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Quantum Curves and Asymptotic Hodge Theory

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Dedicated to Mathematicians, on whose shoulders I have been able to stand and gaze.

ABSTRACT OF THE DISSERTATION

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This dissertation explores a 2015 conjecture of Codesido-Grassi-Marino in topological string theory that relates the enumerative invariants of toric CY 3-folds to the spectra of operators attached to their mirror curves. In the maximally supersymmetric case, our first theorem relates zeroes of the higher normal function associated to an integral K_2-class on the mirror curve to the spectra of the operators for curves of genus one, and suggests a new link between analysis and arithmetic geometry. On the other hand in the 't Hooft limit, [KM, MZ] deduced from the [CGM] conjecture that the limiting values of the local mirror map at the maximal conifold point are given by values of the Bloch-Wigner dilogarithm at algebraic arguments. Our second theorem establishes these assertions by calculating regulator periods on the mirror curves attached to 3-term operators coming from triangles. As a consequence numerous series identities involving the Bloch-Wigner dilogarithm are demonstrated.

Chapter 1

Introduction

The simplest Calabi-Yau threefolds are the noncompact toric CYs X determined by a convex lattice polygon Δ (or more precisely by the fan on a triangulation of $\{1\} \times \Delta$ in $\mathbb{R}^3$). Each such CY has a family of *mirror curves* $\mathcal{C} \subset \mathbb{C}^* \times \mathbb{C}^*$, of genus g equal to the number of interior integer points of Δ, given by the Laurent polynomials $F(x_1, x_2)$ with Newton polygon Δ. Recently a fundamental and novel relationship between **(i)** the enumerative geometry of X and **(ii)** the spectral theory of certain operators $\hat{F}$ on $L^2(\mathbb{R})$ attached to $\mathcal{C}$, has been proposed by M. Mariño and his school, in the context of *non-perturbative* topological string theory [GHM, Ma, CGM]. The goal of this paper is to lay out some mathematical consequences of this meta-conjecture, and provide evidence for it by proving them.

A Laurent polynomial $F = \sum_{\underline{m} \in \Delta \cap \mathbb{Z}^2} a_{\underline{m}} \underline{x}^{\underline{m}}$ is promoted to an operator $\hat{F}$ (or "quantum curve") by a process called *Weyl quantization*, which depends on a real constant $\hbar$. Writing r for the coordinate on $\mathbb{R}$, let $\hat{\mathsf{x}}$ denote multiplication by r, and $\hat{\mathsf{y}} := \mathbf{i}\hbar\partial_r$, so that $[\hat{\mathsf{x}}, \hat{\mathsf{y}}] = \mathbf{i}\hbar$. Taking $\hat{F} := \sum a_{\underline{m}} e^{m_1\hat{\mathsf{x}} + m_2\hat{\mathsf{y}}}$, [CGM] define a *generalized spectral determinant* $\Xi_{\mathcal{C}}(\underline{a}; \hbar)$ whose zero-locus describes those curve moduli $\underline{a}$ for which $\ker(\hat{F}) \neq \{0\}$. They conjecture that under a "quantum mirror map" $\underline{a} \mapsto \mathfrak{t}^{\hbar}(\underline{a})$, $\Xi_{\mathcal{C}}$ is proportional to a *quantum theta function* $\Theta_X(\underline{t}; \hbar)$ derived from the all-genus enumerative invariants of X; see Conjecture 2.2.1. In particular, the zeroes of Θ_X should recover the spectrum of any fixed quantum curve $\hat{F}$.

In the formulation of [BKV], *local mirror symmetry* relates the "maximally supersymmetric" case ($\hbar = 2\pi$) of **(i)** to **(iii)** the Hodge-theoretic invariants (or "regulators") of algebraic K_2-classes on $\mathcal{C}$. This allows us to reformulate this case of the conjecture of Codesido-Grassi-Mariño [CGM] in §2.3 as a putative relationship between quantum curves and regulators (i.e. between **(ii)** and **(iii)**). We do this under the assumption that F ranges only over the *integrally tempered* Laurent polynomials, so that the symbol $\{-x_1, -x_2\} \in K_2(\mathbb{C}(\mathcal{C}))$ extends to motivic cohomology classes on the compactifications $\bar{\mathcal{C}}_{\underline{a}} \subset \mathbb{P}_\Delta$. This smaller moduli space $\boldsymbol{\mathcal{M}}$ has dimension g, and the resulting regulator classes $\frac{1}{4\pi^2}R(\underline{a}) \in H^1(\bar{\mathcal{C}}_{\underline{a}}, \mathbb{C}/\mathbb{Z})$ may be projected modulo $H^{1,0}(\bar{\mathcal{C}}_{\underline{a}})$ to yield a section $\boldsymbol{\nu}$ of the Jacobian bundle $\boldsymbol{\mathcal{J}} \to \boldsymbol{\mathcal{M}}$ of the family $\boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{M}}$, called the *higher normal function.* We deduce from the conjecture of [CGM] that the locus in $\mathcal{M}$ where $\boldsymbol{\nu}$ meets a specific torsion shift of the theta divisor in $\mathcal{J}$ should match the zero-locus of $\Xi_{\mathcal{C}}$ after tweaking the signs of the moduli; this is made precise in Conjecture 2.3.2.

We may further refine this prediction in the genus-1 case, where Δ is now reflexive and the Laurent polynomial $F(\underline{x}) = \varphi(\underline{x}) + a$ now has only one parameter a. In §3.1, we use integral mirror symmetry to compute the torsion shifts, and show that (after a miraculous cancellation) they simply translate the theta divisor to the origin! The prediction is now that the spectrum of the quantum curve is given by[1]

$$\sigma(\hat{\varphi}) = \{a \in \boldsymbol{\mathcal{M}} \mid \nu(a) \equiv 0 \ \in J(\bar{\mathcal{C}}_a)\}. \tag{1.0.1}$$

Keeping in mind that $g = 1$ (Δ reflexive), φ is tempered, and $\hbar = 2\pi$, our first main unconditional result is then the following

Theorem A (Theorems 3.2.2 and 3.3.1)**.** *Assume $\Delta \subset \mathbb{R} \times [-1, 1]$. Then the "$\supseteq$" direction of* (1.0.1) *holds, and the "$\subseteq$" direction holds for "almost all" eigenvalues.*

We prove the "$\supseteq$" statement in §3.2 by explicitly constructing square-integrable eigenfunctions of $\hat{\varphi}$ with eigenvalue a, using vanishing of $\nu(a)$ to show well-definedness. The result (in §3.3) on the "$\subseteq$" inclusion is obtained by using the coherent state representation of $\hat{\varphi}$ to bound

[1]Note the implicit sign flip on a: we are saying that $\ker(\hat{\varphi} - a) \neq \{0\}$ when the regulator associated to $\{-x_1, -x_2\}$ on $\varphi(\underline{x}) + a = 0$ dies in the Jacobian. The notation for the normal function changes from $\boldsymbol{\nu}$ to ν as it no longer has multiple components.

the accumulation of eigenvalues in a manner that matches growth ($\sim$ const. $\times \log^2(a)$) of ν as $a \to \infty$. One perspective on Theorem A is that we may view $\nu(a)$ as a multivalued function solving an inhomogeneous Picard-Fuchs equation, and in effect (1.0.1) states that the eigenvalues of $\hat{\varphi}$ are simply the points where $\nu(a) \in \mathbb{Z}$ (see Remark 3.1.5(i)). The latter condition is a statement about a period of a mixed motive, and combining this with a variant of Grothendieck's period conjecture allows one to show conditionally that the eigenvalues of $\hat{\varphi}$ are transcendental numbers (Prop. 3.3.4).

The conjecture of [CGM] yields a different prediction in the 't Hooft limit $\hbar \to \infty$, which is not empty for $g = 1$ but much more interesting for $g > 1$. Results of Kashaev, Mariño and Zakany [KM, MZ] on the limits of spectral traces of three-term operators can be viewed as providing a general formula for the limiting value of a particular regulator period $R_\gamma(\underline{a}) = \int_\gamma R\{-x_1, -x_2\}|_{\mathcal{C}_{\underline{a}}}$ at the maximal conifold point $\underline{\hat{a}}$, in terms of special values of the Bloch-Wigner ("real single-valued dilogarithm") function. Here "maximal conifold" means a particular point in moduli at which $\mathcal{C}$ acquires g nodes while remaining irreducible; that is, the normalization $\widetilde{\mathcal{C}_{\underline{\hat{a}}}}$ is a $\mathbb{P}^1$. By applying a method from [DK, §6] for computing regulator periods on singular curves of geometric genus zero, we are able to verify this in two infinite families of cases, corresponding to

$$F^{\underline{a}}_{g,g}(\underline{x}) = x_1 + x_2 + x_1^{-g} x_2^{-g} + \textstyle\sum_{j=1}^{g} a_j x_1^{1-j} x_2^{1-j} \quad \text{and}$$
$$F^{\underline{a}}_{2g-1,1}(\underline{x}) = x_1 + x_2 + x_1^{-2g+1} x_2^{-1} + \textstyle\sum_{j=1}^{g} a_j x_1^{1-j}.$$

The $g = 1$ case was already verified in [DK, §6.3], while the $g = 2$ identities were partially verified in [7K, §6].

To give a more explicit statement of this result, write $\tilde{F}^{\underline{a}} := F^{\underline{a}} - a_1$ in either case, and $[\cdot]_{\underline{0}}$ for the operator taking the constant term (in x_1, x_2) in a Laurent polynomial. Then we have:

Theorem B (Theorem 4.1.1)**.** *The regulator periods at the maximal conifold point satisfy*

$$\log(2g+1) - \textstyle\sum_{k>0} \frac{(-1)^{k(g+1)}}{k(2g+1)^k}[(\tilde{F}^{\hat{\underline{a}}}_{g,g})^k]_{\underline{0}} = \frac{1}{2\pi\mathbf{i}} R^{g,g}_{\gamma}(\hat{\underline{a}}) = \frac{(2g+1)}{\pi} D_2(1+e^{\frac{2\pi\mathbf{i}g}{2g+1}})$$

and

$$\log(2g+1) - \textstyle\sum_{k>0} \frac{1}{k(2g+1)^k}[(\tilde{F}^{\hat{\underline{a}}}_{2g-1,1})^k]_{\underline{0}} = \frac{1}{2\pi\mathbf{i}} R^{2g-1,1}_{\gamma}(\hat{\underline{a}}) = \frac{(2g+1)}{\pi} D_2(1+e^{\frac{2\pi\mathbf{i}}{2g+1}}).$$

In fact, the two families are isomorphic under the moduli-map sending $a_j \mapsto a_{g-j+1}$, and the cycles just two amongst g (named $\gamma_1, \ldots, \gamma_g$) for which we can compute the regulator period at $\hat{\underline{a}}$, obtaining g different identities. Part of the proof involves using a method from [Ke2] to determine (from the series expansions of their periods) how many times the "limits" of the $\{\gamma_j\}$ at $\hat{\underline{a}}$ pass through each of the g nodes, cf. Prop. 4.1.4; this method may be of independent interest in the study of monodromy. Incidentally, the identities we prove should have implications for the asymptotic behavior of genus-zero Gromov-Witten numbers of the corresponding CY X, but we do not pursue this direction here.

Chapter 2

A conjecture in topological string theory and its consequences

2.1 Quantum curves.

Let $\Delta \subset \mathbb{R}^2$ be a polygon with vertices in $\mathbb{Z}^2$ whose interior contains the origin $\underline{0}$. Write

$$F(x_1, x_2) = \textstyle\sum_{\underline{m} \in \Delta \cap \mathbb{Z}^2} a_{\underline{m}} \underline{x}^{\underline{m}} \tag{2.1.1}$$

for a general Laurent polynomial with Newton polygon Δ. The affine curve $\mathcal{C} := \{\underline{x} \in (\mathbb{C}^*)^2 \mid F(\underline{x}) = 0\}$ is then smooth of genus $g := |\mathrm{int}(\Delta) \cap \mathbb{Z}^2|$. It admits a smooth compactification $\bar{\mathcal{C}}$ in $\mathbb{P}_\Delta$, which denotes a minimal toric desingularization of the toric surface constructed from the normal fan of Δ. For instance, if Δ is reflexive with polar polygon Δ°, then $g = 1$ and $\mathbb{P}_\Delta$ is constructed from the fan with rays passing through each of the nonzero points of $\Delta^\circ \cap \mathbb{Z}^2$.

Taking a maximal integral triangulation $\mathrm{tr}(\Delta)$, consider the fan Σ on $\{1\} \times \mathrm{tr}(\Delta) \subset \mathbb{R}^3$. The resulting toric variety

$$X := \mathbb{P}_\Sigma \tag{2.1.2}$$

is called a *local CY 3-fold* since $K_X \cong \mathcal{O}_X$.[1] This will be our "A-model", on which we do enumerative geometry and run the Kähler moduli. Such noncompact CY 3-folds often arise

[1] To see this, note that $-c_1(K_X) = c_1(X)$ is the sum of the irreducible divisors corresponding to the elements of $\Delta \cap \mathbb{Z}^2$, which is the divisor of the first toric coordinate w_0 on X hence rationally equivalent to zero.

from the crepant resolution of a finite quotient of $\mathbb{C}^3$. For instance, if $1 \in \mathbb{Z}_{2k+1}$ acts on $\mathbb{C}^3$ by $\mathrm{diag}\{\zeta_{2k+1}, \zeta_{2k+1}^k, \zeta_{2k+1}^k\}$, the resolution X is obtained by taking Δ to be the convex hull of $(1,0)$, $(0,1)$, and $(-k,-k)$ (with $g = k$). Another set of examples (with $g = 1$) arises when Δ is reflexive: in this case, X is just the total space of $K_{\mathbb{P}_{\Delta^\circ}}$. There is some overlap with the quotient construction: for instance, $K_{\mathbb{P}^2}$ [resp. $K_{\mathbb{F}_2}$, $K_{\mathrm{dP}_6'}$[2]] arises from a quotient of $\mathbb{C}^3$ by $\mathbb{Z}_3$ [resp. $\mathbb{Z}_4$, $\mathbb{Z}_6$].

Local mirror symmetry connects the genus-zero enumerative invariants of X to periods of the "B-model"

$$Y := \{(\underline{x}, u, v) \in (\mathbb{C}^*)^2 \times \mathbb{C}^2 \mid F(x_1, x_2) + uv = 0\}, \tag{2.1.3}$$

an open CY 3-fold with K_Y trivialized by the form

$$\eta := \frac{1}{(2\pi \mathbf{i})^2} \mathrm{Res}_Y \left(\frac{dx_1/x_1 \wedge dx_2/x_2 \wedge du \wedge dv}{F(\underline{x}) + uv} \right) \in \Omega^3(Y). \tag{2.1.4}$$

We shall will say more about this in due course. It has been proposed by Mariño and collaborators [GHM, Ma, CGM] that one can capture the higher-genus enumerative invariants of X as well by *quantizing* the curve $\mathcal{C}$ — that is, turning the Laurent polynomial F into an operator and considering its spectral theory. The idea is to write $x_1 = e^{\mathsf{x}}$, $x_2 = e^{\mathsf{y}}$, and promote x, y to noncommuting operators $\hat{\mathsf{x}}, \hat{\mathsf{y}}$ on $L^2(\mathbb{R})$ with $[\hat{\mathsf{x}}, \hat{\mathsf{y}}] = \mathbf{i}\hbar$ ($\hbar \in \mathbb{R}$). More explictly, writing r for the coordinate on $\mathbb{R}$, we take $\hat{\mathsf{x}} = \mu_r$ (multiplication by r) and $\hat{\mathsf{y}} = -\mathbf{i}\hbar\partial_r$; and then we set $\hat{x}_1 = e^{\hat{\mathsf{x}}}$, $\hat{x}_2 = e^{\hat{\mathsf{y}}}$. Notice that if $f \in L^2(\mathbb{R})$ is the restriction of an entire function, then $\hat{x}_2$ is a shift operator, viz. $(e^{-\mathbf{i}\hbar\partial_r} f)(r) = f(r - \mathbf{i}\hbar)$.

The promotion of F to $\hat{F}$ is highly nonunique: for instance, $e^{\hat{\mathsf{x}}} e^{\hat{\mathsf{y}}}$ and $e^{\hat{\mathsf{x}}+\hat{\mathsf{y}}}$ [resp. $e^{\hat{\mathsf{y}}} e^{\hat{\mathsf{x}}}$] differ by a multiplicative factor of $e^{i\hbar/2}$ [resp. $e^{i\hbar}$] by the Campbell-Baker-Hausdorff formula. The

[2]We shall use the notation dP_6' to refer to the generalized del Pezzo of degree 6 defined by the self-dual polygon with vertices $(1,0)$, $(0,1)$, and $(-3,-2)$. (This is called the "E_8 del Pezzo" in [GKMR].

standard way to fix this (before [CGM]) was to employ a perturbative approach called WKB approximation, which works modulo successive powers of $\hbar$. In this context a connection between quantization and $K_2(\mathbb{C}(\mathcal{C}))$ was pointed out in [GS].

So suppose that we want a function ψ on $\mathcal{C}$ (rather than $\mathbb{R}$) and a choice of $\hat{F}$ given by $\hat{F}_0 := F(\hat{x}_1, \hat{x}_2) := F(\mu_{x_1}, e^{-\mathbf{i}\hbar\delta_{x_1}}) \bmod O(\hbar)$, for which $\hat{F}\psi = 0$. (In this case, we will say $\mathcal{C}$ is *quantizable*.) Begin with formal asymptotic expansions $\hat{F} = \sum_{i\geq 0} \hbar^i \hat{F}_i$, and $\psi = e^{\frac{\mathbf{i}}{\hbar}\sum_{j\geq 0}\hbar^j S_j}$. Choosing a base point $p_0 \in \mathcal{C}_F$ with $x_1(p_0) = 1$, we take $S_0(p) = \int_{p_0}^p \log(x_2)\frac{dx_1}{x_1}$ (integral on $\mathcal{C}$), which locally satisfies $\delta_{x_1} S_0 = \log(x_2)$ hence $(\hat{F}\psi)(p) = [F(x_1(p), x_2(p)) + O(\hbar)]\psi(p) = O(\hbar)\psi(p)$. Of course, $e^{\frac{\mathbf{i}}{\hbar}S_0}$ only gives a well-defined function on $\mathcal{C}$ if the integral is path-independent mod $2\pi\hbar\mathbb{Z}$. When this happens, one then solves for the higher-order corrections S_i, by postulating their form in terms of "topological recursion", and finally solves for the $\hat{F}_i$. We remark that for $\hbar = 2\pi$, the well-definedness condition on S_0 is precisely the statement that the regulator class $R\{x_1, x_2\} \in H^1(\mathcal{C}, \mathbb{C}/\mathbb{Z}(2))$ of the coordinate symbol $\{x_1, x_2\} \in K_2(\mathbb{C}(\mathcal{C}))$ is *trivial*. More generally, if the regulator class is *torsion* (which is the quantizability criterion proposed by [GS]), then the well-definedness condition is satisfied for $\hbar = \frac{2\pi}{M}$ for some $M \in \mathbb{Z}$. This is a very different condition on the regulator class than the one appearing in RHS(2.3.13) below, even in the $g = 1$ case.

For the rest of this paper we consider only the non-perturbative (exact) approach pioneered in [GHM]. Namely, we fix the single choice

$$\hat{F} = \textstyle\sum_{\underline{m}\in\Delta\cap\mathbb{Z}^2} a_{\underline{m}} e^{m_1\hat{\mathsf{x}} + m_2\hat{\mathsf{y}}} \tag{2.1.5}$$

and try to describe its spectrum as an operator on $L^2(\mathbb{R})$. A little more precisely, if $\mathrm{int}(\Delta)\cap\mathbb{Z}^2 = \{\underline{m}^{(j)}\}_{j=1,\ldots,g}$, then writing $a_j := a_{\underline{m}^{(j)}}$, $P_j = \underline{x}^{\underline{m}^{(j)}}$, $F_j^{(0)} = P_j^{-1}F|_{a_1=\cdots=a_g=0}$ and $F_j = P_j^{-1}F|_{a_j=0}$, we are interested in determining the eigenvalues $\{e^{E_n^{(j)}(a_1,\ldots,\widehat{a_j},\ldots,a_g)}\}_{n\in\mathbb{N}}$ of $\hat{F}_j$ for

$j = 1, \ldots, g$.[3] We should note here that as long as the $\{a_{\underline{m}}\}$ are all real, the $\hat{F}_j, \hat{F}_j^{(0)}$ are obviously Hermitian; even better, their inverses $\rho_j, \rho_j^{(0)}$ are expected to be bounded self-adjoint and of trace class, with a discrete positive spectrum. These properties, which justify indexing the eigenvalues by $\mathbb{N}$ and make the Fredholm determinants

$$\det(1 + a_j\rho_j) = \textstyle\prod_{n\geq 0}(1 + a_j e^{-E_n^{(j)}(a_1,\ldots,\widehat{a_j},\ldots,a_g)}) \tag{2.1.6}$$

well-defined, are proved in [KM] and [LST] for all the specific operators we will discuss below.

Definition 2.1.1 ([CGM]). The *generalized spectral determinant* is

$$\Xi_{\mathcal{C}}(\underline{a};\hbar) := \det(1 + \textstyle\sum_{j=1}^{g} a_j \hat{P}_j^{-\frac{1}{2}} \rho_j^{(0)} \hat{P}_j^{\frac{1}{2}}). \tag{2.1.7}$$

This function contains all the information we are after. For any fixed $\{a_k\}_{k\neq j}$, we may recover (2.1.6) as $\Xi_{\mathcal{C}}(\underline{a};\hbar)/(\Xi_{\mathcal{C}}(\underline{a};\hbar)|_{a_j=0})$, since their zeroes (in a_j) are the same and both sides are 1 at $a_j = 0$ [CGM, (2.74)]. So the spectra of $\hat{F}_1, \ldots, \hat{F}_g$ are simply slices of the zero-locus of (2.1.7), a union of hypersurfaces in $\mathbb{R}^g$ indexed by $\mathbb{N}$. Note that in the genus one case, (2.1.7) is just $\det(1 + a_1\rho_1)$.

2.2 Local mirror symmetry and Mariño's conjecture

Let $r := |\partial\Delta \cap \mathbb{Z}^2|$, so that $|\Delta \cap \mathbb{Z}^2| = g + r$; and denote by $\mathbb{L} \subset \mathbb{Z}^{g+r}$ the rank-$(g + r - 3)$ lattice of *relations vectors* $\{\ell_{\underline{m}}\}_{\underline{m}\in\Delta\cap\mathbb{Z}^2}$ with $\sum_{\underline{m}} \ell_{\underline{m}}(1, \underline{m}) = \underline{0}$. Each $\underline{m} \in \Delta \cap \mathbb{Z}$ corresponds to a toric divisor $D_{\underline{m}} \subset X$, amongst which we have the g compact $D_j := D_{\underline{m}^{(j)}}$. If $C \subset X$ is any *compact* toric curve (corresponding to any edge of $\mathrm{tr}(\Delta)$), its intersection numbers with the divisors of the toric coordinates w_0, w_1, w_2 are zero, leading to a relations vector

[3] For the time being, one should think of the non-interior parameters $a_{\underline{m}}$ as being fixed. For the assertion that the spectrum is positive and discrete, further restrictions (such as those we impose for temperedness later) should be made.

$\ell_{\underline{m}} = (C \cdot D_{\underline{m}})_X$. Such relations integrally span $\mathbb{L}$, although the (Mori) cone generated by *effective* curves may not be smooth or even simplicial. We will ignore such "finite data" issues here, as we will eventually pass to a slice of the complex-structure moduli space where this is not an issue.

So write $\{C_i\}_{i=1,\ldots,g+r-3}$ for independent generators of this cone (i.e. $H_2(X,\mathbb{Z})_{\text{eff}}$), with corresponding relations $\underline{\ell}^{(i)}$, and define complex structure parameters

$$z_i = z_i(\underline{a}) := \textstyle\prod_{\underline{m}\in\Delta\cap\mathbb{Z}^2} a_{\underline{m}}^{\ell^{(i)}_{\underline{m}}} \tag{2.2.1}$$

for $\mathcal{C}$ and Y. It is convenient at this stage to fix three vertices of Δ and set the corresponding $a_{\underline{m}}$'s equal to 1. We shall mainly work in a neighborhood of the *large complex structure limit* (LCSL) point $\underline{z} = \underline{0}$, though at times will also be concerned with the *maximal conifold* point $\underline{\hat{z}}$ — the unique point (if it exists) on the "boundary" of that neighborhood[4] where $\mathcal{C}$ develops g nodes (while remaining irreducible) hence has geometric genus zero.

What are the periods parametrized by (2.2.1)? We summarize some results from [BKV].[5] One may construct 3-cycles $\mathcal{T}, \mathcal{A}_1, \ldots, \mathcal{A}_{g+r-3}$ on Y such that near the LCSL

$$\int_{\mathcal{T}} \eta = 2\pi \mathbf{i}, \quad -t_i := \int_{\mathcal{A}_i} \eta \;\sim\; \log(z_i). \tag{2.2.2}$$

The *mirror map* $\underline{z} \mapsto e^{\underline{t}}$, which we usually express as $\underline{t}(\underline{z})$ (or $\underline{t}(\underline{a}) := \underline{t}(\underline{z}(\underline{a}))$) then induces a biholomorphism between neighborhoods of the LCSL and the large volume point (in Kähler

[4]i.e., the region of convergence for certain power series representing the periods of $\mathcal{C}$; see §4.

[5]While stated there for $g = 1$, the proof — by "limiting" results of [Ir] for compact CY 3-folds to the local setting — works for any Δ that makes the BKV polytope $\mathbb{A} := \{\text{the convex hull of } (-1,1,0,0), (2,-1,0,0), \text{ and } (-1,-1)\times\Delta \text{ in } \mathbb{R}^4\}$ reflexive. (For instance, take Δ to be the convex hull of $(1,0)$, $(0,1)$, and $(-g,-g)$ [resp. $(-n,-1)$] for $g \mid 6$ [resp. $n \mid 12$]). We also expect these results to hold more generally. A minor difference in formulation here is that instead of applying the BKV limit to derivatives of the prepotential Φ of a compact CY, we can directly take derivatives of F_0.

moduli space[6] of X). Next write

$$\mathcal{F}_0(\underline{t}) := \tfrac{1}{6}\sum_{\underline{i}} c_{i_1 i_2 i_3} t_{i_1} t_{i_2} t_{i_3} + \sum_{\underline{d}\in H_2(X,\mathbb{Z})_{\text{eff}}} N_{0,\underline{d}} e^{-\underline{d}\cdot\underline{t}} \tag{2.2.3}$$

for the *genus-zero free energy* of X, in which the $c_{\underline{i}} \in \mathbb{Q}$ are certain triple intersection numbers[7] and the $N_{0,\underline{d}} \in \mathbb{Q}$ are genus-zero local Gromov-Witten numbers. The basic *Hodge-theoretic* assertion of local mirror symmetry is that there are 3-cycles $\mathcal{B}_1, \ldots, \mathcal{B}_g$ on Y for which[8]

$$\int_{\mathcal{B}_j} \eta = \tfrac{1}{2\pi\mathbf{i}} \sum_{i=1}^{g+r-3} C_{ij} \partial_{t_i} \mathcal{F}_0(\underline{t}) - \tfrac{1}{2}\sum_{i=1}^{g+r-3} A_{ij} t_i + 2\pi\mathbf{i} T_j \tag{2.2.4}$$

under the mirror map, where $-C_{ij} = (\ell^{(i)}_{\underline{m}^{(j)}} =)\, C_i \cdot D_j$, $A_{ij} \underset{(2)}{\equiv}$ the coefficient of C_i in D_j^2, and $T_j \in \mathbb{Q}$.

The 3-cycles are constructed by describing $Y \to (\mathbb{C}^*)^2$ as a conic bundle, with fibers isomorphic to $\mathbb{C}^*$ over $(\mathbb{C}^*)^2 \setminus \mathcal{C}$, and to $\mathbb{C} \cup_0 \mathbb{C}$ (pair of complex lines crossing once) over $\mathcal{C}$. This yields (cf. [DK, §5.1]) an exact sequence of MHS

$$0 \to \mathbb{Q}(3) \overset{\mathtt{A}}{\to} H_3(Y) \overset{\mathtt{B}}{\to} \ker\{H_1(\mathcal{C}) \to H_1((\mathbb{C}^*)^2)\}(1) \to 0 \tag{2.2.5}$$

in which $\text{im}(\mathtt{A}) = \langle \mathcal{T} \rangle$ and the right-hand term has basis ($2\pi\mathbf{i}$ times) $\alpha_1, \ldots, \alpha_{g+r-3}, \beta_1, \ldots, \beta_g$. On the level of $\mathbb{Q}$-vector spaces, $\mathtt{B}$ has a section $\mathcal{M}$ sending this basis to the $\mathcal{A}_i = \mathcal{M}(\alpha_i)$ and $\mathcal{B}_j = \mathcal{M}(\beta_j)$. It is constructed by sending $\varphi \in \ker\{H_1(\mathcal{C},\mathbb{Q}) \to H_1((\mathbb{C}^*)^2,\mathbb{Q})\}$ first to its bounding $\mathbb{Q}$-chain Γ_φ in $(\mathbb{C}^*)^2$ (with $\partial\Gamma_\varphi = \varphi$), over which $\mathcal{M}(\varphi)$ is a 3-cycle with S^1 fibers (shrinking to points over φ). Writing $R\{f,g\} := \log(f)\frac{dg}{g} - 2\pi\mathbf{i}\log(g)\delta_{T_f}$ for the standard regulator current for Milnor K_2-symbols ($T_f := f^{-1}(\mathbb{R}_{<0})$ the cut in branch of log), we have

[6] If $\{\mathcal{J}_i\} \subset H^2(X)$ is a basis dual to $\{C_i\}$, then the Kähler parameter is $\sum_i \frac{-t_i}{2\pi\mathbf{i}}\mathcal{J}_i$.

[7] by interpreting X as a (decompactifying) limit of a compact CY and computing intersections $-\mathcal{J}_{i_1}\mathcal{J}_{i_2}\mathcal{J}_{i_3}$ there.

[8] The 2nd and 3rd terms are required in order for integrality of the periods, and arise from applying the procedure described in [BKV]; the second term arises from the fact that $\text{ch}(\mathcal{O}_{D_j}) \equiv [D_j] - \frac{1}{2}[D_j^2] \bmod \mathbb{Q}[p]$, where $[p]$ is the class of a point.

on $(\mathbb{C}^*)^2$ the relation $d[R\{-x,-y\}] = \frac{dx}{x} \wedge \frac{dy}{y} - (2\pi\mathbf{i})^2 \delta_{(\mathbb{R}_{>0})^2}$. This leads at once to

$$2\pi\mathbf{i} \int_{\mathcal{M}(\varphi)} \eta = \int_{\Gamma_\varphi} \frac{dx}{x} \wedge \frac{dy}{y} = \int_\varphi R\{-x,-y\} =: R_\varphi, \tag{2.2.6}$$

which is to say that $R_{\alpha_i} = -2\pi\mathbf{i}t_i$ and $R_{\beta_j} \equiv \sum_i C_{ij}\partial_{t_i}\mathcal{F}_0 - \pi\mathbf{i}\sum_i A_{ij}t_i \bmod \mathbb{Q}(2)$.

In the physics literature, the nontrivial $a_{\underline{m}}$ on the boundary are called *mass parameters*; if we write these as $a'_1, \ldots, a'_{r-3}$, then our complex structure parameters take the form $z_i = \prod_{j=1}^g a_j^{-C_{ij}} \times \prod_{k=1}^{r-3} a'^{\,C'_{ik}}_k$. Taking the $a_j \gg 0$ large but keeping the a'_k bounded, so that $t_i \sim \sum_{j=1}^g C_{ij}\log(a_j)$, the subleading terms (constant in $\underline{a}$) can be shown[9] to be $\mathbb{Q}$-linear combinations of logarithms of the negative roots $\{q_k\}_{k=1,\ldots,r}$ of the edge polynomials of F. (The latter are defined as follows: if $\mathbf{e}$ is an edge of Δ, with vertex $\underline{\nu}$, and $\underline{m}^{\mathbf{e}} \in \mathbb{Z}^2$ is a primitive lattice vector along $\mathbf{e}$, then put $P_{\mathbf{e}}(w) := \sum_{\underline{m}\in\mathbf{e}\cap\mathbb{Z}^2} a_{\underline{m}} w^{(\underline{m}-\underline{\nu})/\underline{m}^{\mathbf{e}}}$.) The key observation is that each q_k is the Tame symbol of $\{-x,-y\} \in K_2(\mathcal{C})$ at a point $p_k \in \bar{\mathcal{C}} \cap (\mathbb{P}_\Delta \setminus (\mathbb{C}^*)^2)$, so that a loop $\varepsilon_k \subset \mathcal{C}$ around p_k has $\int_{\varepsilon_k} R\{-x,-y\} = 2\pi\mathbf{i}\log(q_k)$.

The physicists have a *grand potential function* $J_X(\underline{t};\hbar)$ which says "everything they know how to say" about enumerative geometry of X, and includes (refinements of) higher-genus GW-invariants. We refer the reader to [CGM] for details, as we shall only discuss two special cases in which those invariants (mostly) drop out. First, in the *maximally supersymmetric* case $\hbar = 2\pi$, we have[10]

$$\begin{aligned} J_X(\underline{t};2\pi) =& \tfrac{1}{8\pi^2}\left\{\textstyle\sum_{i_1,i_2}\delta_{t_{i_1}}\delta_{t_{i_2}} - 3\sum_i \delta_{t_i} + 2\right\}\hat{\mathcal{F}}_0(\underline{t}) \\ &+ \hat{\mathcal{F}}_1(\underline{t}) + \hat{\mathcal{F}}_1^{\mathrm{NS}}(\underline{t}) + A(\underline{q},2\pi), \end{aligned} \tag{2.2.7}$$

[9]Done from a physics perspective in [GKMR], and from a regulator perspective in Appendix A. Here "negative roots" means the roots of $P_{\mathbf{e}}(-w)$. In particular, if edge polynomials are powers of $(1+w)$, the q_k are all 1.

[10]Remark that $\underline{q}$ is an abuse of notation since the q_k are B-model coordinates; one would ideally replace them by monomials in the e^{t_i} which equal q_k under the mirror map. (Similar remarks apply to $\underline{\mathsf{m}}$ in (2.2.8).) But we don't need to be more precise here as these terms quickly become irrelevant.

where $\hat{\mathcal{F}}_0, \hat{\mathcal{F}}_1, \hat{\mathcal{F}}_1^{\mathrm{NS}}$ are free energies in which the instanton part is twisted by a "B-field" $\underline{\mathbb{B}} \in \mathbb{Z}^{g+r-3}$:[11]

- $\hat{\mathcal{F}}_0(\underline{t}) = \frac{1}{6}\sum_{\underline{i}} c_{\underline{i}} t_{i_1} t_{i_2} t_{i_3} + \sum_{\underline{d}} N_{0,\underline{d}} e^{-\underline{d}\cdot(\underline{t}-\pi\mathbf{i}\underline{\mathbb{B}})}$;
- $\hat{\mathcal{F}}_1(\underline{t}) = \sum_i b_i t_i + F_1^{\mathrm{inst}}(\underline{t} - \pi\mathbf{i}\underline{\mathbb{B}})$; and
- $\hat{\mathcal{F}}_1^{\mathrm{NS}}(\underline{t}) = \sum_i b_i^{\mathrm{NS}} t_i + F_1^{\mathrm{NS,\,inst}}(\underline{t} - \pi\mathbf{i}\underline{\mathbb{B}})$.

In the *'t Hooft limit*, where $\hbar \to \infty$ (and $a_j \to \infty$) while $\mathsf{m}_k := e^{-\frac{2\pi}{\hbar}\log(q_k)}$, $\zeta_j := \frac{\log(a_j)}{\hbar}$, and $\tau_i := \frac{2\pi t_i}{\hbar}$ remain finite, one finds that

$$\hbar^{-2} J_X(\underline{t};\hbar) = \underbrace{\left\{\tfrac{1}{16\pi^4}\hat{\mathcal{F}}_0(\underline{\tau}) + \tfrac{1}{4\pi^2}\sum_i b_i^{\mathrm{NS}}\tau_i + A_0(\underline{\mathsf{m}})\right\}}_{=:J_0^X(\underline{\zeta},\underline{\mathsf{m}})} + O(\hbar^{-2}). \tag{2.2.8}$$

We may disregard the unknown functions $A_0(\underline{\mathsf{m}}), A(\underline{q}, 2\pi)$ of the mass parameters.

To state the main physics conjecture, we need two more ingredients. First is the *quantum theta function*

$$\Theta_X(\underline{t};\hbar) := \sum_{\underline{n}\in\mathbb{Z}^g} \exp\left\{J_X(\underline{t} + 2\pi\mathbf{i}[C]\underline{n};\hbar) - J_X(\underline{t};\hbar)\right\}, \tag{2.2.9}$$

where $[C]$ is the matrix C_{ij} (and so $[C]\underline{n}$ is a $(g+r-3)$-vector with entries $\sum_{j=1}^g C_{ij} n_j$). Terms in J_X which are $2\pi\mathbf{i}$-periodic in the $\{t_i\}$, including all but $\sum_i (b_i + b_i^{\mathrm{NS}}) t_i$ in the second line of (2.2.7), drop out. The second is a "quantum deformation" $\underline{\mathsf{t}}^\hbar(\underline{z}) = \underline{t}(\underline{z}) + O(\hbar)$ of the mirror map. (We shall also write $\underline{\mathsf{t}}^\hbar(\underline{a}) := \underline{\mathsf{t}}^\hbar(\underline{z}(\underline{a}))$ where convenient.) Again, we describe this where we need it: at $\hbar = 2\pi$ it is given by

$$\mathsf{t}_i(\underline{z}) := \mathsf{t}_i^{2\pi}(\underline{z}) = t_i((-1)^{\underline{\mathbb{B}}}\underline{z}) + \pi\mathbf{i}\mathbb{B}_i; \tag{2.2.10}$$

[11] In the $g = 1$ case, $\underline{\mathbb{B}}_i$ is just C_{i1}; see §2.3 below and [SWH] for $g > 1$.

like $t_i(z)$, this is asymptotic to $-\log(z_i)$, but the signs are (in general) different in the power-series part. In the 't Hooft limit, the previous asymptotic relation $t_i \sim \sum_j C_{ij}\log(a_j) + \sum_k D_{ik}\log(q_k)$ becomes exact in the sense that

$$\tau_i = 2\pi \textstyle\sum_j C_{ij}\zeta_j - \sum_k D_{ik}\log(\mathsf{m}_k). \tag{2.2.11}$$

Conjecture 2.2.1 ([GHM],[CGM])**.** *Under the quantum mirror map, the generalized spectral determinant of $\mathcal{C}$ is given (up to a nonvanishing factor) by the quantum theta function of its mirror:*

$$\Xi_{\mathcal{C}}(\underline{a};\hbar) = e^{J_X(\underline{\mathfrak{t}}^{\hbar}(\underline{a});\hbar)}\Theta_X(\underline{\mathfrak{t}}^{\hbar}(\underline{a});\hbar). \tag{2.2.12}$$

This postulates a *fundamental and very general* relation between spectral theory (of the B-model) and enumerative geometry (of the A-model). Since local mirror symmetry relates the latter to Hodge theory of the B-model, it should imply relationships between Hodge/K-theory and spectral theory of our curves with no reference to mirror symmetry. We now derive these in our two special cases, under the assumption that F is *integrally tempered*: all $q_k = 1 = \mathsf{m}_k$; equivalently, all edge polynomials of F are powers of $w+1$. Accordingly, by $\underline{a}$ (resp. $\underline{z}(\underline{a})$) we henceforth shall mean just $(a_1,\ldots,a_g)$, with the remaining $\{a_{\underline{m}}\}$ determined uniquely by this constraint.

2.3 Consequences in the "maximal SUSY" case

Of course, the last paragraph was a bit glib, since the classical and quantum mirror maps are not the same. One should rather expect a relation between Hodge theory of $\mathcal{C}_{\underline{z}}$ and spectral theory of a "partner" $\mathcal{C}_{\underline{z}'}$ given by $\underline{z} = \underline{t}^{-1}(\underline{\mathfrak{t}}^{\hbar}(\underline{z}'))$ or some variant thereof. (In fact this is still too vague, since the spectral theory and the regulator class really depend on $\underline{a}$.) We now work this out at $\hbar = 2\pi$.

First we address the nature and significance of $\underline{\mathbb{B}}$. Because the monomials $\underline{x}^{\underline{m}}$ in $\hat{F}$ were quantized as $e^{m_1\hat{\mathsf{x}}+m_2\hat{\mathsf{y}}} = e^{\frac{\mathbf{i}\hbar}{2}m_1m_2}\hat{x}_1^{m_1}\hat{x}_2^{m_2}$, at $\hbar = 2\pi$ we have $\hat{F} = \sum_{\underline{m}}(-1)^{m_1m_2}a_{\underline{m}}\underline{\hat{x}}^{\underline{m}}$. The B-field is determined mod 2 by the effect on the signs of the z_i were we to replace $a_{\underline{m}}$ by $(-1)^{m_1m_2}a_{\underline{m}}$: namely, $\mathbb{B}_i \underset{(2)}{\equiv} \sum_{\underline{m}} m_1m_2\ell^{(i)}_{\underline{m}}$. Under the assumption that

$$\partial\Delta \cap (2\mathbb{Z}\times 2\mathbb{Z}) = \emptyset, \tag{2.3.1}$$

this is compatible with taking $\underline{\mathbb{B}}$ to be in the $\mathbb{Z}$-span of the columns of $[C]$, which we write $\mathbb{B}_i = \sum_{j=1}^{g}\mathbb{A}_jC_{ij}$.[12] Notice that then $\underline{t}((-1)^{\underline{\mathbb{A}}}\underline{a}) = (-1)^{\underline{\mathbb{B}}}\underline{t}(\underline{a})$, so that by (2.2.10) we have $\underline{\mathfrak{t}}^{2\pi}((-1)^{\underline{\mathbb{A}}}\underline{a}) = \underline{t}(\underline{a}) + \pi\mathbf{i}\underline{\mathbb{B}}$ and the conjectured equality (2.2.12) becomes

$$\Xi_{\mathcal{C}}((-1)^{\underline{\mathbb{A}}}\underline{a}; 2\pi) = e^{J_X(\underline{t}(\underline{a})+\pi\mathbf{i}\underline{\mathbb{B}};2\pi)}\Theta_X(\underline{t}(\underline{a}) + \pi\mathbf{i}\underline{\mathbb{B}}; 2\pi). \tag{2.3.2}$$

That is, after absorbing the "$+\pi\mathbf{i}\underline{\mathbb{B}}$" twist into Θ_X and J_X, our Hodge/ spectral "partners" are related by at most a change of sign in the complex structure parameters. The main question is what the *quantization condition* looks like: which values of $\underline{a}$ make $\Theta_X(\underline{t}(\underline{a}) + \pi\mathbf{i}\underline{\mathbb{B}}; 2\pi)$, hence the spectral determinant, zero?

This is where the local mirror symmetry enters. Under our assumption (2.3.1), its previous incarnation in (2.2.4) can (by a tedious intersection theory argument) be expressed as[13]

$$R_{\beta_j}(\underline{a}) = \textstyle\sum_i C_{ij}\partial_{t_i}\hat{\mathcal{F}}_0\left(\underline{t}(\underline{a}) + \pi\mathbf{i}\underline{\mathbb{B}}\right) + (2\pi\mathbf{i})^2\mathtt{B}^{\circ}_j \qquad (\mathtt{B}^{\circ}_j \in \mathbb{Q}). \tag{2.3.3}$$

Next, since our temperedness assumption has eliminated the Tame symbols, the $\{R_{\alpha_i}\}_{i=1}^{g+r-3}$ are no longer independent (unless $r = 3$). More precisely, there are g cycles $\gamma_j \in H_1(\bar{\mathcal{C}}, \mathbb{Z})$

[12] mod 2, $\underline{\mathbb{A}}$ is just the characteristic function of $\Delta \cap (2\mathbb{Z}\times 2\mathbb{Z})$.

[13] Although the regulator periods R_φ [resp. periods $\Omega_{j_1j_2}$ in (2.3.7) below] are infinitely multivalued, they are periods of a class $\mathcal{R}$ [resp. classes $\{\omega_j\}$] which are single-valued in $\underline{a}$ [resp. $\underline{z}$]; so we shall loosely write them as functions thereof.

with regulator periods $R_{\gamma_j} \sim -2\pi\mathbf{i}\log(a_j)$ (cf. Appendix A), whence

$$R_{\alpha_i} = \sum_j C_{ij} R_{\gamma_j}; \tag{2.3.4}$$

and the $\mathbb{A}_j$ can be chosen so that $\{\gamma_j, \beta_j\}_{j=1}^g$ is a symplectic basis.[14] The regulator class $\mathcal{R} = R\{-x_1, -x_2\} \in H^1(\bar{\mathcal{C}}, \mathbb{C}/\mathbb{Z}(2))$ then has a local lift[15] to $H^1(\bar{\mathcal{C}}, \mathbb{C})$ given by

$$\tilde{\mathcal{R}} = \sum_{\ell=1}^g \left(R_{\gamma_\ell}\gamma_\ell^* + R_{\beta_\ell}\beta_\ell^*\right), \tag{2.3.5}$$

whose Gauss-Manin derivatives

$$\omega_j := \nabla_{\partial/\partial R_{\gamma_j}} \tilde{\mathcal{R}} = \gamma_j^* + \sum_{\ell=1}^g \frac{\partial R_{\beta_\ell}}{\partial R_{\gamma_j}} \beta_\ell^* \tag{2.3.6}$$

are classes of holomorphic 1-forms by Griffiths transversality. Evidently these are normalized so that the symmetric $g \times g$ matrix

$$\begin{aligned} \Omega_{j_1 j_2}(\underline{z}) :&= -\tfrac{1}{2\pi\mathbf{i}} \sum_{i_1,i_2} C_{i_1 j_1} C_{i_2 j_2} \partial_{t_{i_1}} \partial_{t_{i_2}} \hat{\mathcal{F}}_0(\underline{t}(\underline{z}) + \pi\mathbf{i}\underline{\mathbb{B}}) \\ &= -\tfrac{1}{2\pi\mathbf{i}} \sum_{i_1} C_{i_1 j_1} \partial_{t_{i_1}} R_{\beta_{j_2}} = \sum_{i_1} C_{i_1 j_1} \frac{\partial R_{\beta_{j_2}}}{\partial R_{\alpha_{i_1}}} \\ &= \frac{\partial R_{\beta_{j_2}}}{\partial R_{\gamma_{j_1}}} = \int_{\gamma_{j_1}} \omega_{j_2} \end{aligned} \tag{2.3.7}$$

is the standard period matrix of $\bar{\mathcal{C}}$.

We have already observed that the isomorphism class of $\bar{\mathcal{C}}$ depends only on $\underline{z}$, which parametrizes the standard coarse moduli space for toric hypersurfaces; and we are restricting to a "tempered slice" of this space. However, $\mathcal{R}$ only becomes single-valued in $\underline{a}$, forcing

[14] This is again by local mirror symmetry: the R_{γ_j} [resp. R_{α_i}] are the A-model periods of flat sections arising from curves dual to the D_j [resp. $\mathcal{J}_i$]; while the R_{β_j} are those arising from $\mathrm{ch}(\mathcal{O}_{D_j}(-E_j)) \cup \hat{\Gamma}(X)$ for suitable curves E_j.

[15] For our purposes, this can be regarded as living on an open neighborhood (in $\underline{z}$-space $\mathbb{C}^g$) of $(0, \epsilon)^g$ for some $\epsilon > 0$.

us to work on the finite cover $\boldsymbol{\mathcal{M}} := \{\underline{a} \in (\mathbb{C}^*)^g \mid \mathcal{C}_{\underline{z}(\underline{a})} \text{ is smooth}\}$ of this slice. Let $\bar{\boldsymbol{\mathcal{C}}} \overset{\pi}{\to} \boldsymbol{\mathcal{M}}$ be the universal (compactified) curve, and set $\mathscr{H} := R^1\pi_*\mathbb{C} \otimes \mathcal{O}_{\boldsymbol{\mathcal{M}}}$, $\mathbb{H} := R^1\pi_*\mathbb{Z}$, and $\mathscr{J} := \boldsymbol{\mathcal{H}}/\{\mathbb{H} + \boldsymbol{\mathcal{F}}^1\boldsymbol{\mathcal{H}}\}$. Then $\mathscr{J}$ is the sheaf of sections of the Jacobian bundle $\boldsymbol{\mathcal{J}} \overset{\rho}{\to} \boldsymbol{\mathcal{M}}$, and $\mathscr{H}/\mathbb{H}$ is the sheaf of sections of the $\mathbb{C}/\mathbb{Z}$ cohomology bundle $\boldsymbol{\mathcal{H}}^1_{\mathbb{C}/\mathbb{Z}} \to \boldsymbol{\mathcal{M}}$, which factors through the obvious $\mathbb{C}^g$-torsor $\boldsymbol{\mathcal{H}}^1_{\mathbb{C}/\mathbb{Z}} \overset{\varpi}{\to} \boldsymbol{\mathcal{J}}$. By temperedness, the symbol $\{-x_1, -x_2\} \in K_2(\mathbb{C}(\boldsymbol{\mathcal{C}}))$ lifts to a motivic cohomology class $\boldsymbol{\mathcal{Z}} \in H^2_{\mathcal{M}}(\bar{\boldsymbol{\mathcal{C}}}, \mathbb{Z}(2))$, and we make the key

Definition 2.3.1. By the *higher normal function* associated to $\boldsymbol{\mathcal{Z}}$, we shall mean the well-defined section $\frac{1}{(2\pi\mathbf{i})^2}\mathcal{R}$ of $\boldsymbol{\mathcal{H}}^1_{\mathbb{C}/\mathbb{Z}}$, or its projection $\boldsymbol{\nu} := \boldsymbol{\varpi}(\frac{1}{(2\pi\mathbf{i})^2}\mathcal{R})$ to a section of $\boldsymbol{\mathcal{J}}$. The latter is computed by evaluating $\mathcal{R}$ as a functional on holomorphic 1-forms (modulo periods), i.e. by the column vector

$$\begin{aligned}\nu_j &:= \tfrac{1}{(2\pi\mathbf{i})^2}\langle \mathcal{R}, \omega_j\rangle \quad (j = 1, \ldots, g)\\ &= \tfrac{-1}{4\pi^2}\textstyle\sum_{\ell=1}^g \langle R_{\gamma_\ell}\gamma_\ell^* + R_{\beta_\ell}\beta_\ell^*,\, \gamma_j^* + \sum_{\ell'} \Omega_{j\ell'}\beta_{\ell'}^*\rangle \\ &= \tfrac{1}{4\pi^2}(\textstyle\sum_{\ell=1}^g R_{\gamma_\ell}\Omega_{j\ell} - R_{\beta_j})\end{aligned} \tag{2.3.8}$$

modulo the $\mathbb{Z}$-span of columns of $(\mathbb{I}_g \mid \Omega)$.

To use mirror symmetry to compute $\underline{\nu}$, put $\tilde{R}_{\beta_j} := R_{\beta_j} - (2\pi\mathbf{i})^2\mathtt{T}_j$, and observe that by (2.3.3) thru (2.3.7) (together with $\Omega_{jj'} = \Omega_{j'j}$)

$$\begin{aligned}\xi_j(\underline{a}) &:= \tfrac{1}{4\pi^2}\textstyle\sum_{i_1} C_{i_1 j}(\sum_{i_2}\delta_{t_{i_2}} - 1)\partial_{t_{i_1}}\hat{\mathcal{F}}_0(\underline{t}(\underline{a}) + \pi\mathbf{i}\mathbb{B})\\ &= \tfrac{1}{4\pi^2}(\textstyle\sum_i \delta_{t_i} - 1)\tilde{R}_{\beta_j} = \tfrac{1}{4\pi^2}(\tfrac{-1}{2\pi\mathbf{i}}\sum_i R_{\alpha_i}\partial_{t_i}R_{\beta_j} - \tilde{R}_{\beta_j})\\ &= \tfrac{1}{4\pi^2}(\tfrac{-1}{2\pi\mathbf{i}}\textstyle\sum_{i,\ell} C_{i\ell}R_{\gamma_j}\partial_{t_i}R_{\beta_j} - \tilde{R}_{\beta_j})\\ &= \tfrac{1}{4\pi^2}(\textstyle\sum_\ell R_{\gamma_\ell}\Omega_{j\ell} - \tilde{R}_{\beta_j}) \;=\; \nu_j - \mathtt{B}_j^\circ.\end{aligned} \tag{2.3.9}$$

Returning to the quantization condition, the exponent in (2.2.9) is

$$J_X(\underline{t}+2\pi\mathbf{i}[C]\underline{n};2\pi)-J_X(\underline{t};2\pi)$$
$$=\pi\mathbf{i}\,{}^t\underline{n}[\hat{\Omega}]\underline{n}+2\pi\mathbf{i}\underline{n}\cdot\hat{\xi}-\tfrac{\pi\mathbf{i}}{3}\textstyle\sum_{\underline{i},\underline{j}}c_{\underline{i}}\prod_{\ell=1}^{3}C_{i_\ell j_\ell}n_{j_\ell}, \quad (2.3.10)$$

where

- $\hat{\Omega}_{j_1j_2}:=\frac{-1}{2\pi\mathbf{i}}\sum_{i_1,i_2}C_{i_1j_1}C_{i_2j_2}\partial_{t_{i_1}}\partial_{t_{i_2}}\hat{\mathcal{F}}_0(\underline{t})$ and
- $\hat{\xi}_j:=\frac{1}{4\pi^2}\sum_{i_1}C_{i_1j}(\sum_{i_2}\delta_{t_{i_2}}-1)\partial_{t_{i_1}}\hat{\mathcal{F}}_0(\underline{t})+\sum_i C_{ij}(b_i+b_i^{\mathrm{NS}})$

by a straightforward computation, cf. [CGM, (3.28)]. Substituting in $\underline{t}=\underline{t}(\underline{a})+\pi\mathbf{i}\underline{\mathbb{B}}$, the first two terms of (2.3.10) become

$$\pi\mathbf{i}\,{}^t\underline{n}[\Omega(\underline{a})]\underline{n}+2\pi\mathbf{i}\underline{n}\cdot(\underline{\nu}(\underline{a})+\underline{\mathtt{B}}+\tfrac{1}{2}[\Omega(\underline{a})]\underline{\mathbb{A}}) \quad (2.3.11)$$

(for $\underline{\mathtt{B}}\in\mathbb{Q}^g$) by (2.3.7)-(2.3.9). By an intersection theory argument and the identity $n^3\underset{(6)}{\equiv}n$, the cubic third term becomes $-\frac{\pi\mathbf{i}}{3}\sum_j n_jD_j^3 \bmod \mathbb{Z}(1)$, which may be absorbed into $\underline{\mathtt{B}}$. Therefore, writing $\underline{\mathtt{A}}:=\frac{1}{2}\underline{\mathbb{A}}$ and θ for the usual Jacobi theta function,

$$\Theta_X(\underline{t}(\underline{a})+\pi\mathbf{i}\underline{\mathbb{B}};2\pi)=\theta(\underline{\nu}(\underline{a})+\underline{\mathtt{B}}+[\Omega(\underline{a})]\underline{\mathtt{A}},[\Omega(\underline{a})]). \quad (2.3.12)$$

We have thus deduced from Conjecture 2.2.1 a striking relationship between the quantization condition and the higher normal function. Let $\boldsymbol{\mathcal{D}}_\theta\subset\boldsymbol{\mathcal{J}}$ be the theta divisor and $\boldsymbol{\mathcal{D}}_\theta[{}^{\underline{\mathtt{A}}}_{\underline{\mathtt{B}}}]$ its translate by (minus) the torsion section $\underline{\mathtt{B}}+[\Omega]\underline{\mathtt{A}}$.

Conjecture 2.3.2. *For Δ satisfying* (2.3.1) *and F integrally tempered, the zero-locus of the twisted spectral determinant $\Xi_{\mathcal{C}}((-1)^{\underline{\mathbb{A}}}\underline{a};2\pi)$ is exactly the locus where the normal function* *meets this torsion shift of the theta divisor: as subsets of $\boldsymbol{\mathcal{M}}$, we have*

$$\mathrm{ZL}\left(\Xi_{\mathcal{C}}((-1)^{\underline{\mathbb{A}}}\underline{a}; 2\pi)\right) = \boldsymbol{\rho}\left(\boldsymbol{\nu}(\boldsymbol{\mathcal{M}}) \cap \boldsymbol{\mathcal{D}}_{\theta}[\begin{smallmatrix}\underline{\mathbb{A}}\\ \underline{\mathbb{B}}\end{smallmatrix}]\right). \tag{2.3.13}$$

In genus $g = 1$, there are 15 reflexive polygons (up to unimodular transformation) which can be presented inside $\mathbb{R} \times [-1, 1]$. After making the torsion shifts completely explicit in §3.1, we prove the "$\supseteq$" direction of (2.3.13) for these cases in §3.2.

2.4 Consequences in the 't Hooft limit.

The spectral determinant $\Xi_{\mathcal{C}}$ has *fermionic spectral traces* which generalize, from the $(g = 1)$ case of a single operator, the traces of $\rho_1^{\otimes N}$ acting on $\bigwedge^N L^2(\mathbb{R})$, cf. [CGM, §2.3]. Defined by

$$\Xi_{\mathcal{C}}(\underline{a}; \hbar) =: \textstyle\sum_{N_1,\ldots,N_g \geq 0} Z_{\mathcal{C}}(\underline{N}, \hbar)\underline{a}^{\underline{N}}, \tag{2.4.1}$$

these can clearly also be expressed in terms of loop integrals about 0:

$$Z_{\mathcal{C}}(\underline{N}, \hbar) = \frac{1}{(2\pi\mathbf{i})^g} \oint \cdots \oint \Xi_{\mathcal{C}}(\underline{a}; \hbar) \frac{da_1}{a_1^{N_1+1}} \wedge \cdots \wedge \frac{da_g}{a_g^{N_g+1}}. \tag{2.4.2}$$

Applying Conjecture 2.2.1 replaces $\Xi_{\mathcal{C}}(\underline{a}; \hbar)$ by $\sum_{\underline{n} \in \mathbb{Z}^g} e^{J_X(\underline{\mathfrak{t}}^{\hbar}(\underline{a}) + 2\pi\mathbf{i}[C]\underline{n}; \hbar)}$, where the $2\pi\mathbf{i}[C]\underline{n}$ simply accounts for the change in $\underline{\mathfrak{t}}^{\hbar}(\underline{a})$ as the a_j go n_j times around 0 — or equivalently, as $\mu_j := \log(a_j)$ increases by $2\pi\mathbf{i}n_j$ (for each j). Accordingly, (2.4.2) becomes

$$\tfrac{1}{(2\pi\mathbf{i})^g} \textstyle\int_{-\mathbf{i}\infty}^{\mathbf{i}\infty} \cdots \int_{-\mathbf{i}\infty}^{\mathbf{i}\infty} e^{J_X(\underline{\mathfrak{t}}^{\hbar}(\underline{a}); \hbar) - \sum_{j=1}^{g} N_j \mu_j} d\mu_1 \wedge \cdots \wedge d\mu_g, \tag{2.4.3}$$

Recall from §2.2 that the 't Hooft limit takes $\hbar \to \infty$ while essentially fixing $\zeta_j = \frac{\mu_j}{\hbar}$ and $\tau_i = \frac{2\pi t_i}{\hbar}$, which we will also impose on $\lambda_j := \frac{N_j}{\hbar}$. As temperedness makes the $q_k = 1$ hence $\mathsf{m}_k = 1$, we write $J_0^X(\underline{\zeta}) := J_0^X(\underline{\zeta}, \underline{1})$, and note that (2.2.11) reduces to $\tau_i = 2\pi \sum_j C_{ij}\zeta_j$.

Remark 2.4.1. In fact, even if we don't assume temperedness, but *fix the edge polynomials* hence the $\{q_k\}$, the effect is the same since $\mathsf{m}_k (= e^{-\frac{2\pi}{\hbar}\log(q_k)}) = 1$ in the limit.

Now by (2.2.8), for $\hbar \gg 0$ (2.4.3) becomes

$$\tfrac{\hbar^g}{(2\pi\mathbf{i})^g} \int_{-\mathbf{i}\infty}^{\mathbf{i}\infty} \cdots \int_{-\mathbf{i}\infty}^{\mathbf{i}\infty} e^{\hbar^2\{J_0^X(\underline{\zeta}) - \sum_j \lambda_j \zeta_j + O(\hbar^{-2})\}} d\zeta_1 \wedge \cdots \wedge d\zeta_g; \tag{2.4.4}$$

and we write $\underline{\hat{\zeta}}(\underline{\lambda})$ for the stationary point of (the leading part of) the exponential, where $0 = \partial_{\zeta_i}(J_0^X(\underline{\zeta}) - \sum_j \lambda_j\zeta_j)$, or equivalently $\lambda_j = \partial_{\zeta_j} J_0^X(\underline{\zeta})$, for each j. By the saddle-point method, we can write (2.4.4) as $\exp(\hbar^2\{J_0^X(\underline{\hat{\zeta}}(\underline{\lambda})) - \sum_j \lambda_j \hat{\zeta}_j(\underline{\lambda}) + O(\hbar^{-2})\})$, which is to say that

$$\lim_{\hbar\to\infty} (\partial_{\lambda_j} \hbar^{-2} \log Z_{\mathcal{C}}(\hbar\underline{\lambda}, \hbar))|_{\underline{\lambda}=\underline{0}} = -\hat{\zeta}_j(\underline{0}). \tag{2.4.5}$$

Moreover, according to [CGM, §2.3], $\hat{\tau}_i(\underline{\lambda}) = 2\pi \sum_j C_{ij}\hat{\zeta}_j(\underline{\lambda})$ is nothing but the classical mirror map in the "conifold frame", with $\underline{\lambda}$ a parameter which vanishes at the maximal conifold point $\underline{\hat{z}}$.[16] In other words, if $\underline{\hat{a}}$ is any preimage of $\underline{\hat{z}}$ in $\overline{\boldsymbol{\mathcal{M}}}$, then we have $R_{\alpha_i}(\underline{\hat{a}}) \equiv -2\pi\mathbf{i}\hat{\tau}_i(\underline{0})$ and

$$R_{\gamma_j}(\underline{\hat{a}}) \equiv -4\pi^2\mathbf{i}\hat{\zeta}_j(\underline{0}) \mod \mathbb{Q}(2). \tag{2.4.6}$$

On the other hand, if we set $N_j = 0$ for $j > 1$, then the asymptotic expansion of $Z_{\mathcal{C}}(N_1, 0\ldots, 0; \hbar) = \mathrm{tr}_{\wedge^{N_1} L^2(\mathbb{R})}((\rho_1^{(0)})^{\otimes N_1})$ can be computed via operator theory and asymptotic properties of the quantum dilogarithm. This is worked out in [KM, MZ] for the three-term operators $(\rho_1^{(0)})^{-1} = e^{\hat{\mathsf{x}}} + e^{\hat{\mathsf{y}}} + e^{-m\hat{\mathsf{x}}-n\hat{\mathsf{y}}}$, corresponding to the Laurent polynomials

$$F^{\circ}_{m,n}(\underline{x}) := x_1 + x_2 + x_1^{-m}x_2^{-n} + \textstyle\sum_{j=1}^g a_j x_1^{m_1^{(j)}} x_2^{m_2^{(j)}}. \tag{2.4.7}$$

[16]We are not aware of a proof of this statement, but there is strong computational evidence; it is also consistent with the observation, in view of (2.3.3), that the vanishing of $\partial_{\zeta_j} J_0^X(\underline{\zeta})$ at $\underline{\hat{\zeta}}(\underline{0})$ is equivalent to that of a $\mathbb{Q}(2)$-translate of $R_{\beta_j}(\underline{a})$ at $\underline{a} \in \underline{t}^{-1}(\underline{\hat{\tau}}(\underline{0}) - \pi\mathbf{i}\mathbb{B})$. This is exactly what should happen at a g-nodal fiber.

(Here we recall that the $\{\underline{m}^{(j)}\}$ index the interior integral points of Δ; for instance, if $m = n = g$, then $\underline{m}^{(j)} = (1-j, 1-j)$.) Note that by Remark 2.4.1, $\hat{\underline{\tau}}(\underline{\lambda})$ will actually compute the mirror map/regulator periods in the conifold frame *for the families defined by the integrally tempered polynomials*[17]

$$\begin{aligned} F_{m,n}(\underline{x}) &:= x_1 + x_2 + x_1^{-m}x_2^{-n} + \textstyle\sum_{j=1}^{g} a_j x_1^{m_1^{(j)}} x_2^{m_2^{(j)}} \\ &+ \textstyle\sum_{\ell=1}^{g_1-1} \binom{g_1}{\ell} x_1^{1-\ell\frac{m+1}{g_1}} x_2^{-\ell\frac{n}{g_1}} + \textstyle\sum_{\ell=1}^{g_2-1} \binom{g_2}{\ell} x_1^{-\ell\frac{m}{g_2}} x_2^{1-\ell\frac{n+1}{g_2}}, \end{aligned} \tag{2.4.8}$$

where $g_1 := \gcd(m+1, n)$ and $g_2 = \gcd(m, n+1)$. Anyway, the result of [op. cit.] (see also [Ma, §4.3]) is that

$$\begin{aligned} &\lim_{\hbar\to\infty} (\partial_{\lambda_1} \hbar^{-2} \log Z_{\mathcal{C}}(\hbar\lambda_1, 0, \ldots, 0; \hbar))|_{\lambda_1=0} \\ &\qquad = \tfrac{m+n+1}{2\pi^2} D_2(-\mathfrak{z}_{m,n}^{m+1}\mathfrak{w}_{m,n}), \end{aligned} \tag{2.4.9}$$

where D_2 is the Bloch-Wigner function, $\mathfrak{z}_{m,n} := e^{\frac{\pi \mathbf{i}}{m+n+1}}$, and $\mathfrak{w}_{m,n} := \frac{\mathfrak{z}_{m,n}^{m} - \mathfrak{z}_{m,n}^{-m}}{\mathfrak{z}_{m,n} - \mathfrak{z}_{m,n}^{-1}}$. Since LHS(2.4.9) must agree with LHS(2.4.5) (with $j = 1$), in view of (2.4.6) we arrive at

Conjecture 2.4.2. *For the families $\mathcal{C}_{m,n}$ arising from* (2.4.8)*, the regulator period R_{γ_1} asymptotic to $-2\pi\mathbf{i}\log(a_1)$ at the origin has value*

$$\tfrac{1}{2\pi\mathbf{i}} R_{\gamma_1}(\hat{\underline{a}}) \equiv \tfrac{m+n+1}{\pi} D_2(-\mathfrak{z}_{m,n}^{m+1}\mathfrak{w}_{m,n}) =: \mathcal{D}_{m,n} \mod \mathbb{Q}(1) \tag{2.4.10}$$

at the maximal conifold point.

Example 2.4.3. A toric coordinate change brings $F_{2,2}$ into the form $F_{3,1}$, but with a_1 and a_2 swapped. So Conjecture 2.4.2 actually yields predictions for both nontrivial regulator periods at $\hat{\underline{a}} = (5, -5)$, namely $\frac{1}{2\pi\mathbf{i}} R_{\gamma_1}(\hat{\underline{a}}) \equiv \mathcal{D}_{2,2} = \frac{5}{\pi} D_2(e^{\frac{2\pi\mathbf{i}}{5}}\mathfrak{w})$ and $\frac{1}{2\pi\mathbf{i}} R_{\gamma_2}(\hat{\underline{a}}) \equiv \mathcal{D}_{3,1} = \frac{5}{\pi} D_2(e^{\frac{\pi\mathbf{i}}{5}}\mathfrak{w})$

[17] Of course, there is no distinction between (2.4.7) and (2.4.8) if $g_1 = 1 = g_2$.

mod $\mathbb{Q}(1)$, where $\mathfrak{w} := \frac{1+\sqrt{5}}{2}$. This assertion was checked in [7K] by a computation we will generalize (and make more rigorous) in §4.

Chapter 3

From higher normal functions to eigenfunctions

In this section we state and prove a precise version of Conjecture 2.3.2 in the genus 1 case.

3.1 Integral mirror symmetry and quantization conditions

The condition $g = 1$ is equivalent to reflexivity of Δ, whereupon X becomes simply the total space of $K_{\mathbb{P}_{\Delta^\circ}}$. There is a unique compact toric divisor $D = D_1 \cong \mathbb{P}_{\Delta^\circ} \subset X$, corresponding to the ray through $(1, 0, 0)$, which amounts to the zero-section of $\rho\colon X \twoheadrightarrow D$. Denoting by $E^\circ \subset D$ a general anticanonical (elliptic) curve, we remark that $D^2 = -E^\circ$ in $H^*_c(X)$.

Let φ be the unique integrally tempered Laurent polynomial with Newton polygon Δ and coefficients 1 at the vertices, and (writing $a = a_1$) take $F = a + \varphi$. After compactifying fibers in $\mathbb{P}_\Delta$ and birationally modifying the total space, this produces a relatively minimal elliptic fibration $\mathcal{E} \to \mathbb{P}^1_a$ with rational total space, fibers E_a, and discriminant locus $\Sigma \cup \{\infty\}$. Writing $r := |\partial\Delta \cap \mathbb{Z}^2|$ and $r^\circ := |\partial\Delta^\circ \cap \mathbb{Z}^2|$, E_∞ has type I_{r°, and Σ is cut out by a polynomial P_Σ of degree $12 - r^\circ = r$.[1]

A section of the relative dualizing sheaf for our family is given by

$$\omega(a) := \tfrac{1}{2\pi\mathbf{i}}\mathrm{Res}_{E_a}\left(\tfrac{dx_1/x_1 \wedge dx_2/x_2}{1 + a^{-1}\varphi(\underline{x})}\right), \tag{3.1.1}$$

[1]For a *generic* choice of φ, the remaining singular fibers of $\mathcal{E}$ are I_1's. Since $\mathcal{E}$ is rational (as a blowup of $\mathbb{P}_\Delta$), the degree of the relative dualizing sheaf must be 1; and as each I_k contributes $\frac{k}{12}$ to this degree, there must be $12 - r^\circ$ I_1's. Each of these contributes 1 to $\deg(P_\Sigma)$, and this degree is invariant as we specialize φ.

with period[2]

$$\omega_\gamma(a) := \int_\gamma \omega(a) = 1 + \textstyle\sum_{k>0}(-1)^k[\varphi^k]_{\underline{0}} a^{-k} \tag{3.1.2}$$

in a neighborhood of the large complex structure point ∞. More precisely, this series converges on $\mathbb{D}^* := \{a \mid |a| > |\hat{a}|\} \subset U := \mathbb{P}^1 \setminus (\Sigma \cup \{\infty\})$, where the *conifold point* $\hat{a}$ can be described by $-\hat{a} := \min(\varphi(\mathbb{R}_+ \times \mathbb{R}_+))$ since the coefficients of φ are all positive [Ga].

By assumption, all the tame symbols of $\{-x_1, -x_2\}$ are trivial, and so the R_{α_i} ($i = 1, \ldots, r-2$) must be integer multiples of $R_\gamma \sim -2\pi\mathbf{i}\log(a)$. More precisely, we have $\frac{-1}{2\pi\mathbf{i}} R_{\alpha_i} = t_i = C_{i1}t = -(C_i \cdot D)t = d_i t$, where $d_i \in [0,4] \cap \mathbb{Z}$ is the lattice-length of the edge of $\partial\Delta$ corresponding to C_i. From Appendix A, we have on the cut disk $\mathbb{D}^- := \mathbb{D}^* \setminus (\mathbb{D}^* \cap \mathbb{R}_-)$

$$t = t(a) := \tfrac{-1}{2\pi\mathbf{i}} R_\gamma(a) = \log(a) + \textstyle\sum_{k>0} \tfrac{(-1)^{k-1}}{k} [\varphi^k]_{\underline{0}} a^{-k}, \tag{3.1.3}$$

which gives $\omega = \frac{-1}{2\pi\mathbf{i}} \nabla_{\delta_a} \mathcal{R}$ hence (in the notation of §2.3) $\omega_1 = \omega/\omega_\gamma$ globally on U. We also see that $e^{-t} \sim a^{-1}$ makes sense as a coordinate on $\mathbb{D} = \mathbb{D}^* \cup \{\infty\}$. The local mirror symmetry results in [BKV] can be made very explicit:[3]

Lemma 3.1.1. *On $\mathbb{D}^-$ we have the following identifications:*

(a) $R_\beta(a) = \frac{r^\circ}{2} t(a)^2 + \pi\mathbf{i} r^\circ t(a) + (2\pi\mathbf{i})^2(\frac{1}{2} + \frac{r^\circ}{12}) - \sum_{k>0} k\mathfrak{N}_k e^{-kt(a)}$,

(b) $\Omega(a)\,(= \frac{\omega_\beta(a)}{\omega_\gamma(a)}) = \frac{\mathbf{i}r^\circ}{2\pi} t(a) - \frac{r^\circ}{2} - \frac{1}{2\pi\mathbf{i}} \sum_{k>0} k^2 \mathfrak{N}_k e^{-kt(a)}$, *and*

(c) $\nu(a) = \frac{r^\circ}{8\pi^2} t(a)^2 + (\frac{1}{2} + \frac{r^\circ}{12}) + \frac{1}{4\pi^2} \sum_{k>0} k(1 + kt(a))\mathfrak{N}_k e^{-kt(a)}$,

where $\mathfrak{N}_k$ is the local GW-invariant for D counting rational curves whose classes $\mathtt{C} \in H_2(D)$ satisfy $(\mathtt{C} \cdot E^\circ)_D = k$.

Proof. X is described in [BKV, §6] as the large-fiber-volume limit of an elliptically-fibered compact CY 3-fold $W \to \mathbb{P}_{\Delta^\circ}$ with section D. Let $C_1, \ldots, C_r$ be the components of $\mathbb{P}_{\Delta^\circ} \setminus (\mathbb{C}^*)^2$

[2] $[\cdot]_{\underline{0}}$ takes the constant term; γ is γ_1 from §2.3.
[3] Here as above $\beta = \beta_1$, $\Omega = \Omega_{11}$, $\nu = \nu_1$.

(and their images in X), $D_i' := \rho^{-1}(C_i)$, and $C_0 := \rho^{-1}(\mathrm{pt})$. Then $\{C_0, C_1, \ldots, C_{r-2}\}$ span $H^4(W,\mathbb{Q})$, $\{D, D_1', \ldots, D_{r-2}'\}$ span $H^2(W,\mathbb{Q})$, and we can write $-D^2 = E^\circ = \sum_{i=1}^r C_i = \sum_{i=1}^{r-2} e_i C_i$ for unique $e_i \in \mathbb{Q}$, whereupon $D^3 = \sum_{i=1}^{r-2} d_i e_i = r^\circ$. Let $J_0, \ldots, J_{r-2}$ denote a basis of $H^2(W,\mathbb{Q})$ dual to $C_0, \ldots, C_{r-2}$, and define $\mathcal{J}_1, \ldots, \mathcal{J}_{r-2}$ by $\mathcal{J}_i := J_i - \frac{e_i}{r^\circ} J_0$. Then the $c_{\underline{i}}$ in (2.2.3) are given by $c_{i_1 i_2 i_3} = -\mathcal{J}_{i_1}\mathcal{J}_{i_2}\mathcal{J}_{i_3}$.[4]

The integral periods of the A-model VHS given by [BKV, (6.13-15)] lead (in the LMHS as $t_0 \to 0$) to the following periods for our A-model VMHS. First, the limit of the Gamma class for W yields $\hat{\Gamma}(X) := 1 - \frac{1}{2}D^2 + (\frac{11r^\circ + r}{24})C_0 = 1 + \sum_{i=1}^{r-2} e_i C_i + (\frac{1}{2} + \frac{5}{12} r^\circ) C_0 \in H^*(X,\mathbb{Q})$. Next, for integral periods we need to compose $\mathrm{ch}(\cdot) \cup \hat{\Gamma}(X) \colon K_0^{c,\mathrm{num}}(X) \to H_c^*(X,\mathbb{Q})$ with the following assignment of periods to cohomology classes: $\mathrm{pt} \mapsto 1$; $C_i \mapsto \frac{1}{2\pi\mathbf{i}} t_i = \frac{-1}{(2\pi\mathbf{i})^2} R_{\alpha_i}$; and $D \mapsto \frac{1}{(2\pi\mathbf{i})^2} \sum_{i=1}^{r-2} d_i \partial_{t_i} \mathcal{F}_0(\underline{t})$. Applying this to $\mathcal{O}_D$, we have $\mathrm{ch}(\mathcal{O}_D) = D - \frac{1}{2}D^2 + \frac{1}{6}D^3$, whence $\mathrm{ch}(\mathcal{O}_D) \cup \hat{\Gamma}(X) = D + \frac{1}{2}\sum_i e_i C_i + (\frac{1}{2} + \frac{r^\circ}{12})$, and finally (after multiplying the resulting integral period by $(2\pi\mathbf{i})^2$)

$$R_\beta = \textstyle\sum_i d_i \partial_{t_i} \mathcal{F}_0(\underline{t}) + \pi\mathbf{i} \sum_i e_i t_i + (2\pi\mathbf{i})^2 (\frac{1}{2} + \frac{r^\circ}{12}). \tag{3.1.4}$$

We also recall from (2.3.7) that the period ratio is given by $\Omega = \frac{-1}{2\pi\mathbf{i}} \sum_i d_i \partial_{t_i} R_\beta$, and the normal function by $\nu = \frac{1}{4\pi^2}(R_\gamma \Omega - R_\beta)$.

The last step is to substitute $t_i = d_i t$, which gives

$$\mathcal{F}_0(t) = -\tfrac{1}{6}(\textstyle\sum_i \mathcal{J}_i t_i)^3 + \sum_{\mathtt{C}} N_{0,\mathtt{C}} e^{-(\mathtt{C}\cdot E^\circ)_D t} = \frac{r^\circ}{6} t^3 + \sum_{k>0} \mathfrak{N}_k e^{-kt} \tag{3.1.5}$$

since $\sum_i \mathcal{J}_i d_i = \sum_i d_i J_i - \sum_i \frac{e_i d_i}{r^\circ} J_0 = (J_0 - D) - J_0 = -D$ [BKV, (6.5)]. Using $d_i \partial_{t_i} = \partial_t$ in (3.1.4)ff now gives (a)-(c). □

[4] The results of [loc. cit.] are stated in terms of derivatives of the prepotential $\Phi(t_0, \underline{t})$ of W in the limit as $t_0 \to \infty$. One can obtain the free energy $\mathcal{F}_0(\underline{t})$ for X by substituting $t_0 = -\sum_{i=1}^{r-2} \frac{e_i}{r^\circ} t_i$ into Φ^{cl} and taking $t_0 \to \infty$ in Φ^{inst}; we then have $\frac{1}{(2\pi\mathbf{i})^3}\partial_D \Phi = \frac{1}{(2\pi\mathbf{i})^2}(-\partial_0 + \sum_i d_i \partial_i)\Phi = \frac{1}{(2\pi\mathbf{i})^2}\sum_i d_i \partial_i \mathcal{F}_0$, hence the version of the A-model periods given here.

Remark 3.1.2. We point out two immediate consequences of Lemma 3.1.1. First, along with (3.1.3), (c) makes it clear that $\nu(a)$ as well as

$$V(a) := \omega_\gamma(a)\nu(a) = \tfrac{1}{4\pi^2}(R_\gamma\omega_\beta - R_\beta\omega_\gamma) \tag{3.1.6}$$

are real-valued on $\mathbb{D}^* \cap \mathbb{R}_+$. Second, notice that $\frac{1}{(2\pi\mathbf{i})^2}\partial_t^2 R_\beta = \partial_{R_\gamma}^2 R_\beta = \partial_{R_\gamma}\frac{\delta_a R_\beta}{\delta_a R_\gamma} = \partial_{R_\gamma}\frac{\omega_\beta}{\omega_\gamma} = \frac{\mathcal{Y}(a)}{\omega_\gamma^3}$, where the Yukawa coupling $\mathcal{Y}(a) = \omega_\gamma\delta_a\omega_\beta - \omega_\beta\delta_a\omega_\gamma$ blows up at $\hat{a}$. Differentiating (a) twice expresses this as a power series in e^{-t}, from which one deduces that

$$\limsup_{k\to\infty} \sqrt[k]{|\mathfrak{N}_k|} = \exp(\Re(t(\hat{a}))). \tag{3.1.7}$$

as in [DK, §5.4] (though this result in now unconditional).

We may now identify all of the torsion constants in §§2.2-2.3:[5]

Lemma 3.1.3. *In $\mathbb{Q}/\mathbb{Z}$ the following equalities hold:*

(i) $b := \sum_i d_i b_i = \frac{r^\circ}{12} - \frac{1}{2}$ *and* $b^{\mathrm{NS}} := \sum_i d_i b_i^{\mathrm{NS}} = \frac{r^\circ}{24} - \frac{1}{2}$.

(ii) $\mathtt{T} = \frac{1}{2} + \frac{r^\circ}{12}$ *and* $\mathtt{B}^\circ = \frac{1}{2} - \frac{r^\circ}{24}$.

(iii) $\mathtt{A} = \frac{1}{2} = \mathtt{B}$, *where* $\mathtt{B}$ *is as in* (2.3.12)-(2.3.13).[6]

Proof. (i) These are the coefficients of t in $\mathcal{F}_1$ and $\mathcal{F}_1^{\mathrm{NS}}$ (after substituting $t_i = d_i t$), which can be derived from [GKMR, (4.18) and (4.21)].[7] Namely, we have $b_i = \frac{1}{24}c_2(X)\cdot\mathcal{J}_i$ [GKMR, (4.18)] and $c_2(X) = (11r^\circ + r)C_0 + 12\sum_i e_i C_i = (10r^\circ + 12)C_0 - 12D^2$ [BKV, §6.2] hence $b = \frac{1}{24}c_2(X)\cdot\sum_i d_i\mathcal{J}_i = -\frac{1}{24}c_2(X)\cdot D = -\frac{10r^\circ+12}{24} + \frac{12r^\circ}{24} = \frac{r^\circ}{12} - \frac{1}{2}$. According to [GKMR, (4.21)], we have $\mathcal{F}_1^{\mathrm{NS}} \sim -\frac{1}{24}\log(P_\Sigma(a)) \sim -\frac{\deg(P_\Sigma)}{24}\log(a) \sim -\frac{r}{24}t \sim (\frac{r^\circ}{24} - \frac{1}{2})t$. (So of course, (i) holds in $\mathbb{Q}$, but we'll only need it mod $\mathbb{Z}$.)

[5] Again, for simplicity writing $\mathtt{T} = \mathtt{T}_1$, $\mathtt{B}^\circ = \mathtt{B}_1^\circ$, $\mathtt{B} = \mathtt{B}_1$, and $\mathtt{A} = \mathtt{A}_1$.

[6] and *not* as in (2.3.11), where $\mathtt{B}$ does not yet incorporate the correction from the cubic term.

[7] We should point out here that our "r" is not the "r" in [GKMR], where it means $\gcd\{d_i\}$. (Moreover, their "t" is r_{GKMR} times our t.)

(ii) The value of $\mathtt{T}$ is immediate from Lemma 3.1.1(a). To compute $\mathtt{B}^\circ = \nu(a) - \xi(a)$, we need to revisit ξ from (2.3.9). The B-field is given by $\mathbb{B}_i = d_i$ (cf. §2.3 above or [GKMR, §3.2]), and $\mathbb{A} = \mathbb{A}_1 = 1$, which means that replacing $\underline{t}$ by $\underline{t} + \pi\mathbf{i}\mathbb{B}$ is equivalent to replacing t by $t + \pi\mathbf{i}$. Together with $\sum_i \delta_{t_i} = t \sum_i d_i \partial_{t_i} = t\partial_t = \delta_t$ and (3.1.5), this gives

$$\begin{aligned}\xi(a) &= \tfrac{1}{4\pi^2}(\delta_t - 1)\partial_t \hat{\mathcal{F}}_0(t(a) + \pi\mathbf{i}) \\ &= \tfrac{r^\circ}{8\pi^2} t(a)^2 + \tfrac{r^\circ}{8} + \tfrac{1}{4\pi^2}\textstyle\sum_{k>0} k(1 + kt(a))\mathfrak{N}_k e^{-kt(a)}\end{aligned} \tag{3.1.8}$$

and, together with Lemma 3.1.1(c), the claimed value of $\mathtt{B}^\circ$.

(iii) We already have $\mathtt{A} = \frac{1}{2}\mathbb{A} = \frac{1}{2}$. For $\mathtt{B}$, we compute

$$\begin{aligned}\hat{\xi}(t(a) + \pi\mathbf{i}) &= \tfrac{1}{4\pi^2}\left((t + \pi\mathbf{i})\partial_t - 1\right)\partial_t \hat{\mathcal{F}}_0(t(a) + \pi\mathbf{i}) + (b + b^{\mathrm{NS}}) \\ &= \xi(a) + \tfrac{\pi\mathbf{i}}{4\pi^2}\partial_t^2 \hat{\mathcal{F}}_0(t(a) + \pi\mathbf{i}) + (b + b^{\mathrm{NS}}) \\ &= \nu(a) + \tfrac{1}{2}\Omega(a) + (b + b^{\mathrm{NS}} - \mathtt{B}^\circ)\end{aligned} \tag{3.1.9}$$

and note that the cubic term in (2.3.10) becomes $-\frac{\pi\mathbf{i}}{3}D^3 n^3 = -\frac{r^\circ}{3}\pi\mathbf{i}n^3 \equiv -\frac{r^\circ}{6}2\pi\mathbf{i}n \bmod \mathbb{Z}(1)$. Together with (i)-(ii), this results in the apparently miraculous cancellation

$$\mathtt{B} = b + b^{\mathrm{NS}} - \mathtt{B}^\circ - \tfrac{r^\circ}{6} = -\tfrac{3}{2} \equiv \tfrac{1}{2} \tag{3.1.10}$$

modulo $\mathbb{Z}$. □

Finally, we turn to the quantization conditions, i.e. to the spectrum (as an operator on $L^2(\mathbb{R})$) of[8]

$$\begin{aligned}\hat{\varphi} &= \textstyle\sum_{\underline{m} \in \partial\Delta \cap \mathbb{Z}^2} (-1)^{m_1 m_2} a_{\underline{m}} \hat{x}_1^{m_1} \hat{x}_2^{m_2} \\ &= \textstyle\sum_{\underline{m} \in \partial\Delta \cap \mathbb{Z}^2} (-1)^{m_1 + m_2 + 1} a_{\underline{m}} \hat{x}_1^{m_1} \hat{x}_2^{m_2} = -\varphi(-\hat{x}_1, -\hat{x}_2)\end{aligned} \tag{3.1.11}$$

[8]Remark that $\varphi = F_1$ and $\rho = \rho_1$ in the notation of §2.1. We have $m_1 m_2 \underset{(2)}{\equiv} m_1 + m_2 + 1$ because (2.3.1) always holds for reflexive polygons.

or $\rho := \hat{\varphi}^{-1}$. Writing $\sigma(\cdot)$ for spectrum and $\Lambda(a) := \mathbb{Z}\langle\omega_\gamma(a), \omega_\beta(a)\rangle$ for the period lattice, we have the

Proposition 3.1.4. *In the genus-1 case, Conjecture 2.3.2 is equivalent to*

$$\sigma(\hat{\varphi}) = \{a \in U \mid V(a) \in \Lambda(a)\}. \tag{3.1.12}$$

Proof. Noting that $\boldsymbol{\mathcal{M}} = U$, in the LHS of (2.3.13) we are taking the zero-locus of $\Xi(-a; 2\pi) = \det(1 - a\rho)$, which is precisely the spectrum of $\hat{\varphi}$. The RHS of (2.3.13) is the locus in U where $\nu(a)$ meets the theta divisor (which is $\frac{1+\Omega(a)}{2}$ mod $\mathbb{Z}\langle 1, \Omega(a)\rangle$) shifted by $\mathtt{A}\Omega(a) + \mathtt{B} = \frac{1+\Omega(a)}{2}$, which is to say *where* $\nu(a)$ *is zero mod* $\mathbb{Z}\langle 1, \Omega(a)\rangle$. Outside of $\mathbb{D}^-$, this condition is only well-defined in the sense of analytic continuation; to fix this, we multiply by ω_γ to get the form displayed in RHS(3.1.12). $\square$

Remark 3.1.5. (i) The condition $V(a) \in \Lambda(a)$, which is well-defined on U, reduces to $\nu(a) \in \mathbb{Z}\langle 1, \Omega(a)\rangle$ for $a \in \mathbb{D}^-$. Moreover, the argument in [LST, §3.1] using the coherent state representation obviously shows more generally (for any φ considered here) that $\sigma(\hat{\varphi})$ belongs to $\mathbb{R}_+$, and is countable with eigenvalues λ_j limiting to ∞ (so that ρ is bounded). In fact, we expect that $\sigma(\hat{\varphi}) \subset (|\hat{a}|, \infty)$, as is clear for $\varphi = x_1 + x_1^{-1} + x_2 + x_2^{-1}$ or $x_1 + x_1^{-1} + x_2 + x_2^{-1} + x_1 x_2^{-1} + x_1^{-1} x_2$ and experimentally observed in other cases. This would mean that the quantization condition "$V \in \Lambda$" reduces not just to $\nu \in \mathbb{Z}\langle 1, \Omega\rangle$, but to

$$\nu(a) \in \mathbb{Z}, \tag{3.1.13}$$

as ν is real by Remark 3.1.2. We'll have more to say about this in §3.2.

(ii) The most crucial "torsion" invariant in Lemma 3.1.3, leading to the cancellation in (3.1.10) and the simple form of (3.1.12), is surely the constant term $\mathtt{T}$ of the regulator period R_β. As an independent check, one can directly compute this constant term without using mirror symmetry and the Gamma class; see Appendix A for examples. Another check on our quantization condition is that it should coincide with that in [GKMR, §3.3.2] when all $Q_{m_k} = 1$ ($\implies D_0(\underline{m}) = 0$ and $B(\underline{m}, 2\pi) = b + b^{\mathrm{NS}} = \frac{r^\circ}{8} - 1$). Since $\mathrm{vol}_0(E)$ in [GKMR, (3.24)] is just R_β, we may also identify "C" there as $\frac{r^\circ}{2}$. Taking $E = \log(a)$ and $E_{\text{eff}} = t(a)$, [GKMR, (3.105)] collapses to $\xi(a) - \frac{r^\circ}{24} \in \mathbb{Z} + \frac{1}{2}$, hence to $\nu(a) \in \mathbb{Z}$.

(iii) There is an interesting sign discrepancy in (3.1.12): quantizability of $\hat{\varphi} - a$ is being linked to a regulator class on the curve $E_a \subset \mathbb{P}_\Delta$ compactifying solutions to $\varphi(\underline{x}) + a = 0$. Blame it on the B-field! Or better yet, proceed to the next section for a more basic reason why it has to be this way.

3.2 Construction of eigenfunctions for difference operators

In this section we assume that Δ is a reflexive polygon satisfying

$$\Delta \subset \mathbb{R} \times [-1, 1], \tag{3.2.1}$$

and φ is as in §3.1, so that

$$\varphi(\underline{x}) = x_1^{m_u}(x_1 + 1)^{d_u} x_2 + \varphi_0(x_1) + x_1^{m_\ell}(x_1 + 1)^{d_\ell} x_2^{-1}. \tag{3.2.2}$$

Remark 3.2.1. Regarding unimodular change of coordinates ($x_1, x_2 \mapsto x_1^{\mathtt{a}} x_2^{\mathtt{b}}, x_1^{\mathtt{c}} x_2^{\mathtt{d}}$ with $\mathtt{ad} - \mathtt{bc} = 1$) as an equivalence relation on reflexive polygons, there are 16 equivalence classes. All but one[9] of these has representatives satisfying (3.2.1).

For each $a \in U$, $E_a \subset \mathbb{P}_\Delta$ denotes as before the Zariski closure of $E_a^* := \{\underline{x} \in (\mathbb{C}^*)^2 \mid \varphi(\underline{x}) + a = 0\}$. Forgetting x_2 produces a $2:1$ map $\pi \colon E_a \to \mathbb{P}^1$ with corresponding involution

[9]represented by Δ = convex hull of $\{(-1, -1), (2, -1), (-1, 2)\}$, with $\mathbb{P}_\Delta = \mathbb{P}^2$

$\iota\colon E_a \to E_a$ and discriminant

$$(\varphi_0(x_1)+a)^2 - 4x_1^{m_u+m_\ell}(x_1+1)^{d_u+d_\ell} =: \mathscr{D}(x_1). \tag{3.2.3}$$

The latter is a Laurent polynomial (in x_1) with "Newton polytope" an interval $[-c_-, c_+]$ containing $[-1,1]$ (and contained in $[-2,2]$), whose length is the number of ramification points of $\pi^{-1}(\mathbb{C}^*) =: E_a^\times \overset{\pi^\times}{\to} \mathbb{C}^*$; denote the set of these by $\mathfrak{B} \subset E_a^\times$, and let $p_0 \in \mathfrak{B}$ be one of them. The holomorphic function

$$\delta(p) := x_1(p)^{m_u}(x_1(p)+1)^{d_u}(x_2(p) - x_2(\iota(p))), \tag{3.2.4}$$

on $E_a^\times$ satisfies $\delta^2 = (\pi^\times)^*\mathscr{D}$, thereby providing a well-defined lift of $\sqrt{\mathscr{D}}$ to $E_a^\times$.

Writing $\tilde{E}_a^\times$ for the fiber product of $\pi^\times$ and $(-\exp)\colon \mathbb{C} \to \mathbb{C}^*$ yields a diagram

$$\begin{array}{ccccccc} E_a & \hookleftarrow & E_a^\times & \overset{\mathcal{P}}{\twoheadleftarrow} & \tilde{E}_a^\times & \ni & \tilde{z} \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi^\times}\downarrow & & {\scriptstyle\Pi}\downarrow & & \downarrow \\ \mathbb{P}^1 & \hookleftarrow & \mathbb{C}^* & \underset{-\exp}{\twoheadleftarrow} & \mathbb{C} & \ni & z \end{array} \tag{3.2.5}$$

with vertical maps of degree 2, and points in $\tilde{E}_a^\times$ [resp. $\mathbb{C}$] denoted by $\tilde{z}$ [resp. $z = \Pi(\tilde{z})$]. We also write $\mathcal{P}(\tilde{z}) =: (x_1(\tilde{z}), x_2(\tilde{z}))$, where $x_1(\tilde{z}) = x_1(z) = -e^z$, and $\tilde{z}_0 \in \tilde{E}_a^\times$ for the point with $\mathcal{P}(\tilde{z}_0) = p_0$ and $\Im(z_0) \in (-\pi, \pi]$. For later reference put $\tilde{E}_a^* := \mathcal{P}^{-1}(E_a^*)$, which is either all of $\tilde{E}_a^\times$ or the complement of $\Pi^{-1}(\mathbb{Z}(1))$.[10]

Now **suppose $\boldsymbol{V(a) \in \Lambda(a)}$.** If $a \in \mathbb{D}^-$, then $\gamma, \beta, \omega_\gamma, \omega_\beta, \Omega, R_\gamma, R_\beta$, and ν are well-defined; if not, we take them to be analytic continuations (along the same path) to a of those objects from $\mathbb{D}^-$. (We will not write $\omega(a)$ etc., just ω, since a is fixed and understood.) Then

[10] There are 4 equivalence classes of ploygons for which $\tilde{E}_a^* = \tilde{E}_a^\times$, corresponding to $X = \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_1$, and $\mathbb{F}_2$. Otherwise, for $\tilde{z} \in \tilde{E}_a^\times \setminus \tilde{E}_a^*$, in view of (3.2.2) we have $-1 = x_1(\tilde{z}) = x_1(z) = -e^z \implies z \in \mathbb{Z}(1)$.

we have

$$\nu = \frac{1}{4\pi^2}(R_\gamma \Omega - R_\beta) = n_1 + n_2 \Omega \tag{3.2.6}$$

for some $n_1, n_2 \in \mathbb{Z}$. Notice that the regulator class $\mathcal{R}$ is only well-defined in $H^1(E_a, \mathbb{C}/\mathbb{Z}(2))$, so its value on γ is still represented by $\mathfrak{R}_\gamma := R_\gamma - 4\pi^2 n_2$. This replaces (3.2.6) by

$$R_\beta - \mathfrak{R}_\gamma \frac{\omega_\beta}{\omega_\gamma} = -4\pi^2 n_1 \in \mathbb{Z}(2), \tag{3.2.7}$$

and we claim this allows us to define a holomorphic function on $\tilde{E}_a^*$ by

$$\chi(\tilde{z}) := \exp\left(\frac{\mathbf{i}}{2\pi} \left\{ \int_{\mathscr{P}_{\tilde{z}_0}^{\tilde{z}}} z \frac{dx_2(\tilde{z})}{x_2(\tilde{z})} - \frac{\mathfrak{R}_\gamma}{\omega_\gamma} \int_{\mathscr{P}_{\tilde{z}_0}^{\tilde{z}}} \mathcal{P}^* \omega \right\} \right), \tag{3.2.8}$$

where ω is as in (3.1.1), and $\mathscr{P}_{\tilde{z}_0}^{\tilde{z}}$ is any path from $\tilde{z}_0$ to $\tilde{z}$.

The issue here is well-definedness, since nothing in the braces blows up on $\tilde{E}_a^*$. To check this, we remind the reader that for a loop $\mathscr{L}$ on E_a^* based at p_0, the value of $\mathcal{R}$ on its homology class is computed by[11]

$$R_{\mathscr{L}} \underset{\mathbb{Z}(2)}{\equiv} \int_{\mathscr{L}} \log(-x_1) \mathrm{dlog}(-x_2) - \log(-x_2(p_0)) \int_{\mathscr{L}} \mathrm{dlog}(-x_1), \tag{3.2.9}$$

where $\log(-x_1)$ is analytically continued along $\mathscr{L}$ [Ke1]. If $\mathscr{L}$ lifts to a loop $\tilde{\mathscr{L}}$ on $\tilde{E}_a^*$, then clearly $\int_{\mathscr{L}} \mathrm{dlog}(x_1) = 0$, and (3.2.9) pulls back to $\int_{\tilde{\mathscr{L}}} z \frac{dx_2(\tilde{z})}{x_2(\tilde{z})}$. Now given two paths $\mathscr{P}, \mathscr{P}'$ from $\tilde{z}_0$ to $\tilde{z}$ on $\tilde{E}_a^*$, take $\tilde{\mathscr{L}}$ to be the loop obtained by composing $\mathscr{P}$ with the "reverse" of $\mathscr{P}'$, and write $\mathscr{L} = k_1 \gamma + k_2 \beta$ in $H_1(E_a, \mathbb{Z})$. (By integral temperedness of $\{-x_1, -x_2\}$, this determines $R_{\mathscr{L}}$ mod $\mathbb{Z}(2)$.) The difference between the braced expression in (3.2.8) for these

[11] Of course, $\mathrm{dlog}(-x) = \mathrm{dlog}(x) = \frac{dx}{x}$. Note that (3.2.9), which is due to Beilinson [Be] and Deligne [unpublished], is different from the regulator formula using the current $R\{-x_1, -x_2\}$ (in which the function "log" is not analytically continued but has a branch cut), but is easily shown to give the same integral regulator.

two paths is then

$$\begin{aligned}\int_{\tilde{\mathscr{L}}} z\tfrac{dx_2(\tilde{z})}{x_2(\tilde{z})} - \tfrac{\mathfrak{R}_\gamma}{\omega_\gamma}\int_{\tilde{\mathscr{L}}} \mathcal{P}^*\omega &= \int_{\mathscr{L}} \log(-x_1)\mathrm{dlog}(x_2) - \tfrac{\mathfrak{R}_\gamma}{\omega_\gamma}\int_{\mathscr{L}}\omega \\ &\underset{\mathbb{Z}(2)}{\equiv} k_1 R_\gamma + k_2 R_\beta - \tfrac{\mathfrak{R}_\gamma}{\omega_\gamma}(k_1\omega_\gamma + k_2\omega_\beta) \\ &= k_1(R_\gamma - \mathfrak{R}_\gamma) + k_2(R_\beta - \mathfrak{R}_\gamma\Omega) \\ &= 4\pi^2(k_1 n_2 - k_2 n_1) \underset{\mathbb{Z}(2)}{\equiv} 0,\end{aligned} \tag{3.2.10}$$

using (3.2.7). After multiplying by $\frac{\mathbf{i}}{2\pi}$, this discrepancy is killed by the exp and the claim is verified.

In fact, $\chi(\tilde{z})$ extends to a meromorphic function on $\tilde{E}_a^\times$ which is holomorphic at $\Pi^{-1}(0)$. Of course, ω has no poles on E_a, and so $\mathcal{P}^*\omega$ has none on $\tilde{E}_a^\times$; the potential culprit is $\frac{dx_2}{x_2}$, when d_u, d_ℓ are not both zero. Writing $z = 2\pi\mathbf{i}n + w + O(w^2)$, $x_2 = w^d$ (for $d = -d_u$ or d_ℓ), we find $\int z\frac{dx_2}{x_2} \sim 2\pi\mathbf{i}dn\log(w)$ hence $\exp(\frac{\mathbf{i}}{2\pi}\int z\frac{dx_2}{x_2}) \sim w^{-nd}$, as desired.

Finally, writing $\tilde{\imath}\colon \tilde{E}_a^\times \to \tilde{E}_a^\times$ for the involution over $\mathbb{C}$, we put

$$\tilde{\Psi}(\tilde{z}) := \frac{\chi(\tilde{z}) - \chi(\tilde{\imath}(\tilde{z}))}{\delta(\mathcal{P}(\tilde{z}))}. \tag{3.2.11}$$

The denominator has zeroes at $\mathcal{P}^{-1}(\mathfrak{B})$, which does not intersect any of the poles of the numerator.[12] Moreover, these are simple zeroes, and the numerator also has zeroes at these points (which are just the fixed points of $\tilde{\imath}$). So $\tilde{\Psi}$ is holomorphic on $\tilde{E}_a^\times$. Notice also that applying $\tilde{\imath}$ to $\tilde{z}$ changes the sign in the numerator and denominator of (3.2.11) (since $\mathcal{P}\circ\tilde{\imath} = \iota\circ\mathcal{P}$). Conclude that there exists a meromorphic function Ψ on $\mathbb{C}$, with (at worst) poles on $2\pi\mathbf{i}(\mathbb{Z}\setminus\{0\})$, such that $\tilde{\Psi} = \Pi^*\Psi$; we write this loosely as

$$\Psi(z) := \frac{\chi(\tilde{z}) - \chi(\tilde{\imath}(\tilde{z}))}{\delta(\mathcal{P}(\tilde{z}))}, \tag{3.2.12}$$

[12] The only way ι has a fixed point at $x_1 = -1$ is if $d_u = d_\ell = 0$.

and denote its restriction to the real line by $\psi(r)$. We are now ready to prove the

Theorem 3.2.2. *For* Δ *satisfying* (3.2.1)*, the "*$\supseteq$*" direction of* (3.1.12) *holds. That is, if* $V(a) \in \Lambda(a)$*, then* $a \in \sigma(\hat{\varphi})$*.*

Proof. First note that $\hat{x}_1 =$ multiplication by e^r (*not* $-e^r$), $\hat{x}_2 = e^{-2\pi\mathbf{i}\partial_r}$, and $\hat{\varphi} = -\varphi(-\hat{x}_1, -\hat{x}_2)$ are *unbounded* operators on $L^2(\mathbb{R})$, whose domains are roughly the *proper* linear subspaces on which each operator preserves square integrability. (See [LST] for details.) In particular, it is possible in this sense to be in the domain of $\hat{\varphi}$ while failing to be in that of $\hat{x}_1^{\pm 1}$ and $\hat{x}_2^{\pm 1}$, which is just what happens for $\psi(r)$. Indeed, assuming $V(a) \in \Lambda(a)$, we claim that $\psi \in L^2(\mathbb{R}) \setminus \{0\}$ and

$$\hat{\varphi}\psi = a\psi, \tag{3.2.13}$$

which will obviously prove the theorem.

As Ψ is holomorphic on $\{z \in \mathbb{C} \mid -2\pi\mathbf{i} < \Im(z) < 2\pi\mathbf{i}\}$, with meromorphic extension to a neighborhood of its closure, we have

$$\begin{aligned} e^{\pm 2\pi\mathbf{i}\partial_r}\psi(r) &= e^{\pm 2\pi\mathbf{i}\partial_z}\Psi(r) = \Psi(r \pm 2\pi\mathbf{i}) \\ &=: \Psi(\tau_\pm(r)) =: (\mathscr{S}_\pm\Psi)(r) =: (\mathscr{S}_\pm\psi)(r). \end{aligned} \tag{3.2.14}$$

Furthermore, $\tau_\pm$ has a unique lift $\tilde{\tau}_\pm \colon \tilde{E}_a^\times \to \tilde{E}_a^\times$ with the property that $\mathcal{P} \circ \tilde{\tau}_\pm = \mathcal{P}$; and so the difference operator $\mathscr{S}_\pm$ lifts to $(\tilde{\mathscr{S}_\pm}\chi)(\tilde{z}) := \chi(\tilde{\tau}_\pm(\tilde{z}))$. By the independence of path in (3.2.8), we can take our path from $\tilde{z}_0$ to $\tilde{\tau}_\pm(\tilde{z})$ to be the composition of $\tilde{\tau}_\pm(\mathscr{P}_{\tilde{z}_0}^{\tilde{z}})$ with a fixed path $\mathscr{P}_0^\pm$ from $\tilde{z}_0$ to $\tilde{\tau}_\pm(\tilde{z}_0)$. That is, writing $\mathcal{P}(\mathscr{P}_0^\pm) =: \mathscr{L}_0^\pm$, we have

$$\begin{aligned} \chi(\tilde{\tau}_\pm(\tilde{z})) &= \exp\left(\tfrac{\mathbf{i}}{2\pi}\left\{\int_{\tilde{\tau}_\pm(\mathscr{P}_{\tilde{z}_0}^{\tilde{z}})+\mathscr{P}_0^\pm} z\tfrac{dx_2(\tilde{z})}{x_2(\tilde{z})} - \tfrac{\Re_\gamma}{\omega_\gamma}\int_{\tilde{\tau}_\pm(\mathscr{P}_{\tilde{z}_0}^{\tilde{z}})+\mathscr{P}_0^\pm} \mathcal{P}^*\omega\right\}\right) \\ &= \exp\left(\tfrac{\mathbf{i}}{2\pi}\left\{\int_{\mathscr{P}_{\tilde{z}_0}^{\tilde{z}}} (z \pm 2\pi\mathbf{i})\tfrac{dx_2(\tilde{z})}{x_2(\tilde{z})} - \tfrac{\Re_\gamma}{\omega_\gamma}\int_{\mathscr{P}_{\tilde{z}_0}^{\tilde{z}}} \mathcal{P}^*\omega\right\}\right) \\ &\qquad \times \exp\left(\tfrac{\mathbf{i}}{2\pi}\left\{\int_{\mathscr{L}_0^\pm} \log(-x_1)\tfrac{dx_2}{x_2} - \tfrac{\Re_\gamma}{\omega_\gamma}\int_{\mathscr{L}_0^\pm}\omega\right\}\right). \end{aligned} \tag{3.2.15}$$

Adding and subtracting $-\log(-x_2(\tilde{z}_0))\int_{\mathscr{L}_0^\pm}\frac{dx_1}{x_1}\,(=\,\mp 2\pi\mathbf{i}\log(-x_2(\tilde{z}_0))\,)$ in the last braced expression, (3.2.15) becomes

$$\chi(\tilde{z})e^{\mp\{\log(-x_2(\tilde{z}))-\log(-x_2(\tilde{z}_0))\}}\times e^{\frac{\mathbf{i}}{2\pi}\{R_{\mathscr{L}_0^\pm}-\frac{\mathfrak{R}_\gamma}{\omega_\gamma}\omega_{\mathscr{L}_0^\pm}\}}e^{\mp\log(-x_2(\tilde{z}_0))}. \tag{3.2.16}$$

By the same calculation as in (3.2.10), we have $R_{\mathscr{L}_0^\pm}-\frac{\mathfrak{R}_\gamma}{\omega_\gamma}\omega_{\mathscr{L}_0^\pm}\in\mathbb{Z}(2)$, and so after cancelling $\log(-x_2(p_0))$'s, we arrive at

$$(\tilde{\mathscr{S}}_\pm\chi)(\tilde{z})=-x_2(\tilde{z})^{\pm 1}\cdot\chi(\tilde{z}). \tag{3.2.17}$$

Since $-\hat{x}_1=-\mu_{e^r}=\mu_{-e^r}=\mu_{x_1(r)}$, $\hat{\varphi}$ acts on ψ as $-\varphi(\mu_{x_1(r)},-\mathscr{S}_-)$, which lifts to $-\varphi(\mu_{x_1(r)},-\tilde{\mathscr{S}}_-)$ for functions on $\tilde{E}_a^\times$. Applying this to $\chi(\tilde{z})$ gives $-\varphi(x_1(z),x_2(\tilde{z}))\cdot\chi(\tilde{z})=a\chi(\tilde{z})$, and applying it to $\chi(\tilde{\imath}(\tilde{z}))$ yields $-\varphi(x_1(z),x_2(\tilde{\imath}(\tilde{z})))\cdot\chi(\tilde{\imath}(\tilde{z}))=a\chi(\tilde{\imath}(\tilde{z}))$. (Here we are just using the equation of the curve, $\varphi(x_1(z),x_2(\tilde{z}))+a=0$; and we can ignore $\delta(\mathcal{P}(\tilde{z}))$ in the denominator of $\tilde{\Psi}$ since $\tilde{\mathscr{S}}_\pm$ doesn't affect it.) So the overall effect on $\tilde{\Psi}$, hence ψ, is multiplication by a. This proves (3.2.13).

We still need to check is that ψ is indeed square-integrable. Clearly $\int\mathcal{P}^*\omega$ has a finite limit as $r\to\pm\infty$, so we consider the behavior of

$$\int r\frac{dz_2(\tilde{r})}{z_2(\tilde{r})}=\int\log(-x_1(r))\mathrm{dlog}(-x_2(\tilde{r})). \tag{3.2.18}$$

Let $q\in E_a\setminus E_a^\times$, and set $o_j:=\mathrm{ord}_q(x_j)$; then $(-1)^{o_1o_2}\lim_{p\to q}\frac{x_1(p)^{o_2}}{x_2(p)^{o_1}}=1$ by integral temperedness. Hence there is a local holomorphic coordinate on E_a vanishing at q, with $-x_1=w^{o_1}$ and $-x_2=\pm w^{o_2}(1+O(w))$, and (3.2.18) $=\frac{o_1o_2}{2}\log^2 w+O(w\log w)$ is just $\frac{o_2}{2o_1}r^2$ (with $o_1\neq 0$) plus terms limiting to zero. Since this is multiplied by $\frac{\mathbf{i}}{2\pi}$ before taking exp, we conclude that $\chi(\tilde{z})$ is bounded on $\Pi^{-1}(\mathbb{R})$. On the other hand, in the denominator $\delta(\mathcal{P}(\tilde{r}))=\sqrt{\mathscr{D}(e^r)}$ of ψ, $\mathscr{D}(e^r)=\sum_{j=-c_-}^{c_+}\mathtt{a}_je^{jr}$ ($\mathtt{a}_{-c_-},\mathtt{a}_{c_+}\neq 0$) is dominated by the e^{c_+r} term as $r \to +\infty$ and the e^{-c-r} term as $r \to -\infty$. That is, $|\psi(r)| \leq Ce^{-|r/2|}$ for some constant C, hence ψ belongs to $L^2(\mathbb{R})$.

Finally, we must show that ψ is not identically zero. If it were, then by basic complex analysis Ψ would be zero; so it suffices to check that (say) $\Psi(z_0 + 2\pi\mathbf{i}n) \neq 0$ for some $n \in \mathbb{Z}$. We may choose a local holomorphic coordinate u on $\tilde{E}_a^\times$ about $\tilde{z}_0$, such that (locally) $\tilde{\iota}$ sends $u \mapsto -u$ and $z = z_0 + u^2$. Clearly $x_2(\tilde{z}) = x_2(p_0)(1 + c_1 u + O(u^2))$ and $\mathcal{P}^*\omega = (c_2 + O(u))du$ for constants $c_1, c_2 \in \mathbb{C}^*$. The expression in braces in (3.2.8) (integrating on a path from $\tilde{z}_0$ to $\tilde{z}(u)$) takes the form $(c_1 z_0 - \frac{\mathfrak{R}_\gamma}{\omega_\gamma} c_2)u + O(u^2)$, and we can ensure the coefficient of u is nonzero by replacing z_0 by $z_0 + 2\pi\mathbf{i}n$ if necessary (since this affects nothing else). So the numerator of (3.2.11) becomes $e^{c_0 u + O(u^2)} - e^{-c_0 u + O(u^2)} \sim 2c_0 u$, and since the denominator also has a simple zero at $u = 0$ we are done. □

Remark 3.2.3. Returning to the "sign flip" between curve and operator highlighted in Remark 3.1.5(iii), we remind the reader that it is $\{-x_1, -x_2\}$, not $\{x_1, x_2\}$, which is integrally tempered for the simplest choices of Laurent polynomial φ.[13] So it is the regulator integral for *this symbol* which produces a well-defined $\tilde{\Psi}(\tilde{z})$. But the signs in the symbol force the shift operator $\hat{x}_2$ to act on $\chi(\tilde{z})$ through multiplication by $-x_2(\tilde{z})$ rather than $x_2(\tilde{z})$, which in turn forced us to use $(-\exp)$ (not exp) in (3.2.5) so that $\hat{x}_1$ acts through multiplication by $-x_1(z)$, resulting in the action of $\hat{\varphi} = -\varphi(-\hat{x}_1, -\hat{x}_2)$ through multiplication by $-\varphi(x_1(z), x_2(\tilde{z}))$. The upshot is that the signs *in the symbol*[14] are ultimately responsible for the presence of the B-field.

Without stating any results formally, we want to briefly address the higher genus hyperelliptic case, where $F_1 = \varphi$ still takes the form in (3.2.1)-(3.2.2) but Δ is no longer reflexive. (Note that φ_0 will have $a_2, \ldots, a_g$ as coefficients.) One easily checks that the construction of ψ and the proof of Theorem 3.2.2 still go through after modifying $\chi(\tilde{z})$, provided we impose

[13] e.g. $x_1 + x_2 + x_1^{-1}x_2^{-1}$, and including the examples studied in [GKMR] with trivial mass invariants $Q_{m_k} = 1$.

[14] along with those in (3.1.11) arising from Weyl quantization and the CBH formula.

a stronger quantization condition than that in RHS(2.3.13). Namely, referring to (2.3.8), suppose that

$$\textit{the normal function vector } \underline{\nu}(\underline{a}) \textit{ belongs to } (\mathbb{I}_g \mid \Omega)\mathbb{Z}^g. \tag{3.2.19}$$

Then replacing the expression in braces in (3.2.8) by

$$\int_{\mathscr{P}_{\tilde{z}_0}^{\tilde{z}}} z\frac{dx_2(\tilde{z})}{x_2(\tilde{z})} - \sum_{j=1}^{g} \mathfrak{R}_{\gamma_j} \int_{\mathscr{P}_{\tilde{z}_0}^{\tilde{z}}} \mathcal{P}^*\omega_j \tag{3.2.20}$$

for appropriate determinations of $\mathfrak{R}_{\gamma_j}$, the obvious generalization of (3.2.10) goes through, ensuring that the generalized $\chi(\tilde{z})$ is well-defined. Under an additional assumption like (2.3.1), and changing the signs in $\hat{\varphi}$ of those a_j's attached to even powers of $\hat{x}_1$, one finds as before that $\hat{\varphi}\psi = a_1\psi$.

The criterion (3.2.19), which we expect corresponds to the *exact NS quantization conditions* of [SWH], will only hold at countably many points in moduli. On the other hand, Conjecture 2.3.2 predicts the *existence* of eigenfunctions for $\underline{a}$ in a codimension-1 subset of moduli. So it stands to reason that there should be something special about the eigenfunctions ψ, which we can only construct for $\underline{a}$ in the smaller locus. In the genus-2 example worked out explicitly in [Za, §4.3], whose "fully on-shell" quantization conditions (cf. [loc. cit., (4.45)]) should agree with (3.2.19), Zakany highlights the *enhanced decay* of his explicit eigenfunctions. Indeed, in our construction, for $g > 1$ the discriminant $\mathscr{D}$ will involve higher powers of both x_1 and x_1^{-1} than for $g = 1$, which leads to decay better than $e^{-|r/2|}$ at infinity for $\psi(r)$; this perhaps begins to explain the discrepancy.

3.3 Remarks on the spectrum of $\hat{\varphi}$

Notably absent from the last section is any discussion of the "converse question", as to whether every eigenfunction of $\hat{\varphi}$ arises from the construction described there. We will prove a fairly strong result in this direction, to the effect that "almost every" eigenvalue λ satisfies $V(\lambda) \in \Lambda(\lambda)$. As already mentioned in Remark 3.1.5,[15] the spectrum $\sigma(\hat{\varphi})$ is a countable subset of $[c, \infty)$ for some $c > 0$, whose elements can be arranged in an increasing sequence $\{\lambda_j\}_{j \geq 1}$ with $\lambda_j \to 0$. We may replace $\hat{\varphi}$ by its self-adjoint Friedrichs extension to $L^2(\mathbb{R})$ without affecting these statements, cf. [LST].

Suppose $\mathbf{P}$ is a proposition (that can be true or false) about elements of $\sigma(\hat{\varphi})$. Write $N(\lambda) := |\{j \in \mathbb{N} \mid \lambda_j \leq \lambda\}|$ and

$$N_{\mathbf{P}}(\lambda) := |\{j \in \mathbb{N} \mid \lambda_j \leq \lambda \text{ and } \mathbf{P}(\lambda_j) \text{ holds}\}|.$$

We will say that $\mathbf{P}$ *holds asymptotically* if

$$\lim_{\lambda \to \infty} \frac{N_{\mathbf{P}}(\lambda)}{N(\lambda)} = 1. \tag{3.3.1}$$

Theorem 3.3.1. *In the setting of Theorem 3.2.2, the* "$\subseteq$" *direction of* (3.1.12) *holds asymptotically.*

Proof. The statement $\mathbf{P}(\lambda_j)$ about eigenvalues here is, of course, that $\nu(\lambda_j) \in \mathbb{Z}$.[16] From Lemma 3.1.1(c), we know that $\nu(a) = \frac{r^\circ}{8\pi^2} \log^2 a + O(\log a)$, whence

$$N(\lambda) \geq N_{\mathbf{P}}(\lambda) \geq \lfloor \nu(\lambda) - \nu(|\hat{a}|) \rfloor \geq \tfrac{r^\circ}{8\pi^2} \log^2 \lambda + O(\log \lambda). \tag{3.3.2}$$

Now given $f, g \in L^2(\mathbb{R})$, write $\langle f, g \rangle := \int_{\mathbb{R}} f(r)\overline{g(r)} dr$, and

$$\tilde{f}(y_1, y_2) := 2^{-5/4} \pi^{-3/2} \int_{\mathbb{R}} e^{-\frac{1}{4\pi}\{(r-y_1)^2 + 2\mathbf{i} y_2 r\}} f(r)\, dr \tag{3.3.3}$$

for the *coherent state transform* of f. Adapting the calculations of [LST, §3.1] to our setting

[15] The point is that the proof of [LST, Prop. 3.4] trivially generalizes to all φ we consider here, because Δ always contains a reflexive triangle (or square). The proof of Theorem 3.3.1 involves, in contrast, a rather nontrivial generalization of [op. cit., §3.2].

[16] We can always throw out a finite set of eigenvalues less than $|\hat{a}|$, if they exist (cf. Remark 3.1.5).

gives

$$\langle \hat{\varphi} f, f \rangle = \iint_{\mathbb{R}^2} \Phi(y_1, y_2)\, |\tilde{f}(y_1, y_2)|^2 dy_1\, dy_2 \tag{3.3.4}$$

where

$$\Phi(y_1, y_2) := \sum_{\underline{m} \in \partial\Delta \cap \mathbb{Z}^2} \underbrace{a_{\underline{m}} e^{-\frac{\pi}{2}(m_1^2 + m_2^2)}}_{=: \tilde{a}_{\underline{m}}} e^{m_1 y_1 + m_2 y_2}. \tag{3.3.5}$$

This implies, for instance, the semi-boundedness of $\hat{\varphi}$, as $\Phi \geq c := \min_{\underline{y} \in \mathbb{R}^2} \Phi(\underline{y}) > 0 \implies \hat{\varphi} \geq c \cdot \mathrm{Id} \implies \sigma(\hat{\varphi}) \subset [c, \infty)$.

Let $(\cdot)_+$ be the function on $\mathbb{R}$ defined by $(s)_+ = s$ for $s \geq 0$ and $(s)_+ = 0$ for $s \leq 0$, and note that

$$\int_0^\lambda N(s)\, ds = \sum_{j \geq 1} (\lambda - \lambda_j)_+. \tag{3.3.6}$$

Reasoning with Jensen's inequality as in [op. cit., §2.2], we have

$$\sum_{j \geq 1} (\lambda - \lambda_j)_+ \leq \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} (\lambda - \Phi(y_1, y_2))_+\, dy_1\, dy_2. \tag{3.3.7}$$

Choose $M > 0$ so that $M \tilde{a}_{\underline{m}} \geq a_{\underline{m}}$ $(\forall \underline{m} \in \partial\Delta \cap \mathbb{Z}^2)$. Writing $Y_j := e^{y_j}$ and $\Gamma_L := \{\underline{Y} \in \mathbb{R}_+^2 \mid L \geq \varphi(Y_1, Y_2)\}$, note that the boundary $\partial\Gamma_L$ is the cycle β on E_{-L}. Together with Lemma 3.1.1(a) and (2.2.6), this gives

$$\begin{aligned} \mathrm{RHS}(3.3.7) &\leq \frac{1}{4\pi^2 M} \iint_{\mathbb{R}^2} (M\lambda - \varphi(Y_1, Y_2))_+ \frac{dY_1}{Y_1} \frac{dY_2}{Y_2} \\ &\leq \frac{\lambda}{4\pi^2} \iint_{\Gamma_{M\lambda}} \frac{dY_1}{Y_1} \frac{dY_2}{Y_2} = \frac{\lambda}{4\pi^2} R_\beta(-M\lambda) \\ &= \frac{r^\circ}{8\pi^2} \lambda \log^2 \lambda + O(\log \lambda). \end{aligned} \tag{3.3.8}$$

Putting the last three equations together, we get

$$\frac{r^\circ}{8\pi^2} \log^2 \lambda + O(\log \lambda) \geq N(\lambda), \tag{3.3.9}$$

which combined with (3.3.2) gives the result. □

The constraints imposed on the zero locus of $\boldsymbol{\rho} \circ \boldsymbol{\nu}$ by its interpretation as eigenvalues of $\hat{\varphi}$ (Theorem 3.2.2), and vice versa (Theorem 3.3.1), seem worth exploring further. For instance, per Remark 3.1.5, we expect (and know in some cases) that $c > |\hat{a}|$; together with the following Lemma, this essentially rules out points $a \in U$ at which $V(a) \in \Lambda(a)$ (the exact quantization condition) and $\mathcal{R}(a)$ is torsion (the perturbative quantization condition proposed in [GS]).

Lemma 3.3.2. *For $a \in (|\hat{a}|, \infty)$, $\mathcal{R}(a) \in H_1(E_a, \mathbb{C}/\mathbb{Z}(2))$ is a nontorsion class.*

Proof. From the known integrality of local instanton numbers of toric CY 3-folds [Ko], it follows that LHS(3.1.7) ≥ 1, hence that $\Re(t(\hat{a})) \geq 0$. From (3.1.3) (and positivity of coefficients of φ, and negativity of $\hat{a}$), it is immediate that $t(|\hat{a}|) > \Re(t(\hat{a}))$, hence $t(a) \in \mathbb{R}_+$ for $a \in (|\hat{a}|, \infty)$. But if $\mathcal{R}(a)$ is torsion, then $R_\gamma(a) \in \mathbb{Q}(2) \implies t(a) \in \mathbb{Q}(1) \subset \mathbf{i}\mathbb{R}$. □

More striking is a conditional transcendence result on the eigenvalues that arises from their asymptotic Hodge-theoretic interpretation in Theorem 3.3.1. A mixed version of the Grothendieck period conjecture (which we will simply call the GPC) says that the transcendence degree of a period point arising from a motive defined over $\bar{\mathbb{Q}}$ is equal to the dimension of the minimal mixed Mumford-Tate domain containing it. The (mixed) motive in question is the K_2-cycle $\{-x_1, -x_2\}$ on E_a, with MHS the extension of $\mathbb{Z}(0)$ by $H^1(E_a, \mathbb{Z}(2))$ given by $\frac{1}{(2\pi\mathbf{i})^2}\mathcal{R}$. The possibillities for the M-T group are an extension of SL_2 or a 1-torus (depending on whether E_a is CM) by $\mathbb{G}_a^{\times 2}$ or $\{1\}$ (depending on whether $\mathcal{R}$ is torsion); the corresponding domain is $\mathfrak{H}$, a CM point in it, or the product of either one with $\mathbb{C}^2$. The coordinates of the period point are $\Omega(a)$ (in $\mathfrak{H}$) and $(\frac{R_\gamma(a)}{(2\pi\mathbf{i})^2}, \frac{R_\beta(a)}{(2\pi\mathbf{i})^2})$ (in $\mathbb{C}^2$).[17]

[17] We have to divide by $(2\pi\mathbf{i})^2$, of course, because a torsion class must have coordinates in $\mathbb{Q}$, not transcendental ones in $\mathbb{Q}(2)$.

Conjecture 3.3.3 (GPC)**.** *If $a \in \bar{\mathbb{Q}}$ and $\mathcal{R}(a)$ is nontorsion, then the transcendence degree of $\bar{\mathbb{Q}}(\Omega(a), \frac{R_\gamma(a)}{(2\pi\mathbf{i})^2}, \frac{R_\beta(a)}{(2\pi\mathbf{i})^2})/\bar{\mathbb{Q}}(\Omega(a))$ is 2.*

Proposition 3.3.4. *Assuming the GPC, asymptotically $\sigma(\hat{\varphi})$ consists of transcendental numbers.*

Proof. Let $\lambda \in \sigma(\hat{\varphi})$ be an eigenvalue for which $\nu(\lambda) \in \mathbb{Z}$. (We may assume $\lambda \in (|\hat{a}|, \infty)$.) That is, we have an algebraic relation $\frac{1}{4\pi^2}(R_\gamma(\lambda)\Omega(\lambda_i) - R_\beta(\lambda)) = n$ on $\frac{R_\gamma(\lambda)}{(2\pi\mathbf{i})^2}$ and $\frac{R_\beta(\lambda)}{(2\pi\mathbf{i})^2}$ over $\bar{\mathbb{Q}}(\Omega(\lambda))$. By the GPC, either $\lambda \notin \bar{\mathbb{Q}}$ or $\mathcal{R}(\lambda)$ is torsion. But the latter possibility is ruled out by Lemma 3.3.2, and so we are done by Theorem 3.3.1. □

We conclude with somthing of a curiosity: in case $\varphi = x_1 + x_1^{-1} + x_2 + x_2^{-1} + x_1x_2^{-1} + x_1^{-1}x_2$, our normal function is closely related to the Feynman integral $\mathcal{I}$ associated to the sunset graph with equal masses [BKV]. This is written in [op. cit.] as a function of $s = \frac{1}{3-a}$ = the inverse norm of the external momentum, but written as a function of a we have $\mathcal{I}(a) = \frac{(2\pi\mathbf{i})^2}{a}V(a)$ (see [op. cit., (7.17)]). The condition that $V(a) \in \Lambda(a)$ means that V, or equivalently $\mathcal{I}$, belongs to its own lattice of ambiguities under monodromy. As we have seen, the values of a at which this happens correspond to eigenvalues of $\hat{\varphi}$. One wonders if there is any deeper physical relation here between Feynman amplitudes and quantum curves.

Chapter 4

Regulator periods at the maximal conifold point.

In this section we prove Conjecture 2.4.2 in the cases $(m,n) = (g,g)$ and $(2g-1,1)$, for every $g \geq 1$.

Because we have to enumerate multiple nodes on the maximal conifold curve, it is better in this section to replace (x_1, x_2) as toric coordinates by (x, y), which we do throughout. We also denote the zero-locus of a polynomial by $\mathbf{Z}(\cdot)$.

4.1 The main result and some preliminaries

Consider the families of genus-g curves cut out of $(\mathbb{C}^*)^2$ by the (integrally tempered) polynomials $F_{g,g}(x,y)$ and $F_{2g-1,1}(x,y)$ from (2.4.8). In contrast to §2, $\mathcal{C}_{g,g}$ and $\mathcal{C}_{2g-1,1}$ will denote their *compactifications* in $\mathbb{P}_\Delta$. There are no mass parameters in either case, so $r = 3$ and the equations take the simpler form (2.4.7). Moreover, $\mathcal{C}_{g,g}$ is torically equivalent to $\mathcal{C}_{2g-1,1}$ via the map $u = x^{-1}y^{-1}$, $v = x^g y^{g-1}$. The effect of this map is straightforward: for $n = 1, \ldots, g$ it simply shifts $n \mapsto g - n + 1$ on the level of indices; that is, if $F_{g,g}(x,y)$ is written with parameters a_n, then the image (under the above map) is precisely $F_{2g-1,g}(u,v)$ with parameters a_{g-n+1}. The upshot of this connection is that statements concerning regulator periods of $\mathcal{C}_{2g-1,1}$ can be pulled back to those corresponding to $\mathcal{C}_{g,g}$, provided we choose the correct cycles. For our purposes here, the important case is that the cycle γ_{g-n+1} of $\mathcal{C}_{2g-1,1}$ giving rise to $R_{\gamma_{g-n+1}} \sim -2\pi\mathbf{i}\log(a_{g-n+1})$ pulls back to the cycle γ_n of $\mathcal{C}_{g,g}$ corresponding to

$R_{\gamma_n} \sim -2\pi\mathbf{i}\log(a_n)$.

Theorem 4.1.1. *Conjecture 2.4.2 holds for the families $\mathcal{C}_{g,g}$ and $\mathcal{C}_{2g-1,1}$; that is,*

$$\tfrac{1}{2\pi\mathbf{i}}R_{\gamma_1}(\underline{\hat{a}}) \underset{\mathbb{Q}(1)}{\equiv} \mathcal{D}_{g,g} \text{ and} \tag{4.1.1}$$

$$\tfrac{1}{2\pi\mathbf{i}}R_{\gamma_g}(\underline{\hat{a}}) \underset{\mathbb{Q}(1)}{\equiv} \mathcal{D}_{2g-1,g}. \tag{4.1.2}$$

Remark 4.1.2. The predictions of [CGM] aligning with Conjecture 2.4.2 are written in terms of the complex structure parameters $z_i := z_i(\underline{a})$. Translated into statements about the corresponding regulator periods (cf. (2.3.4)), these essentially amount to[1]

$$\tfrac{1}{2\pi\mathbf{i}}\textstyle\sum_{i=1}^{g}[C^{-1}]_{1j}R_{\alpha_i}(\underline{\hat{z}}) \underset{\mathbb{Q}(1)}{\equiv} \mathcal{D}_{m,n}, \tag{4.1.3}$$

which of course is equivalent to (2.4.10). While z_i and R_{α_i} are more natural from the standpoint of GKZ systems, the $\{a_j\}$ and the corresponding regulator periods R_{γ_j} simplify the statement of the result, and are more natural to compute directly (cf. Appendix A). As we will see, the $\{\gamma_j\}$ are also the cycles which limit to loops passing through individual nodes at the maximal conifold point $\underline{\hat{a}}$.

Remark 4.1.3. As $R\{-x,-y\} \equiv R\{x,y\}$ mod $\mathbb{Q}(2)$ we may work with the latter. Note also that (2.4.10) is stated in terms of the regulator period asymptotic to $-2\pi\mathbf{i}\log(a_n)$; it is convenient in this section to drop the negative sign and work with one asymptotic to $2\pi\mathbf{i}\log(a_n)$. Thus from now on

$$R_{\gamma_n} \sim 2\pi\mathbf{i}\log(a_n).$$

Furthermore, since we intend to investigate different components of the discriminant locus throughout this section, it will be important to track the moduli; so henceforth we will

[1] Here $[C^{-1}]$ is the inverse of the first $g \times g$ minor of the intersection matrix $[C]$. The R_{α_i} "correspond" to z_i in the sense of being asymptotic to $2\pi\mathbf{i}\log(z_i)$.

rename $F_{g,g}$ and $F_{2g-1,1}$ to $F^{\underline{a}}_{g,g}$ and $F^{\underline{a}}_{2g-1,1}$.

Let us outline a proof of Theorem 4.1.1. Denote by $\hat{\mathcal{C}}_{g,g}$ the fiber of the family over the *maximal conifold* point $\underline{\hat{a}}$. It has g nodes $\{\hat{p}_j\}$, and the cycles $\{\hat{\gamma}_j\}_{j=1}^{g}$ passing through each node generate $H_1(\hat{\mathcal{C}}_{g,g})$; we set $R_{\hat{\gamma}_j} := \int_{\hat{\gamma}_i} R\{x,y\}$. Writing $\boldsymbol{\kappa} = {}_{\underline{\hat{\gamma}}}[\mathrm{Id}]_{\underline{\gamma}(\hat{a})}$ for the change-of-basis matrix, we have

Proposition 4.1.4. *Let* $\kappa_j := \gcd(2j-1, 2g+1)$*. Then*

$$\boldsymbol{\kappa} = \mathrm{diag}(\kappa_1, \ldots, \kappa_g). \tag{4.1.4}$$

It then follows from temperedness that

$$\tfrac{1}{2\pi\mathbf{i}} R_{\gamma_j}(\underline{\hat{a}}) \underset{\mathbb{Q}(1)}{\equiv} \tfrac{\kappa_n}{2\pi\mathbf{i}} R_{\hat{\gamma}_j}. \tag{4.1.5}$$

In §4.2 we detect monodromies via power series representing classical periods, verifying Proposition 4.1.4 in the process. In §4.3 we use a key technique developed in [DK, §6] (cf. Appendix B) that allows us to connect conifold limits of regulator periods to special values of the Bloch-Wigner function; this method coupled with Proposition 4.1.4 settles Theorem 4.1.1. As a consequence *g-many* series identities are borne out in §4.4 — not just the two required for the Theorem.

We conclude this subsection with two preliminary results. The first will help us to control certain power series asymptotics, and the second gives us information on nodal fibers of $\mathcal{C}_{g,g}$.

Lemma 4.1.5. *If* $a, b, c \in \mathbb{R}_{\gg 0}$ *are such that* $a = 2b + c$*, then*

$$\frac{\Gamma(1+a)}{\Gamma^2(1+b)\Gamma(1+c)} \sim \frac{1}{2\pi b}\sqrt{\frac{a}{c}}\left(\frac{a}{c}\left(\frac{c}{b}\right)^{2b/a}\right)^{a}. \tag{4.1.6}$$

Proof. Stirling's approximation yields

$$\frac{\Gamma(1+a)}{\Gamma^2(1+b)\Gamma(1+c)} \sim \frac{1}{2\pi b}\sqrt{\frac{a}{c}}\frac{a^a}{b^{2b}c^c}e^{-a+2b+c} = \frac{1}{2\pi b}\sqrt{\frac{a}{c}}\frac{a^a}{b^{2b}c^{a-2b}}$$
$$= \frac{1}{2\pi b}\sqrt{\frac{a}{c}}\frac{a^a}{c^a}\frac{c^{2b}}{b^{2b}} = \frac{1}{2\pi b}\sqrt{\frac{a}{c}}\left(\frac{a}{c}\left(\frac{c}{b}\right)^{2b/a}\right)^a$$

for $b, c \to \infty$ (and $a = 2b + c$). □

Lemma 4.1.6. *Suppose that the fiber over* $\underline{\tilde{a}} = (\tilde{a}_1, \ldots, \tilde{a}_g)$ *has* g*-many singularities, say* $\tilde{p}_j := (\tilde{x}_j, \tilde{y}_j), n = 1, \ldots, g$*. Then for each* j*,* $\tilde{p}_j$ *is a node, and* $\tilde{x}_j = \tilde{y}_j$*.*

Proof. Since $x\partial_x F^{\underline{a}}_{g,g}(x,y) - y\partial_y F^{\underline{a}}_{g,g}(x,y) = x - y$, any singularity must have symmetric co-ordinates; that is, $\tilde{x}_j = \tilde{y}_j$. By toric equivalence we may replace $F^{\underline{\tilde{a}}}_{g,g}(x,y)$ by

$$F^{\underline{\tilde{a}}}_{2g-1,g}(u,v) = u + v + \textstyle\sum_{\ell=1}^{g} \tilde{a}_\ell u^{-\ell+1} + u^{-2g+1}v^{-1} \tag{4.1.7}$$

(reversing the order of the $\{a_\ell\}$); by abuse of notation we continue to label the singularities of $F^{\underline{\tilde{a}}}_{2g-1,1}$ by $\tilde{p}_j$, but with coordinates $(\tilde{u}_j, \tilde{v}_j)$ satisfying $\tilde{u}_j^{-2g+1} = \tilde{v}_j^2$. Since the edge polynomials of (4.1.7) are all $w + 1$, the curve intersects each component of the toric boundary with multiplicity 1, and so all $\tilde{p}_j \in \mathbb{C}^* \times \mathbb{C}^*$. Moreover, (4.1.7) is irreducible since it is quadratic in v, with discriminant $\mathscr{D}(u)$ of odd degree. As a consequence, the vanishing cycle sequence associated to the smoothing $F^{\underline{\tilde{a}}}_{2g-1,1} + s$ takes the form

$$0 \to H^1(\mathcal{C}^{\underline{\tilde{a}}}_{2g-1,1}) \to H^1_{\lim} \to H^1_{\mathrm{van}} \to 0. \tag{4.1.8}$$

Since $\mathrm{rk}(F^1 H^1_{\lim}) = g$ and the g singularities each contribute nontrivially to $\mathrm{rk}(F^1 H^1_{\mathrm{van}})$, each contribution must be exactly 1. So the $\tilde{p}_j$ are either nodes or cusps, and to show they are nodes it will suffice to show that the Hessians $H_{F^{\underline{\tilde{a}}}_{2g-1,1}}$ is non-degenerate at $\tilde{p}_j$.

To do this, define

$$\tilde{P}(u) := 2g+1+\textstyle\sum_{j=1}^{g}(2g+1-2j)\tilde{a}_j u^{-j}, \tag{4.1.9}$$

and observe that

$$\tilde{P}(\tilde{u}_j) = \tfrac{2g-1}{\tilde{u}_j}F_{2g-1,1}^{\tilde{a}}(\tilde{p}_j) + 2\partial_u F_{2g-1,1}^{\tilde{a}}(\tilde{p}_j) = 0. \tag{4.1.10}$$

Thus $\mathbf{Z}(\tilde{P}) = \{\tilde{u}_1, \ldots, \tilde{u}_g\}$. It follows that $\tilde{P}$ has no repeated roots; that is, $\tilde{P}'(\tilde{u}_j) \neq 0$ $(\forall j)$. To compute the Hessians, write

$$\begin{aligned}\partial_{uu}F_{2g-1,1}^{\tilde{a}}(\tilde{p}_j) &= \textstyle\sum_{\ell=1}^{g}\ell(\ell-1)\tilde{a}_\ell\tilde{u}_j^{-\ell-1} + 2g(2g-1)\tilde{u}_j^{-2g-1}\tilde{v}_n^{-1} \\ &= \textstyle\sum_{\ell=1}^{g}\ell(\ell-1)\tilde{a}_\ell\tilde{u}_j^{-\ell-1} + \tfrac{2g(2g-1)\tilde{y}_j}{\tilde{u}_j^2}, \end{aligned} \tag{4.1.11}$$

$$\partial_{uv}F_{2g-1,1}^{\tilde{a}}(\tilde{p}_j) = (2g-1)\tilde{u}_j^{-2g}\tilde{v}_j^{-2} = \tfrac{2g-1}{\tilde{v}_j}, \text{ and} \tag{4.1.12}$$

$$\partial_{vv}F_{2g-1,1}^{\tilde{a}}(\tilde{p}_j) = 2\tilde{u}_j^{2g-1}\tilde{v}_j^{-3} = \tfrac{2}{\tilde{v}_j}. \tag{4.1.13}$$

At this point a few simplifications can be made. Differentiating the defining equation of $\tilde{P}$ and plugging in $u = \tilde{u}_j$, we obtain,

$$\tilde{P}'(\tilde{u}_j) = 2\textstyle\sum_{\ell=1}^{g}\ell(\ell-1)\tilde{a}_\ell\tilde{u}_j^{-\ell-1} - \sum_{\ell=1}^{g}(2g-1)\ell\tilde{a}_\ell\tilde{u}_j^{-\ell-1} \tag{4.1.14}$$

On the other hand $\partial_u(F_{2g-1,1}^{\tilde{a}}(u,v)/u)$ vanishes at $\tilde{p}_j$, which yields

$$\begin{aligned} -\tfrac{\tilde{v}_j}{\tilde{u}_j^2} - \textstyle\sum_{\ell=1}^{g}\ell\tilde{a}_\ell\tilde{u}_j^{-\ell-1} - 2g\tilde{u}_j^{-2g-1}\tilde{v}_j^{-1} = 0 \\ \implies \textstyle\sum_{\ell=1}^{g}(2g-1)j\tilde{a}_\ell\tilde{u}_j^{-\ell-1} = -\tfrac{(2g-1)(2g+1)\tilde{v}_j}{\tilde{u}_j^2} \end{aligned} \tag{4.1.15}$$

Combining everything, we arrive at

$$\partial_{uu} F^{\tilde{a}}_{2g-1,1}(\tilde{p}_j) = \tfrac{(2g-1)^2 \tilde{v}_j}{2\tilde{u}_j^2} + \tfrac{\tilde{P}'(\tilde{u}_j)}{2} \tag{4.1.16}$$

Therefore,

$$\begin{aligned} H_{F^{\tilde{a}}_{2g-1,1}}(\tilde{p}_j) &= \left(\partial_{uv} F^{\tilde{a}}_{2g-1,1}(\tilde{p}_j)\right)^2 - \partial_{uu} F^{\tilde{a}}_{2g-1,1}(\tilde{p}_j)\partial_{vv} F^{\tilde{a}}_{2g-1,1}(\tilde{p}_j) \\ &= \tfrac{(2g-1)^2}{\tilde{u}_j^2} - \tfrac{(2g-1)^2}{\tilde{u}_j^2} - \tfrac{\tilde{P}'(\tilde{u}_j)}{\tilde{v}_j} \;=\; -\tfrac{\tilde{P}'(\tilde{u}_j)}{\tilde{v}_j} \neq 0 \end{aligned}$$

as was to be shown. □

4.2 Monodromy calculations via power series

Consider a 1-parameter family of curves $\mathcal{C} \to \mathbb{P}^1$ with coordinate t, endowed with a section ω of the relative dualizing sheaf; on smooth fibers $\mathcal{C}_t$, ω_1 is a holomorphic 1-form. Assume that $\mathcal{C}_c$ has a single node p_c (i.e. is a "conifold fiber"), and let δ_0 be the "conifold" vanishing cycle pinched at p_c. Writing ε_0 for a cycle invariant about $t = 0$, its monodromy about $t = c$ is a multiple of δ_0, say $\boldsymbol{k}\delta_0$ for some $\boldsymbol{k} \in \mathbb{Z}_{\geq 0}$. We would like to compute this *conifold multiple* $\boldsymbol{k}$.

Writing $\epsilon_0(t) = \sum_{m \geq 0} b_m t^m := \int_{\varepsilon_0} \omega_t$, we have

$$\int_{\boldsymbol{k}\delta_0} \omega_t = (T_c - I)\epsilon_0 = 2\pi \mathbf{i} C_0 + O(t-c) \tag{4.2.1}$$

for some $C_0 \in \mathbb{C}$. Observe that

$$\int_{\boldsymbol{k}\delta_0} \omega_c = \boldsymbol{k} \int_{\delta_0} \omega_c = \boldsymbol{k} \cdot 2\pi \mathbf{i} \cdot \operatorname*{Res}_{p_c} \omega_c \implies C_0 = \boldsymbol{k} \cdot \operatorname*{Res}_{p_c} \omega_c. \tag{4.2.2}$$

On the other hand, [Ke2, Lemma 6.4] (with $B(t) = \epsilon_0(t)$, $\lambda = 2\pi\mathbf{i}C_0$, and $w = 1$) yields

$$b_m \sim \frac{C_0}{c^m \cdot m}. \tag{4.2.3}$$

provided $C_0 \neq 0$.[2] Therefore we have proven

Lemma 4.2.1. *The conifold multiple is computed by*

$$\boldsymbol{k} = \frac{\lim_{m\to\infty} b_m \cdot c^m \cdot m}{\mathrm{Res}_{p_c}\ \omega_c}. \tag{4.2.4}$$

Example 4.2.2. Consider the Legendre family, $y^2 = x(x-1)(x-t)$. Setting $c = 1$ gives rise to a node at $(1, 0)$. Taking $\omega_t = \frac{dx}{y}$, we have

$$\mathrm{Res}_{(1,0)}\ \omega_c = \mathrm{Res}_{x=1} \tfrac{dx}{(x-1)\sqrt{x}} = 1. \tag{4.2.5}$$

Moreover $b_m = 2\pi\binom{-1/2}{m}^2$, hence (4.2.4) implies

$$\boldsymbol{k} = \lim_{m\to\infty} 2\pi m \binom{-1/2}{m}^2 = 2. \tag{4.2.6}$$

Example 4.2.3. Now consider the family $\mathcal{C}_t$ defined by $f_t(x, y) = xy - t^{1/3}(x^3 + y^3 + 1)$. In this case $c = \frac{1}{3^3}$ and $b_m = \frac{(3m)!}{m!^3}$, but $\mathcal{C}_c = \mathbf{Z}(\prod_{\ell=1}^{3}(1 + \zeta_3^i x + \zeta_3^{2i} y))$ is a Néron 3-gon with *three* nodes p_i. But since $\varepsilon_0(c)$ will pass through each p_i the same number $\boldsymbol{k_0}$ of times, and ω_c must have the same residue at each, (4.2.4) holds (taking say $p_c = p_1 := (1, 1)$) provided we interpret $\boldsymbol{k}$ as $3\boldsymbol{k_0}$. For the residue of

$$2\pi\mathbf{i}\omega_c = \mathrm{Res}_{\mathcal{C}_c} \frac{dx \wedge dy}{f_c} = \frac{dx}{\partial_y f_c} = \frac{dx}{x - y^2} \tag{4.2.7}$$

[2] Otherwise, B_m has a smaller exponential growth-rate and RHS(4.2.4) is zero, which confirms the Lemma when $C_0 = 0$ as well.

at p_1, we can restrict to the component $X_c := \mathbf{Z}(1+\zeta_3 x+\zeta_3^2 y)$:

$$\begin{aligned}\mathrm{Res}_{p_1}\omega_c &= \frac{1}{2\pi\mathbf{i}}\mathrm{Res}_{(1,1)}\left(\frac{dx}{x-y^2}\bigg|_{X_c}\right) = \frac{1}{2\pi\mathbf{i}}\mathrm{Res}_{y=1}\left(\frac{\zeta_3 dy}{y^2+\zeta_3 y+\zeta_3^2}\right) \\ &= \frac{1}{2\pi\mathbf{i}}\frac{\zeta_3}{1-\zeta_3^2} = \frac{1}{2\pi\sqrt{3}}. \end{aligned} \tag{4.2.8}$$

Since $b_m = \frac{(3m)!}{m!^3}$ we get

$$\boldsymbol{k} = \lim_{m\to\infty}\frac{1}{3^{3m}}\cdot m\cdot\frac{(3m)!}{m!^3}\cdot 2\pi\sqrt{3} = 3, \tag{4.2.9}$$

which means that $\varepsilon_0(c)$ winds once around the Néron 3-gon.

For the proof of Proposition 4.1.4, we need to compute the Picard-Lefschetz matrix $\boldsymbol{\kappa}$, whose entries $\boldsymbol{\kappa}_{ij}$ tell how many times the specialization $\gamma_i(\underline{\hat{a}})$ passes through $\hat{p}_j$. In order to invoke Lemma 4.2.1 for this purpose, we should reinterpret these numbers as (roughly speaking) conifold multiples for 1-parameter subfamilies of $\mathcal{C}_{\underline{a}}$ acquiring a *single* node. The idea is that $\underline{\hat{a}}$ is a normal-crossing point of the discriminant locus, whose g local-analytic irreducible components each parametrize fibers carrying a single node p_j. These are labeled in such a way that the j^{th} component can be followed out to where it meets the a_j-axis at $a_j = \mathring{a}_j$. Call this fiber $\mathcal{C}_{g,g}^{\mathring{a}_j}$, and $\mathring{p}_j = (\mathring{x}_j, \mathring{x}_j)$ for the limit of the node to it.

From Appendix A we have the 1-forms

$$\varpi_j = \tfrac{1}{2\pi\mathbf{i}}\nabla_{\delta_{a_j}}R\{x,y\} = \frac{-a_j}{2\pi\mathbf{i}}\mathrm{Res}_{\mathcal{C}_{g,g}}\left(\frac{dx\wedge dy}{x^j y^j F_{g,g}(x,y)}\right) \tag{4.2.10}$$

and 1-cycles γ_j $(j=1,\ldots,g)$. The computation that follows will consider periods $\Pi_{jj} = \int_{\gamma_j}\varpi_j$ on the 1-parameter families over the a_j-axes (acquiring a single node at $a_j = \mathring{a}_j$), which will suffice to determine the diagonal terms $\boldsymbol{\kappa}_{jj}$. That the remaining, off-diagonal terms are actually zero follows from the fact (cf. Appendix A) that each γ_j is well-defined on a tubular neighborhood of the hyperplane in (compactified) moduli defined by $z_j = 0$, which is cut by the conifold components carrying p_i for every $i \neq j$.

Now $\mathcal{C}^{\mathring{a}_j}_{g,g}$ is defined by

$$f^{(j)}_{g,g} := F^{\mathring{a}_j}_{g,g}(x,y) = x + y + \mathring{a}_n x^{1-j} y^{1-j} + x^{-g} y^{-g}, \tag{4.2.11}$$

and to find the node $\mathring{p}_j$ we solve

$$\mathring{x}_j^{2g} f^{(j)}_{g,g}\Big|_{x=y=\mathring{x}_j} = 2\mathring{x}_j^{2g+1} + 1 + \mathring{a}_j \mathring{x}_j^{2g-2j+2} = 0, \tag{4.2.12}$$

$$\mathring{x}_j^{2g+1} \partial_x f^{(j)}_{g,g}\Big|_{x=y=\mathring{x}_j} = \mathring{x}_j^{2g+1} - g - (j-1)\mathring{a}_j \mathring{x}_j^{2g-2j+2} = 0. \tag{4.2.13}$$

to obtain

$$\mathring{x}_j = \sqrt[2g+1]{\frac{g-j+1}{2j-1}}, \tag{4.2.14}$$

$$\mathring{a}_j = -\frac{2g+1}{2j-1}\left(\frac{2g+1}{g-j+1}\right)^{\frac{2(g-j+1)}{2g+1}}. \tag{4.2.15}$$

In particular, we have the relation

$$\mathring{a}_j \mathring{x}_j^{2(g-j+1)} = -\frac{2g+1}{2j-1}. \tag{4.2.16}$$

In order to calculate the residue of ϖ_j at $\mathring{p}_j$, recall that for any $f(x,y) = Ax^2 + Bxy + Cy^2 +$ higher order terms $\in \mathbb{C}[x,y]$, we have

$$\mathrm{Res}^2_{\underline{0}} \frac{dx \wedge dy}{f} := \mathrm{Res}_{\underline{0}} \left(\mathrm{Res}_{\boldsymbol{Z}(f)} \frac{dx \wedge dy}{f} \right) = \frac{1}{\sqrt{B^2 - 4AC}}. \tag{4.2.17}$$

Changing variables to $X := x - \mathring{x}_j$, $Y := y - \mathring{x}_j$ in $f_{g,g}^{(j)}(x,y)$ leads to the equation

$$\begin{aligned} x^g y^g f_{g,g}^{(j)} =& \tfrac{\mathring{x}_j^{2g-1}(2g^2+2g+1-(g-j+1)(2g+1))}{2} X^2 + \mathring{x}_j^{2g-1}\big(2g^2+2g-(g-j+1)(2g+1)\big)XY \\ &+ \tfrac{\mathring{x}_j^{2g-1}(2g^2+2g+1-(g-j+1)(2g+1))}{2} Y^2 + \text{ higher order terms.} \end{aligned} \tag{4.2.18}$$

Therefore

$$\begin{aligned} \operatorname{Res}^2_{\mathring{p}_j} \frac{dx \wedge dy}{x^g y^g f_{g,g}^{(j)}} &= \tfrac{1}{\mathring{x}_j^{2g-1}\sqrt{(2g^2+2g-(g-j+1)(2g+1))^2-(2g^2+2g+1-(g-j+1)(2g+1))^2}} \\ &= \tfrac{1}{\mathring{x}_j^{2g-1}\sqrt{(2g-2g-1)(4g^2+4g+1-2(g-j+1)(2g+1))}} \\ &= \tfrac{1}{\mathring{x}_j^{2g-1}\sqrt{-(2g+1)(2g+1-2g+2j-2)}} \\ &= \tfrac{\mathbf{i}}{\mathring{x}_j^{2g-1}\sqrt{(2g+1)(2j-1)}}. \end{aligned} \tag{4.2.19}$$

Consequently the residue of ϖ_j may now be found:

$$\begin{aligned} \operatorname{Res}_{\mathring{p}_j} \varpi_j &= \frac{-\mathring{a}_j}{2\pi\mathbf{i}} \operatorname{Res}^2_{\mathring{p}_j} \frac{dx \wedge dy}{x^j y^j f_{g,g}^{(j)}} \\ &= \frac{-\mathring{a}_j}{2\pi\mathbf{i}} \cdot \mathring{x}_j^{2(g-j)} \cdot \operatorname{Res}^2_{\mathring{p}_j} \frac{dx \wedge dy}{x^g y^g f_{g,g}^{(j)}} \\ &= \frac{-1}{2\pi} \cdot (\mathring{a}_j \mathring{x}_j^{2(g-j+1)}) \cdot \frac{1}{\mathring{x}_j^{2g+1}\sqrt{(2g+1)(2j-1)}} \\ &= \frac{\sqrt{2g+1}}{2\pi(g-j+1)\sqrt{(2j-1)}}. \end{aligned} \tag{4.2.20}$$

For the periods of ϖ_j, we start as in Appendix A with those of the regulator class.

Writing $\varphi_j := x^{j-1}y^{j-1}F^{\underline{a}}_{g,g}(x,y) - a_j$, (A.0.3) (with the sign flip from our choice of γ_j) yields

$$\frac{1}{2\pi\mathbf{i}}R_{\gamma_j}(\underline{a}) \underset{\mathbb{Q}(1)}{\equiv} \log(a_j) - \sum_{m>0} \frac{(-a_j)^{-m}}{m}[\varphi_j^m]_{\underline{0}}$$

$$= \log(a_j) - \sum_{m>0} \frac{(-a_j)^{-m}}{m} \times \tag{4.2.21}$$

$$[(\underbrace{x^j y^{j-1}}_{=:A_j} + \underbrace{x^{j-1}y^j}_{=:B_j} + \textstyle\sum_{\substack{k=1\\k\neq j}}^{g} a_k \underbrace{x^{j-k}y^{j-k}}_{=:C_j^k} + \underbrace{x^{j-g-1}y^{j-g-1}}_{=:D_j})^m]_{\underline{0}}$$

where $[L]_{\underline{0}}$ stands for the constant term (in x, y) appearing in the Laurent polynomial L. Now, given $l_1, l_2, \cdots, l_g \in \mathbb{Z}$, we define

$$\mathfrak{l}_j := \frac{1}{2j-1}\left((2g+1)l_j + \sum_{\substack{k=1\\k\neq j}}^{g}(2k-1)l_k\right) \tag{4.2.22}$$

$$\mathfrak{l}'_j := \frac{1}{2j-1}\left((g-j+1)l_j + \sum_{\substack{k=1\\k\neq j}}^{g}(k-j)l_k\right), \text{ and put} \tag{4.2.23}$$

$$\mathcal{L}_j := \{(l_1, l_2, \cdots, l_g) \in \mathbb{Z}^g_{\geq 0} \mid \mathfrak{l}'_j \in \mathbb{Z}_{\geq 0}\} \setminus \{(0, \cdots, 0)\} \tag{4.2.24}$$

Note that $\mathfrak{l}'_j \in \mathbb{Z}_{\geq 0} \implies \mathfrak{l}_j \in \mathbb{Z}_{\geq 0}$. The upshot of this construction is if $L_j, L'_j \in \mathbb{Z}_{\geq 0}$ are such that

$$A_j^{L_j} B_j^{L'_j} \prod_{\substack{k=1\\k\neq j}}^{g-1} (C_j^k)^{l_k} D_j^{l_j} = 1 \text{ and} \tag{4.2.25}$$

$$L_j + L'_j + \sum_{k=1}^{g} l_k = m \tag{4.2.26}$$

then $L_j = L'_j = \mathfrak{l}'_j$ (by symmetry) and $m = \mathfrak{l}_j$. Thus the lattice $\mathcal{L}_j \subset \mathbb{Z}^g$ encodes all possible constant terms appearing in (4.2.21), giving

$$\frac{1}{2\pi\mathbf{i}}R_{\gamma_j}(\underline{a}) \underset{\mathbb{Q}(1)}{\equiv} \log(a_j) \ - \sum_{\mathcal{L}_j} \frac{\Gamma(\mathfrak{l}_j)}{\Gamma^2(1+\mathfrak{l}'_j)\prod_{k=1}^{g}\Gamma(1+l_k)} \frac{(-a_j)^{-\mathfrak{l}_j}}{\mathfrak{l}_j} \prod_{\substack{k=1\\k\neq j}}^{g} a_k^{l_k}. \tag{4.2.27}$$

For the classical periods $\Pi_{j\ell} = \int_{\gamma_j} \varpi_\ell = \frac{1}{2\pi\mathbf{i}} \delta_{a_\ell} R_{\gamma_j}$, it is clear from (4.2.27) that $\Pi_{j\ell}$ vanishes on the a_j-axis for $\ell \neq j$. Focusing then on

$$\Pi_{jj}(\underline{a}) = \int_{\gamma_j} \varpi_j = 1 + \sum_{\mathcal{L}_j} \frac{\Gamma(1+\mathfrak{l}_j)}{\Gamma^2(1+\mathfrak{l}'_j)\prod_{k=1}^{g}\Gamma(1+l_k)} (-a_j)^{-\mathfrak{l}_j} \prod_{\substack{k=1 \\ k\neq j}}^{g} a_k^{l_k}, \tag{4.2.28}$$

we set $\underline{a}_i = 0$ for $i \neq j$ to obtain

$$\mathcal{S} := 1 + \sum_{\frac{g-j+1}{2j-1} l_j \in \mathbb{Z}_{>0}} \frac{\Gamma(1+\frac{2g+1}{2j-1} l_j)}{\Gamma^2(1+\frac{g-j+1}{2j-1} l_j)\Gamma(1+l_j)} (-a_j)^{-\frac{2g+1}{2j-1} l_j}. \tag{4.2.29}$$

Recall that $\kappa_j := \gcd(2j-1, 2g+1)$, and set

$$\begin{aligned} n_j &:= \frac{2j-1}{\kappa_j}, \qquad & m_j &:= \frac{2g+1}{\kappa_j} = \frac{(2g+1)n_j}{2j-1}, \\ r_j &:= \frac{l_j}{n_j}, \qquad \text{and} \qquad & s_j &:= a_j^{-m_j}. \end{aligned} \tag{4.2.30}$$

Clearly $n_j, m_j, r_j \in \mathbb{Z}_{>0}$. Now we have a power series of the form

$$\mathcal{S} = 1 + \sum_{r_j \in \mathbb{N}} \frac{(-1)^{m_j r_j}\Gamma(1+m_j r_j)}{\Gamma^2(1+\frac{m_j-n_j}{2} r_j)\Gamma(1+n_j r_j)} s_j^{r_j} =: \sum_{r_j} b_{r_j} s_j^{r_j}. \tag{4.2.31}$$

Let $\mathring{s}_j := \mathring{a}_j^{-m_j}$. Applying Lemma 4.1.5,

$$\frac{\Gamma(1+m_j r_j)}{\Gamma^2(1+\frac{m_j-n_j}{2} r_j)\Gamma(1+n_j r_j)} \approx \frac{(-1)^{m_j r_j} 2\sqrt{m_j}}{2\pi r_j (m_j - n_j)\sqrt{n_j}} \mathring{s}_j^{r_j} \tag{4.2.32}$$

from which we may conclude that

$$\lim_{r_j \to \infty} b_{r_j} \cdot r_j \cdot \mathring{s}_j^{r_j} = \frac{2\sqrt{m_j}}{2\pi(m_j - n_j)\sqrt{n_j}}. \tag{4.2.33}$$

Observing that

$$\operatorname{Res}_{\mathring{p}_j}\varpi_j = \frac{\sqrt{2g+1}}{2\pi(g-j+1)\sqrt{(2j-1)}} = \frac{\sqrt{n_j}}{2\pi n_j(g-j+1)}\cdot\sqrt{\frac{(2g+1)n_j}{2j-1}}$$
$$= \frac{2\sqrt{m_j n_j}}{2\pi(m_j-n_j)(2j-1)}. \tag{4.2.34}$$

we apply (4.2.4) to obtain

$$\boldsymbol{\kappa}_{jj} = \frac{\lim_{r_j\to\infty} b_{r_j}\cdot r_j\cdot \mathring{s}_j^{r_j}}{\operatorname{Res}_{\mathring{p}_j}\varpi_j} = \frac{2j-1}{n_j} = \kappa_j. \tag{4.2.35}$$

This concludes the proof of Theorem 4.1.4.

Remark 4.2.4. Notice that $\kappa_1 = \kappa_g = 1$. We document $\underline{\kappa} := (\kappa_1, \ldots, \kappa_n)$ for $g = 1, \ldots, 10$ in Table 4.1. The lack of symmetry for $g \geq 4$ should not be surprising given the shape of the Newton polygon.

g	$\underline{\kappa}$
1	1
2	(1,1)
3	(1,1,1)
4	(1,3,1,1)
5	(1,1,1,1,1)
6	(1,1,1,1,1,1)
7	(1,3,5,1,3,1,1)
8	(1,1,1,1,1,1,1,1)
9	(1,1,1,1,1,1,1,1,1)
10	(1,3,1,7,3,1,1,3,1,1)

Table 4.1: Conifold multiples for small genera

4.3 Normalization of the conifold fibers

For the family $F^{\underline{a}}_{m,n} := F_{m,n}(\underline{a})$ the *maximal conifold* point $\hat{\underline{a}} \in \mathbb{C}^{g+r-3}$ is defined to be the unique point (if it exists) on the boundary of the region of convergence of the series (4.2.27) where $F^{\underline{a}}_{m,n}$ acquires g nodes (labeled by $\hat{p}_j := (\hat{x}_j, \hat{y}_j)$).

Remark 4.3.1. Strictly speaking, it is only $\hat{\underline{z}}$ which is unique, with finitely many preimages in $\underline{a}$, one of which has real coordinates; it is this one which we call $\hat{\underline{a}}$. Given existence, we refer to [Tyomkin, Prop. 7] for an argument proving uniqueness of $\hat{\underline{z}}$ - essentially, the variety V parameterizing all irreducible nodal rational curves of the (untempered) family $\hat{\mathcal{C}}^{\hat{\underline{a}}}_{m,n}$ is either empty or irreducible, and is isomorphic to a subgroup of $(\mathbb{C}^*)^2 \times (\mathbb{P}^1)^3 \big/ \mathrm{PGL}_2(\mathbb{C})$. Hence V is of dimension 2. However by passing from $\underline{a}$ to $\underline{z}$ parameters we cut down two degrees of freedom via toric automorphisms (namely $(\mathbb{C}^*)^2$), and as such the projection of V, being 0-dimensional and irreducible, is a single point.

Remark 4.3.2. The convergence issues can be taken care of by transforming the series (4.2.27) into one in terms of the GKZ variables $\underline{z}$. We claim that $\tilde{R}(\underline{a}) = (2\pi\mathbf{i})^{-1}R_{\gamma_1}(\underline{a}) + \log(a_1)$ has no monodromy for $\underline{z} = \underline{z}(t) := (t^m, t, \ldots, t)$ if $m \gg 0$ and $|t| < 1$. It is enough to check that there is no monodromy on $z_1 = 0$ (obvious, as the power series is identically zero there) or when $|z_1| < 1$ and $z_i = \hat{z}_i (i \geq 2)$. For the latter, note that the discriminant of (4.3.3) is a power of $z_1 - 1$.

So $B(t) := \tilde{R}(\underline{a}(\underline{z}(t)))$ is represented by a power series $\sum_m B_m t^m$ on the unit disk, is bounded on $\{|t| < 1 + \epsilon\} \setminus [1, 1 + \epsilon)$ (as the K_2 symbol is nonsingular at $t = 1$), and has monodromy about t $= 1$ $(T_1 - I)B \sim$ cst. $\times (t - 1)$ (since $(T_1 - I)\gamma_1$ is a vanishing cycle with trivial regulator). We are now in the situation of [Ke2, Lemma 6.4] with $w = 2$, so that $B_m \sim$ cst. $\times\, m - 2$. The power series thus converges at $t = 1$, and must evaluate to $B(1)$ by Tauber's theorem.

The standard way to find $\underline{\hat{a}}$ is via the discriminant locus: we look for transverse intersections amongst its local analytic branches. This is a viable strategy in particular cases; however, it requires careful analysis even in genus 2.

Example 4.3.3. The case $\mathcal{C}_{2,2}$ giving rise to a resolved $\mathbb{C}^3/\mathbb{Z}_5$ orbifold was extensively studied in [CGM, §4.1]. The discriminant locus is described by the equation (using z_i's),

$$3125z_1^2z_2^3 + 500z_1z_2^2 + 16z_2^2 - 225z_1z_2 - 8z_2 + 27z_1 + 1 = 0 \tag{4.3.1}$$

where

$$z_1 = \frac{a_2}{a_1^3}, \quad z_2 = \frac{a_1}{a_2^2}. \tag{4.3.2}$$

Figure 4.3.1 illustrates the intersection that gives rise to the maximal conifold point $\underline{\hat{z}} = (-\frac{1}{25}, \frac{1}{5})$, which lifts to $\underline{\hat{a}} = (5, -5)$.

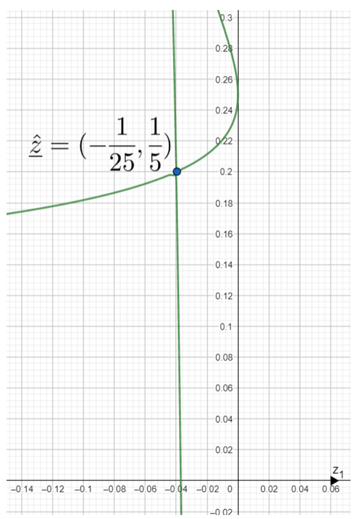

Figure 4.3.1: Discriminant locus of resolved $\mathbb{C}^3/\mathbb{Z}_5$, axes are z_i's.

It is clear that for the family $\mathcal{C}_{g,g}$, the discriminant locus is described by a degree $2g+1$ polynomial in g variables; so that approach quickly becomes untenable. However, a close study of the $g = 1$ and $g = 2$ cases suggested a "constructive" approach to producing g-nodal fibers, which generalized well and leads to the following:

Proposition 4.3.4. *Let $\mathcal{T}_m$ denote the m^{th}* Chebyshev Polynomial *of the first kind; this is a degree-m polynomial characterized by $\mathcal{T}_m(\cos\theta) = \cos m\theta$. Then we have*

$$F^{\hat{\underline{a}}}_{g,g}(x,x) = 2x(\mathcal{T}_{2g+1}(\tfrac{1}{2x}) + 1). \tag{4.3.3}$$

It follows that

$$\hat{a}_j = (-1)^{g-j+1}\frac{2g+1}{2j-1}\binom{g+j-1}{g-j+1} \quad \textit{and} \tag{4.3.4}$$

$$\hat{x}_j = \hat{y}_j = -\frac{1}{2}\sec\left(\frac{2\pi j}{2g+1}\right) \tag{4.3.5}$$

for $j = 1, \ldots, g$. In particular, $\hat{\underline{a}} \in \mathbb{Z}^g$.

Proof. That $\hat{x}_j \in \mathbf{Z}(\mathrm{RHS}(4.3.3))$ is immediate from the defining property of $\mathcal{T}_{2g+1}$, and the $\hat{x}_j$ are distinct and different from $-\frac{1}{2}$. Moreover, writing $\mathcal{U}_m$ for the m^{th} Chebyshev polynomial of the second kind, the relation $(\mathcal{T}_{2g+1}(w) - 1)(\mathcal{T}_{2g+1}(w) + 1) = (w^2 - 1)U_{2g}(w)$ guarantees that all roots other than $-\frac{1}{2}$ of $(\mathcal{T}_{2g+1}(\frac{1}{2x}) + 1)$ have even multiplicity. So they all have multiplicity 2 and are precisely the $\{\hat{x}_j\}$.

The polynomial $\hat{F}(x,y) := x+y+\sum_{j=1}^{g} \hat{a}_j x^{1-j}y^{1-j} + x^{-g}y^{-g}$, with $\hat{a}_j$ as in (4.3.4), satisfies $\hat{F}(x,x) = \mathrm{RHS}(4.3.3)$ by standard results on coefficients of $\mathcal{T}_m$. Clearly $\hat{F}(\hat{p}_j) = 0$, and the $\{\hat{p}_j\}$ are in fact singularities of $\mathbf{Z}(\hat{F})$ since $\frac{\partial \hat{F}}{\partial x}(x,x) = \frac{1}{2}\frac{d}{dx}(\hat{F}(x,x))$ and they are double roots of $\hat{F}(x,x)$. Therefore, by Proposition 4.1.6, they are all nodes. Since one can also check that (4.2.27) converges at $\hat{p}_j$, $\mathbf{Z}(\hat{F})$ is the maximal conifold curve. □

Remark 4.3.5. Of course, Proposition 4.3.4 recovers the known maximal conifold points for the families $\mathcal{C}_{1,1}, \mathcal{C}_{2,2}$ ($\hat{a}_1 = -3$ for $g = 1$ and $\hat{a}_1 = 5, \hat{a}_2 = -5$ for $g = 2$). Table 5.2 gathers $\mathcal{T}_{2g+1}$ and $\hat{\underline{a}}$ for a few low genus cases.

Being of geometric genus zero, the maximal conifold fiber $\hat{\mathcal{C}}_{g,g}$ admits uniformizations by

g	$\mathcal{T}_{2g+1}(x)$	$\underline{\hat{a}}$
1	$4x^3 - 3x$	-3
2	$16x^5 - 20x^3 + 5x$	(5,-5)
3	$64x^7 - 112x^5 + 56x^3 - 7x$	(-7,14,-7)
4	$256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$	(9,-30,27,-9)
5	$1024x^{11} - 2916x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x$	(-11, 55, -77, 44, -11)

Table 4.2: Maximal conifold points for low genera.

$\mathbb{P}^1$. In particular, we have the g distinct parametrizations $z \mapsto (\hat{X}_j(z), \hat{Y}_j(z))$, with

$$\hat{X}_j(z) = \frac{\hat{x}_j \left(1 - \frac{1}{z}\right)^{g+1}}{\left(1 - \frac{\zeta_{2g+1}^{g-j+1}}{z}\right)\left(1 - \frac{\zeta_{2g+1}^{2(g-j+1)}}{z}\right)^g} \quad \text{and} \tag{4.3.6}$$

$$\hat{Y}_j(z) = \frac{\hat{y}_j \left(1 - \frac{z}{\zeta_{2g+1}^{2(g-j+1)}}\right)^{g+1}}{\left(1 - \frac{z}{\zeta_{2g+1}^{g-j+1}}\right)(1-z)^g}, \tag{4.3.7}$$

having the property that $z = 0, \infty$ are mapped to $\hat{p}_j$. Hence the image of the path from $z = 0$ to $z = \infty$ on $\mathbb{P}^1$ is sent (by the j^{th} map) to $\hat{\gamma}_j$. As dictated by [DK, §6.2], we assign a formal divisor $\hat{\mathcal{N}}_j$ on $\mathbb{P}^1 \setminus \{0, \infty\}$ to each uniformization: for $X(z) = c_1 \prod_j (1 - \frac{\alpha_j}{z})^{d_j}$ and $Y(z) = c_2 \prod_k (1 - \frac{z}{\beta_k})^{e_k}$, this divisor is $\mathcal{N} := \sum_{j,k} d_j e_k [\frac{\alpha_j}{\beta_k}]$. According to [loc. cit.], the imaginary part of $\int_0^\infty R\{X(z), Y(z)\}$ is then given by $D_2(\mathcal{N}) := \sum_{j,k} d_j e_k D_2(\frac{\alpha_j}{\beta_k})$.

In present case,

$$\begin{aligned}
\hat{\mathcal{N}}_j &= g^2[\zeta_{2g+1}^{2(g-j+1)}] + 2g[\zeta_{2g+1}^{g-j+1}] - (2g^2 + 2g - 1)[1] \\
&\qquad - 2(g+1)[\zeta_{2g+1}^{-(g-j+1)}] + (g+1)^2[\zeta_{2g+1}^{-2(g-j+1)}] \\
&= 2(2g+1)[\zeta_{2g+1}^{g-j+1}] - (2g+1)[\zeta_{2g+1}^{2(g-j+1)}] \\
&= 2(2g+1)[1 + \zeta_{2g+1}^{g-j+1}],
\end{aligned} \tag{4.3.8}$$

where we are working modulo the scissors congruence relations

$$[\xi]+[\tfrac{1}{\xi}]=0,\ [\xi]+[\overline{\xi}]=0,\ [\xi]+[1-\xi]=0 \text{ and} \tag{4.3.9}$$

$$[\xi_1]+[\xi_2]+[\tfrac{1-\xi_1}{1-\xi_1\xi_2}]+[\tfrac{1-\xi_2}{1-\xi_1\xi_2}]+[1-\xi_1\xi_2]=0 \tag{4.3.10}$$

of the Bloch Group $\mathcal{B}_2(\mathbb{C})$. Consequently we have the identity

$$D_2(\hat{\mathcal{N}}_j)=2(2g+1)D_2(1+\zeta_{2g+1}^{g-j+1}), \tag{4.3.11}$$

of which two particular cases are of note: we claim that

$$D_2(\hat{\mathcal{N}}_1)=-2\pi\mathcal{D}_{g,g} \quad \text{and} \tag{4.3.12}$$

$$D_2(\hat{\mathcal{N}}_g)=-2\pi\mathcal{D}_{2g-1,1}. \tag{4.3.13}$$

(See §2.4 for notation.) In fact, we can say something even more general. Given $m\in\mathbb{Z}_{>0}$, we have

$$\begin{aligned}-\mathfrak{z}^{m+1}\mathfrak{w}_{m,1} &= -\mathfrak{z}^{m+1}\frac{\mathfrak{z}_{m,1}^{m}-\mathfrak{z}_{m,1}^{-m}}{\mathfrak{z}_{m,1}-\mathfrak{z}_{m,1}^{-1}} = -\zeta_{2(m+2)}^{m+1}\sum_{k=0}^{m-1}\zeta_{2(m+2)}^{k}\zeta_{2(m+2)}^{-(m-1-k)}\\ &= -\frac{\zeta_{2(m+2)}^{m+1}}{\zeta_{2(m+2)}^{m-1}}\sum_{k=0}^{m-1}\zeta_{2(m+2)}^{2k} = -\zeta_{2(m+2)}^{2}\sum_{k=0}^{m-1}\zeta_{m+2}^{k}\\ &= \zeta_{m+2}\left(\zeta_{m+2}^{m}+\zeta_{m+2}^{m+1}\right) = 1+\zeta_{m+2}^{m+1}.\end{aligned} \tag{4.3.14}$$

Therefore, taking conjugates,

$$\begin{aligned}2(m+2)D_2(1+\zeta_{m+2}) &= -2(m+2)D_2(1+\zeta_{m+2}^{m+1})\\ &= -2\pi\mathcal{D}_{m,1}\end{aligned} \tag{4.3.15}$$

which implies (4.3.13) upon setting $m = 2g-1$. Similarly one can see that

$$\mathfrak{w}_{g,g} = \zeta_{2(2g+1)}^{1-g} \sum_{k=0}^{g-1} \zeta_{2g+1}^{k} \tag{4.3.16}$$

and thus

$$\begin{aligned} 2(2g+1)D_2(1+\zeta_{2g+1}^{g}) &= -2(2g+1)D_2\left(-\sum_{k=1}^{g}\zeta_{2g+1}^{k}\right) \\ &= -2(2g+1)D_2\left(-\zeta_{2(2g+1)}^{2}\sum_{k=0}^{g-1}\zeta_{2g+1}^{k}\right) \\ &= -2\pi\mathcal{D}_{g,g}, \end{aligned} \tag{4.3.17}$$

as was to be shown.

We are now ready to prove Theorem 4.1.1. By the previously mentioned result of [DK, §6.2], we know that $\Im(R_{\hat{\gamma}_j}) = D_2(\hat{\mathcal{N}}_j)$ or

$$\Re(\tfrac{1}{2\pi\mathbf{i}}R_{\hat{\gamma}_j}) = \tfrac{1}{2\pi}D_2(\hat{\mathcal{N}}_j). \tag{4.3.18}$$

Next, Proposition 4.1.4 tells us that $R_{\gamma_j}(\underline{\hat{a}}) = \kappa_j R_{\hat{\gamma}_j}$, while (4.3.4) and (4.2.27) ensure that (mod $\mathbb{Q}(1)$) $\frac{1}{2\pi\mathbf{i}}R_{\gamma_j}(\underline{\hat{a}})$ hence $\frac{1}{2\pi\mathbf{i}}R_{\hat{\gamma}_j}$ is real. Combining this with (4.3.11) gives

$$\frac{1}{2\pi\mathbf{i}}R_{\gamma_j}(\underline{\hat{a}}) = \frac{1}{2\pi\mathbf{i}}\kappa_j R_{\hat{\gamma}_j} \underset{\mathbb{Q}(1)}{\equiv} \frac{(2g+1)\kappa_j}{\pi}D_2(1+\zeta_{2g+1}^{g-j+1}), \tag{4.3.19}$$

whence (4.1.1) [resp. (4.1.2)] follows from (4.3.12) [resp. (4.3.15)] by setting $j=1$ [resp. $j=g$] in (4.3.19).

4.4 Explicit series identities

Spelling out (4.3.19) in light of (4.2.27) kills any torsion modulo $\mathbb{Q}(1)$ as both sides are real,[3] and yields the relationship

$$\frac{(2g+1)\cdot\gcd(2j-1,2g+1)}{\pi}D_2(1+\zeta_{2g+1}^{g-j+1}) = \log(|\hat{a}_j|) - \sum_{\mathcal{L}_j}\frac{\Gamma(\mathfrak{l}_j)}{\Gamma^2(1+\mathfrak{l}'_j)\prod_{k=1}^{g}\Gamma(1+l_k)}(-\hat{a}_j)^{-\mathfrak{l}_j}\sum_{\substack{k=1\\k\neq j}}^{g}\hat{a}_k^{l_k} \tag{4.4.1}$$

valid for $j=1,\ldots,g$. The LHS can be shifted to a different avatar via the formula

$$D_2(1+\zeta_{2g+1}^{g-j}) = D_2\left(2\cos(\tfrac{\pi}{2g+1})e^{\pi i(g-j)/(2g+1)}\right). \tag{4.4.2}$$

Let us consider some applications of (4.4.1). For the family $\mathcal{C}_{2,2}$ Table 4.1 and Table 4.2 say that $\underline{\kappa}=(1,1)$ and $\underline{\hat{a}}=(5,-5)$. Recalling that $\mathfrak{w} := \frac{1+\sqrt{5}}{2} = 2\cos(\pi/5)$ and plugging in $j=1$ in (4.4.1) gives

$$\begin{aligned}\frac{5}{\pi}D_2(\mathfrak{w}e^{2\pi\mathbf{i}/5}) &= \log 5 - \sum_{l_1,l_2\in\mathbb{Z}_{\geq 0}}{}'\frac{\Gamma(5l_1+3l_2)(-5)^{-5l_1-3l_2}5^{l_2}}{\Gamma^2(1+2l_1+l_2)\Gamma(1+l_1)\Gamma(1+l_2)}\\ &= \log 5 - \sum_{m,r\in\mathbb{Z}_{\geq 0}}{}'\frac{(-1)^m\Gamma(5m+3r)5^{-5m-2r}}{\Gamma^2(1+2m+r)\Gamma(1+m)\Gamma(1+r)}.\end{aligned}$$

On the other hand for $j=2$,

$$\frac{5}{\pi}D_2(\mathfrak{w}e^{\pi\mathbf{i}/5}) = \log 5 - \sum_{l_1,l_2\in\mathbb{Z}_{\geq 0}}{}'\frac{\Gamma(\frac{5l_2+l_1}{3})5^{-\frac{5l_2+l_1}{3}}5^{l_1}}{\Gamma^2(1+\frac{l_2-l_1}{3})\Gamma(1+l_1)\Gamma(1+l_2)}. \tag{4.4.3}$$

[3] after changing $\log(\hat{a}_j)$ to $\log(|\hat{a}_j|)$

Defining $r := l_1, m := (l_2 - l_1)/3$,

$$\frac{5}{\pi} D_2(\mathfrak{w} e^{\pi i/5}) = \log 5 - \sum_{m,r \in \mathbb{Z}_{\geq 0}}' \frac{\Gamma(5m+2r) 5^{-5m-r}}{\Gamma^2(1+m)\Gamma(1+r)\Gamma(1+3m+r)}. \tag{4.4.4}$$

These identities, conjectured in [CGM, A.10], match the identities [7K, (6.13)-(6.14)].[4] Likewise, for $\mathcal{C}_{3,3}$ we have $\underline{\hat{a}} = (-7, 14, -7)$ and $\underline{k} = (1,1,1)$, and thus

$$\frac{7}{\pi} D_2(1+\zeta_7^3) = \log 7 - \sum_{m,r,p \in \mathbb{Z}_{\geq 0}}' \frac{(-1)^{m+p}\Gamma(7m+5r+3p) 7^{-7m-4r-p} 2^{a_2}}{\Gamma^2(1+3m+2r+p)\Gamma(1+m)\Gamma(1+r)\Gamma(1+p)} \tag{4.4.5}$$

$$\frac{7}{\pi} D_2(1+\zeta_7^2) = \log 7 - \sum_{m,r,p \in \mathbb{Z}_{\geq 0}}' \frac{(-1)^{r}\Gamma(7m+5r+p) 7^{-4m-5r+2p} 2^{-7m-5r-p}}{\Gamma^2(1+2m+r-p)\Gamma(1+3m)\Gamma(1+3r)\Gamma(1+3p)} \tag{4.4.6}$$

$$\frac{7}{\pi} D_2(1+\zeta_7) = \log 7 - \sum_{m,r,p \in \mathbb{Z}_{\geq 0}}' \frac{(-1)^{m}\Gamma(7m+3r+p) 7^{-7m+2p} 2^{3r}}{\Gamma^2(1+m-r-2p)\Gamma(1+3m)\Gamma(1+3r)\Gamma(1+3p)}. \tag{4.4.7}$$

More generally, for the family $\mathcal{C}_{g,g}$, $\mathcal{L}_1$ becomes the lattice $\mathbb{Z}^g_{\geq 0} \setminus \{0, \ldots, 0\}$ and we end up with a tidy expression,

$$\begin{aligned} \frac{(2g+1)}{\pi} D_2(1+\zeta_{2g+1}^g) \quad &= \quad \log(|\hat{a}_1|) - \\ \sum_{\substack{l_k \in \mathbb{Z}_{\geq 0} \\ 1 \leq k \leq g}}' (-1)^{\sum_{k=1}^{g} l_k} & \frac{\Gamma\left((2g+1)l_1 + \sum_{k=1}^{g}(2k-1)l_k\right)}{\Gamma^2\left(1+gl_1+\sum_{k=2}^{g}(k-1)l_k\right)\prod_{k=1}^{g}\Gamma(1+l_k)} \hat{a}_1^{-(2g+1)l_1 - \sum_{k=1}^{g}(2k-1)l_k} \prod_{k=1}^{g} \hat{a}_k^{l_k}, \end{aligned} \tag{4.4.8}$$

where $\sum_{l_k}'$ means that we omit the term corresponding to $\{0, \ldots, 0\}$.

[4] The proof there was incomplete as it did not address $\boldsymbol{\kappa}$.

Chapter 5

Recent advances in the $(2g, 1)$ case

In this section we prove Conjecture 2.4.2 in the cases $(m, n) = (2g, 1)$, for every $g \geq 1$. Much of the analysis from §4 goes through verbatim; however presence of the mass parameter significantly alters asymptotics of the regulator periods as well as the ansatz developed in §4.3.

5.1 The main result and some preliminaries

Consider the families of genus-g curves cut out of $(\mathbb{C}^*)^2$ by the (integrally tempered) polynomial $F_{2g,1}(x, y)$ from (2.4.8). As before $\mathcal{C}_{2g,1}$ will denote its *compactifications* in $\mathbb{P}_\Delta$. In this case there is precisely one mass parameter, namely a_{g+1} placed at the point $(-g, 0)$, so $r = 4$ and the equations take the simpler form (2.4.7). Moreover, temperedness fixes $a_{g+1} = 2$.

Theorem 5.1.1. *Conjecture 2.4.2 holds for the family $\mathcal{C}_{2g,1}$; that is,*

$$\tfrac{1}{2\pi \mathbf{i}} R_{\gamma_1}(\underline{\hat{a}}) \underset{\mathbb{Q}(1)}{\equiv} \mathcal{D}_{2g,1} \tag{5.1.1}$$

Remark 5.1.2. Note that the Milnor symbol $\{x, y\}$ on the curve defined by substituting $-x, -y$ for x, y resp. then multiplying the equation by -1, being a pullback, is integrally tempered with the same integral regulator as $\{-x, -y\}$. The new equation replaces a_{g+1} by $-a_{g+1}$, and also changes the sign of $a_1, a_3, a_5, \ldots$; it is this new equation which we will use going forward. Note also that (2.4.10) is stated in terms of the regulator period asymptotic to $-2\pi\mathbf{i}\log(a_n)$; it is convenient in this section to drop the negative sign and work with one asymptotic to $2\pi\mathbf{i}\log(a_n)$. Thus from now on

$$R_{\gamma_n} \sim 2\pi\mathbf{i}\log(a_n).$$

We will borrow some notations from §4.1 - let us rename $F_{2g,1}$ to $F^{\underline{a}}_{2g,1}$. Denote by $\hat{\mathcal{C}}_{2g,1}$ the fiber of the family over the *maximal conifold* point $\underline{\hat{a}}$. It has g nodes $\{\hat{p}_j\}$, and the cycles $\{\hat{\gamma}_j\}_{j=1}^{g}$ passing through each node generate $H_1(\hat{\mathcal{C}}_{g,g})$; we set $R_{\hat{\gamma}_j} := \int_{\hat{\gamma}_i} R\{x,y\}$. Writing $\boldsymbol{\kappa} = {}_{\underline{\hat{\gamma}}}[\mathrm{Id}]_{\underline{\gamma}(\hat{a})}$ for the change-of-basis matrix, we have

Proposition 5.1.3. *Let* $\kappa_j := \gcd(j, g+1)$. *Then*

$$\boldsymbol{\kappa} = \mathrm{diag}(\kappa_1, \ldots, \kappa_g). \tag{5.1.2}$$

It then follows from temperedness that

$$\tfrac{1}{2\pi\mathbf{i}} R_{\gamma_j}(\underline{\hat{a}}) \underset{\mathbb{Q}(1)}{\equiv} \tfrac{\kappa_j}{2\pi\mathbf{i}} R_{\hat{\gamma}_j}. \tag{5.1.3}$$

To prove 5.1.1 we will proceed in the same vein as described in §4.1, beginning with a few preliminary results. The first two help us to control certain power series asymptotics, while the third gives us information on nodal fibers of $\mathcal{C}_{2g,1}$.

Lemma 5.1.4. *The following identity holds,*

$$\sum_{k=0}^{m/2} \frac{2^{-2k}}{\Gamma(1+k)^2\Gamma(1+m-2k)} = \frac{\Gamma\left(\dfrac{1+2m}{2}\right)}{\Gamma(1+m)\Gamma\left(\dfrac{2+m}{2}\right)\Gamma\left(\dfrac{1+m}{2}\right)}. \tag{5.1.4}$$

Proof. We reduce the given series into a hypergeometric series, and apply **Gauss' summation theorem** as follows,

$$\sum_{k=0}^{m/2} \frac{2^{-2k}}{\Gamma(1+k)^2\Gamma(1+m-2k)} = \frac{1}{m!}\ {}_2F_1\left[\begin{matrix}\frac{1-m}{2}, -\frac{m}{2} \\ 1\end{matrix}; 1\right]$$
$$= \frac{\Gamma\left(\frac{1+2m}{2}\right)}{\Gamma(1+m)\Gamma\left(\frac{2+m}{2}\right)\Gamma\left(\frac{1+m}{2}\right)}. \tag{5.1.5}$$

□

Lemma 5.1.5. *If* $a, b, c \in \mathbb{R}_{>>>0}$ *are such that* $a = b + c$*, then*

$$\underbrace{\frac{2^c\Gamma(1+a)\Gamma\left(\frac{1+2c}{2}\right)}{\Gamma(1+b)\Gamma(1+c)\Gamma\left(\frac{1+c}{2}\right)\Gamma\left(\frac{2+c}{2}\right)}}_{=:\mathcal{A}_{a,b,c}} \approx \frac{1}{\sqrt{2}\pi c}\sqrt{\frac{a}{b}}\left(\frac{a}{b}\left(\frac{4b}{c}\right)^{c/a}\right)^a. \tag{5.1.6}$$

Proof. Using **Duplication formula**,

$$\frac{1}{\Gamma\left(\frac{1+c}{2}\right)\Gamma\left(\frac{2+c}{2}\right)} = \frac{2^c}{\sqrt{\pi}\Gamma(1+c)} \tag{5.1.7}$$

Thus

$$\frac{\Gamma\left(\frac{1+2c}{2}\right)}{\Gamma\left(\frac{1+c}{2}\right)\Gamma\left(\frac{2+c}{2}\right)} = \frac{2^c\Gamma\left(c+\frac{1}{2}\right)}{\Gamma(c+1)} \approx 2^c\sqrt{\frac{1}{c}}. \tag{5.1.8}$$

wherein we have used a modified **Sterling's approximation** which says that for large $x \in \mathbb{R}_{\geq 0}$ and $\alpha, \beta \in \mathbb{R}_{>0}$,

$$\frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)} \approx x^{\alpha-\beta}. \tag{5.1.9}$$

It follows that

$$\mathcal{A}_{a,b,c} \approx \frac{4^c\Gamma(1+a)}{\sqrt{\pi}\Gamma(1+b)\Gamma(1+c)\sqrt{c}} \approx \frac{1}{\sqrt{2\pi c}}\sqrt{\frac{a}{b}}\frac{4^c a^a}{b^b c^c}e^{-a+b+c}$$
$$= \frac{1}{\sqrt{2\pi c}}\sqrt{\frac{a}{b}}\frac{4^c a^a}{b^{a-c}c^c} \tag{5.1.10}$$
$$= \frac{1}{\sqrt{2\pi c}}\sqrt{\frac{a}{b}}\left(\frac{a}{b}\left(\frac{4b}{c}\right)^{c/a}\right)^a$$

as was to be shown.

□

Lemma 5.1.6. *Suppose that the fiber over $\underline{\tilde{a}} = (\tilde{a}_1, \ldots, \tilde{a}_{g+1})$ has g-many singularities, say $\tilde{p}_j := (\tilde{x}_j, \tilde{y}_j), n = 1, \ldots, g$. Then for each j, $\tilde{p}_j$ is a node.*

Proof. Due to Prop. 4.1.6 the result becomes immediate modulo the hessian calculation, which in this case boils down to the following - we begin by defining

$$\tilde{P}(x) := g + 1 + \textstyle\sum_{j=1}^{g}(g+1-j)\tilde{a}_j x^{-j}, \tag{5.1.11}$$

and observing that

$$\tilde{P}(\tilde{p}_j) = \tfrac{g}{\tilde{x}_j}F^{\tilde{a}}_{2g,1}(\tilde{p}_j) + \partial_x F^{\tilde{a}}_{2g,1}(\tilde{p}_j) = 0. \tag{5.1.12}$$

Thus $\mathbf{Z}(\tilde{P}) = \{\tilde{p}_1, \ldots, \tilde{p}_g\}$, i.e., $\tilde{P}$ has no repeated roots; that is, $\tilde{P}'(\tilde{p}_j) \neq 0$ $(\forall j)$. To compute the Hessians, write

$$\begin{aligned}\partial_{xx}F^{\tilde{a}}_{2g,1}(\tilde{p}_j) &= \textstyle\sum_{\ell=1}^{g+1}\ell(\ell-1)\tilde{a}_\ell\tilde{x}_j^{-\ell-1} + 2g(2g+1)\tilde{x}_j^{-2g-2}\tilde{y}_j^{-1}\\ &= \textstyle\sum_{\ell=1}^{g+1}\ell(\ell-1)\tilde{a}_\ell\tilde{x}_j^{-\ell-1} + \frac{2g(2g+1)\tilde{y}_j}{\tilde{x}_j^2}, \qquad (5.1.13)\end{aligned}$$

$$\partial_{xy}F^{\tilde{a}}_{2g,1}(\tilde{p}_j) = 2g\tilde{x}_j^{-2g-1}\tilde{y}_j^{-2} = \tfrac{2g}{\tilde{y}_j}, \text{ and} \qquad (5.1.14)$$

$$\partial_{yy}F^{\tilde{a}}_{2g,1}(\tilde{p}_j) = 2\tilde{x}_j^{2g}\tilde{y}_j^{-3} = \tfrac{2}{\tilde{y}_j}. \qquad (5.1.15)$$

It can be shown that

$$\partial_{xx}F^{\tilde{a}}_{2g,1}(\tilde{p}_j) = \tfrac{2g^2\tilde{y}_j}{2\tilde{x}_j^2} + \tfrac{\tilde{P}'(\tilde{x}_j)}{2}, \qquad (5.1.16)$$

therefore,

$$\begin{aligned}H_{F^{\tilde{a}}_{2g,1}}(\tilde{p}_j) &= \Big(\partial_{xy}F^{\tilde{a}}_{2g,1}(\tilde{p}_j)\Big)^2 - \partial_{xx}F^{\tilde{a}}_{2g,1}(\tilde{p}_j)\partial_{yy}F^{\tilde{a}}_{2g,1}(\tilde{p}_j)\\ &= \tfrac{4g^2}{\tilde{x}_j^2} - \tfrac{4g^2}{\tilde{x}_j^2} - \tfrac{\tilde{P}'(\tilde{x}_j)}{\tilde{y}_j} \;=\; -\tfrac{\tilde{P}'(\tilde{x}_j)}{\tilde{y}_j} \neq 0\end{aligned}$$

as was to be shown. □

5.2 Monodromy calculations via power series

The essence of the monodromy calculation was already captured in Lemma 4.2.1. In this case we have the 1-forms

$$\varpi_j = \tfrac{1}{2\pi\mathbf{i}}\nabla_{\delta_{a_j}}R\{x,y\} = \frac{-a_j}{2\pi\mathbf{i}}\mathrm{Res}_{\mathcal{C}_{2g,1}}\left(\frac{dx\wedge dy}{x^{2j}yF_{2g,1}(x,y)}\right). \qquad (5.2.1)$$

$\mathcal{C}^{\mathring{a}_j}_{2g,1}$ is defined by

$$f^{(j)}_{2g,1} := F^{\mathring{a}_j}_{2g,1}(x,y) = x + y + \mathring{a}_j x^{1-j} + a_{g+1}x^{-g} + x^{-2g}y^{-1}, \qquad (5.2.2)$$

and to find the node we set $f^{(j)}_{2g,1}(\mathring{x}_j,\mathring{y}_j) = \mathring{x}_j\partial_x f^{(j)}_{2g,1}(\mathring{x}_j,\mathring{y}_j)$ which gives rise to equations of the form,

$$2\mathring{y}_j + \mathring{x}_j + \mathring{a}_j{\mathring{x}_j}^{1-j} + a_{g+1}\mathring{x}_j^{-g} = 0, \tag{5.2.3}$$

$$\mathring{x}_j + (1-j)\mathring{x}_j^{-j} - g\mathring{a}_j{\mathring{x}_j}^{-g-1} - 2g{\mathring{x}_j}^{-2g-1}\mathring{y}_j^{-1} = 0. \tag{5.2.4}$$

This yields

$$\mathring{x}_j = \sqrt[g+1]{\frac{4(g-j+1)}{j}}, \tag{5.2.5}$$

$$\mathring{a}_j = -\frac{g+1}{g-j+1}\left(\frac{4(g-j+1)}{j}\right)^{\frac{j}{g+1}}. \tag{5.2.6}$$

In particular, we have the relation

$$\mathring{a}_j\mathring{x}_j^{g-j+1} = -\frac{4(g+1)}{j}. \tag{5.2.7}$$

Changing variables to $X := x - \mathring{x}_j$, $Y := y - \mathring{x}_j$ in $f^{(j)}_{2g,1}(x,y)$ leads to the equation

$$x^{2g}yf^{(j)}_{2g,1} = \tfrac{(6g^2-4(j+1)-4g(j-1)}{{\mathring{x}_j}^2}X^2 - 2g\mathring{x}_j^{g-1}XY + \mathring{x}_j^{2g}Y^2 + \text{higher order terms.} \tag{5.2.8}$$

Therefore

$$\begin{aligned}\mathrm{Res}^2_{\mathring{p}_j}\frac{dx\wedge dy}{x^{2g}yf^{(j)}_{2g,1}} &= \frac{1}{\mathring{x}_j^{g-1}\sqrt{4g^2-2\left(6g^2-4(j+1)-4g(j-1)\right)^2}}\\ &= \frac{(-1)^{g+1}}{\mathbf{i}\mathring{x}_j^{g-1}\sqrt{8(g-j+1)(g+1)}}\end{aligned} \tag{5.2.9}$$

Consequently the residue of ϖ_j may now be found:

$$\begin{aligned}\mathrm{Res}_{\mathring{p}_j}\varpi_j &= \frac{-\mathring{a}_j}{2\pi\mathbf{i}}\mathrm{Res}^2_{\mathring{p}_j}\frac{dx\wedge dy}{x^{2j}yf^{(j)}_{2g,1}} \\ &= \frac{-\mathring{a}_j}{2\pi\mathbf{i}}\cdot\mathring{x}_j^{2g-j}\cdot\mathrm{Res}^2_{\mathring{p}_j}\frac{dx\wedge dy}{x^{2g}yf^{(j)}_{2g,1}} \\ &= \frac{-1}{2\pi}\cdot(\mathring{a}_j\mathring{x}_j^{2(g-j+1)})\cdot\frac{1}{\mathbf{i}\mathring{x}_j^{g-1}\sqrt{8(g-j+1)(g+1)}} \\ &= \frac{\sqrt{g+1}}{\pi j\sqrt{2(g-j+1)}}.\end{aligned} \tag{5.2.10}$$

Writing $\varphi_j := x^{j-1}F^{\underline{a}}_{2g,1}(x,y)-a_j$, (A.0.3) (with the sign flip from our choice of γ_j) yields

$$\begin{aligned}\frac{1}{2\pi\mathbf{i}}R_{\gamma_j}(\underline{a}) &\underset{\mathbb{Q}(1)}{\equiv} \log(a_j)-\sum_{m>0}\frac{(-a_j)^{-m}}{m}[\varphi_j^m]_{\underline{0}} \\ &= \log(a_j)-\sum_{m>0}\frac{(-a_j)^{-m}}{m}\times \\ &[(\underbrace{x^j}_{=:A_j}+\underbrace{x^{j-1}y}_{=:B_j}+\textstyle\sum_{\substack{k=1\\k\neq j}}^{g+1}a_k\underbrace{x^{j-k}}_{=:C_j^k}+\underbrace{x^{j-2g-1}y^{-1}}_{=:D_j})^m]_{\underline{0}}\end{aligned} \tag{5.2.11}$$

where $[L]_{\underline{0}}$ stands for the constant term (in x,y) appearing in the Laurent polynomial L. Now, given $l_1,l_2,\cdots,l_g\in\mathbb{Z}$, we define

$$\mathfrak{l}_j := \frac{1}{j}\left((g+1)(2l_j+l_{g+1})+\sum_{\substack{k=1\\k\neq j}}^{g}kl_k\right) \tag{5.2.12}$$

$$\mathfrak{l}'_j := \frac{1}{j}\left((g-j+1)(2l_j+l_{g+1})+\sum_{\substack{k=1\\k\neq j}}^{g}(k-j)l_k\right),\ \text{and put} \tag{5.2.13}$$

$$\mathcal{L}_j := \{(l_1,l_2,\cdots,l_g)\in\mathbb{Z}^g_{\geq 0}\mid \mathfrak{l}'_j\in\mathbb{Z}_{\geq 0}\}\setminus\{(0,\cdots,0)\} \tag{5.2.14}$$

Note that $\mathfrak{l}'_j\in\mathbb{Z}_{\geq 0}\implies \mathfrak{l}_j\in\mathbb{Z}_{\geq 0}$. The upshot of this construction is if $L_j,L'_j\in\mathbb{Z}_{\geq 0}$ are such that

$$A_j^{L_j} B_j^{L'_j} \prod_{\substack{k=1 \\ k\neq j}}^{g+1} (C_j^k)^{l_k} D_j^{l_j} = 1 \text{ and} \tag{5.2.15}$$

$$L_j + L'_j + \sum_{k=1}^{g} l_k = m \tag{5.2.16}$$

then $L_j = L'_j = \mathfrak{l}'_j$ (by symmetry) and $m = \mathfrak{l}_j$. Thus the lattice $\mathcal{L}_j \subset \mathbb{Z}^g$ encodes all possible constant terms appearing in (5.2.11), giving

$$\frac{1}{2\pi\mathbf{i}} R_{\gamma_j}(\underline{a}) \underset{\mathbb{Q}(1)}{\equiv} \log(a_j) \ - \sum_{\mathcal{L}_j} \frac{\Gamma(\mathfrak{l}_j)}{\Gamma(1+\mathfrak{l}'_j)\Gamma^2(1+l_j) \prod\limits_{\substack{k=1 \\ k\neq j}}^{g+1} \Gamma(1+l_k)} (-a_j)^{-\mathfrak{l}_j} \prod_{\substack{k=1 \\ k\neq j}}^{g+1} a_k^{l_k}. \tag{5.2.17}$$

For the classical periods $\Pi_{j\ell} = \int_{\gamma_j} \varpi_\ell = \frac{1}{2\pi\mathbf{i}} \delta_{a_\ell} R_{\gamma_j}$, it is clear from (5.2.17) that $\Pi_{j\ell}$ vanishes on the a_j-axis for $\ell \neq j$. Focusing then on

$$\Pi_{jj}(\underline{a}) = \int_{\gamma_j} \varpi_j = 1 + \sum_{\mathcal{L}_j} \frac{\Gamma(1+\mathfrak{l}_j)}{\Gamma(1+\mathfrak{l}'_j)\Gamma^2(1+l_j) \prod\limits_{\substack{k=1 \\ k\neq j}}^{g+1} \Gamma(1+l_k)} (-a_j)^{-\mathfrak{l}_j} \prod_{\substack{k=1 \\ k\neq j}}^{g+1} a_k^{l_k}, \tag{5.2.18}$$

we set $\underline{a}_i = 0$ for $i \neq j, g+1$ to obtain

$$\mathcal{S} := 1 + \sum_{l_j, l_{g+1} \in \mathbb{Z}_{>0}} \frac{\Gamma(1 + \frac{g+1}{j} l_j)}{\Gamma(1 + \frac{g+1}{j} l_j)\Gamma^2(1+l_j)\Gamma(1+l_{g+1})} (-a_j)^{-\frac{g+1}{j} l_j} a_{g+1}^{l_{g+1}}. \tag{5.2.19}$$

Recall that $\kappa_j := \gcd(j, g+1)$. Let us shift indices by renaming $l_n \to l_{g+1} + 2l_n$ and define,

$$\begin{aligned} n_j &:= \frac{j}{\kappa_j}, & m_j &:= \frac{g+1}{\kappa_j} = \frac{(g+1)n_j}{j}, \\ r_j &:= \frac{l_j}{n_j}, & \text{and} \quad s_j &:= a_j^{-m_j}. \end{aligned} \tag{5.2.20}$$

Clearly $n_j, m_j, r_j \in \mathbb{Z}_{>0}$. Now we have a power series of the form

$$\mathcal{S} = 1 + \sum_{r_j \in \mathbb{N}} \frac{(-1)^{m_j r_j} \Gamma(1 + m_j r_j) a_{g+1}^{n_j r_j - 2l_j}}{\Gamma^2(1 + \frac{m_j - n_j}{2} r_j) \Gamma^2(1 + l_j) \Gamma(1 + n_j r_j - 2l_j)} s_j^{r_j} =: \sum_{r_j} b_{r_j} s_j^{r_j}. \tag{5.2.21}$$

Let $\mathring{s}_j := \mathring{a}_j^{-m_j}$. Setting $a_{g+1} = 2$ and applying Lemma 4.1.5,

$$\frac{\Gamma(1 + m_j r_j)}{\Gamma^2(1 + \frac{m_j - n_j}{2} r_j) \Gamma(1 + n_j r_j)} \approx \frac{(-1)^{m_j r_j} \sqrt{2m_j}}{2\pi r_j n_j \sqrt{m_j - n_j}} \mathring{s}_j^{r_j} \tag{5.2.22}$$

from which we may conclude that

$$\lim_{r_j \to \infty} b_{r_j} \cdot r_j \cdot \mathring{s}_j^{-r_j} = \frac{\sqrt{2m_j}}{2\pi n_j \sqrt{m_j - n_j}}. \tag{5.2.23}$$

Observing that

$$\mathrm{Res}_{\mathring{p}_j} \varpi_j = \frac{\sqrt{g+1}}{\pi j \sqrt{2(g - j + 1)}} = \frac{\sqrt{2m_j}}{2\pi j \sqrt{m_j - n_j}}, \tag{5.2.24}$$

we apply (4.2.4) to obtain

$$\boldsymbol{\kappa}_{jj} = \frac{\lim_{r_j \to \infty} b_{r_j} \cdot r_j \cdot \mathring{s}_j^{r_j}}{\mathrm{Res}_{\mathring{p}_j} \varpi_j} = \frac{j}{n_j} = \kappa_j. \tag{5.2.25}$$

This concludes the proof of Proposition 5.1.3.

Remark 5.2.1. Notice that $\kappa_1 = \kappa_g = 1$. We document $\underline{\kappa} := (\kappa_1, \ldots, \kappa_n)$ for $g = 2, \ldots, 10$ in Table 5.1.

g	$\underline{\kappa}$
2	(1,1)
3	(1,2,1)
4	(1,1,1,1)
5	(1,2,3,2,1)
6	(1,1,1,1,1,1)
7	(1,2,1,4,1,2,1)
8	(1,1,3,1,1,3,1,1)
9	(1,2,1,2,5,2,1,2,1)
10	(1,1,1,1,1,1,1,1,1)

Table 5.1: Conifold multiples for small genera

5.3 Normalization of the conifold fibers

Recall that for the family $\mathcal{C}_{m,n}$ determined by the $\{F^{\underline{a}}_{m,n}\}$, the *maximal conifold* point $\underline{\hat{a}} \in (\mathbb{C}^*)^g$ is defined to be the unique[1] point (if it exists) on the boundary of the region of convergence of the series (5.2.17) where $\hat{\mathcal{C}}^{\underline{\hat{a}}}_{m,n}$ (given by $F^{\underline{\hat{a}}}_{m,n} = 0$) acquires g nodes (labeled by $\hat{p}_j := (\hat{x}_j, \hat{y}_j)$).

We demonstrate an example that underlines difficulties in finding $\underline{\hat{a}}$ in case of a mass parameter being present.

Example 5.3.1. Consider the (untempered) family $\mathcal{C}_{4,1}$ corresponding to a local $\mathbb{C}^3/\mathbb{Z}_6$ geometry cut out by

$$F_{4,1}(x,y) = x + y + a_1 + a_2 x^{-1} + a_3 x^{-2} + x^{-4} y^{-1} = 0. \tag{5.3.1}$$

a_3 is a mass parameter, and

$$z_1 = \frac{a_1 a_3}{a_2^2}, \quad z_2 = \frac{a_2}{a_1^2}, \quad z_3 = \frac{1}{a_3^2}. \tag{5.3.2}$$

[1]Strictly speaking, it is only $\underline{\hat{z}}$ which is unique, with finitely many preimages in $\underline{a}$, one of which has real coordinates; it is this one which we call $\underline{\hat{a}}$.

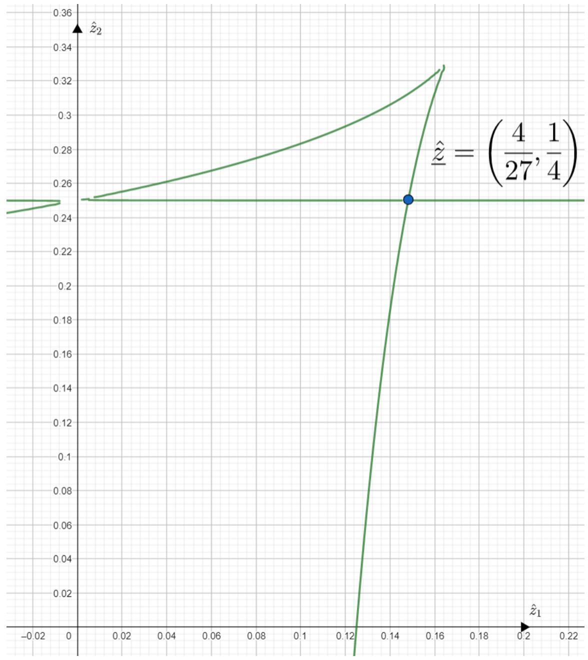

Figure 5.3.1: Discriminant locus of resolved $\mathbb{C}^3/\mathbb{Z}_6$ with $z_3 = 1/4$, axes are z_i-s.

The discriminant locus is obtained by setting [CGM2, 4.1],

$$\begin{aligned} &729z_1^4(1-4z_3)^2z_2^4 + 108z_1^3(9z_2-2)(4z_3-1)z_2^2 + (1-4z_2)^2 \\ &-4z_1(36z_2^2-17z_2+2) + 2z_1^2(108(4z_3+1)z_2^3 - 27(28z_3-5)z_2^2) \\ &+72((4z_3-1)z_2 - 32z_3+8) = 0. \end{aligned} \tag{5.3.3}$$

This is significantly harder to analyze compared to the situation with $\mathcal{C}_{2,2}$, however much of the complexity goes away when we enforce temperedness, which amounts to letting $z_3 = \frac{1}{4}$, and the maximal conifold point

$$\hat{\underline{z}} = \left(\frac{4}{27}, \frac{1}{4}\right) \tag{5.3.4}$$

can once again be recovered from transverse intersection.

The *ansatz* in present case takes the form of

Proposition 5.3.2. *Let $\mathcal{T}_m$ denote the m^{th}* Chebyshev polynomial *of the first kind; this is* *a degree-m polynomial characterized by* $\mathcal{T}_m(\cos\theta) = \cos m\theta$. *Then we have*

$$F^{\hat{a}}_{2g,1}(x, (-1)^j x^{-g}) = \frac{2}{x^{g+1}}(\mathcal{T}_{2g+2}(\tfrac{\sqrt{x}}{2}) + (-1)^j). \tag{5.3.5}$$

It follows that

$$\hat{a}_j = (-1)^j \frac{(2g+2)(2g-j+1)!}{j!(2g-2j+2)!} \quad \textit{and} \tag{5.3.6}$$

$$\hat{x}_j = (-1)^{j/g}\hat{y}_j^{-1/g} = (-1)^j 4\cos^2\left(\frac{\pi j}{2g+2}\right) \tag{5.3.7}$$

for $j = 1, \ldots, g$. *In particular,* $\underline{\hat{a}} \in \mathbb{Z}^g$.

Proof. That $\hat{x}_j \in \mathbf{Z}(\mathrm{RHS}(5.3.5))$ is immediate from the defining property of $\mathcal{T}_{2g+1}$, and the $\hat{x}_j$ are distinct and different from 4. Moreover, writing $\mathcal{U}_m$ for the m^{th} Chebyshev polynomial of the second kind, the relation $(\mathcal{T}_{2g+2}(w)-1)(\mathcal{T}_{2g+2}(w)+1) = (w^2-1)U_{2g+1}(w)$ guarantees that all roots other than 4 of $(\mathcal{T}_{2g+2}(\frac{1}{2x})+1)$ have even multiplicity. So they all have multiplicity 2 and are precisely the $\{\hat{x}_j\}$.

The polynomial $\hat{F}(x,y) := x + y + \sum_{j=1}^{g} \hat{a}_j x^{1-j} + a_{g+1}x^{-g} + x^{-2g}y^{-1}$, with $\hat{a}_j$ as in (5.3.6), satisfies $\hat{F}(x,(-1)^j x^{-g}) = \mathrm{RHS}(5.3.5)$ by standard results on coefficients of $\mathcal{T}_m$. Clearly $\hat{F}(\hat{p}_j) = 0$, and the $\{\hat{p}_j\}$ are in fact singularities of $\mathbf{Z}(\hat{F})$ since $\frac{\partial \hat{F}}{\partial x}(x,(-1)^j x^{-g}) = 2\frac{d}{dx}(\hat{F}(x,(-1)^j x^{-g}))$ and they are double roots of $\hat{F}(x,(-1)^j x^{-g})$. Therefore, by Proposition 4.1.6, they are all nodes. Since one can also check that (5.2.17) converges at $\hat{p}_j$, $\mathbf{Z}(\hat{F})$ is the maximal conifold curve. □

Remark 5.3.3. Of course, Proposition 5.3.2 recovers the predicted maximal conifold point for the $g = 2$ family $\mathcal{C}_{4,1}$, namely $\hat{a}_1 = -6, \hat{a}_2 = 9$. Table 5.2 gathers $\mathcal{T}_{2g+2}$ and $\underline{\hat{a}}$ for a few low genus cases.

$\hat{\mathcal{C}}_{2g,1}$ admits uniformizations by $\mathbb{P}^1$ via the g distinct parametrizations $z \mapsto (\hat{X}_j(z), \hat{Y}_j(z))$,

g	$\mathcal{T}_{2g+2}(x)$	$\underline{\hat{a}}$
2	$8x^4 - 8x^2+1$	(-6,9)
3	$32x^6 - 48x^4 + 18x^2 - 1$	(-8,20,-16)
4	$128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$	(-10,35,-50,25)
5	$512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$	(-12,54,-112,105,-36)

Table 5.2: Maximal conifold points for low genera.

with

$$\hat{X}_j(z) = \frac{\hat{x}_j\left(1 - \frac{\zeta_{2g+2}^j}{z}\right)^2}{\left(1 - \frac{\zeta_{2g+2}^{2j}}{z}\right)\left(1 - \frac{1}{z}\right)} \quad \text{and} \tag{5.3.8}$$

$$\hat{Y}_j(z) = \frac{\hat{y}_j\,(1-z)^{2g+1}}{\left(1 - \frac{z}{\zeta_{2g+2}^j}\right)^{2g}\left(1 - \frac{z}{\zeta_{2g+2}^{2j}}\right)}, \tag{5.3.9}$$

$z = 0, \infty$ being mapped to $\hat{p}_j$, while the image of the path from $z = 0$ to $z = \infty$ on $\mathbb{P}^1$ is sent (by the j^{th} map) to $\hat{\gamma}_j$. As dictated by [DK, §6.2], we assign a formal divisor $\hat{\mathcal{N}_j}$ on $\mathbb{P}^1 \setminus \{0, \infty\}$ to each uniformization: in this case,

$$\hat{\mathcal{N}_j} = 2(2g+2)[1 + \zeta_{2g+2}^j],$$

where we are working modulo the. Hence we have the identity

$$D_2(\hat{\mathcal{N}_j}) = 2(2g+2)D_2(1 + \zeta_{2g+2}^j), \tag{5.3.10}$$

of which one particular case is of note: we claim that

$$D_2(\hat{\mathcal{N}_g}) = -2\pi\mathcal{D}_{2g,1}, \tag{5.3.11}$$

(See §4.4 for notation.) a fact that follows from 4.3.15.

We are now ready to prove Theorem 5.1.1. By the previously mentioned result of [DK,

§6.2], we know that $\Im(R_{\hat{\gamma}_j}) = D_2(\hat{\mathcal{N}}_j)$ or

$$\Re(\tfrac{1}{2\pi\mathbf{i}} R_{\hat{\gamma}_j}) = \tfrac{1}{2\pi} D_2(\hat{\mathcal{N}}_j). \tag{5.3.12}$$

Next, Proposition 5.1.3 tells us that $R_{\gamma_j}(\underline{\hat{a}}) = \kappa_j R_{\hat{\gamma}_j}$, while (5.3.6) and (5.2.17) ensure that (mod $\mathbb{Q}(1)$) $\frac{1}{2\pi\mathbf{i}} R_{\gamma_j}(\underline{\hat{a}})$ hence $\frac{1}{2\pi\mathbf{i}} R_{\hat{\gamma}_j}$ is real. Combining this with (5.3.10) gives

$$\frac{1}{2\pi\mathbf{i}} R_{\gamma_j}(\underline{\hat{a}}) = \frac{1}{2\pi\mathbf{i}} \kappa_j R_{\hat{\gamma}_j} \underset{\mathbb{Q}(1)}{\equiv} \frac{(2g+2)\kappa_j}{\pi} D_2(1+\zeta_{2g+2}^j), \tag{5.3.13}$$

whence (5.1.1) follows from (5.3.11) by setting $j = 1$ in (5.3.13).

5.4 Explicit series identities

(5.3.13) combined with (5.2.17) gets rid of any torsion modulo $\mathbb{Q}(1)$ as both sides are real,[2] and it follows that

$$\frac{(2g+2)\cdot \gcd(j, g+1)}{\pi} D_2(1+\zeta_{2g+2}^j) = \log(|\hat{a}_j|) - \sum_{\mathcal{L}_j} \frac{\Gamma(\mathfrak{l}_j)}{\Gamma(1+\mathfrak{l}'_j)\Gamma^2(1+l_j)\prod_{\substack{k=1\\k\neq j}}^{g+1}\Gamma(1+l_k)} (-\hat{a}_j)^{-\mathfrak{l}_j} \sum_{\substack{k=1\\k\neq j}}^{g+1} \hat{a}_k^{l_k} \tag{5.4.1}$$

valid for $j = 1, \ldots, g$. As an application consider the case of $\mathcal{C}_{4,1}$. Table 5.1 and Table 5.2 say that $\underline{\kappa} = (1,1)$ and $\underline{\hat{a}} = (-6, 9)$. Plugging in $j = 1$ in (5.4.1) gives

$$\frac{6}{\pi} D_2(1+e^{\pi\mathbf{i}/3}) = \log 6 - \sum_{l_1,l_2,l_3\in\mathbb{Z}_{\geq 0}}{}' \frac{\Gamma(6l_1+3l_3+2l_2)(-6)^{-6l_1-3l_3-2l_2} 9^{l_2} (-2)^{l_3}}{\Gamma(1+4l_1+2l_3+l_2)\Gamma^2(1+l_1)\Gamma(1+l_2)\Gamma(1+l_3)} \tag{5.4.2}$$

[2] after changing $\log(\hat{a}_j)$ to $\log(|\hat{a}_j|)$

On the other hand for $j=2$,

$$\frac{6}{\pi}D_2(1+e^{2\pi \mathbf{i}/3}) = \log 9 - \sum_{l_1,l_2,l_3\in\mathbb{Z}_{\geq 0}}{}' \frac{\Gamma(\frac{6l_2+3l_3+l_1}{2})9^{-\frac{6l_2+3l_3+l_1}{2}}(-6)^{l_1}(-2)^{l_3}}{\Gamma(1+\frac{2l_2+l_3-l_1}{2})\Gamma^2(1+l_2)\Gamma(1+l_1)\Gamma(1+l_3)}. \tag{5.4.3}$$

These identities, although not explicitly derived, were conjectured and computationally verified in [CGM2, 4.31]. We conclude by observing that for the family $\mathcal{C}_{2g,1}$, $\mathcal{L}_1 = \mathbb{Z}^g_{\geq 0} \setminus \{0,\ldots,0\}$ and we end up with

$$\begin{aligned}\frac{(2g+2)}{\pi}D_2(1+\zeta_{2g+2}) \quad &= \quad \log(|\hat{a}_1|) - \\ \sum_{\substack{l_k\in\mathbb{Z}_{\geq 0}\\ 1\leq k\leq g+1}}{}'(-1)^{\sum_{k=1}^{g} l_k} \frac{\Gamma\left((g+1)(2l_1+l_{g+1})+\sum_{k=2}^{g} kl_k\right)}{\Gamma\left(1+g(2l_1+l_{g+1})+\sum_{k=2}^{g}(k-1)l_k\right)\Gamma^2(1+l_1)\prod_{k=2}^{g+1}\Gamma(1+l_k)} &\hat{a}_1^{-(g+1)(2l_1+l_{g+1})-\sum_{k=2}^{g} kl_k}\prod_{k=2}^{g+1}\hat{a}_k^{l_k},\end{aligned} \tag{5.4.4}$$

where $\sum'_{l_k}$ means that we omit the term corresponding to $\{0,\ldots,0\}$.

Appendix A

Some regulator calculations

Here we demonstrate the existence of integral 1-cycles $\{\gamma_j\}_{j=1}^g$ on $\mathcal{C}$ with regulator periods behaving as $R_{\gamma_j} \sim -2\pi\mathbf{i}\log(a_j)$ for large a_j, as claimed in §4.3. We refer the reader to [DK] or [KLi] for background on regulator currents.

We start by defining the 1-cycles in distinct regions of moduli. We will need some notation. Set $\mathbb{T} := \{\underline{x} \in (\mathbb{C}^*)^2 \big| |x_1| = 1 = |x_2|\}$ (with the standard orientation as a 2-cycle) and let $\Gamma \subset \mathbb{P}_\Delta$ be a 3-chain bounding on $\mathbb{T}$ (but avoiding $\bar{\mathcal{C}} \setminus \mathcal{C}$). Write $x^{\mathbf{e}} := \underline{x}^{\underline{m}^{\mathbf{e}}}$ for the toric coordinate along the boundary component $\mathbb{D}_{\mathbf{e}} \subset \mathbb{P}_\Delta$ corresponding to an edge $\mathbf{e} \subset \partial\Delta$, and $\{q_{\mathbf{e},\ell}\}$ for the roots of $P(-x_{\mathbf{e}})$ (amongst the $\{q_k\}$), repeated with multiplicity; we have $P_{\mathbf{e}}(x_{\mathbf{e}}) = \prod_\ell (1 + \frac{x_{\mathbf{e}}}{q_{\mathbf{e},\ell}})$, with $\prod_\ell q_{\mathbf{e},\ell} = 1$. Also, $\log_{\mathbf{e}}(\xi)$ will mean $\log(\xi)$ for ξ enclosed (counterclockwise on $\mathbb{D}_{\mathbf{e}}$) by $\Gamma \cap \mathbb{D}_{\mathbf{e}}$ and 0 otherwise.

Now, fixing $j \in \{1, \ldots, g\}$, take $\mathbf{i}a_j \in \mathfrak{H}$ and $|a_j| \gg \max_{i \neq j} |a_i|$; and note that then $F(\mathbb{T}) \cap \mathbb{R}_- = \emptyset$. *In this region*, define $\gamma_j := \Gamma \cap \mathcal{C}$, and use the current coboundary

$$\tfrac{1}{2\pi\mathbf{i}} d[R\{F(\underline{x}), \text{-}x_1, \text{-}x_2\}] = \textstyle\sum_{\mathbf{e}} R\{P_{\mathbf{e}}(x_{\mathbf{e}}), \text{-}x_{\mathbf{e}}\} \cdot \delta_{\mathbb{D}_{\mathbf{e}}} - R\{\text{-}x_1, \text{-}x_2\} \cdot \delta_{\bar{\mathcal{C}}} \tag{A.0.1}$$

together with the Tame symbols of $R\{P(x_{\mathbf{e}}), -x_{\mathbf{e}}\}$ (which are just the $\{q_{\mathbf{e},\ell}^{-1}\}$) and the Cauchy integral formula to compute

$$
\begin{aligned}
R_{\gamma_j} &= \int_{\gamma_j} R\{\text{-}x_1,\text{-}x_2\} = \int_\Gamma R\{\text{-}x_1,\text{-}x_2\}\cdot\delta_{\bar{\mathcal{C}}} \\
&= \tfrac{-1}{2\pi\mathbf{i}}\int_{\mathbb{T}} R\{F(\underline{x}),\text{-}x_1,\text{-}x_2\} + \textstyle\sum_{\mathbf{e}}\int_{\Gamma\cap\mathbb{D}_{\mathbf{e}}} R\{P_{\mathbf{e}}(x_{\mathbf{e}}),\text{-}x_{\mathbf{e}}\} \\
&= \tfrac{-1}{2\pi\mathbf{i}}\int_{\mathbb{T}} \log(a_j(1+a_j^{-1}F_j(\underline{x})))\tfrac{dx_1}{x_1}\wedge\tfrac{dx_2}{x_2} + \textstyle\sum_{\mathbf{e}}\int_{\Gamma\cap\mathbb{D}_{\mathbf{e}}} R\{P_{\mathbf{e}}(x_{\mathbf{e}}),\text{-}x_{\mathbf{e}}\} \\
&= 2\pi\mathbf{i}\left(-\log(a_j) + \textstyle\sum_k \tfrac{(-1)^k}{k}[(F_j(\underline{x}))^k]_{\underline{0}}a_j^{-k} - \textstyle\sum_{\mathbf{e},\ell}\log_{\mathbf{e}}(q_{\mathbf{e},\ell})\right).
\end{aligned}
\tag{A.0.2}
$$

In the tempered case, the $\{q_k\}$ are of course all 1, and the last term vanishes. We are then left with[1]

$$
\tfrac{1}{2\pi\mathbf{i}}R_{\gamma_j}(\underline{a}) = -\log(a_j) + \textstyle\sum_{k>0}\tfrac{(-1)^k}{k}[F_j^k]_{\underline{0}}a_j^{-k}, \tag{A.0.3}
$$

in which (by virtue of the GKZ theory) the sum can always be written as a power series in $z_1,\ldots,z_g$.[2] This gives a common region of convergence for the series for all j (where the z-coordinates are small), to which the γ_j admit well-defined continuation from the regions on which they were originally defined: namely, they are the cycles with these regulator periods. Moreover, they are clearly independent due to the asymptotic behaviors of these periods in the $\{a_j\}$.

In addition, (A.0.2)-(A.0.3) lead to formulas for periods of 1-forms. Noting that $d[R\{F(\underline{x}),-x_1,-x_2\}] = \frac{dF}{F}\wedge\frac{dx_1}{x_1}\wedge\frac{dx_2}{x_2}$, one introduces

$$
\varpi_\ell := \tfrac{1}{2\pi\mathbf{i}}\nabla_{\delta_{a_\ell}}\mathcal{R} = \tfrac{-1}{2\pi\mathbf{i}}\mathrm{Res}_{\mathcal{C}}\left(\tfrac{\delta_{a_\ell}F}{F}\tfrac{dx_1}{x_1}\wedge\tfrac{dx_2}{x_2}\right) \tag{A.0.4}
$$

and computes

[1] Note that the version of this formula in [KLi, Prop. 6.2] is missing a $\pm\pi\mathbf{i}$ ("2-torsion") term: the λ_j parameter there is $-a_j$, so the leading term should have read $-\log(-\lambda_j)$ or $-\log(\lambda_j)+\pi\mathbf{i}$.

[2] Essentially, this is just because in order to contribute to the constant term in $(F_j(\underline{x}))^k$, a product of monomials must correspond to a sum of relations on points of $\Delta\cap\mathbb{Z}^2$, and the relations are how we defined the $\{z_i\}$.

$$-\Pi_{j\ell} := -\int_{\gamma_j} \varpi_\ell = \tfrac{-1}{2\pi\mathbf{i}}\delta_{a_\ell} R_{\gamma_j} = \boldsymbol{\delta}_{\ell j} + \textstyle\sum_{k>0}(-1)^k [F_j^k]_{\underline{0}} a_j^{-k}, \tag{A.0.5}$$

where $\boldsymbol{\delta}_{\ell j}$ is the Kronecker delta. This formula proves useful in §5.2 where we change the sign of γ_j.

Turning to the $g = 1$ case and the computation of R_β, it is more convenient to work with $u = -a \gg 0$. In this coordinate, (3.1.3) becomes $t = \log(u) - \pi\mathbf{i} + O(u^{-1})$. Substituting this in Lemma 3.1.1(a) and using $12 - r^\circ = r$ yields

$$R_\beta = \tfrac{r^\circ}{2}\log^2 u - \tfrac{r}{6}\pi^2 + O(u^{-1}\log u). \tag{A.0.6}$$

Consider the Laurent polynomial $\varphi = x_1 + x_1^{-1} + x_2 + x_2^{-1}$, which corresponds to local ($\mathbb{P}_{\Delta^\circ} =$) $\mathbb{P}^1 \times \mathbb{P}^1$. The discriminant (over the x_1-axis) of the equation $x_2 + (x_1 + x_1^{-1} - u) + x_2^{-1} = 0$ has roots $\xi_1 \sim \frac{1}{u+2}$, $\xi_2 \sim \frac{1}{u-2}$, $\xi_3 \sim u - 2$, and $\xi_4 \sim u + 2$ (in increasing order). Introduce $2x_{2,\pm}(x_1) := u - x_1 - x_1^{-1} \pm \sqrt{(x_1 + x_1^{-1} - u)^2 - 4}$ and $w(x_1) := \frac{4}{(u - x_1 - x_1^{-1})^2}$. For $x_1 \in (\xi_2, \xi_3)$, w lies in $(0, 1)$, and we write $\log(\frac{4}{w} \cdot \frac{1-\sqrt{1-w}}{1+\sqrt{1-w}}) =: \sum_{m\geq 1}\theta_m w^m = \frac{1}{2}w + \frac{3}{16}w^2 + \cdots$. Now we compute

$$\begin{aligned}
R_\beta &= -\textstyle\int_\beta R\{-x_2, -x_1\} = \int_{\xi_2}^{\xi_3} \log(\frac{x_{2,+}}{x_{2,-}})\frac{dx_1}{x_1} = \int_{\xi_2}^{\xi_3}\log(\frac{1+\sqrt{1-w}}{1-\sqrt{1-w}})\frac{dx_1}{x_1} \\
&= -\textstyle\int_{\xi_2}^{\xi_3}\log(\frac{w}{4})\frac{dx_1}{x_1} - \sum_{m\geq 1}\theta_m \int_{\xi_2}^{\xi_3} w^m \frac{dx_1}{x_1} \\
&= 2\log(u)\textstyle\int_{\xi_2}^{\xi_3}\frac{dx_1}{x_1} + 2\int_{\xi_2}^{\xi_3}\log(1 - u^{-1}(x_1 + x_1^{-1}))\frac{dx_1}{x_1} + O(u^{-1}\log u) \\
&= 4\log^2 u - 2\textstyle\sum_{k>0}\frac{u^{-k}}{k}\int_{\xi_2}^{\xi_3}(x_1 + x_1^{-1})^k \frac{dx_1}{x_1} + O(u^{-1}\log u) \\
&= 4\log^2 u - \tfrac{2\pi^2}{3} + O(u^{-1}\log u),
\end{aligned} \tag{A.0.7}$$

at the end using the approximations $\int_{\xi_2}^{\xi_3}(x_1 + x_1^{-1})^k \frac{dx_1}{x_1} \sim \frac{2\xi_3^k}{k} \sim \frac{2u^k}{k}$ to rewrite the sum as $-4\sum\frac{1}{k^2} = -\frac{2}{3}\pi^2$ up to $O(u^{-1}\log u)$. The point is that since $r = 4$, this agrees with the result (A.0.6) from integral local mirror symmetry. A similar computation in [KLi, §6] for

$\varphi = x_1 + x_2 + x_1^{-1}x_2^{-1}$ (mirror to local $\mathbb{P}^2$) gives $R_\beta = \frac{9}{2}\log^2 u - \frac{\pi^2}{2} + O(u^{-1}\log u)$, where the $-\frac{\pi^2}{2}$ arises as $-2\mathrm{Li}_2(\frac{1}{2}) - 2\mathrm{Li}_2(1) - \log^2 2$. Since $r = 3$, this agrees once more with (A.0.6) (as it must).

The crucial constant term in R_β has a nice interpretation via the LMHS at $a = \infty$ of the VMHS $\mathcal{V}$ attached to $\mathcal{R} \in H^1(E_a, \mathbb{C}/\mathbb{Z}(2))$, the regulator class of $\{-x_1, -x_2\} \in H^2_{\mathrm{M}}(E_a, \mathbb{Z}(2))$. (Note that the LMHS depends on a choice of a local coordinate, which we take to be a^{-1} or equivalently $Q := e^{-t} = a^{-1}(1 + O(a^{-1}))$.) We can present $\mathcal{V}$ and its dual as extensions

$$H^1(E, \mathbb{Z}(2)) \to \mathcal{V}_\mathbb{Z} \to \mathbb{Z}(0) \;\text{ and }\; \mathbb{Z}(0) \to \mathcal{V}_\mathbb{Z}^\vee \to H_1(E, \mathbb{Z}(-2)). \tag{A.0.8}$$

On the left, a unique class $\mathfrak{R} \in F^0\mathcal{V}_\mathbb{C}$ maps to $1 \in \mathbb{Z}(0)$; on the right, let $\tau \in \mathcal{V}_\mathbb{Z}^\vee$ be the image of 1, and $\tilde{\gamma}, \tilde{\beta} \in \mathcal{V}_\mathbb{Z}^\vee$ classes mapping to $\frac{1}{(2\pi\mathbf{i})^2}\gamma$, $\frac{1}{(2\pi\mathbf{i})^2}\beta$. Writing $\ell(Q) := \frac{\log(Q)}{2\pi\mathbf{i}}$, we have

$$\tilde{R}_\beta := \langle \mathfrak{R}, \tilde{\beta} \rangle = \tfrac{1}{(2\pi\mathbf{i})^2} R_\beta = \tfrac{r^\circ}{2}\ell(Q)^2 - \tfrac{r^\circ}{2}\ell(Q) + \mathtt{T} + O(Q), \tag{A.0.9}$$

where $\mathtt{T} = \frac{1}{2} + \frac{r^\circ}{12}$ (cf. Lemma 3.1.1(a)), as well as $\tilde{R}_\gamma := \langle \mathfrak{R}, \tilde{\gamma} \rangle = \frac{1}{(2\pi\mathbf{i})^2} R_\gamma = \ell(Q)$ and $\langle \mathfrak{R}, \tau \rangle = 1$.

To obtain a period matrix for $\mathcal{V}$, we compare Hodge and Betti bases as follows. Writing ∇ for $\nabla_{\partial_{\ell(Q)}}$, the change-of-basis matrix from $\{\mathfrak{R}, \nabla\mathfrak{R}, \frac{1}{r^\circ}\nabla^2\mathfrak{R}\}$ to $\{\tau^\vee, \tilde{\gamma}^\vee, \tilde{\beta}^\vee\}$ is

$$\mathbf{\Omega} := \begin{pmatrix} 1 & & \\ \tilde{R}_\gamma & 1 & \\ \tilde{R}_\beta & \partial_{\ell(Q)}\tilde{R}_\beta & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ \ell(Q) & 1 & \\ \frac{r^\circ}{2}\ell(Q)^2 - \frac{r^\circ}{2}\ell(Q) + \mathtt{T} & r^\circ\ell(Q) - \frac{r^\circ}{2} & 1 \end{pmatrix} + O(Q). \tag{A.0.10}$$

From (A.0.10) one easily deduces the monodromies $T \in \mathrm{Aut}(\mathcal{V})$ and $T^\vee \in \mathrm{Aut}(\mathcal{V}^\vee)$ about $Q = 0$:

$$[T^\vee]_{\{\tilde{\beta}, \tilde{\gamma}, \tau\}} = \begin{pmatrix} 1 & & \\ r^\circ & 1 & \\ 0 & 1 & 1 \end{pmatrix} \quad \Longrightarrow \quad \boldsymbol{T} := [T]_{\{\tau^\vee, \tilde{\gamma}^\vee, \tilde{\beta}^\vee\}} = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ 0 & r^\circ & 1 \end{pmatrix}. \tag{A.0.11}$$

Consequently the *limiting* period matrix is

$$\mathbf{\Omega}_{\lim,Q} := \lim_{Q\to 0} e^{-\ell(Q)\log(\boldsymbol{T})}\mathbf{\Omega} = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ \mathtt{T} & -\frac{r^\circ}{2} & 1 \end{pmatrix}. \tag{A.0.12}$$

The LMHS with respect to a^{-1}, as mentioned above, gives the same result; but if we change local coordinate to $-Q$ or (equivalently) u^{-1}, we get

$$\mathbf{\Omega}_{\lim,-Q} := \lim_{Q\to 0} e^{-\ell(-Q)\log(\boldsymbol{T})}\mathbf{\Omega} = \begin{pmatrix} 1 & & \\ \frac{1}{2} & 1 & \\ \mathtt{B}^\circ & 0 & 1 \end{pmatrix}, \tag{A.0.13}$$

where $\mathtt{B}^\circ = \frac{1}{2} - \frac{r^\circ}{24} = \mathtt{T} - \frac{r^\circ}{8}$. So we see that both of the constants appearing in Lemma 3.1.3(ii) have a standard asymptotic Hodge-theoretic meaning, in terms of (torsion) extension classes in the LMHS of $\mathcal{V}$ in the large complex structure limit.

Appendix B

Degenerations and limiting regulator periods

In [7K] the concept of "going up in K-theory" is established in order to capture limits of higher normal functions. Let $\mathcal{X} \to \mathcal{S}$ be a dominant morphism of smooth varieties with generic fiber of genus g and a singular nodal fiber X_0 embedded via $i : X_0 \mathcal{X}$. Let $\Xi \in \mathrm{CH}^p(\mathcal{X}, r)$ be a higher cycle, with fiberwise restrictions $\Xi_s \in H^{2p-r}_{\mathrm{M}}(X_s, \mathbb{Z}(p))$; taking fiberwise Abel-Jacobi (integral regulator) classes leads to a higher normal function

$$\mathrm{AJ}_{X_s}(\Xi_s) \in J(H^n(X_s)(p)) := \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H^p(X_s, \mathbb{Z}(p))),$$

where $n = 2p - r - 1$. From the Clemens-Schimd exact sequence one has a morphism of MHS $\mathfrak{r}^* : H^n(X_0) \to H^n_{lim}(X_s)$, with induced morphism $J(\mathfrak{r}^*)\colon J(H^n(X_0)(p)) \to J(H^n_{\mathrm{lim}}(X_s)(p)$, and according to [loc. cit.] we have

$$\lim_{s\to 0} \mathrm{AJ}_{X_s}(\Xi_s) = J(\mathfrak{r}^*)\mathrm{AJ}_{X_0}(i^*\Xi) \tag{B.0.1}$$

in $J(H^n_{\mathrm{lim}}(X_s)(p))$.

The upshot of this result is that the left-hand side of (B.0.1) are direct representatives of the regulator periods R_{γ_j}. On the other hand, the right-hand side can be worked out, using techniques developed in [DK], from the g-many normalizations of X_0 by $\mathbb{P}^1$: one assigns a carefully constructed divisor $\mathcal{N}_j$ on $\mathbb{G}_m$ to each such normalization, and applies the Bloch-Wigner function D_2. In this way we arrive at

$$\mathrm{Im} R_{\hat{\gamma}_j} := \mathrm{Im}\langle \mathrm{AJ}^{2,2}_{X_0}(i^*\Xi), \hat{\gamma}_j \rangle = D_2(\mathcal{N}_j), \tag{B.0.2}$$

where $\hat{\gamma}_j \in H_1(X_0, \mathbb{Z})$ is a cycle passing once through the j^{th} node.

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