p/q}\sigma(z)\,dS(z)$$
$$\leq M^p\|\sigma\|_{L^1(bD)}\|\sigma^{\frac{-1}{p-1}}\|^{p-1}_{L^1(bD)}$$
$$< \infty$$

since $\sigma, \sigma^{-\frac{1}{p-1}}$ are integrable on bD. Thus the theorem holds in this case.

To pass to the case where K is unbounded, let $\delta_j = \frac{1}{j}$ and

$$K_j(z,w) = \begin{cases} K(z,w) & d(z,w) \geq \delta_j \\ 0 & d(z,w) < \delta_j. \end{cases}$$

Let $\mathcal{K}_j$ be the integral operator with kernel K_j. Then, by hypothesis K_j is bounded on $bD \times bD$ and by the argument above, $\mathcal{K}_j$ is compact on $L^p_\sigma(bD)$. Since the compact operators are a closed subspace of the Banach space of bounded linear operators on $L^p_\sigma(bD)$, if we can show that the operators $\mathcal{K}_j$ approach $\mathcal{K}$ in operator norm, we will be done.

To this end, let $f \in L^p_\sigma(bD)$ with $\|f\|_{L^p_\sigma(bD)} \leq 1$. Note that as $\sigma \in A_p$, we can write

$$\sigma = \frac{\sigma_1}{\sigma_2^{p-1}}$$

where $\sigma_1, \sigma_2 \in A_1$ by the factorization of A_p weights in the setting of spaces of homogeneous type (see, for example, [62] for a proof of this well-known fact). By Hölder's Inequality applied to the functions $|K(z,w) - K_j(z,w)|^{1/q}\sigma_2(w)^{1/q}$ and $|K(z,w) - K_j(z,w)|^{1/p}\sigma_2(w)^{-1/q}|f(w)|$

and then applying Proposition 3.3.6, we obtain the estimate:

$$
\begin{aligned}
|(\mathcal{K}-\mathcal{K}_j)(f)(z)| &\leq \int\limits_{bD} |K(z,w)-K_j(z,w)||f(w)|\,dS(w) \\
&= \left(\int\limits_{B(z,\delta_j)} |K(z,w)|\sigma_2(w)\,dS(w)\right)^{1/q} \left(\int\limits_{B(z,\delta_j)} |K(z,w)|(\sigma_2(w))^{1-p}|f(w)|^p\,dS(w)\right)^{1/p} \\
&\lesssim C(\delta_j)^{1/q}\sigma_2(z)^{1/q} \left(\int\limits_{B(z,\delta_j)} |K(z,w)|(\sigma_2(w))^{1-p}|f(w)|^p\,dS(w)\right)^{1/p}.
\end{aligned}
$$

Thus, we obtain, applying the proceeding estimate, Fubini, and Proposition 3.3.6 again:

$$
\begin{aligned}
\|(\mathcal{K}-\mathcal{K}_j)f\|^p_{L^p_\sigma(bD)} &\leq \int\limits_{bD} C(\delta_j)^{\frac{p}{q}}\sigma_2(z)^{\frac{p}{q}} \left(\int\limits_{B(z,\delta_j)} |K(z,w)|(\sigma_2(w))^{1-p}|f(w)|^p\,dS(w)\right) \frac{\sigma_1(z)}{\sigma_2(z)^{p-1}}\,dS(z) \\
&= C(\delta_j)^{\frac{p}{q}} \int\limits_{bD}\int\limits_{B(z,\delta_j)} |K(z,w)|(\sigma_2(w))^{1-p}|f(w)|^p\,dS(w)\,\sigma_1(z)\,dS(z) \\
&= C(\delta_j)^{\frac{p}{q}} \int\limits_{bD} \left(\int\limits_{B(w,\delta_j)} |K(z,w)|\sigma_1(z)\,dS(z)\right)|f(w)|^p(\sigma_2(w))^{1-p}\,dS(w) \\
&\lesssim C(\delta_j)^p \int\limits_{bD} \sigma_1(w)|f(w)|^p(\sigma_2(w))^{1-p}\,dS(w) \\
&= C(\delta_j)^p\|f\|^p_{L^p_\sigma(bD)} \\
&\leq C(\delta_j)^p.
\end{aligned}
$$

Letting $j \to \infty$, we have $\delta_j \to 0$ and $C(\delta_j) \to 0$. Thus, it immediately follows that the operators $\mathcal{K}_j$ approach $\mathcal{K}$ in operator norm and hence $\mathcal{K}$ is compact.

□

The preceding lemma admits the following, very useful corollary:

Corollary 3.3.2. The operators $\mathcal{R}$, $\mathcal{R}^*$, and $(\mathcal{C}^\sharp)^* - \mathcal{C}^\sharp$ are compact on $L^p_\sigma(bD)$ for $\sigma \in A_p$.

We need one more crucial lemma to conclude our analysis of the error terms and allow us to present the proof of Theorem 3.3.1 in the next section.

Lemma 3.3.3. Let $\mathcal{K}$ be an integral operator with a kernel $K(z,w)$ that satisfies the size estimate $|K(z,w)| \lesssim d(z,w)^{-2n+1}$. Further suppose that $i\mathcal{K}$ is self-adjoint on $L^2(bD)$. Then 1 is not in the spectrum of $\mathcal{K}$ considered as an operator on $L^p_\sigma(bD)$, where σ is an A_p weight.

Proof. First, note that 1 is not an eigenvalue of $\mathcal{K}$ considered as an operator on (unweighted) $L^2(bD)$. So suppose to the contrary that there exists an eigenfunction $f \in L^p_\sigma(bD)$ such that $\mathcal{K}f = f$. We assert $f \in L^1(bD)$. To see this, note by Hölder

$$\begin{aligned}\int_{bD} |f(w)|\, dS(w) &= \int_{bD} |f(w)|\sigma(w)^{1/p}\sigma(w)^{-1/p}\, dS(w) \\ &\leq \|f\|_{L^p_\sigma(bD)} \|\sigma^{-\frac{1}{p-1}}\|^{1/q}_{L^1(bD)} \\ &< \infty.\end{aligned}$$

Fix any $\varepsilon \in (0, \frac{1}{2n-1})$. Then, by Corollary 3.3.1, $f \in L^{1+\varepsilon}(bD)$. In particular, we have

$$\begin{aligned}\|f\|_{L^{1+\varepsilon}(bD)} &= \|\mathcal{K}f\|_{L^{1+\varepsilon}(bD)} \\ &\lesssim \|f\|_{L^1(bD)} \\ &< \infty.\end{aligned}$$

But since $\mathcal{K}f = f$, we can repeat this argument to obtain $f \in L^{1+2\varepsilon}$. In fact, we can iterate this argument arbitrarily many times to obtain that $f \in L^p(bD)$ for all $p \geq 1$! In particular, $f \in L^2(bD)$. This contradicts the fact that 1 is not an eigenvalue of $\mathcal{K}$ on $L^2(bD)$. Since $\mathcal{K}$ is compact on $L^p_\sigma(bD)$ by Corollary 3.3.2 (or rather the arguments leading to this corollary), this implies 1 is not in the spectrum of $\mathcal{K}$ on $L^p_\sigma(bD)$, as required.

□

3.3.4 Proof of Theorem 3.3.1

Equipped with these definitions and results, we are in a position to prove Theorem 3.3.1. As discussed, the essential ideas in this proof have been around for a long time and can be found, for example, in [28, 29].

Proof of Theorem 3.3.1. First, note that both $\mathcal{S}$ and $\mathcal{C}$ essentially produce and reproduce boundary values of holomorphic functions: they are projections onto $H^2(bD)$ (this is proven precisely in [39]). Consequently, we obtain the following two operator identities on $L^2(bD)$: $\mathcal{S}\mathcal{C} = \mathcal{C}$ and $\mathcal{C}\mathcal{S} = \mathcal{S}$. Taking adjoints of the second identity and using the fact that the Szegő projection is self-adjoint, we get $\mathcal{S}\mathcal{C}^* = \mathcal{S}$, and further manipulation yields $\mathcal{S}(\mathcal{C}^* - \mathcal{C}) = \mathcal{S} - \mathcal{C}$, or $\mathcal{S}(I - \mathcal{A}) = \mathcal{C}$ where I denotes the identity operator and $\mathcal{A} = \mathcal{C}^* - \mathcal{C}$. By Theorem 3.3.3 and Corollary 3.3.2 and, we know that $\mathcal{C} = \mathcal{C}^\sharp + \mathcal{R}$ is bounded on $L^p_\sigma(bD)$ for $\sigma \in A_p$.

Next, we assert that the operator $\mathcal{A}$ is compact on $L^p_\sigma(bD)$. To see this, write

$$\mathcal{A} = (\mathcal{C}^\sharp)^* - \mathcal{C}^\sharp + (\mathcal{C}^\sharp - \mathcal{C}) + (\mathcal{C}^* - (\mathcal{C}^\sharp)^*) = ((\mathcal{C}^\sharp)^* - \mathcal{C}^\sharp) - \mathcal{R} + \mathcal{R}^*$$

and appeal to Corollary 3.3.2. Next, an easy computation shows that $i\mathcal{A}$ is self-adjoint on $L^2(bD)$. It follows from Lemma 3.3.3 that 1 is not in the spectrum of $\mathcal{A}$ considered as an operator on $L^p_\sigma(bD)$ and hence the operator $(I - \mathcal{A})$ is invertible on $L^p_\sigma(bD)$. Thus, we may write

$$\mathcal{S} = \mathcal{C}(I - \mathcal{A})^{-1}$$

and conclude that $\mathcal{S}$ extends to a bounded operator on $L^p_\sigma(bD)$ since both $\mathcal{C}$ and $(I - \mathcal{A})^{-1}$ are bounded on $L^p_\sigma(bD)$. Thus, we have established all parts of Theorem 3.3.1. □

3.4 The Szegő Projection on C^2 domains

3.4.1 Preliminaries for C^2 domains

We now consider what modifications are necessary to prove Theorem 3.1.1, as in [40]. From now on we assume D has boundary of class C^2, but all the other assumptions about D and ρ from before remain in force. We shall be brief, as basically the same setup applies with one crucial change. This involves uniformly approximating the second derivatives of ρ by differentiable functions. In particular, since that boundary is of class C^2, we must replace the second derivatives $\frac{\partial^2 \rho}{\partial w_j \partial w_k}$ by an $n \times n$ matrix of $\{\tau^{\varepsilon}_{j,k}\}$ of C^1 functions satisfying

$$\sup_{w \in bD} \left| \frac{\partial \rho}{\partial w_j \partial w_k}(w) - \tau^{\varepsilon}_{j,k}(w) \right| \leq \varepsilon \quad 1 \leq j, k \leq n.$$

Now we define the analogs of $g(z,w)$, $G(z,w)$, and $\eta(z,w)$. In particular, define

$$g_\varepsilon(z,w) := \chi \left(\sum_{j=1}^{n} \frac{\partial \rho}{\partial w_j}(w)(w_j - z_j) - \frac{1}{2} \sum_{j,k=1}^{n} \tau^{\varepsilon}_{j,k}(w)(w_j - z_j)(w_k - z_k) \right) + (1-\chi)|w-z|^2$$

where χ is the same C^∞ cutoff function as in the C^3 case. If ε is taken sufficiently small, we have the analogous estimate

$$\mathrm{Re}(g_\varepsilon(z,w)) \gtrsim -\rho(z) + |z-w|^2,$$

where the implicit constant is independent of ε.

In the same way, we define the $(1,0)$ form in w $G_\varepsilon(z,w)$ as follows:

$$G_\varepsilon(z,w) := \chi\left(\sum_{j=1}^n \frac{\partial\rho}{\partial w_j}(w)dw_j - \frac{1}{2}\sum_{j,k=1}^n \tau_{j,k}^\varepsilon(w)(w_k - z_k)dw_j\right) + (1-\chi)\sum_{j=1}^n(\bar{w}_j - \bar{z}_j)dw_j.$$

As before, we define for $w \in bD, z \in D$:

$$\eta_\varepsilon(z,w) := \frac{G_\varepsilon(z,w)}{g_\varepsilon(z,w)}.$$

Then of course η_ε is again a generating form. Therefore, we can construct the associated Cauchy-Fantappié integral operator $\mathbf{C}_\varepsilon^1$ in exactly the same way as we constructed $\mathbf{C}_1$, with η_ε playing the role of η. In particular, the analog of Proposition 3.3.1 holds for $\mathbf{C}_\varepsilon^1$.

The issue, again, is that $\mathbf{C}_\varepsilon^1$ reproduces but does not produce holomorphic functions. Again, we can introduce a correction operator $\mathbf{C}_\varepsilon^2$ and consider the operator $\mathbf{C} = \mathbf{C}_\varepsilon^1 + \mathbf{C}_\varepsilon^2$. Proposition 3.3.2 will hold in this case; the operator $\mathbf{C}_\varepsilon$ will reproduce and produce holomorphic functions.

The rest of the setup follows basically identical. The definition of the Leray-Levi measure $d\lambda$ does not change, except now Λ will merely be a continuous rather than a C^1 function. The quasi-metric d will be defined in the same way, namely

$$d(z,w) = |g_\varepsilon(z,w)|^{1/2}$$

and will satisfy the same properties, including (bD, d, S) being a space of homogeneous type.

We can again consider the operator

$$\mathcal{C}_\varepsilon(f)(z) = \mathbf{C}_\varepsilon(f)(z)|_{bD}$$

and this definition makes sense when the function f is Hölder continuous with respect to d as before. We also can obtain the decomposition

$$\mathbf{C}_\varepsilon = \mathbf{C}^\sharp_\varepsilon + \mathbf{R}_\varepsilon$$

where

$$\mathbf{C}^\sharp_\varepsilon(f)(z) = \int\limits_{bD} \frac{f(w)}{g_\varepsilon(z,w)^n}\, d\lambda(w)$$

and the kernel $R_\varepsilon(w,z)$ of the operator $\mathbf{R}_\varepsilon$ satisfies

$$|R_\varepsilon(z,w)| \leq c_\varepsilon d(z,w)^{-2n+1}.$$

Here c_ε denotes a constant that can depend on ε.

Restricting this decomposition to the boundary, it is possible to obtain the following operator equation, acting on an appropriate class of functions:

$$\mathcal{C}_\varepsilon = \mathcal{C}^\sharp_\varepsilon + \mathcal{R}_\varepsilon.$$

The class of A_p weights and the maximal function are defined in the exact same manner as before.

This concludes our reiteration of the preliminaries for the C^2 case. The reader is invited to consult [40] for more details.

3.4.2 Weighted estimates in the C^2 case

We now demonstrate how weighted L^p bounds can be obtained in the C^2 case. Throughout we closely follow the arguments in [40]. First, note that we can still obtain the Kerzman-Stein equation in the same way as before. Thus, we have on $L^2(bD)$:

$$\mathcal{S}(I - (\mathcal{C}^*_\varepsilon - \mathcal{C}_\varepsilon)) = \mathcal{C}_\varepsilon. \tag{3.4.1}$$

In this case, we will be unable to invert the operator $(I - (\mathcal{C}_\varepsilon^* - \mathcal{C}_\varepsilon))$. It suffices to prove that $\mathcal{S}$ is bounded on $L^2_\sigma(bD)$ for all $\sigma \in A_2$; then we can appeal to extrapolation. To begin with, we have:

Lemma 3.4.1. For $\sigma \in A_2$ the operator $\mathcal{C}_\varepsilon$ extends to a bounded operator on $L^2_\sigma(bD)$ and in particular satisfies

$$\|\mathcal{C}_\varepsilon f\|_{L^2_\sigma(bD)} \leq c_{\varepsilon,\sigma}\|f\|_{L^2_\sigma(bD)},$$

where $c_{\varepsilon,\sigma}$ is a constant that depends on ε and the weight σ.

Proof. First, the operator $\mathcal{C}_\varepsilon^\sharp$ is Calderón-Zygmund (see the proof of [40, Theorem 7]); however, the constants in its smoothness estimates do depend on ε. The bound on the kernel of $\mathcal{R}_\varepsilon$ in fact implies that it is compact on $L^2_\sigma(bD)$ by the arguments in Lemma 3.3.2. This finishes the proof. □

The dependence of the constant on ε turns out not to be an issue because ultimately in the course of the proof we will fix ε sufficiently small and do not need to take a limit as $\varepsilon \to 0$.

Next, we need to break up the operator $\mathcal{C}_\varepsilon^* - \mathcal{C}_\varepsilon$. Roughly, we break the kernel of $\mathcal{C}_\varepsilon$ into pieces supported on and off the diagonal $z = w$. Let $s = s(\varepsilon)$ be a parameter chosen depending on ε. We write

$$\mathcal{C}_\varepsilon = \mathcal{C}_\varepsilon^s + \mathcal{R}_\varepsilon^s$$

where

$$\mathcal{C}_\varepsilon^s(f) = \mathcal{C}_\varepsilon(f\chi_s)$$

and $\chi_s(z,w)$ is a symmetrized smooth cutoff function that is 1 when $d(z,w) \leq cs$ and 0 when $d(z,w) \geq s$ (see [40] for details). Thus,

$$\mathcal{C}_\varepsilon^* - \mathcal{C}_\varepsilon = [(\mathcal{C}_\varepsilon^s)^* - \mathcal{C}_\varepsilon^s] + [(\mathcal{R}_\varepsilon^s)^* - \mathcal{R}_\varepsilon^s] := \mathcal{A}_\varepsilon + \mathcal{D}_\varepsilon.$$

It is immediate from previous discussions that for fixed ε, s, the kernel of $\mathcal{R}_\varepsilon^s$ (and $(\mathcal{R}_\varepsilon^s)^*$ is bounded. It is then an entirely straightforward exercise using Hölder's inequality and the integrability of σ that $\mathcal{D}_\varepsilon$ boundedly maps $L^2_\sigma(bD)$ to $L^\infty(bD)$, with an operator norm that can depend on s and ε.

We now need to deal with the other term. First, we state a lemma ([40, Lemma 24]) that we will later need. It is a decomposition lemma that partitions $\mathbb{C}^n = \mathbb{R}^{2n}$ into cubes at various levels. In particular, let Q_0^1 denote unit cube centered at the origin in $\mathbb{C}^n$, and for $k \in \mathbb{Z}^n$, let $Q_k^1 = k + Q_0^1$ be its integer translates. For $\gamma > 0$, let $Q_k^\gamma = \gamma Q_k^1$. Note that for a given cube Q_k^γ, there are at most $N = 3^{2n}$ cubes Q_j^γ that touch it; i.e whose closures have non-empty intersection. When γ is fixed, we write $\mathbb{1}_k$ for the indicator function on the set $Q_k^\gamma \cap bD$.

Lemma 3.4.2. Fix $\gamma > 0$ and suppose σ is a weight. Suppose T is a bounded operator on $L^2_\sigma(bD)$ that satisfies:

1. $\mathbb{1}_j T \mathbb{1}_k = 0$ if the cubes Q_j^γ and Q_k^γ do not touch.

2. $\|\mathbb{1}_j T \mathbb{1}_k\|_{L^2_\sigma(bD)\to L^2_\sigma(bD)} \leq A$ otherwise.

Then T satisfies

$$\|T\|_{L^2_\sigma(bD)\to L^2_\sigma(bD)} \leq AN.$$

Proof. The proof is identical to the one given in [40]. The underlying measure is now $\sigma\, dS$ as opposed to just Lebesgue measure, but the argument is the same. □

We have the following theorem:

Lemma 3.4.3. Given $\varepsilon > 0$, there exists an $s = s(\varepsilon)$ so the following holds:

$$\|(\mathcal{C}_\varepsilon^s)^* - \mathcal{C}_\varepsilon^s\|_{L^2_\sigma(bD)\to L^2_\sigma(bD)} \leq \varepsilon^{1/2} M_\sigma$$

where the constant M_σ depends on the weight σ but not ε.

Proof. Here the distinction between the Leray-Levi measure and Lebesgue measure becomes important. As before, let $\dagger$ denote the adjoint of an operator taken with respect to Leray-Levi measure, and write

$$(\mathcal{C}_\varepsilon^s)^* - \mathcal{C}_\varepsilon = [(\mathcal{C}_\varepsilon^s)^\dagger - \mathcal{C}_\varepsilon] + [(\mathcal{C}_\varepsilon^s)^* - (\mathcal{C}_\varepsilon^s)^\dagger].$$

We will first show

$$\|(\mathcal{C}_\varepsilon^s)^\dagger - \mathcal{C}_\varepsilon^s\|_{L^2_\sigma(bD)\to L^2_\sigma(bD)} \leq \varepsilon^{1/2} M_\sigma.$$

Note as before we decomposed $\mathcal{C}_\varepsilon$, we can write $\mathcal{C}_\varepsilon^s = \mathcal{C}_\varepsilon^{\sharp,s} + \mathcal{R}_\varepsilon^{\sharp,s}$, where $\mathcal{C}_\varepsilon^{\sharp,s}$ is the corresponding truncation of the operator $\mathcal{C}_\varepsilon^\sharp$. Write

$$(\mathcal{C}_\varepsilon^s)^\dagger - \mathcal{C}_\varepsilon = [(\mathcal{C}_\varepsilon^{\sharp,s})^\dagger - \mathcal{C}_\varepsilon^{\sharp,s}] + [(\mathcal{R}_\varepsilon^{\sharp,s})^\dagger - \mathcal{R}_\varepsilon^{\sharp,s}] = \mathcal{A}_\varepsilon^s + \mathcal{B}_\varepsilon^s.$$

Recall that the kernel of $\mathcal{R}_\varepsilon$ is majorized by $c_\varepsilon d(w,z)^{-2n+1}$. Using basically the arguments of Proposition 3.3.6, we have, for any $\sigma' \in A_1$:

$$\mathcal{R}_\varepsilon^{\sharp,s}(\sigma')(z) \lesssim s\sigma'(z)$$

and

$$(\mathcal{R}_\varepsilon^{\sharp,s})^*(\sigma')(z) \lesssim s\sigma'(z)$$

where the implicit constants depend on the weight σ' and ε. Then, by writing $\sigma \in A_2$

as a quotient of A_1 weights and applying the reasoning in the proof of Lemma 3.3.2, it is straightforward to show that $\|\mathcal{R}_\varepsilon^{\sharp,s}\|_{L^2_\sigma(bD)\to L^2_\sigma(bD)} \leq c_\varepsilon s M_\sigma$. Choosing s appropriately small in terms of ε, we obtain the estimate

$$\|\mathcal{R}_\varepsilon^{\sharp,s}\|_{L^2_\sigma(bD)\to L^2_\sigma(bD)} \leq \varepsilon^{1/2} M_\sigma,$$

as desired. The same estimate is easily seen to hold for $(\mathcal{R}_\varepsilon^{\sharp,s})^\dagger$, proving the estimate for $\mathcal{B}_\varepsilon^s$.

We now turn to $\mathcal{A}_\varepsilon^s$. It is proven in [40] that the operators $\varepsilon^{-1/2}\mathcal{A}_\varepsilon^s$ satisfy smoothness and cancellation conditions that are uniform in ε. Lanzani and Stein apply the $T(1)$ theorem to show that $\|\mathcal{A}_\varepsilon^s\|_{L^p(bD)\to L^p(bD)} \leq \varepsilon^{1/2} M_p$, where M_p is independent of ε. But the same Calderón-Zygmund theory shows that

$$\|\mathcal{A}_\varepsilon^s\|_{L^2_\sigma(bD)\to L^2_\sigma(bD)} \leq \varepsilon^{1/2} M_\sigma,$$

as we sought to show. We have thus demonstrated the result for $(\mathcal{C}_\varepsilon^s)^\dagger - \mathcal{C}_\varepsilon^s$.

We now turn to the operator $(\mathcal{C}_\varepsilon^s)^* - (\mathcal{C}_\varepsilon^s)^\dagger$. Estimating the norm of this operator turns out to involve estimating the norm of a commutator. In particular, $(\mathcal{C}_\varepsilon^s)^* - (\mathcal{C}_\varepsilon^s)^\dagger = (\mathcal{C}_\varepsilon^s)^* - \Lambda^{-1}(\mathcal{C}_\varepsilon^s)^*\Lambda$, where $d\lambda = \Lambda\, dS$ and Λ is a continuous, positive function that is bounded above and below. Thus, the L^2_σ norm of this operator is controlled by

$$\|\Lambda^{-1}\|_{L^\infty(bD)}\, \|[\Lambda, (\mathcal{C}_\varepsilon^s)^*]\|_{L^2_\sigma(bD)\to L^2_\sigma(bD)}\,,$$

where $[A, B] = AB - BA$.

Notice by a simple computation,

$$([\Lambda, (\mathcal{C}_\varepsilon^s)^*])^* = \mathcal{C}_\varepsilon^s\Lambda - \Lambda\mathcal{C}_\varepsilon^s = [\mathcal{C}_\varepsilon^s, \Lambda]\,,$$

so by duality it suffices to estimate the norm of a commutator $[\mathcal{C}_\varepsilon^s, \phi]$ on $L^2_\sigma(bD)$ for any $\sigma \in A_2$, where ϕ is an arbitrary continuous map $bD \to \mathbb{C}$. In particular, we claim for fixed ϕ:

$$\|[\mathcal{C}_\varepsilon^s, \phi]\|_{L^2_\sigma(bD)\to L^2_\sigma(bD)} \leq \varepsilon M_\sigma.$$

This is exactly proven in [40], but for unweighted L^p. A key ingredient in the proof is contained in [40, Proposition 19], which states that we can get a uniform bound $\|\mathcal{C}_\varepsilon^s\|_{L^p(bD)\to L^p(bD)} \leq M_p$ for ε and s chosen sufficiently small. This is proven using the $T(1)$ theorem with estimates uniform in ε, but then of course the same proof implies

$$\|\mathcal{C}_\varepsilon^s\|_{L^2_\sigma(bD)\to L^2_\sigma(bD)} \leq M_\sigma.$$

Now we provide a short sketch of how Lemma 3.4.2 leads to the desired conclusion again following the arguments from [40]. In particular, we apply the lemma to the operator $[\mathcal{C}_\varepsilon^s, \phi]$ with ε and s chosen appropriately. The first condition of Lemma 3.4.2 basically follows because $\mathcal{C}_\varepsilon^s$ has a kernel that is supported in a small neighborhood of the diagonal (in particular, we take $\gamma = cs$).

The second condition follows from the (uniform) continuity of ϕ. For a cube Q_k^γ, denote its center by z_k. If s is chosen sufficiently small, then by continuity, if $z \in Q_j^\gamma$, where Q_j^γ touches Q_k, we have

$$|\phi(z) - \phi(z_k)| < \varepsilon.$$

Now write $\phi = \phi_k + \psi_k$, where $\phi_k(z) = \phi(z) - \phi(z_k)$ and $\psi_k(z) = \phi(z_k)$. Obviously, $[\mathcal{C}_\varepsilon^s, \phi] = [\mathcal{C}_\varepsilon^s, \phi_k] + [\mathcal{C}_\varepsilon^s, \psi_k]$, but $[\mathcal{C}_\varepsilon^s, \psi_k] = 0$ as ψ_k is constant. Therefore, we have for any cube Q_j^γ that touches Q_k^γ:

$$\begin{aligned} \|\mathbb{1}_j[\mathcal{C}_\varepsilon^s,\phi]\mathbb{1}_k\|_{L^2_\sigma(bD)} &= \|\mathbb{1}_j[\mathcal{C}_\varepsilon^s,\phi_k]\mathbb{1}_k\|_{L^2_\sigma(bD)} \\ &\leq \|\mathbb{1}_j\mathcal{C}_\varepsilon^s\phi_k\mathbb{1}_k\|_{L^2_\sigma(bD)} + \|\mathbb{1}_j\phi_k\mathcal{C}_\varepsilon^s\mathbb{1}_k\|_{L^2_\sigma(bD)} \\ &\lesssim 2\varepsilon\|\mathcal{C}_\varepsilon^s\|_{L^2_\sigma(bD)} \\ &\leq 2\varepsilon M_\sigma. \end{aligned}$$

This completes the proof.

□

The following proposition is an immediate consequence of the well-known reverse Hölder property of A_p weights.

Proposition 3.4.1. Let $1 < p < \infty$ and suppose $\sigma \in A_p$. Then there exists a $\delta > 0$ so $\sigma^{1+\delta} \in L^1(bD)$.

We are now finally ready to prove the main theorem.

Proof of Theorem 3.1.1. As noted before, it suffices to prove the result for $p = 2$. Recall $\mathcal{A}_\varepsilon = (\mathcal{C}_\varepsilon^s)^* - \mathcal{C}_\varepsilon^s$ and $\mathcal{D}_\varepsilon = (\mathcal{R}_\varepsilon^s)^* - \mathcal{R}_\varepsilon^s$. Thus, the Kerzman-Stein equation takes the form

$$\mathcal{S}(I - \mathcal{A}_\varepsilon) - \mathcal{S}\mathcal{D}_\varepsilon = \mathcal{C}_\varepsilon.$$

By Lemma 3.4.3, if ε and s are chosen sufficiently small, then $\|\mathcal{A}_\varepsilon\|_{L^2_\sigma(bD)} < 1$. Inverting $\mathcal{A}_\varepsilon$ using a Neumann series yields:

$$\mathcal{S} = \mathcal{C}_\varepsilon(I - \mathcal{A}_\varepsilon)^{-1} + \mathcal{S}\mathcal{D}_\varepsilon(I - \mathcal{A}_\varepsilon)^{-1}.$$

By Lemma 3.4.1, the operator $\mathcal{C}_\varepsilon(I - \mathcal{A}_\varepsilon)^{-1}$ maps $L^2_\sigma(bD)$ to itself. Now, by discussions above $\mathcal{D}_\varepsilon(I - \mathcal{A}_\varepsilon)^{-1}$ maps $L^2_\sigma(bD)$ to $L^\infty(bD)$, and hence maps $L^2_\sigma(bD)$ to $L^p(bD)$ boundedly for any p, $1 < p < \infty$. Additionally, by the principle result in [40], $\mathcal{S}$ extends to a bounded operator on L^p. So in particular $\mathcal{S}\mathcal{D}_\varepsilon(I - \mathcal{A}_\varepsilon)^{-1}$ maps $L^2_\sigma(bD)$ to L^p for all p, $1 < p < \infty$. We claim that if p is chosen sufficiently large (depending on σ), then $\|g\|_{L^2_\sigma(bD)} \lesssim \|g\|_{L^p(bD)}$ for all measurable functions g. Then

$$\|\mathcal{S}\mathcal{D}_\varepsilon(I - \mathcal{A}_\varepsilon(f))\|_{L^2_\sigma(bD)} \lesssim \|\mathcal{S}\mathcal{D}_\varepsilon(I - \mathcal{A}_\varepsilon(f))\|_{L^p(bD)}$$

for all measurable f, which will then establish the result.

To prove the claim, we use Proposition 3.4.1. In particular, we have, using Hölder's inequality with exponents $\frac{p}{2}$ and $r = \left(\frac{p}{2}\right)'$:

$$\begin{aligned} \|g\|^2_{L^2_\sigma(bD)} &= \int_{bD} |g|^2 \sigma \, dS \\ &\leq \left(\int_{bD} |g|^p \, dS \right)^{\frac{2}{p}} \left(\int_{bD} \sigma^r \, dS \right)^{\frac{1}{r}} \\ &\lesssim \|g\|^2_{L^p(bD)} \end{aligned}$$

provided p is chosen sufficiently large so $r < 1 + \delta$. This completes the proof.

$\square$

Chapter 4

The Bergman Projection on Near Minimally Smooth Domains

4.1 Summary of Main Results and Outline of Proof

In this chapter, which again mainly consists of material from [67], we prove that the B_p condition (see Definition 1.5.1 in Chapter 1) is sufficient for the boundedness of the Bergman projection $\mathcal{B}$ on $L^p_\sigma(D)$ on strongly pseudoconvex domains with C^4 boundary smoothness. In doing so, we generalize some of the results of Chapter 2, in particular Theorem 2.1.1 in the case of strongly pseudoconvex domains, to domains with substantially less regularity. In particular, the following is the main result of this chapter:

Theorem 4.1.1. Let D be strongly pseudoconvex with C^4 boundary and $1 < p < \infty$. If $\sigma \in B_p$, then there exists $C > 0$ so that $\|\mathcal{B}f\|_{L^p_\sigma(D)} \leq C\|f\|_{L^p_\sigma(D)}$.

The techniques involved are very similar to those in Chapter 3 for the Szegő projection on C^3 domains. In this case the relevant non-orthogonal projection, which we denote by $\mathcal{T}$, is given by a Cauchy-Fantappié intgeral taken over the solid domain rather than the boundary plus a correction term (this approach was used to prove certain regularity properties of the Bergman projection; see for example [41, 42]). In particular, we obtain the Kerzman-Stein equation $\mathcal{B}(I - (\mathcal{T}^* - \mathcal{T})) = \mathcal{T}$. The L^p boundedness of the operator $\mathcal{T}$ can be established using Schur's Test, and it remains to establish the invertibility of $(I - (\mathcal{T}^* - \mathcal{T}))$ on L^p. This task with completed by Lanzani and Stein in [37], and in fact their methods extend to the minimal smoothness (C^2) case by again using a more refined truncation argument and a family $\{\mathcal{T}_\varepsilon\}$ of projections.

In the case of weighted estimates, Schur's Test is ill-equipped to deal with weights other than "radial" weights, so a new approach is needed. In particular, to prove the operator $\mathcal{T}$ is bounded on $L^p_\sigma(D)$, we must use a modified singular integral theory and view the projection $\mathcal{T}$ as a kind of Calderón-Zygmund operator with respect to the quasi-metric d introduced in Chapter 2. Proving the invertibility of the operator $(I - (\mathcal{T}^* - \mathcal{T}))$ on $L^p_\sigma(D)$ proceeds via the compactness of $\mathcal{T}^* - \mathcal{T}$ in a very similar manner to the arguments for the Szegő projection on C^3 domains in Chapter 3. We should also remark that in the unweighted case that $\sigma = 1$, our result can be seen as alternate proof of the main theorem in [37] for domains with sufficient (C^4) regularity.

We also provide some results in the C^2 case. In particular, we prove an analog of Theorem 4.1.1 when σ is "radial", and connect regularity properties of the Bergman projection and $\mathcal{T}$ to Toeplitz operators with "radial" symbols.

This chapter is organized as follows. Section 4.2 goes over the relevant background material, in particular the construction of the non-orthogonal projection $\mathcal{T}$ and the form of the quasi-metric. In Section 4.3, we prove Theorem 4.1.1, and in fact provide a more precise version of the theorem. In Section 4.4, we establish weighted estimates for the Bergman projection on minimally smooth strongly pseudoconvex domains for a special class of weights. Finally, in Section 4.5, we provide an additional application of the Kerzman-Stein equation to Toeplitz operators.

4.2 Preliminaries

Now we let D be a strongly pseudoconvex domain with C^4 defining function ρ that is strictly plurisubharmonic. As in Lanzani-Stein [37], we can construct an integral operator

$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ that integrates over the interior of the domain D, where $\mathcal{T}_1$ is constructed using Cauchy-Fantappié theory and $\mathcal{T}_2$ is obtained by solving a $\bar{\partial}$ problem. The operator $\mathcal{T}$ has the property that it produces and reproduces holomorphic functions.

We now make several definitions that are analogous to our treatment in Chapter 3 of the Szegő projection. We will slightly abuse notation by reusing certain letters to represent analogous objects in the Bergman case. Define

$$g(z,w) := -\rho(w) - \chi(P_w(z)) + (1-\chi)|z-w|^2$$

where $P_w(z)$ denotes the Levi polynomial at w and χ is an appropriately chosen C^∞ cutoff function. In particular, using the strict pseudoconvexity of D, χ can be chosen so

$$\text{Re } g(z,w) \gtrsim -\rho(w) - \rho(z) + c|z-w|^2.$$

Now, as before define the $(1,0)$ form in w

$$G(z,w) := \chi\left(\sum_{j=1}^{n} \frac{\partial\rho}{\partial w_j}(w)\, dw_j - \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial w_j \partial w_k}(w)(w_k - z_k)\, dw_j\right) + (1-\chi)\sum_{j=1}^{n}(\bar{w}_j - \bar{z}_j)\, dw_j\,.$$

Note that G has the property that if we let

$$\widehat{\eta}(z,w) = \frac{G(z,w)}{g(z,w)+\rho(w)}, \tag{4.2.1}$$

then

$$\langle \widehat{\eta}(z,w), w-z\rangle = 1$$

for all $z \in D$ and w in neighborhood of bD. Note that (4.2.1) indicates $\widehat{\eta}$ is a generating form. However, we instead define the $(1,0)$ form in w:

$$\eta(z,w) = \frac{G(z,w)}{g(z,w)}$$

and associated integral operator

$$\mathcal{T}_1(f)(z) := \frac{1}{(2\pi i)^n} \int_D (\bar{\partial}_w \eta)^n(z,w) f(w),$$

where $(\bar{\partial}_w \eta)^n$ denotes the wedge product taken n times. We have the following proposition (see [37, Proposition 3.1]), which uses the C^4 regularity of the domain D:

Proposition 4.2.1. Suppose f is holomorphic on D and belongs to $L^1(D)$. Then for all $z \in D$, one has

$$\mathcal{T}_1(f)(z) = f(z).$$

A computation shows the operator $\mathcal{T}_1$ has kernel

$$K_1(z,w) = \frac{N(z,w)}{(2\pi i)^n (g(z,w))^{n+1}} \tag{4.2.2}$$

where $N(z,w)$ is an (n,n) form of class C^1 (in w) and smooth in the parameter z. In particular, we have:

$$N(z,w) = -\left(n(\bar{\partial}_w G)^{n-1} \wedge \bar{\partial}_w g \wedge G - g(\bar{\partial}_w G)^n \right)(z,w). \tag{4.2.3}$$

We write $N(z,w) = \mathcal{N}(z,w)dV(w)$, where dV denotes the Euclidean volume form. Notice the fact that $\mathcal{N}(z,w)$ is of class C^1 in w is a direct consequence of the fact that D has C^4 boundary.

Proposition 4.2.1 guarantees that $\mathcal{T}_1$ reproduces holomorphic functions, but as in the Szegő case we need to add a correction operator to ensure that it produces holomorphic functions. The details can be found in [37], and again involve solving a $\bar{\partial}$ problem on a strongly pseudoconvex domain that contains D. We have the following proposition concerning $\mathcal{T}_2$ and the operator $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$ (see [37, Proposition 3.2]):

Proposition 4.2.2. There is an integral operator $\mathcal{T}_2$ defined

$$\mathcal{T}_2 f(z) := \int_D K_2(z,w) f(w)\, dV(w)$$

with

$$\sup_{z,w\in\overline{D}} |K(z,w)| < \infty$$

that satisfies:

1. If $f \in L^1(D)$, then $\mathcal{T}(f) = \mathcal{T}_1(f) + \mathcal{T}_2(f)$ is holomorphic on D.

2. If, in addition, f is holomorphic on D, then $\mathcal{T}(f)(z) = f(z)$ for $z \in D$.

We now review the specific construction of the quasi-metric d in the strongly pseudoconvex case. This metric can be defined using polydiscs introduced by McNeal (see [50]) and is defined locally at first on a neighborhood U of a point $p \in bD$. Fix a point $q \in U$. First, we may by a unitary rotation (plus a normalization) and translation assume $\partial\rho(q) = dz_1$ and $q = 0$. Then, define holomorphic coordinates $\zeta = (\zeta_1, \ldots, \zeta_n)$ as follows:

$$\zeta_1 = z_1 + \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 \rho(w)}{\partial z_j \partial z_k}(z_j)(z_k), \;\; \zeta_j = z_j, j = 2, \ldots n.$$

Note if $\Phi : U \to \Phi(U)$ denotes this coordinate map, Φ is a biholomorphism if U is chosen small enough.

Consider the polydisc:

$$P(q,\delta) = \{z : |z_1| < \delta, |z_j| < \delta^{1/2}, 2 \le j \le n\},$$

where again z_j denotes the special holomorphic coordinates centered at q.

These polydiscs are precisely those introduced in Chapter 2 in the special case of strongly pseudoconvex domains. The polydiscs satisfy certain types of doubling properties (see [50]). We include a proof for completeness, since the result was stated in the smooth case (but in fact C^2 boundary is sufficient).

Proposition 4.2.3. There exist independent constants C_1, C_2 so the following hold for the polydiscs:

1. If $P(q_1,\delta) \cap P(q_2,\delta) \neq \emptyset$, then $P(q_1,\delta) \subset C_1 P(q_2,\delta)$ and $P(q_2,\delta) \subset C_1 P(q_1,\delta)$.

2. There holds $P(q_1, 2\delta) \subset C_2 P(q_1,\delta)$.

Proof. The second property is essentially immediate from the definition of P, so we focus on the first property. Suppose $P(q_1,\delta) \cap P(q_2,\delta) \neq \emptyset$. Let $z_1, \dots, z_n$ denote the holomorphic coordinates centered at q_1 and $\zeta_1, \dots, \zeta_n$ denote the holomorphic coordinates centered at q_2. The general idea is that these holomorphic coordinates do not differ greatly. We need to take an arbitrary point $p \in P(q_1,\delta)$ and show there exists a constant C_1 so $p \in C_1 P(q_2,\delta)$. Let $r \in P(q_1,\delta) \cap P(q_2,\delta)$. Write the coordinates of p relative to the coordinate system of the second polydisc as $(\zeta_1(p), \dots, \zeta_n(p))$. First observe that the definition of the polydiscs implies

$$|p - q_2| \le |p - r| + |r - q_2| \lesssim \delta^{1/2}$$

and the same bound holds for the quantities $|q_1 - q_2|$ and $|p - q_1|$. Then we have

$$\begin{aligned}
|\zeta_1(p)| &\approx \left|\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(q_2)(p_j - q_{2,j})\right| + \mathcal{O}(|p-q_2|^2) \\
&\lesssim |z_1(p)| + \left|\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(q_2)(p_j - q_{2,j}) - \sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(q_1)(p_j - q_{1,j})\right| + \delta \\
&\lesssim \delta + |\langle \partial\rho(q_2) - \partial\rho(q_1), p - q_2\rangle| + |\langle \partial\rho(q_1), q_2 - q_1\rangle| \\
&\lesssim \delta + |q_2 - q_1||p - q_2| + |\langle \partial\rho(q_1), q_2 - q_1\rangle| \\
&\lesssim \delta + |\langle \partial\rho(q_1), q_2 - q_1\rangle| \,.
\end{aligned}$$

We control $|\langle \partial\rho(q_1), q_2 - q_1\rangle|$ as follows:

$$\begin{aligned}
|\langle \partial\rho(q_1), q_2 - q_1\rangle| &\leq |\langle \partial\rho(q_1), r - q_1\rangle| + |\langle \partial\rho(q_1), q_2 - r\rangle| \\
&\leq z_1(r) + |\langle \partial\rho(q_1) - \partial\rho(q_2), q_2 - r\rangle| + |\langle \partial\rho(q_2), r - q_2\rangle| \\
&\lesssim \delta + |q_1 - q_2||q_2 - r| + \zeta_1(r) \\
&\lesssim \delta.
\end{aligned}$$

It is easy to verify all the implicit constants are independent of q_1, q_2. So there exists a constant C_1 so $|\zeta_1(p)| < C_1\delta$.

On the other hand, for $2 \leq j \leq n$, we have

$$|\zeta_j(p)| \lesssim |p - q_2| \lesssim \delta^{1/2},$$

so if C_1 is chosen appropriately large, then $|\zeta_j(p)| < C_1\delta^{1/2}$. Then $p \in C_1 P(q_1, \delta)$, as we sought to show.

The other conclusion is immediate by symmetry. This completes the proof. □

Use these coordinates to construct a global quasi-metric $d(z, w)$ with the construction given in Chapter 2, Section 2.2 (thus, we will not repeat it here). Technically, this metric is only defined on a tubular neighborhood of the boundary, but this presents us with no issues and we abuse notation by writing it to be defined on D.

Recall that (D, d, V) is a space of homogeneous type in the sense of Definition 1.2.1. It is also a fact that $V(B(z,r)) \approx r^{n+1}$. Note that locally,

$$d(z,w) \approx |z_1 - w_1| + \sum_{j=2}^{n} |z_j - w_j|^2$$

where the components of z and w are computed in the special coordinates at w.

We have the following relation between the quasi-metric d and the Euclidean distance:

Proposition 4.2.4. We have, for $z', z \in D$:

$$|z' - z|^2 \lesssim d(z', z) \lesssim |z' - z|.$$

Proof. It suffices to work locally, so we may assume d coincides with one of the local quasi-metrics on a neighborhood U. Let $\Phi(z) = \zeta(z) = (\zeta_1, \dots, \zeta_n)$ denote the biholomorphic coordinate change described in detail above in the construction of d. Because the coordinate change is biholomorphic with Jacobian uniformly bounded above and below, we have the following bounds:

$$\begin{aligned} |z - z'|^2 &= \sum_{j=1}^{n} |z_j - z_j'|^2 \\ &\lesssim \sum_{j=1}^{n} |\zeta_j - \zeta_j'|^2 \\ &\leq d(z, z'). \end{aligned}$$

The proof of the upper bound is similar. □

We now show that when we restrict d to $bD \times bD$, we obtain a quantity comparable in size to $|g(z,w)|$, which establishes a natural connection between the Szegő and Bergman cases.

Proposition 4.2.5. If $z, w \in bD$, then we have

$$d(z,w) \approx |g(z,w)|.$$

Proof. Let $z = (\zeta_1, \ldots, \zeta_n)$ in the special holomorphic coordinates centered at w. Note

$$d(z,w) \approx |\zeta_1| + \sum_{j=2}^{n} |\zeta_j|^2.$$

Also, we have by [37, Proposition 2.1],

$$|g(z,w)| \approx |\mathrm{Im}\langle \partial\rho(w), w - z\rangle| + |z - w|^2.$$

But notice that

$$|\langle \partial\rho(w), w - z\rangle| \lesssim |\zeta_1| + |z - w|^2$$

and moreover

$$|z - w|^2 \lesssim \sum_{j=1}^{n} |\zeta_j|^2 \lesssim d(z,w)$$

since the coordinate change is biholomorphic. This shows $|g(z,w)| \lesssim d(z,w)$. To see the reverse, note that if $|z - w|$ is small enough, then $g(z,w) = P_w(z)$ and

$$|\zeta_1| \lesssim |P_w(z)| + |z - w|^2$$

which combined with the estimates above gives $d(z,w) \lesssim |g(z,w)|$.

□

We maintain the definition for the B_p class from Chapter 1, Definition 1.5.1, and the maximal function $\mathcal{M}$ given in Definition 2.2.2. Recall Theorem 2.2.3 states that $\mathcal{M}$ is bounded on $L^p_\sigma(D)$ for $\sigma \in B_p$ (the proof does not depend in any way on the boundary smoothness).

Recall that B_1 weights were defined in Chapter 2, Definition 1.5.2. It is easy to see that an equivalent definition is as follows: we say $\sigma \in B_1$ if σ is integrable and bounded below, and satisfies

$$\mathcal{M}\sigma(z) \lesssim \sigma(z)$$

for almost every $z \in D$.

4.3 Proofs of Theorems

In this section, we will study the Bergman projection on weighted spaces under the assumption that D is a C^4 domain. Our main goal is to prove the following theorem, which is a more detailed version of Theorem 4.1.1.

Theorem 4.3.1. Let D be strongly pseudoconvex with C^4 boundary. Then for $1 < p < \infty$ and $\sigma \in B_p$, the following hold:

1. The operator $\mathcal{T}^* - \mathcal{T}$ is compact on $L^p_\sigma(D)$.

2. The operator $I - (\mathcal{T}^* - \mathcal{T})$ is invertible on $L^p_\sigma(D)$.

3. The Bergman projection $\mathcal{B}$ extends to a bounded operator on $L^p_\sigma(D)$ and satisfies

$$\mathcal{B} = \mathcal{T}(I - (\mathcal{T}^* - \mathcal{T}))^{-1}.$$

4.3.1 The Main Term

We follow the following general outline to prove Theorem 4.3.1. First, we obtain size and smoothness estimates for $K_1(z,w)$, the kernel of $\mathcal{T}_1$. This enables us to prove that $\mathcal{T}$ maps $L^p_\sigma(D)$ to $L^p_\sigma(D)$. We then proceed to show that $\mathcal{T}^* - \mathcal{T}$ is compact on $L^2_\sigma(D)$ and improves L^p spaces. These properties allow us to use the Kerzman-Stein equation to extract the $L^p_\sigma(D)$ boundedness of $\mathcal{B}$ from the $L^p_\sigma(D)$ boundedness of $\mathcal{T}$.

The following proposition follows immediately from the fact that $\mathcal{T}_2$ has a bounded kernel and D is a bounded domain.

Proposition 4.3.1. Let $1 < p < \infty$. If $\sigma \in B_p$, then the operator $\mathcal{T}_2$ is bounded on $L^p_\sigma(D)$.

Proof. Take $f \in L^p_\sigma(D)$. Then we have

$$\begin{aligned}
\|\mathcal{T}_2 f\|^p_{L^p_\sigma(D)} &= \int_D \left| \int_D K_2(z,w) f(w)\, dV(w) \right|^p \sigma(z)\, dV(z) \\
&\lesssim \left(\int_D |f(w)|\, dV(w) \right)^p \left(\int_D \sigma(z)\, dV(z) \right) \\
&\leq \|f\|^p_{L^p_\sigma(D)} \left(\int_D \sigma(z)\, dV(z) \right) \left(\int_D \sigma(w)^{-\frac{1}{p-1}}\, dV(w) \right)^{p-1} \\
&\leq [\sigma]_{B_p} \|f\|^p_{L^p_\sigma(D)}.
\end{aligned}$$

□

We now work to prove the following theorem:

Theorem 4.3.2. Let $1 < p < \infty$. If $\sigma \in B_p$, then there exists a constant $C > 0$ so that $\|\mathcal{T} f\|_{L^p_\sigma(D)} \leq C\|f\|_{L^p_\sigma(D)}$ and $\|\mathcal{T}^* f\|_{L^p_\sigma(D)} \leq C\|f\|_{L^p_\sigma(D)}$.

In light of the previous proposition, which clearly also works for $\mathcal{T}_2^*$, it is sufficient to show that $\mathcal{T}_1$ and $\mathcal{T}_1^*$ are bounded on $L^p_\sigma(D)$. To this end, we define the following comparison operator:

$$\Gamma(f)(z) = \int_D \frac{1}{|g(z,w)|^{n+1}} f(w)\, dV(w)\,.$$

Note that in light of (4.2.2), we have the pointwise domination:

$$|\mathcal{T}_1(f)(z)| \lesssim \Gamma(|f|)(z).$$

To prove the weighted L^p regularity of Γ, we follow Békollè's approach of using singular integral theory that was also undertaken in Chapter 2. In particular, we obtain the following size and smoothness estimates on the kernel of Γ:

Lemma 4.3.1. There exist positive constants C_3, C_4 so the following hold:

1. $$\frac{1}{|g(z,w)|^{n+1}} \leq C_3 \min\left\{\frac{1}{V(B(z,d(z,bD)))}, \frac{1}{V(B(w,d(w,bD)))}\right\}.$$

2. If $d(z,w) \geq C_4 d(z,z')$, then

$$\left|\frac{1}{(g(z,w))^{n+1}} - \frac{1}{(g(z',w))^{n+1}}\right| \leq C_3 \left(\frac{d(z,z')}{d(z,w)}\right)^{1/2} \frac{1}{V(B(z,d(z,w)))}.$$

3. If $d(z,w) \geq C_4 d(w,w')$, then

$$\left|\frac{1}{(g(z,w))^{n+1}} - \frac{1}{(g(z,w'))^{n+1}}\right| \leq C_3 \left(\frac{d(w,w')}{d(z,w)}\right)^{1/2} \frac{1}{V(B(w,d(z,w)))}.$$

Proof. For the first statement, it suffices to prove

$$\frac{1}{|g(z,w)|^{n+1}} \lesssim \frac{1}{V(B(z,d(z,bD)))},$$

since $|g(z,w)| \approx |g(w,z)|$ by [37, Proposition 2.1]. Since $V(B(z, d(z, bD))) \approx [d(z, bD)]^{n+1}$, it is enough to show $d(z, bD) \lesssim |g(z,w)|$. We have $d(z, bD) \approx \mathrm{dist}(z, bD) \approx |\rho(z)|$. On the other hand, $|g(z,w)| \gtrsim |\rho(z)|$ by [37, Proposition 2.1]). This proves the size estimate.

For the smoothness estimate, we first prove as a preliminary fact that $d(z,w) \lesssim |g(z,w)|$. We may assume $|z-w|$ is small enough so that $g(z,w) = -\rho(w) - P_w(z)$. By definition we have

$$d(z,w) \approx |\zeta_1| + \sum_{j=2}^{n} |\zeta_j|^2$$

where $\zeta_1, \dots, \zeta_n$ are the components of z in the holomorphic coordinates centered at w. Using the triangle inequality and the definition of the biholomorphic coordinates, we obtain

$$|\zeta_1| \lesssim \left| \sum_{j=1}^{n} \frac{\partial \rho(w)}{\partial z_j}(z_j - w_j) \right| + \mathcal{O}(|z-w|^2) \lesssim |\rho(w)| + |-\rho(w) - P_w(z)| + \mathcal{O}(|z-w|^2).$$

Then, appeal to the fact that $|g(z,w)| \gtrsim |\rho(w)| + |z-w|^2$ by [37, Proposition 2.1] and the fact that the coordinate change is biholomorphic to obtain the desired conclusion.

We only prove the first smoothness estimate; the second one is proven similarly and is only slightly more complicated. We use similar ideas as in [40]. We first prove the estimate

$$|g(z,w) - g(z',w)| \lesssim d(z,z')^{1/2} d(z,w)^{1,2} + d(z,z').$$

To begin with, note that we have

$$\begin{aligned} |g(z,w) - g(z',w)| \;\; &\leq \;\; |\langle \partial\rho(w), w - z\rangle - \langle \partial\rho(w), w - z'\rangle| \\ &+ \;\; \frac{1}{2} \left| \sum_{j,k=1}^{n} \frac{\partial^2 \rho(w)}{\partial w_j \partial w_k} \left[(w_j - z_j)(w_k - z_k) - (w_j - z'_j)(w_k - z'_k) \right] \right|. \end{aligned}$$

We deal with the first term, $|\langle \partial\rho(w), w-z\rangle - \langle \partial\rho(w), w-z'\rangle| = |\langle \partial\rho(w), z'-z\rangle|$. We then have, using Proposition 4.2.4:

$$\begin{aligned} |\langle \partial\rho(w), z'-z\rangle| &\leq |\langle \partial\rho(z), z'-z\rangle| + |\langle \partial\rho(w) - \partial\rho(z), z'-z\rangle| \\ &\lesssim d(z,z') + |z-w||z-z'| \\ &\lesssim d(z,z') + d(z,w)^{1/2} d(z,z')^{1/2}. \end{aligned}$$

Now we handle the second term. Notice that we have

$$\begin{aligned} |(w_j - z_j)(w_k - z_k) - (w_j - z_j')(w_k - z_k')| &\leq |(w_j - z_j)(w_k - z_k) - (w_j - z_j')(w_k - z_k)| \\ &+ |(w_j - z_j')(w_k - z_k) - (w_j - z_j')(w_k - z_k')| \\ &\leq |w_k - z_k||z_j - z_j'| + |w_j - z_j'||z_k - z_k'| \\ &\leq |w-z||z-z'| + (|w-z| + |z-z'|)|z-z'| \\ &\lesssim d(z,w)^{1/2} d(z,z')^{1/2} + (d(z,w)^{1/2} + d(z,z')^{1/2}) d(z,z')^{1/2} \\ &\lesssim d(z,w)^{1/2} d(z,z')^{1/2} \end{aligned}$$

which proves the required bound for the second piece.

Now, we show $|g(z,w)| \approx |g(z',w)|$ if $d(z,w) \geq C_4 d(z,z')$. We estimate, using the work previously done:

$$\begin{aligned} |g(z,w)| &\leq |g(z,w)| + |g(z',w) - g(z,w)| \\ &\lesssim |g(z',w)| + d(z,w)^{1/2} d(z,z')^{1/2} + d(z,z') \\ &\lesssim |g(z',w)| + (C_4^{-1/2} + C_4^{-1}) d(z,w) \\ &\lesssim |g(z',w)| + (C_4^{-1/2} + C_4^{-1}) |g(z,w)|. \end{aligned}$$

Thus, if C_4 is chosen appropriately large, we can subtract the $|g(z,w)|$ term to the other side and obtain $|g(z,w)| \lesssim |g(z',w)|$. The bound $|g(z',w)| \lesssim |g(z,w)|$ is obtained similarly.

Finally, we obtain, using our assumption $d(z,w) \geq C_4 d(z,z')$:

$$\begin{aligned}\left|\frac{1}{(g(z,w))^{n+1}} - \frac{1}{(g(z',w))^{n+1}}\right| &\leq \frac{|g(z,w)-g(z',w)|\left(\sum_{t=0}^{n}|g(z,w)|^t|g(z',w)|^{n-t}\right)}{|g(z,w)|^{n+1}|g(z',w)|^{n+1}} \\ &\lesssim \frac{|g(z,w)-g(z',w)|}{|g(w,z)|^{n+2}} \\ &\lesssim \frac{1}{d(z,w)^{n+1}}\frac{d(z,w)^{1/2}d(z,z')^{1/2}}{d(z,w)} \\ &\lesssim \left(\frac{d(z,z')}{d(z,w)}\right)^{1/2}\frac{1}{V(B(z,d(z,w)))}\end{aligned}$$

which establishes the smoothness estimate. □

As a consequence of the size and smoothness estimates obtained on the kernel of the positive operator Γ, we get the following theorem (one can follow the arguments verbatim contained in Theorem 2.1.1 in Chapter 2).

Theorem 4.3.3. Let $1 < p < \infty$. If $\sigma \in B_p$, then the operators Γ, Γ^* are bounded on $L^p_\sigma(D)$.

Now we can prove Theorem 4.3.2 as follows:

Proof of Theorem 4.3.2. Note that Theorem 4.3.3 implies the operators $\mathcal{T}_1, \mathcal{T}_1^*$ map $L^p_\sigma(D)$ to $L^p_\sigma(D)$ boundedly, which together with Proposition 4.3.1 establishes the result. □

4.3.2 The Error Term

We now proceed to deal with the "error term" $\mathcal{T}^* - \mathcal{T}$. In light of the arguments above, we already know $\mathcal{T}^* - \mathcal{T}$ is bounded on $L^p_\sigma(D)$, but in fact this operator exhibits much better behavior. In analogy with the approach taken in Chapter 3 for the Szegő operator, we show that this operator is compact on $L^p_\sigma(D)$ for $\sigma \in B_p$ and improves L^p spaces. We conclude by applying the Kerzman-Stein trick to deduce the boundedness of $\mathcal{B}$ from this information.

Lemma 4.3.2. Let $K(z,w)$ denote the kernel of the integral operator $\mathcal{T}^* - \mathcal{T}$. Then we have the size estimates:

$$|K(z,w)| \lesssim d(z,w)^{-(n+\frac{1}{2})}$$

and

$$|K(z,w)| \lesssim \min\left\{d(z,bD)^{-(n+\frac{1}{2})}, d(w,bD)^{-(n+\frac{1}{2})}\right\}.$$

Proof. It is proven in [60, VII, Theorem 7.6] that $|K(z,w)| \lesssim |g(z,w)|^{-(n+\frac{1}{2})}$, so using the fact, contained in the proof of Lemma 2.2.2, that $d(z,w) \lesssim |g(z,w)|$, we deduce that $|K(z,w)| \lesssim d(z,w)^{-(n+\frac{1}{2})}$. For completeness, we sketch the argument given in [60].

First, note from (4.2.3) that we can write $N(z,w) = N_0(z,w)+N_1(z,w)$, where $N_0(z,w) = -n\left((\bar{\partial}_w G)^{n-1} \wedge \bar{\partial}_w g \wedge G\right)(z,w)$ and $N_1(z,w) = g(z,w)\left((\bar{\partial}_w G)^n\right)(z,w)$. Note that

$$N_0(w,w) = -n\left((\bar{\partial}_w \partial_w \rho)^{n-1} \wedge \bar{\partial}_w \rho \wedge \partial_w \rho\right)(w).$$

Write $N_0(z,w) = \mathcal{N}_0(z,w)dV(w)$ and $N_1(z,w) = \mathcal{N}_1(z,w)dV(w)$, so in particular $\frac{\mathcal{N}_0(w,w)}{(2\pi i)^n}$ is a real-valued function. Moreover, it is clear $\mathcal{N}_0(z,w) = \mathcal{N}_0(w,w) + \mathcal{O}(|z-w|)$ by our smoothness assumptions and the same is true of $\mathcal{N}_0(w,z)$. Thus, we have, using the fact that $|g(z,w)| \approx |g(w,z)|$ and that the kernel of $\mathcal{T}_2$ is uniformly bounded by a constant C:

$$\begin{aligned}
|K(z,w)| &\lesssim \left|\frac{1}{(-2\pi i)^n}\left(\frac{\overline{\mathcal{N}_0(w,z)}}{\overline{g(w,z)}^{n+1}} + \frac{\overline{\mathcal{N}_1(w,z)}}{\overline{g(w,z)}^{n+1}}\right) - \frac{1}{(2\pi i)^n}\left(\frac{\mathcal{N}_0(z,w)}{g(z,w)^{n+1}} + \frac{\mathcal{N}_1(z,w)}{g(z,w)^{n+1}}\right)\right| + C \\
&\lesssim \left|\frac{1}{(-2\pi i)^n}\left(\frac{\overline{\mathcal{N}_0(z,w)}}{\overline{g(w,z)}^{n+1}}\right) - \frac{1}{(2\pi i)^n}\left(\frac{\mathcal{N}_0(w,z)}{g(z,w)^{n+1}}\right)\right| + \frac{1}{|g(z,w)|^n} \\
&\lesssim \left|\mathcal{N}_0(w,w)\left(\frac{1}{\overline{g(w,z)}^{n+1}} - \frac{1}{g(z,w)^{n+1}}\right)\right| + \frac{|z-w|}{|g(z,w)|^{n+1}} + \frac{1}{|g(z,w)|^n}.
\end{aligned}$$

Moreover, [60, Lemma 7.4] gives that $|g(z,w) - \overline{g(w,z)}| = \mathcal{O}(|z-w|^3)$ with an argument very similar to Proposition 3.3.5. Then proceeding as in Lemma 3.3.1 and using the fact that $|z-w| \lesssim |g(z,w)|^{1/2}$ yields the desired conclusion.

The other estimate is proven in the same way, using the fact that $d(z, bD) \lesssim |g(z,w)|$ and $d(w, bD) \lesssim |g(z,w)|$. □

We have the following lemma concerning the behavior of B_1 weights when integrated against this kernel:

Lemma 4.3.3. Let $K(z,w)$ be a kernel measurable on $D \times D$ that satisfies the size estimate $|K(z,w)| \lesssim d(z,w)^{-(n+1/2)}$, and let $\sigma \in B_1$. Then the following estimates hold for all $z, w \in D$ and $\delta > 0$:

$$\int_{B(z,\delta)} |K(z,w)|\sigma(w)\, dV(w) \lesssim (\delta^{1/2} + d(z,bD)^{1/2})\sigma(z)$$

and

$$\int_{B(w,\delta)} |K(z,w)|\sigma(z)\, dV(z) \lesssim (\delta^{1/2} + d(w,bD)^{1/2})\sigma(w).$$

Proof. By symmetry, it clearly suffices to prove the first assertion. Let N be the largest non-negative integer so that $2^{-N}\delta > d(z, bD)$. If there is no such N, make the obvious modifications. We have, integrating over dyadic "annuli"

$$
\begin{aligned}
\int\limits_{B(z,\delta)} |K(z,w)|\sigma(w)\,dV(w) &= \sum_{j=0}^{\infty} \int\limits_{B(z,2^{-j}\delta)\setminus B(z,2^{-(j+1)}\delta)} |K(z,w)|\sigma(w)\,dV(w) \\
&= \sum_{j=0}^{N} \int\limits_{B(z,2^{-j}\delta)\setminus B(z,2^{-(j+1)}\delta)} |K(z,w)|\sigma(w)\,dV(w) \\
&+ \sum_{j=N+1}^{\infty} \int\limits_{B(z,2^{-j}\delta)\setminus B(z,2^{-(j+1)}\delta)} |K(z,w)|\sigma(w)\,dV(w)\,.
\end{aligned}
$$

We deal with the first summation first. We have

$$
\begin{aligned}
\sum_{j=0}^{N} \int\limits_{B(z,2^{-j}\delta)\setminus B(z,2^{-(j+1)}\delta)} |K(z,w)|\sigma(w)\,dV(w) &\lesssim \sum_{j=0}^{N} \int\limits_{B(z,2^{-j}\delta)\setminus B(z,2^{-(j+1)}\delta)} d(z,w)^{-(n+1/2)}\sigma(w)\,dV(w) \\
&\leq \sum_{j=0}^{N} \int\limits_{B(z,2^{-j}\delta)} 2^{(j+1)(n+1/2)}\delta^{-(n+1/2)}\sigma(w)\,dV(w) \\
&\lesssim \sum_{j=0}^{N} \delta^{1/2}2^{-j/2}\frac{1}{V(B(z,2^{-j}\delta))} \int\limits_{B(z,2^{-j}\delta)} \sigma(w)\,dV(w) \\
&\leq \sum_{j=0}^{N} \delta^{1/2}2^{-j/2}\mathcal{M}(\sigma)(z) \\
&\lesssim \delta^{1/2}\mathcal{M}(\sigma)(z) \\
&\lesssim \delta^{1/2}\sigma(z).
\end{aligned}
$$

Note the implicit constant is independent of N. We now proceed to deal with the second summation:

$$\begin{aligned}
\sum_{j=N+1}^{\infty} \int_{B(z,2^{-j}\delta)\setminus B(z,2^{-(j+1)}\delta)} |K(z,w)|\sigma(w)\,dV(w) &\leq \int_{B(z,d(z,bD))} |K(z,w)|\sigma(w)\,dV(w) \\
&\lesssim \int_{B(z,d(z,bD))} d(z,bD)^{-(n+1/2)}\sigma(w)\,dV(w) \\
&= \frac{d(z,bD)^{1/2}}{V(B(z,d(z,bD)))} \int_{B(z,d(z,bD))} \sigma(w)\,dV(w) \\
&\leq d(z,bD)^{1/2}\mathcal{M}(\sigma)(z) \\
&\lesssim d(z,bD)^{1/2}\sigma(z).
\end{aligned}$$

This establishes the result.

□

Now we will engage in a series of arguments very similar to what is proven in the Szegő section. We first note that $\mathcal{T}^* - \mathcal{T}$ improves L^p spaces. The proof of this fact is basically identical to that of Proposition 3.3.8 and stems from the fact that $\mathcal{T}^* - \mathcal{T}$ has an "integrable kernel", so we omit it.

Proposition 4.3.2. The operator $\mathcal{T}^* - \mathcal{T}$ maps $L^p(D)$ to $L^{p+\varepsilon}(D)$ boundedly for $p \geq 1$ and $\varepsilon \in [0, \frac{1}{2n+1})$.

The exact same reasoning from Lemma 3.3.3 yields the following:

Corollary 4.3.1. If $\sigma \in B_p$, then 1 is not an eigenvalue of $\mathcal{T}^* - \mathcal{T}$ considered as an operator on $L^p_\sigma(D)$.

It remains to prove that $\mathcal{T}^* - \mathcal{T}$ is compact on $L^p_\sigma(D)$ for $\sigma \in B_p$. The argument is again a reprise of the reasoning in the preceding chapter, namely Lemma 3.3.2.

Lemma 4.3.4. The operator $\mathcal{T}^* - \mathcal{T}$ is compact on on $L^p_\sigma(D)$ for $\sigma \in B_p$.

Proof. We first note that an integral operator with kernel K bounded on $D \times D$ is automatically compact on $L^p_\sigma(D)$ for $\sigma \in B_p$; the proof follows as in Theorem 3.3.1.

To pass to the case where K is unbounded, let $\delta_j = \frac{1}{j}$ and

$$K_j(z,w) = \begin{cases} K(z,w) & d(z,w) \geq \delta_j, d(z,bD) \geq \delta_j \text{ or } d(w,bD) \geq \delta_j \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{T}_j$ be the integral operator with kernel K_j. Note that K_j is bounded on $D \times D$ because $|K(z,w)| \lesssim \frac{1}{|g(z,w)|^{n+1/2}}$ and $|g(z,w)| \gtrsim |\rho(w)| + |\rho(z)| + |z-w|^2$ by [37, Proposition 2.1]. Thus $\mathcal{T}_j$ is compact on $L^p_\sigma(D)$. To show $\mathcal{T}$ is compact, it suffices to show $\mathcal{T}_j \to \mathcal{T}$ in operator norm.

To this end, let $f \in L^p_\sigma(D)$ with $\|f\|_{L^p_\sigma(D)} \leq 1$. Note that as $\sigma \in B_p$, we can write

$$\sigma = \frac{\sigma_1}{\sigma_2^{p-1}}$$

where $\sigma_1, \sigma_2 \in B_1$ by the factorization of B_p weights. This factorization of B_p weights holds by the arguments in [62]. It should also be noted that this factorization appears in the literature in the context of the unit disk $\mathbb{D}$; see [6]. By Hölder's Inequality applied to the functions

$$|K(z,w) - K_j(z,w)|^{1/q}\sigma_2(w)^{1/q} \quad \text{and} \quad |K(z,w) - K_j(z,w)|^{1/p}\sigma_2(w)^{-1/q}|f(w)|$$

and then applying Proposition 4.3.3, we obtain the estimate:

$$
\begin{aligned}
&|(\mathcal{T}-\mathcal{T}_j)(f)(z)| \\
&\leq \int\limits_{bD} |K(z,w)-K_j(z,w)||f(w)|\,dV(w) \\
&\leq \chi_{d(z,bD)<\delta_j} \left(\int\limits_{B(z,\delta_j)} |K(z,w)|\sigma_2(w)\,dV(w) \right)^{\frac{1}{q}} \left(\int\limits_{B(z,\delta_j)\cap\{d(w,bD)<\delta_j\}} |K(z,w)|(\sigma_2(w))^{1-p}|f(w)|^p\,dV(w) \right)^{\frac{1}{p}} \\
&\lesssim \delta_j^{1/2q}\sigma_2(z)^{\frac{1}{q}} \left(\int\limits_{B(z,\delta_j)\cap\{d(w,bD)<\delta_j\}} |K(z,w)|(\sigma_2(w))^{1-p}|f(w)|^p\,dV(w) \right)^{\frac{1}{p}}.
\end{aligned}
$$

Thus, we obtain, applying the proceeding estimate, Fubini, and Proposition 4.3.3 again:

$$
\begin{aligned}
&\|(T-T_j)f\|^p_{L^p_\sigma(bD)} \\
&\leq \int\limits_D \delta_j^{\frac{p}{2q}}\sigma_2(z)^{p-1} \left(\int\limits_{B(z,\delta_j)\cap\{d(w,bD)<\delta_j\}} |K(z,w)|(\sigma_2(w))^{1-p}|f(w)|^p\,dV(w) \right) \frac{\sigma_1(z)}{\sigma_2(z)^{p-1}}\,dV(z) \\
&= \delta_j^{\frac{p}{2q}} \int\limits_D \int\limits_{B(z,\delta_j)\cap\{d(w,bD)<\delta_j\}} |K(z,w)|(\sigma_2(w))^{1-p}|f(w)|^p\,dV(w)\,\sigma_1(z)\,dV(z) \\
&= \delta_j^{\frac{p}{2q}} \int\limits_D \chi_{d(w,bD)<\delta_j}(w) \left(\int\limits_{B(w,\delta_j)} |K(z,w)|\sigma_1(z)\,dV(z) \right) |f(w)|^p(\sigma_2(w))^{1-p}\,dV(w) \\
&\lesssim \delta_j^{p/2} \int\limits_D \sigma_1(w)|f(w)|^p(\sigma_2(w))^{1-p}\,dV(w) \\
&= \delta_j^{p/2}\|f\|^p_{L^p_\sigma(D)} \\
&\leq \delta_j^{p/2}.
\end{aligned}
$$

Letting $j \to \infty$, we have $\delta_j \to 0$ and thus it immediately follows that the operators $\mathcal{T}_j$ approach $\mathcal{T}$ in operator norm and hence $\mathcal{T}$ is compact.

□

4.3.3 Proof of Main Theorem

We now can finally prove Theorem 4.3.1, using the Kerzman-Stein operator equation trick.

Proof of Theorem 4.3.1. The proof is virtually identical to that of Theorem 3.3.1. Again, the starting point is the Kerzman-Stein equation, and the invertibility of $(I - (\mathcal{T}^* - \mathcal{T}))$ on $L^p_\sigma(D)$ is granted by Corollary 4.3.1 and Lemma 4.3.4 using the spectral theorem. The boundedness of $\mathcal{T}$ on $L^p_\sigma(D)$ is given by Theorem 4.3.2. □

4.4 Radial Weights

The lack of a reverse Hölder inequality for B_p weights was an obstruction to obtaining the full analog of Theorem 4.1.1 for domains with C^2 boundary (see the proof of Theorem 3.1.1. However, we can prove weighted estimates for $\mathcal{B}$ in this setting for a special class of weights that might be called "radial" or "power" weights. In particular, we prove an analog of a result stated in [70, Theorem 2.10] for the unit ball (at least the sufficiency).

Throughout this section, D is a strongly pseudoconvex domain with C^2 boundary. For $t \in \mathbb{R}$ we define the weights $\sigma_t(z) = |\rho(z)|^t$, where ρ is the (fixed) defining function for the domain D. Note that these weights are pointwise comparable to a power of the distance to the boundary. In this section, we prove the following theorem:

Theorem 4.4.1. Let D be a strongly pseudoconvex domain with C^2 boundary and $1 < p < \infty$. Then if $-1 < t < p-1$, there exists a constant $C > 0$ so $\|\mathcal{B}f\|_{L^p_{\sigma_t}(D)} \leq C\|f\|_{L^p_{\sigma_t}(D)}$.

This result heuristically suggests that Theorem 4.1.1 probably holds in the C^2 setting, since the range of t in Theorem 4.4.1 is precisely the range for which the weight σ_t belongs to B_p, as can readily be checked (we omit the details).

To begin with, we define an analog of the function $g(z,w)$ that is suitable for the C^2 setting. In particular, as in [37], for $\varepsilon > 0$ we define the modified Levi polynomial

$$P_w^\varepsilon(z) := \sum_{j=1}^{n} \frac{\partial \rho}{\partial w_j}(w)(z_j - w_j) + \frac{1}{2} \sum_{j,k=1}^{n} \tau_{j,k}^\varepsilon(w)(z_j - w_j)(z_k - w_k).$$

where the $\tau_{j,k}^\varepsilon$ are C^2 functions satisfying

$$\sup_{w \in \overline{D}} \left| \frac{\partial \rho(w)}{\partial w_j \partial w_k} - \tau_{j,k}^\varepsilon(w) \right| \leq \varepsilon \quad 1 \leq j,k \leq n.$$

Accordingly, we define

$$g_\varepsilon(z,w) := -\rho(w) - \chi(P_w^\varepsilon(z)) + (1-\chi)|z-w|^2,$$

where χ is an appropriately chosen smooth cut-off function. We invite the reader to consult [37] for additional details.

We define the corresponding operator Γ_ε as an analog of the operator Γ in Section 4.3 in the obvious way:

$$\Gamma_\varepsilon(f)(z) = \int_D |g_\varepsilon(z,w)|^{-n-1} f(w)\, dV(w).$$

The aim of the next proposition is to provide sufficient conditions on t for the operators Γ and Γ_ε to be bounded on $L^p_{\sigma_t}(D)$.

Proposition 4.4.1. Let $1 < p < \infty$. If $-1 < t < p-1$, there exists $C > 0$ so $\|\Gamma(f)\|_{L^p_{\sigma_t}(D)} \leq C\|f\|_{L^p_{\sigma_t}(D)}$, and also $\|\Gamma_\varepsilon(f)\|_{L^p_{\sigma_t}(D)} \leq C\|f\|_{L^p_{\sigma_t}(D)}$.

Proof. By [37, Proposition 2.1], it is clearly sufficient to establish the result for Γ, since the functions $|g(z,w)|$ and $|g_\varepsilon(z,w)|$ are comparable. Write

$$\Gamma(f)(z) = \int\limits_D |g(z,w)|^{-n-1} f(w) |\rho(w)|^{-t} \sigma_t(w)\, dV(w).$$

Let q be the dual exponent to p and consider the intervals $\left(0, \frac{1}{q}\right)$ and $\left(\frac{t}{p}, \frac{t+1}{p}\right)$. The conditions on t imply $0 < \frac{t+1}{p}$ and $\frac{t}{p} < \frac{1}{q}$, so $\left(0, \frac{1}{q}\right) \cap \left(\frac{t}{p}, \frac{t+1}{p}\right) \neq \emptyset$ and we may choose $s \in \left(0, \frac{1}{q}\right) \cap \left(\frac{t}{p}, \frac{t+1}{p}\right)$. We will establish the L^p boundedness of Γ using Schur's Test with test function $|\rho|^{-s}$.

Notice

$$\begin{aligned} \int\limits_D |g(z,w)|^{-n-1} (|\rho(w)|^{-s})^q |\rho(w)|^{-t} \sigma_t(w)\, dV(w) &= \int\limits_D |g(z,w)|^{-n-1} |\rho(w)|^{-sq} dV(w) \\ &\leq c_1 (|\rho(z)|^{-s})^q \end{aligned}$$

using [37, Lemma 4.1] and the fact that $0 < sq < 1$.

On the other hand,

$$\begin{aligned} \int\limits_D |g(z,w)|^{-n-1} (|\rho(z)|^{-s})^p |\rho(w)|^{-t} |\rho(z)|^t\, dV(z) &= |\rho(w)|^{-t} \int\limits_D |g(z,w)|^{-n-1} |\rho(z)|^{-sp+t} dV(z) \\ &\leq c_2 |\rho(w)|^{-t} |\rho(w)|^{-sp+t} \\ &= c_2 |\rho(w)|^{-sp} \end{aligned}$$

again by [37, Lemma 4.1] and the fact that $0 < sp - t < 1$.

The boundedness of the operator Γ on $L^p_{\sigma_t}(D)$ then follows immediately by Schur's Test. In particular, $\|\Gamma\|_{L^p_{\sigma_t}(D) \to L^p_{\sigma_t}(D)} \leq c_1^{1/q} c_2^{1/p}$.

$\square$

Using the function g_ε, we can analogously construct a non-orthogonal projection operator $\mathcal{T}_\varepsilon = \mathcal{T}_\varepsilon^1 + \mathcal{T}_\varepsilon^2$, where $\mathcal{T}_\varepsilon^1$ is the corresponding Cauchy-Fantappié integral, and $\mathcal{T}_\varepsilon^2$ is the suitable correction operator to make the kernel of $\mathcal{T}_\varepsilon$ holomorphic in the variable z. It is a fact that $|\mathcal{T}_\varepsilon^1 f(z)| \lesssim \Gamma_\varepsilon(|f|)(z)$ and the kernel $K_2(z,w)$ of $\mathcal{T}_\varepsilon^2$ is bounded on $\overline{D} \times \overline{D}$, say by a positive constant M. We omit the details and refer the interested reader to [37].

We now prove the $L^p_{\sigma_t}(D)$ boundedness of the operator $\mathcal{T}_\varepsilon^2$.

Proposition 4.4.2. Let $1 < p < \infty$. Then if $-1 < t < p-1$, there exists a constant $C > 0$ so $\|\mathcal{T}_\varepsilon^2 f\|_{L^p_{\sigma_t}(D)} \leq C\|f\|_{L^p_{\sigma_t}(D)}$.

Proof. We have, for $f \in L^p_{\sigma_t}(D)$:

$$\begin{aligned}
\left(\int_D \left|\int_D f(w)K_2(z,w)dV(w)\right|^p \sigma_t(z)\, dV(z)\right)^{1/p} &\leq (\sigma_t(D))^{1/p} \sup_{z\in D}\left(\int_D |f(w)||K_2(z,w)|\, dV(w)\right) \\
&\leq (\sigma_t(D))^{1/p} M \int_D |f(w)|\, dV(w) \\
&= (\sigma_t(D))^{1/p} M \int_D |f(w)||\rho(w)|^{t/p}|\rho(w)|^{-t/p}\, dV(w) \\
&\leq (\sigma_t(D))^{1/p} M \|f\|_{L^p_{\sigma_t}(D)} \left(\int_D |\rho(w)|^{-tq/p}\, dV(w)\right)^{1/q} \\
&< \infty
\end{aligned}$$

where we have used Holder's inequality, the fact that $-tq/p > -1$, and the fact that $\int_D |\rho(w)|^\alpha dV(w) < \infty$ for $\alpha > -1$, a fact which is contained in the proof of [37, Lemma 4.1]. □

Corollary 4.4.1. Let $1 < p < \infty$. If $-1 < t < p-1$, then there exists $C > 0$ so that $\|\mathcal{T}_\varepsilon f\|_{L^p_{\sigma_t}(D)} \leq C\|f\|_{L^p_{\sigma_t}(D)}$.

Proof. Recall $\mathcal{T}_\varepsilon = \mathcal{T}_\varepsilon^1 + \mathcal{T}_\varepsilon^2$. The operator $\mathcal{T}_\varepsilon^1$ is majorized by Γ_ε, which is bounded on $L^p_{\sigma_t}(D)$ according to Proposition 4.4.1. The operator $\mathcal{T}_\varepsilon^2$ is bounded in virtue of Proposition 4.4.2, which then grants the boundedness of $\mathcal{T}_\varepsilon$. □

The next lemma allows us to prove the boundedness of $\mathcal{B}$ on $L^p_{\sigma_t}(D)$.

Lemma 4.4.1. For each $\varepsilon > 0$, we can decompose $\mathcal{T}_\varepsilon - \mathcal{T}_\varepsilon^* = \mathcal{A}_\varepsilon + \mathcal{D}_\varepsilon$ where the decomposition satisfies the following properties:

1. For each $1 < p < \infty$, there exists $C > 0$ (independent of ε) so $\|\mathcal{A}_\varepsilon\|_{L^p_{\sigma_t}(D) \to L^p_{\sigma_t}(D)} \leq \varepsilon C$.
2. The operator $\mathcal{D}_\varepsilon$ has a kernel which is continuous on $\overline{D} \times \overline{D}$, and hence maps $L^p_{\sigma_t}(D)$ to $C(\overline{D})$ for $1 < p < \infty$.

Proof. These statements follow more or less directly from [37, Lemma 5.1] and the work done above. The decomposition proceeds identically. For the first part, note that in the proof of Lemma 5.1, it is established that the operator $\mathcal{A}_\varepsilon$ has a kernel that is controlled in absolute value by a constant multiple of $\frac{\varepsilon}{|g(z,w)|^{n+1}}$, and thus the estimate follows directly from the proof of Proposition 4.4.1. The second part follows by virtue of the fact that $L^p_{\sigma_t}(D) \subset L^1(D)$ with bounded inclusion, a fact which is contained in the proof of Proposition 4.4.2. □

We now proceed to prove Theorem 4.4.1

Proof of Theorem 4.4.1. Fix $\varepsilon > 0$ sufficiently small so that $\|\mathcal{A}_\varepsilon\|_{L^p_{\sigma_t}(D) \to L^p_{\sigma_t}(D)} < \frac{1}{2}$. We follow a similar approach as [37]. The Kerzman-Stein equation in this case is

$$\mathcal{B}(I - (\mathcal{T}_\varepsilon^* - \mathcal{T}_\varepsilon)) = \mathcal{T}_\varepsilon.$$

Using the decomposition provided in Lemma 4.4.1 and rearranging, we obtain

$$\mathcal{B}(I - \mathcal{A}_\varepsilon) = \mathcal{T}_\varepsilon + \mathcal{B}\mathcal{D}_\varepsilon$$

as an identity on $L^2(D)$.

Now, we know that $(I - \mathcal{A}_\varepsilon)$ is invertible on $L^p_{\sigma_t}(D)$ using a Neumann series by virtue of Lemma 4.4.1 and our choice of ε. Thus, to prove that $\mathcal{B}$ extends to a bounded operator on $L^p_{\sigma_t}(D)$, we only need to prove that $\mathcal{B}\mathcal{D}_\varepsilon$ extends to a bounded operator on $L^p_{\sigma_t}(D)$, since we have already proven that $\mathcal{T}_\varepsilon$ does. Choose a large exponent $\widetilde{p}$ so that $t\frac{\widetilde{p}}{\widetilde{p}-p} > -1$. This is clearly possible since the target interval is open, $t \in (-1, p-1)$ by hypothesis, and $\frac{\widetilde{p}}{\widetilde{p}-p}$ tends to 1 as $\widetilde{p} \to \infty$. Now, note that $\mathcal{D}_\varepsilon$ maps $L^p_{\sigma_t}(D)$ to $C(\overline{D})$ by virtue of Lemma 4.4.1, and hence maps $L^p_{\sigma_t}(D)$ to $L^{\widetilde{p}}(D)$, since D is a finite measure space. Then, appeal to the main theorem in [37] which says $\mathcal{B}$ maps $L^p(D)$ to $L^p(D)$ for all $1 < p < \infty$, so in particular it follows $\mathcal{B}\mathcal{D}\varepsilon$ maps $L^p_{\sigma_t}(D)$ to $L^{\widetilde{p}}(D)$ boundedly. The result follows by observing that $L^{\widetilde{p}}(D) \subset L^p_{\sigma_t}(D)$ for this choice of $\widetilde{p}$: indeed, for $f \in L^{\widetilde{p}}(D)$

$$\begin{aligned}
\int_D |f|^p \sigma_t \, dV &= \int_D |f|^p |\rho|^t \, dV \\
&\leq \left(\int_D |f|^{\widetilde{p}} \, dV \right)^{p/\widetilde{p}} \left(\int_D |\rho|^{t\frac{\widetilde{p}}{\widetilde{p}-p}} \, dV \right)^{\frac{p(\widetilde{p}-1)}{\widetilde{p}}} \\
&\lesssim \|f\|^p_{L^{\widetilde{p}}(D)}
\end{aligned}$$

since $|\rho|^\alpha$ is integrable if $\alpha > -1$.

□

4.5 Application to Toeplitz Operators

In this section, we assume D is strongly pseudoconvex with C^2 boundary. We give an additional application of the Kerzman-Stein technique to the boundedness of Toeplitz operators. We now define *Toeplitz operators* on the Bergman space $A^2(D)$:

Definition 4.5.1. Given a symbol function $u \in L^\infty(D)$, define the *Toeplitz operator* $T_u : A^2(D) \to A^2(D)$ by $T_u(f) = \mathcal{B}(uf)$.

It is immediate that any Toeplitz operator is bounded on $A^2(D)$. There has been an extensive study of Toeplitz operators on Bergman spaces, particularly the question of when such operators are compact; we refer the reader to [1, 51, 65] for some important results. Although Toeplitz operators are typically studied from the operator-theoretic perspective of their restriction to the Bergman space (or other holomorphic function spaces), it also makes sense to study their behavior on $L^p(D)$, especially in a context when the operator is "smoothing" and the symbol improves the behavior of the original projection (similar to the study of fractional integral operators in harmonic analysis). It is clear, from the main result in [37], that Toeplitz operators are bounded on L^p for $1 < p < \infty$.

We consider the special case when $u = \delta^\eta$, where η is a positive power and $\delta(z) = \text{dist}(z, bD)$. These symbols were considered by McNeal and Cuckovic in [66] and it was shown that the associated Toeplitz operators have smoothing properties and improve L^p spaces. The point is that the decay of the symbol near the boundary "cancels out" some of the singularities of the Bergman kernel on the boundary diagonal in a precise and quantitative sense. However, their proof depends crucially on estimates for the Bergman kernel available only in the smooth case.

We show below that their result can be partially recovered in the minimal smoothness case using the Kerzman-Stein operator theory trick. Let $\mathcal{T}_\varepsilon$ denote the Kerzman Stein operator introduced in Section 4.4. In particular, the relation $\mathcal{T}_\varepsilon \mathcal{B} = \mathcal{B}$ yields, taking adjoints and using the fact that the Bergman projection is self-adjoint, the relation $\mathcal{B}\mathcal{T}_\varepsilon^* = \mathcal{B}$ on $L^2(D)$. Note that the Toeplitz operator T_u can be written $T_u = \mathcal{B}M_u$, where M_u denotes the operator of multiplication by u. Because the operator $\mathcal{T}_\varepsilon^*$ can be inserted after $\mathcal{B}$, the Toeplitz operator T_u can be rewritten (on $L^2(D)$) as $\mathcal{B}\mathcal{T}_\varepsilon^* M_u$. Then, the operator $\mathcal{T}_\varepsilon^* M_u$ is much more tractable than the original Toeplitz operator because more precise information is known about the kernel of $\mathcal{T}_\varepsilon$ (and hence $\mathcal{T}_\varepsilon^*$).

For the operator $\mathcal{T}_\varepsilon$, one can choose any real parameter $\varepsilon > 0$ sufficiently small so that $|g(z,w)| \approx |g_\varepsilon(z,w)|$. In fact, really the only properties of $\mathcal{T}_\varepsilon$ that are important in our proof are a bound on the size of its kernel and the fact that $\mathcal{T}_\varepsilon$ is bounded on $L^2(D)$ and reproduces holomorphic functions, so that we obtain the identity $\mathcal{T}_\varepsilon\mathcal{B} = \mathcal{B}$. Therefore, in what follows, we assume the parameter ε has been fixed once and for all. However, since we are appealing to the result that $\mathcal{B}$ is bounded on $L^p(D)$, we are implicitly using deeper properties of the operator T_ε, including a quasi-cancellation of singularities in $\mathcal{T}_\varepsilon^* - \mathcal{T}_\varepsilon$ (see [37] for more details).

The following lemma is the main tool in the proof of the theorem. We note that this result was given in [66, Proposition 2.1].

Lemma 4.5.1. Let $\mathcal{K}$ denote the integral operator on acting on measurable functions on D with kernel $K(z,w)$; that is

$$\mathcal{K}f(z) = \int_D K(z,w)f(w)\,dV(w).$$

Fix $1 < s < \infty$ and $\gamma > 0$ small. Let $p \in (1,\infty)$ and denote by q the Hölder conjugate exponent to p. Suppose there exists a $t \in (0,1)$ and a finite, positive constant C so that

$$\int_D |K(z,w)|^{tq}|\rho(w)|^{-\gamma}\,dV(w) \le C|\rho(z)|^{-\gamma}$$

$$\int_D |K(z,w)|^{(1-t)s}|\rho(w)|^{-\gamma}\,dV(w) \le C|\rho(z)|^{-\gamma}.$$

Then the integral operator $\mathcal{K}$ is bounded as a map from $L^p(D)$ to $L^s(D)$ with operator norm at most C.

The following theorem is an extension of [66, Theorem 1.2] to the C^2 setting.

Theorem 4.5.1. Let D be a strongly pseudoconvex domain with C^2 boundary. Let T_η denote the Toeplitz operator with symbol δ^η. For $0 \le \eta < n+1$, let $E = \frac{n+1}{n+1-\eta}$. Then the following mapping properties hold:

1. If $1 < p < \infty$ and $E < \frac{p}{p-1}$, then $T_\eta : L^p(D) \to L^{p+G}(D)$, where $G = \frac{p^2}{\frac{n+1}{\eta} - p}$, boundedly.

2. If $1 < p < \infty$ and $E \ge \frac{p}{p-1}$, then $T_\eta : L^p(\Omega) \to L^s(\Omega)$ boundedly for any $s < \infty$.

Proof. To prove this theorem, we will use the following facts:

1. The Toeplitz operator T_η can be written, as a map on L^p

$$T_\eta = \mathcal{B}\mathcal{T}_\varepsilon^* M_{\delta^\eta}.$$

2. The operator $\mathcal{T}_\varepsilon^* M_{\delta^\eta}$ has the mapping properties above.

3. The Bergman projection $\mathcal{B}$ extends to a bounded map on $L^p(D)$ for $1 < p < \infty$.

The first item was essentially proven in the discussion before the proof, because we already know the operator $\mathcal{T}_\varepsilon^*$ is bounded on $L^p(D)$ by the same arguments leading to Corollary 4.4.1; so the operator relation will extend from L^2 to L^p by density. The third item was proven in [37]. So it remains to show that $\mathcal{T}_\varepsilon^* M_{\delta^\eta}$ has the desired mapping properties.

The major step in proving this result is to prove the analog of Proposition 3.4 in this case. First, note that we may replace the symbol δ^η by $|\rho|^\eta$, where ρ is the (fixed) C^2 defining function for D since the two functions are uniformly comparable. Write $K_\eta(z,w) = K(z,w)|\rho(w)|^\eta$, where $K(z,w)$ denotes the kernel for $\mathcal{T}_\varepsilon^*$. Then we claim that for all $0 \le \eta < n+1$ and $0 \le r \le \frac{n+1}{n+1-\eta}$, the following estimate holds for $\gamma \in (0,1)$:

$$\int_D |K_\eta(z,w)|^r |\rho(w)|^{-\gamma}\, dV(w) \le C_\gamma |\rho(z)|^{-\gamma}.$$

This estimate can be shown in straightforward manner by appealing to known facts. First, we recall that it is possible to decompose $\mathcal{T}_\varepsilon = \mathcal{T}_\varepsilon^1 + \mathcal{T}_\varepsilon^2$, where $\mathcal{T}_\varepsilon^2$ has a kernel that is bounded on $\overline{D} \times \overline{D}$, while $\mathcal{T}_\varepsilon^1$ has a kernel controlled by a multiple of $\frac{1}{|g(z,w)|^{n+1}}$. Thus, we can write $\mathcal{T}_\varepsilon^* = (\mathcal{T}_\varepsilon^1)^* + (\mathcal{T}_\varepsilon^2)^*$, where the kernel of $(\mathcal{T}_\varepsilon^2)^*$ is still bounded and the kernel of $(\mathcal{T}_\varepsilon^1)^*$ is controlled by a multiple of $\frac{1}{|g(w,z)|^{n+1}}$. But the fact that $|g(z,w)| \approx |g(w,z)|$ by [37, Proposition 2.1] implies that we can replace $g(w,z)$ by $g(z,w)$ in the kernel bound. Altogether, we deduce $|K(z,w)| \lesssim \frac{1}{|g(z,w)|^{n+1}}$.

We then write

$$\begin{aligned}\int_D |K_\eta(z,w)|^r |\rho(w)|^{-\gamma}\, dV(w) &\lesssim \int_D \frac{|\rho(w)|^{\eta r}|\rho(w)|^{-\gamma}}{|g(z,w)|^{r(n+1)}}\, dV(w) \\ &= \int_D \frac{|\rho(w)|^{-\gamma}}{|g(z,w)|^{n+1}} \frac{|\rho(w)|^{\eta r}}{|g(z,w)|^{(n+1)(r-1)}}\, dV(w)\,.\end{aligned}$$

If $r \leq 1$, then the last display is controlled by an independent constant times

$$\int_D \frac{|\rho(w)|^{-\gamma}}{|g(z,w)|^{n+1}}\, dV(w)$$

since $\rho(w)$ and $g(z,w)$ are both bounded functions. On the other hand, if $r > 1$, then use the fact that $|\rho(w)| \lesssim |g(z,w)|$ uniformly in $z, w \in D$ (see [37, Proposition 2.1]). In this case, we control the integral by a constant times

$$\int_D \frac{|\rho(w)|^{-\gamma}|\rho(w)|^{\eta r-(n+1)(r-1)}}{|g(z,w)|^{n+1}}\, dV(w)\,.$$

Finally, compute to see

$$
\begin{aligned}
\eta r - (n+1)(r-1) &= r(\eta - (n+1)) + (n+1) \\
&\geq \frac{n+1}{n+1-\eta}(\eta - (n+1)) + (n+1) \\
&= 0
\end{aligned}
$$

to conclude that in either case, the original integral is dominated by a constant times

$$
\int_D \frac{|\rho(w)|^{-\gamma}}{|g(z,w)|^{n+1}}\, dV(w)\,.
$$

By [37, Lemma 4.1], this integral is dominated by $C_\gamma \rho(z)^{-\gamma}$, which proves the assertion.

The theorem is now a straightforward consequence of the Lemma 4.5.1 above. In particular, in the case $E \geq \frac{p}{p-1}$, then the above claim holds for $r = \frac{p}{p-1} = q$. Then, given any fixed $s \in (1, \infty)$ we can choose t arbitrarily close to 1 so that $(1-t)s \leq E$, so that both hypotheses of the lemma are satisfied. In the other case, note that in order for the hypotheses of the lemma to hold we need $(1-t)s \leq E$ and $tq \leq E$. Solving for t in the second inequality and substituting into the first, we obtain the correct value for s. □

Chapter 5

Endpoint Estimates

5.1 Summary of Main Results

In this chapter, our main results are weighted weak-type estimates for the Bergman and Szegő projections on strongly pseudoconvex domains with near-minimal smoothness as an additional appliation of the Kerzman-Stein equation. The majority of the material in this chapter appears in [64]. The first main result of this chapter is the weighted weak-type $(1,1)$ estimate for the Bergman projection on C^4 domains. Recall the definition of B_1 weights given in Definition 1.5.2. The quasi-metric d is constructed using the special coordinate system discussed in Chapter 4.

Theorem 5.1.1. Let D be strongly pseudoconvex with C^4 boundary. If $\sigma \in B_1$, then the Bergman projection $\mathcal{B}$ extends boundedly from $L^1_\sigma(D)$ to $L^{1,\infty}_\sigma(D)$. That is, there exists $C > 0$ such that

$$\|\mathcal{B}f\|_{L^{1,\infty}_\sigma(D)} := \sup_{\lambda>0} \lambda\sigma(\{z \in D : |\mathcal{B}f(z)| > \lambda\}) \leq C\|f\|_{L^1_\sigma(D)}$$

for all $f \in L^1_\sigma(D)$.

We remark that Theorem 5.1.1 is new even in the unweighted setting ($\sigma = 1$). In this case, Theorem 5.1.1 can be viewed as an extension of McNeal's results of [47] to domains with near minimal smoothness and also of the work of Lanzani and Stein in [37] to address the behavior at the $p = 1$ endpoint. In fact, Theorem 5.1.1 and an interpolation argument imply the $L^p(D)$, $1 < p < \infty$, boundedness result of [37] in the case of D having C^4 boundary. With B_1 weights, Theorem 5.1.1 generalizes Bekollé's endpoint weak-type result of [3] to domains with near minimal smoothness and extends the work in Chapter 4 to address the $p = 1$ endpoint.

The weak-type estimate of Theorem 5.1.1 implies some other useful endpoint bounds, generalizing results in [15]. Recall that for $0 < p < 1$, the space $L^p_\sigma(D)$ is a quasi-Banach space. In particular, one has the following weighted Kolmogorov inequality:

Corollary 5.1.1. Let D be strongly pseudoconvex with C^4 boundary and $0 < p < 1$. If $\sigma \in B_1$, then the Bergman projection $\mathcal{B}$ extends boundedly from $L^1_\sigma(D)$ to $L^p_\sigma(D)$. That is, there exists $C > 0$ such that

$$\|\mathcal{B}f\|_{L^p_\sigma(D)} \leq C\|f\|_{L^1_\sigma(D)}$$

for all $f \in L^1_\sigma(D)$.

Additionally, one also gets the following Zygmund inequality as a corollary:

Corollary 5.1.2. Let D be strongly pseudoconvex with C^4 boundary. If $\sigma \in B_1$, then the Bergman projection $\mathcal{B}$ extends boundedly from $(L\log^+ L)_\sigma(D)$ to $L^1_\sigma(D)$. That is, there exists $C > 0$ such that

$$\|\mathcal{B}f\|_{L^1_\sigma(D)} \leq C\|f\|_{(L\log^+ L)_\sigma(D)}$$

for all $f \in (L\log^+ L)_\sigma(D)$.

Refer to Section 5.4 for a precise definition of the Zygmund spaces $L\log^+ L$ and their norms.

The second main result of this chapter is the weighted weak-type $(1, 1)$ estimate for the Szegő projection on domains with near minimal smoothness.

Theorem 5.1.2. Let D be strongly pseudoconvex with C^3 boundary. If $\sigma \in A_1$, then the Szegő projection $\mathcal{S}$ extends boundedly from $L^1_\sigma(bD)$ to $L^{1,\infty}_\sigma(bD)$. That is, there exists $C > 0$ such that $\|\mathcal{S}f\|_{L^{1,\infty}_\sigma(bD)} \leq C\|f\|_{L^1_\sigma(bD)}$ for all $f \in L^1_\sigma(bD)$.

We remark that Theorem 5.1.2 is new even in the unweighted setting. Theorem 5.1.2 can be viewed as a weighted extension of the work of Lanzani and Stein in [37] and of the results in Chapter 3 to address the behavior at the $p = 1$ endpoint.

Similar to the case of the Bergman projection, we obtain a weighted Kolmogorov inequality and a weighted Zygmund inequality for the Szegő projection.

Corollary 5.1.3. Let D be strongly pseudoconvex with C^3 boundary and $0 < p < 1$. If $\sigma \in A_1$, then the Szegő projection $\mathcal{S}$ extends boundedly from $L^1_\sigma(bD)$ to $L^p_\sigma(bD)$. That is, there exists $C > 0$ such that

$$\|\mathcal{S}f\|_{L^p_\sigma(bD)} \leq C\|f\|_{L^1_\sigma(bD)}$$

for all $f \in L^1_\sigma(bD)$.

Corollary 5.1.4. Let D be strongly pseudoconvex with C^3 boundary. If $\sigma \in A_1$, then the Szegő projection $\mathcal{S}$ extends boundedly from $(L \log L)_\sigma(bD)$ to $L^1_\sigma(bD)$. That is, there exists $C > 0$ such that

$$\|\mathcal{S}f\|_{L^1_\sigma(bD)} \leq C\|f\|_{(L \log L)_\sigma(bD)}$$

for all $f \in (L \log L)_\sigma(bD)$.

We can also obtain an (unweighted) estimate for the Bergman projection on C^4 domains at the endpoint $p = \infty$. In particular, we prove that the Bergman projection boundedly maps $L^\infty(D)$ into a space of holomorphic functions known as the Bloch space, defined precisely in Section 5.5.

Theorem 5.1.3. Let D be strongly pseudoconvex with C^4 boundary. The Bergman projection $\mathcal{B}$ maps $L^\infty(D)$ to the Bloch space $\mathscr{B}(D)$. That is, there exists a constant $C > 0$ so that $\|\mathcal{B}f\|_{\mathscr{B}(D)} \leq C\|f\|_{L^\infty(D)}$.

This chapter is organized as follows. In Section 5.2, we deal with the weighted weak-type estimates for the Bergman projection on C^4 domains with B_1 weights. In Section 5.3, we deal with the weighted weak-type estimates for the Szegő projection on C^3 domains with A_1 weights. In Section 5.4, we prove the Kolmogorov and Zygmund inequalities. Finally, in Section 5.5, we prove that the Bergman projection on C^4 domains maps $L^\infty(D)$ to the Bloch space boundedly.

5.2 The Bergman Projection

5.2.1 Preliminaries

Let $D \subseteq \mathbb{C}^n$ be a strongly pseudoconvex bounded domain with C^4 boundary. Via the ideas of Kerzman, Stein, and Ligocka, we construct a non-orthogonal projection operator $\mathcal{T}$ using Cauchy-Fantappié theory. This operator is defined exactly the same way as in Section 4.2 in Chapter 4, and the details of its construction are identical. Therefore, we use the same notation as Chapter 4 and refer the reader to the relevant section and [37] for additional details without revisiting the construction.

Recall we also have the Kerzman-Stein equation:

$$\mathcal{B}(I - (\mathcal{T}^* - \mathcal{T})) = \mathcal{T}. \tag{5.2.1}$$

The proof of Theorem 5.1.1 follows easily from the following two facts.

Proposition 5.2.1. If σ is a B_1 weight, then the operator $I - (\mathcal{T}^* - \mathcal{T})$ is invertible on $L^1_\sigma(D)$.

Proposition 5.2.2. If σ is a B_1 weight, then $\mathcal{T}$ maps $L^1_\sigma(D)$ to $L^{1,\infty}_\sigma(D)$ boundedly.

Proof of Theorem 1.5.5. Using Proposition 5.2.1, we may rewrite (5.2.1) as

$$\mathcal{B} = \mathcal{T}(I - (\mathcal{T}^* - \mathcal{T}))^{-1}.$$

The bound of $\mathcal{B}$ from $L^1_\sigma(D)$ to $L^{1,\infty}_\sigma(D)$ follows from Proposition 5.2.1 and Proposition 5.2.2. □

The remainder of this section is devoted to proving Proposition 5.2.1 and Proposition 5.2.2. Proposition 5.2.1 will follow from the spectral theorem for compact operators on a Banach space. In particular, we will show that $\mathcal{T}^* - \mathcal{T}$ is compact on $L^1_\sigma(D)$ and also that 1 is not an eigenvalue of $\mathcal{T}^* - \mathcal{T}$ on $L^1_\sigma(D)$. Proposition 5.2.2 relies on methods from Calderón-Zygmund theory reminiscent of the ideas in [3].

Recall the arguments in Chapter 2 and Chapter 4 as well as [47, 50] make use of an appropriately constructed quasi-metric d that reflects the geometry of the boundary. Technically, the quasi-metric D is only defined for points z, w sufficiently close to the boundary, but we will abuse notation and define objects as if d were defined globally. This reduction is possible because the kernels of all the relevant operators are uniformly continuous on compact subsets of $\overline{D} \times \overline{D}$ off the boundary diagonal and all the necessary properties will hold for trivial reasons. See Lemma 2.2.1 for an example of how such a reduction works in the smooth setting (similar reasoning is followed for the operator $\mathcal{T}$). For relevant properties of this quasi-metric, including comparability to a "distance-like" quantity in terms of coordinates, homogeneous structure, and doubling, we refer the reader to Chapter 2 Section 2.2 and Chapter 4 Section 4.2 for all the relevant details.

As in Chapter 2 we use the constant $c > 0$ to denote the implicit constant in the triangle inequality for d:

$$d(z, w) \leq c(d(z, \zeta) + d(\zeta, w)).$$

If B is a quasi-ball, then its center and radius are represented by $c(B)$ and $r(B)$ respectively, meaning $B = \{w \in D : d(c(B), w) < r(B)\}$. We also write kB to denote the k-fold dilate of B, that is $kB := \{w \in D : d(c(B), w) < kr(B)\}$.

Notice that for a B_1 weight σ, $\sigma\, dV$ also satisfies a particular doubling property for quasi-balls close to the boundary:

$$\sigma(B(z,2r)) \lesssim \left(\inf_{w\in B(z,2r)} \sigma(w)\right) V(B(z,2r)) \lesssim \sigma(B(z,r))$$

for any $z \in D$ and $r > 0$ such that $r > kd(z, bD)$ for some absolute $k > 0$ (the first inequality above depends on $[\sigma]_{B_1}$; this is an analog of Proposition 2.3.3 in Chapter 2 for B_1 weights). For sets $E, F \subseteq \overline{D}$, we write $d(E,F) := \inf_{\substack{z\in E\\ w\in F}} d(z,w)$.

We work with the maximal operator $\mathcal{M}$ adapted to our setting. In particular, see Chapter 2 Definition 2.2.2 for the definition of $\mathcal{M}$.

5.2.2 Inversion of the "mild" operator

To deduce the compactness of $\mathcal{T}^* - \mathcal{T}$, we use a more general result which follows from [17, Corollary 4.1]. In the following lemma, $\mathcal{K}$ is an integral operator given by

$$\mathcal{K}f(x) = \int_X k(x,y) f(y)\, d\mu(y)$$

and $k_y(x) = k(x,y)$.

Lemma 5.2.1. Let (X, μ) be a positive measure space. Suppose that $k : X \times X \to \mathbb{R}$ is a measurable function such that $\| \int_X k(x,\cdot)\, d\mu(x)\|_{L^\infty(X,\mu)} < \infty$. If the set $\{k_y\}_{y\in X}$ is relatively compact in $L^1(X,\mu)$, then $\mathcal{K}$ and $\mathcal{K}^*$ are compact operators on $L^1(X,\mu)$ and $L^\infty(X,\mu)$ respectively.

To justify the relative compactness of $\{k_y\}$ in our application of Lemma 5.2.1, we use the following characterization for relatively compact sets, which can be viewed as a generalization of the classical Riesz-Kolmogorov theorem.

Lemma 5.2.2. Let μ be a finite Borel measure on X such that $\inf_{x\in X} \mu(B(x,r)) > 0$ for any $r > 0$ and let $1 \leq p < \infty$. If $K \subseteq L^p(X, \mu)$ is a bounded set satisfying

$$\lim_{r \to 0^+} \sup_{f \in K} \int_X |f(x) - \langle f \rangle_{B(x,r),\mu}|^p \, d\mu(x) = 0,$$

then K is relatively compact in $L^p(X, \mu)$.

Lemma 5.2.2 was originally stated for the case of metric spaces in [27], but we will need a version from [26, Lemma 1] in the case where we only have a quasi-metric.

We next apply Lemma 5.2.1 and Lemma 5.2.2 to prove the following result.

Lemma 5.2.3. If σ is a B_1 weight, then the operator $\mathcal{T}^* - \mathcal{T}$ is compact on $L^1_\sigma(D)$.

Proof. First, we note that $\sigma \, dV$ is a finite Borel measure on D. Using the B_1 condition and the fact that $B(z, R) = D$ for $z \in D$ and sufficiently large R, one has

$$\sigma(D) \lesssim \left(\inf_{w \in D} \sigma(w) \right) V(D).$$

The infimum condition on the measure σdV can be verified using a compactness argument and the fact that $B(z, r)$ contains a Euclidean ball intersected with D with radius comparable to $r^{1/2}$, which was proved in Proposition 4.2.4. Let $k(z, w)$ denote the kernel of $\mathcal{T}^* - \mathcal{T}$ with respect to Lebesgue measure. Recall the following key properties of $k(z, w)$ were proved in Lemma 4.3.2:

$$|k(z, w)| \lesssim |g(z, w)|^{-\left(n+\frac{1}{2}\right)} \lesssim d(z, w)^{-\left(n+\frac{1}{2}\right)}$$

as well as

$$|k(z, w)| \lesssim \min \left\{ d(z, bD)^{-\left(n+\frac{1}{2}\right)}, d(w, bD)^{-\left(n+\frac{1}{2}\right)} \right\}.$$

Here, the assumption that the boundary of D is of class C^4 is in fact crucial. Let $\tilde{k}(z, w)$ denote the kernel of $\mathcal{T}^* - \mathcal{T}$ with respect to the weighted measure $\sigma \, dV$ and notice $\tilde{k}(z, w) = k(z, w)\sigma(w)^{-1}$.

We claim that there exists $M > 0$ such that $\sup_{w\in D} \int_D |\tilde{k}(z,w)|\sigma(z)\, dV(z) < M$. To see this, fix $w \in D$ and integrate over dyadic annuli, choosing R so that $B(w,R) = D$ and letting N be the largest positive integer such that $B(w, 2^{-N}R)$ meets the boundary of D. We use the above control of $|k(z,w)|$ and the fact that $V(B(z,r)) \approx r^{n+1}$ to obtain

$$\begin{aligned}
\int_D |\tilde{k}(z,w)|\sigma(z)\, dV(z) &= \sigma(w)^{-1} \int_D |k(z,w)|\sigma(z)\, dV(z) \\
&= \sigma(w)^{-1} \sum_{j=0}^{N} \int_{B(w,2^{-j}R)\setminus B(w,2^{-(j+1)}R)} d(z,w)^{-\left(n+\frac{1}{2}\right)}\sigma(z)\, dV(z) \\
&+ \sigma(w)^{-1} \int_{B(w,2^{-(N+1)}R)} d(w,bD)^{-(n+\frac{1}{2})}\sigma(z)\, dV(z) \\
&\lesssim \sigma(w)^{-1} \sum_{j=0}^{N} \frac{2^{-j/2}R^{1/2}}{V(B(w,2^{-j}R))} \int_{B(w,2^{-j}R)} \sigma(z)\, dV(z) \\
&+ \sigma(w)^{-1} \frac{d(w,bD)^{1/2}}{V(B(w,d(w,bD)))} \int_{B(w,d(w,bD))} \sigma(z)\, dV(z) \\
&\leq \sigma(w)^{-1} \sum_{j=0}^{N} 2^{-j/2}R^{1/2}\mathcal{M}\sigma(w) + \sigma(w)^{-1} d(w,bD)^{1/2}\mathcal{M}\sigma(w) \\
&\lesssim \sigma(w)^{-1}(R^{1/2} + d(w,bD)^{1/2})\mathcal{M}\sigma(w) \\
&\lesssim R^{1/2}.
\end{aligned}$$

Note that we used the B_1 condition in the last line above. All the implicit constants are independent of w, and R is also independent of w since we can just take R to be the diameter of D in the quasi-metric. This establishes the claim. Notice that this argument also shows that replacing the region of integration by a quasi-ball $B(w,\delta)$ yields

$$\int_{B(w,\delta)} |\tilde{k}(z,w)|\sigma(z)\, dV(z) \lesssim \delta^{1/2} + d(w,bD)^{1/2}, \tag{5.2.2}$$

where the implicit constant is independent of w (see also Lemma 4.3.3).

Now we must show the crucial condition

$$\lim_{r\to 0^+} \sup_{w\in D} \sigma(w)^{-1} \int_D |k_w(z) - \langle k_w\rangle_{B(z,r),\sigma dV}|\sigma(z)\, dV(z) = 0,$$

where $k_w(z) = k(z,w)$. Fix $\varepsilon > 0$, $w \in D$, and let $\delta > 0$ and $0 < r < \delta$ be constants to be fixed later. We emphasize all constants obtained will ultimately be independent of w.

Let $G := \{z \in D : d(z,w) \geq \delta \text{ or } d(z,bD) \geq \delta\}$. We will first estimate

$$\sigma^{-1}(w) \int_G |k_w(z) - \langle k_w\rangle_{B(z,r),\sigma dV}|\sigma(z)\, dV(z).$$

Recall that the kernel function $k(z,w)$ is uniformly continuous on compact subsets off the boundary diagonal, so in particular the function $k_w(z)$ is uniformly continuous on G with a modulus of continuity independent of w. We can choose r sufficiently small relative to δ and independent of w so that we have $|k_w(z) - \langle k_w\rangle_{B(z,r),\sigma dV}| < \varepsilon$ for $z \in G$ and hence,

$$\begin{aligned}
\sigma(w)^{-1} \int_G |k_w(z) - \langle k_w\rangle_{B(z,r),\sigma dV}|\sigma(z)\, dV(z) &\leq \varepsilon\sigma(w)^{-1} \int_D \sigma(z)\, dV(z) \\
&\lesssim \varepsilon\sigma(w)^{-1}\mathcal{M}\sigma(w) \\
&\lesssim \varepsilon
\end{aligned}$$

as required. We used the B_1 condition of σ in the last inequality above.

Now we need to estimate the integral on $D \setminus G$. Note $D \setminus G = B(w,\delta) \cap A$, where

$A := \{z : d(z, bD) < r\}$. We have

$$\sigma(w)^{-1} \int\limits_{D\setminus G} |k_w(z) - \langle k_w \rangle_{B(z,r),\sigma dV}|\sigma(z)\, dV(z) \leq \sigma(w)^{-1} \left(\int\limits_{D\setminus G} |k_w(z)|\sigma(z)\, dV(z) + \int\limits_{D\setminus G} |\langle k_w \rangle_{B(z,r),\sigma dV}|\sigma(z)\, dV(z) \right).$$

By (5.2.2), it is easy to deduce

$$\sigma(w)^{-1} \int\limits_{D\setminus G} |k_w(z)|\sigma(z)\, dV(z) \lesssim \delta^{1/2}.$$

We will also show

$$\sigma(w)^{-1} \int\limits_{D\setminus G} |\langle k_w \rangle_{B(z,r),\sigma dV}|\sigma(z)\, dV(z) \lesssim \delta^{1/2}$$

using similar methods. We consider two separate regions of integration based on the relative positions of z and w. First, suppose that $cr < \frac{1}{2} d(z, w)$. One can show that if $\zeta \in B(z, r)$, then $d(z, w) \lesssim d(\zeta, w)$ with an implicit constant independent of z and w. We then estimate

$$\begin{aligned}
&\sigma(w)^{-1} \int\limits_{(B(w,\delta)\setminus B(w,2cr))\cap A} |\langle k_w \rangle_{B(z,r),\sigma dV}|\sigma(z)\, dV(z) \\
&\leq \sigma(w)^{-1} \int\limits_{(B(w,\delta)\setminus B(w,2cr))\cap A} \frac{1}{\sigma(B(z,r))} \int\limits_{B(z,r)} d(\zeta, w)^{-\left(n+\frac{1}{2}\right)} \sigma(\zeta)\, dV(\zeta)\, \sigma(z)\, dV(z) \\
&\lesssim \sigma(w)^{-1} \int\limits_{B(w,\delta)\cap A} d(z, w)^{-\left(n+\frac{1}{2}\right)} \sigma(z)\, dV(z) \\
&\lesssim \delta^{1/2}
\end{aligned}$$

as before. We have used the B_1 condition of σ in the third inequality above.

On the other hand, if $d(z,w) \le 2cr$, then $B(z,r) \subseteq B(w,Cr)$ and $B(w,r) \subseteq B(z,Cr)$, where $C = 2c^2 + c$. We first consider a further subcase where $d(w,bD) < r$. In this case, note $d(z,bD) \lesssim r$ on this set as well by the quasi-triangle inequality. Thus, we calculate:

$$
\begin{aligned}
&\sigma(w)^{-1} \int\limits_{B(w,2cr)\cap A} \frac{1}{\sigma(B(z,r))} \int\limits_{B(z,r)} d(\zeta,w)^{-\left(n+\frac{1}{2}\right)} \sigma(\zeta)\, dV(\zeta)\, \sigma(z)\, dV(z) \\
&\le \sigma(w)^{-1} \frac{1}{\sigma(B(w,r))} \int\limits_{B(w,2cr)\cap A} \frac{\sigma(B(z,Cr))}{\sigma(B(z,r))} \int\limits_{B(w,Cr)} d(\zeta,w)^{-\left(n+\frac{1}{2}\right)} \sigma(\zeta)\, dV(\zeta)\, \sigma(z)\, dV(z) \\
&\lesssim \delta^{1/2} \frac{1}{\sigma(B(w,r))} \int\limits_{B(w,2cr)\cap A} \frac{\sigma(B(z,Cr))}{\sigma(B(z,r))} \sigma(z)\, dV(z) \\
&\lesssim \delta^{1/2}
\end{aligned}
$$

using the B_1 condition in the second inequality and the doubling property of σ in the third inequality. For the second subcase, suppose $d(w,bD) \ge r$ and note that we still assume $d(z,w) \le 2cr$, so we in fact have $d(w,bD)^{-(n+1/2)} \lesssim d(z,w)^{-(n+1/2)}$. We estimate

$$
\begin{aligned}
&\sigma(w)^{-1} \int\limits_{B(w,2cr)\cap A} |\langle k_w \rangle_{B(z,r),\sigma dV}| \sigma(z)\, dV(z) \\
&\le \sigma(w)^{-1} \int\limits_{B(w,2cr)\cap A} \frac{1}{\sigma(B(z,r))} \int\limits_{B(z,r)} d(w,bD)^{-\left(n+\frac{1}{2}\right)} \sigma(\zeta)\, dV(\zeta)\, \sigma(z)\, dV(z) \\
&\lesssim \sigma(w)^{-1} \int\limits_{B(w,\delta)\cap A} d(z,w)^{-\left(n+\frac{1}{2}\right)} \sigma(z)\, dV(z) \\
&\lesssim \delta^{1/2},
\end{aligned}
$$

where we have used the B_1 condition in the third inequality.

Thus, we obtain

$$\sigma(w)^{-1} \int_{D \setminus G} |k_w(z) - \langle k_w \rangle_{B(z,r),\sigma dV}| \sigma(z)\, dV(z) \lesssim \delta^{1/2}$$

with an independent implicit constant. This can be made less than ε by making an appropriately small choice of δ, completing the proof. □

We need the following lemma to conclude that $(I - (\mathcal{T}^* - \mathcal{T}))$ is invertible on $L^1_\sigma(D)$.

Lemma 5.2.4. If $\sigma \in B_1$, the number 1 is not an eigenvalue of $\mathcal{T}^* - \mathcal{T}$ considered as an operator on $L^1_\sigma(D)$.

Proof. The proof proceeds in the same way as Corollary 4.3.1 in Chapter 4. In particular, it was proved in Proposition 4.3.2 in Chapter 4 that there exists $\varepsilon > 0$ so that $\mathcal{T}^* - \mathcal{T}$ maps $L^p(D)$ to $L^{p+\varepsilon}(D)$ boundedly for $p \geq 1$. Thus, if 1 were an eigenvalue for $\mathcal{T}^* - \mathcal{T}$ with eigenvector $f \in L^1_\sigma(D)$, then we would have

$$\|f\|_{L^{1+\varepsilon}(D)} = \|(\mathcal{T}^* - \mathcal{T})f\|_{L^{1+\varepsilon}(D)} \lesssim \|f\|_{L^1(D)} \lesssim \|f\|_{L^1_\sigma(D)},$$

noting that a weight in B_1 is bounded below. If we repeat this argument a second time, we get $f \in L^{1+2\varepsilon}(D)$. In fact, we can iterate arbitrarily many times to obtain $f \in L^p(D)$ for all $p \geq 1$. In particular, $f \in L^2(D)$. This is a contradiction because 1 is not an eigenvalue of $\mathcal{T}^* - \mathcal{T}$ on $L^2(D)$, since all of these eigenvalues are purely imaginary. □

Proof of Proposition 5.2.1. This follows immediately from Lemma 5.2.3 and Lemma 5.2.4 using the spectral theorem for compact operators. □

5.2.3 Weak-type estimate for the auxiliary operator

To show the weighted weak-type $(1,1)$ property for $\mathcal{T}$, we first prove the analogous bound for our maximal operator $\mathcal{M}$.

Lemma 5.2.5. If σ is a B_1 weight, then $\mathcal{M}$ maps $L^1_\sigma(D)$ into $L^{1,\infty}_\sigma(D)$ boundedly.

Proof. It suffices to prove the estimate for the centered version of $\mathcal{M}$,

$$\widetilde{\mathcal{M}}f(z) := \sup_{r>d(z,bD)} \langle |f| \rangle_{B(z,r)},$$

since we have the pointwise equivalence $\widetilde{\mathcal{M}}f \leq \mathcal{M}f \lesssim \widetilde{\mathcal{M}}f$. Indeed, the first inequality is clear, and the second is justified by the fact that $\langle |f| \rangle_B \lesssim \langle |f| \rangle_{B(z,2cr(B))}$ for any $z \in D$ and quasi-ball B containing z.

Let $f \in L^1_\sigma(D)$, $\lambda > 0$, and $E_\lambda := \{\widetilde{\mathcal{M}}f > \lambda\}$. We show that

$$\sigma\left(E_\lambda\right) \lesssim \frac{1}{\lambda}\|f\|_{L^1_\sigma(D)}.$$

For each $z \in E_\lambda$, let B_z be a quasi-ball centered at z such that $r(B_z) > d(z,bD)$ and $\langle |f| \rangle_{B_z} > \lambda$. Apply a Vitali-type lemma to obtain a subcollection $\{B_j\}_{j=1}^\infty$ of $\{B_z\}_{z\in E_\lambda}$ consisting of pairwise disjoint quasi-balls such that there exists $R \geq 1$ with $E_\lambda \subseteq \bigcup_{j=1}^\infty RB_j$. Use the doubling property of σ, the B_1 property of σ, and the selection property of the B_j to conclude

$$\sigma\left(E_\lambda\right) \leq \sum_{j=1}^\infty \sigma(RB_j) \lesssim \sum_{j=1}^\infty \sigma(B_j)$$

$$\lesssim \sum_{j=1}^\infty \left(\frac{1}{\|\sigma^{-1}\|_{L^\infty(B_j)}}\right) V(B_j) < \sum_{j=1}^\infty \left(\inf_{w\in B_j} \sigma(w)\right) \frac{1}{\lambda}\int_{B_j} |f|\, dV \leq \frac{1}{\lambda}\|f\|_{L^1_\sigma(D)}.$$

□

Lemma 5.2.6. There exists $k > 0$ such that $|g(z,w)| \approx |g(z,w')|$ for all $z, w, w' \in D$ satisfying $d(w,w') \leq kd(w,bD)$.

Proof. From the proof of Lemma 4.3.1 in Chapter 4, we know $|g(z,w)| \approx |g(z,w')|$ whenever $d(w,w') \leq Cd(z,w)$, where $C > 0$ is an absolute constant. If $d(w,bD) < \frac{C}{k}d(z,w)$, then $d(w,w') \leq kd(w,bD) < Cd(z,w)$, and hence $|g(z,w)| \approx |g(z,w')|$.

We may now assume that $d(z,w) \leq \frac{k}{C}d(w,bD)$. In this case, we use the triangle inequality, the fact that $|g(z,w) - g(z,w')| \lesssim d(w,w')^{\frac{1}{2}}d(z,w)^{\frac{1}{2}} + d(w,w')$ (which is similar to arguments in Lemma 4.3.1 as well), and the assumptions to get

$$\begin{aligned} |g(z,w)| &\leq |g(z,w) - g(z,w')| + |g(z,w')| \\ &\lesssim d(w,w')^{\frac{1}{2}}d(z,w)^{\frac{1}{2}} + d(w,w') + |g(z,w')| \\ &\leq \left(\frac{k}{C^{1/2}} + k\right) d(w,bD) + |g(z,w')|. \end{aligned}$$

Now using the triangle inequality and the hypothesis, we have

$$d(w,bD) \leq cd(w,w') + cd(w',bD) \leq ckd(w,bD) + cd(w',bD).$$

Choosing k sufficiently small, the above line implies $d(w,bD) \lesssim d(w',bD)$, and so

$$|g(z,w)| \lesssim d(w',bD) + |g(z,w')|.$$

Again referring to Lemma 4.3.1, we have $d(w',bD) \lesssim |g(z,w')|$, and we conclude

$$|g(z,w)| \lesssim |g(z,w')|.$$

A symmetric argument proves the reverse inequality, establishing the lemma. □

The following lemma is a modified version of the Calderón-Zygmund decomposition.

Recall the definition of the regularizing operator R_k that was given in Chapter 2, Section 2.3.

Lemma 5.2.7. For any $\lambda > 0$, $k \in (0,1)$, and nonnegative $f \in L^1(D)$, we can write $f \approx f_1 + f_2$, where

1. $R_k f_1 \lesssim \lambda$,
2. there exists a countable collection of almost disjoint quasi-balls $\mathcal{F}$ such that $r(B) \geq \frac{1}{2}d(B, bD)$ for each $B \in \mathcal{F}$ and $f_2 \approx \sum_{B\in\mathcal{F}} f_{2,B}$ where the $f_{2,B}$ are supported in B with $\langle |f_{2,B}| \rangle_B \leq \lambda$, and
3. $\sum_{B\in\mathcal{F}} \sigma(B) \lesssim \frac{1}{\lambda}\|f\|_{L^1_\sigma(D)}$.

Proof. Apply a Whitney decomposition to write

$$\{\mathcal{M}f > \lambda\} = \bigcup_{B\in\mathcal{F}'} B,$$

where $\mathcal{F}'$ is a countable collection of almost disjoint quasi-balls for which there exists $K > 1$ such that $KB \cap \{\mathcal{M}f \leq \lambda\} \neq \emptyset$ for all $B \in \mathcal{F}'$. We take

$$\mathcal{F} := \left\{ B \in \mathcal{F}' : r(B) \geq \frac{1}{2}d(B, bD) \right\}.$$

Put

$$f_1 := f\chi_{\{\mathcal{M}f\leq\lambda\}\cup\bigcup_{B\in\mathcal{F}'\setminus\mathcal{F}} B} \quad \text{and} \quad f_2 := f\chi_{\bigcup_{B\in\mathcal{F}} B}.$$

Clearly, $f \approx f_1 + f_2$.

To show (1), we first claim that $R_k f_1(z) \lesssim \mathcal{M}f_1(z)$ for any $z \in D$. Indeed, since the radius of $\frac{k+1}{k}B_k(z)$ is greater than $d(z, bD)$ and using the fact that $V(B(z,r)) \approx r^{n+1}$, we have

$$R_k f_1(z) = \langle f_1 \rangle_{B_k(z)} \lesssim \langle |f_1| \rangle_{\frac{k+1}{k}B_k(z)} \leq \mathcal{M}f_1(z).$$

Therefore it is enough to prove $\mathcal{M}f_1 \lesssim \lambda$. To this end, fix $z \in D$ and let B_0 be a quasi-ball containing z that intersects bD. If either $B_0 \cap \{\mathcal{M}f \leq \lambda\} \neq \emptyset$ or if $f \equiv 0$ on B_0, then $\langle |f_1| \rangle_{B_0} \leq \lambda$. Otherwise, $B_0 \cap B \neq \emptyset$ for some $B \in \mathcal{F}' \setminus \mathcal{F}$. Notice that $CB_0 \supseteq KB$ with $C = c^3(K+1) + c$, since $d(c(B_0), bD) < r(B_0)$ and $r(B) < \frac{1}{2}d(B, bD)$. Since $KB \cap \{\mathcal{M}f \leq \lambda\} \neq \emptyset$ and using the polynomial growth condition on the quasi-balls, we have

$$\langle |f_1| \rangle_{B_0} \lesssim \langle |f_1| \rangle_{CB_0} \leq \lambda.$$

Therefore (1) holds.

For (2), note that the properties of the collection $\mathcal{F}$ are satisfied by construction. Take $f_{2,B} := f\chi_B$ for $B \in \mathcal{F}$. Since $KB \cap \{\mathcal{M}f \leq \lambda\} \neq \emptyset$, we have $\langle |f_{2,B}| \rangle_B \lesssim \lambda$.

Finally, (3) follows from the almost disjointness of the quasi-balls in $\mathcal{F}$ and Lemma 5.2.5

$$\sum_{B \in \mathcal{F}} \sigma(B) \lesssim \sigma\left(\bigcup_{B \in \mathcal{F}} B\right) \leq \sigma(\{\mathcal{M}f > \lambda\}) \lesssim \frac{1}{\lambda}\|f\|_{L^1_\sigma(D)}.$$

□

Proof of Proposition 5.2.2. Since $\mathcal{T}_2$ has a bounded kernel and $\sigma(D) < \infty$ (using the B_1 condition), it is immediate that $\mathcal{T}_2$ is bounded on $L^1_\sigma(D)$, and hence from $L^1_\sigma(D)$ to $L^{1,\infty}_\sigma(D)$. It is thus sufficient to prove the estimate for $\mathcal{T}_1$.

Recall the comparison operator Γ that was introduced in Chapter 1 Section 4.3:

$$\Gamma f(z) := \int_D \frac{f(w)}{|g(z,w)|^{n+1}}\, dV(w).$$

Recall it can easily be shown that

$$|\mathcal{T}_1 f(z)| \lesssim \Gamma|f|(z),$$

so it suffices to prove that Γ maps $L^1_\sigma(D)$ to $L^{1,\infty}_\sigma(D)$.

Let f be a nonnegative and continuous function on D and let $\lambda > 0$. We will show that

$$\sigma(\{\Gamma f > \lambda\}) \lesssim \frac{1}{\lambda}\|f\|_{L^1_\sigma(D)}.$$

A density argument and doubling the implied constant in the display above yields the result for general $f \in L^1_\sigma(D)$.

Apply Lemma 5.2.7 to write

$$f \approx f_1 + f_2 \approx f_1 + \sum_{B\in\mathcal{F}} f_{2,B},$$

where the properties and notations from the lemma hold. Then

$$\begin{aligned}\sigma(\{\Gamma f > \lambda\}) &\le \sigma\left(\left\{\Gamma f_1 > \frac{\lambda}{C}\right\}\right) + \sigma\left(\left\{\Gamma f_2 > \frac{\lambda}{C}\right\}\right)\\ &\le \sigma\left(\left\{\Gamma f_1 > \frac{\lambda}{C}\right\}\right) + \sigma\left(\bigcup_{B\in\mathcal{F}} RB\right) + \sigma\left(\left\{z \in D \setminus \bigcup_{B\in\mathcal{F}} RB : \Gamma f_2(z) > \frac{\lambda}{C}\right\}\right)\end{aligned}$$

for some $C > 0$ and where $R > 1$ will be fixed later. Therefore it is enough to bound

$$\begin{aligned}&\mathrm{I} := \sigma(\{\Gamma f_1 > \lambda\}),\\ &\mathrm{II} := \sigma\left(\bigcup_{B\in\mathcal{F}} RB\right), \quad \text{and}\\ &\mathrm{III} := \sigma\left(\left\{z \in D \setminus \bigcup_{B\in\mathcal{F}} RB : \Gamma f_2(z) > \lambda\right\}\right)\end{aligned}$$

by constants multiplied by $\frac{1}{\lambda}\|f\|_{L^1_\sigma(D)}$.

To address I, we first claim that there exists $k > 0$ such that for all integrable and nonnegative u, we have

$$\Gamma u(z) \lesssim \Gamma(R_{k'} u)(z),$$

where $k' = \frac{ck}{1-ck}$. Indeed, by Lemma 5.2.6, we have $|g(z,w)| \approx |g(z,w')|$ for all $z \in D$ and $w' \in B_k(w)$. Using the above and Lemma 2.3.2 in Chapter 2, we deduce

$$\begin{aligned}
\Gamma u(z) &= \int_D \frac{1}{|g(z,w)|^{n+1}} u(w)\, dV(w) \\
&\approx \int_D \left(\frac{1}{V(B_k(w))} \int_{B_k(w)} \frac{1}{|g(z,w')|^{n+1}}\, dV(w') \right) u(w)\, dV(w) \\
&\lesssim \int_D \frac{1}{|g(z,w)|^{n+1}} \left(\frac{1}{V(B_{k'}(w))} \int_{B_{k'}(w)} u(w')\, dV(w') \right) dV(w) \\
&= \Gamma(R_{k'}u)(z).
\end{aligned}$$

Therefore, using Chebyshev's inequality, the above claim, the $L^2_\sigma(D)$ bound of Γ (see Chapter 4 Theorem 4.3.3), property (1) of Lemma 5.2.7, Lemma 2.3.2, and the B_1 condition of σ we have

$$\begin{aligned}
\mathrm{I} &\lesssim \frac{1}{\lambda^2} \int_D (\Gamma f_1)^2 \sigma\, dV \\
&\lesssim \frac{1}{\lambda^2} \int_D (\Gamma(R_{k'} f_1))^2 \sigma\, dV \\
&\lesssim \frac{1}{\lambda^2} \int_D (R_{k'} f_1)^2 \sigma\, dV \\
&\lesssim \frac{1}{\lambda} \int_D (R_{k'} f_1) \sigma\, dV \\
&\lesssim \frac{1}{\lambda} \int_D f_1 (R_{k''}\sigma)\, dV \\
&\lesssim \frac{1}{\lambda} \int_D f_1 \sigma\, dV \\
&\leq \frac{1}{\lambda} \|f\|_{L^1_\sigma(D)}.
\end{aligned}$$

The control of II follows from the doubling property of σ and property (3) of Lemma 5.2.7:

$$\mathrm{II} \le \sum_{B\in\mathcal{F}} \sigma(RB) \lesssim \sum_{B\in\mathcal{F}} \sigma(B) \lesssim \frac{1}{\lambda}\|f\|_{L^1_\sigma(D)}.$$

For III, we claim that if $R > 1$ is sufficiently large and u is supported on a quasi-ball B, then

$$\Gamma u(z) \lesssim \Gamma\left(\langle u\rangle_B \chi_B\right)(z)$$

for all $z \in D \setminus RB$. Indeed, as stated in the proof of Lemma 5.2.6, we have $|g(z,w)| \approx |g(z,w')|$ whenever $d(w,w') \le Cd(z,w)$. For $z \in D \setminus RB$ and $w, w' \in B$, we use the triangle inequality to obtain $d(w,w') < 2cr(B)$ and $\frac{R-c}{c} r(B) < d(z,w)$. Thus, if R is chosen large enough so that $2c \le C\frac{R-c}{c}$, we have $d(w,w') \le Cd(z,w)$, and hence $|g(z,w)| \approx |g(z,w')|$. The claim follows via using Fubini's theorem

$$\begin{aligned}
\Gamma u(z) &= \int_B \frac{1}{|g(z,w)|^{n+1}} u(w)\, dV(w) \\
&\approx \int_B \left(\frac{1}{V(B)} \int_B \frac{1}{|g(z,w')|^{n+1}}\, dV(w') \right) u(w)\, dV(w) \\
&= \int_B \frac{1}{|g(z,w')|^{n+1}} \left(\frac{1}{V(B)} \int_B u(w)\, dV(w) \right) dV(w') \\
&= \Gamma\left(\langle u\rangle_B \chi_B\right)(z).
\end{aligned}$$

Using the above claim, we have

$$\Gamma f_2(z) \approx \sum_{B\in\mathcal{F}} \Gamma f_{2,B}(z) \lesssim \sum_{B\in\mathcal{F}} \Gamma\left(\langle f_{2,B}\rangle_B \chi_B\right)(z) = \Gamma \tilde{f}_2(z)$$

for $z \in D \setminus \bigcup_{B\in\mathcal{F}} RB$, where $\tilde{f}_2 := \sum_{B\in\mathcal{F}} \langle f_{2,B}\rangle_B \chi_B$. Therefore, to control III, it suffices to prove

$$\sigma(\{\Gamma\tilde{f}_2 > \lambda\}) \lesssim \frac{1}{\lambda}\|f\|_{L^1_\sigma(D)}.$$

To accomplish this, apply Chebyshev's inequality, the bound of Γ on $L^2_\sigma(D)$, property (2) of Lemma 5.2.7, the B_1 condition of σ, and the almost disjointness of the quasi-balls in $\mathcal{F}$:

$$\begin{aligned}\sigma(\{\Gamma\tilde{f}_2 > \lambda\}) &\lesssim \frac{1}{\lambda^2}\int_D (\Gamma\tilde{f}_2)^2\sigma\,dV \\ &\lesssim \frac{1}{\lambda^2}\int_D \tilde{f}_2^2\sigma\,dV \\ &\lesssim \frac{1}{\lambda}\sum_{B\in\mathcal{F}}\int_B \langle f\rangle_B\sigma\,dV \\ &\lesssim \frac{1}{\lambda}\sum_{B\in\mathcal{F}}\int_B f\sigma\,dV \\ &\lesssim \frac{1}{\lambda}\|f\|_{L^1_\sigma(D)}.\end{aligned}$$

□

5.3 The Szegő Projection

Throughout this section, we assume that the domain D has class C^3 boundary. We recall that in Chapter 3 Section 3.3, we constructed a "Cauchy-type" non-orthogonal projection $\mathcal{C}$ using Cauchy-Fantappié theory that, roughly speaking, reproduces and produces holomorphic functions on the boundary. We maintain all the notation from Section 3.3. It was proved in [40] that the operator $\mathcal{C}$ extends boundedly on $L^p(bD)$ for all $1 < p < \infty$, and weighted bounds for $1 < p < \infty$ were established in Section 3.3. Recall the Kerzman-Stein equation now takes the following form:

$$\mathcal{S}(I - (\mathcal{C}^* - \mathcal{C})) = \mathcal{C}.$$

To prove that $\mathcal{S}$ is weak-type $(1,1)$ with respect to the A_1 weight σ, we proceed in two steps as before. In particular, we have the following two propositions:

Proposition 5.3.1. If σ is an A_1 weight, then the operator $I - (\mathcal{C}^* - \mathcal{C})$ is invertible on $L^1_\sigma(bD)$.

Proposition 5.3.2. If σ is an A_1 weight, then $\mathcal{C}$ maps $L^1_\sigma(bD)$ to $L^{1,\infty}_\sigma(bD)$ boundedly.

We can now prove Theorem 1.5.6.

Proof of Theorem 1.5.6. This follows directly from Proposition 5.3.1 and Proposition 5.3.2. □

The proof of Proposition 5.3.1 proceeds as in the Bergman case. We again appeal to Lemma 5.2.1 and Lemma 5.2.2 to prove a compactness result. In this case, the underlying space is $X = bD$ and the finite Borel measure is $\sigma\, dS$. Recall we can define the appropriate quasi-metric:

$$d(z,w) := |g(z,w)|^{\frac{1}{2}},$$

as in Section 3.3. It was proved in [40] that d is indeed a quasi-metric and that (D, d, S) is a space of homogeneous type. Additionally, we have $S(B(z,r)) \approx r^{2n}$.

Lemma 5.3.1. If σ is an A_1 weight, then the operator $\mathcal{C}^* - \mathcal{C}$ is compact on $L^1_\sigma(bD)$.

Proof. Let $k(z,w)$ denote the kernel of the operator $\mathcal{C}^* - \mathcal{C}$. We consider the family of functions $k_w(z) = k(z,w)$ for $w \in bD$, and let $\tilde{k}(z,w) = \tilde{k}_w(z) = k(z,w)\sigma^{-1}(w)$. By Lemma 5.2.1, it suffices to show that $\{\tilde{k}_w : w \in bD\}$ is relatively compact in $L^1_\sigma(bD)$, which we can do by verifying the criteria of Lemma 5.2.2. The infimum condition can be verified as before. This set is clearly bounded in $L^1_\sigma(bD)$; this follows by observing as in Chapter 3 that

$$\int_{bD} |k(z,w)|\sigma(z)\, dS(z) \lesssim \sigma(w).$$

In particular, this is deduced from the bound $|k(z,w)| \lesssim d(z,w)^{-2n+1}$ (which relies on the domain having boundary of class C^3) and a dyadic integration argument similar to the one presented in Lemma 5.2.3. Similarly, we obtain that

$$\int_{B(w,\delta)} |k(z,w)|\sigma(z)\, dS(z) \lesssim \delta\sigma(w).$$

Notice that this bound does not involve a $d(z, bD)$ term which highlights a key difference from the case of the Bergman projection.

The second conclusion mirrors very closely the argument in Section 5.2, so we only sketch the ideas. Namely, for a fixed function k_w, we excise a small ball about w and integrate the function $|k_w(z) - \langle k_w\rangle_{B(z,r),\sigma dS}|$ over this ball and its complement. The integral on the complement of the ball can be controlled by uniform continuity, since $k(z,w)$ is continuous off the boundary diagonal. The integral over the ball is controlled via the triangle inequality and splitting into regions as in the proof of Lemma 5.2.3. It should be noted that it is not necessary to split into subcases based on the distance of points z and w to bD because all the integration occurs on the boundary and A_1 weights satisfy a true doubling property. □

The following lemma follows the exact same argument as Lemma 5.2.4.

Lemma 5.3.2. If $\sigma \in A_1$, the number 1 is not an eigenvalue of $\mathcal{C}^* - \mathcal{C}$ considered as an operator on $L^1_\sigma(bD)$.

Therefore, we can prove Proposition 5.3.1:

Proof of Proposition 5.3.1. This proposition follows from Lemma 5.3.1, Lemma 5.3.2, and the spectral theorem for compact operators on a Banach space. □

To complete the proof of Theorem 1.5.6, it remains to prove Proposition 5.3.2.

Proof of Proposition 5.3.2. As in [40] or Section 3.3, write $\mathcal{C} = \mathcal{C}^\sharp + \mathcal{R}$. It is proven in [40] that $\mathcal{C}^\sharp$ is a Calderón-Zygmund operator with respect to the quasi-metric d. Thus, by standard theory, $\mathcal{C}^\sharp$ maps $L^1_\sigma(bD)$ to $L^{1,\infty}_\sigma(bD)$ boundedly for $\sigma \in A_1$.

On the other hand, the operator $\mathcal{R}$ has a kernel $R(z,w)$ that satisfies

$$\int_{bD} |R(z,w)|\sigma(w)\, dS(w) \lesssim \sigma(z)$$

and

$$\int_{bD} |R(z,w)|\sigma(z)\, dS(z) \lesssim \sigma(w)$$

for $\sigma \in A_1$ (see Section 3.3 and Proposition 3.3.6). A simple argument using Fubini's theorem shows that $\mathcal{R}$ is bounded on $L^1_\sigma(bD)$. This completes the proof. □

5.4 Kolmogorov and Zygmund Inequalities

We first prove the general fact that the weak-type $(1,1)$ estimate implies the Kolmogorov inequality on finite measure spaces.

Theorem 5.4.1. Let T be a linear operator and (X,μ) a finite measure space. If T maps $L^1(X,\mu)$ to $L^{1,\infty}(X,\mu)$ boundedly and $0 < p < 1$, then T extends boundedly from $L^1(X,\mu)$ to $L^p(X,\mu)$.

Proof. Using the distribution function and the weak-type $(1,1)$ assumption, we have for any

$t > 0$:

$$\|Tf\|^p_{L^p(X,\mu)} = \int_0^\infty p\lambda^{p-1}\mu(\{x \in X : |Tf(x)| > \lambda\})\, d\lambda$$

$$= \int_0^t p\lambda^{p-1}\mu(\{x \in X : |Tf(x)| > \lambda\})\, d\lambda + \int_t^\infty p\lambda^{p-1}\mu(\{x \in X : |Tf(x)| > \lambda\})\, d\lambda$$

$$\leq t^p\mu(X) + \frac{p}{1-p}t^{p-1}\|f\|_{L^1(X,\mu)}.$$

Taking $t = \|f\|_{L^1(X,\mu)}$ completes the proof. □

Proof of Corollary 5.1.1. This follows immediately from Theorem 5.1.1 and Theorem 5.4.1. □

Proof of Corollary 5.1.3. This follows immediately from Theorem 5.1.2 and Theorem 5.4.1. □

Before proving our Zygmund inequalities, we first define the space $L\log^+ L$, which falls within the scope of Orlicz spaces. We call a function $\Phi : [0,\infty] \to [0,\infty]$ a Young function if Φ is continuous, convex, increasing, and satisfies $\Phi(0) = 0$. Given a measure space (X,μ) and a Young function Φ, the associated Orlicz space, $L^\Phi(X,\mu)$, is the linear hull of all measurable functions on X satisfying

$$\int_X \Phi(|f|)\, d\mu < \infty$$

equipped with the following Luxemburg norm:

$$\|f\|_{L^\Phi(X,\mu)} := \inf\left\{\lambda > 0 : \int_X \Phi\left(\frac{|f|}{\lambda}\right)\, d\mu \leq 1\right\}.$$

The Zygmund space $L\log^+ L(X,\mu)$ is defined to be the Orlicz space $L^\Psi(X,\mu)$ associated with the Young function $\Psi(t) = t\log^+ t$, where $\log^+(t) := \max\{\log(t), 0\}$. We use the notation $(L\log^+ L)_\sigma(D)$ to represent $L\log^+ L(D, \sigma\, dV)$ for a domain $D \subseteq \mathbb{C}^n$ and a weight σ on D and we similarly write $(L\log^+ L)_\sigma(bD)$ for $L\log^+ L(bD, \sigma\, dS)$ with σ a weight on bD. We refer to [35, 61] for thorough treatments of Orlicz spaces.

We next prove that the weak-type $(1,1)$ and L^2 bounds imply the Zygmund inequality on general finite measure spaces.

Theorem 5.4.2. Let T be a linear operator and (X,μ) a finite measure space. If T is bounded on $L^2(X,\mu)$ and maps $L^1(X,\mu)$ to $L^{1,\infty}(X,\mu)$ boundedly, then T extends boundedly from $L\log^+ L(X,\mu)$ to $L^1(X,\mu)$.

Proof. Let $f \in L\log^+ L(X,\mu)$ be given and normalized to assume $\|f\|_{L\log^+ L(X,\mu)} = 1$. Observe that $L^1(X,\mu)$ is the Orlicz space $L^\Phi(X,\mu)$ with Young function $\Phi(t) = t$. Define Φ_1 by

$$\Phi_1(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < 2 \\ t-2 & \text{if} \quad 2 \le t \le \infty \end{cases}$$

and notice that Φ and Φ_1 are equivalent Young functions in the sense that

$$\Phi_1(t) \le \Phi(t) \le \Phi_1(2t)$$

for all $t \ge 2$. Therefore by [35, Theorem 13.2 and Theorem 13.3], it suffices to prove

$$\|Tf\|_{L^{\Phi_1}(X,\mu)} \lesssim 1.$$

For a fixed $\lambda > 0$, write $f = f_0 + f_\infty$, where $f_0 := f\chi_{\{|f|\le\lambda\}}$ and $f_\infty := f\chi_{\{|f|>\lambda\}}$. Using the assumed bounds of T and the distribution function, we have

$$\begin{aligned}
\mu(\{|Tf| > 2\lambda\}) &\leq \mu(\{|Tf_0| > \lambda\}) + \mu(\{|Tf_\infty| > \lambda\}) \\
&\leq \frac{1}{\lambda^2}\|f_0\|_{L^2(X,\mu)}^2 + \frac{1}{\lambda}\|f_\infty\|_{L^1(X,\mu)} \\
&\approx \frac{1}{\lambda^2}\int_0^\lambda s\mu(\{|f| > s\})\,ds + \frac{1}{\lambda}\int_\lambda^\infty \mu(\{|f| > s\})\,ds.
\end{aligned}$$

Use the distribution function, a change of variables, the above estimate, and Fubini's Theorem, direct estimates, and the normalization $\|f\|_{L\log^+ L(X,\mu)} = 1$ to deduce

$$\begin{aligned}
\int_X \Phi_1(|Tf|)\,d\mu &= \int_2^\infty \mu(\{|Tf| > \lambda\})\,d\lambda \approx \int_1^\infty \mu(\{|Tf| > 2\lambda\})\,d\lambda \\
&\leq \int_1^\infty \frac{1}{\lambda^2}\int_0^\lambda s\mu(\{|f| > s\})\,dsd\lambda + \int_1^\infty \frac{1}{\lambda}\int_\lambda^\infty \mu(\{|f| > s\})\,dsd\lambda \\
&= \int_0^1 s\mu(\{|f| > s\})\int_1^\infty \frac{1}{\lambda^2}\,d\lambda ds + \int_1^\infty s\mu(\{|f| > s\})\int_s^\infty \frac{1}{\lambda^2}\,d\lambda ds \\
&\quad + \int_1^\infty \mu(\{|f| > s\})\int_1^s \frac{1}{\lambda}\,d\lambda ds \\
&= \int_0^1 s\mu(\{|f| > s\})\,ds + \int_1^\infty (1 + \log s)\mu(\{|f| > s\})\,ds \\
&\leq \mu(X) + \int_X \Psi(|f|)\,d\mu \\
&\lesssim 1,
\end{aligned}$$

where $\Psi(t) = t\log^+(t)$. Thus $\|Tf\|_{L^{\Phi_1}(X,\mu)} \lesssim 1$ as desired. □

Proof of Corollary 5.1.2. This follows immediately from Theorem 5.1.1 and Theorem 5.4.2. □

Proof of Corollary 5.1.4. This follows immediately from Theorem 5.1.2 and Theorem 5.4.2. □

5.5 The Bloch Space

In this section, we assume D is a strongly pseudoconvex domain with C^4 boundary. Our goal in this section is to prove Theorem 5.1.3.

We define the Bloch space $\mathscr{B}(D)$ as follows (see [34] for many equivalent definitions of the Bloch space in this context).

Definition 5.5.1. The *Bloch space* $\mathscr{B}(D)$ is defined

$$\mathscr{B}(D) := \left\{ f \in \mathrm{Hol}(D) : \sup_{z \in D} \mathrm{dist}(z, bD) |\nabla_\nu f(z)| < \infty \right\},$$

where ∇_ν denotes the complex normal derivative. Note that the quantity $\|f\|_{\mathscr{B}(D)} := \sup_{z \in D} \mathrm{dist}(z, bD) |\nabla_\nu f(z)|$ defines a semi-norm on the Bloch space.

We remark that Theorem 5.1.3 is established in [34, Theorem 3.19]. The proof there uses duality and an appropriate definition of BMOA spaces, while we make use of the Kerzman-Stein equation.

Theorem 5.1.3 will be an immediate consequence of the following two lemmas, together with 5.2.1.

Lemma 5.5.1. The operator $(I - (\mathcal{T}^* - \mathcal{T}))$ is invertible on $L^\infty(D)$.

Proof. It is clear from previous discussions that $\mathcal{T}$ and $\mathcal{T}^*$ are both bounded on $L^\infty(D)$. It is immediate that 1 cannot be an eigenvalue of $(\mathcal{T}^* - \mathcal{T})$ on $L^\infty(D)$, since $L^\infty(D) \subset L^2(D)$. Therefore, to prove the invertibility of $(I - (\mathcal{T}^* - \mathcal{T}))$ on $L^\infty(D)$, it suffices to show that $\mathcal{T}^* - \mathcal{T}$ is compact on $L^\infty(D)$. This follows easily from Lemma 5.2.1 and Lemma 5.2.2 in

Section 5.2 by simply interchanging the roles of the variables. Indeed, if $k(z,w)$ denotes the kernel of $\mathcal{T}^* - \mathcal{T}$, then $\widetilde{k}(z,w) := \overline{k(w,z)}$ is the kernel for $\mathcal{T} - \mathcal{T}^*$. Moreover, since $|k(z,w)| \approx |k(w,z)| = |\widetilde{k}(z,w)|$, all of the same arguments in Lemma 5.2.3 go through. □

Lemma 5.5.2. The operator $\mathcal{T}$ maps $L^\infty(D)$ to $\mathcal{B}(D)$ boundedly.

Proof. Write $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$, where $\mathcal{T}_1$ has kernel $K_1(z,w)$ and $\mathcal{T}_2$ has kernel $K_2(z,w)$. Examining the construction of $\mathcal{T}_2$ in [37, Proposition 3.2] and using regularity properties of the $\overline{\partial}$-Neumann operator in [10, 5.2.7], one can verify

$$\sup_{z,w\in D} |\nabla_z K_2(z,w)| < \infty,$$

a fact which we will use below (here, ∇_z denotes the gradient taken in the z variable). Moreover, the kernel K_1 is smooth in the z variable.

It clearly suffices for us to show

$$|\nabla \mathcal{T} f(z)| \lesssim \operatorname{dist}(z, bD)^{-1} \|f\|_{L^\infty(D)}.$$

We have

$$\begin{aligned}
|\nabla \mathcal{T} f(z)| &\lesssim \int_D |\nabla_z (K_1(z,w)) + \nabla_z (K_2(z,w)| \, |f(w)| \, dV(w) \\
&\leq \int_D |\nabla_z (K_1(z,w))| |f(w)| \, dV(w) + \int_D |\nabla_z (K_2(z,w))| |f(w)| \, dV(w) \\
&\leq \|f\|_{L^\infty(D)} \int_D |\nabla_z (K_1(z,w))| \, dV(w) + \sup_{z,w\in D} |\nabla_z (K_2(z,w))| \, \|f\|_{L^\infty} V(D).
\end{aligned}$$

Therefore, it is clear we must show

$$\int_D |\nabla_z (K_1(z,w))| \, dV(w) \lesssim \operatorname{dist}(z, bD)^{-1}.$$

Recall from (4.2.2) that $K_1(z,w) = \frac{\mathcal{N}(z,w)}{(2\pi i)^n (g(z,w))^{n+1}}$ where $\mathcal{N}(z,w)$ is a function of class C^1 (in w) with coefficients smooth in z. Moreover, it is easy to see that $\sup_{z,w\in D} |\nabla_z \mathcal{N}(z,w)| < \infty$. Thus, we compute

$$\nabla_z(K_1(z,w)) = \frac{\nabla_z \mathcal{N}(z,w)}{(g(z,w))^{n+1}} - (n+1)\frac{\mathcal{N}(z,w)}{(g(z,w))^{n+2}} \nabla_z(g(z,w))$$

and it is thus clear

$$|\nabla_z(K_1(z,w))| \lesssim \frac{1}{|g(z,w)|^{n+2}}.$$

Arguments and several changes of variables given in [37, Lemma 4.1]

$$\int_D \frac{1}{|g(z,w)|^{n+2}}\, dV(w) \lesssim \int_{\mathbb{R}} \int_{\mathbb{C}^{n-1}} \int_{\mathbb{R}^+} \frac{1}{(|\rho(z)| + s + |u_n| + |w'|^2)^{n+2}}\, ds\, dV(w')\, du_n\,.$$

Thus, we compute, using polar coordinates

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{C}^{n-1}} \int_{\mathbb{R}^+} \frac{1}{(|\rho(z)| + s + |u_n| + |w'|^2)^{n+2}}\, ds\, dV(w)\, du_n &\lesssim \int_{\mathbb{R}} \int_{\mathbb{C}^{n-1}} \frac{1}{(|\rho(z)| + |u_n| + |w'|^2)^{n+1}}\, dV(w')\, du_n \\ &\lesssim \int_{\mathbb{R}} \int_0^\infty \frac{r^{2n-3}}{(\rho(z) + |u_n| + r^2)^{n+1}}\, dr\, du_n\,.\end{aligned}$$

Now let $X = |\rho(z)| + |u_n|$. Applying a change of variable,we can estimate the inner integral:

$$\int_0^\infty \frac{r^{2n-3}}{(X+r^2)^{n+1}}\,dr = \frac{1}{2}\int_X^\infty \frac{(v-X)^{n-2}}{v^{n+1}}\,dv$$
$$\lesssim X^{-2}.$$

And finally, we are reduced to computing

$$\int_{\mathbb{R}} \frac{1}{(|\rho(z)|+|u_n|)^2} \lesssim \frac{1}{|\rho(z)|} \lesssim \operatorname{dist}(z,bD)^{-1},$$

which is exactly what we wanted to show.

□

Proof of Theorem 5.1.3. This is an immediate consequence of Lemma 5.5.1, Lemma 5.5.2, and (5.2.1).

□

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