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Properties of Cyclic Functions

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Dedicated to

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ABSTRACT OF THE DISSERTATION

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For $1 \leq p < \infty$, consider the Hardy space $H^p(\mathbb{D}^n)$ on the unit polydisk. Beurling's theorem characterizes all shift cyclic functions in the Hardy spaces when $n = 1$. Such a theorem is not known to exist in most other analytic function spaces, even in the one variable case. Therefore, it becomes natural to ask what properties these functions satisfy in order to understand them better. The goal of this thesis is to showcase some important properties of cyclic functions in two different settings.

1. Fix $1 \leq p, q < \infty$ and $m, n \in \mathbb{N}$. Let $T : H^p(\mathbb{D}^n) \to H^q(\mathbb{D}^m)$ be a bounded linear operator. Then T preserves cyclic functions, i.e. Tf is cyclic whenever f is, if and only if T is a weighted composition operator.

2. Let $\mathcal{H}$ be a normalized complete Nevanlinna-Pick space, and let $f, g \in \mathcal{H}$ be such that $fg \in \mathcal{H}$. Then f and g are multiplier cyclic if and only if their product fg is.

We also extend (1) to a large class of analytic function spaces that includes the Dirichlet space, and the Drury-Arveson space on the unit ball $\mathbb{B}_n$ among others. Both of these properties generalize all previously known results of this type.

Chapter 1

Cyclicity Preserving Operators

Cyclic functions have been well-studied in a variety of different contexts, thanks to their association with many deep and interesting questions in mathematics. Let $\mathcal{X}$ be a Banach space and let $T : \mathcal{X} \to \mathcal{X}$ be a bounded linear map. A vector $v \in \mathcal{X}$ is said to be *T-cyclic* if

$$T[v] := \overline{\text{span}} \left\{ v,\, Tv,\, T^2 v,\, \dots \right\} = \mathcal{X}.$$

When $\mathcal{X}$ is a finite dimensional vector space, we have a generalization of the Jordan decomposition theorem called the *cyclic decomposition theorem*, which states that there exist vectors $v_1, v_2, \dots, v_n \in \mathcal{X}$ such that $\mathcal{X} = \bigoplus_{k=1}^{n} T[v_k]$. K. Hoffman and R. Kunze refer to it as "one of the deepest results in linear algebra" (*Section 7.2*, [19]).

When $\mathcal{X}$ is infinite dimensional, it is generally difficult to determine all cyclic vectors. As infinite dimensional Banach spaces usually arise in function spaces, one can hope to use tools from functional analysis and operator theory to answer questions about cyclicity. A fundamental object in function theory is the *shift operator*. We typically refer to cyclic vectors of the shift operator as *shift-cyclic functions*. Shift-cyclicity is known to be associated with a famous open problem in harmonic analysis namely the *Periodic Dilation Completeness Problem (PDCP)*, which is in turn somewhat loosely associated with the Riemann Hypothesis (see [9], [28], and [27] for more details on this subject). In this chapter we shall introduce shift-cyclic functions for a large class of Banach spaces of analytic functions, and explore their behaviour under weighted composition operators. We shall also consider situations in which one may extend the results obtained to certain non-Banach spaces as well.

1.1 Introduction

Fix $n \in \mathbb{N}$ and let $\mathbb{D}^n := \{z \in \mathbb{C}^n \,\big|\, |z_i| < 1, \forall\, 1 \le i \le n\}$ be the unit polydisk in $\mathbb{C}^n$. For $0 < p < \infty$, we define the Hardy space as

$$H^p(\mathbb{D}^n) := \left\{ f \in \mathrm{Hol}(\mathbb{D}^n) \,\middle|\, ||f||_p^p := \sup_{0<r<1} \int_{\mathbb{T}^n} |f(rw)|^p \, d\sigma_n(w) < \infty \right\}.$$

Here, for an open set $D \subset \mathbb{C}^n$, $\mathrm{Hol}(D)$ is the set of holomorphic functions on D. Also, σ_n is the normalized Lebesgue measure on the unit n-torus $\mathbb{T}^n := \big\{z \in \mathbb{C}^n \,\big|\, |z_i| = 1,\ \forall\, 1 \le i \le n\big\}$. It is known that $H^p(\mathbb{D}^n)$ is a Banach space for all $1 \le p < \infty$ with norm $||\cdot||_p$. We also have the space of all bounded analytic functions defined on $\mathbb{D}^n$,

$$H^\infty(\mathbb{D}^n) := \left\{ f \in \mathrm{Hol}(\mathbb{D}^n) \,\middle|\, ||f||_\infty := \sup_{w \in \mathbb{D}^n} |f(w)| < \infty \right\}.$$

Just like $H^p(\mathbb{D}^n)$ for $1 \le p < \infty$, $H^\infty(\mathbb{D}^n)$ is a Banach space with the supremum norm $||\cdot||_\infty$. It is important to note that for $n \in \mathbb{N}$ and $0 < p \le \infty$, every $f \in H^p(\mathbb{D})^n$ has *radial limits*

$$f^*(w) := \lim_{r \longrightarrow 1} f(rw)$$

for almost all $w \in \mathbb{T}^n$ with respect to σ_n. See *Section 2.3* in [30] for more details.

Definition 1.1.1. A function $f \in H^p(\mathbb{D}^n)$ is said to be *(shift) cyclic* if

$$\mathrm{S}[f] := \overline{\mathrm{span}}\,\big\{z^\alpha f(z) \,\big|\, \alpha \in \mathbb{Z}^+(n)\big\} = \overline{\big\{pf \,\big|\, p \text{ - polynomial}\big\}} = H^p(\mathbb{D}^n).$$

$\mathbb{Z}^+(n)$ is the set of all tuples $\alpha = (\alpha_i)_{i=1}^n$ of non-negative integers, and z^α denotes the monomial $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ for each $\alpha \in \mathbb{Z}^+(n)$.

When $n = 1$, cyclic functions in $H^p(\mathbb{D})$ have been characterized using Beurling's theorem (see *Theorem 7.4* in [13] and *Theorem 4* in [14]). When $n > 1$, we do not have a version of Beurling's theorem or the canonical factorization theorem (see *Section 4.4* in [30] for more details). Several sufficient conditions for cyclicity in $H^p(\mathbb{D}^n)$ were provided by N. Nikolski (*Theorems 3.3* and *3.4*, [27]) but in general, not a lot is known about cyclicity when $n > 1$.

A common way of obtaining cyclic functions when $n > 1$ is through operators that preserve cyclicity, i.e. linear maps $T : H^p(\mathbb{D}^n) \to H^q(\mathbb{D}^m)$ such that Tf is cyclic whenever f is cyclic. When $p = q = 2$ and $n = m = 1$, a result of P. Gibson and M. Lamoureux shows that all such operators have to be weighted composition operators (*Theorem 4*, [15]). Gibson and Lamoureux's result was directly motivated by the work of J. Borcea and P. Brändén on *stability preserving operators* in [5] and [6]. In [16] Gibson and Lamoureux, along with G. Margrave, showcase an application of cyclicity preserving operators in signal processing pertaining to a certain geophysical problem. Cyclic functions and their monomial multiples represent *minimum-phase* signals, which correspond to short but powerful bursts of energy (like a dynamite blast). Therefore, operators that preserve such functions can help model physical phenomenons that preserve short high-energy bursts.

The result of Gibson and Lamoureux was generalized to all $0 < p, q \leq \infty$ by J. Mashreghi and T. Ransford for *'outer-preserving operators'* in [24].

Definition 1.1.2. For $0 < p \leq \infty$, a function $f \in H^p(\mathbb{D}^n)$ is said to be *outer* if it is non-vanishing and satisfies

$$\log |f(0)| = \int_{\mathbb{T}^n} \log |f^*|.$$

In $H^p(\mathbb{D})$, using Beurling's theorem, it is known that the class of cyclic functions coincides with that of outer functions. The outerness property captures the necessary lack of zeros of f in $\mathbb{D}$, and a subtle non-vanishing property of f^* on $\mathbb{T}$. More details on outer functions can be found in *Chapter 4* of [30].

With this in mind, the following theorem of J. Mashreghi and T. Ransford (*Theorem 2.2*, [24]) is a generalization of the result of Gibson and Lamoureux.

Theorem 1.1.3 (Mashreghi, Ransford 2015)**.** *Let $0 < p \leq \infty$ and let $T : H^p(\mathbb{D}) \to \mathrm{Hol}(\mathbb{D})$ be a linear map such that $Tg(z) \neq 0$ for all outer functions $g \in H^p(\mathbb{D})$ and all $z \in \mathbb{D}$. Then there exist holomorphic maps $\phi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C} \setminus \{0\}$ such that*

$$Tf = \psi \cdot (f \circ \phi) \;\; (\forall\, f \in H^p(\mathbb{D}))\,.$$

It is important to note that continuity of T is not assumed in the above theorem.

For more general spaces over $\mathbb{D}$, J. Mashreghi and T. Ransford prove a similar result. Let $X \subset \mathrm{Hol}(\mathbb{D})$ be a Banach space that satisfies the following properties :

$(X1)$ X contains the set of polynomials, and they form a dense subspace of X.

$(X2)$ For each $w \in \mathbb{D}$, the evaluation map $f \mapsto f(w) : X \to \mathbb{C}$ is continuous.

$(X3)$ X is shift-invariant, i.e. $f \in X \Rightarrow zf \in X$.

We also need a subset $Y \subset X$ that satisfies the following properties.

$(Y1)$ If $g \in X$ and $0 < \inf_{\mathbb{D}}|g| \leq \sup_{\mathbb{D}}|g| < \infty$, then $g \in Y$.

$(Y2)$ If $g(z) = z - \lambda$ where $\lambda \in \mathbb{T}$, then $g \in Y$.

For these spaces, we have the following theorem (*Theorem 3.2*, [24]).

Theorem 1.1.4 (Mashreghi, Ransford 2015)**.** *Let $T : X \to \mathrm{Hol}(\mathbb{D})$ be a continuous linear map such that $Tg(z) \neq 0$ for every $g \in Y$ and $z \in \mathbb{D}$. Then there exist holomorphic functions $\phi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C} \setminus \{0\}$ such that*

$$Tf(z) = \psi(z)\, f(\phi(z)), \, \forall f \in \mathcal{X}.$$

The proof of *Theorem 1.1.4* relies on classifying $\Lambda \in X^*$ such that $\Lambda(g) \neq 0, \forall g \in Y$ (*Theorem 3.1*, [24]). This is similar to a result now known as the Gleason-Kahane-Żelazko (GKŻ) theorem (see [17] and [22]), which identifies multiplicative linear functionals in a complex unital Banach algebra through its action on invertible elements (see *Theorem 1.5.1* below). In [24], it is shown that a version of the GKŻ theorem holds for modules of a complex unital Banach algebra and it can be applied to the multiplier algebra of the space X satisfying properties $(X1)$-$(X3)$ to obtain *Theorem 1.1.4*. In [23], K. Kou and J. Liu provide a similar argument for $H^p(\mathbb{D})$ when $1 < p < \infty$. It is essentially the same as that of *Theorem 1.1.4*, but instead of the subset Y they consider the set $\{e^{w \cdot z} \mid w \in \mathbb{C}\}$ (see *Theorem 2*, [23]). They also showed that the converse of *Theorem 1.1.4* is true when $1 < p < \infty$, i.e. all weighted composition operators on $H^p(\mathbb{D})$ for $1 < p < \infty$ also preserve outer (and thus, cyclic) functions.

Using techniques similar to those in [23] and [24], we can generalize *Theorem 1.1.4* to spaces of analytic functions in more than one variable, and also over arbitrary domains. To that end, we shall work with spaces $\mathcal{X}$ consisting of functions defined on a set $D \subset \mathbb{C}^n$ for some $n \in \mathbb{N}$, and that are holomorphic on an open subset of D. Furthermore, $\mathcal{X}$ satisfies the following properties.

Q1 The set of polynomials $\mathcal{P}$ is dense in $\mathcal{X}$.

Q2 The point evaluation map $\Lambda_z : \mathcal{X} \to \mathbb{C}$, defined as $\Lambda_z f := f(z)$, is a bounded linear functional on $\mathcal{X}$ for all $z \in D$. Furthermore, if for some $z \in \mathbb{C}^n$ the map $\Lambda_z p := p(z)$ defined on $\mathcal{P}$ extends to a bounded linear functional on all of $\mathcal{X}$, then $z \in D$.

Q3 The i^{th}-shift operator $S_i : \mathcal{X} \to \mathcal{X}$, defined as $S_i f(z) := z_i f(z)$ for every $(z_i)_{i=1}^n = z \in D$ and $f \in \mathcal{X}$, is bounded for every $1 \leq i \leq n$.

The domain D, in this case, is called the *maximal domain* of $\mathcal{X}$. Here, the maximality is with respect to bounded extension of point evaluations on the set of polynomials.

Maximal domains are generally considered in the setting of reproducing kernel Hilbert spaces, and are closely related to the notion of *algebraic consistency*. See [11], [26], and [18] for more background on maximal domains. For the reader's convenience, we now list all of the main results. These can also be found in *Section 1* of [31].

Theorem 1.1.5. *Suppose $\mathcal{X}$ satisfies **Q1-Q3** over a set $D \subset \mathbb{C}^n$. Let $\Lambda \in \mathcal{X}^*$ be such that $\Lambda(e^{w \cdot z}) \neq 0$ for every $w \in \mathbb{C}^n$. Then, there exist $a \in \mathbb{C} \setminus \{0\}$ and $b \in D$ such that $\Lambda(f) = a \cdot f(b)$.*

Using *Theorem 1.1.5*, we will obtain the following generalization of *Theorem 1.1.4*.

Theorem 1.1.6. *Suppose $\mathcal{X}$ satisfies **Q1-Q3** over a set $D \subset \mathbb{C}^n$. Let $\mathcal{Y}$ be a topological vector space of functions, defined on a set E, such that $\Gamma_u g := g(u)$, $g \in \mathcal{Y}$ defines a continuous linear functional for all $u \in E$. Let $T : \mathcal{X} \to \mathcal{Y}$ be a continuous linear operator. Then, the following are equivalent :*

(1) *$T(e^{w \cdot z})$ is non-vanishing for every $w \in \mathbb{C}^n$.*

(2) *$Tf(u) = a(u)f(b(u))$ for some non-vanishing function $a \in \mathcal{Y}$, and a map $b : E \to D$.*

Furthermore $a = T1$ and $b = \frac{T(z)}{T(1)}$, where $T(z) = \big(T(z_i)\big)_{i=1}^n$.

Using *Theorem 1.1.6*, we will prove the following generalization of *Theorem 1.1.3*.

Theorem 1.1.7. (1) *Fix $0 < p, q < \infty$ and $m, n \in \mathbb{N}$. Let $T : H^p(\mathbb{D}^n) \to H^q(\mathbb{D}^m)$ be a bounded linear operator such that Tf is cyclic whenever f is cyclic. Then, there exist analytic functions $a \in H^q(\mathbb{D}^m)$ and $b : \mathbb{D}^m \to \mathbb{D}^n$ such that*

$$Tf(z) = a(z)f(b(z)),\ \forall z \in \mathbb{D}^n,\ f \in H^p(\mathbb{D}^n).$$

Furthermore, $a = T1$ is cyclic and $b = \frac{T(z)}{T1}$, where $T(z) = \big(T(z_i)\big)_{i=1}^n$.

(2) *Fix* $0 < p, q \leq \infty$ *and* $m, n \in \mathbb{N}$. *Then, the conclusion of part* (1) *holds if we replace* 'cyclic' *with* 'outer'.

For $1 \leq q < \infty$, *the converse of part* (1) *is also true. That is, all bounded weighted composition operators from* $H^p(\mathbb{D}^n)$ *into* $H^q(\mathbb{D}^m)$ *also preserve cyclicity.*

It is important to note here that the boundedness of T in *Theorem 1.1.7* is a crucial non-trivial assumption, since it is difficult to characterize when a weighted composition operator is bounded when $n > 1$. *Theorems 1–6* in [12] explore the boundedness and compactness of weighted composition operators for various spaces when $n = 1$.

In *Section 1.5*, as an interesting byproduct of the main results, we will prove the following version of the GKŻ theorem for Banach spaces of analytic functions.

Theorem 1.1.8. *Suppose* $\mathcal{X}$ *satisfies* ***Q1-Q3*** *over* $D \subset \mathbb{C}^n$. *Let* $\Lambda \in \mathcal{X}^*$ *such that* $\Lambda(1) = 1$, *and let* $\mathcal{M}(\mathcal{X}) := \{\phi : D \to \mathbb{C} \mid \phi f \in \mathcal{X}, \forall f \in \mathcal{X}\}$ *be the multiplier algebra of* $\mathcal{X}$ *Then, the following are equivalent :*

(*i*) $\Lambda(e^{w \cdot z}) \neq 0$ *for every* $w \in \mathbb{C}^n$.

(*ii*) $\Lambda \equiv \Lambda_z$ *for some* $z \in D$.

(*iii*) $\Lambda(fg) = \Lambda(f)\Lambda(g)$ *for all* $f, g \in \mathcal{X}$ *such that* $fg \in \mathcal{X}$.

(*iv*) $\Lambda(\phi f) = \Lambda(\phi)\Lambda(f)$ *for all* $\phi \in \mathcal{M}(\mathcal{X})$ *and* $f \in \mathcal{X}$.

Theorem 1.1.8 was partially motivated by *Corollary 1.3* in [3].

1.2 Notations and preliminary results

Before we consider spaces of functions defined over its maximal domain, we will work with spaces of holomorphic functions defined on an open set in $\mathbb{C}^n$ for some $n \in \mathbb{N}$ as the notation is much simpler in this case.

Fix $n \in \mathbb{N}$. For an open set $D \subset \mathbb{C}^n$, let $\mathcal{X} \subset \mathrm{Hol}(D)$ be a Banach space satisfying:

P1 The set of polynomials $\mathcal{P}$ is dense in $\mathcal{X}$.

P2 The point-evaluation map $\Lambda_z : \mathcal{X} \to \mathbb{C}$, defined as $\Lambda_z(f) := f(z)$ for every $f \in \mathcal{X}$, is a bounded linear functional on $\mathcal{X}$ for every $z \in D$.

P3 The i^{th}-shift operator $S_i : \mathcal{X} \to \mathcal{X}$, defined as $S_i f(z) := z_i f(z)$ for every $z \in D$ and $f \in \mathcal{X}$, is a bounded linear operator for every $1 \leq i \leq n$.

Examples of spaces that satisfy **P1-P3** include the Hardy space $H^p(\mathbb{D}^n)$ for $1 \leq p < \infty$, the Drury-Arveson space $\mathcal{H}_n^2$ on the unit ball $\mathbb{B}_n := \left\{ z \in \mathbb{C}^n \,\middle|\, \sum_{i=1}^n |z_i|^2 < 1 \right\}$, and the Dirichlet-type spaces $\mathcal{D}_\alpha$ for $\alpha \in \mathbb{R}$.

$$\mathcal{H}_n^2 = \left\{ f \sim \sum \hat{f}(a) z^a \in \mathrm{Hol}(\mathbb{B}_n) \,\middle|\, \sum_{a \in \mathbb{Z}^+(n)} \frac{a_1!\, a_2! \cdots a_n!}{(a_1 + a_2 + \cdots + a_n)!} |\hat{f}(a)|^2 < \infty \right\}$$

$$\mathcal{D}_\alpha = \left\{ f \sim \sum \hat{f}(a) z^a \in \mathrm{Hol}(\mathbb{D}^n) \,\middle|\, \sum_{a \in \mathbb{Z}^+(n)} \big((a_1+1) \cdots (a_n+1)\big)^\alpha |\hat{f}(a)|^2 < \infty \right\}$$

The list of Dirichlet-type spaces consists of many important spaces like the usual Dirichlet space ($\alpha = 1$), the Hardy space $H^2(\mathbb{D}^n)$ ($\alpha = 0$), and also the Bergman space ($\alpha = -1$). For these spaces, we prove the following preliminary result.

Theorem 1.2.1. *Suppose $\mathcal{X}$ satisfies **P1-P3** over an open set $D \subset \mathbb{C}^n$. Let $\Lambda \in \mathcal{X}^*$ be such that $\Lambda(e^{w \cdot z}) \neq 0$ for every $w \in \mathbb{C}^n$. Then, there exist $a \in \mathbb{C} \setminus \{0\}$ and $b \in \sigma_r(S)$ such that $\Lambda p = a \cdot p(b)$ for every $p \in \mathcal{P}$. Here, $\sigma_r(S)$ is the right Harte spectrum of $S = (S_i)_{i=1}^n$.*

Recall that $\sigma_r(S)$ is the complement in $\mathbb{C}^n$ of $\rho_r(S)$, where

$$\rho_r(S) := \left\{ \lambda \in \mathbb{C}^n \,\middle|\, \exists \, \{A_i\}_{i=1}^n \subset \mathcal{B}(\mathcal{X}) \text{ such that } \sum_{i=1}^n (S_i - \lambda_i I) A_i = I \right\}.$$

Note that it is not immediate from **P1-P3** that $e^{w\cdot z} \in \mathcal{X}$. We address this separately as a lemma before we prove *Theorem 1.2.1*.

Lemma 1.2.2. *For each $w \in \mathbb{C}^n$, we have $e^{w\cdot z} \in \mathcal{X}$. In fact, $p_k := \sum_{|\alpha|\leq k} \frac{w^\alpha z^\alpha}{\alpha!} \to e^{w\cdot z}$ in $\mathcal{X}$ as $k \to \infty$, where $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\alpha! := \alpha_1!\, \alpha_2! \cdots \alpha_n!$.*

Proof. Fix $w \in \mathbb{C}^n$. We show that $\lim_{k\to\infty} p_k$ exists. This follows from the fact that $\mathcal{X}$ is a Banach space and

$$\sum_{\alpha\in\mathbb{Z}^+(n)} \left|\left|\frac{w^\alpha z^\alpha}{\alpha!}\right|\right| \leq \sum_{\alpha\in\mathbb{Z}^+(n)} \frac{|w|^\alpha\, ||S||^\alpha\, ||1||}{\alpha!} = ||1||e^{|w|\cdot||S||}$$

where $|w| := \big(|w_1|, \ldots, |w_n|\big)$ and $||S|| := \big(||S_1||, \ldots, ||S_n||\big)$. Let $g = \lim_{k\to\infty} p_n$ in $\mathcal{X}$. Note that p_k converges to $e^{w\cdot z}$ point-wise. By **P2**, this implies $g(z) = e^{w\cdot z}$. ☐

Proof of Theorem 1.2.1. Since $\Lambda(e^{z\cdot w}) \neq 0$ for all $w \in \mathbb{C}^n$, and Λ is continuous, we get

$$\sum_{\alpha\in\mathbb{Z}^+(n)} \frac{\Lambda(z^\alpha)w^\alpha}{\alpha!} \neq 0,\ \forall w \in \mathbb{C}^n.$$

Let $\lambda_\alpha := \Lambda(z^\alpha)$, $\forall\alpha \in \mathbb{Z}^+(n)$. Now, $|\lambda_\alpha| = ||\Lambda(z^\alpha)|| \leq ||\Lambda|| \cdot ||z^\alpha||$ implies

$$|\lambda_\alpha| \leq ||\Lambda|| \cdot ||S_1||^{\alpha_1} \cdot ||S_2||^{\alpha_2} \cdots ||S_n||^{\alpha_n} \cdot ||1|| \text{ for every } \alpha \in \mathbb{Z}^+(n).$$

Let $F(w) := \sum_{\alpha\in\mathbb{Z}^+(n)} \frac{\lambda_\alpha w^\alpha}{\alpha!}$, and note that F is a non-vanishing entire function such that

$$|F(w)| \leq ||\Lambda|| \cdot ||1|| \cdot e^{|w|\cdot||S||}.$$

When $n = 1$, it is well-known that all such F are of the form $e^{a_0+b\cdot w}$ for some $a_0 \in \mathbb{C}$ and $b \in \mathbb{C}^n$ (see *Section 3.2, Chapter 5* in [2]). We will show that this is true for all values of n.

Lemma 1.2.3. *Fix $n \in \mathbb{N}$. Let $F \in \mathrm{Hol}(\mathbb{C}^n)$ be a non-vanishing entire function for which there exist constants A, B such that $|F(z)| \leq Ae^{Br^m}$ for all z in $(r\mathbb{D})^n$, and for all $r > 0$. Then, there exists a polynomial p with $\deg(p) \leq m$ such that $F(z) = e^{p(z)}$ for all $z \in \mathbb{C}^n$.*

Proof. Since F is non-vanishing, there exists an entire function G such that $F = e^G$. Note that the hypothesis then implies $\mathrm{Re}(G) \leq \ln A + Br^m$ in $(r\mathbb{D})^n$. We need to show that G is a polynomial with $\deg(G) \leq m$. The case $n = 1$ is known (see *Section 3.2*, *Chapter 5* in [2]), so assume $n > 1$. Let $G(z) = \sum G_k(z)$ be the homogeneous expansion of G. Fix $z \in \mathbb{C}^n$ and let $g_z(\lambda) := G(\lambda z) = \sum \lambda^k G_k(z)$ for $\lambda \in \mathbb{C}$. Notice that

$$\mathrm{Re}\big(g_z(\lambda)\big) = \mathrm{Re}\big(G(\lambda z)\big) \leq \ln A + B \cdot C^m |\lambda|^m$$

where $C = \sup_{1 \leq j \leq n} |z_j|$, since $z \in (r\mathbb{D})^n$ for every $r > C$. Thus, for $\lambda \in r\mathbb{D}$

$$\mathrm{Re}\big(g_z(\lambda)\big) \leq \ln A + B \cdot C^m r^m.$$

Applying the one variable case to g_z, we get $G_k(z) = 0$ for all $k > m$. As the choice of $z \in \mathbb{C}^n$ was arbitrary, this means $G_k(z) = 0$ for all $z \in \mathbb{C}^n$ and $k > m$. Therefore, G is a polynomial with $\deg(G) \leq m$ as required. □

Proof of Theorem 1.2.1 (cont.) By *Lemma 1.2.3*, we get that $F(w) = e^{a_0 + b \cdot w}$ for some $a_0 \in \mathbb{C}$ and $b \in \mathbb{C}^n$. Using the definition of $F(w)$, and comparing power-series coefficients, we get $\lambda_\alpha = e^{a_0} b^\alpha$, $\forall \alpha \in \mathbb{Z}^+(n)$. Let $a := e^{a_0} \in \mathbb{C} \setminus \{0\}$. This means $\Lambda(z^\alpha) = a \cdot b^\alpha$, $\forall \alpha \in \mathbb{Z}^+(n)$.

Note that we have shown $\Lambda p = a \cdot p(b)$ for every polynomial p. It only remains to show that $b \in \sigma_r(S)$. For the sake of contradiction, suppose $b \notin \sigma_r(S)$. Therefore there exists $\{A_i\}_{i=1}^n \subset \mathcal{B}(\mathcal{X})$ such that

$$\sum_{i=1}^{n} (S_i - b_i I) A_i = I.$$

In particular,

$$\sum_{i=1}^{n}(z_i - b_i)A_i 1 = 1.$$

Fix an $\epsilon > 0$. Since $\mathcal{X}$ satisfies **P1**, we can pick $p_i \in \mathcal{P}$ for each $1 \leq i \leq n$ such that

$$||A_i 1 - p_i|| < \frac{\epsilon}{n \cdot ||\Lambda|| \cdot ||S_i - b_i I||}.$$

Note that,

$$\left|\left|1 - \sum_{i=1}^{n}(z_i - b_i)p_i\right|\right| = \left|\left|\sum_{i=1}^{n}(z_i - b_i)(A_i 1 - p_i)\right|\right| \leq \sum_{i=1}^{n} ||S_i - b_i I||\, ||A_i 1 - p_i|| < \frac{\epsilon}{||\Lambda||}.$$

Based on the representation of Λ on polynomials, we know that

$$\Lambda\left(\sum_{i=1}^{n}(z_i - b_i)p_i\right) = 0.$$

This means

$$|a| = |\Lambda 1| = \left|\Lambda 1 - \Lambda\left(\sum_{i=1}^{n}(z_i - b_i)p_i\right)\right| \leq ||\Lambda|| \cdot \left|\left|1 - \sum_{i=1}^{n}(z_i - b_i)p_i\right|\right| < \epsilon.$$

As $\epsilon > 0$ was arbitrarily chosen and $a \neq 0$, we get a contradiction. Hence, $b \in \sigma_r(S)$. □

Remark 1.2.4. It would be great if we could show that $b \in D$, but that need not be the case. It is obvious that $D \subset \sigma_r(S)$, but it may not be possible to extend the domain of every function in $\mathcal{X}$ to the whole of $\sigma_r(S)$ in order to extend the functional in the theorem to all of $\mathcal{X}$. Also, $\sigma_r(S)$ was chosen solely for the above proof to work. One may also work with the Taylor spectrum $\sigma_{Tay}(S)$ if needed, since (v) in *Theorem 14.53*, [1] shows that

$$\sigma_r(S) \subset \sigma_{Tay}(S).$$

Example 1. When $\mathcal{X} = H^p(\mathbb{D}^n)$, for some $1 \leq p < \infty$, it is easy to check that $\sigma_r(S) = \overline{\mathbb{D}^n}$. So, b obtained in *Theorem 1.2.1* lies in $\overline{\mathbb{D}^n}$. We claim that in this case, b lies in $\mathbb{D}^n$. For the sake of argument, assume $b = (b_i)_{i=1}^n \in \partial\mathbb{D}^n$ with $b_j \in \mathbb{T}$ for some $1 \leq j \leq n$. Consider $q(z) := z_j - b_j$. Since $z - \beta$ is cyclic in $H^p(\mathbb{D})$ for all $1 \leq p < \infty$ and $\beta \notin \mathbb{D}$, q is cyclic in $H^p(\mathbb{D}^n)$. This means that for any given $f \in H^p(\mathbb{D}^n)$, there exist polynomials $\{q_k\}_{k\in\mathbb{N}}$ such that $q_k q \to f$. Note that since $q(b) = 0$ we have that for every $k \in \mathbb{N}$,

$$\Lambda(q_k q) = a \cdot q_k(b)q(b) = 0.$$

Thus, $\Lambda(f) = 0$ which implies $\Lambda \equiv 0$, a contradiction. So, $b \in \mathbb{D}^n$ and $\Lambda \equiv a\Lambda_b$.

A similar argument can be made for spaces that have an *envelope of cyclic polynomials over* D. Recall that $f \in \mathcal{X}$ is cyclic if the *shift-invariant subspace* $\mathrm{S}[f]$ generated by f is all of $\mathcal{X}$.

$$\mathrm{S}[f] := \overline{\operatorname{span}}\left\{z^\alpha f(z) \;\middle|\; \alpha \in \mathbb{Z}^+(n)\right\} = \overline{\left\{pf \;\middle|\; p \in \mathcal{P}\right\}} = \mathcal{X}.$$

By **P1**, it follows that $f \in \mathcal{X}$ is cyclic if and only if $1 \in \mathrm{S}[f]$. It is then easy to see that all cyclic functions are non-vanishing.

Definition 1.2.5. $\mathcal{X}$ has an *envelope of cyclic polynomials* over D if there is a family $\mathcal{F} \subset \mathcal{P}$ of cyclic polynomials such that $\widetilde{D}_{\mathcal{F}} := \bigcap_{q\in\mathcal{F}} \left(\mathbb{C}^n \setminus \mathcal{Z}(q)\right) \subseteq D$, where $\mathcal{Z}(q)$ is the zero-set of q.

Proposition 1.2.6. *Suppose $\mathcal{X}$ satisfies **P1-P3** over an open set $D \subset \mathbb{C}^n$, and also has an envelope of cyclic polynomials with $\mathcal{F} \subset \mathcal{P}$. Let $\Lambda \in \mathcal{X}^*$ be such that $\Lambda(e^{w\cdot z}) \neq 0$ for every $w \in \mathbb{C}^n$. Then, there exist $a \in \mathbb{C} \setminus \{0\}$ and $b \in D$, such that $\Lambda f = a \cdot f(b)$ for all $f \in \mathcal{X}$.*

Proof. We only need to show that $b \in D$ since in that case, we get $\Lambda \equiv a\Lambda_b$ on $\mathcal{X}$. For this, let $q \in \mathcal{F}$ be arbitrary and suppose $q(b) = 0$. Since q is cyclic, for every $f \in \mathcal{X}$ we obtain a sequence of polynomials $\{q_k\}_{k\in\mathbb{N}}$ such that $q_k q \to f$.

This means that

$$0 = a \cdot q_k(b)q(b) = \Lambda(q_k q) \to \Lambda(f).$$

Thus, $\Lambda \equiv 0$ and we get a contradiction. So $q(b) \neq 0$ for every $q \in \mathcal{F}$, and $b \in \widetilde{D}_{\mathcal{F}} \subseteq D$. $\square$

Example 2. For $H^p(\mathbb{D}^n)$ when $1 \leq p < \infty$,

$$\mathcal{F} := \{z_i - \beta \mid 1 \leq i \leq n \text{ and } \beta \notin \mathbb{D}\}$$

is an envelope of cyclic polynomials over $\mathbb{D}^n$ (see *Example 1*). The same set of polynomials works for the Dirichlet-type spaces $\mathcal{D}_\alpha$ when $\alpha \leq 1$. For $\alpha > 1$ and $1 \leq i \leq n$, the polynomial $z_i - w$ is not cyclic in $\mathcal{D}_\alpha$ for all $w \in \mathbb{T}$, and hence the same example does not work. In fact, every $f \in \mathcal{D}_\alpha$ is continuous up to the boundary when $\alpha > 1$. Plus Λ_b is a bounded linear functional on $\mathcal{D}_\alpha$ even when $b \in \partial\mathbb{D}^n$. Therefore $\mathcal{D}_\alpha$ cannot have an envelope of cyclic polynomials over $\mathbb{D}^n$. A detailed discussion on cyclicity of polynomials in the Dirichlet-type spaces can be found in [4].

1.3 Maximal domains

Let us try to make sense of how big the domain of functions in a general space $\mathcal{X}$ that satisfies properties **P1-P3** can become. We also want to make sure that shift operators and cyclic functions are well-defined in this extension.

Definition 1.3.1. Given $\mathcal{X}$ satisfying **P1-P3** over an open set $D \subset \mathbb{C}^n$, we define the *maximal domain* of functions in $\mathcal{X}$ to be the set

$$\widehat{D} := \left\{ w \in \mathbb{C}^n \;\middle|\; \Lambda_w p := p(w), \forall p \in \mathcal{P} \text{ has a bounded linear extension to } \mathcal{X} \right\}.$$

First, we prove an important property of the maximal domain.

Theorem 1.3.2. *Suppose $\mathcal{X}$ satisfies **P1-P3** over an open set $D \subset \mathbb{C}^n$. Then, we have*

$$D \subset \widehat{D} \subset \sigma_r(S).$$

Proof. $D \subset \widehat{D}$ follows from **P1** and **P2**. To show $\widehat{D} \subset \sigma_r(S)$, let $b \in \widehat{D}$. By **P1** and *Lemma 1.2.2*, we get

$$\Lambda_b(e^{w \cdot z}) = e^{w \cdot b} \neq 0 \text{ for all } w \in \mathbb{C}^n.$$

By *Theorem 1.2.1*, there exists $\hat{b} \in \sigma_r(S)$ such that $\Lambda_b|_{\mathcal{P}} \equiv \Lambda_{\hat{b}}|_{\mathcal{P}}$. Evaluating both functionals at z_i for each $1 \leq i \leq n$, we get $b = \hat{b} \in \sigma_r(S)$ as needed. □

Remark. This shows that the maximal domain is not a very large set, since it is contained in a nice compact set. In the case of $H^p(\mathbb{D}^n)$ for $1 \leq p < \infty$ and $\mathcal{D}_\alpha$ for $\alpha \leq 1$, we saw earlier in *Example 2* that $\widehat{D} = \mathbb{D}^n$. However for $\mathcal{D}_\alpha$ when $\alpha > 1$, $\widehat{D} = \overline{\mathbb{D}^n}$. Therefore, both inclusions in the theorem can be proper.

We now show that in general, $\mathcal{X}$ can be identified with a space $\widehat{\mathcal{X}}$ of functions over $\widehat{D}$, which satisfies **Q1-Q3**. The following discussion is similar to that of *Section 5* in [18], where the author talks about the idea of *'algebraic consistency'* and considers a couple different notions of maximal domains. Our notion of maximal domain is different from those discussed in [18], so we will provide all the details here for the sake of completeness.

Let us begin with some notation before proving the identification. For every $f \in \mathcal{X}$, define $\hat{f}(\hat{z}) := \Lambda_{\hat{z}} f$ for every $\hat{z} \in \widehat{D}$ where, with the abuse of notation, we write $\Lambda_{\hat{z}} f$ to represent the extension of $\Lambda_{\hat{z}}|_{\mathcal{P}}$ on $\mathcal{X}$ evaluated at f. Notice that for $z \in D$, $\hat{f}(z) = f(z)$ for every $f \in \mathcal{X}$. This also implies $\hat{f}|_D \in \mathrm{Hol}(D)$. Also, for $p \in \mathcal{P}$, we have $\hat{p}(\hat{z}) = p(\hat{z})$ for every $\hat{z} \in \widehat{D}$. Thus, $\hat{\mathcal{P}} := \{\hat{p} \,|\, p \in \mathcal{P}\}$ is the same set as $\mathcal{P}$. Now, let

$$\widehat{X} := \{\hat{f} : \widehat{D} \to \mathbb{C} \mid f \in \mathcal{X}\}.$$

We endow $\hat{\mathcal{X}}$ with the natural vector space structure of point-wise addition and scalar multiplication. This can be done because it is obvious that $\hat{f} + \hat{g} = \widehat{f+g}$, and $\alpha\hat{f} = \widehat{\alpha f}$ for every $\alpha \in \mathbb{C}$, $f, g \in \mathcal{X}$.

Define the map $\iota : \mathcal{X} \to \widehat{\mathcal{X}}$ as $\iota(f) := \hat{f}$ for every $f \in \mathcal{X}$. ι is clearly a vector space isomorphism, and we can define $||\hat{f}||_{\widehat{\mathcal{X}}} := ||f||_{\mathcal{X}}$ for every $\hat{f} \in \hat{\mathcal{X}}$. This implies

$$f_k \to f \text{ in } \mathcal{X} \iff \hat{f}_k \to \hat{f} \text{ in } \widehat{\mathcal{X}}.$$

So, $\widehat{\mathcal{X}}$ turns into a Banach space, and ι becomes an isometric isomorphism of Banach spaces. Note that since

$$\widehat{\mathcal{X}}|_D := \left\{ \hat{f}|_D \mid \hat{f} \in \widehat{\mathcal{X}} \right\} = \mathcal{X},$$

we can say that $\widehat{\mathcal{X}}$ is an extension of $\mathcal{X}$ to $\widehat{D}$.

Proposition 1.3.3. *$\widehat{\mathcal{X}}$ satisfies **Q1** and **Q2** over $\widehat{D}$.*

Proof. In order to show **Q1**, first recall that $f_k \to f$ in $\mathcal{X}$ if and only if $\hat{f}_k \to \hat{f}$ in $\hat{\mathcal{X}}$. Since $\mathcal{P}$ is dense in $\mathcal{X}$ by **P1**, it implies easily that the set of polynomials $\hat{\mathcal{P}}$ is dense in $\hat{\mathcal{X}}$.

In order to show **Q2**, notice that the map $\Lambda_{\hat{z}}\hat{f} := \hat{f}(\hat{z})$ is bounded for every $\hat{z} \in \widehat{D}$ since

$$|\Lambda_{\hat{z}}\hat{f}| = |\hat{f}(\hat{z})| = |\Lambda_{\hat{z}}f| \leq ||\Lambda_{\hat{z}}||_{\mathcal{X}^*}||f|| = ||\Lambda_{\hat{z}}||_{\mathcal{X}^*}||\hat{f}||.$$

For the second part of **Q2**, suppose for some $\hat{z} \in \mathbb{C}^n$, $\Lambda_{\hat{z}}$ defined as above extends to all of $\widehat{\mathcal{X}}$. As $\mathcal{P}$ and $\hat{\mathcal{P}}$ are identical, we can evaluate $\Lambda_{\hat{z}}$ on polynomials in $\mathcal{P}$ to get

$$|\Lambda_{\hat{z}}p| = |p(\hat{z})| = |\hat{p}(\hat{z})| \leq ||\Lambda_{\hat{z}}||_{\widehat{\mathcal{X}}^*}||\hat{p}|| \leq ||\Lambda_{\hat{z}}||_{\widehat{\mathcal{X}}^*}||p||.$$

By **P1**, $\Lambda_{\hat{z}}$ extends to a bounded functional on $\mathcal{X}$, and by definition of $\widehat{D}$, we get $\hat{z} \in \widehat{D}$. $\square$

Instead of showing that $\widehat{\mathcal{X}}$ satisfies **Q3** directly, we will prove a general result about multipliers. Recall that $\phi : D \to \mathbb{C}$ is a *multiplier* of $\mathcal{X}$, if $\phi f \in \mathcal{X}$ for every $f \in \mathcal{X}$. Denote the set of multipliers by $\mathcal{M}(\mathcal{X})$. Using closed graph theorem it is easy to check that if ϕ is a multiplier, then $M_\phi : \mathcal{X} \to \mathcal{X}$ defined as $M_\phi f := \phi f$, $\forall f \in \mathcal{X}$ is a bounded linear operator on $\mathcal{X}$. The norm

$$||\phi||_{\mathcal{M}(\mathcal{X})} := ||M_\phi||$$

turns $\mathcal{M}(\mathcal{X})$ into a Banach algebra. As $1 \in \mathcal{X}$, we get that $\mathcal{M}(\mathcal{X}) \subset \mathcal{X}$. Using **P2** and the above identification, it is easy to check that $|\phi(z)| \leq ||\phi||_{\mathcal{M}(\mathcal{X})}$ for all $z \in D$ and $\phi \in \mathcal{M}(\mathcal{X})$. Thus, $\mathcal{M}(\mathcal{X}) \subset L^\infty(D)$.

Proposition 1.3.4. *$\phi \in \mathcal{M}(\mathcal{X})$ if and only if $\hat{\phi} \in \mathcal{M}(\widehat{\mathcal{X}})$.*

Proof. First, note that for every choice of polynomials p, q we have $\widehat{pq} = \hat{p}\hat{q}$. Let $f \in \mathcal{X}$ be arbitrary, and let $\{q_k\}_{k\in\mathbb{N}}$ be a sequence of polynomials that converges to f in $\mathcal{X}$. Then for every $\hat{z} \in \widehat{D}$, since $pq_k \to pf$ implies $\widehat{pq_k} \to \widehat{pf}$, we get

$$\widehat{pf}(\hat{z}) = \lim_{k\to\infty} \widehat{pq_k}(\hat{z}) = \lim_{k\to\infty} \hat{p}(\hat{z})\hat{q}_k(\hat{z}) = \hat{p}(\hat{z}) \lim_{k\to\infty} \hat{q}_k(\hat{z}) = \hat{p}(\hat{z})\hat{f}(\hat{z}).$$

Thus $\hat{p}\hat{f} = \widehat{pf} \in \mathcal{X}$ for every $p \in \mathcal{P}$, $f \in \mathcal{X}$. This implies $\hat{p} \in \mathcal{M}(\widehat{\mathcal{X}})$.

Suppose now that $\phi \in \mathcal{M}(\mathcal{X})$. We already know $\widehat{\phi q} = \hat{\phi}\hat{q}$ for every $q \in \mathcal{P}$. Let $f \in \mathcal{X}$ and suppose again that $q_k \to f$ for some polynomials q_k. It is now easy to see for every $\hat{z} \in \widehat{D}$,

$$\widehat{\phi f}(\hat{z}) = \lim_{k\to\infty} \widehat{\phi q_k}(\hat{z}) = \lim_{k\to\infty} \hat{\phi}(\hat{z})\hat{q}_k(\hat{z}) = \hat{\phi}(\hat{z}) \lim_{k\to\infty} \hat{q}_k(\hat{z}) = \hat{\phi}(\hat{z})\hat{f}(\hat{z}).$$

Therefore $\hat{\phi}\hat{f} = \widehat{\phi f} \in \widehat{\mathcal{X}}$ for every $\phi \in \mathcal{M}(\mathcal{X})$, $f \in \mathcal{X}$. This implies $\hat{\phi} \in \mathcal{M}(\widehat{\mathcal{X}})$.

The converse is easy since $\hat{\phi}\hat{f} \in \widehat{\mathcal{X}}$ implies there exists $g \in \mathcal{X}$ such that $\hat{\phi}\hat{f} = \hat{g}$. This means $g = \hat{g}|_D = \phi f$ and so, $\phi f \in \mathcal{X}$. Thus $\phi \in \mathcal{M}(\mathcal{X})$ whenever $\hat{\phi} \in \mathcal{M}(\widehat{\mathcal{X}})$. □

Corollary 1.3.5. *$\widehat{\mathcal{X}}$ satisfies **Q3** over $\widehat{D}$.*

Proof. This follows from *Proposition 1.3.4* as shift operators are multiplication operators. □

Now that the shift operators are bounded, we can talk about cyclic functions in $\widehat{\mathcal{X}}$. However, the way we have defined the norm in $\widehat{\mathcal{X}}$, it is obvious that $f \in \mathcal{X}$ is cyclic if and only if $\hat{f} \in \widehat{\mathcal{X}}$ is cyclic. This and the propositions above prove the following identification theorem.

Theorem 1.3.6. *Given a space $\mathcal{X}$ that satisfies **P1-P3** over an open set $D \subset \mathbb{C}^n$, there exists a space $\widehat{\mathcal{X}}$, consisting of functions defined over the maximal domain $\widehat{D}$ of functions in $\mathcal{X}$, that satisfies **Q1-Q3** and is isometrically isomorphic to $\mathcal{X}$ with the map $\iota(f) := \hat{f}$, for $f \in \mathcal{X}$.*

Furthermore $\widehat{\mathcal{X}}|_D := \left\{ \hat{f}|_D \,\middle|\, \hat{f} \in \widehat{\mathcal{X}} \right\} = \mathcal{X}$, and $\widehat{\mathcal{X}}$ has the same set of multipliers and cyclic functions as $\mathcal{X}$. That is, $\phi \in \mathcal{M}(\mathcal{X})$ if and only if $\hat{\phi} \in \mathcal{M}(\widehat{\mathcal{X}})$, and f is cyclic in $\mathcal{X}$ if and only if $\hat{f}$ is cyclic in $\widehat{\mathcal{X}}$.

With the help of *Theorem 1.2.1* and *Theorem 1.3.6*, we can easily prove *Theorem 1.1.5*.

Proof of Theorem 1.1.5. The proof of this theorem is the same as that of *Theorem 1.2.1* except, by **Q2** we directly obtain $b \in D$ instead of having to show that $b \in \sigma_r(S)$. □

It should be noted that while *Theorem 1.1.5* is technically not an improvement to *Theorem 1.2.1*, it shows that the point b is not completely arbitrary; functions in $\mathcal{X}$ are well-behaved around b, and most of the structure we need can be extended to it.

1.4 Cyclicity preserving operators

We have now covered all the preliminaries required to identify all cyclicity preserving operators on these spaces. First, we prove *Theorem 1.1.6*.

Proof of Theorem 1.1.6. (2) $\Rightarrow$ (1) is follows from the fact that $a(u)e^{b(u)\cdot w} \neq 0$ for all non-vanishing $a \in \mathcal{Y}$ and $b : E \to D$, and for all $u \in E$.

Suppose now that (1) holds. Fix $u \in E$ and define $\Lambda := \Gamma_u \circ T \in \mathcal{X}^*$. Note that for every $w \in \mathbb{C}^n$, as $T(e^{w\cdot z})$ is non-vanishing, we get

$$\Lambda(e^{w\cdot z}) = \Gamma_u\big(T(e^{w\cdot z})\big) = T(e^{w\cdot z})(u) \neq 0.$$

By *Theorem 1.1.5*, we get that $\Lambda f = a(u)f(b(u))$ for some $a(u) \in \mathbb{C} \setminus \{0\}$, and $b(u) \in D$.

As the choice of $u \in E$ was arbitrary, we get the functions $a = T(1) \in \mathcal{Y}$ and $b = \frac{T(z)}{T(1)} : E \to D$ as desired. Also, $Tf(u) = a(u)f(b(u))$ for every $u \in E$. $\square$

The only thing we require to identify cyclicity preserving operators is the following lemma.

Lemma 1.4.1. *$e^{w\cdot z}$ is a cyclic multiplier in $\mathcal{X}$ for every $w \in \mathbb{C}^n$.*

Proof. Fix $w \in \mathbb{C}^n$. We need to find polynomials p_k so that $||p_k e^{w\cdot z} - 1|| \to 0$ as $k \to \infty$. Let p_k be truncations of the power-series of $e^{-w\cdot z}$. By *Lemma 1.2.2*, $p_k \to e^{-w\cdot z}$ in $\mathcal{X}$.

First, we show that $e^{w\cdot z}$ is a multiplier. Let q_k be truncations of the power-series of $e^{w\cdot z}$. Given $f \in \mathcal{X}$, we need to show $e^{w\cdot z}f$ lies in $\mathcal{X}$. Note that by the triangle inequality, we get

$$||q_l f - q_k f|| \leq \left(\sum_{k<|\alpha|\leq l} \frac{|w|^\alpha\, ||S||^\alpha\, ||1||}{\alpha!}\right) ||f||, \text{ for every } k \leq l.$$

Therefore $q_k f$ is a Cauchy sequence and thus, converges to some function $g \in \mathcal{X}$. As $q_k \to e^{w\cdot z}$ point-wise, by **Q2** we get that $q_k f \to e^{w\cdot z} f$, which implies $e^{w\cdot z} \in \mathcal{M}(\mathcal{X})$. This means,

$$\lim_{k\longrightarrow\infty} p_k e^{w\cdot z} = \lim_{k\longrightarrow\infty} M_{e^{w\cdot z}}(p_k) = M_{e^{w\cdot z}}(e^{-w\cdot z}) = 1.$$

That is, $p_k e^{w\cdot z} \to 1$ as $k \to \infty$ and thus, $e^{w\cdot z}$ is cyclic. $\square$

With this in mind, the following is a trivial consequence of *Theorem 1.1.6*.

Theorem 1.4.2 (Cyclicity Preserving Operators). *Let $m, n \in \mathbb{N}$. Suppose $\mathcal{X}$ and $\mathcal{Y}$ satisfy **Q1-Q3** over $D \subset \mathbb{C}^n$ and $E \subset \mathbb{C}^m$ respectively. Let $T : \mathcal{X} \to \mathcal{Y}$ be such that Tf is cyclic whenever f is cyclic. Then, there exist analytic functions $a \in \mathcal{Y}$ and $b : E \to D$ such that $Tf(u) = a(u)f(b(u))$ for every $u \in E$.*

Moreover, $a = T(1)$ is cyclic and $b = \frac{T(z)}{T(1)}$, where $T(z) = \big(T(z_i)\big)_{i=1}^{n}$.

Remark. One can immediately observe in *Theorems 1.1.6* and *1.4.2*, that the spaces $\mathcal{X}$ and $\mathcal{Y}$ may be defined for functions in different number of variables. Also note that for *Theorem 1.4.2*, we do not get a proper equivalence easily as in *Theorem 1.1.6* since it is not at all trivial to determine when a weighted composition operator preserves cyclicity.

In the case when $\mathcal{X} = H^p(\mathbb{D}^n)$ and $\mathcal{Y} = H^q(\mathbb{D}^m)$ for some $1 \leq p, q < \infty$, we get that all operators that preserve cyclicity are weighted composition operators. The same is true for the Dirichlet-type spaces $\mathcal{D}_\alpha$ when $\alpha \leq 1$. When $\alpha > 1$, we need to consider the space over its maximal domain $\overline{\mathbb{D}^n}$.

1.4.1 Cyclicity preserving operators on Hardy spaces

The aim of this subsection is to provide a proof of *Theorem 1.1.7.* We will start by showing that the converse of *Theorem 1.4.2* is true whenever $\mathcal{Y} = H^q(\mathbb{D}^m)$ for some $1 \leq q < \infty$. First, we prove some interesting properties of $\mathrm{S}[f]$ for functions in $H^p(\mathbb{D}^n)$.

Lemma 1.4.3. *If $f \in H^p(\mathbb{D}^n)$ for some $1 \leq p < \infty$, then $\phi f \in \mathrm{S}[f]$, $\forall \phi \in H^\infty(\mathbb{D}^n)$.*

Proof. For the sake of contradiction, let $\phi f \notin \mathrm{S}[f]$. By the Hahn-Banach theorem, there exists $\Gamma \in (H^p(\mathbb{D}^n))^*$ such that $\Gamma(\phi f) \neq 0$ and $\Gamma|_{\mathrm{S}[f]} \equiv 0$. Since $H^p(\mathbb{D}^n) \subset L^p(\mathbb{T}^n)$ is a closed subspace, by duality of $L^p(\mathbb{T}^n)$ there exists $h \in L^{p'}(\mathbb{T}^n)$ such that for every $g \in H^p(\mathbb{D}^n)$

$$\Gamma(g) = \int_{\mathbb{T}^n} g\overline{h}.$$

Here, p' is the exponent dual to p (see *Theorem 7.1* in [13] for more details). As ϕ is the weak*-limit of some sequence of analytic polynomials p_k in $L^\infty(\mathbb{T}^n)$ (take Fejér means, for example), and $f\overline{h} \in L^1(\mathbb{T}^n)$ for $f \in H^p(\mathbb{D}^n)$, we get that

$$\Gamma(\phi f) = \int_{\mathbb{T}^n} \phi f \overline{h} = \lim_{k \to \infty} \int_{\mathbb{T}^n} p_k f \overline{h} = 0.$$

The last equality follows from the fact that $p_k f \in \mathrm{S}[f]$ for each k, and

$$\int_{\mathbb{T}^n} g\overline{h} = \Gamma(g) = 0, \forall g \in \mathrm{S}[f].$$

Thus, we reach a contradiction since Γ was chosen so that $\Gamma(\phi f) \neq 0$. □

Proposition 1.4.4. *Let $f \in H^p(\mathbb{D}^n)$, $1 \leq p < \infty$. Let $\{f_k\}_{k\in\mathbb{N}} \subset H^\infty(\mathbb{D}^n)$ be such that $f_k f \to g$ for some $g \in \mathcal{X}$. Then, $g \in \mathrm{S}[f]$. In particular, if there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset H^\infty(\mathbb{D}^n)$ such that $f_k f \to g$ for some cyclic $g \in H^p(\mathbb{D}^n)$, then f is cyclic.*

Proof. The first part of the proposition follows easily from *Lemma 1.4.3*, since $f_k f \in \mathrm{S}[f]$ for each $k \in \mathbb{N}$, and $\mathrm{S}[f]$ is closed implies $g = \lim_{k\to\infty} f_k f \in \mathrm{S}[f]$.

For the second part, note that $g \in \mathrm{S}[f]$ implies $\mathrm{S}[g] \subset \mathrm{S}[f]$. Since g is assumed to be cyclic, $\mathrm{S}[g] = H^p(\mathbb{D}^n)$ which means $\mathrm{S}[f] = H^p(\mathbb{D}^n)$. Therefore in this case, f is also cyclic. □

Theorem 1.4.5. *Suppose $\mathcal{X}$ satisfies properties **Q1-Q3** over $D \subset \mathbb{C}^n$. Let $T : \mathcal{X} \to H^q(\mathbb{D}^m)$ be a bounded linear map for some $1 \leq q < \infty$. Then, the following are equivalent :*

(1) *T preserves cyclicity.*

(2) *$Tf = a \cdot (f \circ b)$, $f \in \mathcal{X}$ for some cyclic $a \in H^q(\mathbb{D}^m)$, and analytic $b : \mathbb{D}^m \to D$.*

Proof. (1) ⇒ (2) follows from *Theorem 1.4.2.*

For the converse, let $a \in H^q(\mathbb{D}^m)$ and $b : \mathbb{D}^m \to D$ be as in (2). We show that for every cyclic $f \in \mathcal{X}$, $Tf = a \cdot (f \circ b)$ is cyclic in $H^q(\mathbb{D}^m)$.

As f is cyclic, there exist polynomials p_k such that $p_k f \to 1$ in $\mathcal{X}$. Since T is a bounded operator, $T(p_k f) \to T(1)$ in $H^q(\mathbb{D}^m)$. Note that $T(1) = a$ is cyclic and that

$$T(p_k f) = a \cdot (p_k \circ b) \cdot (f \circ b) = (p_k \circ b) \cdot \big(a \cdot (f \circ b)\big).$$

It follows that $(p_k \circ b) \in H^\infty(\mathbb{D}^m)$ for each n, since the image of b lies in $D \subset \sigma_r(S)$ by *Theorem 1.3.2*. From the second part of *Proposition 1.4.4*, as

$$(p_k \circ b) \cdot \big(a \cdot (f \circ b)\big) \to a,$$

and a is cyclic, we get that $Tf = a \cdot (f \circ b)$ is cyclic in $H^q(\mathbb{D}^m)$. Thus, (2) $\Rightarrow$ (1). $\square$

Remark 1.4.6. (*i*) The proof of (2) $\Rightarrow$ (1) relies on *Proposition 1.4.4*, which further relies on the fact that the dual of $L^p(\mathbb{T}^n)$ for $1 \leq p < \infty$ is $L^{p'}(\mathbb{T}^n)$ where $1/p + 1/p' = 1$ and thus, does not translate easily to other general spaces of analytic functions.

(*ii*) Note that the proof of *Theorem 1.1.3* and *Theorem 2* in [23] uses the canonical factorization theorem for Hardy spaces on the unit disc $\mathbb{D}$ (*Theorem 2.8*, [13]). We do not have such a result when $n > 1$ (see *Section 4.2* in [30]), hence a different approach was needed.

(*iii*) Recall that *Theorem 1.1.3* does not require boundedness of T for the proof of (1) $\Rightarrow$ (2) to work when $\mathcal{X} = H^p(\mathbb{D})$. Plus, *Theorem 1.1.3* is valid even for $0 < p < 1$. This is because its proof also depends on the canonical factorization theorem as mentioned above.

(*iv*) We will see later in this section that (1) $\Rightarrow$ (2) is still valid for the case when $\mathcal{X} = H^p(\mathbb{D}^n)$ and $\mathcal{Y} = H^q(\mathbb{D}^m)$ for $0 < p, q < 1$ even though they are not Banach spaces. The case $p, q = \infty$ shall be treated separately as well since $H^\infty(\mathbb{D}^n)$ is not separable and hence the standard notion of cyclicity does not make any sense. As outer functions do make sense for $H^\infty(\mathbb{D}^n)$, we will consider outer preserving operators instead, and show that these have to be weighted composition operators as well.

We now show that the assumption 'T is a bounded operator' can be dropped in a specific case for the Hardy spaces. First, we need the following fact.

Proposition 1.4.7. *For $1 \leq p < \infty$ and a given analytic function $b : \mathbb{D}^m \to \mathbb{D}$, the map $T : H^p(\mathbb{D}) \to H^p(\mathbb{D}^m)$ defined as $Tf := f \circ b$ is a well-defined bounded linear operator.*

Proof. First, we show that $f \circ b \in H^p(\mathbb{D}^m)$ for every $f \in H^p(\mathbb{D})$, which shows T is well-defined. The linearity of T is immediate after that. We use the existence of harmonic majorants for functions in the Hardy spaces and their properties for the rest of the proof. See *Section 3.2* in [30] for more details. The argument presented here is inspired by the one given in the corollary of *Theorem 2.12* in [13] for the case $m = 1$. Let U be the smallest harmonic majorant of $|f|^p$, i.e. the Poisson integral of $|f(e^{i\theta})|^p$,

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t)\,|f(e^{it})|^p\,dt, \text{ where } P(r,\theta) := \operatorname{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right).$$

Then, $|f(u)|^p \leq U(u)$ for all $u \in \mathbb{D}$, which implies $|Tf(z)|^p \leq U\big(b(z)\big)$ for every $z \in \mathbb{D}^m$. Since U is harmonic, $U = \operatorname{Re}(g)$ for some analytic function $g : \mathbb{D} \to \mathbb{C}$. This means that $U \circ b = \operatorname{Re}(g \circ b)$ is an m-harmonic function and thus, a harmonic majorant for $|Tf|^p = |f \circ b|^p$. This proves that $f \circ b \in H^p(\mathbb{D}^m)$ and so, T is well-defined. To show T is bounded, observe that

$$M_p(r, f \circ b)^p \leq U(b(0)) \leq \left(\frac{1 + |b(0)|}{1 - |b(0)|}\right) ||f||^p,$$

where

$$M_p(r, f \circ b) := \left(\int\limits_{r\mathbb{T}^m} |f \circ b|^p d\sigma_m\right)^{\frac{1}{p}}.$$

The first inequality follows from the mean value property of m-harmonic functions. The second inequality follows from the fact that

$$P(r,\theta) \leq \frac{1+r}{1-r}, \,\forall\, r < 1,\, \theta \in [0, 2\pi].$$

Taking supremum over r in the above inequality, we get

$$||f \circ b|| \leq \left(\frac{1+|b(0)|}{1-|b(0)|}\right)^{\frac{1}{p}} ||f|| \text{ for every } f \in H^p(\mathbb{D}).$$

Thus, T is bounded. □

Theorem 1.4.8. *Fix $1 \leq p < \infty$ and let $T : H^p(\mathbb{D}) \to H^p(\mathbb{D}^m)$ be a linear map such that $T1 = 1$. Then, the following are equivalent :*

(1) *T is a bounded linear map that preserves cyclicity.*

(2) *$Tf = f \circ b$, $f \in H^p(\mathbb{D})$ for some analytic $b : \mathbb{D}^m \to \mathbb{D}$.*

Proof. As before, (1) ⇒ (2) follows directly from *Theorem 1.4.5*.

For the converse, let $b : \mathbb{D}^m \to \mathbb{D}$ be an analytic function such that $Tf = f \circ b$ for each $f \in H^p(\mathbb{D})$. By *Proposition 1.4.7*, T is a bounded linear operator. (2) ⇒ (1) in *Theorem 1.4.5* shows that T preserves cyclicity. □

Remark 1.4.9. Note that the only place we use that the domain of $H^p(\mathbb{D})$ is in one variable, is to show boundedness of $f \mapsto f \circ b$ for every $b : \mathbb{D}^m \to \mathbb{D}$. More precisely, we use the fact that any harmonic function U in one variable is the real part of some holomorphic function. This is not true for $n > 1$ (see *Section 2.4* in [30]).

As mentioned in *Remark (iv)* under *Theorem 1.4.5*, we now consider the cases $0 < p < 1$ and $p = \infty$. First, we address the case $\mathcal{X} = H^p(\mathbb{D}^n)$ for $0 < p < 1$.

Example 3 ($0 < p < 1$). Note that $H^p(\mathbb{D}^n)$ satisfies **P1-P3** if we replace boundedness with continuity. The issue is that $H^p(\mathbb{D}^n)$ is not a Banach space. Even though $H^p(\mathbb{D}^n)$ is not normable, it is still a complete metric space under the metric

$$d_p(f, g) := ||f - g||_p^p.$$

Here, $||\cdot||_p$ is as defined in *Section 1.1.* Using this, its bounded linear functionals can be defined in the usual manner. That is, we say that $\Lambda : H^p(\mathbb{D}^n) \to \mathbb{C}$ is bounded if

$$||\Lambda|| := \sup_{||f||_p=1} |\Lambda(f)| < \infty.$$

This means that $|\Lambda(f)| \leq ||\Lambda|| \cdot ||f||$ for all bounded Λ, and $f \in H^p(\mathbb{D}^n)$. It is easy to verify that this notion of boundedness is equivalent to the continuity of Λ. Similarly, we say an operator $T : H^p(\mathbb{D}^n) \to H^q(\mathbb{D}^m)$ for some $0 < q \leq \infty$ is bounded if

$$||T|| := \sup_{||f||_p=1} ||Tf||_q < \infty.$$

As was the case with linear functionals, it is easy to verify that this notion of boundedness is equivalent to the continuity of T. This implies that for all bounded linear operators T on $H^p(\mathbb{D}^n)$ and $f \in H^p(\mathbb{D}^n)$,

$$||Tf|| \leq ||T|| \cdot ||f||.$$

So, *Lemmas 1.2.2* and *1.4.1* hold for $H^p(\mathbb{D}^n)$ even when $0 < p < 1$. In order to show that *Theorem 1.4.2* holds for $\mathcal{X} = H^p(\mathbb{D}^n)$, we only need to show that *Theorem 1.1.5* holds since the arguments in the proof of *Theorem 1.4.2* do not rely on the Banach space structure of $\mathcal{X}$ except when *Theorem 1.1.5* is applied.

First, we show that the maximal domain for functions in $H^p(\mathbb{D}^n)$ when $0 < p < 1$ is also $\mathbb{D}^n$. We will show as in *Example 2* that the family

$$\mathcal{F} := \{z_i - \beta \,\big|\, 1 \leq i \leq n,\, \beta \notin \mathbb{D}\}$$

is an envelope of cyclic polynomials in $H^p(\mathbb{D}^n)$ for $0 < p < 1$. Let $b \in \overline{\mathbb{D}^n} \setminus \mathbb{D}^n$ be such that $\Lambda_b|_{\mathcal{P}}$ extends to a bounded linear functional $\Lambda \in H^p(\mathbb{D}^n)$. Thus $b_j \in \mathbb{T}$ for some $1 \leq j \leq n$.

It is known that outer functions are cyclic in $H^p(\mathbb{D})$ for $0 < p < 1$ (see *Theorem 4*, [14]). This implies $z - b_j$ is cyclic in $H^p(\mathbb{D})$ and thus, $q(z) := z_j - b_j$ is cyclic in $H^p(\mathbb{D}^n)$. Clearly $\mathcal{F}$ defined above is then an envelope of cyclic polynomials. This means that for any given $f \in H^p(\mathbb{D}^n)$, there exists a sequence of polynomials $\{p_k\}_{k\in\mathbb{N}}$ such that $p_k q \to f$. Since $q(b) = 0$, we get

$$\Lambda(f) = \lim_{k\longrightarrow\infty} \Lambda(p_k q) = \lim_{k\longrightarrow\infty} p_k(b)q(b) = 0.$$

This means $\Lambda \equiv 0$, a contradiction. So, $b \in \mathbb{D}^n$ and we get $\widehat{D} = \mathbb{D}^n$.

Notice that the only other place we use the norm in the proof of *Theorem 1.2.1* (and hence *Theorem 1.1.5*) is to obtain the non-vanishing entire function $F(w)$ using

$$|\Lambda(z^\alpha)| \leq ||\Lambda|| \cdot ||z^\alpha|| \leq ||\Lambda|| \cdot ||S_1||^{\alpha_1} \dots ||S_n||^{\alpha_n} \cdot ||1||, \text{ for every } \alpha \in \mathbb{Z}^+(n).$$

As we saw above, this should not be an issue for $H^p(\mathbb{D}^n)$ since $||\Lambda||$ makes just as much sense and $||z^\alpha|| = 1$ for all $\alpha \in \mathbb{Z}^+(n)$. This gives us $|\Lambda(z^\alpha)| \leq ||\Lambda||$, which is good enough for the rest of the proof to work. Therefore *Theorem 1.1.5* holds for $\mathcal{X} = H^p(\mathbb{D}^n)$ and $\mathcal{Y} = H^q(\mathbb{D}^m)$, and so does *Theorem 1.4.2* even when $0 < p, q < 1$.

Example 4 ($p = \infty$). $H^\infty(\mathbb{D}^n)$ is different from *Example 3* as it is a Banach space, but it does not satisfy **Q1** over $\mathbb{D}^n$. In fact $H^\infty(\mathbb{D}^n)$ is not separable, so cyclicity of functions does not make sense. Since outer functions (*see Definition 1.1.2*) do make sense for $n \geq 1$ and $0 < p \leq \infty$, we can talk about outer functions instead of cyclic functions in this case.

Note that for $p = \infty$, the hypothesis of *Theorem 1.1.5* does not make sense. In fact, we will completely avoid using maximal domains for $H^\infty(\mathbb{D}^n)$ since without cyclicity, we cannot even determine if $\widehat{D} \subset \sigma_r(S)$. Instead, consider $\Lambda \in \left(H^\infty(\mathbb{D}^n)\right)^*$ such that $\Lambda(f) \neq 0$ for all outer functions $f \in H^\infty(\mathbb{D}^n)$. Since $e^{w\cdot z}$ is an outer function for all $w \in \mathbb{C}^n$, we proceed as in the proof of *Theorem 1.2.1* to obtain $\Lambda|_{\mathcal{P}} \equiv a\Lambda_b|_{\mathcal{P}}$ for some $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}^n$.

Now, proceed as in *Example 2* and instead of having an envelope of cyclic polynomials, we have an envelope of outer polynomials which is the same set

$$\{z_i - \beta \mid 1 \leq i \leq n,\ \beta \notin \mathbb{D}.\}$$

Since $\Lambda(f) \neq 0$ for all outer functions f, we get $b_i - \beta \neq 0$ for all $1 \leq i \leq n$ and $\beta \notin \mathbb{D}$ which implies $b_i \in \mathbb{D}$ for every $1 \leq i \leq n$. Therefore, $b \in \mathbb{D}^n$ and $\Lambda f = a \cdot f(b)$ for all $f \in H^\infty(\mathbb{D}^n)$.

Thus, the conclusion of *Theorem 1.2.1* is valid for $H^\infty(\mathbb{D}^n)$ if we consider all $\Lambda \in \big(H^\infty(\mathbb{D}^n)\big)^*$ that act on outer functions as above and so, *Theorem 1.4.2* is valid for $\mathcal{X} = H^\infty(\mathbb{D}^n)$ if we replace cyclic functions with outer functions. A similar logic can be applied to operators that preserve outer functions in $H^p(\mathbb{D}^n)$ for $0 < p < \infty$.

This discussion about Hardy spaces above yields the proof of *Theorem 1.1.7*.

Proof of Theorem 1.1.7. (1) For $1 \leq p, q < \infty$, this follows from *Theorems 1.4.2* and *1.4.5*. For $0 < p < 1$ or $0 < q < 1$, this follows from the discussion in *Example 3*.

(2) This follows from the discussion in *Example 4*. □

Remark 1.4.10. (i) Note that the proof of *Proposition 1.4.4* above is not valid for $0 < q < 1$ or $q = \infty$ since we use the duality of $L^q(\mathbb{T}^m)$ when $1 \leq q < \infty$. Therefore, we do not obtain a result like *Theorem 1.4.5* when $0 < q < 1$ or $q = \infty$. *Theorem 1.1.7* is probably the best we can expect in these cases with our techniques.

(ii) If all bounded weighted composition operators preserve outer functions as well, we get a kind of '*linear rigidity*' between outer and cyclic functions. The following result shows that this is not the case when $n > 1$.

Theorem 1.4.11. *Let* $0 < q < 1/2$. *There exists a bounded linear map* $T : H^2(\mathbb{D}^2) \to H^q(\mathbb{D})$ *such that it preserves cyclicity, but not outer functions.*

Proof. This example is from [30] but it was used in a different context: to obtain an outer function in $H^2(\mathbb{D}^2)$ which is not cyclic. We refer the reader to the discussion surrounding *Theorem 4.4.8* in [30] for details on the facts mentioned below.

Fix $0 < q < 1/2$. Let $T : H^2(\mathbb{D}^2) \to H^q(\mathbb{D})$ be defined as

$$Tf(z) = f\left(\frac{1+z}{2}, \frac{1+z}{2}\right) \text{ for every } z \in \mathbb{D} \text{ and } f \in H^2(\mathbb{D}^2).$$

T is a bounded linear operator (*Theorem 4.4.8* (a), [30]), and therefore preserves cyclicity. Also, $f \in H^2(\mathbb{D}^2)$ defined below is outer (*Theorem 4.4.8* (b), [30]), but Tf is not.

$$f(z_1, z_2) = \exp\left(\frac{z_1 + z_2 + 2}{z_1 + z_2 - 2}\right).$$

$$Tf(z) = \exp\left(\frac{z+3}{z-1}\right) = \frac{1}{e} \cdot \left(\exp\left(\frac{z+1}{z-1}\right)\right)^2.$$

Therefore, T does not preserve outer functions. □

It would be interesting to characterize all operators that preserve outer functions as it might help us understand the difference between outer and cyclic functions when $n > 1$.

(*iii*) Notice that the proof of (1) ⇒ (2) in *Theorems 1.1.6* and *1.4.2* depends mostly on the properties of $\mathcal{X}$, since $\mathcal{Y}$ can be chosen to be fairly general. On the other hand, all the discussion about Hardy spaces shows that the proof of (2) ⇒ (1) depends on the properties of $\mathcal{Y}$. In *Proposition 1.4.4*, we saw that the proof relies heavily on the properties of $H^p(\mathbb{D}^n)$ and might not work for other spaces. This shows that it is not completely obvious what properties $\mathcal{Y}$ needs to have generally in order for the converse of *Theorem 1.4.2* to hold.

To show some different application of the results proved in *Sections 1.2* and *1.3*, we conclude our discussion by proving a GKŻ-type theorem (*Theorem 1.1.8*) for spaces of analytic functions.

1.5 GKŻ-type theorem for spaces of analytic functions

The following result was proved independently by A. Gleason (*Theorem 1*, [17]), and J.-P. Kahane and W. Żelazko (*Theorem 1*, [22]) for commutative Banach algebras. Żelazko extended the result to non-commutative Banach algebras shortly after in [34].

Theorem 1.5.1. *Let $\mathcal{B}$ be a complex unital Banach algebra, and let $\Lambda \in \mathcal{B}^*$ be such that $\Lambda(1) = 1$. Then $\Lambda(a) \neq 0$ for every a which is invertible in $\mathcal{B}$ if and only if*

$$\Lambda(ab) = \Lambda(a)\Lambda(b), \, \forall \, a, b \in \mathcal{B}$$

We shall prove a similar result about *partially multiplicative linear functionals* on spaces of analytic functions as an interesting byproduct of topics discussed in *Sections 1.2* and *1.3*.

Definition 1.5.2. Suppose $\mathcal{X}$ is a space of functions that satisfies **Q1-Q3** over $D \subset \mathbb{C}^n$. We will consider two types of partially multiplicative linear functionals $\Lambda \in \mathcal{X}^*$ as follows.

M1 $\Lambda(\phi f) = \Lambda(\phi)\Lambda(f), \, \forall \, \phi \in \mathcal{M}(\mathcal{X}), \, f \in \mathcal{X}$.

M2 $\Lambda(fg) = \Lambda(f)\Lambda(g), \, \forall \, f, g \in \mathcal{X}$ such that $fg \in \mathcal{X}$.

Note that **M2** $\Rightarrow$ **M1**, but it is not obvious if the converse is true in general.

Theorem 1.1.8 states that when $\mathcal{X}$ satisfies **Q1-Q3** over its maximal domain $D \subset \mathbb{C}^n$, both **M1** and **M2** are equivalent. Not only that, but they are precisely the set of point evaluations on D, and can be identified by their action on a certain set of exponentials.

Proof of Theorem 1.1.8. $(i) \Rightarrow (ii)$ follows from *Theorem 1.1.5*, and $(ii) \Rightarrow (iii) \Rightarrow (iv)$ follows from *Definition 1.5.2*. For $(iv) \Rightarrow (i)$, assume Λ is **M1** and note that $e^{w \cdot z} \in \mathcal{M}(\mathcal{X})$ for every $w \in \mathbb{C}^n$. Thus for every $w \in \mathbb{C}^n$, we get $\Lambda(e^{w \cdot z}) \neq 0$ as required, since

$$\Lambda(e^{w \cdot z})\Lambda(e^{-w \cdot z}) = \Lambda(e^{w \cdot z} \cdot e^{-w \cdot z}) = \Lambda(1) = 1. \qquad \square$$

This shows that all reasonable notions of partially multiplicative linear functionals align when we consider these nice spaces of analytic functions. A similar result for reproducing kernel Hilbert spaces with complete Pick property was recently proved (*Corollary 3.4*, [3]). It was shown that in the case of a complete Pick space, **M1** and **M2** are equivalent. It should be noted that this is not a special case of *Theorem 1.1.8* since it covers Hilbert spaces of functions that are not necessarily analytic. On the other hand, *Theorem 1.1.8* covers certain Banach spaces of analytic functions and not just Hilbert spaces.

It is worth mentioning that, just as we devised a maximal domain from point evaluations on polynomials that extend to $\mathcal{X}$, one can construct a different notion of maximal domain from **M1** and **M2**. We end this section by showing that our notion of maximal domain can also be identified with some form of partially multiplicative functionals.

Suppose $\mathcal{X}$ satisfies **P1**-**P3** over an open set $D \subset \mathbb{C}^n$. We say $\Lambda \in \mathcal{X}^*$ is **M0** if it satisfies the following property.

M0 $\Lambda(pq) = \Lambda(p)\Lambda(q)$, for every $p, q \in \mathcal{P}$.

Proposition 1.5.3. *Λ is **M0** if and only if $\Lambda|_{\mathcal{P}} \equiv \Lambda_b|_{\mathcal{P}}$ for some $b \in \widehat{D}$.*

Proof. If Λ is **M0**, then $\Lambda(z_i^k) = \big(\Lambda(z_i)\big)^k$ for all $1 \leq i \leq n$ and $k \in \mathbb{N}$. Pick $b = \big(\Lambda(z_i)\big)_{i=1}^n$ and note that $\Lambda(p) = p(b)$ for all $p \in \mathcal{P}$. As $\Lambda \in \mathcal{X}^*$, and $\mathcal{X}$ satisfies **P1**, this means $\Lambda|_{\mathcal{P}}$ extends to $\mathcal{X}$. Thus $b \in \widehat{D}$, and $\Lambda|_{\mathcal{P}} \equiv \Lambda_b|_{\mathcal{P}}$ as required. The converse is trivial. □

Depending on what properties we want the extension $\widehat{\mathcal{X}}$ to have, we may want to choose from **M0**-**M2** accordingly. For more details, refer to *Section 2* in [26], and *Section 5* in [18], and [25].

Chapter 2

Algebraic Properties of Cyclic Functions

In this chapter, we study certain properties of the cyclic functions that are intrinsic to the space in consideration. We start by exploring the situation in Hardy spaces, and move to more general function spaces later. We shall generalize the notion of cyclicity to non-analytic function spaces, and show that these cyclic functions satisfy some nice algebraic properties.

2.1 Hardy spaces

Consider a cyclic function $f \in H^p(\mathbb{D})$ for some $1 \leq p < \infty$. By definition, there exists a sequence of polynomials $\{p_k\}_{k\in\mathbb{N}}$ such that $||1 - p_k f|| \to 0$ as $k \to \infty$. Therefore, one can think of cyclic functions as *'almost invertible'* elements in the space. Recall that cyclic functions in $H^p(\mathbb{D})$ are outer and vice versa. This enables us to show that cyclic functions have certain algebraic properties. We mention the properties that will be of primary interest.

C1 If f, g, $fg \in \mathcal{X}$, then fg is cyclic if and only if both f, g are cyclic.

C2 If f, $1/f \in \mathcal{X}$, then f is cyclic.

In $H^p(\mathbb{D})$, both **C1** and **C2** are known to hold. One can check that **C1** follows from the definition of outerness. For **C2**, note that

$$\log|f(0)| \leq \int_{\mathbb{T}} \log|f^*|$$

is true for all $f \in H^p(\mathbb{D})$ for all values of p (see *Theorem 3.3.5*, [30]). Thus if f, $1/f \in H^p(\mathbb{D})$, the above inequality turns into an equality and both f and $1/f$ become outer.

The same logic can be applied to $H^p(\mathbb{D}^n)$ in order to show that **C1** and **C2** hold for $H^p(\mathbb{D}^n)$ if we replace cyclicity with outerness. However, it is not known if **C1** or **C2** hold for $H^p(\mathbb{D}^n)$ when $n > 1$. This is particularly interesting for the following reason. Suppose $f \in H^p(\mathbb{D}^n)$. A quick application of Jensen's inequality shows that $f \in H^r(\mathbb{D}^n)$ for all $r < p$. Now, the outerness property does not rely on the norm of the space. Therefore, f is outer in $H^p(\mathbb{D}^n)$ if and only if f is outer in $H^r(\mathbb{D}^n)$. The same cannot be said about cyclicity. If f is cyclic in $H^p(\mathbb{D}^n)$, then there exist polynomials $\{p_k\}_{k\in\mathbb{N}}$ such that as $k \to \infty$,

$$||1 - p_k f||_p \to 0.$$

Let $0 < r < p$ be arbitrary. By a version of Jensen's inequality we get that as $k \to \infty$,

$$||1 - p_k f||_r \leq ||1 - p_k f||_p \to 0.$$

Thus, f is cyclic in $H^r(\mathbb{D}^n)$. The converse is not at all obvious, and we do not have a counter example nor a proof for it. Although, we can connect this question to **C1** and **C2**.

Proposition 2.1.1. *Let $n > 1$ be a natural number, and let $1 \leq p < \infty$. For $H^p(\mathbb{D}^n)$ consider the following statements.*

(1) If $f \in H^p(\mathbb{D}^n)$, then f is cyclic in $H^p(\mathbb{D}^n)$ if and only if f is cyclic in $H^{\frac{p}{2}}(\mathbb{D}^n)$.

(2) If $f, g, fg \in H^p(\mathbb{D}^n)$, then fg is cyclic in $H^p(\mathbb{D}^n)$ if and only if both f, g are cyclic.

(3) If $f, 1/f \in H^p(\mathbb{D}^n)$, then f and $1/f$ are cyclic in $H^p(\mathbb{D}^n)$.

In this case, we have that $(1) \implies (2) \implies (3)$.

Note: We do not know if (1) holds for $H^p(\mathbb{D}^n)$, but if it did, it would imply the other two properties as well.

Proof. (1) $\Rightarrow$ (2)

Suppose (1) holds, and let f, g, $fg \in H^p(\mathbb{D}^n)$. Suppose first that both f and g are cyclic in $H^p(\mathbb{D}^n)$. Therefore, there exist polynomials $\{p_k\}_{k\in\mathbb{N}}$ such that $p_k f \xrightarrow{H^p(\mathbb{D}^n)} 1$ as $k \to \infty$. Using the generalized Hölder's inequality we get that as $k \to \infty$,

$$||g - p_k fg||_{\frac{p}{2}} \leq ||g||_p \cdot ||1 - p_k f||_p \to 0$$

Thus $g \in \mathrm{S}^{(\frac{p}{2})}[fg]$, where $\mathrm{S}^{(\frac{p}{2})}[fg]$ denotes the shift-invariant subspace of fg in $H^{\frac{p}{2}}(\mathbb{D}^n)$. As g is cyclic in $H^p(\mathbb{D}^n)$, (1) implies that g is cyclic in $H^{\frac{p}{2}}(\mathbb{D}^n)$ as well. As $g \in \mathrm{S}^{\frac{p}{2}}[fg]$, this means fg is cyclic in $H^{\frac{p}{2}}(\mathbb{D}^n)$. Using (1) again, we get that fg is cyclic in $H^p(\mathbb{D}^n)$ as required.

Conversely, suppose fg is cyclic in $H^p(\mathbb{D}^n)$. As polynomials are dense in every Hardy space, there exist polynomials $\{p_k\}_{k\in\mathbb{N}}$ such that $p_k \xrightarrow{H^p(\mathbb{D}^n)} g$ as $k \to \infty$. Using the generalized Hölder's inequality as before, we get that as $k \to \infty$,

$$||p_k f - fg||_{\frac{p}{2}} \leq ||f||_p \cdot ||p_k - g||_p \to 0$$

Thus, $fg \in \mathrm{S}^{(\frac{p}{2})}[f]$. since fg is cyclic in $H^p(\mathbb{D}^n)$, it is cyclic in $H^{\frac{p}{2}}(\mathbb{D}^n)$ by (1). Thus, f is cyclic in $H^{\frac{p}{2}}(\mathbb{D}^n)$. Using (1) once again, we get that f is cyclic in $H^p(\mathbb{D}^n)$.

The argument to show g is cyclic in $H^p(\mathbb{D}^n)$ is symmetric. Thus, (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3)

Suppose (2) holds, and let f, $1/f \in H^p(\mathbb{D}^n)$. Clearly,

$$f \cdot (1/f) = 1 \in H^p(\mathbb{D}^n)$$

and also 1 is cyclic. Thus by (2), both f and $1/f$ are cyclic in $H^p(\mathbb{D}^n)$. $\square$

Proposition 2.1.1 shows that counter-examples to **C1** or **C2** can also serve as counter-examples for the norm independence of cyclicity in the Hardy spaces when $n > 1$. It is believed that **C1** and **C2** do not hold for $H^p(\mathbb{D}^n)$, but the lack of examples of non-polynomial cyclic functions in the Hardy spaces make it difficult to answer. We can show that something very close to **C2** holds for the Hardy spaces.

Proposition 2.1.2. *If $f \in H^p(\mathbb{D}^n)$ and $1/f \in H^q(\mathbb{D}^n)$ for some p, $q > 0$, then f is cyclic in $H^r(\mathbb{D}^n)$ for all $r < p$.*

Proof. This proof is similar to, and is inspired by the proof of (3) in *Theorem 3.3*, [27].

Fix p, q, r as given in the statement of the proposition. Let $\alpha = \frac{pr}{p-r}$ and fix $N \in \mathbb{N}$ such that $N > \alpha/q$. Now,

$$\begin{aligned} \frac{\alpha}{N} < q &\implies 1/f \in H^{\frac{\alpha}{N}}(\mathbb{D}^n) \\ &\implies 1/f^{1/N} \in H^{\alpha}(\mathbb{D}^n) \end{aligned}$$

By the density of polynomials, we can choose polynomials $\{p_k\}_{k\in\mathbb{N}}$ such that as $k \to \infty$,

$$\left|\left|\frac{1}{f^{1/N}} - p_k\right|\right|_{\alpha} \to 0$$

Note that $\frac{1}{p} + \frac{1}{\alpha} = \frac{1}{r}$, therefore by a generalized Hölder's inequality we get that as $k \to \infty$,

$$||f^{1-\frac{1}{N}} - p_k f||_r \leq ||f||_p \left|\left|\frac{1}{f^{1/N}} - p_k\right|\right|_{\alpha} \to 0$$

Thus, $f^{1-\frac{1}{N}} \in \mathrm{S}^{(r)}[f]$. Next, we can show that $f^{1-\frac{2}{N}} \in \mathrm{S}^{(r)}[f]$ using the generalized Hölder's inequality on $||f^{1-\frac{2}{N}} - p_k f^{1-\frac{1}{N}}||_r$. After N such steps, we get that $1 \in \mathrm{S}^{(r)}[f]$ and thus, f is cyclic in $H^r(\mathbb{D}^n)$. $\square$

2.2 Motivation for more general function spaces

C1 and **C2** hold in the Hardy spaces when $n = 1$ because of the various ways in which we can represent functions. For example, every $f \in H^p(\mathbb{D})$ can be represented as $f = \phi/\psi$ such that $\phi, \psi \in H^\infty(\mathbb{D})$ (*Theorem 2.1*, [13]). Moreover, ψ can be chosen to be cyclic in $H^p(\mathbb{D})$. This is known as a *Smirnov representation* for functions in $H^p(\mathbb{D})$.

A Smirnov representation does not exist in many other analytic function spaces, even in the one variable case. $H^p(\mathbb{D}^n)$ when $n > 1$ does not have a Smirnov representation (see *Theorem 4.1.1*, [30]). In the one variable setting, consider the Bergman space

$$L_a^2(\mathbb{D}) := \left\{ f \in \mathrm{Hol}(\mathbb{D}) \,\middle|\, \int_{\mathbb{D}} |f(z)|^2 dS(z) < \infty \right\}$$

where dS represents the area measure of the unit disk. It is known that functions in $L_a^2(\mathbb{D})$ do not have a Smirnov representation. A famous result of A. Borichev and H. Hedenmalm shows that **C2** is not satisfied in $L_a^2(\mathbb{D})$ (*Theorem 1.4*, [7]). It is easy to check that **C1** implies **C2**, therefore **C1** is also not satisfied in $L_a^2(\mathbb{D})$. In their paper, Borichev and Hedenmalm suggest that the lack of a Smirnov representation could be a contributing factor to this phenomenon.

A result of S. Richter and J. Sunkes shows that **C1** and **C2** hold for the Drury-Arveson space $\mathcal{H}_n^2$ on the unit ball $\mathbb{B}_n$ (*Theorem 4.6*, [29]). As $\mathcal{H}_n^2$ somewhat serves as a model for *complete Nevanlinna-Pick (CNP) spaces*, it is natural to ask if **C1** and **C2** extend to these spaces. It should be noted that the notion of shift-cyclicity does not make sense for an arbitrary CNP space, but we do have a generalized notion of cyclicity. The goal of the rest of this chapter is to establish that **C1** and **C2** hold for this generalized notion of cyclicity in CNP spaces. Let us start by introducing complete Nevanlinna-Pick spaces.

2.3 Complete Nevanlinna-Pick Spaces

First introduced in the context of the Pick interpolation problem, CNP spaces have now become an important part of function theory. See [1] for a detailed discussion on these spaces. Before we can properly understand CNP spaces, we must first introduce *reproducing kernel Hilbert spaces.*

2.3.1 Reproducing Kernel Hilbert Spaces

Let X be a set, and let $\mathcal{H}$ be a Hilbert space consisting of complex-valued functions defined on X. $\mathcal{H}$ is said to be a *reproducing kernel Hilbert space* (RKHS) if for each $x \in X$, the map $\Lambda_x : \mathcal{H} \to \mathbb{C}$ defined as

$$\Lambda_x f = f(x), \, \forall f \in \mathcal{H}$$

is a bounded functional on $\mathcal{H}$. Given $x \in X$, by Riesz representation theorem, there exists a unique $k_x \in \mathcal{H}$ such that $\Lambda_x f = \langle f, k_x \rangle$ for each $f \in \mathcal{H}$. The map $K : X \times X \to \mathbb{C}$ given by

$$K(x, y) = \langle k_y, k_x \rangle, \, \forall x, y \in X$$

is called the *reproducing kernel* of $\mathcal{H}$. The function k_x is typically referred to as the *kernel function at x*. It is easy to check that K is positive semi-definite. That is, for every choice of N points $\{x_1, x_2, \ldots, x_N\} \subset X$, the matrix $\big(K(x_i, x_j)\big)_{N \times N}$ is positive semi-definite. Just like spaces that we considered in *Chapter 1*, every RKHS has its *multiplier algebra*

$$\mathcal{M} := \{\phi : X \to \mathbb{C} \,|\, \phi f \in \mathcal{H}, \, \forall f \in \mathcal{H}\} \, .$$

The elements of $\mathcal{M}$ are called *multipliers.* Using the closed graph theorem, it can be shown that if ϕ is a multiplier, then the map $M_\phi : \mathcal{H} \to \mathcal{H}$ given by $M_\phi f = \phi f, \forall f \in \mathcal{H}$ is a bounded operator. The norm $\|\phi\|_{\mathcal{M}} := \|M_\phi\|_{\mathcal{B}(\mathcal{H})}$ then turns $\mathcal{M}$ into a Banach algebra. For the Hardy-Hilbert space $H^2(\mathbb{D})$, the multiplier algebra turns out to be $H^\infty(\mathbb{D})$, but for a general RKHS $\mathcal{H}$, the multiplier algebra $\mathcal{M}$ can be very difficult to determine. However, there are some important properties that every multiplier has. For instance, every kernel function is an eigenvector for the adjoint of each multiplier.

$$M_\phi^* k_x = \overline{\phi(x)} k_x, \ \forall x \in X, \ \phi \in \mathcal{M}. \tag{2.3.1}$$

Using (2.3.1), it can be easily shown that for all $\phi \in \mathcal{M}$,

$$\sup_{x \in X} |\phi(x)| \ \leq \ \|\phi\|_{\mathcal{M}} .$$

Thus, every multiplier is a bounded function. The converse need not be true in general (see *Theorem A* in [33]). Another important property of multipliers hinges on the following fact. Given $T \in \mathcal{B}(\mathcal{H})$,

$$\|T\| \leq c \iff c^2 I - T^* T \geq 0. \tag{2.3.2}$$

Using $T = M_\phi$ in (2.3.2) shows that $\phi \in \mathcal{M}(\mathcal{X})$ with $\|\phi\|_{\mathcal{M}} \leq c$ if and only if

$$\left(c^2 - \phi(z)\overline{\phi(w)}\right) K(z,w) \geq 0. \tag{2.3.3}$$

With (2.3.3) in mind, we can easily define operator-valued multipliers as well.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be Hilbert spaces. A function $\Phi : X \to \mathcal{B}(\mathcal{L}_1, \mathcal{L}_2)$ is said to be a multiplier, if there exists $c > 0$ such that, for all $\{x_1, x_2, \ldots, x_N\} \subset X$

$$\left(c^2 I - \Phi(x_i)\Phi(x_j)^*\right) K(x_i, x_j) \geq 0. \tag{2.3.4}$$

The space of all such multipliers is denoted by $Mult(\mathcal{H} \otimes \mathcal{L}_1, \mathcal{H} \otimes \mathcal{L}_2)$, and is a Banach algebra under the norm $\|\Phi\| := \inf\{c > 0 \,|\, (2.3.4) \text{ holds for } \Phi\}$.

2.3.2 The complete Pick property

Consider the following problem.

Q. Given $\{x_1, \ldots, x_N\} \subset X$, and $\{W_1, \ldots, W_N\} \subset \mathbb{C}^{s \times t}$, find $\Phi \in Mult(\mathcal{H} \otimes \mathbb{C}^t, \mathcal{H} \otimes \mathbb{C}^s)$ such that

$$\Phi(x_i) = W_i, \, \forall \, 1 \leq i \leq N.$$

By (2.3.4), it is obvious that a necessary condition for such a Φ to exist is

$$\left(c^2 I - W_i W_j^*\right) K(x_i, x_j) \geq 0 \tag{2.3.5}$$

for some $c > 0$. If for all s, t, $\{x_1, \ldots, x_N\} \subset X$, and $\{W_1, \ldots, W_N\} \subset \mathbb{C}^{s \times t}$, (2.3.5) becomes a sufficient condition for such a Φ to exist, then K is said to have the *complete Pick property*, and is called a *complete Pick kernel.*

There are several equivalent ways of describing the complete Pick property, but adding small assumptions to K can simplify it.

Definition 2.3.1. K is said to be *irreducible* if it satisfies the following properties.

(i) For $x, y \in X$ distinct, k_x and k_y are linearly independent.

(ii) For all $x, y \in X$, $K(x, y) \neq 0$.

If K is irreducible, then it can be *normalized* at a point $x_0 \in X$ such that $k_{x_0} \equiv 1$ without changing the topology of $\mathcal{H}$, or its multipliers. See *Section 2.6* in [1] for more details. With this in mind, we have the following characterization of irreducible complete Pick kernels.

Proposition 2.3.2. *Let K be an irreducible positive semi-definite kernel, normalized at some point $x_0 \in X$. Then, K is a complete Pick kernel if and only if there exists an auxillary Hilbert space $\mathcal{E}$, and a map $b : X \to \mathcal{E}$ such that $\|b(x)\|_{\mathcal{E}} < 1$ for all $x \in X$, $b(x_0) = 0$, and*

$$K(x, y) = \frac{1}{1 - \langle b(x), b(y)\rangle_{\mathcal{E}}}, \ \forall x, y \in X. \tag{2.3.6}$$

See the discussion surrounding *Theorem 7.31* in [1] for a proof. Note that using (2.3.4) and (2.3.6), we can show that $b_y(\cdot) := \langle b(\cdot), b(y)\rangle$ is a multiplier with $\|b_y\|_{\mathcal{M}} < 1$.

Two of the most important examples of complete Pick spaces in function theory are the Dirichlet space $\mathcal{D}$ on the unit disk, and the Drury-Arveson space $\mathcal{H}^2_n$ on the unit-ball $\mathbb{B}_n$. It should be noted that here, $n = \infty$ is allowed. Their reproducing kernels are

$$K_{\mathcal{D}}(z, w) = \frac{1}{z\overline{w}} \log\left(\frac{1}{1 - z\overline{w}}\right), \qquad \forall z, w \in \mathbb{D}. \tag{2.3.7}$$

$$K_{\mathcal{H}^2_n}(z, w) = \frac{1}{1 - \langle z, w\rangle}, \qquad \forall z, w \in \mathbb{B}_n. \tag{2.3.8}$$

It is a little tricky to show that $\mathcal{D}$ is a complete Pick space. See *Corollary 7.41* in [1] for a proof using *Kaluza's lemma*. One can check using *Proposition 2.3.2*, that $\mathcal{H}^2_n$ is a complete Pick space for all $n \in \mathbb{N} \cup \{\infty\}$. See [32] for a detailed discussion on function theory in $\mathcal{H}^2_n$. When $n = 1$, $\mathcal{H}^2_n$ is the Hardy-Hilbert space $H^2(\mathbb{D})$. The case $n = \infty$ is perhaps the most important because, in a way, every complete Pick space can be realized as a subspace of $\mathcal{H}^2_\infty$ (see *Chapter 8*, [1]). In fact, this realization is helpful in obtaining the following fundamental fact about complete Pick spaces which we shall require.

Proposition 2.3.3. *Let $\mathcal{H}$ be a RKHS with a normalized complete Pick kernel, and let $\mathcal{M}$ be its multiplier algebra. Then $\overline{\mathcal{M}} = \mathcal{H}$.*

See *Proposition 2.1* in [10] for a proof.

2.4 Cyclicity and Smirnov representation

Let $\mathcal{H}$ be a RKHS, and let $\mathcal{M}$ be its multiplier algebra.

Definition 2.4.1. $f \in \mathcal{H}$ is said to be cyclic if $f\mathcal{M} := \{\phi f \mid \phi \in \mathcal{M}\}$ is dense in $\mathcal{H}$.

As we saw in *Chapter 1*, cyclicity in spaces of analytic functions is understood in the context of shift operators. As every polynomial is a bounded analytic map on the unit disk, every polynomial is a multiplier of $H^2(\mathbb{D})$. Therefore for spaces that do not necessarily contain polynomials, it makes sense to consider cyclicity with respect to multipliers.

These two notions of cyclicity coincide in the case of $H^2(\mathbb{D})$ (apply *Theorem 7.4, [13]* to $g = \phi f$ where $\phi \in H^2(\mathbb{D})$). This is also true for both $\mathcal{D}$ and $\mathcal{H}^2_n$. For the $\mathcal{D}$, the problem of characterizing shift-cyclic functions is now a celebrated conjecture of Brown and Shields (see *Question 12*, [8]).

The *Smirnov class* $N^+(\mathbb{D})$ is defined as

$$N^+(\mathbb{D}) := \left\{ \phi/\psi \,\middle|\, \phi, \psi \in H^\infty(\mathbb{D}),\ \psi \text{ is cyclic in } H^2(\mathbb{D}) \right\}.$$

It can be shown that $H^2(\mathbb{D}) \subset N^+(\mathbb{D})$ (*Section 2.5*, [13]). This observation is helpful in determining several properties of functions in the Hardy space. Recently, a similar property for complete Pick spaces was discovered by A. Aleman, M. Hartz, J. McCarthy, and S. Richter (*Theorem 1.1*, [3]). M. Jury and R. Martin provided a slightly stronger result with a different proof in [21], which we present here for the reader's convenience.

Theorem 2.4.2 (Jury, Martin 2018)**.** *Let $\mathcal{H}$ be a normalized complete Pick space, and let $\mathcal{M}$ be its multiplier algebra. For every $f \in \mathcal{H}$, there exists $\phi, \psi \in \mathcal{M}$ such that ψ is cyclic, $1/\psi \in \mathcal{H}$, and $f = \phi/\psi$.*

Note: The notion of cyclicity used in [21] is different from the one that we use, but for normalized complete Pick spaces, these two notions coincide using *Proposition 2.3.3*.

Our goal is to utilize this representation to prove certain algebraic properties of cyclic functions in complete Pick spaces.

2.5 Algebraic properties of cyclic functions

We start with a simple observation.

Proposition 2.5.1. *Let $\mathcal{H}$ be a RKHS with a dense multiplier algebra $\mathcal{M}$. That is, $\overline{\mathcal{M}} = \mathcal{H}$ Let $\phi \in \mathcal{M}$ and $f \in \mathcal{H}$. Then ϕf is cyclic if and only if both ϕ and f are cyclic.*

Proof. "$\Leftarrow$" Suppose ϕ, f are cyclic. Fix $g \in \mathcal{H}$, and an $\epsilon > 0$. To show that ϕf is cyclic, we need to find $\psi \in \mathcal{M}$ such that $\|\psi\phi f - g\| < \epsilon$. As ϕ is cyclic, there exists $\psi_0 \in \mathcal{M}$ such that

$$\|\psi_0\phi - g\| < \frac{\epsilon}{2}. \tag{2.5.1}$$

As f is also cyclic, there exists $\psi \in \mathcal{M}$ such that

$$\|\psi f - \psi_0\| < \frac{\epsilon}{2\,\|M_\phi\|}, \tag{2.5.2}$$

$$\Rightarrow \|\psi\phi f - \psi_0\phi\| < \frac{\epsilon}{2}. \tag{2.5.3}$$

Combining (2.5.1) and (2.5.3), we get

$$\|\psi\phi f - g\| < \epsilon$$

as desired. Since $\epsilon > 0$ and $g \in \mathcal{H}$ were arbitrarily chosen, ϕf is cyclic.

"⇒" Suppose ϕf is cyclic. Note that

$$\begin{aligned} f\mathcal{M} &\supset f(\phi\mathcal{M}), \\ \Rightarrow \overline{f\mathcal{M}} &\supset \overline{f(\phi\mathcal{M})} = \overline{\phi f\mathcal{M}} = \mathcal{H}. \end{aligned}$$

Therefore, f is cyclic. To show ϕ is cyclic, note that the density of $\mathcal{M}$ implies

$$\overline{\phi\mathcal{M}} \supset \phi\mathcal{H} \supset \phi f\mathcal{M}.$$

As ϕf is cyclic, $\phi f\mathcal{M}$ is dense in $\mathcal{H}$. Therefore $\overline{\phi\mathcal{M}} = \mathcal{H}$, and ϕ is cyclic as well. □

If we assume that $\mathcal{H}$ is a normalized complete Pick space, we can combine *Theorem 2.4.2* and *Proposition 2.5.1* to obtain the following corollary.

Corollary 2.5.2. *Let $\mathcal{H}$ be a normalized complete Pick space, and let $f = \phi/\psi$ be as in* Theorem 2.4.2. *Then,*

$$f \text{ is cyclic} \iff \phi \text{ is cyclic.} \tag{2.5.4}$$

Proof. Note that $f = \phi(1/\psi)$. It suffices to show that $1/\psi$ is cyclic, which is true because

$$\frac{1}{\psi}\mathcal{M} \supset \frac{1}{\psi}(\psi\mathcal{M}) \supset \mathcal{M}.$$ □

Note: The proof of the fact that $1/\psi$ is cyclic does not require the assumption that ψ is cyclic. This works the other way around as well. If $\psi \in \mathcal{M}$ is such that $1/\psi \in \mathcal{H}$, then using *Proposition 2.3.3* repeatedly, we get

$$\begin{aligned} \overline{\psi\mathcal{M}} &\supset \psi\mathcal{H} \supset \psi\left(\frac{1}{\psi}\mathcal{M}\right) \supset \mathcal{M}, \\ \Longrightarrow \overline{\psi\mathcal{M}} &= \mathcal{H}. \end{aligned}$$

Therefore, ψ is cyclic. With this in mind, *Corollary 2.5.2* also reduces the problem of characterizing cyclic functions in a complete Pick space to the following two problems about multipliers.

1. Determine all $\phi \in \mathcal{M}$ that are cyclic.

2. Determine all $\psi \in \mathcal{M}$ such that $1/\psi$ lies in $\mathcal{H}$.

For H^2_n, several sufficient conditions for **Q2** can be found in [29].

Corollary 2.5.3. *Let $\mathcal{H}$ be a normalized complete Pick space with kernel $K : X \times X \to \mathbb{C}$. Then, k_y is a cyclic multiplier for all $y \in X$.*

Proof. Let $b : X \to \mathcal{E}$ be the map as in *Proposition 2.3.2* for some Hilbert space $\mathcal{E}$. Then,

$$k_y = \frac{1}{1 - b_y}, \ \forall y \in X.$$

The proof of *Proposition 2.3.3* (see *Proposition 2.1*, [10]) shows that k_y is a multiplier for every $y \in X$. As b_y is a multiplier for every y (see the discussion under *Proposition 2.3.3*), so is $1 - b_y$. Therefore, the note under *Corollary 2.5.2* shows that k_y is cyclic. □

Note: It may not necessarily be the case for a general RKHS to have cyclic kernel functions. See for instance *Theorem 13* in [20], where the author constructs a weighted Hardy space on the bidisk $\mathbb{D}^2$ with a non-cyclic reproducing kernel.

Using the above results, we can now show that **C1** and **C2** hold for complete Nevanlinna-Pick spaces.

Theorem 2.5.4. *Let $\mathcal{H}$ be a normalized complete Pick space, and let $f_1, f_2 \in \mathcal{H}$ be such that $f_1 f_2 \in \mathcal{H}$. Then,*

$$f_1 f_2 \text{ is cyclic} \iff f_1,\ f_2 \text{ are cyclic.}$$

Proof. "$\Leftarrow$" Suppose f_1, f_2 are cyclic. Let $f_1 = \phi_1/\psi_1$, and $f_2 = \phi_2/\psi_2$ for some $\phi_i, \psi_i \in \mathcal{M}$ $(i = 1, 2)$ as in *Theorem 2.4.2*. Therefore, by *Corollary 2.5.2*, ϕ_1 and ϕ_2 are cyclic. Now,

$$f_1 f_2 = \frac{\phi_1 \phi_2}{\psi_1 \psi_2}.$$

However, this is not a Smirnov representation, because we do not know if $\frac{1}{\psi_1 \psi_2} \in \mathcal{H}$. Even then, note that

$$f_1 f_2 \mathcal{M} = \frac{\phi_1 \phi_2}{\psi_1 \psi_2} \mathcal{M} \supset \frac{\phi_1 \phi_2}{\psi_1 \psi_2} (\psi_1 \psi_2 \mathcal{M}) \supset \phi_1 \phi_2 \mathcal{M}.$$

As ϕ_1 and ϕ_2 are both cyclic, $\phi_1 \phi_2$ is also cyclic (by *Proposition 2.5.1*). Thus, $f_1 f_2$ is cyclic.

"$\Rightarrow$" Suppose now $f_1 f_2$ is cyclic. Let $f_1 = \phi_1/\psi_1$ and $f_2 = \phi_2/\psi_2$ be as in *Theorem 2.4.2*. By *Corollary 2.5.2*, it suffices to show that ϕ_1 and ϕ_2 are cyclic. Now,

$$\phi_1 \phi_2 \left(\frac{1}{\psi_1} \right) = \psi_2 \left(f_1 f_2 \right).$$

As ψ_2 and $f_1 f_2$ are both cyclic, $\phi_1 \phi_2 \left(1/\psi_1 \right)$ is cyclic by *Proposition 2.5.1*. Using the reverse implication in *Proposition 2.5.1* twice, we get that ϕ_1 and ϕ_2 are both cyclic, and hence f_1 and f_2 are also cyclic. □

Note: *Theorem 2.5.4* generalizes *Theorem 4.6* in [29] to all CNP spaces for which multiplier and shift cyclicity coincide. In particular, this serves as another proof of *Theorem 4.6* in [29]. As a natural corollary of *Theorem 2.5.4*, we show that all CNP spaces satisfy **C2**.

Corollary 2.5.5. *Let $\mathcal{H}$ be a normalized complete Pick space. If f, $1/f \in \mathcal{H}$, then f and $1/f$ are cyclic.*

Proof. If f and $1/f$ lie in $\mathcal{H}$, then $f\left(1/f\right) = 1 = k_{x_0}$, where $x_0 \in X$ is the point where the kernel K is normalized. Therefore, $f\left(1/f\right)$ is cyclic. Using the reverse implication in *Theorem 2.5.4*, it is clear that both f and $1/f$ are cyclic. □

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