Spring 5-15-2022

Some Problems in Reproducing Kernel Spaces

Christopher Felder
Washington University in St. Louis

WASHINGTON UNIVERSITY IN ST. LOUIS
Department of Mathematics and Statistics

Dissertation Examination Committee:
John E. M^{c}Carthy, Chair
Carl C. Cowen
Greg Knese
Henri Martikainen
Brett D. Wick

Some Problems in Reproducing Kernel Spaces
by
Christopher Felder

A dissertation presented to
the Graduate School
of Washington University in
partial fulfillment of the
requirements for the degree
of Doctor of Philosophy

May 2022
St. Louis, Missouri

Table of Contents

Acknowledgments

As is customary, I give my first gratitude to my advisor, John McCarthy. I am fortunate that this initial acknowledgment is both sincere and fitting. John has provided mathematical insight and academic guidance that is unmatched. More importantly, every ounce of wisdom and rigor that John possesses is matched in equal parts with compassion. I can convey this with only an anecdote: upon learning of a death in my family, John immediately produced his wallet, handed me his credit card, and told me to make travel arrangements for the funeral– we would worry about repayment later.

My next acknowledgement goes to the other committee members: Carl Cowen, Greg Knese, Henri Martikainen, and Brett Wick. Special thanks to Greg and Brett for teaching me throughout graduate school. I attribute the bulk of my mathematical understanding to their lectures over the years.

I am also indebted to the many great analysts with which I have interacted while in St. Louis: Kelly Bickel, Alberto Dayan, Francesco DiPlinio, Michael Hartz, James Pascoe, Meredith Sargent, Cody Stockdale, and Nathan Wagner. Thank you all for sharing.

Eternal thanks go to those that set me down this path: Carl Cowen, Eva Gallardo-Gutiérrez, Bill Johnston, and Becky Wahl. I am also grateful for Mohan Kumar.

For support, both mathematical and personal, I owe a very special note to Catherine Bénéteau, Dima Khavinson, Myrto Manolaki, Daniel Seco, and Alan Sola. Thank you for your guidance. I hold dear the many great conference experiences we have shared.

Any perceived progress I have made belongs not only to me, but also to my collaborators. Trieu Le made me a better mathematician and taught me more about mathematical exposition than I thought possible. Ray Cheng is a constant source of encouragement and wisdom. I have been inspired by his unending enthusiasm and commitment. I am ever grateful to have found kindred spirits in Ben Russo and Doug Pfeffer. Our follies and successes have been pure camaraderie, and a constant reminder that mathematics is *fun*.

This work is dedicated to my family. To my father, and the memory of my mother. Steve, Janice, Kyle, and Dillon for pure love and firm ground. And again to Janice– for unending wisdom and support.

My time in St. Louis has been filled with an immense amount of love and friendship, of which I am surely undeserving. I owe my most special thanks to Janine St. Gemme (and co.), Dan Turner, Dan Lowery, Billy Croghan/Tony Compton and the Les Gruff crew, Nick Gusman and the Coyotes, Greg Belter, Jakob Baxter and the Yard Eagle boys, the Venice Cafe crew (esp. Neil and Zack), Alan Murray, Sarah Ronquillio, Irene Allen, Billy Tomber, Lauren Olin, and The River Kittens. You have all played a very special part in this journey.

The last line is reserved for Mr. Nate Lowery, who has been an exemplary friend and constant source of grace and love. Thank you for all of the great times, all of the songs, and all of the beautiful exchange of ideas.

Christopher Felder

Washington University in St. Louis

May 2022

To live is to fly.

ABSTRACT OF THE DISSERTATION

Some Problems in Reproducing Kernel Spaces

by

Christopher Felder

Doctor of Philosophy in Mathematics

Washington University in St. Louis, 2022

Professor John E. M^{c}Carthy, Chair

The two chapters of this thesis are comprised of work in the setting of reproducing kernel (Hilbert) spaces. These are Banach (or Hilbert) spaces of functions defined on some set, with the special property that point evaluation, on the underlying set, is bounded.

The first chapter deals with the study of *inner functions*. These functions have a rich history in function and operator theory in the Hardy spaces of the unit disk. The first section of this chapter studies the relationship between generalized inner functions and optimal polynomial approximants. The second section, which is joint work with Trieu Le, deals with a generalization of a classical type of inner function (finite Blaschke product). The last section, which is joint work with Raymond Cheng, considers the (Banach) space ℓ^p_A– the space of analytic functions on the disk with p-summable Maclaurin coefficients. We consider the geometry of the multiplier algebra of this space and characterize extremal multipliers.

The second chapter considers the geometry of two planar sets associated to linear operators acting on reproducing kernel Hilbert spaces. The first section of this chapter, which is joint work with Carl Cowen, considers the convexity of the Berezin range of an operator on a reproducing kernel Hilbert space. We focus primarily on a class of composition operators acting on the Hardy space of the unit disk. The final section of the chapter, and the thesis, joint work with Benjamin Russo and Douglas Pfeffer, deals with the connectedness of various spectra of certain Toeplitz acting on a family of sub-Hardy Hilbert spaces.

Chapter 1

Introduction

Throughout this thesis, we will work in *reproducing kernel spaces*. This terminology may be deemed non-standard to some, but is necessary in order to encompass all settings in which we work.

1.1 Reproducing Kernel Spaces

Definition 1.1.1 (Reproducing kernel space)**.** Let X be a set and let $\mathcal{B}$ be a Banach space of complex-valued functions defined on X. We say that $\mathcal{B}$ is a *reproducing kernel space on* X if, for each $x \in X$, the linear functional of point evaluation, given by

$$f \mapsto f(x),$$

is a bounded linear functional on $\mathcal{B}$.

This definition is a generalization of the classical setting when the Banach space $\mathcal{B}$ is a Hilbert space (which we will typically denote by $\mathcal{H}$). In this case, by the Riesz representation theorem, for each point $x \in X$, there is unique element $k_x \in \mathcal{H}$ such that

$$f(x) = \langle f, k_x \rangle_{\mathcal{H}} \quad \forall f \in \mathcal{H}.$$

The element k_x is referred to as the *reproducing kernel at* x (for $\mathcal{H}$). We call $\mathcal{H}$ a *reproducing kernel Hilbert space* (RKHS) and any use of $\mathcal{H}$ will be reserved for this meaning. These spaces have a deep mathematical history and continue to attract the attention of mathematicians and data scientists (see, e.g., [64, 141]). We point to [129] for a thorough introduction to abstract RKHSs and to [3] for more general theory.

In either case, these spaces should be thought of as complete normed vector spaces of functions where point evaluation on some underlying set is bounded. We turn to provide several examples reproducing kernel spaces, which come naturally from spaces of analytic functions. We focus on these spaces as they permit an interplay between operator theory and complex-function theory– two subjects at the base of much work over the past century, and this thesis. When $\Omega \subset \mathbb{C}^n$ is a domain, we will use $\mathrm{Hol}(\Omega)$ to denote the space of holomorphic functions on Ω.

1.1.1 ℓ^p_A

For $1 \leq p \leq \infty$, the space ℓ^p_A is defined to be collection of analytic functions on the open unit disk $\mathbb{D}$ of the complex plane for which the Maclaurin coefficients are p-summable, i.e.,

$$\ell^p_A := \left\{ f(z) = \sum_{k \geq 0} a_k z^k \in \mathrm{Hol}(\mathbb{D}) : \sum_{k \geq 0} |a_k|^p < \infty \right\}.$$

This definition makes sense when $0 < p < 1$, but our attention will be limited to the range $1 \leq p \leq \infty$. This function space is endowed with the norm it inherits from the sequence space ℓ^p. Thus, we write

$$\|f\|_p = \|(a_k)_{k=0}^{\infty}\|_{\ell^p}$$

for

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

belonging to ℓ^p_A. We begin with this space as an example (when $p \neq 2$) of a reproducing kernel space that is not a Hilbert space, although point evaluation on $\mathbb{D}$ is still a bounded linear functional. When $p = 2$, we have the classical Hardy space, which we discuss now.

1.1.2 Weighted Hardy Spaces on the Unit Disk

Given a sequence of positive numbers $w = \{w_k\}_{k\geq 0}$, with $\lim_{k\to\infty} w_k/w_{k+1} = 1$, define the weighted Hardy space H^2_w as

$$H^2_w := \left\{ f(z) = \sum_{k\geq 0} a_k z^k \in \mathrm{Hol}(\mathbb{D}) : \sum_{k\geq 0} w_k |a_k|^2 < \infty \right\}.$$

For $f(z) = \sum_{k\geq 0} a_k z^k$ and $g(z) = \sum_{k\geq 0} b_k z^k$ in H^2_w, their inner product in H^2_w is given by

$$\langle f, g\rangle_w = \sum_{k\geq 0} w_k a_k \overline{b_k}.$$

One may verify (e.g. see [63, Section 2.1]) that these spaces are reproducing kernel Hilbert spaces on $\mathbb{D}$ with reproducing kernel given by

$$k_\beta(z) = \sum_{n\geq 0} \frac{1}{w_k} \left(\bar{\beta} z\right)^n.$$

If we let $\alpha \in \mathbb{R}$ and take $w = \{(k+1)^\alpha\}_{k\geq 0}$, we recover the so-called Dirichlet-type space $\mathcal{D}_\alpha$. There are three classically important sets of weights in this setting.

- ($\alpha = 0$) The Hardy space,

$$H^2 := \left\{ f(z) = \sum_{k\geq 0} a_k z^k \in \mathrm{Hol}(\mathbb{D}) : \sum_{k\geq 0} |a_k|^2 < \infty \right\}.$$

- ($\alpha = -1$) The Bergman space,

$$A^2 := \left\{ f(z) = \sum_{k\geq 0} a_k z^k \in \mathrm{Hol}(\mathbb{D}) : \sum_{k\geq 0} (k+1)^{-1} |a_k|^2 < \infty \right\}.$$

- ($\alpha = 1$) The Dirchlet space,

$$\mathcal{D} := \left\{ f(z) = \sum_{k\geq 0} a_k z^k \in \mathrm{Hol}(\mathbb{D}) : \sum_{k\geq 0} (k+1)|a_k|^2 < \infty \right\}.$$

These spaces also have the following equivalent norms, which will be useful at times.

H^2	$\\|f\\|_{H^2} = \sup_{0\leq r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|^2 \, d\theta \right)^{1/2}$
A^2	$\\|f\\|_{A^2} = \left(\frac{1}{\pi} \int_{\mathbb{D}} \|f(z)\|^2 \, dz \right)^{1/2}$
$\mathcal{D}$	$\\|f\\|_{\mathcal{D}} = \left(\frac{1}{\pi} \int_{\mathbb{D}} \|f'(z)\|^2 \, dz \right)^{1/2} + \sup_{0\leq r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|^2 \, d\theta \right)^{1/2}$

In the coming sections, more examples of reproducing kernel spaces will be discussed, as they become relevant.

1.2 Cyclic and Inner functions

Much of the work compiled in this thesis begins with results in H^2 and aims to generalize these results, or obtain a similar understanding of certain concepts, but in other reproducing kernel spaces. A cornerstone question in functional analysis is to ask whether a given a subspace $\mathcal{M}$ of a Banach space $\mathcal{B}$ is dense in $\mathcal{B}$. If we suppose that $\mathcal{B}$ is a space of complex-valued functions defined on some planar domain, where the forward shift

$$(Sf)(z) = zf(z)$$

is bounded, then particular attention has been given to the following problem of this type:

Question 1.2.1. *Given $f \in \mathcal{B}$, when are the polynomial multiples of f dense in $\mathcal{B}$? Conversely, if the polynomial multiples of f are dense, what can be said of f?*

In 1948, Arne Beurling [31] studied this problem, and, in doing so, proved several seminal results regarding function and operator theory in H^2. One result showed that every function $f \in H^2$ may be factored as $f = \theta U$, where $|\theta| = 1$ almost everywhere on the unit circle $\mathbb{T}$ (coined *inner*), and U is such that $\log|U(0)| = \int_0^{2\pi} \log|U(e^{i\theta})|$ (coined *outer*). This result allowed Beurling to characterize cyclic functions and shift-invariant subspaces of H^2. Namely, the cyclic functions in H^2 are those that are outer, and shift-invariant subspaces must coincide with θH^2, for some inner function θ.

Although analogous definitions of Beurling's inner and outer functions in spaces other than H^2 have been made, the objects they describe are not as well understood. We give these definitions now (here, $\mathcal{H}$ is an RKHS and $\mathrm{cl}_{\mathcal{H}}(V)$ refers to the norm-closure of a subspace $V \subseteq \mathcal{H}$).

Definition 1.2.1 (Cyclic function)**.** Say that $f \in \mathcal{H}$ is cyclic (for S in $\mathcal{H}$) if

$$[f] := \mathrm{cl}_{\mathcal{H}}\left(\mathrm{span}\{S^k f : k \geq 0\}\right)$$

is equal to all of $\mathcal{H}$.

Note that the operator semigroup $(S^n)_{n\geq 0}$ acting on f generates the polynomial multiples of the function f. Note also that $[f]$ is always a shift-invariant subspace, i.e. $S[f] \subseteq [f]$, and is the smallest such containing f. The space $[f]$ is read "bracket f" or "the (shift-invariant) subspace generated by f." Again, Beurling's theorem says that, in H^2, the cyclic functions are precisely the outer functions. Although we make no major contributions to the characterization of cyclic functions here, the tools and themes of Section 2.1 are geared to address such problems.

The other factor in Beurling's factorization– the inner part– also plays a critical role in function and operator theory on Hardy spaces; see [42] for a recent survey of classical and new results linking inner functions and operator theory. In addition to the above definition, inner functions can also be realized via the inner product in H^2. Indeed, it can be checked that a function $f \in H^2$ is inner if and only if $\|f\|_{H^2} = 1$ and $\langle z^m f, f\rangle = 0$ for all integers $m \geq 1$.

In the case of the Dirichlet space $\mathcal{D}$, Richter [133] showed that any shift-invariant subspace is also generated by a single function that satisfies the same orthogonality properties as above. Aleman, Richter, and Sundberg [6] proved an analogue of Beurling's Theorem for the Bergman space A^2; any invariant subspace $\mathcal{M}$ of A^2 is generated by the so-called wandering subspace $\mathcal{M} \ominus z\mathcal{M}$. Any unit norm function in this subspace satisfies $\|f\|_{A^2} = 1$ and $z^m f \perp f$ for all $m \geq 1$ and is called an A^2-inner function. Prior to this work, Hedenmalm [100] showed the existence of so-called contractive zero-divisors, which play the role of Blaschke products in the Bergman space. In certain cases, explicit formulas for these functions have been given, e.g. see MacGregor and Stessin [121] and Hansbo [96]. These results are phrased in the language of *extremal functions*. Although the work here will not explicitly cover this aspect, it is well known that (normalized) inner functions are solutions to the extremal problem

$$\sup\left\{\operatorname{Re}(g^{(d)}(0)) : g \in \mathcal{M}, \|g\| \leq 1\right\},$$

where $\mathcal{M}$ is a shift-invariant subspace and d is the smallest integer so that $z^d \notin \mathcal{M}^\perp$. See [75, Chapters 5 and 9] and [101, Chapter 3] for a detailed discussion of inner functions on Bergman spaces A^p. Thus, the notion of inner functions in more general reproducing kernel Hilbert spaces has been formulated in the following way.

Definition 1.2.2 (Inner function)**.** Say that $f \in \mathcal{H} \setminus \{0\}$ is $\mathcal{H}$-inner if, for all integers $k \geq 1$,

$$\langle f, z^k f\rangle = 0.$$

This definition was originally considered in [6] for the Bergman space. The authors there also require an inner function to be of unit norm, as well as other authors. In a recent paper, Cheng, Mashreghi and Ross [50] introduced and studied the notion of inner functions with respect to a bounded linear operator. However, they do not require unit norm, which turns out more convenient in several situations. We will follow their approach in this work. Although no function-theoretic description of inner functions is known in general reproducing Hilbert spaces, there are known constructions of certain types of inner functions. We will introduce one of these constructions at the end of Section 2.2.3.

Bénéteau et al. [23, 27] studied inner functions and examined the connections between them and optimal polynomial approximants on weighted Hardy spaces. They also described a method to construct inner functions that are analogues of finite Blaschke products with simple zeroes. In [137], Seco discussed inner functions on Dirichlet-type spaces and characterized such functions as those whose norm and multiplier norm are equal. In [114], Le studied inner functions on weighted Hardy spaces and obtained generalizations of several results from [23, 137]. In a recent paper [24], Bénéteau et al. investigated inner functions on general simply connected domains in the complex plane. It should also be mentioned that operator-valued inner functions on vector-valued weighted Hardy spaces have also been defined and studied [16, 17, 128]. In particular, Ball and Bolotnikov [17] obtained a realization of inner functions on vector-valued weighted Hardy spaces. In [18], they investigated the expansive multiplier property of inner functions. They obtained a sufficient condition on the weight sequences for which any inner function has the expansive multiplier property. Recently, Cheng, Mashreghi, and Ross considered inner vectors for Toeplitz operators [52] and for the shift operator in the Banach space setting of ℓ^p_A [49]. Additional background and information on cyclic and inner functions will be provided as it becomes relevant in each section of Chapter 2.

The main results of each section of Chapter 2 are outlined below.

1.2.1 Optimal Polynomial Approximants

Throughout Section 2.1, $\mathcal{H}$ is assumed to be an RKHS of analytic functions on $\mathbb{D}$ for which the shift is bounded and the polynomials are dense. For $n \in \mathbb{N}$, we will denote by $\mathcal{P}_n$ the set of complex polynomials of degree less than or equal to n. For $f \in \mathcal{H}$, we define $f\mathcal{P}_n := \{pf : p \in \mathcal{P}_n\}$. Noting that $f\mathcal{P}_n$ is always a closed finite-dimensional subspace of $\mathcal{H}$, we will use $\Pi_n : \mathcal{H} \to f\mathcal{P}_n$ to denote the orthogonal projection onto $f\mathcal{P}_n$.

In [36], it was pointed out that $f \in \mathcal{H}$ is cyclic if and only if, for any cyclic function $g \in \mathcal{H}$, there exist polynomials $(p_n)_{n\geq 0}$ so that $\|p_n f - g\|_{\mathcal{H}} \to 0$. From this equivalence, and taking $g = 1$ in spaces where $1 = k_0$, the study of *optimal polynomial approximants* has arisen. The optimality referred to here is with respect to the distance between $f\mathcal{P}_n$ and 1, i.e.,

$$\min_{p\in\mathcal{P}_n} \|pf - 1\|_{\mathcal{H}}.$$

The element of $f\mathcal{P}_n$ minimizing this distance will be denoted $p_n^* f$ (details to come in Section 2.1.2).

Approximation problems of this kind were first studied under the engineering lens of filter design in the 1970's and 80's, referred to as *least squares inverses* (e.g. see [56, 57, 138]). It seems this body of work was not known to mathematicians prior to the discussion in [28].

A modern jumping off point for optimal approximants could be considered the work in [84]; the authors study the optimal approximants of the function $1-z$ in order to characterize the cyclicity of holomorphic functions on the closed unit disk. In [28], the authors compute Taylor coefficients of $1 - p_n^* f$ in weighted Hardy spaces when f is a polynomial, proving results about the convergence of $(1 - p_n^* f)$.

In [26], the authors study a larger class of reproducing kernel Hilbert spaces and give results on accumulation points, along with lower bounds on the moduli of zeros of optimal approximants. Then in [27], the authors dive into orthogonal polynomials and reproducing kernels in order to get lower bounds on the moduli of zeros of optimal approximants in Dirichlet-type spaces.

Following these themes, we would like to develop some theory for different choices of g (cyclic or not) in minimizing $\|pf - g\|_{\mathcal{H}}$, and then explore the relationship between optimal approximants and $\mathcal{H}$-inner functions (this relationship first studied in [23]). This will then yield some observations which allow us to explicitly compute the orthogonal projection of 1 onto $[f]$ when f is a polynomial (note that f is cyclic if and only if this projection is equal to 1 itself).

In particular:

- Section 2.1.2 develops the framework necessary for handling general optimal approximants.

- Section 2.1.3 deals with *stabilization* of optimal approximants to $\hat{k}_0/f$, with Theorem 2.1.1 characterizing when $p_n^* f = p_M^* f$ for all n great than some fixed $M \geq 0$.

- Section 2.1.5 discusses stabilization of general optimal approximants, with Theorem 2.1.2 giving a version of Theorem 2.1.1 for general approximants.

- Section 2.1.6 develops the theory of *reproducible points*, and then returns to certain spaces where $\hat{k}_0 = 1$, with Theorem 2.1.3 providing an explicit description of the projection of 1 onto the shift invariant subspace generated by a polynomial.

Many of the themes of this section follow from those in [23]. The authors there show that inner functions correspond to constant optimal approximants and investigate certain inner functions that arise as linear combinations of reproducing kernels.

1.2.2 Analogues of Finite Blashcke Products

The simplest H^2-inner functions are called *finite Blaschke products*, given for $\beta_1, \ldots, \beta_n \in \mathbb{D}$ as

$$B(z) = \lambda \prod_{j=1}^{n} \frac{z - \beta_j}{1 - \overline{\beta_j} z},$$

where $\lambda \in \mathbb{T}$. One may check that $|B| = 1$ on $\mathbb{T}$, and so B is in fact H^2-inner. We recall that *Blaschke factors*, given when $n = 1$ in B above, define the automorphisms of the unit disk. See the recent book [89] for a nice treatment of finite Blaschke products and their applications.

The work of Section 2.2, which is joint work with Trieu Le, begins with a simple observation. Applying a partial fractions decomposition, and assuming simple zeros different from the origin, one can check that any finite Blaschke product can be expressed as

$$B(z) = c_0 - \sum_{j=1}^{n} \frac{c_j}{1 - \overline{\beta_j} z}$$

for some constants $c_0, c_1, \ldots, c_n \in \mathbb{C}$. Further, each term in the sum can be seen as a scalar multiple of the Szegő kernel, $s_\lambda(z) = 1/(1 - \overline{\lambda} z)$, $\lambda \in \mathbb{D}$, which is the *reproducing kernel* for H^2. Noting that $s_0(z) = 1$, we have

$$B(z) = c_0 s_0(z) - \sum_{j=1}^{n} c_j s_{\beta_j}(z).$$

Consequently, every Blaschke product with simple zeros can be seen as a linear combination of reproducing kernels. If B also has repeated zeros, certain derivatives of kernel functions will also be needed in the linear combination. Nonetheless, it turns out this observation actually characterizes finite Blaschke products among inner functions: an H^2-inner function f is a finite Blaschke product if and only if f is a linear combination of reproducing kernels and their derivatives.

We will extend this result to a more general setting in Theorem 2.2.2. In this setting, $\mathcal{H}$ is assumed to be an RKHS of analytic functions on a planar domain Ω containing the origin, for which the shift is bounded and the polynomials are dense.

We call functions characterized in this way *analogues of finite Blaschke products*. Further, we show precisely how these functions arise as certain Gram determinants, or as certain projections onto shift-invariant subspaces generated by polynomials.

1.2.3 Extremal Multipliers of ℓ^p_A

Sequence spaces play an important role in functional analysis. Indeed, the theory of Banach spaces arose from early studies of the sequence space ℓ^p. The case ℓ^1 is connected to the Wiener Algebra, and its additional structure has made deeper inroads possible. The case of ℓ^2 is particularly well understood, having been studied by Hilbert himself, and serving as a launching point for the spaces that bear his name. Moreover, as previously mentioned, ℓ^2 is isometrically isomorphic to the Hardy space H^2. In this situation, the interplay between the analytical properties of the functions and the behavior of the space has given rise to a deep and extensive body of results, one of the great triumphs of the past century of mathematical analysis.

In Section 2.3, we uncover some geometric properties of the multiplier space of ℓ^p_A. These include the failure of the weak parallelogram laws and the Pythagorean inequalities. Further, we characterize extremal multipliers of ℓ^p_A. We use $\mathscr{M}_p$ to denote the multiplier algebra of ℓ^p_A,

$$\mathscr{M}_p := \{\phi \in \ell^p_A : \phi f \in \ell^p_A \ \text{ for all } f \in \ell^p_A\},$$

endowed with the norm

$$\|\phi\|_{\mathscr{M}_p} := \sup\{\|\phi f\|_p : f \in \ell^p_A, \|f\|_p \le 1\}.$$

It is elementary to see that if $\phi \in \mathscr{M}_p$, then $\|\phi\|_{\mathscr{M}_p} \ge \|\phi\|_p$. We say that the multiplier ϕ is *extremal* if equality holds, that is,

$$\|\phi\|_{\mathscr{M}_p} = \|\phi\|_p.$$

For $\ell^2_A = H^2$, the multipliers are the bounded analytic functions on $\mathbb{D}$, and the extremal multipliers are exactly the constant multiples of inner functions. Indeed, if

$$\sup_{z\in\mathbb{D}} |\phi(z)| = \Big(\sup_{0<r<1} \int_{\mathbb{T}} |\phi(re^{i\theta})|^2 \,\frac{d\theta}{2\pi}\Big)^{1/2},$$

then $|\phi(e^{i\theta})| = \|\phi\|_{H^\infty}$ a.e. is forced. The reverse implication is similarly trivial.

However, relatively little is known about the multipliers on ℓ^p_A, except when $p = 1$ or $p = 2$. In the former case, we know that $\mathscr{M}_1 = \ell^1_A$, and in the latter, $\mathscr{M}_2 = H^\infty$ (as mentioned above). We will accordingly concentrate our efforts on the range $1 < p < \infty$, with $p \ne 2$.

Despite the lack of an inner product when $p \ne 2$, there are analogous definitions of inner functions in ℓ^p_A using Birkhoff-James orthogonality. These functions are called *p-inner functions* (see [49, 50] or Section 2.3 here). It would be plausible to guess that the extremal multipliers are the p-inner functions. However, this turns out to be incorrect.

Instead, we show that when $p \in (1,\infty) \setminus \{2\}$, a multiplier ϕ is extremal precisely if it is of the form

$$\phi(z) = \gamma z^k$$

for some $\gamma \in \mathbb{C}$ and nonnegative integer k. Again, this is quite distinct from the $p = 2$ case, in which the extremal multipliers consist of the constant multiples of inner functions.

1.3 The Berezin Range

The penultimate section of this thesis deals with the convexity of the Berezin range, which we define now.

Definition 1.3.1. Let $\mathcal{H}$ be an RKHS on a set X and let T be a bounded linear operator on $\mathcal{H}$.

1. For $x \in X$, the *Berezin transform* of T at x (or *Berezin symbol of* T) is

$$\tilde{T}(x) := \langle T\hat{k}_x, \hat{k}_x \rangle_{\mathcal{H}}.$$

2. The *Berezin range* of T (or *Berezin set of* T) is

$$B(T) := \left\{ \langle T\hat{k}_x, \hat{k}_x \rangle_{\mathcal{H}} : x \in X \right\}.$$

3. The *Berezin radius of* T (or *Berezin number of* T) is

$$b(T) := \sup_{x \in X} |\tilde{T}(x)|.$$

The Berezin set and number, also denoted by $\mathrm{Ber}(T)$ and $\mathrm{ber}(T)$, respectively, were purportedly first formally introduced by Karaev in [108]. The Berezin transform itself was introduced by F. Berezin in [29] and has proven to be a critical tool in operator theory, as many foundational properties of important operators are encoded in their Berezin transforms.

One of the first important results involving the Berezin transform involves the invertibility of Toeplitz operators acting on H^2. In [70], R.G. Douglas asked the following: if $\varphi \in L^\infty(\mathbb{T})$

with $|\tilde{T_\varphi}| \geq \delta > 0$, is the Toeplitz operator T_φ invertible? Tolokonnikov [146], and then Wolff [157], showed that when δ is sufficiently close to 1, the answer is affirmative. The lower bound on δ was then improved by Nikolskii [126]. Much later, Karaev proved similar results for certain general operators acting on RKHS's: if the modulus of the Berezin transform of a suitably nice operator is sufficiently bounded away from zero, then the invertibility of the operator is ensured [108, Theorem 3.4]. Similar recent results for closed range type properties of Toeplitz operators can be found in [164].

Following Douglas' question, Berger and Coburn [30] asked something similar: if the Berezin symbol of an operator on the Hardy or Bergman space vanishes on the boundary of the disk, must the operator be compact? This question was addressed by Nordgren and Rosenthal [127], where they presented several counterexamples. However, Nordgren and Rosenthal showed, on a so-called *standard* RKHS, that if the Berezin symbols of all unitary equivalents of an operator vanish on the boundary, then the operator is compact. The counterexamples presented come in the form of composition operators, and motivate the study in Section 3.1.4.

Another important theorem here, due to Axler and Zheng [13], is that if T is a finite sum of finite products of Toeplitz operators acting on the Bergman space of the unit disk, then T is compact if and only if the Berezin transform of T vanishes as it approaches the boundary of the disk. Shortly after, Engliš [78] generalized this result to weighted Bergman spaces on bounded symmetric domains in several variables. Later, Suárez [143] proved an analogous result for *any operator in the Toeplitz algebra* of the Bergman space on the unit ball. Some other Axler-Zheng type results for various spaces over several types of domains can be found in [21, 103, 123, 147, 149, 163].

Some other quite interesting results involving applications of the Berezin transform include the characterization of invertible operators which are unitary [111], characterizations of Schatten-von Neumann class membership [41, 105, 109], Beurling-Arveson type theorems for some RKHSs [106], a characterization of skew-symmetric operators [8], and the characterization of truncated Toeplitz operators with bounded symbols, along with descriptions of invariant subspaces of isometric composition operators [87]. See also [107], which will be discussed in Section 3.1.5. A detailed introduction to the Berezin transform of operators on spaces of analytic functions can be found in [142].

We note that the range of the Berezin transform has been studied from a function theoretic viewpoint, for example in work by Ahern, which establishes a Brown-Halmos type theorem for the Bergman space [4] (see also [132, 148]). However, apart from some examples due to Karaev [107, Section 2.1], it does not appear that the Berezin range has been studied from a set-theoretic or geometric viewpoint, as we will discuss in Section 3.1.

The Berezin range of an operator T is a subset of the numerical range of T

$$W(T) := \{\langle Tu, u\rangle : \|u\| = 1\},$$

which is convex (this result due independently to both Toeplitz and Hausdorff). It is natural to ask when the Berezin range of an operator is convex. Section 3.1 gives characterizations for certain concrete operators to have convex Berezin range. Namely, we characterize convex Berezin range for matrices and multiplication operators in Section 3.1.3, and composition operators with automorphic symbol (acting on H^2) in Section 3.1.4. We conclude with some open questions in Section 3.1.5.

1.4 Spectra of Toeplitz Operators

The final section of this thesis deals with various spectra of certain Toeplitz operators acting on sub-Hardy Hilbert spaces. Recall the spectrum of a linear operator T is defined as

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

We will use the standard notation $\sigma_p(T)$ to denote the point spectrum, or eigenvalues, of T.

For $1 \leq p \leq \infty$, we let $L^p = L^p(\mathbb{T}, \mu)$ denote the classical Lebesgue spaces on the unit circle $\mathbb{T}$ with respect to normalized Lebesgue measure μ. Similarly, we let H^p denote the usual Hardy spaces, with the standard identification of H^p as spaces of analytic functions on the unit disk such that

$$\sup_{0<r<1} \int_0^{2\pi} \left| f\left(re^{it}\right) \right|^p \, dt < \infty.$$

When $p = \infty$, these spaces are comprised of (essentially) bounded functions.

For $\phi \in L^\infty$, the classical *Toeplitz operator with symbol* ϕ acts on H^2 by

$$T_\phi f = P_+(\phi f),$$

where P_+ is the orthogonal projection from L^2 onto H^2.

In 1963, Halmos asked if the spectrum of every Toeplitz operator is connected [95]. At that time, the following facts were known about the spectrum of Toeplitz operators, due to Hartman and Wintner (e.g. see [71, Chapter 7]):

- If $\phi \in L^\infty$ is real-valued, then $\sigma(T_\phi) = [\operatorname{ess\,inf}(\phi), \operatorname{ess\,sup}(\phi)]$.

- If $\phi \in H^\infty$, then $\sigma(T_\phi) = \overline{\phi(\mathbb{D})}$.

Shortly after, Widom gave an affirmative answer to Halmos' question with the following result [154]:

- Every Toeplitz operator has connected spectrum and connected essential spectrum.

It can also be shown that when the symbol of the Toeplitz operator is real-valued, its point spectrum is empty, and therefore connected (e.g. see [83, Exercise 12.4.3]). We point the reader to [12] for a well-written account of this history.

The aim of the last section of this thesis is to present results in this vein, but for Toeplitz operators acting on sub-Hardy Hilbert spaces arising naturally from certain finite codimension subalgebras of H^∞; so-called *constrained subalgebras.* The most well-known constrained subalgebra of H^∞ is the Neil algebra,

$$\mathfrak{A} := \{f \in H^\infty : f'(0) = 0\}.$$

Note again that H^∞ acts as a multiplier algebra for H^2; for each $\phi \in H^\infty$, we have $\phi H^2 \subseteq H^2$. When moving to the Neil setting, we have that $\mathfrak{A}$ serves as a multiplier algebra not for a single space, but for a continuum of spaces

$$H^2_{\alpha,\beta} = \{f \in H^2 : \alpha f(0) = \beta f'(0) \text{ for } (\alpha, \beta) \in \mathbb{S}^2\},$$

where $\mathbb{S}^2$ is the complex unit-sphere in $\mathbb{C}^2$ (see [66] for details). Although $\mathfrak{A}$ serves as a multiplier algebra for many Hilbert spaces of analytic functions, it is canonical to consider $\mathfrak{A}$ acting on the spaces $H^2_{\alpha,\beta}$, as the associated representations of $\mathfrak{A}$ are rank one bundle shifts in this setting (see [34, Section 4]). This perspective is useful in many contexts, however, is not needed here.

For $\phi \in L^\infty$, the Toeplitz operator $T^{\alpha,\beta}_\phi$ acts on $H^2_{\alpha,\beta}$ by

$$T^{\alpha,\beta}_\phi f = P_{\alpha,\beta}(\phi f),$$

where $P_{\alpha,\beta}$ is the orthogonal projection from L^2 onto $H^2_{\alpha,\beta}$. In the Neil setting, Broschinski [34] observed that the point spectrum of $T^{\alpha,\beta}_\phi$ might be non-empty; a stark difference from the classical setting. However, in a Widomesque quest, Broschinksi showed that the point spectrum of these operators *relative to the algebra* $\mathfrak{A}$ is indeed connected for certain symbols. In particular, Broschinski proved that when $\phi \in L^\infty$ is real-valued, the set of *eigenvalues of*

$T_\phi^{\alpha,\beta}$ *relative to* $\mathfrak{A}$,

$$\Lambda_\phi^{\mathfrak{A}} := \bigcup_{(\alpha,\beta)\in\mathbb{S}^2} \sigma_p(T_\phi^{\alpha,\beta})$$

is either empty, a point, or an interval. Heuristically speaking, when working in settings that involve infinite families of representation-carrying Hardy spaces, it is typical for results to involve the entire family (e.g. see discussion in Section 3.2.1).

Section 3.2 provides, among other results, an analogue of Broschinski's result in the setting of a so-called two-point algebra. The two-point algebra associated to fixed points $a, b \in \mathbb{D}$ is

$$\mathscr{A}_{a,b} := \{f \in H^\infty(\mathbb{D}) : f(a) = f(b)\}.$$

Similar to the Neil algebra, there is an associated infinite family of sub-Hilbert Hardy spaces that each carry a representation for $\mathscr{A}_{a,b}$. For fixed $a, b \in \mathbb{D}$, define

$$H_t^2 := \{f \in H^2 : f(a) = tf(b)\},$$

where $t \in \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Note that, while the parameter space $\mathbb{C} \setminus \{0\}$ could also be used to yield the main results in this section, we instead use the Riemann sphere $\widehat{\mathbb{C}}$ to be consistent with the notation introduced in [130], where compactness of the parameter space was necessary. To this end, this section considers $t \neq 0, \infty$, and leaves the details of these cases to the interested reader. For each choice of $t \in \widehat{\mathbb{C}}$, the space H_t^2 carries a representation for $\mathscr{A}_{a,b}$. Specifically, the map taking $h \in \mathscr{A}_{a,b}$ to the operator on H_t^2 given by $f \mapsto hf$, is an isometric homomorphism from $\mathscr{A}_{a,b}$ to the collection of bounded linear operators on H_t^2. In particular, we have $\mathscr{A}_{a,b}H_t^2 \subseteq H_t^2$ for every $t \in \widehat{\mathbb{C}}$. As with the Neil algebra, the representations associated with H_t^2 can be seen as rank one bundle shifts for $\mathscr{A}_{a,b}$.

For $\phi \in L^\infty$, we define the Toeplitz operator $T_\phi^t \colon H_t^2 \to H_t^2$ by

$$T_\phi^t f = P_t(\phi f),$$

where P_t is the orthogonal projection from L^2 onto H_t^2. We denote the *eigenvalues of* T_ϕ^t *relative to* $\mathscr{A}_{a,b}$ as

$$\Lambda_\phi^{a,b} := \bigcup_{t \in \widehat{\mathbb{C}}} \sigma_p(T_\phi^t).$$

In Section 3.2, we establish the following main results, regarding analytic and real-valued symbols, respectively:

Theorem 3.2.1. *If* $\phi \in \mathscr{A}_{a,b}$, *then*

(i) $\sigma(T_\phi^t) = \overline{\phi(\mathbb{D})}$ *and*

(ii) $\Lambda_\phi^{a,b} = \{\phi(a)\}$.

In particular, both $\sigma(T_\phi^t)$ *and* $\Lambda_\phi^{a,b}$ *are connected.*

Theorem 3.2.2. *If* $\phi \in L^\infty$ *is real-valued, then* $\Lambda_\phi^{a,b}$ *is either empty, a point, or an interval. In particular,* $\Lambda_\phi^{a,b}$ *is connected.*

Chapter 2

Inner Functions

For the next two sections, 2.1 and 2.2, $\mathcal{H}$ will be an RKHS of analytic functions. The following notation will be used throughout:

(a) When $V \subseteq \mathcal{H}$ is a closed subspace, we will use $\Pi_V : \mathcal{H} \to V$ to denote the orthogonal projection from $\mathcal{H}$ onto V.

(b) When $X \subseteq \mathcal{H}$ is a subset, we will use $\mathrm{cl}_{\mathcal{H}}(X)$ to denote the norm closure of X in $\mathcal{H}$.

(c) For $f \in \mathcal{H}$, we will use the standard notation $\mathrm{ord}_0(f)$ to denote the order of the zero of f at the origin.

(d) For a polynomial f, we will let $Z(f)$ be the multiset containing the zeros f, i.e. $Z(f)$ is the zero set of f, each zero listed with its multiplicity.

2.1 Optimal Polynomial Approximants

The work in this section can be found in preprint form in [80]. For various Hilbert spaces of analytic functions on the unit disk, we characterize when a function has optimal polynomial approximants given by truncations of a single power series. We also introduce a generalized notion of optimal approximant and use this to explicitly compute orthogonal projections of 1 onto certain shift invariant subspaces.

2.1.1 Preliminaries

Throughout this section, $\mathcal{H}$ will be a reproducing kernel Hilbert space of analytic functions on the unit disk $\mathbb{D}$. Further, we will assume that $\mathcal{H}$ satisfies the following:

1. The polynomials $\mathcal{P}$ are dense in $\mathcal{H}$.

2. The forward shift S, mapping $f(z) \mapsto zf(z)$, is a bounded operator on $\mathcal{H}$.

As is standard, we will denote the reproducing kernel for $\mathcal{H}$ as $k_\lambda(z) = k(z, \lambda)$ and the normalized reproducing kernel as $\hat{k}_\lambda = k_\lambda / \|k_\lambda\|_{\mathcal{H}}$.

In addition to the weighted Hardy spaces mentioned in Section 1.1.2, we begin this section by mentioning some spaces where assumptions 1 and 2 from above hold.

Example 2.1.1 (Szegő's Theorem and $\frac{1}{m}H^2$)**.** A classical theorem of Szegő says that for $v \in L^1(\mathbb{T})$ positive, the closure of the analytic polynomials in $L^2(v)$ coincides with all of $L^2(v)$ if and only if $\int_{\mathbb{T}} \log v = -\infty$ (e.g., see [61]). In the case that $\int_{\mathbb{T}} \log v > -\infty$, there exists an outer (i.e., H^2-cyclic) function m such that $v = |m|^2$. Further, $P^2(v) := \overline{\text{span}\{z^k : k \geq 0\}}^{L^2(v)}$ is isomorphic to $\frac{1}{m}H^2 := \{f/m : f \in H^2\}$ (which we endow with the H^2 norm). It follows that multiplication by $1/m$ is an isometry and for all $f \in P^2(v)$, we have $\|f\|_{P^2(v)} = \|f/m\|_{H^2}$. A distinctive characteristic of these spaces is that the monomials are not pairwise orthogonal (as they are in the weighted Hardy spaces).

Example 2.1.2 (de Branges-Rovnyak Spaces)**.** Another example comes from considering H^∞– the set of bounded analytic functions on $\mathbb{D}$. If b is a function in the unit ball of H^∞ (i.e., $\sup_{z\in\mathbb{D}} |b(z)| \leq 1$), then there exists a reproducing kernel Hilbert space on $\mathbb{D}$, denoted $\mathcal{H}(b)$ so that the reproducing kernel for this space is given by

$$k_\lambda(z) = \frac{1 - \overline{b(\lambda)}b(z)}{1 - \overline{\lambda}z}.$$

These spaces are called *de Branges-Rovnyak spaces* (see [144] for an introduction). The structure of these spaces varies with the choice of b; we would like to keep in mind the spaces for which the reproducing kernel at zero is not equal to 1 (i.e., when $b(0) \neq 0$). We will generalize some ideas from the existing body of work, for example in the Dirichlet-type spaces, where the function 1 is the reproducing kernel at zero. We will not dig into the study of de Branges-Rovnyak spaces here, but the authors in [85] have characterized cyclicity when b is non-extreme.

2.1.2 General Optimal Approximants

Definition 2.1.1 (Optimal Polynomial Approximant)**.** Let $f, g \in \mathcal{H}$ and $n \in \mathbb{N}$. Define the nth *optimal polynomial approximant* to g/f as

$$p_n^* := \arg\min_{p \in \mathcal{P}_n} \|pf - g\|_{\mathcal{H}}.$$

Here, $\arg\min$ is the argument of the minimum, i.e.,

$$p_n^* = \{p \in \mathcal{P}_n : \|pf - g\|_{\mathcal{H}} \leq \|qf - g\|_{\mathcal{H}} \text{ for all } q \in \mathcal{P}_n\}.$$

We make the distinction of *general* optimal polynomial approximant to generalize the case when $g = 1$ in studying $\|pf - g\|_{\mathcal{H}}$. Any further use of g will be in this context.

Given the Hilbert space structure, the above minimization is immediate– simply project g onto the closed subspace $f\mathcal{P}_n$, i.e.,

$$p_n^* f = \Pi_{f\mathcal{P}_n}(g).$$

Hence, the solution to the minimization problem uniquely exists so long as f is not identically zero, and is non-zero so long as g is not orthogonal to $f\mathcal{P}_n$. In turn, we will be mostly concerned with the cases where $f \not\equiv 0$ and g is not orthogonal to $f\mathcal{P}_n$ for some $n \geq 0$. We note that when g is chosen to be the reproducing kernel at the origin, we have that k_0 is orthogonal to $f\mathcal{P}_n$ (for any $n \geq 0$, and in the limit) if and only if f and k_0 are orthogonal, i.e., $f(0) = 0$. Intuitively, if $\lim_{n\to\infty} p_n^*$ looks like g/f, then the above norm goes to zero and does so *optimally.* In this sense, we are trying to approximate g/f with polynomials.

In [84] (Theorem 2.1), an algorithm for finding optimal polynomial approximants is given for $g = 1$ in spaces where k_0, the reproducing kernel at zero, is equal to 1. We generalize the ideas from this algorithm below.

Definition 2.1.2 (Optimal System)**.** For $f, g \in \mathcal{H}$, define the nth *optimal matrix* of f in $\mathcal{H}$ as

$$G_n := \left(\langle z^i f, z^j f\rangle_{\mathcal{H}}\right)_{0\leq i,j\leq n}$$

and the nth *optimal system* of g/f as

$$G_n\vec{x} = \left(\langle g, f\rangle, \langle g, zf\rangle, \dots, \langle g, z^n f\rangle\right)^T.$$

The following proposition will shed light on these definitions.

Proposition 2.1.1. Let $f, g \in \mathcal{H}$. The vector $\vec{a}_n = (a_0, a_1, \dots, a_n)^T$ solving the optimal system

$$G_n\vec{x} = \left(\langle g, f\rangle, \langle g, zf\rangle, \dots, \langle g, z^n f\rangle\right)^T$$

gives the coefficients of the nth optimal approximant to g/f. That is, the nth optimal approximant to g/f is $p_n^*(z) = a_0 + a_1 z + \cdots + a_n z^n$.

Proof. The optimality of p_n^* means for all $q \in \mathcal{P}_n$

$$\|p_n^* f - g\|_{\mathcal{H}}^2 \leq \|qf - g\|_{\mathcal{H}}^2.$$

This occurs if and only if $p_n^* f - g \perp qf$. Equivalently, for $j = 0, \ldots, n$, we must have

$$\langle p_n^* f - g, z^j f \rangle_{\mathcal{H}} = 0.$$

Moving $\langle g, z^j f \rangle_{\mathcal{H}}$ to the right hand side of the above equation and putting $p_n^*(z) = \sum_{j=0}^{n} a_j z^j$ gives the proposed system. $\square$

Our next proposition is well-known and will be important in our work and discussion; for posterity, we provide a proof.

Proposition 2.1.2. For $f \in \mathcal{H}$, the orthogonal projections $\Pi_n : \mathcal{H} \to f\mathcal{P}_n$ converge to the orthogonal projection $\Pi_{[f]} : \mathcal{H} \to [f]$ in the strong operator topology. Further, if $f, g \in \mathcal{H}$ with $f \not\equiv 0$, and $(p_n^*)_{n \geq 0}$ the optimal approximants to g/f, then $\varphi := \Pi_{[f]}(g)$ is the unique function such that

$$\|p_n^* f - \varphi\|_{\mathcal{H}} \to 0.$$

Proof. Let $u \in \mathcal{H}$ and put $u = \Pi_{[f]}(u) + v$. Then v is orthogonal to $[f]$, and hence orthogonal to $f\mathcal{P}_n$, so $\Pi_n(v) = 0$ for all $n \geq 0$. Since $\cup_n f\mathcal{P}_n$ is dense in $[f]$, given $\epsilon > 0$, there exists N such that $\operatorname{dist}(\Pi_{[f]}(u), f\mathcal{P}_N) < \epsilon$. Then, for all $n \geq N$, we have

$$\begin{aligned}
\|\Pi_{[f]}(u) - \Pi_n(u)\|_{\mathcal{H}} &= \operatorname{dist}(\Pi_{[f]}(u), f\mathcal{P}_n) \\
&\leq \operatorname{dist}(\Pi_{[f]}(u), f\mathcal{P}_N) \\
&< \epsilon.
\end{aligned}$$

Since u was arbitrary, we have that $\Pi_n \to \Pi_{[f]}$ strongly.

Further, take $u = g$ to get $\|\Pi_n(g) - \Pi_{[f]}(g)\|_{\mathcal{H}} = \|p_n^* f - \varphi\|_{\mathcal{H}} \to 0.$ $\square$

Again, note that if g is cyclic, then f is cyclic if and only if $p_n^* f \to g$, where $(p_n^*)_{n \geq 0}$ are the optimal approximants to g/f. We will now make some observations and motivate a few questions surrounding the behavior of optimal approximants.

2.1.3 Truncations of Power Series and Stabilization of OPAs

Let h be analytic on some planar domain containing the origin. We will denote the nth Taylor polynomial of h as

$$T_n(h) := \sum_{k=0}^{n} \frac{h^{(k)}(0)}{k!} z^k.$$

For $f \in \mathcal{H}$ with $f(0) \neq 0$, a first natural guess might be that the optimal approximants to g/f are $T_n(g/f)$. However, it turns out that Taylor polynomials are a poor guess. For example, in the Dirichlet space $\mathcal{D}$, the cyclic function $1 - z$ was studied in [25], and there it was pointed out that

$$\begin{aligned} \|T_n(1/f)f - 1\|_{\mathcal{D}} &= \|(1 + z + \ldots + z^n)(1 - z) - 1\|_{\mathcal{D}} \\ &= \|z^{n+1}\|_{\mathcal{D}} \\ &= n + 1, \end{aligned}$$

which is unbounded as $n \to \infty$. In this case, $T_n(1/f)$ is neither optimal nor provides a sequence that proves f to be cyclic (even though $T_n(1/f)f \to 1$ pointwise in $\mathbb{D}$). Instead of using Taylor polynomials, we ask a couple of more general questions:

(Q1) Given $g \in \mathcal{H}$ and a power series $\varphi(z) = \sum_{k=0}^{\infty} a_k z^k$, can we characterize $f \in \mathcal{H}$ such that the nth optimal polynomial approximants to g/f are given by $T_n(\varphi)$ for all n greater than some $M > 0$?

(Q2) Given $g \in \mathcal{H}$ and supposing p is a polynomial, can we characterize f such that $\Pi_{[f]}(g) = pf$?

We will proceed by first answering these questions when $g = \hat{k}_0$.

2.1.4 The Reproducing Kernel at Zero and Inner Functions

As mentioned previously, much of the existing literature on optimal approximants has been centered around approximating $1/f$ in spaces where 1 is the reproducing kernel at zero. In the present section, we will make a few observations and generalize these results, beginning with the interaction between optimal approximants and inner functions. Again, we point to [23] for further discussion on this topic, where it was first studied.

Proposition 2.1.3. If there is a function in $\mathcal{H}$ that is both cyclic and $\mathcal{H}$-inner, then, up to a unimodular constant, this function is unique, and is the normalized reproducing kernel at zero.

Proof. Let $\theta \in \mathcal{H}$ be cyclic and $\mathcal{H}$-inner. Then for all $h \in \mathcal{H}$, there exist polynomials p_n such that $p_n\theta \to h$ and as θ is $\mathcal{H}$-inner, $\langle p_n\theta, \theta\rangle_{\mathcal{H}} = p_n(0)$. Taking limits, and noting $\theta(0) \neq 0$ by cyclicity, we have $\langle h, \theta\rangle_{\mathcal{H}} = h(0)/\theta(0)$. This implies that $\overline{\theta(0)}\theta$ is the reproducing kernel at zero. Thus, by the Riesz representation theorem, this function is well-defined for any choice of θ and must be k_0. Normalizing $\overline{\theta(0)}\theta$ then concludes the proof. □

In general, the kernel at the origin is always $\mathcal{H}$-inner, but it is not known if it must also be cyclic (hence, the existence hypothesis in the above proposition). Note that in the Dirichlet-type spaces, the functions θ above are just unimodular constants, and $k_0 = 1$ is clearly cyclic. However, as noted previously, in DeBrange-Rovnyak spaces $\mathcal{H}(b)$, unless $b(0) = 0$, the reproducing kernel at zero is non-constant and is given by $\overline{\theta(0)}\theta = 1 - \overline{b(0)}b$. Even in this case, it is not known if the kernel at zero must always be cyclic.

We mention that the optimal approximants to $\hat{k}_0/f$ are non-zero if and only if $f(0) \neq 0$, which informs the hypotheses in the following results.

Lemma 2.1.1. Let $f \in \mathcal{H}$ with $f(0) \neq 0$. Let φ be the orthogonal projection of k_0 onto $[f]$. Then $\varphi/\sqrt{\varphi(0)}$ is $\mathcal{H}$-inner and has norm one.

Proof. Notice that $k_0 - \varphi \perp [f]$ and $[f]$ is shift invariant, so for all $j \geq 1$ we have

$$0 = \langle z^j \varphi, k_0 - \varphi \rangle_{\mathcal{H}} = -\langle z^j \varphi, \varphi \rangle_{\mathcal{H}}.$$

Further, $\langle \varphi, \varphi \rangle_{\mathcal{H}} = \langle k_0, \varphi \rangle_{\mathcal{H}} = \varphi(0)$ which gives $\|\varphi\|_{\mathcal{H}} = \sqrt{\varphi(0)}$. Thus

$$\left\langle \frac{\varphi}{\sqrt{\varphi(0)}}, z^j \frac{\varphi}{\sqrt{\varphi(0)}} \right\rangle_{\mathcal{H}} = \delta_{j0},$$

so $\varphi/\sqrt{\varphi(0)}$ is $\mathcal{H}$-inner with norm one. $\square$

Lemma 2.1.2. Let $f \in \mathcal{H}$ with $f(0) \neq 0$ and let (p_n^*) be the optimal approximants to $\hat{k}_0/f$. Let $\varphi(z) = \sum_{k=0}^{\infty} a_k z^k$ and suppose that $p_n^* = T_n(\varphi)$ for all $n \geq M$. Then $p_n^* = p_M^*$ for all $n \geq M$. That is, $\varphi = p_M^*$.

Proof. By hypothesis, for all $n \geq M$, $\varphi(0) = (p_n^* f)(0) = (p_M^* f)(0)$. Now notice, for all $n \geq M$,

$$\begin{aligned}
\|p_n^* f - p_M^* f\|_{\mathcal{H}}^2 &= \|p_n^* f\|_{\mathcal{H}}^2 - 2\mathrm{Re}\{\langle p_n^* f, p_M^* f \rangle_{\mathcal{H}}\} + \|p_m^* f\|_{\mathcal{H}}^2 \\
&= (p_n^* f)(0) - 2(p_M^* f)(0) + (p_M^* f)(0) \\
&= 0
\end{aligned}$$

Hence, $p_n^* f = p_M^* f$ for all $n \geq M$, and as f is not identically zero, $p_n^* = p_M^*$ for all $n \geq M$.

$\square$

Remark 2.1.1. It should be pointed out that Lemma 2.1.2 says that there are no functions f for which the optimal approximants to $\hat{k}_0/f$ come from truncations of a single power series with finitely many zero coefficients. This lemma can also be seen as a consequence of the simple exercise showing that $\mathrm{dist}^2(\hat{k}_0, f\mathcal{P}_n) = 1 - (p_n^* f)(0)$. This also tells us that for $g = \hat{k}_0$, (Q1) and (Q2) are equivalent. The following definition is now natural.

Definition 2.1.3 (Stabilizing approximants)**.** Let $f, g \in \mathcal{H}$ with g not orthogonal to $[f]$ and let $(p_n^*)_{n\geq 0}$ be the optimal approximants to g/f. Say that the optimal approximants *stabilize* at p_M^* if M is the smallest non-negative integer such that $p_n^* = p_M^*$ for all $n \geq M$.

Lemma 2.1.3. Let $f \in \mathcal{H}$ with $f(0) \neq 0$ and let $(p_n^*)_{n\geq 0}$ be the optimal approximants to $\hat{k}_0/f$. Then f is $\mathcal{H}$-inner (up to a constant multiple) if and only if, for all $n \geq 0$,

$$p_n^* = \frac{\overline{f(0)}}{\|k_0\|\|f\|^2}.$$

Proof. For the forward direction, suppose f is a constant multiple of an $\mathcal{H}$-inner function. For any $n \geq 0$, consider the optimal system for $\hat{k}_0/f$:

$$G_n\vec{x} = \left(\langle \hat{k}_0, f\rangle, 0, \dots, 0\right)^T = \left(\|k_0\|^{-1}\overline{f(0)}, 0, \dots, 0\right)^T.$$

As $\langle f, z^k f\rangle = 0$ for all $k \geq 1$, the entries in the first row and column of G_n, except the (0,0) entry, are all zero. It follows that the inverse of G_n must also satisfy this property. Now, considering $G_n^{-1}\left(\|k_0\|^{-1}\overline{f(0)}, 0, \dots, 0\right)^T$ to recover the coefficients of p_n^*, we see that p_n^* is the constant $\frac{\overline{f(0)}}{\|k_0\|\|f\|^2}$ for any $n \geq 0$.

Now suppose $p_n^*(z) = \frac{\overline{f(0)}}{\|k_0\|\|f\|^2}$ for all $n \geq 0$. Considering the optimal system

$$G_1\left(\frac{\overline{f(0)}}{\|k_0\|\|f\|^2}, 0\right)^T = \left(\frac{\overline{f(0)}}{\|k_0\|}, 0\right)^T$$

quickly yields that $\langle f, zf\rangle_{\mathcal{H}} = 0$. As the coefficients of p_n^* are stable, a simple induction argument then shows that $\langle f, z^k f\rangle_{\mathcal{H}} = 0$ for all $k \geq 1$. Thus, f is a constant multiple of an $\mathcal{H}$-inner function. $\square$

The forward implication of this lemma was given in [23] for spaces where $\hat{k}_0 = 1$. We now give a characterization of stabilizing approximants, which answers (Q2) when $g = \hat{k}_0$.

Theorem 2.1.1. Let $f \in \mathcal{H}$ with $f(0) \neq 0$ and let (p_n^*) be the optimal polynomial approximants to $\hat{k}_0/f$. The following are equivalent, and the smallest M for which each of the statements hold is the same:

1. There exists a function $\varphi(z) = \sum_{k \geq 0} a_k z^k$ such that $p_n^* = T_n(\varphi)$ for all $n \geq M$.

2. The optimal approximants to $\hat{k}_0/f$ stabilize at p_M^*.

3. $p_M^* f$ is the orthogonal projection of $\hat{k}_0$ onto $[f]$.

4. $f = cu/p_M^*$, where $c = \sqrt{(p_M^* f)(0)}$ and u is $\mathcal{H}$-inner with norm one.

Proof. The equivalence of (1) and (2) is given by Lemma 2.1.2 and taking $p_M^* = \varphi$ for the backward implication. The equivalence of (2) and (3) follows by definition. The fact that (3) implies (4) is given by Lemma 2.1.1. The unique minimality of M until now follows by definition and trivial arguments.

Now let us assume (4), putting $p_M^*(z) = \sum_{k=0}^{M} a_k z^k$ and assuming that M is minimal. Then,

$$\begin{aligned} 0 &= \left\langle z \, \frac{p_M^* f}{\sqrt{(p_M^* f)(0)}}, \frac{p_M^* f}{\sqrt{(p_M^* f)(0)}} \right\rangle_{\mathcal{H}} \\ &= \langle z p_M^* f, p_M^* f \rangle_{\mathcal{H}} \\ &= \sum_{k=0}^{M} a_k \langle z^{k+1} f, p_M^* f \rangle_{\mathcal{H}} \\ &= a_M \langle z^{M+1} f, p_M^* f \rangle_{\mathcal{H}} \end{aligned}$$

where the last equality holds by optimality of p_M^*. By the minimality of M, $a_M \neq 0$ so we must have $\langle z^{M+1} f, p_M^* f \rangle_{\mathcal{H}} = 0$. A simple induction argument shows that $\langle z^{M+k} f, p_M^* f \rangle_{\mathcal{H}} = 0$ for all $k \geq 1$. It follows that

$$\langle qf, p_M^* f \rangle_{\mathcal{H}} = q(0) f(0)$$

for all $q \in \mathcal{P}$. In other words, $p_M^* f$ is the orthogonal projection of $\hat{k}_0$ onto $[f]$, i.e., (3) holds. $\square$

As previously mentioned, much effort has gone into understanding the location of zeros of optimal approximants. We end this section by showing that if the kernel at the origin is cyclic, then stable approximants must have zeros which are outside of the open unit disk.

Corollary 2.1.1. Let $f \in \mathcal{H}$ be cyclic and suppose that k_0 is cyclic in $\mathcal{H}$. If the optimal polynomial approximants to $\hat{k}_0/f$ stabilize at p_M^*, then $f = \hat{k}_0/p_M^*$, and p_M^* has no zeros inside $\mathbb{D}$.

Proof. Since f is cyclic, $f(0) \neq 0$. By optimality, we have

$$\langle p_M^* f, qf \rangle_{\mathcal{H}} = \langle \hat{k}_0, qf \rangle_{\mathcal{H}}$$

for all $q \in \mathcal{P}$. As f is cyclic, $\{qf : q \in \mathcal{P}\}$ is dense in $\mathcal{H}$. It follows immediately that $p_M^* f = \hat{k}_0$. Lastly, as $\hat{k}_0$ is assumed cyclic, and therefore zero-free in $\mathbb{D}$, and f is analytic in $\mathbb{D}$, p_M^* must not have any zeros in $\mathbb{D}$. $\square$

Remark 2.1.2. Recall, for a polynomial p, we deonte zero set of p as

$$Z(p) := \{\beta \in \mathbb{C} : p(\beta) = 0\}.$$

It was shown in [27] that in the Dirichlet-type spaces D_α, $Z(p_n^*) \cap \overline{\mathbb{D}} = \emptyset$ when $\alpha \geq 0$ and $Z(p_n^*) \cap \overline{D}(0, 2^{-\alpha/2}) = \emptyset$ when $\alpha < 0$. The above corollary improves this result for $\alpha < 0$ when f is cyclic and has stabilizing approximants. However, it should be noted that, a priori, p_m^* may have zeros on the unit circle.

2.1.5 General Approximants and Stabilization

We now return to the case of approximating some arbitrary g/f with $g, f \in \mathcal{H}$. Recalling the $\frac{1}{m}H^2$ spaces from Example 2.1.1, which serve as one motivation for studying general approximants, we have the following proposition.

Proposition 2.1.4. Let $f \in \frac{1}{m}H^2 \setminus \{0\}$. Put $f = h/m$ with $h \in H^2$. Then the optimal polynomial approximants to $1/f$ in $\frac{1}{m}H^2$ correspond to the optimal polynomial approximants to m/h in H^2.

Proof. Recall that multiplication by m is an isometry from $\frac{1}{m}H^2$ to H^2, and notice that for any polynomial p we have

$$\|pf - 1\|_{\frac{1}{m}H^2} = \|ph - m\|_{H^2}.$$

Minimizing each side of the equality above we see that

$$\left(\Pi_{f\mathcal{P}_n}(1)\right)/f = \left(\Pi_{h\mathcal{P}_n}(m)\right)/h,$$

where the projections on the left and right hand sides above are taken in $\frac{1}{m}H^2$ and H^2, respectively. Lastly, as $f \not\equiv 0$, these projections are unique and represent the optimal approximants. □

We can now reframe questions about cyclicity in $\frac{1}{m}H^2$ as questions in H^2 via general optimal approximants. This is advantageous because H^2 has nicer structural properties than $\frac{1}{m}H^2$ (e.g., the monomials are orthogonal in H^2 but not in $\frac{1}{m}H^2$).

Let us now give some results pertaining to $\mathcal{H}$-inner functions and general optimal approximants. In general, (Q1) and (Q2) are not equivalent. For example, if $f(z) = 1$ and $g(z) = \sum_{k\geq 0} b_k z^k$, then the optimal approximants to g/f are just $T_n(g)$, since $\Pi_{f\mathcal{P}_n}(g) = \Pi_{\mathcal{P}_n}(g) = T_n(g)$.

The remainder of this section aims to provide a stabilization theorem for a certain class of general approximants. We will be able to do so with the help of the following proposition, which deals with the orthogonal complement of the subspace generated by zf. When $f \in H^2$ is inner, these spaces are examples of *model spaces* (see, e.g., [144] for an introduction).

Proposition 2.1.5. Let $f \in \mathcal{H}$ and define

$$\mathcal{K}_{Sf} := \mathcal{H} \ominus [Sf].$$

For any $h \in \mathcal{H}$, we have $h \in \mathcal{K}_{Sf}$ if and only if $\Pi_{[f]}(h) \in \mathcal{K}_{Sf}$. Further, $\hat{k}_0$ and $\Pi_{[f]}(\hat{k}_0)$ are always elements of $\mathcal{K}_{Sf}$.

Proof. Note that $\mathcal{K}_{Sf}$ can also be expressed as

$$\mathcal{K}_{Sf} = \{h \in \mathcal{H} : \langle h, z^k f\rangle_{\mathcal{H}} = 0 \text{ for all } k \geq 1\}.$$

Simply observe that $\langle h, z^k f\rangle_{\mathcal{H}} = \langle h, \Pi_{[f]}(z^k f)\rangle_{\mathcal{H}} = \langle \Pi_{[f]}(h), z^k f\rangle_{\mathcal{H}}$ and that $\langle k_0, z^k f\rangle_{\mathcal{H}} = 0$ for all $k \geq 1$. □

We can now prove the main result of this section.

Theorem 2.1.2. Let $f, g \in \mathcal{H}$ with g not orthogonal to $[f]$. Let (q_n^*) be the optimal approximants to g/f. The following are equivalent, and the smallest M for which each of the statements hold is the same:

1. $g \in \mathcal{K}_{Sf}$ and $\Pi_{[f]}(g) = q_M^* f$.
2. $q_M^* f \in \mathcal{K}_{Sf}$.
3. $q_M^* f / \|q_M^* f\|_{\mathcal{H}}$ is $\mathcal{H}$-inner with norm one and $\langle q_M^* f, z^k f\rangle_{\mathcal{H}} = 0$ for $k = 1, \ldots, M$.

Proof. To see (1) implies (2), note that if $\Pi_{[f]}(g) = q_M^* f$ then $\langle q_M^* f, z^k f\rangle_{\mathcal{H}} = \langle g, z^k f\rangle_{\mathcal{H}}$. So if $g \in \mathcal{K}_{Sf}$, then $\langle q_M^* f, z^k f\rangle_{\mathcal{H}} = 0$ for all $k \geq 1$.

For (2) implies (3), the fact that $\langle q_M^* f, z^k f\rangle_{\mathcal{H}} = 0$ for $k = 1, \dots, M$ follows by definition of $q_M^* f \in \mathcal{K}_{Sf}$. To see $q_M^* f/\|q_M^* f\|_{\mathcal{H}}$ is $\mathcal{H}$-inner, put $q_M^*(z) = \sum_{j=0}^{M} b_k z^j$ and observe, for all $k \geq 1$,

$$\langle q_M^* f, z^k q_M^* f\rangle_{\mathcal{H}} = \sum_{j=0}^{M} \overline{b_j} \langle q_M^* f, z^{j+k} f\rangle_{\mathcal{H}} = 0$$

where the second equality holds because $q_M^* f \in \mathcal{K}_{Sf}$. Thus, $q_M^* f/\|q_M^* f\|_{\mathcal{H}}$ is $\mathcal{H}$-inner. Further, the unique minimality of M in the above statements is immediate.

For (3) implies (1), we use the same idea as the last part of Theorem 2.1.1. Put $q_M^*(z) = \sum_{j=0}^{M} b_k z^j$ and assume M is minimal. Since $q_M^* f/\|q_M^* f\|_{\mathcal{H}}$ is $\mathcal{H}$-inner, we have

$$\begin{aligned} 0 &= \langle z q_M^* f, q_M^* f\rangle_{\mathcal{H}} \\ &= \sum_{j=0}^{M} b_k \langle z^{j+1} f, q_M^* f\rangle_{\mathcal{H}} \\ &= b_M \langle z^{M+1} f, q_M^* f\rangle_{\mathcal{H}} \end{aligned}$$

where the last equality holds by the assumption that $q_m^* f$ is orthogonal to $z^k f$ for $k = 1, \dots, M$. By the minimality of M, $b_M \neq 0$ so we must have $\langle z^{M+1} f, q_M^* f\rangle_{\mathcal{H}} = 0$. A simple induction argument shows that $\langle z^{M+k} f, q_M^* f\rangle_{\mathcal{H}} = 0$ for all $k \geq 1$. Thus, $q_M^* f \in \mathcal{K}_{Sf}$. Further, if $q_M^* f \in \mathcal{K}_{Sf}$, then $g - q_M^* f = g - \Pi_M(g)$ is orthogonal to $[f]$. It follows that $\Pi_{[f]}(g) = q_M^* f$, thus the approximants to g/f stabilize at q_M^*. Lastly, $g \in \mathcal{K}_{Sf}$ since now $\langle q_M^* f, z^k f\rangle_{\mathcal{H}} = \langle g, z^k f\rangle_{\mathcal{H}} = 0$ for all $k \geq 1$. □

When k_0 is cyclic in $\mathcal{H}$, we can also characterize cyclicity in terms of $\mathcal{K}_{Sf}$.

Proposition 2.1.6. Let $f \in \mathcal{H}$ and suppose that k_0 is cyclic in $\mathcal{H}$. Then f is cyclic if and only if $\mathcal{K}_{Sf} = \text{span}\{k_0\}$.

Proof. Suppose f is cyclic. Then for any $h \in \mathcal{H}$, we can find polynomials (p_n) so that $p_n f \to h$. Letting $g \in \mathcal{K}_{Sf}$ we have

$$\begin{aligned} \langle g, h \rangle_{\mathcal{H}} &= \lim_{n\to\infty} \langle g, p_n f \rangle_{\mathcal{H}} \\ &= \lim_{n\to\infty} \overline{(p_n f)(0)} \, \langle g, 1 \rangle_{\mathcal{H}} \\ &= \overline{h(0)} \, \langle g, 1 \rangle_{\mathcal{H}}. \end{aligned}$$

Thus, g reproduces, up to a constant, the value of h at zero so $g \in \operatorname{span}\{k_0\}$.

Conversely, let $\mathcal{K}_{Sf} = \operatorname{span}\{k_0\}$. Since $\Pi_{[f]}(k_0) \in \mathcal{K}_{Sf}$, there exists some constant λ so that $\Pi_{[f]}(k_0) = \lambda k_0$. This means that the cyclic function $k_0 \in [f]$ so f is cyclic. $\square$

One may compare this with the well-known "codimension one" property of shift invariant subspaces (e.g., see [134]).

2.1.6 Projections of Unity

In light of Proposition 2.1.5, we will compute $\Pi_{[f]}(1)$ (i.e., a projection of unity) when $f \in \mathcal{P} \subset H^2_w$. Note that in our definition of H^2_w from Section 1.1.2, we have $\hat{k}_0 = 1$. As we will see, these projections are linear combinations of reproducing kernels. This idea goes back to a construction of Shapiro and Shields in the Bergman space [139] involving certain Gram determinants, later modified by the authors in [23] to produce examples of H^2_w-inner functions. The inner functions constructed there are associated to a finite set of distinct points in the open unit disk. We would like to generalize this theory by considering finite sets of points in the plane with any multiplicity. This motivates the following definition, which will be discussed more thoroughly in Section 2.2.3.

Definition 2.1.4 (Reproducible point/order)**.** Say $\beta \in \mathbb{C}$ is a reproducible point of order

$m \in \mathbb{Z}_{\geq 0}$ in H^2_w if the linear functional

$$p \mapsto p^{(m)}(\beta)$$

extends from $\mathcal{P}$ to a bounded linear functional on H^2_w. If no such m exists, say that β is not reproducible. Denote the collection of reproducible points of order m for H^2_w as $\Omega_m(H^2_w)$.

Notice that $\Omega_0(H^2_w)$ is just the set of points for which point evaluation is bounded in H^2_w. Since H^2_w is a reproducing kernel Hilbert space on $\mathbb{D}$, we always have $\mathbb{D} \subseteq \Omega_0(H^2_w)$. But $\Omega_0(H^2_w)$ could be a strictly larger set. For example, a routine exercise shows that when $\alpha > 1$, $\Omega_0(\mathcal{D}_\alpha) = \overline{\mathbb{D}}$ and $\Omega_m(\mathcal{D}_\alpha) \subseteq \overline{\mathbb{D}}$ for all $m \geq 1$ (the proper inclusion depending on m and α). If $|\beta| > 1$, then β is not reproducible in $\mathcal{D}_\alpha$. In H^2, $\Omega_m(H^2) = \mathbb{D}$ for all m.

We need one more observation and lemma before stating our last theorem. Let $s_\beta(z) = \frac{1}{1-\overline{\beta}z}$ denote the Szegő kernel, which is the reproducing kernel in H^2. Let $s_\beta^{(n)}$ denote the n-th derivative of s_β and let s_β^n denote the reproducing kernel for n-th derivatives in H^2, i.e., $\langle f, s_\beta^n \rangle_{H^2} = f^{(n)}(\beta)$ for all $f \in H^2$. Such an element exists because f is assumed to be analytic, and is unique by the Riesz representation theorem. A simple exercise shows that

$$s_\beta^{(n)}(z) = \sum_{j \geq 0} (j+1)(j+2)\dots(j+n)\overline{\beta}^{j+n} z^j$$

and

$$s_\beta^n(z) = \sum_{k \geq 0} j(j-1)\dots(j-n+1)\overline{\beta}^{j-n} z^j.$$

Further, in H^2_w, we have

$$k_\beta^{(n)}(z) = \sum_{j \geq 0} (j+1)(j+2)\dots(j+n)\frac{\overline{\beta}^{j+n} z^j}{w_j}$$

and

$$k_\beta^n(z) = \sum_{j\geq 0} j(j-1)\dots(j-n+1)\frac{\overline{\beta}^{j-n}z^j}{w_j}.$$

Let us now relate the Maclaurin series coefficients of $s_\beta^{(n)}$ and s_β^n.

Lemma 2.1.4. Let $F_0(j) = P_0(j) = 1$. For each $N \in \mathbb{Z}^+$, define $F_N(j) := \prod_{n=1}^{N}(j+n)$ and $P_N(j) := \prod_{n=0}^{N-1}(j-n)$. Then $F_N \in \operatorname{span}\{P_0, \dots, P_N\}$ for all $N \in \mathbb{Z}^+ \cup \{0\}$.

Proof. We will proceed by induction. Let $N = 1$ and observe $F_1(j) = j+1 = P_1(j) + P_0(j)$, so the base case holds. Now suppose $F_N \in \operatorname{span}\{P_0, \dots, P_N\}$ and note that $F_{N+1}(j) = (j+N+1)F_N(j)$. By the induction hypothesis, we can find constants c_i such that

$$\begin{aligned} F_{N+1}(j) &= (j+N+1)F_N(j) \\ &= j\sum_{i=0}^{N} c_i P_i(j) + (N+1)\sum_{i=0}^{N} c_i P_i(j). \end{aligned}$$

Observe, for any $n \geq 0$, that $jP_n(j) = (j-n)P_n(j) + nP_n(j) = P_{n+1}(j) + nP_n(j)$. Hence, $jP_n \in \operatorname{span}\{P_0, \dots, P_{n+1}\}$ and also $j\sum_{i=0}^{N} c_i P_i \in \operatorname{span}\{P_0, \dots, P_{N+1}\}$. Thus, $F_{N+1} \in \operatorname{span}\{P_0, \dots, P_{N+1}\}$. $\square$

Remark 2.1.3. The purpose of this lemma, as an immediate corollary, is that

$$s_\beta^{(n)} \in \operatorname{span}\{s_\beta, s_\beta^1, \dots, s_\beta^n\}.$$

We may now state and prove our final theorem.

Theorem 2.1.3. Let $f \in H_w^2$ be a monic polynomial with $f(0) \neq 0$. Suppose f has zeros $\beta_1, \dots, \beta_r$ with multiplicities $m_1, \dots, m_r$, respectively. Let $Z_j := \{\beta_i \in Z(f) \cap \Omega_j : m_i > j\}$ be the set of zeros of f that are reproducible of order j and have multiplicity greater than j. Let $I_j := \{i \in \{1, \dots, r\} : \beta_i \in Z_j\}$ be the set of indices appearing in Z_j. Let $R := \max(\{j :$

$Z_j \neq \emptyset\}$) be the largest value of j such that Z_j is non-empty. Let φ be the orthogonal projection of 1 onto $[f]$. Then

$$\varphi(z) = 1 + \sum_{j=0}^{R} \sum_{i \in I_j} C_{i,j} k_{\beta_i}^{j}(z),$$

where k_β^i denotes the reproducing kernel for i-th derivatives in H_w^2 at β and $C_{i,j}$ are constants determined by $\langle \varphi, k_{\beta_i}^j \rangle_w = 0$ for each $i \in I_j$ and $0 \leq j \leq R$.

Proof. Put $f(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_0$ and denote the Maclaurin coefficients of φ as $\varphi_n = \langle \varphi, z^n \rangle_w / \|z^n\|_w^2$. Since $\varphi \in \mathcal{K}_{Sf}$, we have $\langle \varphi, z^{n+d} + a_{d-1}z^{n+d-1} + \cdots + a_0 z^n \rangle_w = 0$ for all $n \geq 1$. This gives the recurrence relation

$$w_{n+d}\varphi_{n+d} = \sum_{j=0}^{d-1} -w_{n+j}\ \overline{a}_j\ \varphi_{n+j}.$$

Now let us use $\Phi_n := w_n \varphi_n$ to obtain the constant coefficient recurrence relation

$$\Phi_{n+d} = \sum_{j=0}^{d-1} -\overline{a}_j \Phi_{n+j}.$$

We will now find the generating function $\Phi(z)$ (viewed as a *formal* power series) by summing over all n (see, e.g., [79, Chapter 2] for more on solving recurrence relations and generating functions):

$$\begin{aligned}
\Phi(z) &= p(z) + \sum_{n \geq 0} -\overline{a_{d-1}}\Phi_{n-1}z^n + \cdots + \sum_{n \geq 0} -\overline{a_0}\Phi_{n-d}z^n \\
&= p(z) - \overline{a_{n-1}}z\Phi(z) - \cdots - \overline{a_0}z^d\Phi(z).
\end{aligned}$$

where p is a polynomial of degree d given by the initial conditions of the relation. Solving for $\Phi(z)$ gives

$$\begin{aligned}\Phi(z) &= \frac{p(z)}{1+\overline{a_{d-1}}z+\dots\overline{a_0}z^d} \\ &= \frac{p(z)}{z^d\overline{f(1/\overline{z})}} \\ &= \frac{p(z)}{\prod_{i=1}^r(1-\overline{\beta_i}z)^{m_i}}.\end{aligned}$$

After doing long division (because $\deg p = d$) and using partial fractions, with constants C and $c_{i,j}$, we may put

$$\begin{aligned}\Phi(z) &= C+\sum_{i=1}^r\sum_{j=1}^{m_i}\frac{c_{i,j}}{(1-\overline{\beta_i}z)^j} \\ &= C+\sum_{i=1}^r\sum_{j=1}^{m_i}\frac{c_{i,j}}{\overline{\beta_i}^{j-1}(j-1)!}\ D^{j-1}\left(\frac{1}{1-\overline{\beta_i}z}\right),\end{aligned}$$

where D is the derivative operator with respect to z. Putting $\tilde{C}_{i,j}=\frac{c_{i,j}}{\overline{\beta_i}^{j-1}(j-1)!}$ and substituting in with terms of $s_\beta^{(j)}$, we get

$$\Phi(z)=C+\sum_{i=1}^r\sum_{j=1}^{m_i}\tilde{C}_{i,j}s_{\beta_i}^{(j-1)}(z).$$

By Lemma 2.1.4, we can find constants $C_{i,j}$ such that

$$\Phi(z)=C+\sum_{i=1}^r\sum_{j=1}^{m_i}C_{i,j}s_{\beta_i}^{j-1}(z).$$

The upshot of going through the trouble of writing Φ in this way is that when substituting back in with φ_n, each term of the form $s_{\beta_i}^{j-1}$ becomes $k_{\beta_i}^{j-1}$. Doing so, we find the formal power series

$$\tilde{\varphi}(z)=C+\sum_{i=1}^r\sum_{j=1}^{m_i}C_{i,j}k_{\beta_i}^{j-1}(z).$$

In order to find φ, we must determine which terms above converge in H^2_w. This is precisely when $\beta_i \in Z_j$, for appropriate i, j. Namely,

$$\varphi(z) = C + \sum_{j=0}^{R} \sum_{i \in I_j} C_{i,j} k^j_{\beta_i}(z).$$

Lastly, the claim about the constants follows by noting that any function in $[f]$ must vanish, with proper multiplicity, at the reproducible zeros of f. If we let $F := \sum_{j=0}^{R} \sum_{i \in I_j} C_{i,j} k^j_{\beta_i}$ and note that $\Pi_{[f]} F = 0$, then $\varphi = \Pi_{[f]}\varphi = \Pi_{[f]} C + \Pi_{[f]} F = C\varphi$, so $C = 1$. As $\varphi \in [f]$, the other constants $C_{i,j}$ can also be determined by using the fact that $\varphi^{(j)}(\beta) = \langle \varphi, k^j_\beta \rangle_w = 0$ for $\beta \in Z_j$. □

Remark 2.1.4. The above theorem shows something stronger than what is stated; we have actually shown that $\mathcal{K}_{Sf} = \text{span}\{1, k^j_\beta : \beta \in Z_j\}$. Example 2.1.4 below gives an explicit linear system whose solution gives the constants appearing in φ when f has simple zeros. An immediate corollary of the above theorem is that if $f, q \in \mathcal{P} \subset H^2_w$ with $Z(q) \cap (\cup_{m \geq 0} \Omega_m) = \emptyset$, then $\Pi_{[f]}(1) = \Pi_{[qf]}(1)$. This also tells us that the optimal approximants to $1/f$ and $1/(qf)$ form an equivalence class with respect to the limit of their approximants. That is, the equivalence $f \sim h$ if and only if $\Pi_{[f]}(1) = \Pi_{[h]}(1)$. We will call this the Roman equivalence relation; the approximants of two different functions in the same equivalence class travel along different roads, but end up in the same place. This observation, along with Theorem 2.1.3, motivates much of the work in Section 2.2. Another observation worth noting is that $\text{dist}^2(1, [f]) = 1 - \varphi(0) = \sum_i C_{i,0}$. This is due to the fact that $k_\beta(0) = 1$ and $k^j_\beta(0) = 0$ for all $j \geq 1$.

2.1.7 Examples, Further Questions, and Discussion

An immediate corollary of Theorem 2.1.3 is that a polynomial is cyclic in $\mathcal{H}$ if and only if it has no reproducible zeros. A natural desire would be to extend Theorem 2.1.3 to any function, not just a polynomial. Such an extension could possibly provide new information helpful for understanding cyclicity.

We also mention again the hypotheses of Proposition 2.1.3; it is not known if the existence assumption is needed. Namely, one may ask, what additional hypotheses, if any, are required of $\mathcal{H}$ so that k_0 is cyclic?

We conclude this section with a couple of examples. It is well known that for a function $f \in H^2$, $\Pi_{[f]}(1)$ is precisely the inner part of f. The first of these examples communicates this by applying Theorem 2.1.3.

Example 2.1.3. Let us consider $f(z) = \prod_{i=1}^{d}(z-\beta_i) \in H^2$ with $f(0) \neq 0$. Let $\Omega = Z(f) \cap \mathbb{D}$. Since $\Omega_m(H^2) = \mathbb{D}$ for all $m \geq 0$, we know from Theorem 2.1.3 that $\varphi := \Pi_{[f]}1$ is given by

$$\varphi(z) = \frac{p(z)}{\prod_{\beta \in \Omega}(1-\overline{\beta}z)}.$$

We also know that p must vanish at each $\beta \in \Omega$ so we get, for some constant C,

$$\varphi(z) = C \prod_{\beta \in \Omega} \frac{(z-\beta)}{(1-\overline{\beta}z)}.$$

The fact that $\langle \varphi, \varphi \rangle = \varphi(0)$ implies

$$|C|^2 \left\| \prod_{\beta \in \Omega} \frac{(z-\beta)}{(1-\overline{\beta}z)} \right\|_{H^2}^2 = C \prod_{\beta \in \Omega} \left(-\overline{\beta}\right).$$

In turn, we have $C = \prod_{\beta \in \Omega}(-\overline{\beta})$. This gives φ as a multiple of a familiar Blaschke product (an H^2-inner function):

$$\varphi(z) = \prod_{\beta \in \Omega}(-\overline{\beta}) \frac{(z-\beta)}{(1-\overline{\beta}z)}.$$

This also tells us that $\operatorname{dist}^2(1, [f]) = 1 - \varphi(0) = 1 - \prod_{\beta \in \Omega} |\beta|^2$. Further, when Ω is empty, we have that $\varphi \equiv 1$, and f is cyclic.

Our last example is another application of Theorem 2.1.3, which points out a computational improvement; finding an optimal approximant requires solving a linear system, while, in the polynomial case, invoking Theorem 2.1.3 allows us to compute the *limit* of optimal approximants by solving a linear system.

Example 2.1.4. Suppose $f \in H^2_w$ is a monic polynomial with simple zeros and $f(0) \neq 0$. Let $\{\beta_i\}_1^d = Z(f) \cap \Omega_0(H^2_w)$. Theorem 2.1.3 says the orthogonal projection of 1 onto $[f]$ is given by

$$\varphi(z) = 1 + \sum_{i=1}^{d} C_i k_{\beta_i}(z)$$

for some constants C_i. Since φ vanishes at each β_i, we get, for $1 \leq j \leq d$,

$$0 = \varphi(\beta_j) = 1 + \sum_{i=1}^{d} C_i k_{\beta_i}(\beta_j) = 1 + \sum_{i=1}^{d} C_i \langle k_{\beta_i}, k_{\beta_j} \rangle_w.$$

Moving the independent term 1 to the left-hand side in each equation expressed above gives the linear system

$$\left(\langle k_{\beta_i}, k_{\beta_j} \rangle_w \right)_{1 \leq i,j \leq d} (C_1, \ldots, C_d)^T = (-1, \ldots, -1)^T.$$

2.2 Analogues of Finite Blaschke Products

The present section consists of joint work with Trieu Le, which can be found in preprint form in [81], and will appear in *The Bulletin of the London Mathematical Society*. We give a generalization of the notion of finite Blaschke products from the perspective of generalized inner functions in various reproducing kernel Hilbert spaces. Further, we study precisely how these functions relate to the so-called Shapiro–Shields functions and shift-invariant subspaces generated by polynomials.

2.2.1 Preliminaries

Throughout this section, $\mathcal{H}$ will denote a reproducing kernel Hilbert space of analytic functions on some planar domain $\Omega \subset \mathbb{C}$ with $0 \in \Omega$. Again, we also ask that $\mathcal{H}$ has the following properties:

(i) The forward shift operator S, given by $f(z) \mapsto zf(z)$, is bounded on $\mathcal{H}$.

(ii) The analytic polynomials $\mathcal{P}$ form a dense subset of $\mathcal{H}$.

We begin with a proposition whose backward direction is a generalization of Lemma 2.1.1.

Proposition 2.2.1. Let $f \in \mathcal{H}$ and suppose $d = \mathrm{ord}_0(f)$. Then f is $\mathcal{H}$-inner if and only if f is a non-zero constant multiple of $\Pi_{[f]}(k_0^{(d)})$. Moreover, $\Pi_{[f]}(k_0^{(d)})$ is always an $\mathcal{H}$-inner function.

Proof. Without loss of generality, suppose $\|f\| = 1$ and consider the forward implication. For all analytic polynomials p, we have

$$\begin{aligned}\left\langle pf, \overline{f^{(d)}(0)}f - \Pi_{[f]}(k_0^{(d)})\right\rangle &= p(0)f^{(d)}(0) - \sum_{k=0}^{d}\binom{d}{k}p^{(k)}(0)f^{(d-k)}(0)\\ &= p(0)f^{(d)}(0) - p(0)f^{(d)}(0) = 0.\end{aligned}$$

Thus, $\overline{f^{(d)}(0)}f = \Pi_{[f]}(k_0^{(d)})$.

Conversely, suppose $f = c\Pi_{[f]}(k_0^{(d)})$ for some non-zero constant c. Then, for $k \geq 1$, we have

$$\langle z^k f, f\rangle = \langle z^k f, c\Pi_{[f]}(k_0^{(d)})\rangle = \bar{c}\langle z^k f, k_0^{(d)}\rangle = 0$$

since $z^k f(z)$ vanishes at the origin with order at least $d+1$. Thus, f is $\mathcal{H}$-inner. Further, this shows that $\Pi_{[f]}(k_0^{(d)})$ is alway $\mathcal{H}$-inner. $\square$

In [50], the authors conducted a robust exploration of inner functions. It was shown there [50, Proposition 3.1] that every inner function is given by $\Pi_{[Sf]^\perp}(f)$. We show here that, up to a constant, this function is the same as a projection of a kernel onto a shift invariant subspace.

Proposition 2.2.2. Let $f \in \mathcal{H}$ and let $d = \mathrm{ord}_0(f)$. Put

$$J = f - \Pi_{[Sf]}(f) = \Pi_{[Sf]^\perp}(f)$$

and $v = \Pi_{[f]}(k_0^{(d)})$. Then we have

$$v = \frac{v^{(d)}(0)}{f^{(d)}(0)} J.$$

Proof. Note that any element of $[Sf]$ vanishes at the origin with multiplicities at least $d+1$. It follows that $k_0^{(d)} \perp [Sf]$ and hence $v \perp [Sf]$, which implies that $v \in [f] \ominus [Sf]$. On the other hand, we also have $J \in [f] \ominus [Sf]$. Since $[f] \ominus [Sf]$ is a one dimensional space, we conclude that $v = \lambda J$ for some constant λ. To find the constant λ, let us compute the inner product

$$J^{(d)}(0) = \langle J, k_0^{(d)} \rangle = \langle f, k_0^{(d)} \rangle - \langle \Pi_{[Sf]}(f), k_0^{(d)} \rangle = f^{(d)}(0)$$

because $k_0^{(d)} \perp [Sf]$. It then follows that

$$\lambda = \frac{\langle v, k_0^{(d)} \rangle}{\langle J, k_0^{(d)} \rangle} = \frac{v^{(d)}(0)}{f^{(d)}(0)}$$

and the conclusion follows. $\square$

In [139], Shapiro and Shields used Gram determinants to produce linear combinations of reproducing kernels that are Dirichlet-inner functions. This was then generalized by

Bénéteau et al. in [23], and further by Le [114] to weighted Hardy spaces over the unit disk. Surprisingly, as we will see later, even in general RKHSs in which monomials are not necessarily orthogonal, such a construction (Definition 2.2.5) is the only way to produce inner functions that are linear combinations of kernels (see Theorem 2.2.2).

2.2.2 Gram Determinants

Let $v_1, \ldots, v_n$ be vectors in an inner product space. We will denote the associated *Gram matrix* by

$$G(v_1, \ldots, v_n) = (\langle v_i, v_j \rangle)_{1 \le i,j \le n} = \begin{bmatrix} \langle v_1, v_1 \rangle & \ldots & \langle v_1, v_n \rangle \\ \vdots & \ddots & \vdots \\ \langle v_n, v_1 \rangle & \ldots & \langle v_n, v_n \rangle \end{bmatrix}.$$

The *Gram determinant* is then $\det(G(v_1, \ldots, v_n))$. Note that for any $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$,

$$\langle G(v_1, \ldots, v_n)x, x \rangle = \left\langle \sum_{i=1}^{n} x_i v_i, \sum_{j=1}^{n} x_j v_j \right\rangle \ge 0.$$

Hence, every Gram matrix is positive semidefinite. Moreover, the vectors $\{v_1, \ldots v_n\}$ are linearly independent if and only if $G(v_1, \ldots, v_n)$ has full rank, and equivalently, if and only if $\det(G(v_1, \ldots, v_n)) > 0$.

Similarly, for any vector u, we define

$$D(u; v_1, \ldots, v_n) := \det \begin{bmatrix} u & \langle u, v_1 \rangle & \ldots & \langle u, v_n \rangle \\ v_1 & \langle v_1, v_1 \rangle & \ldots & \langle v_1, v_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ v_n & \langle v_n, v_1 \rangle & \ldots & \langle v_n, v_n \rangle \end{bmatrix}.$$

Note that $D(u; v_1, \ldots, v_n)$ is a linear combination of $u, v_1, \ldots, v_n$.

For our purposes, it is critical to note that $D(u; v_1, \ldots, v_n)$ is orthogonal to each $v_j, 1 \le$

$j \leq n$. When $\delta := \det(G(v_1, \dots, v_n)) > 0$, we have that

$$u - \delta^{-1} D(u; v_1, \dots, v_n)$$

is a linear combination of the vectors $\{v_1, \dots, v_n\}$, since the coefficient of u in $D(u; v_1, \dots, v_n)$ is δ. Consequently, we have the following well-known lemma.

Lemma 2.2.1. Let $v_1, \dots, v_n$ be linearly independent vectors in an inner product space $\mathcal{V}$. Then for any $u \in \mathcal{V}$,

$$\frac{D(u; v_1, \dots, v_n)}{\det(G(v_1, \dots, v_n))}$$

is the orthogonal projection of u onto $(\operatorname{span}\{v_1, \dots, v_n\})^{\perp}$.

We note here that for any vectors $u, v_1, \dots, v_n$ with $n \geq 2$, the coefficient of v_n in $D(u; v_1, \dots, v_n)$ is exactly $-\big\langle D(u; v_1, \dots, v_{n-1}), v_n \big\rangle$.

For a set of distinct points $\beta_1, \dots, \beta_n \subset \mathbb{D}$, the authors in [23] coined the *Shapiro–Shields function* as the normalization of $D(1; k_{\beta_1}, \dots, k_{\beta_n})$, where the inner product is taken in some H^2_w. This follows the construction of Shapiro and Shields in [139]. They also showed that this function is always inner.

2.2.3 Reproducible Points

We now return to the theory of reproducible points, as introduced in Section 2.1.6. Again, it is important to note that the spaces in which we are working are RKHSs on Ω but may also have kernels that extend to points outside of Ω. For example, the linear functional of point evaluation extends boundedly to the unit circle in the Dirichlet-type spaces when $\alpha > 1$ (more on this below in Example 2.2.1).

It is our aim to uncover the relationship between $\mathcal{H}$-inner functions, linear combinations of reproducing kernels, generalized Shapiro–Shields functions, and projections of kernels onto shift-invariant subspaces generated by polynomials. If $\beta \notin \Omega$, but point evaluation is bounded at β, we can still make sense of, for example, the function $D(u; k_\beta)$ when $u \in \mathcal{H}$. We recall the framework to handle such points.

Definition 2.2.1 (Reproducible point/order)**.** Say $\beta \in \mathbb{C}$ is a reproducible point of order m in $\mathcal{H}$ if the linear functional

$$p \mapsto p^{(m)}(\beta)$$

extends from $\mathcal{P}$ to a bounded linear functional on $\mathcal{H}$.

It is evident that any $\beta \in \Omega$ is reproducible of order m for all $m \geq 0$. On the other hand, points outside of Ω may only be reproducible up to a certain finite order. First we establish a simple fact about the orders of a reproducible point.

Lemma 2.2.2. If β is reproducible of order m, then it is also reproducible of all orders $0 \leq j \leq m$.

Proof. By assumption, there exists a constant $C_m > 0$ such that for all polynomials q, one has

$$|q^{(m)}(\beta)| \leq C_m \|q\|.$$

For any $0 \leq j \leq m$, since multiplication by z is bounded, there exists a constant $B_j > 0$ such that $\|(z-\beta)^{m-j} p\| \leq B_j \|p\|$ for all polynomials p. On the other hand,

$$(m-j)! \binom{m}{j} p^{(j)}(\beta) = \frac{d^m}{dz^m} \left((z-\beta)^{m-j} p(z) \right) \Big|_{z=\beta}.$$

It then follows that

$$|p^{(j)}(\beta)| \leq \frac{C_m B_j}{(m-j)! \binom{m}{j}} \|p\|.$$

Therefore the map $p \mapsto p^{(j)}(\beta)$ extends to a bounded linear functional on $\mathcal{H}$. $\square$

Definition 2.2.2. Let $\beta \in \mathbb{C}$ be reproducible of some order. Define the reproducible order of β (in $\mathcal{H}$) as

$$\mathrm{ro}(\beta) := \sup_m \{p \mapsto p^{(m)}(\beta) \text{ extends as a bounded functional from } \mathcal{P} \text{ to } \mathcal{H}\}.$$

Example 2.2.1 (D_α, $\alpha > 1$)**.** Consider the Dirichlet-type spaces $\mathcal{D}_\alpha$. Recall that monomials have norm $\|z^n\|_\alpha = (n+1)^{\alpha/2}$ and that for $|\beta| < 1$,

$$k_\beta(z) = \sum_{n\geq 0} \frac{\bar{\beta}^n}{(n+1)^{\alpha/2}} \frac{z^n}{(n+1)^{\alpha/2}} = \sum_{n\geq 0} \frac{(\bar{\beta}z)^n}{(n+1)^\alpha}.$$

For $\alpha > 1$, it is evident that k_β is a function in D_α even for $|\beta| = 1$, which implies that all points on the unit circle are reproducible points.

In addition, it is well known that for $|\beta| < 1$, the linear functional given by $f \mapsto f^{(m)}(\beta)$ has a reproducing kernel given by

$$k_\beta^{(m)}(z) = \frac{\partial^m}{\partial \bar{\beta}^m} k_\beta(z) = \sum_{n\geq m} \frac{n(n-1)\cdots(n-m+1)}{(n+1)^\alpha} \bar{\beta}^{n-m}\, z^n.$$

The above series expansion shows that $k_\beta^{(m)}$ belongs to D_α for some (and hence all) $|\beta| = 1$ if and only if $\alpha > 2m+1$. As a consequence, all points on the unit circle are reproducible of order r in D_α, where r is the largest natural number strictly less than $\frac{\alpha-1}{2}$.

Example 2.2.2 (Local Dirichlet spaces)**.** Let $\zeta \in \mathbb{T}$ and let δ_ζ denote the Dirac measure at ζ. One may form the *local Dirichlet space at* ζ

$$\mathcal{D}_{\delta_\zeta} := \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^2 \frac{1-|z|^2}{|z-\zeta|^2} dA(z) < \infty \right\}$$

where dA is normalized area measure on $\mathbb{D}$. It is well known that the polynomials are dense in $\mathcal{D}_{\delta_\zeta}$ and that it is a reproducing kernel Hilbert space on $\mathbb{D}$. Additionally, $\mathcal{D}_{\delta_\zeta}$ has the property that point evaluation is bounded at $\zeta \in \mathbb{T}$ but not at any other point on $\mathbb{T}$ (see [77, Theorems 7.2.1 and 8.1.2 (ii)]).

Example 2.2.3 (L regions)**.** Let Δ be an infinite sequence of disjoint closed discs whose centers lie on the positive real axis and decrease monotonically to zero. By deleting Δ from $\mathbb{D}$, one obtains an infinitely connected region, known as an L region (see [122, 161]). The origin is a boundary point of the region, and in [122] it was shown that for certain reproducing kernel Hilbert spaces of analytic functions on the region, where the polynomials are dense, point evaluation is bounded at the origin, dependent on the rate of decay of the radii of the disks in Δ. Uniformly perturbing the disks of Δ to the right by a fixed positive amount $\epsilon > 0$, we obtain an infinitely connected L like region containing zero, where the results describing bounded point evaluation at the origin hold then for the boundary point ϵ. We communicate this example to highlight that we need no hypotheses on the connectedness of Ω and that there is interesting reproducible behavior in this case.

The previous examples show that spaces with bounded point evaluation can behave very differently, and point evaluation can extend outside of the domain Ω in various ways.

As we will see, Shapiro–Shields functions can be viewed as projections of a kernel at zero onto certain shift invariant subspaces generated by polynomials. Consequently, we would like to connect reproducibility with the zeros of a polynomial.

Definition 2.2.3 (Reproducible zeros)**.** Let $p \in \mathcal{P}$. The multiset of reproducible zeros of p (in $\mathcal{H}$) is

$$R(p) := \{\beta \in \mathbb{C} : p^{(m)}(\beta) = 0 \text{ and } \beta \text{ is reproducible of order } m \text{ in } \mathcal{H}\},$$

listed with multiplicity.

Namely, by multiset and "listed with multiplicity," we require that if p has a zero of order m at β, then β appears in $R(p)$ with multiplicity $\min\{\mathrm{ro}(\beta), m\}$. For example, if the point 1 is reproducible of order 2, but not of order 3 (i.e. $\mathrm{ro}(1) = 2$), and the point πi is not reproducible in $\mathcal{H}$, then for $p(z) = z(z-1)^3(z-\pi i)$, we have $R(p) = \{0, 1, 1\}$. Although the multiset of reproducible zeros of a polynomial depends on $\mathcal{H}$, for convenience, we have not included $\mathcal{H}$ in our notation. However, all use of this notation will be clear.

We would also like to study the multisets of reproducible points that coincide with reproducible zero multisets of polynomials. The following definition allows us to do that.

Definition 2.2.4 (Reproducible multiset)**.** A finite multiset Z is a reproducible multiset (for $\mathcal{H}$) if it can be written as

$$Z = \Big\{ \underbrace{0 = \beta_0, \ldots, \beta_0}_{m_0 \text{ times}}, \underbrace{\beta_1, \ldots, \beta_1}_{m_1 \text{ times}}, \ldots, \underbrace{\beta_s, \ldots, \beta_s}_{m_s \text{ times}} \Big\}$$

where each β_j is a distinct reproducible point, $\beta_0 = 0$ appears with multiplicity m_0 (possibly zero), and for each $1 \leq j \leq s$, β_j appears with at least multiplicity 1, but no multiplicity higher than its reproducible order (in $\mathcal{H}$), i.e. $1 \leq m_j \leq \mathrm{ro}(\beta_j)$.

Note that for any polynomial p, $R(p)$ is a reproducible multiset. We can now make a full generalization of the Shapiro–Shields function.

Definition 2.2.5 (Shapiro–Shields function)**.** Let Z be a reproducible multiset and put

$$Z = \Big\{ \underbrace{0 = \beta_0, \ldots, \beta_0}_{m_0 \text{ times}}, \underbrace{\beta_1, \ldots, \beta_1}_{m_1 \text{ times}}, \ldots, \underbrace{\beta_s, \ldots, \beta_s}_{m_s \text{ times}} \Big\}.$$

The *Shapiro–Shields function* associated to Z is then defined as

$$\S_Z = D(k_0^{(m_0)}; k_0^{(m_0-1)}, \ldots, k_0, k_{\beta_1}^{(m_1-1)}, \ldots, k_{\beta_1}, k_{\beta_s}^{(m_s-1)}, \ldots, k_{\beta_s}).$$

It is imperative to note that $\S_Z$ vanishes at each β_j with multiplicity m_j when $0 \le j \le s$. We would also like to view a Shapiro–Shields function as an orthogonal projection of a reproducing kernel at the origin onto the orthogonal complement of the span of some other kernels. In order to do this, along with use in later applications, we need the following lemma.

Lemma 2.2.3. Let $\beta_1, \dots, \beta_s$ be distinct reproducible points of $\mathcal{H}$. If $m_1, \dots, m_s$ are non-negative integers such that $m_j \le \mathrm{ro}(\beta_j)$ for $1 \le j \le s$, then the set

$$\left\{ k_{\beta_j}^{(\ell)} : \quad 0 \le \ell \le m_j,\ 1 \le j \le s \right\}$$

is linearly independent in $\mathcal{H}$.

Proof. Suppose $\{c_{j,\ell} :\ 1 \le j \le s,\ 0 \le \ell \le m_j\}$ are complex numbers such that

$$\sum_{j=1}^{s} \sum_{\ell=0}^{m_j} c_{j,\ell}\, k_{\beta_j}^{(\ell)} = 0$$

in $\mathcal{H}$. We need to prove that $c_{j,\ell} = 0$ for all such j and ℓ. It suffices to show $c_{j,m_j} = 0$ for all $1 \le j \le s$. Fix such an index j. Define the polynomial

$$p(z) = (z - \beta_j)^{m_j} \cdot \prod_{t \ne j} (z - \beta_t)^{m_t + 1}.$$

Note that $p^{(\ell)}(\beta_t) = 0$ for all $0 \le \ell \le m_t$ and $1 \le t \le s$ with $t \ne j$. On the other hand, $p^{(m_j)}(\beta_j) \ne 0$ but

$$p^{(\ell)}(\beta_j) = 0 \text{ for all } 0 \le \ell < m_j.$$

It follows that

$$\bar{c}_{j,m_j} p^{(m_j)}(\beta_j) = \Big\langle p,\ \sum_{j=1}^{s} \sum_{\ell=0}^{m_j} c_{j,\ell}\, k_{\beta_j}^{(\ell)} \Big\rangle = 0,$$

which forces $c_{j,m_j} = 0$ because $p^{(m_j)}(\beta_j) \neq 0$. $\square$

In light of Lemma 2.2.1, the above result tells us that a Shapiro–Shields function $\S_Z$, associated to the reproducible multiset

$$Z = \Big\{ \underbrace{0 = \beta_0, \dots, \beta_0}_{m_0 \text{ times}}, \underbrace{\beta_1, \dots, \beta_1}_{m_1 \text{ times}}, \dots, \underbrace{\beta_s, \dots, \beta_s}_{m_s \text{ times}} \Big\},$$

is a nonzero constant multiple of the projection of $k_0^{(m_0)}$ onto the orthogonal complement of

$$\operatorname{span}\{k_0^{(m_0-1)}, \dots, k_0, k_{\beta_1}^{(m_1-1)}, \dots, k_{\beta_1}, k_{\beta_s}^{(m_s-1)}, \dots, k_{\beta_s}\}.$$

2.2.4 Analogues of Finite Blaschke Products

In this section, we will show that for $p, q \in \mathcal{P}$, $[p] = [q]$ if and only if $R(p) = R(q)$. We will then unify the perspective of Shapiro–Shields functions and projections of kernels at the origin onto shift-invariant subspaces generated by polynomials, giving *analogues of finite Blaschke products.*

2.2.5 Shift Invariant Subspaces

Note that $k_0^{(d)} - \Pi_{[p]}(k_0^{(d)}) \perp [p]$, so in order to understand $\Pi_{[p]}(k_0^{(d)})$, it is useful to have a characterization of $[p]^\perp$. We do this first when all the zeros of p are reproducible.

Proposition 2.2.3. Let $f \in \mathcal{P}$ and suppose that

$$R(f) = Z(f) = \Big\{ \underbrace{\beta_1, \ldots, \beta_1}_{r_1 \text{ times}}, \underbrace{\beta_2, \ldots, \beta_2}_{r_2 \text{ times}}, \ldots, \underbrace{\beta_n, \ldots, \beta_n}_{r_n \text{ times}} \Big\}.$$

Then

$$[f] = \Big(\operatorname{span} \big\{ k_{\beta_j}^{(\ell)} : 0 \le \ell \le r_j - 1,\ 1 \le j \le n \big\} \Big)^{\perp}.$$

Proof. Let $\mathcal{M}$ denote the right hand-side. Then $\mathcal{M}$ consists of all functions h in $\mathcal{H}$ for which $h^{(\ell)}(\beta_j) = 0$ for all $0 \le \ell \le r_j - 1$ and $1 \le j \le n$. Note that for each $p \in \mathcal{P}$, the polynomial fp belongs to $\mathcal{M}$. It follows that $f\mathcal{P} \subseteq \mathcal{M}$ and hence $[f] \subseteq \mathcal{M}$, which implies $\mathcal{M}^{\perp} \subseteq [f]^{\perp}$. On the other hand, since kernel functions are linearly independent by Lemma 2.2.3, the space $\mathcal{M}^{\perp}$ is of dimension $d = r_1 + \cdots + r_n$. To prove the equality, we only need to show that the dimension of $[f]^{\perp}$ is at most d.

We have $f\mathcal{P} + \mathcal{P}_{d-1} = \mathcal{P}$, where $\mathcal{P}_{d-1}$ denotes the space of all polynomials of degree at most $d-1$. Taking closure and using the fact that the sum of a closed subspace with a finite dimensional subspace is closed, we have

$$\mathcal{H} = \mathrm{cl}_{\mathcal{H}}(\mathcal{P}) = \mathrm{cl}_{\mathcal{H}}(f\mathcal{P} + \mathcal{P}_{d-1}) = [f] + \mathcal{P}_{d-1}.$$

As a result, the dimension of $[f]^{\perp}$ is at most that of $\mathcal{P}_{d-1}$, which is d. Therefore, we have $\mathcal{M}^{\perp} = [f]^{\perp}$, which implies $[f] = \mathcal{M}$ as required. $\square$

This proposition generalizes [50, Lemma 4.7] where the authors require the zeros of the polynomial to be contained in Ω and additional properties are imposed on the space.

We will also show that if f has zeros that are not reproducible, this does not change the structure of $[f]$. First though, we need a proposition.

Proposition 2.2.4. Let β be a complex number and m be a non-negative integer. Then the following statements hold.

(a) β is not a reproducible point if and only if $(z-\beta)$ is cyclic.

(b) β is a reproducible point with $\mathrm{ro}(\beta) \leq m$ if and only if $(z-\beta)$ is not cyclic and

$$\mathrm{cl}_{\mathcal{H}}\left((z-\beta)^{m+2}\mathcal{P}\right) = \mathrm{cl}_{\mathcal{H}}\left((z-\beta)^{m+1}\mathcal{P}\right).$$

Proof. For any integer $k \geq 0$, define $\mathcal{X}_k = (z-\beta)^k\mathcal{P}$. It is clear that $\mathcal{X}_{k+1} \subset \mathcal{X}_k$, which shows that the identity $\mathrm{cl}_{\mathcal{H}}(\mathcal{X}_{k+1}) = \mathrm{cl}_{\mathcal{H}}(\mathcal{X}_k)$ holds if and only if $\mathcal{X}_{k+1}$ is dense in $\mathcal{X}_k$ with respect to the norm induced from $\mathcal{H}$.

On the other hand, define $\Lambda_k : \mathcal{P} \to \mathbb{C}$ by $\Lambda_k(p) = p^{(k)}(\beta)$ for $p \in \mathcal{P}$. Observe that

$$\begin{aligned}
\ker(\Lambda_k|_{\mathcal{X}_k}) &= \Big\{(z-\beta)^k q(z) : \ q \in \mathcal{P} \text{ such that } \Lambda_k\big((z-\beta)^k q(z)\big) = 0\Big\} \\
&= \Big\{(z-\beta)^k q(z) : \ q \in \mathcal{P} \text{ such that } q(\beta) = 0\Big\} \\
&= \mathcal{X}_{k+1}.
\end{aligned}$$

It follows from a well-known result in functional analysis (e.g., see Proposition 5.2 and Theorem 5.3 in [60, Chapter III]) that $\Lambda_k|_{\mathcal{X}_k}$ (being a nonzero functional) is unbounded if and only if $\mathcal{X}_{k+1} = \ker(\Lambda|_{\mathcal{X}_k})$ is dense in $\mathcal{X}_k$.

Therefore, we have just showed that for any $k \geq 0$, $\mathrm{cl}_{\mathcal{H}}(\mathcal{X}_{k+1}) = \mathrm{cl}_{\mathcal{H}}(\mathcal{X}_k)$ if and only if the linear function $\Lambda_k|_{\mathcal{X}_k}$ is unbounded.

(a) Since $\mathcal{X}_0 = \mathcal{P}$ and $\mathcal{X}_1 = (z-\beta)\mathcal{P}$, the function $(z-\beta)$ is cyclic if and only if $\mathrm{cl}_{\mathcal{H}}(\mathcal{X}_1) = \mathrm{cl}_{\mathcal{H}}(\mathcal{X}_0)$, which, from the argument above, is equivalent to the fact that $\Lambda_0|_{\mathcal{X}_0}$ is unbounded. Since $\Lambda_0(h) = h(\beta)$ for all $h \in \mathcal{P}$, the unboundedness of Λ_0 means exactly that β is not a reproducible point.

(b) Suppose first $(z-\beta)$ is not cyclic and $\mathrm{cl}_{\mathcal{H}}(\mathcal{X}_{m+2}) = \mathrm{cl}_{\mathcal{H}}(\mathcal{X}_{m+1})$. Then β is a reproducible point and the linear functional Λ_{m+1} is unbounded on $\mathcal{X}_{m+1}$, hence, unbounded on

$\mathcal{P}$. This implies that β is not reproducible of order $m+1$. That is, $\mathrm{ro}(\beta) \leq m$.

Let us now prove the converse. Suppose that β is a reproducible point and $\mathrm{ro}(\beta) \leq m$. By (a), $(z-\beta)$ is not cyclic. To simplify the notation, define $n = \mathrm{ro}(\beta)$. Then the linear functional Λ_k is bounded for each $0 \leq k \leq n$ but Λ_{n+1} is unbounded on $\mathcal{P}$. We show that actually $\Lambda_{n+1}|_{\mathcal{X}_{n+1}}$ is unbounded, which implies that $\mathrm{cl}_{\mathcal{H}}(\mathcal{X}_{n+2}) = \mathrm{cl}_{\mathcal{H}}(\mathcal{X}_{n+1})$.

Suppose, for the purpose of obtaining a contradiction, that $\Lambda_{n+1}|_{\mathcal{X}_{n+1}}$ is bounded. For any $h \in \mathcal{P}$, we write

$$h = \sum_{0 \leq j \leq n} \frac{h^{(j)}(\beta)}{j!}(z-\beta)^j + \tilde{h},$$

where $\tilde{h} \in \mathcal{X}_{n+1}$. Then

$$\Lambda_{n+1}(h) = \Lambda_{n+1}|_{\mathcal{X}_{n+1}}(\tilde{h})$$

and by triangle inequality,

$$\begin{aligned}
\|\tilde{h}\| = \left\| h - \sum_{0 \leq j \leq n} \frac{h^{(j)}(\beta)}{j!}(z-\beta)^j \right\| &\leq \|h\| + \sum_{0 \leq j \leq n} \frac{|h^{(j)}(\beta)|}{j!} \|(z-\beta)^j\| \\
&\leq \|h\| + \sum_{0 \leq j \leq n} \frac{\|\Lambda_j\|}{j!} \|(z-\beta)^j\| \cdot \|h\|.
\end{aligned}$$

Therefore,

$$|\Lambda_{n+1}(h)| = \|\Lambda_{n+1}|_{\mathcal{X}_{n+1}}(\tilde{h})\| \leq \|\Lambda_{n+1}|_{\mathcal{X}_{n+1}}\| \cdot \|\tilde{h}\| \leq C\|h\|,$$

which implies that Λ_{n+1} is bounded on $\mathcal{P}$, a contradiction.

We have thus showed that $\mathrm{cl}_{\mathcal{H}}(\mathcal{X}_{n+2}) = \mathrm{cl}_{\mathcal{H}}(\mathcal{X}_{n+1})$. Now,

$$\begin{aligned}
\mathrm{cl}_{\mathcal{H}}(\mathcal{X}_{n+3}) &= \mathrm{cl}_{\mathcal{H}}\left((z-\beta) \cdot \mathrm{cl}_{\mathcal{H}}(\mathcal{X}_{n+2})\right) \\
&= \mathrm{cl}_{\mathcal{H}}\left((z-\beta) \cdot \mathrm{cl}_{\mathcal{H}}(\mathcal{X}_{n+1})\right) \\
&= \mathrm{cl}_{\mathcal{H}}(\mathcal{X}_{n+2}).
\end{aligned}$$

It then follows inductively that $\mathrm{cl}_{\mathcal{H}}(\mathcal{X}_{m+2}) = \mathrm{cl}_{\mathcal{H}}(\mathcal{X}_{m+1})$. $\square$

The propositions above allows us to provide a complete description of $[f]$ whenever f is a polynomial.

Theorem 2.2.1. Let $f \in \mathcal{P}$. For each distinct $\beta_j \in R(f)$, let r_j be the multiplicity of β_j, i.e.

$$R(f) = \{\underbrace{\beta_1, \dots, \beta_1}_{r_1 \text{ times}}, \underbrace{\beta_2, \dots, \beta_2}_{r_2 \text{ times}}, \dots, \underbrace{\beta_n, \dots, \beta_n}_{r_n \text{ times}}\}.$$

Then

$$[f] = \left[\prod_{\beta \in R(f)} (z - \beta)\right] = \left(\mathrm{span}\left\{k_{\beta_j}^{(\ell)} : 0 \le \ell \le r_j - 1,\ 1 \le j \le n\right\}\right)^{\perp}.$$

Proof. We first recall the fact that for any multipliers g, h of $\mathcal{H}$, we have

$$[g \cdot h] = \mathrm{cl}_{\mathcal{H}}(g \cdot [h]).$$

If β is a non-reproducible zero of f, then Proposition 2.2.4 gives $[(z - \beta)] = \mathcal{H}$. Applying the above identity with $h = z - \beta$ and $g = f/(z - \beta)$, we conclude that $[f] = [f/(z - \beta)]$. So it suffices to consider only the zeros of f with some reproducible order. Put $f(z) = p(z)\prod_{j=1}^{n}(z - \beta_j)^{d_j}$, each β_j distinct with $d_j \ge r_j$, $1 \le j \le n$, and with the zeros of $p \in \mathcal{P}$ being all of the non-reproducible zeros of f (i.e. p is cyclic). Then $[f] = [f/p]$. So without loss of generality, we may assume that $p(z)$ is identically one. Let $h(z) = (z - \beta_1)^{d_1}$. Then by Lemma 2.2.4, we have $[h] = [(z - \beta_1)^{d_1}] = [(z - \beta_1)^{r_1}]$. Letting $g = f/h$, we have

$$\begin{aligned} [f] = [g \cdot h] &= \mathrm{cl}_{\mathcal{H}}(g \cdot [h]) \\ &= \mathrm{cl}_{\mathcal{H}}\left(g \cdot [(z - \beta_1)^{d_1}]\right) = [g \cdot (z - \beta_1)^{r_1}] \\ &= \left[\frac{f}{(z - \beta_1)^{d_1 - r_1}}\right]. \end{aligned}$$

Repeating this argument for each β_j, $2 \le j \le n$, we have

$$[f] = \left[\frac{f}{\prod_{j=1}^{n}(z-\beta_j)^{d_j - r_j}}\right] = \Big[\prod_{j=1}^{n}(z-\beta_j)^{r_j}\Big] = \left[\prod_{\beta \in R(f)}(z-\beta)\right].$$

Applying Proposition 2.2.3 then gives the result. □

As an immediate corollary, this theorem gives a generalization of Theorem 2.1.3, proved in the previous section for H^2_w; namely, when f is a polynomial, having a description of $[f]$ allows us to compute $\Pi_{[f]}(k_0)$. Let us illustrate this theorem by applying it to an example in various spaces.

Example 2.2.4. Let $f(z) = z^2(z - \frac{i}{2})(z^2-1)^2$. Then the zero multiset of f is

$$Z(f) = \Big\{0, 0, \frac{i}{2}, -1, -1, 1, 1\Big\}.$$

Interestingly, by Theorem 2.2.1, the shift-invariant subspace $[f]$ depends greatly on the underlying Hilbert space.

If $\mathcal{H} = D_\alpha$ for $\alpha \le 1$ (which includes the Hardy, Bergman and Dirichlet spaces), then $R(f) = \{0, 0, i/2\}$ and

$$[f] = \Big(\operatorname{span}\big\{k_0, k_0^{(1)}, k_{i/2}\big\}\Big)^{\perp}.$$

If $\mathcal{H} = D_\alpha$ for $1 < \alpha \le 3$, then by Example 2.2.1, we have $R(f) = \{0, 0, i/2, -1, 1\}$, which then implies that

$$[f] = \Big(\operatorname{span}\big\{k_0, k_0^{(1)}, k_{i/2}, k_{-1}, k_1\big\}\Big)^{\perp}.$$

If $\mathcal{H} = D_\alpha$ for $\alpha > 3$, then by Example 2.2.1 again, $R(f) = Z(f)$ and so

$$[f] = \Big(\operatorname{span}\big\{k_0, k_0^{(1)}, k_{i/2}, k_{-1}, k_{-1}^{(1)}, k_1, k_1^{(1)}\big\}\Big)^{\perp}.$$

On the other hand, if $\mathcal{H} = \mathcal{D}_{\delta_1}$, the local Dirichlet space at 1, then by Example 2.2.2, we have $R(f) = \{0, 0, i/2, 1\}$ and hence,

$$[f] = \left(\text{span}\left\{k_0, k_0^{(1)}, k_{i/2}, k_1\right\}\right)^{\perp}.$$

Theorem 2.2.1 also has an immediate and useful corollary.

Corollary 2.2.1. Let $p, q \in \mathcal{P}$. Then $[p] = [q]$ if and only if $R(p) = R(q)$.

Proof. The backward implication is given directly by Theorem 2.2.1. So let $[p] = [q]$ and suppose for contradiction that $R(p) \neq R(q)$. WLOG, there exists $\beta \in R(p)$ with $\beta \notin R(q)$ or β having greater multiplicity in $R(p)$ than in $R(q)$. In either case, Theorem 2.2.1 implies there is some $n \geq 0$ so that $k_\beta^{(n)} \perp [p] = [q]$. But then $\langle p, k_\beta^{(n)} \rangle = \langle q, k_\beta^{(n)} \rangle = 0$, which is a contradiction, since $\beta \notin R(q)$ or β has multiplicity strictly less than $n + 1$ in $R(q)$. $\square$

2.2.6 Inner Functions and Linear Combinations of Kernels

We now show that each inner function that arises as a certain linear combination of reproducing kernels can be identified with a shift invariant subspace and a Shapiro–Shields function. The following theorem generalizes a result of Le [114, Theorem 3.7], proved initially in the H^2_w spaces, and serves as a converse to Theorem 2.1.3. The significance of our contribution is that we do not require monomials be orthogonal and make almost no geometric assumptions on the underlying set for which $\mathcal{H}$ is an RKHS, providing a very general setting for which these results hold. When $\mathcal{H} = H^2$, this result describes classical Blaschke products and in general gives our *analogues of finite Blaschke products.*

Theorem 2.2.2. Suppose that

$$B = \sum_{j=0}^{s} \sum_{\ell=0}^{m_j} c_{j,\ell}\, k_{\lambda_j}^{(\ell)}$$

is an $\mathcal{H}$-inner function, with $c_{j,m_j} \neq 0$. Then B is a constant multiple of $\Pi_{[f]}(k_0^{(d)})$ for some d and some polynomial f. Further, B must also be a Shapiro–Shields function.

The function B here is what we call an *analogue of a finite Blaschke product.*

Proof. Take B as above and without loss of generality, assume that $\|B\| = 1$. For any $g \in \mathcal{P}$,

$$\begin{aligned}
\langle g, k_0 \rangle = g(0) = \langle gB, B \rangle &= \sum_{j=0}^{s} \sum_{\ell=0}^{m_j} \bar{c}_{j,\ell} \langle gB, k_{\lambda_j}^{(\ell)} \rangle \\
&= \sum_{j=0}^{s} \sum_{\ell=0}^{m_j} \bar{c}_{j,\ell} \sum_{m=0}^{\ell} \binom{\ell}{m} g^{(m)}(\lambda_j) \cdot B^{(\ell-m)}(\lambda_j) \\
&= \sum_{j=0}^{s} \sum_{m=0}^{m_j} \Big\{ \sum_{\ell=m}^{m_j} \bar{c}_{j,\ell} \binom{\ell}{m} B^{(\ell-m)}(\lambda_j) \Big\} g^{(m)}(\lambda_j) \\
&= \sum_{j=0}^{s} \sum_{m=0}^{m_j} \Big\{ \sum_{\ell=m}^{m_j} \bar{c}_{j,\ell} \binom{\ell}{m} B^{(\ell-m)}(\lambda_j) \Big\} \langle g, k_{\lambda_j}^{(m)} \rangle.
\end{aligned}$$

Since the set of polynomials is dense in $\mathcal{H}$, we conclude that

$$k_0 = \sum_{j=0}^{s} \sum_{m=0}^{m_j} \Big\{ \sum_{\ell=m}^{m_j} c_{j,\ell} \binom{\ell}{m} \overline{B^{(\ell-m)}(\lambda_j)} \Big\} k_{\lambda_j}^{(m)}.$$

It then follows from Lemma 2.2.3 that $0 \in \{\lambda_0, \ldots, \lambda_s\}$ and for all j and m,

$$\sum_{\ell=m}^{m_j} \bar{c}_{j,\ell} \binom{\ell}{m} B^{(\ell-m)}(\lambda_j) = \begin{cases} 0, & \text{if } \lambda_j \neq 0 \ \text{ or } \ m \geq 1, \\ 1, & \text{if } \lambda_j = 0 \ \text{ and } \ m = 0. \end{cases} \tag{2.2.1}$$

Without loss of generality, we shall always assume that $\lambda_0 = 0$. Then for $j = 0$ and $0 \leq m \leq m_0$, equation (2.2.1) gives

$$\sum_{\ell=m}^{m_0} \bar{c}_{0,\ell} \binom{\ell}{m} B^{(\ell-m)}(0) = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{if } m \geq 1. \end{cases}$$

Since $c_{0,m_0} \neq 0$, we conclude that either $m_0 = 0$, or $m_0 \geq 1$ and $B^{(\ell)}(0) = 0$ for all $0 \leq \ell \leq m_0 - 1$. That is, $B \perp \{k_0^{(\ell)} : 0 \leq \ell \leq m_0 - 1\}$.

On the other hand, for $j \neq 0$ and $0 \leq m \leq m_j$, by equation (2.2.1),

$$\sum_{\ell=m}^{m_j} \bar{c}_{j,\ell} \binom{\ell}{m} B^{(\ell-m)}(\lambda_j) = 0.$$

Since $c_{j,m_j} \neq 0$, it follows that $B^{(\ell)}(\lambda_j) = 0$ for all $0 \leq \ell \leq m_j$. That is, $B \perp \{k_{\lambda_j}^{(\ell)} : 0 \leq \ell \leq m_j\}$. Let $\mathcal{M}$ be the subspace spanned by the functions

$$\left\{k_0^{(\ell)} : 0 \leq \ell \leq m_0 - 1\right\} \cup \left\{k_{\lambda_j}^{(\ell)} : 1 \leq j \leq s, 0 \leq \ell \leq m_j\right\},$$

where the first set is considered to be empty if $m_0 = 0$. Then we have $B = c_{0,m_0} \Pi_{\mathcal{M}^\perp}(k_0^{(m_0)})$. By Proposition 2.2.3, $\mathcal{M}^\perp$ can be recognized as $[f]$, where

$$f(z) = z^{m_0} \prod_{j=1}^{s} (z - \lambda_j)^{m_j+1}.$$

As a result, we have shown that B is a projection of $k_0^{(d)}$ onto $[f]$ for some d and some polynomial f. Further, by Lemma 2.2.1, we know that B must also equal the Shapiro–Shields function $\S_{Z(f)}$. $\square$

2.3 Multipliers of ℓ^p_A

The work in this section is joint with Raymond Cheng, which can be found in preprint form in [45], and will appear in *Concrete Operators.* For $p \in (1, \infty)\backslash\{2\}$, we derive some properties of the space $\mathscr{M}_p$ of multipliers on ℓ^p_A. In particular, the failure of the weak parallelogram laws and the Pythagorean inequalities is demonstrated for $\mathscr{M}_p$. It is also shown that the extremal multipliers on the ℓ^p_A spaces are exactly the monomials, in stark contrast to the

$p = 2$ case.

2.3.1 Preliminaries

For $1 \leq p \leq \infty$, we recall ℓ^p_A, the space of analytic functions on $\mathbb{D}$ for which the corresponding Maclaurin coefficients are p-summable, endowed with the norm

$$\|f\|_p = \|(a_k)_{k=0}^{\infty}\|_{\ell^p}$$

for

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

belonging to ℓ^p_A. We stress that $\|\cdot\|_p$ refers to the norm on ℓ^p_A, and not the norm on H^p, or some other function space.

When $p \neq 1$ and $p \neq 2$, relatively little is known about the space ℓ^p_A. For $1 < p < \infty$, there is a notion of a p-inner function, in terms of which the zero sets of ℓ^p_A can be described [49]. Unlike H^2, however, the analogous inner-outer factorization can fail when $p \neq 2$ [44]. Whereas the multiplier algebra of H^2 is the familiar space H^∞, the multipliers on ℓ^p_A have not been completely characterized.

The following property is elementary, and will be essential for identifying the extremal multipliers of ℓ^p_A (for a proof, see [51, Proposition 1.5.2]).

Proposition 2.3.1. If $1 \leq p_1 < p_2 \leq \infty$, then $\ell^{p_1}_A \subset \ell^{p_2}_A$, and $\|f\|_{p_2} \leq \|f\|_{p_1}$ for all $f \in \ell^{p_1}_A$. Furthermore, $\|f\|_{p_2} = \|f\|_{p_1}$ holds if and only if

$$f(z) = \gamma z^k$$

for some $\gamma \in \mathbb{C}$ and nonnegative integer k.

Throughout this section, if $1 \leq p \leq \infty$, then p' will be the Hölder conjugate to p, that is, $1/p + 1/p' = 1$ holds. We recall that for $1 \leq p < \infty$, $p \neq 2$, the dual space of ℓ^p_A can be identified with $\ell^{p'}_A$, under the pairing

$$\langle f, g \rangle = \sum_{k=0}^{\infty} f_k g_k, \tag{2.3.1}$$

where $f(z) = \sum_{k=0}^{\infty} f_k z^k$ and $g(z) = \sum_{k=0}^{\infty} g_k z^k$. Let us retain this notation for the bilinear form $\langle \cdot, \cdot \rangle$ even when $p = 2$.

For further exploration of ℓ^p_A, we refer to the paper [48] or the book [51].

2.3.2 Orthogonality

There is a natural way to define "inner functions" in the context of ℓ^p_A, that makes use of a notion of orthogonality in general normed linear spaces.

Let $\mathbf{x}$ and $\mathbf{y}$ be vectors belonging to a normed linear space $\mathcal{X}$. We say that $\mathbf{x}$ is *orthogonal* to $\mathbf{y}$ in the Birkhoff-James sense [7, 104] if

$$\|\mathbf{x} + \beta \mathbf{y}\|_{\mathcal{X}} \geq \|\mathbf{x}\|_{\mathcal{X}} \tag{2.3.2}$$

for all scalars β, and in this case we write $\mathbf{x} \perp_{\mathcal{X}} \mathbf{y}$.

Birkhoff-James orthogonality extends the concept of orthogonality from an inner product space to normed spaces. There are other ways to generalize orthogonality, but this approach is particularly fruitful since it is connected to an extremal condition via (2.3.2).

It is straightforward to check that if $\mathcal{X}$ is a Hilbert space, then the usual orthogonality relation $\mathbf{x} \perp \mathbf{y}$ is equivalent to $\mathbf{x} \perp_{\mathcal{X}} \mathbf{y}$. More typically, however, the relation $\perp_{\mathcal{X}}$ is neither symmetric nor linear. When $\mathcal{X} = \ell^p_A$, let us write $\perp_p$ instead of $\perp_{\ell^p_A}$.

There is an analytical criterion for the relation $\perp_p$ when $p \in (1, \infty)$.

Theorem 2.3.1 (James [104]). Suppose that $1 < p < \infty$. Then for $f(z) = \sum_{k=0}^{\infty} f_k z^k$ and $g(z) = \sum_{k=0}^{\infty} g_k z^k$ belonging to ℓ_A^p we have

$$f \perp_p g \iff \sum_{k=0}^{\infty} |f_k|^{p-2}\overline{f}_k g_k = 0, \tag{2.3.3}$$

where any incidence of "$|0|^{p-2}0$" in the above sum is interpreted as zero.

In light of (2.3.3) we define, for a complex number $\alpha = re^{i\theta}$, and any $s > 0$, the quantity

$$\alpha^{\langle s \rangle} = (re^{i\theta})^{\langle s \rangle} := r^s e^{-i\theta}. \tag{2.3.4}$$

It is readily seen that for any complex numbers α and β, exponent $s > 0$, and integer $n \geq 0$, we have

$$\begin{aligned}
(\alpha\beta)^{\langle s \rangle} &= \alpha^{\langle s \rangle}\beta^{\langle s \rangle} \\
|\alpha^{\langle s \rangle}| &= |\alpha|^s \\
\alpha^{\langle s \rangle}\alpha &= |\alpha|^{s+1} \\
(\alpha^{\langle s \rangle})^n &= (\alpha^n)^{\langle s \rangle} \\
(\alpha^{\langle p-1 \rangle})^{\langle p'-1 \rangle} &= \alpha.
\end{aligned}$$

Notice that $\alpha^{\langle 1 \rangle} = \bar{\alpha}$. Thus, by comparing with the case $p = 2$, we can think of taking the $\langle p-1 \rangle$ power as generalizing complex conjugation.

Further to the notation (2.3.4), for $f(z) = \sum_{k=0}^{\infty} f_k z^k$, let us write

$$f^{\langle s \rangle}(z) := \sum_{k=0}^{\infty} f_k^{\langle s \rangle} z^k \tag{2.3.5}$$

for any $s > 0$.

If $f \in \ell^p_A$, it is easy to verify that $f^{\langle p-1 \rangle} \in \ell^{p'}$. Thus from (2.3.3) we get

$$f \perp_p g \iff \langle g, f^{\langle p-1 \rangle} \rangle = 0. \tag{2.3.6}$$

Consequently the relation $\perp_p$ is linear in its second argument, when $p \in (1, \infty)$, and it then makes sense to speak of a vector being orthogonal to a subspace of ℓ^p_A. In particular, if $f \perp_p g$ for all g belonging to a subspace $\mathscr{X}$ of ℓ^p_A, then

$$\|f + g\|_p \geq \|f\|_p$$

for all $g \in \mathscr{X}$. That is, f solves an extremal problem in relation to the subspace $\mathscr{X}$.

Direct calculation will also confirm that

$$\langle f, f^{\langle p-1 \rangle} \rangle = \|f\|_p^p.$$

With this concept of orthogonality established, we may now define what it means for a function in ℓ^p_A to be inner in a related sense.

Definition 2.3.1. Let $1 < p < \infty$. A function $f \in \ell^p_A$ is said to be p-inner if it is not identically zero and it satisfies

$$f(z) \ \perp_p \ z^k f(z)$$

for all positive integers k.

That is, f is nontrivially orthogonal to all of its forward shifts. Apart from a harmless multiplicative constant, this definition is equivalent to the traditional meaning of "inner" when $p = 2$. Furthermore, this approach to defining an inner property is consistent with that taken in other function spaces [6, 23, 50, 52, 73, 74, 76, 133, 137].

Birkhoff-James Orthogonality also plays a role when we examine a version of the Pythagorean theorem for normed spaces in Section 2.3.4.

2.3.3 Multipliers on ℓ^p_A

An analytic function ϕ on $\mathbb{D}$ is said to be a *multiplier* of ℓ^p_A if

$$f \in \ell^p_A \implies \phi f \in \ell^p_A.$$

The set of multipliers of ℓ^p_A will be denoted by $\mathscr{M}_p$.

For $\phi \in \mathscr{M}_p$, an application of the closed graph theorem shows that the linear mapping

$$M_\phi : \ell^p_A \to \ell^p_A, \quad M_\phi f = \phi f$$

is continuous. Thus we can define the *multiplier norm* of ϕ by

$$\|\phi\|_{\mathscr{M}_p} := \sup\{\|\phi f\|_p : f \in \ell^p_A, \|f\|_p \leq 1\}.$$

In other words, the multiplier norm of ϕ coincides with the operator norm of M_ϕ on ℓ^p_A. Henceforth we identify the multiplication operator M_ϕ with its symbol ϕ.

We note again that relatively little is known about the multipliers on ℓ^p_A, except when $p = 1$ or $p = 2$. We will accordingly concentrate our efforts on the range $1 < p < \infty$, with $p \neq 2$. The following basic results have been established in the literature.

Proposition 2.3.2. Let $1 < p < \infty$. If $\phi \in \mathscr{M}_p$, then $\phi \in H^\infty \cap \ell^p_A \cap \ell^{p'}_A$, and $\mathscr{M}_p = \mathscr{M}_{p'}$, with $\|\phi\|_{\mathscr{M}_p} = \|\phi\|_{\mathscr{M}_{p'}}$.

Proposition 2.3.3. Let $1 < p < \infty$. If $\phi(z) = \sum_{k=0}^\infty \phi_k z^k \in \mathscr{M}_p$, then $\|\phi\|_p \leq \|\phi\|_{\mathscr{M}_p} \leq \|\phi\|_1$ (with $\|\phi\|_1 = \infty$ being possible), and

$$|\phi_0| + |\phi_1| + \cdots + |\phi_n| \leq \|\phi\|_{\mathscr{M}_p}(n+1)^{1/p'}.$$

If all of the coefficients of ϕ are nonnegative, then $\phi \in \ell^1_A$, and $\|\phi\|_1 = \|\phi\|_{\mathscr{M}_p}$.

Define the difference quotient mapping Q_w by

$$Q_w f(z) := \frac{f(z) - f(w)}{z - w}$$

for any $w \in \mathbb{D}$ and analytic function f on $\mathbb{D}$.

Difference quotients are (bounded) operators on $\mathscr{M}_p$. In fact, for any multiplier ϕ on ℓ^p_A, and $w \in \mathbb{D}$,

$$\|Q_w \phi\|_{\mathscr{M}_p} \leq \frac{1}{1 - |w|}(\|\phi\|_{\mathscr{M}_p} + \phi(w)).$$

For proofs of these multiplier properties, see [51, Chapter 12], which has references to original sources.

To extract some geometric information about $\mathscr{M}_p$, we will rely on the following observation.

Corollary 2.3.1. For any complex numbers α and β, the multiplier $\phi(z) = \alpha + \beta z$ satisfies

$$\|\phi\|_{\mathscr{M}_p} = \|\phi\|_1 = |\alpha| + |\beta|.$$

Proof. The claim is trivial if $\alpha = 0$ or $\beta = 0$. Otherwise, the mapping

$$f(z) \longmapsto f\left(\frac{\alpha\bar{\beta}}{|\alpha\bar{\beta}|} z\right)$$

determines a linear isometry on ℓ^p_A (in fact it is unitary).

Consequently, the multiplier ϕ has the same norm as the multiplier

$$\phi\left(\frac{\alpha\bar{\beta}}{|\alpha\bar{\beta}|} z\right) = \frac{1}{\bar{\alpha}}\left(|\alpha|^2 + \beta\bar{\alpha}\left(\frac{\alpha\bar{\beta}}{|\alpha\bar{\beta}|} z\right)\right) = \frac{1}{\bar{\alpha}}\big(|\alpha|^2 + |\beta\bar{\alpha}| z\big),$$

which is $|\alpha| + |\beta|$, according to the last part of Proposition 2.3.3. $\square$

Already this delivers some information about the geometry of $\mathscr{M}_p$. Recall that a normed space is strictly convex if

$$\|\mathbf{x} + \mathbf{y}\| < \|\mathbf{x}\| + \|\mathbf{y}\| \tag{2.3.7}$$

whenever the vectors $\mathbf{x}$ and $\mathbf{y}$ are not parallel [40, p. 108].

Corollary 2.3.2. If $1 < p < \infty$, then $\mathscr{M}_p$ fails to be strictly convex.

Proof. Consider the multipliers $\phi_t(z) = t + (1-t)z$ for $0 \le t \le 1$. By Corollary 2.3.1, we have $\|\phi\|_{\mathscr{M}_p} = t + (1-t) = 1$ for all t. But ϕ_0 and ϕ_1 are not parallel, and hence condition (2.3.7) fails when $0 < t < 1$. $\square$

It is known that certain Blaschke products are multipliers of ℓ^p_A (e.g., if the zeros converge to the boundary rapidly enough), and that certain other classes of functions are multipliers. However, there does not yet exist a complete characterization of $\mathscr{M}_p$ in terms of the coefficients, or of the boundary function. Our sources on the subject include [67, 102, 115–119, 125, 150–152], along with the survey paper [48].

2.3.4 The Geometry of $\mathscr{M}_p$

It is well known that when $1 < p < \infty$, the spaces ℓ^p (and hence also ℓ^p_A) are uniformly convex and uniformly smooth (see, for example, [22, 40]). In fact, more can be said. A normed space $\mathscr{X}$ is said to satisfy the Lower Weak Parallelogram property (LWP) with constant $C > 0$ and exponent $r > 1$, if

$$\|x + y\|^r_{\mathscr{X}} + C\|x - y\|^r_{\mathscr{X}} \le 2^{r-1}(\|x\|^r_{\mathscr{X}} + \|y\|^r_{\mathscr{X}})$$

for all x and y in $\mathscr{X}$; it satisfies the Upper Weak Parallelogram property (UWP) if for some (possibly different) constant and exponent the reverse inequality holds for all x and y in $\mathscr{X}$.

If $\mathscr{X}$ is a Hilbert space, then the parallelogram law holds, corresponding to $r = 2$ and $C = 1$. Otherwise, these inequalities generalize Clarkson's inequalities [59], and the parameters r and C give a sense of how far the space $\mathscr{X}$ departs from behaving like a Hilbert space.

It was shown in [54] that the L^p spaces satisfy LWP and UWP when $1 < p < \infty$, and the full ranges of parameters C and r were identified (see also [37, 38, 46, 50, 53]). More generally, a space satisfying LWP is uniformly convex, and a space satisfying UWP is uniformly smooth [54, Proposition 3.1]. From this it could be further surmised that the dual of a LWP space is an UWP space, and vice-versa; this is made precise in [46, Theorem 3.1].

Another useful consequence of the weak parallelogram laws is a version of the Pythagorean Theorem for normed spaces, where orthogonality is in the Birkhoff-James sense. It takes the form of a family of inequalities relating the lengths of orthogonal vectors with that of their sum [54, Theorem 3.3].

Theorem 2.3.2 ([54]). If a smooth Banach space $\mathscr{X}$ satisfies LWP with constant $C > 0$ and exponent $r > 1$, then there exists $K > 0$ such that

$$\|x\|_{\mathscr{X}}^r + K\|y\|_{\mathscr{X}}^r \leq \|x + y\|_{\mathscr{X}}^r \tag{2.3.8}$$

whenever $x \perp_{\mathscr{X}} y$; if $\mathscr{X}$ satisfies UWP with constant $C > 0$ and exponent $r > 1$, then there exists a positive constant K such that

$$\|x\|_{\mathscr{X}}^r + K\|y\|_{\mathscr{X}}^r \geq \|x + y\|_{\mathscr{X}}^r \tag{2.3.9}$$

whenever $x \perp_{\mathscr{X}} y$. In either case, the constant K can be chosen to be $C/(2^{r-1} - 1)$.

When $\mathscr{X}$ is any Hilbert space, the parameters are $K = 1$ and $r = 2$, and the Pythagorean inequalities reduce to the familiar Pythagorean theorem. More generally, these Pythagorean inequalities enable the application of some Hilbert space methods and techniques to smooth

Banach spaces satisfying LWP or UWP; see, for example, [51, Proposition 4.8.1 and Proposition 4.8.3; Theorem 8.8.1].

The weak parallelogram laws and the Pythagorean inequalities fail on L^1 and L^∞. We previously saw in Corollary 2.3.1 that $\mathscr{M}_p$ contains a subspace, consisting of the linear functions, that behaves geometrically like ℓ^1_A. Consequently we would expect the weak parallelogram laws and the Pythagorean inequalities to fail on $\mathscr{M}_p$, and indeed that is the case.

Theorem 2.3.3. Let $1 < p < \infty$. The space $\mathscr{M}_p$ fails to satisfy LWP or UWP for any constant $C > 0$ or exponent $r > 1$.

Proof. If

$$\|1+z\|^r_{\mathscr{M}_p} + C\|1-z\|^r_{\mathscr{M}_p} \leq 2^{r-1}(\|1\|^r_{\mathscr{M}_p} + \|z\|^r_{\mathscr{M}_p})$$

holds, then an application of Corollary 2.3.1 yields

$$(1+C)2^r \leq 2^{r-1}(2),$$

which forces $C \leq 0$. Thus LWP fails.

Similarly, for $C > 0$ we have

$$\|1\|^r_{\mathscr{M}_p} + C^r\|z/C\|^r_{\mathscr{M}_p} \geq 2^{r-1}(\|1+z/C\|^r_{\mathscr{M}_p} + \|1-z/C\|^r_{\mathscr{M}_p})$$

implies

$$2 \geq 2^{r-1} \cdot 2 \cdot (1+1/C)^r,$$

which is absurd when $1 < r < \infty$. Therefore UWP also fails. □

Theorem 2.3.4. Let $1 < p < \infty$. The space $\mathscr{M}_p$ fails to satisfy either of the Pythagorean inequalities for any parameters $r > 1$ and $K > 0$.

Proof. Fix $c \neq 0$. Let $\phi(z) = 1 + cz$, and consider $f(z) \in \ell^p_A$ of the form $f(z) = f_0 + f_2 z^2 + f_4 z^4 + \cdots$. Then

$$\begin{aligned}
\|\phi(z)f(z)\|_p^p &= \|(1+cz)(f_0 + f_2 z^2 + f_4 z^4 + \cdots)\|_p^p \\
&= \|(f_0 + f_2 z^2 + f_4 z^4 + \cdots) + cz(f_0 + f_2 z^2 + f_4 z^4 + \cdots)\|_p^p \\
&= |f_0|^p + |c|^p |f_0|^p + |f_1|^p + |c|^p |f_1|^p + |f_2|^p + |c|^p |f_2|^p + \cdots \\
&= \|f\|_p^p + |c|^p \|f\|_p^p \\
&\geq \|f\|_p^p.
\end{aligned}$$

This shows that $\|1 + cz\|_{\mathscr{M}_p} \geq \|1\|_{\mathscr{M}_p}$ for all constants c, or $1 \perp_{\mathscr{M}_p} z$. By considering the limit

$$\lim_{c \to 0} \frac{\|1 + cz\|^r_{\mathscr{M}_p} - \|1\|^r_{\mathscr{M}_p}}{\|cz\|^r_{\mathscr{M}_p}} = \lim_{c \to 0} \frac{(1 + |c|)^r - 1^r}{|c|^r},$$

we see that $\mathscr{M}_p$ fails to satisfy (2.3.9), as $K = \infty$ would be forced.

Next, note that for $c \neq 0$, we have

$$\begin{aligned}
\|(1+z) + c(1-z)\|_{\mathscr{M}_p} &= \|(1+c) + (1-c)z\|_{\mathscr{M}_p} \\
&= |1+c| + |1-c| \\
&\geq 2 \\
&= \|1+z\|_1 \\
&= \|1+z\|_{\mathscr{M}_p}.
\end{aligned}$$

This shows that $1 + z \perp_{\mathscr{M}_p} 1 - z$. Next, consider

$$\frac{\|(1+z) + c(1-z)\|^r_{\mathscr{M}_p} - \|1+z\|^r_{\mathscr{M}_p}}{\|c(1-z)\|^r_{\mathscr{M}_p}} = \frac{(|1+c| + |1-c|)^r - 2^r}{2|c|^r},$$

where $1 < r < \infty$.

This tends toward zero as $c \to 0^+$, which would require $K = 0$. Thus, (2.3.8) fails to hold. $\square$

2.3.5 Functionals on $\mathscr{M}_p$

Let $1 < p < \infty$. Suppose that $\lambda = (\lambda_0, \lambda_1, \lambda_2, \ldots)$ is a sequence of complex numbers such that for some $C > 0$ we have

$$|\lambda_0\phi_0 + \lambda_1\phi_1 + \lambda_2\phi_2 + \cdots| \leq C\|\phi\|_{\mathscr{M}_p}$$

for all $\phi(z) = \sum_{k=0}^{\infty} \phi_k z^k \in \mathscr{M}_p$. Then λ determines a bounded linear functional on $\mathscr{M}_p$ with norm at most C. Let us give the name $\mathscr{S} = \mathscr{S}_p$ to the collection of functionals arising in this manner. It is a nonempty collection, since it contains all of $\ell_A^{p'}$. Thus $\mathscr{S}$ is a linear manifold within $\mathscr{M}_p^*$, the continuous dual space of $\mathscr{M}_p$.

If $\lambda = (\lambda_0, \lambda_1, \lambda_2, \ldots) \in \mathscr{S}$, then $\lambda_k = \lambda(z^k)$ and the pairing

$$\lambda(\phi) = \sum_{k=0}^{\infty} \lambda_k \phi_k.$$

applies for all $\phi \in \mathscr{M}_p$.

Trivially, we can bound the norm of λ as follows:

$$\|\lambda\|_{p'} = \sup_{\phi \neq 0} \frac{|\lambda(\phi)|}{\|\phi\|_p} \geq \sup_{\phi \neq 0} \frac{|\lambda(\phi)|}{\|\phi\|_{\mathscr{M}_p}} = \|\lambda\|_{\mathscr{M}_p^*} \geq \sup_{\phi \neq 0} \frac{|\lambda(\phi)|}{\|\phi\|_1} = \|\lambda\|_\infty, \tag{2.3.10}$$

possibly with ∞ on the left side.

Since $\mathscr{M}_p = \mathscr{M}_{p'}$ with equal norms (Proposition 4.1), we also have

$$\|\lambda\|_p = \sup_{\phi \neq 0} \frac{|\lambda(\phi)|}{\|\phi\|_{p'}} \geq \sup_{\phi \neq 0} \frac{|\lambda(\phi)|}{\|\phi\|_{\mathscr{M}_{p'}}} = \sup_{\phi \neq 0} \frac{|\lambda(\phi)|}{\|\phi\|_{\mathscr{M}_p}} = \|\lambda\|_{\mathscr{M}_p^*}.$$

Consequently,

$$\|\lambda\|_p \geq \|\lambda\|_{\mathscr{M}_p^*},$$

again, with the left side possibly being infinite.

Taking the $\langle p-1\rangle$ power does something natural in this context.

Proposition 2.3.4. Let $1 < p < \infty$. If $\phi \in \mathscr{M}_p$, then $\phi^{\langle p-1\rangle} \in \mathscr{S}$.

Proof. In this situation, $\phi^{\langle p-1\rangle} \in \ell_A^{p'}$, and hence $\phi^{\langle p-1\rangle} \in \mathscr{S}$, by (2.3.10). ☐

Members of $\mathscr{S}$ might not have radial boundary limits, but they do satisfy the following growth condition, which can also be interpreted as boundedness of point evaluation.

Proposition 2.3.5. Let $1 < p < \infty$. If $\lambda \in \mathscr{S}$, then

$$|\lambda(w)| \leq \frac{\|\lambda\|_{\mathscr{M}_p^*}}{(1-|w|)}, \quad w \in \mathbb{D}.$$

Proof. For $w \in \mathbb{D}$, let us write $k_w(z) = \sum_{k=0}^{\infty} w^k z^k$ for the point evaluation functional at w. We then have

$$\begin{aligned}
|\lambda(w)| &= \Big|\sum_{k=0}^{\infty} \lambda_k w^k\Big| \\
&\leq \|\lambda\|_{\mathscr{M}_p^*} \|k_w\|_{\mathscr{M}_p} \\
&\leq \|\lambda\|_{\mathscr{M}_p^*} \|k_w\|_1 \\
&= \frac{\|\lambda\|_{\mathscr{M}_p^*}}{1-|w|}.
\end{aligned}$$

☐

It turns out that difference quotients are bounded on $\mathscr{S}$. Again, let us denote by S the mapping

$$f(z) \longmapsto zf(z),$$

where f is analytic in the open unit disk. It is straightforward to verify that S determines a bounded linear operator on $\mathscr{M}_p$, with $\|S^k\phi\|_{\mathscr{M}_p} = \|\phi\|_{\mathscr{M}_p}$ for all $k \in \mathbb{N}$ and $\phi \in \mathscr{M}_p$.

Proposition 2.3.6. Let $1 < p < \infty$. If $\lambda \in \mathscr{S}$, and $w \in \mathbb{D}$, then $Q_w\lambda \in \mathscr{S}$, and

$$\|Q_w\lambda\|_{\mathscr{M}_p^*} \leq \frac{\|\lambda\|_{\mathscr{M}_p^*}}{1-|w|}.$$

Proof. Suppose that $\lambda \in \mathscr{S}$, and $w \in \mathbb{D}$. We now calculate

$$\begin{aligned}
(Q_w\lambda)(\phi) &= \left(\frac{\sum \lambda_k z^k - \sum \lambda_k w^k}{z-w}\right)(\phi) \\
&= \Big(\sum_{k=1}^{\infty} \lambda_k(z^{k-1} + z^{k-2}w + \cdots + w^{k-1})\Big)(\phi) \\
&= \Big(\sum_{k=1}^{\infty}\sum_{j=0}^{k-1} \lambda_k z^j w^{k-j-1}\Big)(\phi) \\
&= \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} \lambda_k \phi_j w^{k-j-1} \\
&= \lambda_1(\phi_0) \\
&\quad + \lambda_2(\phi_0 w + \phi_1) \\
&\quad + \lambda_3(\phi_0 w^2 + \phi_1 w + \phi_2) \\
&\quad + \cdots \\
&= \lambda_1\phi_0 + \lambda_2\phi_1 + \lambda_3\phi_2 + \cdots \\
&\quad + w(\lambda_2\phi_0 + \lambda_3\phi_1 + \lambda_4\phi_2 + \cdots) \\
&\quad + w^2(\lambda_3\phi_0 + \lambda_4\phi_1 + \lambda_5\phi_2 + \cdots) \\
&\quad + \cdots \\
&= \lambda(S\phi) + w\lambda(S^2\phi) + w^2\lambda(S^3\phi) + \cdots .
\end{aligned}$$

From this we obtain

$$\begin{aligned}|(Q_w\lambda)(\phi)| &\leq \|\lambda\|_{\mathscr{M}_p^*}\|S\phi\|_{\mathscr{M}_p} + |w|\|\lambda\|_{\mathscr{M}_p^*}\|S^2\phi\|_{\mathscr{M}_p} + |w|^2\|\lambda\|_{\mathscr{M}_p^*}\|S^3\phi\|_{\mathscr{M}_p} + \cdots \\ &= \frac{\|\lambda\|_{\mathscr{M}_p^*}\|\phi\|_{\mathscr{M}_p}}{1-|w|},\end{aligned}$$

which proves the claim. □

Let us add that the weak parallelogram laws and the Pythagorean inequalities must fail on $\mathscr{M}_p^*$ as well. This is because it contains a subspace that is isomorphic to $\ell^\infty_A(\{0,1\})$. Furthermore we see that $\mathscr{M}_p$ fails to be smooth. For example, the multiplier 1 is normed by both 1 and $1+z$ in $\mathscr{M}_p^*$.

2.3.6 The Extremal Multipliers of ℓ^p_A

Recall that if $\phi \in \mathscr{M}_p$, then $\|\phi\|_{\mathscr{M}_p} \geq \|\phi\|_p$. We say that the multiplier ϕ is *extremal* if equality holds, that is,

$$\|\phi\|_{\mathscr{M}_p} = \|\phi\|_p.$$

We mention again for $\ell^2_A = H^2$, the extremal multipliers precisely characterize inner functions. For ℓ^p_A, $p \neq 2$, it would therefore be plausible to guess that the extremal multipliers are the p-inner functions. However, this is incorrect, as the following example illustrates.

Example 2.3.1. If $1 < p < \infty$ and $0 < |w| < 1$, then the function

$$B(z) := \frac{1 - z/w}{1 - w^{\langle p'-1\rangle} z},$$

turns out to be p-inner [47, Lemma 3.2], and

$$\|B\|_p^p = 1 + \frac{(1-|w|^{p'})^{p-1}}{|w|^p}.$$

Note, in particular, that when $p = 2$ the function B is the Blaschke factor, possibly apart from a multiplicative constant, with its root at w.

Since B is analytic in a neighborhood of the closed disk $\overline{\mathbb{D}}$, it is a multiplier. Let us show directly that for $p = 4$ it fails to be extremal. We will take as a test function

$$f(z) := 1 - w^{\langle p'-1 \rangle} z,$$

so that $f \in \ell^p_A$ and

$$\|f\|_p^p = 1 + |w|^{p/(p-1)}.$$

Now fix $p = 4$, so that $p' = 4/3$. For $0 < a < 1$ we have the elementary inequalities

$$\begin{aligned}
a - a^2 &> 0 \\
3(a^2 - a) &< a^2 - a \\
a^3 + 1 - 3a + 3a^2 - a^3 &< a^2 - a + 1 \\
a^3 + (1-a)^3 &< \frac{a^3 + 1}{1 + a} \\
1 + \frac{(1-a)^3}{a^3} &< \frac{1 + 1/a^3}{1 + a}.
\end{aligned}$$

Substitute $a = |w|^{4/3}$ to obtain

$$1 + \frac{(1 - |w|^{4/3})^{4-1}}{|w|^4} < \frac{1 + 1/|w|^4}{1 + |w|^{4/3}}.$$

This yields the bound

$$\begin{aligned}\|B\|_p^p &= 1 + \frac{(1-|w|^{4/3})^{p-1}}{|w|^4} \\ &< \frac{1+1/|w|^4}{1+|w|^{4/3}} \\ &= \frac{\|Bf\|_p^p}{\|f\|_p^p} \\ &\le \|B\|_{\mathscr{M}_p}^p.\end{aligned}$$

This verifies that B fails to be an extremal multiplier.

Furthermore, it was shown in [49] that for $2 < p < \infty$, there are p-inner functions whose zero sets fail to be Blaschke sequences. Such a p-inner function cannot be a multiplier of ℓ^p_A, since it would also have to belong to $\ell^{p'}_A$. In the paper [43] p-inner functions are constructed whose zero sets accumulate at every point of the boundary circle $\mathbb{T}$. However, by [51, Corollary 12.6.3], a multiplier on ℓ^p_A for $p \in [1, 2)$ has unrestricted limits almost everywhere on $\mathbb{T}$. A p-inner function thus described cannot therefore be a multiplier on ℓ^p_A.

More can be said when $p \neq 2$. First, the extremality of a multiplier is inherited by its conjugate in the following sense.

Proposition 2.3.7. Let $\phi \in \mathscr{M}_p$. If $\|\phi\|_{\mathscr{M}_p} = \|\phi\|_p$, then $\|\lambda\|_{\mathscr{M}_p^*} = \|\lambda\|_{p'}$, where $\lambda = \phi^{\langle p-1 \rangle}$.

Proof. Put $g = \phi^{\langle p-1 \rangle}$. By hypothesis,

$$\|\phi\|_{\mathscr{M}_p} = \|\phi\|_p = \frac{|\langle \phi 1, g \rangle|}{\|1\|_p \|g\|_{p'}}.$$

Since $g \in \ell^{p'}$, we also have $g \in \mathscr{S}$ by Proposition 6.2. Relabeling g as the functional λ, we have

$$\|\phi\|_p = \frac{|\langle \phi, g \rangle|}{\|g\|_{p'}} \le \frac{|\langle \phi, \lambda \rangle|}{\|\lambda\|_{\mathscr{M}_p^*}} \le \|\phi\|_{\mathscr{M}_p}.$$

Equality is forced throughout, and we conclude that

$$\|\lambda\|_{p'} = \|\lambda\|_{\mathscr{M}_p^*}.$$

$\square$

This comes into play in the main result, to which we presently turn.

Theorem 2.3.5. Let $p \in (1, \infty) \setminus \{2\}$. A multiplier $\phi \in \mathscr{M}_p$ satisfies $\|\phi\|_{\mathscr{M}_p} = \|\phi\|_p$ if and only if ϕ is a monomial.

Proof. First, the claim is trivial if ϕ is identically zero, so let us suppose otherwise. Also, since $\mathscr{M}_p = \mathscr{M}_{p'}$ as point sets and with equal norms, it follows $\mathscr{M}_p^* = \mathscr{M}_{p'}^*$ with equal norms as well.

Now suppose that $2 < p < \infty$. Then $1 < p' < 2$, and we have

$$\|\phi\|_{p'} \geq \|\phi\|_p = \|\phi\|_{\mathscr{M}_p} = \|\phi\|_{\mathscr{M}_{p'}} \geq \|\phi\|_{p'}.$$

Equality is forced throughout. In particular, $\|\phi\|_p = \|\phi\|_{p'}$, which implies that ϕ is a monomial, according to Proposition 2.3.1 (this step fails if $p = p' = 2$).

Finally, let $1 < p < 2$, and suppose that $\phi \in \mathscr{M}_p$ is extremal; that is, $\|\phi\|_{\mathscr{M}_p} = \|\phi\|_p$.

Then

$$\begin{aligned}
\|\phi\|_{\mathscr{M}_p} &= \|\phi\|_p \\
&\geq \|\phi\|_{p'} \\
&\geq \|\phi\|_{\mathscr{M}_{p'}^*} \\
&\geq \frac{|\langle \phi^{\langle p-1\rangle}, \phi\rangle|}{\|\phi^{\langle p-1\rangle}\|_{\mathscr{M}_{p'}}} \\
&= \frac{\|\phi\|_p^p}{\|\phi^{\langle p-1\rangle}\|_{\mathscr{M}_{p'}}} \qquad (\star) \\
&= \frac{\|\phi\|_p^p}{\|\phi^{\langle p-1\rangle}\|_{p'}} \\
&= \|\phi\|_p.
\end{aligned}$$

This forces ϕ to be a monomial.

From the line $(\star)$ to the next, we used $\|\phi^{\langle p-1\rangle}\|_{\mathscr{M}_{p'}} = \|\phi^{\langle p-1\rangle}\|_{p'}$, which we derive as follows:

$$\begin{aligned}
\|\phi\|_p^{p-1} &= \frac{|\langle \phi^{\langle p-1\rangle}, \phi\rangle|}{\|\phi\|_p} \\
&= \frac{|\langle \phi^{\langle p-1\rangle}, \phi\rangle|}{\|\phi\|_{\mathscr{M}_p}} \\
&\leq \|\phi^{\langle p-1\rangle}\|_{\mathscr{M}_p^*} \\
&= \|\phi^{\langle p-1\rangle}\|_{\mathscr{M}_{p'}^*} \\
&\leq \|\phi^{\langle p-1\rangle}\|_{p'} \\
&= \|\phi\|_p^{p-1}
\end{aligned}$$

and equality must hold throughout.

Conversely, any monomial is a multiplier, and it can be checked by inspection that it is extremal. $\square$

Chapter 3

Planar Sets associated to Linear Operators

The final chapter of this thesis focuses on two planar sets associated to linear operators. The first is the Berezin range of an operator acting on an RKHS and the second is the spectrum of a Toeplitz operator acting on H^2_t.

3.1 Convexity of the Berezin Range

This section, which is joint work with Carl Cowen, discusses the convexity of the range of the Berezin transform. The work here can be found in preprint form in [62] and will appear in *Linear Algebra and its Applications*. Primarily, we focus on characterizing convexity of this range for a class of composition operators acting on the Hardy space of the unit disk.

3.1.1 Preliminaries

Throughout this section, $\mathcal{H}$ will be an RKHS on a set X; we do not impose any additional assumptions on $\mathcal{H}$. We recall the following definitions.

Definition 3.1.1. Let $\mathcal{H}$ be an RKHS on a set X and let T be a bounded linear operator on $\mathcal{H}$.

1. For $x \in X$, the *Berezin transform* of T at x (or *Berezin symbol of* T) is

$$\tilde{T}(x) := \langle T\hat{k}_x, \hat{k}_x \rangle_{\mathcal{H}}.$$

2. The *Berezin range* of T (or *Berezin set of* T) is

$$B(T) := \left\{ \langle T\hat{k}_x, \hat{k}_x \rangle_{\mathcal{H}} : x \in X \right\}.$$

3. The *Berezin radius of* T (or *Berezin number of* T) is

$$b(T) := \sup_{x \in X} |\tilde{T}(x)|.$$

3.1.2 Numerical Range and the Toeplitz-Hausdorff Theorem

In an RKHS, the Berezin range of an operator T is a subset of the numerical range of T,

$$W(T) := \{ \langle Tu, u \rangle : \|u\| = 1 \}.$$

The numerical range of an operator has some interesting properties. For example, it is well known that the spectrum of an operator is contained in the closure of its numerical range. Further, the numerical range of an operator is always *convex*− this results is known as the Toeplitz-Hausdorff Theorem [99, 145]. For more background on the numerical range, we point the reader to [94]. Much effort has gone into describing the geometry of the numerical range (e.g. see [65, 112]), but to the knowledge of the authors, there are only a handful of results describing the geometry of the Berezin range [107, Section 2.1], none of which address convexity.

As the convexity of the numerical range is arguably its most enigmatic property, we are motivated to ask the main question addressed in this section:

Given a bounded operator T acting on an RKHS $\mathcal{H}$, is $B(T)$ convex? Conversely, if $B(T)$ is convex, what can be said of T?

This question was initially pointed out by Karaev [107], and we give answers for a few classes of concrete operators. In general, as we will see, the Berezin range of an operator is *not* convex.

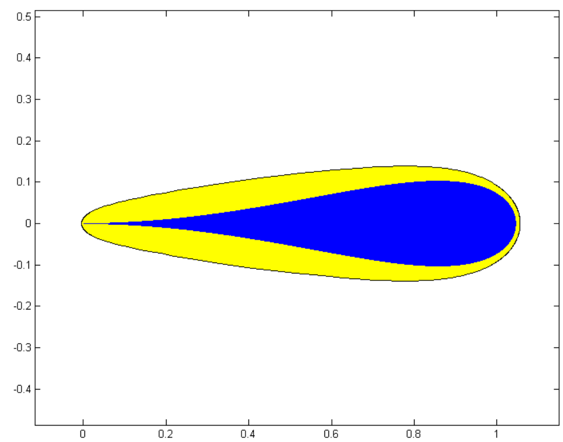

Figure 3.1: The numerical and Berezin ranges of the composition operator with symbol $\frac{1}{4}(1+z)^2$ acting on H^2, with the Berezin range appearing to be non-convex.

What fails in the Toeplitz-Hausdorff Theorem when restricting to normalized reproducing kernels? The proof of the theorem relies on the following result:

Elliptical Range Theorem. *Let A be a 2×2 matrix with complex entries and eigenvalues λ_1 and λ_2. Then $W(A)$ is an elliptical disk with λ_1 and λ_2 as foci, and $\{tr(A^*A)-|\lambda_1|^2-|\lambda_2|^2\}^{1/2}$ as its minor axis.*

There are several proofs of this theorem (e.g. [69, 120]) but a common thread is to produce a linear combination of unit vectors, that is, again, a unit vector. In general, this is simply impossible to do for normalized reproducing kernels; for any two points x_1 and x_2 and constants c_1 and c_2,

$$\frac{c_1\hat{k}_{x_1}+c_2\hat{k}_{x_2}}{\|c_1\hat{k}_{x_1}+c_2\hat{k}_{x_2}\|}$$

is not a normalized reproducing kernel. Another simple proof of the Toeplitz-Hausdorff theorem using an adaptation of the idea above can be found in [93].

3.1.3 Finite Dimensions and Multiplication Operators

In this section, we will give characterizations for convexity of the Berezin transform of some easily understood classes of bounded linear transformations. As a primer, let us consider the finite dimensional setting. Let $v = (v_1, \dots, v_n) \in \mathbb{C}^n$ and $X = \{1, \dots, n\}$. We can consider $\mathbb{C}^n$ as the set of all functions mapping $X \to \mathbb{C}$ by $v(j) = v_j$. Letting e_j be the jth standard basis vector for $\mathbb{C}^n$ under the standard inner product, we can view $\mathbb{C}^n$ as an RKHS with kernel

$$k(i,j) = \langle e_j, e_i \rangle.$$

Note that $k_j = \hat{k}_j$ for each $j = 1, \dots, n$. For any complex $n \times n$ matrix $A = (a_{jk})_{j,k=1}^n$, we have $\langle Ae_j, e_j \rangle = a_{jj}$. Thus, the Berezin range of A is simply

$$B(A) = \{a_{jj} : j = 1, \dots, n\},$$

which is just the collection of diagonal elements of A. It is immediate that the only way this set can be convex is if the diagonal elements of A are all equal:

Proposition 3.1.1. Let A be an $n \times n$ matrix with complex entries. Under the standard inner product for $\mathbb{C}^n$, the Berezin range of A is convex if and only if A has constant diagonal.

This proposition shows that the geometry of the Berezin range of a matrix is remarkably simple compared to the numerical range of the matrix. We also point out that the trace of a matrix can be recovered as the sum over the elements of its Berezin range. More generally, it is known in some spaces (e.g. see [162, Proposition 3.3]) that when T is a trace-class (or positive) operator, the trace of T can be recovered using the Berezin transform. We also point to the reader to [162, Chapter 3] for connections of the Berezin transform with the Fock space and BMO.

As the Berezin transform is immediately understood in the finite dimensional setting, we quickly shift to infinite dimensional spaces.

Recall that for any Hilbert (or Banach) space of functions $\mathcal{H}$,

$$\mathrm{Mult}(\mathcal{H}) := \{g \in \mathcal{H} : gf \in \mathcal{H} \text{ for all } f \in \mathcal{H}\}.$$

For $g \in \mathrm{Mult}(\mathcal{H})$, define the multiplication operator M_g on $\mathcal{H}$ by $M_g f = gf$.

Proposition 3.1.2. Let $\mathcal{H}$ be an RKHS on a set X and $g \in \mathrm{Mult}(\mathcal{H})$. Then the Berezin range of M_g is convex if and only if $g(X)$ is convex.

Proof. Let $x \in X$ and observe that

$$\tilde{M_g}(x) = \langle g\hat{k}_x, \hat{k}_x\rangle_{\mathcal{H}} = \frac{1}{\|k_x\|_{\mathcal{H}}^2} \langle gk_x, k_x\rangle_{\mathcal{H}} = \frac{1}{\|k_x\|_{\mathcal{H}}^2} g(x)k_x(x) = g(x).$$

Thus, $B(M_g) = g(X)$, and the result follows. □

Similar to the matricial case, this characterization of convexity is exceptionally simple. This may lead one to think that convexity of the Berezin range could be easily understood. However, this is not generally the case. In order to demonstrate this, we move to some classes of operators acting on H^2 where the characterization of convexity becomes more technically involved.

3.1.4 Composition Operators on H^2

A composition operator C_φ, induced by a complex-valued function $\varphi : X \to X$ (known as the *symbol* of the operator, not to be confused with the Berezin symbol of the operator), acts on a space of functions defined on X by

$$C_\varphi f := f \circ \varphi.$$

These operators are beloved by many and have a long and important history in function and operator theory (e.g. see the monographs [63, 140]). One motivation for studying the Berezin range of these operators is that they often elude Axler-Zheng type results; e.g. there are composition operators such that $\tilde{C}_\varphi(z) \to 0$ as $z \to \partial X$, but C_φ is *not compact* (see [127, Theorem 2.3]).

We begin by considering a very elementary class of composition operators acting on H^2. For $\zeta \in \mathbb{T}$ (the complex unit circle), consider the elliptic automorphism of the disk

$$\varphi(z) = \zeta z.$$

Acting on H^2, these operators have Berezin transform

$$\begin{aligned}\tilde{C}_\varphi(z) &= \langle C_\varphi \hat{k}_z, \hat{k}_z \rangle \\ &= \left(1 - |z|^2\right) \langle C_\varphi k_z, k_z \rangle \\ &= \frac{1 - |z|^2}{1 - |z|^2 \zeta}.\end{aligned}$$

With a little work, we come to a characterization of the convexity of $B(C_\varphi)$.

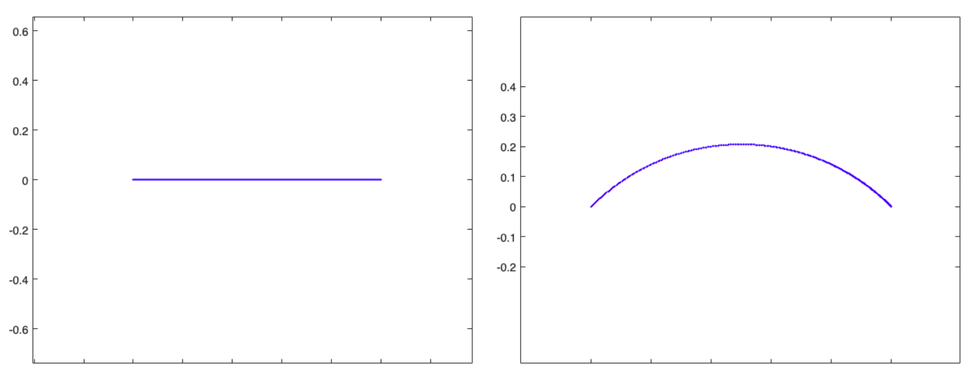

Figure 3.2: $B(C_\varphi)$ on H^2 for $\zeta = -1$ (left, apparently convex) and $\zeta = i\pi/4$ (right, apparently not convex).

Theorem 3.1.1. Let $\zeta \in \mathbb{T}$ and $\varphi(z) = \zeta z$. Then the Berezin range of C_φ acting on H^2 is convex if and only if $\zeta = 1$ or $\zeta = -1$.

Proof. Let us prove the backward implication first. Suppose first that $\zeta = 1$ so $\varphi(z) = z$. Putting $z = re^{i\theta}$ with $0 \le r < 1$, the calculations above show that

$$\tilde{C}_\varphi(re^{i\theta}) = \frac{1-r^2}{1-r^2} = 1$$

so $B(C_\varphi) = \{1\}$, which is convex. Similarly, for $\varphi(z) = -z$, we obtain

$$B(C_\varphi) = \left\{ \frac{1-r^2}{1+r^2} : r \in [0,1) \right\} = (0,1],$$

which is also convex.

Conversely, suppose that $B(C_\varphi)$ is convex. We have that

$$\tilde{C}_\varphi(re^{i\theta}) = \frac{1-r^2}{1+r^2\zeta},$$

which is a function independent of θ. Hence, by definition, $B(C_\varphi)$ is just a path in $\mathbb{C}$. By convexity, $B(C_\varphi)$ must then be either a point or a line segment. It is immediate that $B(C_\varphi)$ is a point if and only if $\zeta = 1$, so let us assume $B(C_\varphi)$ is a line segment. Note that $\tilde{C}_\varphi(0) = 1$ and that $\lim_{r\to 1^-} \tilde{C}_\varphi(re^{i\theta}) = 0$. This tells us that $B(C_\varphi)$ must be a line segment passing through the point 1 and approaching the origin. Consequently, we must have $\Im\{B(C_\varphi)\} = \{0\}$, which can happen if and only if $\Im\{\zeta\} = 0$. Thus, as $\zeta \in \mathbb{T}$, we have $\zeta = 1$ or $\zeta = -1$, the former of which was handled. $\square$

This theorem characterizes the convexity of the Berezin transform for composition operators with the elliptic automorphisms, which belong to a wider class of composition operators with automorphic symbols. We turn to characterize the convexity of the Berezin range of another such class.

For $\alpha \in \mathbb{D}$, consider the automorphism of the unit disk (known as a Blaschke factor)

$$\varphi_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z}$$

and the composition operator

$$C_{\varphi_\alpha} f = f \circ \varphi_\alpha.$$

Acting on H^2, we have

$$\begin{aligned}
\tilde{C}_{\varphi_\alpha}(z) &= \langle C_{\varphi_\alpha}\hat{k}_z, \hat{k}_z\rangle \\
&= \left(1-|z|^2\right)\langle C_{\varphi_\alpha} k_z, k_z\rangle \\
&= \left(1-|z|^2\right) k_z(\varphi_\alpha(z)) \\
&= \frac{1-|z|^2}{1-\overline{z}\varphi_\alpha(z)}.
\end{aligned}$$

With the aid of a computer, we can plot an example:

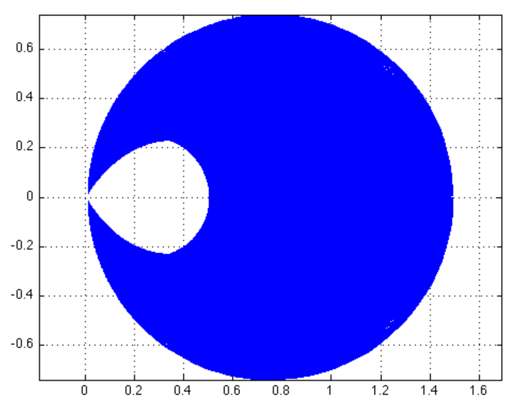

Figure 3.3: $B(C_{\varphi_\alpha})$ on H^2 for $\alpha = -1/2$

In the above example, the Berezin range is more geometrically interesting than in the elliptic case, and is clearly not convex(!). We will ultimately give a characterization for convexity of the Berezin range in this case, but require some calculations and observations first.

Lemma 3.1.1. On H^2, the real and imaginary parts of $\tilde{C}_{\varphi_\alpha}$ are given by

$$\Re\left\{\tilde{C}_{\varphi_\alpha}(z)\right\} = c_{\alpha,z}\left(1-|z|^2\right)\left(1-\Re\{\overline{\alpha}z\}) + 2(\Im\{\overline{\alpha}z\})^2\right)$$

and

$$\Im\left\{\tilde{C}_{\varphi_\alpha}(z)\right\} = c_{\alpha,z}\Im\{\overline{\alpha}z\}\left(1+|z|^2 - 2\Re\{\overline{\alpha}z\}\right),$$

where

$$c_{\alpha,z} = \frac{1-|z|^2}{|1-|z|^2+2i\Im\{\alpha\overline{z}\}|^2}.$$

Proof. Let us make some computations:

$$\begin{aligned}\tilde{C}_{\varphi_\alpha}(z) &= \frac{1-|z|^2}{1-\overline{z}\varphi_\alpha(z)}\\ &= \frac{(1-|z|^2)(1-\overline{\alpha}z)}{1-\overline{\alpha}z-\overline{z}(z-\alpha)}\\ &= \frac{(1-|z|^2)(1-\overline{\alpha}z)}{1-|z|^2+2i\Im\{\alpha\overline{z}\}}.\end{aligned}$$

Multiplying by a complex conjugate in the denominator, we have

$$\begin{aligned}\tilde{C}_{\varphi_\alpha}(z) &= c_{\alpha,z}\left(1-\overline{\alpha}z\right)\left(1-|z|^2-2i\Im\{\alpha\overline{z}\}\right)\\ &= c_{\alpha,z}\left(1-|z|^2+2i\Im\{\overline{\alpha}z\}-\overline{\alpha}z(1-|z|^2)+2i\Im\{\alpha\overline{z}\}\overline{\alpha}z\right)\\ &= c_{\alpha,z}\big[1-|z|^2+2i\Im\{\overline{\alpha}z\}-\left(\Re\{\overline{\alpha}z\}+i\Im\{\overline{\alpha}z\}\right)(1-|z|^2)\\ &\qquad -2i\Im\{\overline{\alpha}z\}(\Re\{\overline{\alpha}z\}+i\Im\{\overline{\alpha}z\})\big]\\ &= c_{\alpha,z}\big[1-|z|^2+2i\Im\{\overline{\alpha}z\}-(1-|z|^2)\Re\{\overline{\alpha}z\}-i(1-|z|^2)\Im\{\overline{\alpha}z\}\\ &\qquad -2i\Im\{\overline{\alpha}z\}\Re\{\overline{\alpha}z\}+2(\Im\{\overline{\alpha}z\})^2\big]\end{aligned}$$

Combining real and imaginary terms, then simplifying, gives

$$\begin{aligned}\tilde{C}_{\varphi_\alpha}(z) &= c_{\alpha,z}\left(1-|z|^2-(1-|z|^2)\Re\{\overline{\alpha}z\}+2(\Im\{\overline{\alpha}z\})^2\right)\\ &\qquad + ic_{\alpha,z}\left(2\Im\{\overline{\alpha}z\}-(1-|z|^2)\Im\{\overline{\alpha}z\}-2\Im\{\overline{\alpha}z\}\Re\{\overline{\alpha}z\}\right)\\ &= c_{\alpha,z}\left((1-|z|^2)(1-\Re\{\overline{\alpha}z\})+2(\Im\{\overline{\alpha}z\})^2\right)\\ &\qquad + ic_{\alpha,z}\Im\{\overline{\alpha}z\}\left(1+|z|^2-2\Re\{\overline{\alpha}z\}\right).\end{aligned}$$

Noting that $c_{\alpha,z}\in\mathbb{R}$ gives the result. □

We can use this information to gather some facts about the geometry of $B(C_{\varphi_\alpha})$.

Proposition 3.1.3. The Berezin range of C_{φ_α} on H^2 is closed under complex conjugation, and therefore symmetric about the real line.

Proof. Put $z = re^{i\theta}$ and $\alpha = \rho e^{i\psi}$. We claim that $\tilde{C}_{\varphi_\alpha}\left(re^{i\theta}\right) = \overline{\tilde{C}_{\varphi_\alpha}(re^{i(2\psi-\theta)})}$. This is the case if and only if $re^{i(2\psi-\theta)}\overline{\varphi_\alpha(re^{i(2\psi-\theta)})} = re^{-i\theta}\varphi_\alpha(re^{i\theta})$ or, equivalently, if and only if $e^{i2\psi}\overline{\varphi_\alpha(re^{i(2\psi-\theta)})} = \varphi_\alpha(re^{i\theta})$. So let us compute:

$$\begin{aligned}e^{2\psi i}\overline{\varphi_\alpha(re^{i(2\psi-\theta)})} &= e^{i2\psi}\,\frac{re^{i(\theta-2\psi)}-\rho e^{-i\psi}}{1-\rho e^{i\psi}re^{i(\theta-2\psi)}}\\ &= \frac{re^{i\theta}-\rho e^{i\psi}}{1-\rho e^{-i\psi}re^{i\theta}}\\ &= \varphi_\alpha(re^{i\theta}).\end{aligned}$$

□

We point out a corollary of this result that will be important in establishing the characterization of convexity.

Corollary 3.1.1. If the Berezin range of C_{φ_α} on H^2 is convex, then $\Re\left\{\tilde{C}_{\varphi_\alpha}(z)\right\}\in B(\tilde{C}_{\varphi_\alpha})$ for each $z\in\mathbb{D}$.

Proof. Suppose $B(\tilde{C}_{\varphi_\alpha})$ is convex. Then since $B(C_{\varphi_\alpha})$ is closed under complex conjugation (by Proposition 3.1.3), we have

$$\frac{1}{2}\tilde{C}_{\varphi_\alpha}(z) + \frac{1}{2}\overline{\tilde{C}_{\varphi_\alpha}(z)} = \Re\left\{\tilde{C}_{\varphi_\alpha}(z)\right\} \in B(\tilde{C}_{\varphi_\alpha}).$$

□

We these tools in hand, we can provide a characterization of convexity.

Theorem 3.1.2. The Berezin range of C_{φ_α} on H^2 is convex if and only if $\alpha = 0$.

Proof. If $\alpha = 0$, then $B(C_{\varphi_\alpha}) = \{1\}$, which is convex. Conversely, suppose $B(\tilde{C}_{\varphi_\alpha})$ is convex. Then since $B(C_{\varphi_\alpha})$ is closed under complex conjugation, we have

$$\frac{1}{2}\tilde{C}_{\varphi_\alpha}(z) + \frac{1}{2}\overline{\tilde{C}_{\varphi_\alpha}(z)} = \Re\left\{\tilde{C}_{\varphi_\alpha}(z)\right\} \in B(\tilde{C}_{\varphi_\alpha}).$$

Accordingly, for each $z \in \mathbb{D}$, we can find $w \in \mathbb{D}$ such that

$$\tilde{C}_{\varphi_\alpha}(w) = \Re\left\{\tilde{C}_{\varphi_\alpha}(z)\right\}.$$

In turn, $\Im\left\{\tilde{C}_{\varphi_\alpha}(w)\right\} = c_{\alpha,w}\Im\{\overline{\alpha}w\}\left(1 + |w|^2 - 2\Re\{\overline{\alpha}w\}\right) = 0$, where $c_{\alpha,w}$ is defined as in Lemma 3.1.1. But because $c_{\alpha,w} > 0$ and $(1 + |w|^2 - 2\Re(\overline{\alpha}w)) > 0$ for any $\alpha, w \in \mathbb{D}$, we have $\Im\left\{\tilde{C}_{\varphi_\alpha}(w)\right\} = 0$ if and only if $\Im\{\overline{\alpha}w\} = 0$. This says that α and w lie on a line passing through the origin. So we can put $w = r\alpha$ for some $r \in (-1/|\alpha|, 1/|\alpha|)$. Now we have

$$\begin{aligned}
\tilde{C}_{\varphi_\alpha}(w) &= \Re\left\{\tilde{C}_{\varphi_\alpha}(r\alpha)\right\} \\
&= \frac{(1-|r\alpha|^2)}{|1-|r\alpha|^2 + 2i\Im(\alpha\overline{r\alpha})|^2}\left((1-|r\alpha|^2)(1-\Re\{\overline{\alpha}r\alpha\}) + 2(\Im\{\overline{\alpha}r\alpha\})^2\right) \\
&= 1 - r|\alpha|^2.
\end{aligned}$$

Consequently, $\left\{\tilde{C}_{\varphi_\alpha}(r\alpha) : r \in (-1/|\alpha|, 1/|\alpha|)\right\} = (1-|\alpha|, 1+|\alpha|)$. However, putting $z = \rho e^{i\theta}$, an elementary exercise shows that

$$\lim_{\rho \to 1^-} \tilde{C}_{\varphi_\alpha}(\rho e^{i\theta}) = \begin{cases} 0, & \alpha \neq 0 \\ 1, & \alpha = 0 \end{cases}$$

This tells us that when $\alpha \neq 0$, given ϵ with $0 < \epsilon < 1 - |\alpha|$, there exists a point z such that $\left|\Re\left\{\tilde{C}_{\varphi_\alpha}(z)\right\}\right| < \epsilon$. But if $\tilde{C}_{\varphi_\alpha}(w) = \Re\left\{\tilde{C}_{\varphi_\alpha}(z)\right\}$, this is a contradiction since $\tilde{C}_{\varphi_\alpha}(w) \in (1-|\alpha|, 1+|\alpha|)$. Thus, $B(\tilde{C}_{\varphi_\alpha})$ cannot be convex unless $\alpha = 0$. □

3.1.5 Other Examples and Further Directions

We conclude this section with a few questions and remarks, motivated by some examples. We start with questions that naturally follow the results regarding composition operators on H^2 in the previous section. These operators have symbols which are automorphisms of the disk, taking the general form

$$b(z) = \zeta \frac{z-\alpha}{1-\overline{\alpha}z},$$

where $\zeta \in \mathbb{T}$ and $\alpha \in \mathbb{D}$.

Question 3.1.1. In H^2, can one characterize the convexity of the Berezin range for the composition operator with symbol b, with b defined as above, by combining the results in Theorems 3.1.1 and 3.1.2?

Even more generally, the examples in Section 3.1.4 belong to the class of composition operators having symbols known as Möbius transformations

$$M(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Another natural step would be to consider the class of operators C_M acting on H^2.

Question 3.1.2. Given the composition operator C_M acting on H^2, with M defined as above, what are necessary and sufficient conditions for $B(C_M)$ to be convex?

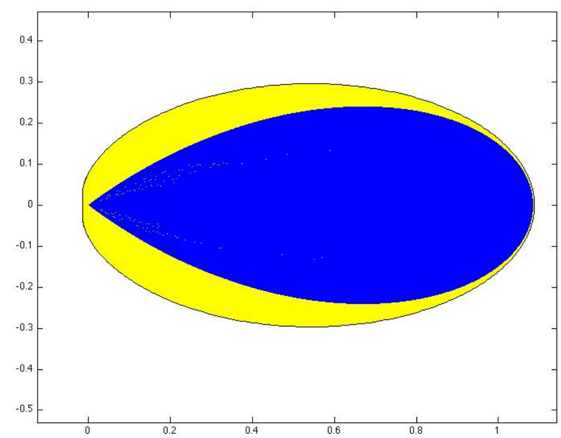

Figure 3.4: $W(C_M)$ and $B(C_M)$ on H^2 for $M(z) = \frac{4+2z}{9-z}$, which appears to be convex.

The Berezin range in the above figure seems to be symmetric about the real line, as was also seen in Figure 3 and in the first plot in Figure 2, but not in the second plot of Figure 2. This begs a question:

Question 3.1.3. For a composition operator C_φ acting on H^2, when is $B(C_\varphi)$ symmetric about the real line?

Recall that symmetry described in Proposition 3.1.3 was a key part in proving Theorem 3.1.2. It has also been of interest to understand when the numerical range of an operator can be a disk or an ellipse (cf. the Elliptical Range Theorem), and to deduce other properties concerning the circular symmetry of the numerical range (see [55, 91, 158]). Similar questions can be asked of the Berezin range:

Question 3.1.4. Given a bounded operator T on $\mathcal{H}$, when is $B(T)$ an ellipse or circular disk?

In fact, Karaev showed the Berezin range of multiplication by z^n acting on a certain model space is a disk [107, Example 2.1(a)]. The following example points toward the possibility that the Berezin range can be a *circular disk* for other operators.

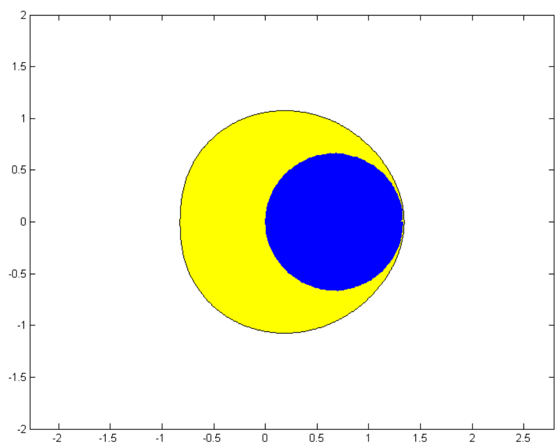

Figure 3.5: $W(C_\varphi)$ and $B(C_\varphi)$ on H^2 for $\varphi(z) = \frac{1+z}{2}$.

In light of the Axler-Zheng Theorem and its descendants, pointed out in Section **??**, the Berezin transform is arguably most useful in the Bergman space, and, admittedly, the Berezin range may be more natural to consider in this setting.

Question 3.1.5. Given a class of concrete operators acting on the Bergman space, what can be said about the convexity of the Berezin range of these operators?

Of the composition operator results presented here, the obstruction to providing immediate analogous results on the Bergman space is the increased complexity of the reproducing kernel, which is given by $k_w(z) = \frac{1}{(1-\overline{w}z)^2}$. In general, one may replace the Bergman space in the question above with any RKHS of holomorphic functions.

Looking back to Theorem 3.1.2, the multiplication operators considered were uncomplicated. In general, if $\mathcal{H}$ is a closed subspace of a Banach space Y, one may take a function $g \in \text{Mult}(Y)$ and consider the generalized *Toeplitz operator* T_g on $\mathcal{H}$, given by

$$T_g f = P_{\mathcal{H}} g f,$$

where $P_{\mathcal{H}}$ is the orthogonal projection from Y onto $\mathcal{H}$.

Question 3.1.6. Given a (generalized) Toeplitz operator T_g acting on an RKHS $\mathcal{H}$, what can be said about the convexity of $B(T_g)$?

We point here to the case where $g(z) = z$ and $\mathcal{H}$ is a model space (see [90] for background on model spaces). The Toeplitz operator in this case is known as a *compression of the shift* and the numerical range of this operator has been studied in both one and two variables [32, 33]. In general, these types of operators are known as truncated Toeplitz operators, and it has been shown that the compactness of these operators can be characterized in terms of the vanishing boundary behavior of the Berezin transform [110].

On many RKHSs, it is well known that various properties (e.g. boundedness or compactness) of certain operators can be deduced from considering only the action of the operator on the set of normalized reproducing kernels. Results of this type are known as *reproducing kernel theses*. The literature surrounding this idea is extensive, so we point to [124] for introduction and further reading. Many of the results mentioned in Section **??** can be likened to reproducing kernel theses. One may ask if such results exist between the Berezin and numerical ranges.

Question 3.1.7. Given an operator T on an RKHS $\mathcal{H}$, are there any properties of $W(T)$ that can deduced from $B(T)$?

For example, can one relate the Berezin radius and the numerical radius of an operator? An elementary estimate gives $b(T) \leq w(T)$. However, one might ask for a sharp constant

C (depending on T) so that $w(T) \leq Cb(T)$. In many of the examples we have presented, it seems that the quantities are equal. Can one characterize when this is the case? The upshot in proving an equality would be that the radius of the Berezin range is much easier to compute than the numerical radius. In the case of Toeplitz operators acting on H^2, and for some truncated Toeplitz operators, it is known that these quantities are equal [107]. However, these quantities are not equal in general [107, Example 2]. On H^2, it is known that the numerical radius of an operator can be bounded above and below by the Berezin radius of certain conjugates of the operator [87, Proposition 1]. In this vein, we end by mentioning recent interest in establishing inequalities for the Berezin radius (e.g. see [14, 88, 159, 160]).

3.2 Spectra for Toeplitz Operators associated with a Constrained Subalgebra

This section contains joint work with Benjamin Russo and Douglas Pfeffer, which can be found in preprint form in [82], and will appear in *Integral Equations and Operator Theory*. We show that for certain symbols, Toeplitz operators acting on H^2_t spaces have (relative point) spectrum which is connected.

3.2.1 Preliminaries

We begin by recalling some definitions and background information. The two-point algebra associated to fixed points $a, b \in \mathbb{D}$ is

$$\mathscr{A}_{a,b} := \{f \in H^\infty(\mathbb{D}) : f(a) = f(b)\}.$$

For fixed $a, b \in \mathbb{D}$, define

$$H^2_t := \{f \in H^2 : f(a) = tf(b)\},$$

where $t \in \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Recall that $\mathscr{A}_{a,b} H^2_t \subseteq H^2_t$ for every $t \in \widehat{\mathbb{C}}$.

For $\phi \in L^\infty$, we define the Toeplitz operator $T^t_\phi \colon H^2_t \to H^2_t$ by

$$T^t_\phi f = P_t(\phi f),$$

where P_t is the orthogonal projection from L^2 onto H^2_t. We denote the *eigenvalues of* T^t_ϕ *relative to* $\mathscr{A}_{a,b}$ as

$$\Lambda^{a,b}_\phi := \bigcup_{t \in \widehat{\mathbb{C}}} \sigma_p(T^t_\phi).$$

In order to avoid trivialities when discussing $\Lambda^{a,b}_\phi$, it is always assumed that the symbol ϕ is non-constant.

Before moving to results, we provide further background on the spectral theory of Toeplitz operators, as well as some relevant information on constrained subalgebras.

Investigations into the spectra of Toeplitz operators are diverse and far-reaching, beginning with spectral inclusion theorems for self-adjoint Toeplitz operators, which originated in [35, 97, 98]. Results involving the spectra of Toeplitz operators with continuous or piecewise continuous symbols can be found in [39, 68, 92, 113, 153]. The spectrum of a Toeplitz operator with analytic symbol was first investigated in [156]. Halmos initially posed the connected-spectrum question in [95], which was answered by Widom in [154, 155].

Following these investigations, there were several works aimed at exploring the spectra of Toeplitz operators acting on spaces other than H^2. For example, Devinatz [68] examined the spectra of real-symboled Toeplitz operators defined on a Hilbert space associated with a Dirichlet algebra. Another natural theme in this arena has been to consider Toeplitz operators acting on Hilbert spaces of analytic functions defined on assorted domains. Early work on multiply-connected planar domains was due to Abrahamse, where he established the existence of self-adjoint Toeplitz operators (relative to a single model space) with disconnected point spectrum [1, Section 5]. Further investigations include [5, 10, 11, 58], where more in-depth spectral analyses were conducted, including the study of large numbers of isolated eigenvalues in the gaps of the range of the symbol of the operator. The case of the annulus was further discussed in [34], where a version of Theorem 3.2.2 is established. Additional work on multiply connected domains can be found in [2, 19].

The introduction of Toeplitz operators to the setting of constrained subalgebras came about, in part, through the study of Pick interpolation. In short, the Pick interpolation problem asks for a multiplier, of norm less than or equal to one, that maps a set of initial points to a set of target points. It is natural to ask the interpolating function to obey additional algebraic constraints. In this direction, Pick interpolation on the Neil algebra was investigated in [66], and on a generalized two-point algebra in [131]. As a result, further work on the associated Toeplitz operators has been carried out. In [9], Anderson and Rochberg established a Widom-type invertibility theorem for Toeplitz operators associated with constrained subalgebras of the unit disk. In [130], Jury and Pfeffer established Widom and Szegő theorems for finite codimensional subalgebras of a class of uniform algebras defined on finite (connected) Riemann surfaces– of which the Neil and two-point algebras are the prototypical examples. Such Szegő and Widom theorems for Toeplitz operators on the Neil algebra were first discussed in [15]. For more results related to constrained algebras, see [19, 20, 72, 131, 135, 136].

3.2.2 Structure of H_t^2

We now move to establish the results of this section, starting with some structural observations of H_t^2 spaces. First, recall for each $f \in H^2$ and $w \in \mathbb{D}$, we have

$$f(w) = \langle f, k_w \rangle := \int_{\mathbb{T}} f \overline{k_w} \, d\mu,$$

where $d\mu$ is normalized Lebesgue measure on the unit circle $\mathbb{T}$, and

$$k_w(z) = \frac{1}{1-\overline{w}z}, \quad w, z \in \mathbb{D}$$

is the Szegő kernel. The spaces H_t^2 inherit the reproducing property from H^2, and we denote their reproducing kernels by k_w^t. We will also need the Blaschke product at a and b, given by

$$B_{a,b}(z) = \frac{z-a}{1-\bar{a}z}\frac{z-b}{1-\bar{b}z}.$$

We take the convention that $0 \cdot \infty = 0$ on $\widehat{\mathbb{C}}$.

Proposition 3.2.1. Let $t \in \mathbb{C} \setminus \{0\}$. For each H_t^2 space associated to $a, b \in \mathbb{D}$, we have

(i) $k_a^t = \bar{t}k_b^t$.

(ii) $H_t^2 = \mathbb{C}k_a^t \oplus B_{a,b}H^2$.

(iii) The set $\{k_a^t/\|k_a^t\|\} \cup \{B_{a,b}z^n\}_{n\geq 0}$ is an orthonormal basis for H_t^2.

Proof. Point ((i)) follows directly from definition. In order to establish ((ii)), we show $B_{a,b}H^2$ is the orthogonal complement of $\mathbb{C}k_a^t$ in H_t^2. Suppose $f \in H_t^2$ with $\langle f, k_a^t\rangle = 0$. Then $\langle f, \bar{t}k_b^t\rangle = 0$, and we see f vanishes at a and b, so $f \in B_{a,b}H^2$. Conversely, suppose $f \in B_{a,b}H^2$. Since $B_{a,b}H^2 \subset H_t^2$, we have $f \in H_t^2$ and $f(a) = 0$. Thus, f is orthogonal to $\mathbb{C}k_a^t$, and the result follows. Point ((iii)) now follows immediately by noting that, in H^2, multiplication by $B_{a,b}$ is an isometry and the monomials are orthogonal.

□

Using the above proposition, a precise formula for the reproducing kernel can be given. Specifically,

$$k_a^t(z) = \frac{\bar{t}}{\bar{t}-\tau}(k_a(z) - \tau k_b(z)),$$

where $\tau = \frac{k_a(a) - tk_a(b)}{k_b(a) - tk_b(b)}$. Further, for $z, w \in \mathbb{D}$, the reproducing kernel for H^2_t is given by

$$k^t_w(z) = \frac{\overline{k^t_a(w)}}{\|k^t_a\|^2} k^t_a(z) + \overline{B_{a,b}(w)} B_{a,b}(z) k_w(z).$$

We end this section with a proposition about the inner-outer factorization of a function $g \in \ker(T^t_\phi)$. Specifically, we note that the outer factor of such a function g must also be in the kernel of a Toeplitz operator (with a possibly different parameter t).

Proposition 3.2.2. Let $\phi \in L^\infty$ and suppose $g \in \ker(T^t_\phi)$. Put $g = \theta G$ with θ inner and G outer. Then $G \in \ker(T^s_\phi)$, where $s = G(a)/G(b)$ (i.e. $G \in H^2_s$).

Proof. Let $h \in H^2_s$ and note as G is outer, it is non-vanishing on $\mathbb{D}$, which means $s \in \mathbb{C} \setminus \{0\}$. Since $g = \theta G \in H^2_t$ and $G \in H^2_s$, it follows that $\theta \in H^2_{t/s}$. In turn, $\theta h \in H^2_t$ and we have

$$\left\langle T^s_\phi G,\ h \right\rangle = \int_{\mathbb{T}} \phi G \overline{h} \, d\mu = \int_{\mathbb{T}} \phi g \overline{\theta h} \, d\mu = \left\langle T^t_\phi g,\ \theta h \right\rangle = 0.$$

As this holds for all $h \in H^2_s$, we have $G \in \ker(T^s_\phi)$. $\square$

3.2.3 Toeplitz Operators with Analytic Symbols

In this section, we provide a proof of Theorem 3.2.1. We begin by showing that the point spectrum of T^t_ϕ with analytic symbol is a singleton. Recall that P_t is the orthogonal projection from L^2 onto H^2_t.

Proposition 3.2.3. If $\phi \in H^\infty$ is non-constant, then

$$\sigma_p(T^t_\phi) = \langle \phi k^t_a, k^t_a \rangle / \|k^t_a\|^2.$$

Proof. Suppose that $(T^t_\phi - \lambda I)g = 0$ for $\lambda \in \mathbb{C}$ and $g \in H^2_t \setminus \{0\}$. Using Proposition 3.2.1, we can put $g = ck_a^t + B_{a,b}h$, for some $c \in \mathbb{C}$ and $h \in H^2$ to obtain

$$\begin{aligned}
0 &= T_\phi^t(ck_a^t + B_{a,b}h) - \lambda(ck_a^t + B_{a,b}h) \\
&= P_t(\phi(ck_a^t + B_{a,b}h)) - \lambda(ck_a^t + B_{a,b}h) \\
&= cP_t(\phi k_a^t) + P_t(\phi B_{a,b}h) - \lambda(ck_a^t + B_{a,b}h) \\
&= cP_t(\phi k_a^t) + \phi B_{a,b}h - \lambda(ck_a^t + B_{a,b}h),
\end{aligned}$$

where the last equality follows from the fact that $\phi B_{a,b}h \in B_{a,b}H^2 \subseteq H_t^2$. Now put $P_t(\phi k_a^t) = c'k_a^t + B_{a,b}f$ where $c' := \langle \phi k_a^t, k_a^t\rangle / \|k_a^t\|^2$ and $f \in H^2$. Then from the above equality we have

$$0 = c(c'k_a^t + B_{a,b}f) + \phi B_{a,b}h - \lambda(ck_a^t + B_{a,b}h)$$

which implies that

$$B_{a,b}((\lambda - \phi)h - cf) = c(c' - \lambda)k_a^t.$$

As the left hand side is in $B_{a,b}H^2$ and right hand side is in $\operatorname{span}\{k_a^t\}$, which are orthogonal, we see that both sides must equal zero. Considering that $c(c' - \lambda)k_a^t = 0$, i.e. $c(c' - \lambda) = 0$, we have $c = 0$ or $(c' - \lambda) = 0$.

If $c = 0$, this means that $g = B_{a,b}h$, which we will show cannot happen. Referring back to the first set of equalities in this proof, we then get that

$$0 = \phi B_{a,b}h - \lambda B_{a,b}h = (\phi - \lambda)B_{a,b}h.$$

This now says that either $\phi = \lambda$, which is a contradiction, or that $h = 0$. But if $h = 0$, then $B_{a,b}h = g = 0$, which is also a contradiction.

In turn, it must be that $c' - \lambda = 0$, which implies that $\lambda = c' = \langle \phi k_a^t, k_a^t\rangle / \|k_a^t\|^2$, as claimed. $\square$

Remark 3.2.1. We point out that the quantity $\langle \phi k_a^t, k_a^t\rangle / \|k_a^t\|^2$ is the Berezin transform of T_ϕ^t at a, as discussed in the previous section here (or [62, Section 2]).

We now provide the main result of this section, which completely characterizes the spectrum and relative eigenvalues of T_ϕ^t when the symbol is in our multiplier algebra.

Theorem 3.2.1. If $\phi \in \mathscr{A}_{a,b}$, then

(i) $\sigma(T_\phi^t) = \overline{\phi(\mathbb{D})}$ and

(ii) $\Lambda_\phi^{a,b} = \{\phi(a)\}$.

In particular, both $\sigma(T_\phi^t)$ and $\Lambda_\phi^{a,b}$ are connected.

Proof. To see ((i)), note as $\phi \in \mathscr{A}_{a,b}$, we have $T_\phi^t f = \phi f$. Now let $z_0 \in \mathbb{D}$ and suppose that $\lambda = \phi(z_0)$. Observe, for any $f \in H_t^2$, that

$$\left((T_\phi^t - \lambda I)f\right)(z_0) = (\phi(z_0) - \lambda)f(z_0) = 0.$$

Thus, $T_\phi^t - \lambda$ cannot be surjective, since all functions in its range must have a zero at z_0. This shows $\phi(\mathbb{D}) \subseteq \sigma(T_\phi^t)$. Since the spectrum is compact and hence closed, we have $\overline{\phi(\mathbb{D})} \subseteq \sigma(T_\phi^t)$. To see the inclusion $\sigma(T_\phi^t) \subseteq \overline{\phi(\mathbb{D})}$, assume that $\lambda \notin \overline{\mathbb{D}}$. In turn, $\operatorname{dist}(\lambda, \overline{\mathbb{D}}) := \delta > 0$. Since $|\phi(z) - \lambda| \geq \delta$ for all $z \in \mathbb{D}$, we have $1/(\phi(z) - \lambda)$ is analytic and bounded by $1/\delta$ on $\mathbb{D}$. Hence, we have $(T_\phi^t - \lambda I)^{-1} = T_{1/(\phi-\lambda)}$, and so $\lambda \notin \sigma(T_\phi^t)$.

To see ((ii)), note first that $\mathscr{A}_{a,b} \subseteq H^\infty$ and therefore Proposition 3.2.3 guarantees that $\sigma_p(T_\phi^t) = \langle \phi k_a^t, k_a^t\rangle / \|k_a^t\|^2$. However, since $\phi \in \mathscr{A}_{a,b}$, we have $\phi k_a^t \in H_t^2$, and therefore $\langle \phi k_a^t, k_a^t\rangle = \phi(a) k_a^t(a)$. This gives $\sigma_p(T_\phi^t) = \phi(a)$, which is independent of t, so we have $\Lambda_\phi^{a,b} = \{\phi(a)\}$. □

Before we turn to real-valued symbols, we provide a brief discussion of arbitrary symbols to establish that $\sigma(T_\phi^t)$ is contained in the closed convex hull of the essential range of ϕ, a result true for *any* $\phi \in L^\infty$. This result is directly analogous to the classical result by Brown and Halmos for unconstrained Toeplitz operators (see [71, Corollary 7.19]). An important part of the proof of the following result is [130, Lemma 4.5], which observes that $\|\phi\|_\infty = \|T^t_\phi\|$.

In order to show this spectral inclusion, we first note that the closed convex hull of a set $E \subseteq \mathbb{C}$ is the intersection of all open half-planes that contain E (e.g. see [71, Lemma 7.17].) We denote the closed convex hull of the essential range of a function ϕ by

$$\mathcal{R}_e(\phi) := \bigcap \{S : \operatorname{essran}(\phi) \subseteq S \text{ and } S \text{ is a half-plane in } \mathbb{C}\}.$$

Note that $\mathcal{R}_e(\phi)$ is necessarily a closed subset of $\mathbb{C}$. Thus, if ϕ is bounded, the above set is compact. We now have the following spectral inclusion.

Proposition 3.2.4. If $\phi \in L^\infty$, then $\sigma(T^t_\phi)$ is contained in $\mathcal{R}_e(\phi)$.

Proof. We show that if $\tau \notin \mathcal{R}_e(\phi)$, then $\tau \notin \sigma(T^t_\phi)$. Accordingly, suppose $\tau \in \mathbb{C} \setminus \mathcal{R}_e(\phi)$. Then $\tau \notin S$ for some open half-plane $S \subseteq \mathbb{C}$ containing $\operatorname{essran}(\phi)$. After translation and rotation, we can assume S is the right half-plane $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$. Thus, since $\phi - \tau$ is bounded, we have that $\mathcal{R}_e(\phi - \tau)$ is a compact subset of S. Hence, there exists $\varepsilon > 0$ such that

$$\varepsilon \mathcal{R}_e(\phi - \tau) \subseteq \{z \in \mathbb{C} : |z - 1| < 1\},$$

and therefore $\|1 - \varepsilon(\phi - \tau)\|_\infty < 1$. It is known that $\|1 - \varepsilon(\phi - \tau)\|_\infty = \|I - T^t_{\varepsilon(\phi-\tau)}\|$ (see [130, Lemma 4.5]), which gives $\|I - T^t_{\varepsilon(\phi-\tau)}\| < 1$. This implies that $T^t_{\varepsilon(\phi-\tau)} = \varepsilon T^t_{\phi-\tau}$ is invertible. Therefore $\tau \notin \sigma(T^t_\phi)$. $\square$

We now turn to results regarding real-valued symbols. Without surprise, this setting poses a greater challenge than the analytic setting.

3.2.4 Toeplitz Operators with Real-valued Symbols

In this section, we provide a proof of Theorem 3.2.2. We begin by noting that, for $\phi \in L^\infty$ real-valued, the space $\ker(T^t_\phi)$ is naturally associated with the annihilator of $\mathscr{A}_{a,b} + \mathscr{A}^*_{a,b} := \{h_1 + \overline{h_2} : h_1, h_2 \in \mathscr{A}_{a,b}\}$. In order to establish this fact, we first record a few propositions.

Proposition 3.2.5. Let $\phi \in L^\infty$ be real-valued. If $g \in \ker(T^t_\phi)$, then $\phi|g|^2$ annihilates $\mathscr{A}_{a,b} + \mathscr{A}^*_{a,b}$.

Proof. Suppose $T^t_\phi g = 0$ and let $h \in \mathscr{A}_{a,b}$. Then $hg \in H^2_t$ and we have

$$0 = \langle T^t_\phi g, hg \rangle = \int_{\mathbb{T}} \phi g \overline{hg}\, d\mu = \int_{\mathbb{T}} \phi |g|^2 \overline{h}\, d\mu.$$

As $\phi|g|^2$ is real-valued, we also have $\int_{\mathbb{T}} \phi|g|^2 h\, d\mu = 0$, which establishes the claim. $\square$

We now characterize annihilating measures for $\mathscr{A}_{a,b} + \mathscr{A}^*_{a,b}$, considered as a subspace of L^∞.

Proposition 3.2.6. Let $\nu << \mu$ be a measure whose Radon-Nikodym derivative with respect to μ is in L^1. Then $\frac{d\nu}{d\mu}$ annihilates $\mathscr{A}_{a,b} + \mathscr{A}^*_{a,b}$ if and only if there exists $d_1, d_2 \in \mathbb{C}$ such that

$$\frac{d\nu}{d\mu} = d_1(k_a - k_b) + \overline{d_2(k_a - k_b)} \quad \textit{a.e. on } \mathbb{T}.$$

Proof. We begin with the backward implication. Assume that there exists $d_1, d_2 \in \mathbb{C}$ such that $\frac{d\nu}{d\mu} = d_1(k_a - k_b) + \overline{d_2(k_a - k_b)}$ and let $h_1 + \overline{h_2} \in \mathscr{A}_{a,b} + \mathscr{A}^*_{a,b}$. It follows that

$$\begin{aligned}\int_{\mathbb{T}} (h_1 + \overline{h_2}) \frac{d\nu}{d\mu}\, d\mu &= \int_{\mathbb{T}} (h_1 + \overline{h_2})(d_1(k_a - k_b) + \overline{d_2(k_a - k_b)})\, d\mu \\ &= d_1 \int_{\mathbb{T}} h_1(k_a - k_b)\, d\mu + \overline{d_2} \int_{\mathbb{T}} h_1 \overline{k_a - k_b}\, d\mu \\ &\quad + d_1 \int_{\mathbb{T}} \overline{h_2}(k_a - k_b)\, d\mu + \overline{d_2} \int_{\mathbb{T}} \overline{h_2 k_a - k_b}\, d\mu\,.\end{aligned}$$

The first and fourth terms are both zero since $(k_a - k_b)(0) = 0$ and the integrands are entirely analytic and anti-analytic, respectively. Further, since h_1 and h_2 are both taken from $\mathscr{A}_{a,b}$, we have

$$\begin{aligned}\int_{\mathbb{T}} (h_1 + \overline{h_2}) \frac{d\nu}{d\mu}\, d\mu &= \overline{d_2} \int_{\mathbb{T}} h_1 \overline{k_a - k_b}\, d\mu + d_1 \int_{\mathbb{T}} \overline{h_2}(k_a - k_b)\, d\mu \\ &= \overline{d_2}\langle h_1, k_a - k_b\rangle + d_1 \overline{\langle h_2, k_a - k_b\rangle} \\ &= 0.\end{aligned}$$

Thus, $\frac{d\nu}{d\mu}$ annihilates $\mathscr{A}_{a,b} + \mathscr{A}^*_{a,b}$.

For the forward implication, assume $\frac{d\nu}{d\mu}$ annihilates $\mathscr{A}_{a,b} + \mathscr{A}^*_{a,b}$ and let $p_n(z) = z^n(z - a)(z - b)$ for $n \geq 0$. Since p_n vanishes at a and b, we have $p_n, \overline{p_n} \in \mathscr{A}_{a,b} + \mathscr{A}^*_{a,b}$. Hence,

$$0 = \int_{\mathbb{T}} p_n \frac{d\nu}{d\mu} d\mu = \int_{\mathbb{T}} z^{n+2} \frac{d\nu}{d\mu} d\mu - \int_{\mathbb{T}} (a + b) z^{n+1} \frac{d\nu}{d\mu} d\mu + \int_{\mathbb{T}} ab z^n \frac{d\nu}{d\mu} d\mu.$$

This gives that the negative Fourier coefficients of $\frac{d\nu}{d\mu}$ must satisfy the linear recurrence relation

$$\widehat{\frac{d\nu}{d\mu}}(n + 2) = (a + b)\widehat{\frac{d\nu}{d\mu}}(n + 1) - ab\widehat{\frac{d\nu}{d\mu}}(n).$$

One can readily verify that this recurrence relation is solved by $\widehat{\frac{d\nu}{d\mu}}(n) = c_1 a^n + c_2 b^n$, for some constants $c_1, c_2 \in \mathbb{C}$. Similarly, considering $0 = \int_{\mathbb{T}} \overline{p_n} \frac{d\nu}{d\mu} d\mu$, we see that the positive Fourier coefficients of $\frac{d\nu}{d\mu}$ must satisfy the recurrence relation

$$\widehat{\frac{d\nu}{d\mu}}(n-2) = (\overline{a+b})\widehat{\frac{d\nu}{d\mu}}(n-1) - \overline{ab}\widehat{\frac{d\nu}{d\mu}}(n).$$

This recurrence relation is solved by $\widehat{\frac{d\nu}{d\mu}}(n) = c_3\overline{a}^n + c_4\overline{b}^n$, for some constants $c_3, c_4 \in \mathbb{C}$. Noting that $\widehat{\frac{d\nu}{d\mu}}(0) = 0$, we find that, almost everywhere on $\mathbb{T}$, we have

$$\begin{aligned}
\frac{d\nu}{d\mu}(z) &= \sum_{n<0} \widehat{\frac{d\nu}{d\mu}}(n) z^n + \sum_{n>0} \widehat{\frac{d\nu}{d\mu}}(n) z^n \\
&= \sum_{n<0} (c_1 a^n + c_2 b^n)\overline{z}^n + \sum_{n>0} (c_3 \overline{a}^n + c_4 \overline{b}^n) z^n \\
&= c_1 \sum_{n\leq 0} (a\overline{z})^n - c_1 + c_2 \sum_{n\leq 0} (b\overline{z})^n - c_2 \\
&\quad + c_3 \sum_{n\geq 0} (\overline{a}z)^n - c_3 + c_4 \sum_{n\geq 0} (\overline{b}z)^n - c_4 \\
&= c_1 \overline{k_a(z)} - c_1 + c_2 \overline{k_b(z)} - c_2 + c_3 k_a(z) - c_3 + c_4 k_b(z) - c_4.
\end{aligned}$$

For any $h \in \mathscr{A}_{a,b}$ with $h(0) = 0$, we now have

$$0 = \int_{\mathbb{T}} h \frac{d\nu}{d\mu} d\mu = c_3 h(a) + c_4 h(b).$$

Since $h(a) = h(b)$, we have $c_4 = -c_3$. Similarly, for $\overline{h} \in \mathscr{A}_{a,b}^*$ with $h(0) = 0$, we have

$$0 = \int_{\mathbb{T}} \overline{h} \frac{d\nu}{d\mu} d\mu = c_1 \overline{h(a)} + c_2 \overline{h(b)}.$$

Again, since $\overline{h(a)} = \overline{h(b)}$, we have $c_2 = -c_1$. Letting $d_1 = c_3$ and $d_2 = c_1$, we have

$$\frac{d\nu}{d\mu} = d_1(k_a - k_b) + \overline{d_2(k_a - k_b)}.$$

$\square$

The upshot now is that, by passing to the outer factor G of $g \in \ker(T_\phi^t)$, we have an explicit description of $\phi|G|^2$. First, however, we need the following lemma.

Lemma 3.2.1. If $G \in H_t^2$ is outer, then $G\mathscr{A}_{a,b}$ is dense in H_t^2.

Proof. Suppose $f, G \in H_t^2$ with G outer. Since G is outer, there exists a sequence of polynomials $\{p_n\}$ such that $p_n G \to f$. Let $P_{a,b} : H^2 \to (H_{t=1}^2)^\perp$ be the projection onto the span$\{k_a - k_b\}$ and define

$$q_n := p_n - P_{a,b}(p_n) = p_n - \left\langle p_n, \ \frac{k_a - k_b}{\|k_a - k_b\|} \right\rangle \frac{k_a - k_b}{\|k_a - k_b\|}.$$

We claim that $q_n G \to f$. Observe that $q_n \in H_{t=1}^2$, and since q_n is a linear combination of H^∞ functions, we have that $q_n \in \mathscr{A}_{a,b}$. Observe that $q_n G \to f$ when

$$\left\langle p_n, \ \frac{k_a - k_b}{\|k_a - k_b\|} \right\rangle \frac{k_a - k_b}{\|k_a - k_b\|} = (p_n(a) - p_n(b)) \frac{k_a - k_b}{\|k_a - k_b\|^2} \to 0,$$

which happens if and only if $(p_n(a) - p_n(b)) \to 0$. Since $p_n G \to f$ in norm, it also converges pointwise. Therefore,

$$p_n(a)G(a) \to f(a) \quad \text{and} \quad p_n(b)G(b) \to f(b).$$

Since G and f are in H_t^2 and G is outer (therefore $G(a) \neq 0$), we have

$$\begin{aligned}\lim_{n\to\infty}(p_n(a) - p_n(b)) &= \lim_{n\to\infty}\frac{1}{G(a)}\left(p_n(a)G(a) - tG(b)p_n(b)\right)\\ &= \frac{1}{G(a)}\left(f(a) - tf(b)\right) = 0.\end{aligned}$$

$\square$

We now prove the aforementioned identification between $\ker(T_\phi^t)$ and the annihilator measures of $\mathscr{A}_{a,b} + \mathscr{A}_{a,b}^*$.

Proposition 3.2.7. Let $\phi \in L^\infty$ be real-valued and suppose $G \in H_t^2$ is outer. Then $G \in \ker(T_\phi^t)$ if and only if there exists a constant $c \in \mathbb{C}$ so that

$$\phi|G|^2 = c(k_a - k_b) + \overline{c(k_a - k_b)}.$$

Proof. We prove the backward direction first. Suppose $G \in H_t^2$ is outer and let $h \in \mathscr{A}_{a,b}$. Observe that

$$\langle T_\phi^t G, hG\rangle = \int_{\mathbb{T}} \phi|G|^2\overline{h}\,d\mu = \int_{\mathbb{T}} (c(k_a - k_b) + \overline{c(k_a - k_b)})\overline{h}\,d\mu = 0.$$

In view of Lemma 3.2.1, $T_\phi^t G \equiv 0$.

To see the forward direction, observe that if $T_\phi^t G = 0$, then it follows from Proposition 3.2.5 that $\phi|G|^2$ annihilates $\mathscr{A}_{a,b} + \mathscr{A}_{a,b}^*$. Further, since $G \in L^2$ and $\phi \in L^\infty$, we have that $\phi|G|^2 \in L^1$. Thus, the measure $\phi|G|^2\,d\mu$ is a measure that is absolutely continuous with respect to μ and whose Radon-Nikodym derivative with respect to μ is in L^1. It now follows from Proposition 3.2.6 that there exists $d_1, d_2 \in \mathbb{C}$ such that

$$\phi|G|^2 = d_1(k_a - k_b) + \overline{d_2(k_a - k_b)}.$$

However, since $\phi|G|^2$ is real-valued, this is only possible if $d_1 = d_2$. Putting c as this common value, the result follows. $\square$

Remark 3.2.2. In the backward direction of Proposition 3.2.7, we require G to be outer in order to use Lemma 3.2.1. However, the forward direction does not require G to be outer.

We will now have an interlude to discuss the behavior of $c(k_a - k_b) + \overline{c(k_a - k_b)}$ on $\mathbb{T}$.

Lemma 3.2.2. For any choice of $a, b \in \mathbb{D}$ and $c \in \mathbb{C} \setminus \{0\}$, the function

$$\operatorname{Re}\left(c(k_a(e^{it}) - k_b(e^{it}))\right)$$

is positive (negative) on precisely one proper sub-arc of $\mathbb{T}$.

Proof. Begin by letting $u(z) := k_a(z) - k_b(z)$ and noting

$$u(z) = \frac{\overline{(a-b)}z}{(1-\overline{a}z)(1-\overline{b}z)}.$$

In turn, we see that cu fixes the origin for any value of $c \in \mathbb{C}$ and is analytic on a disk containing $\overline{\mathbb{D}}$. By the open mapping theorem, we have 0 is in the interior of $cu(\mathbb{D})$. Further, and again by the open mapping theorem, we have that the boundary of $cu(\mathbb{D})$ is contained in $cu(\mathbb{T})$. But since 0 is in the interior of $cu(\mathbb{D})$, it must be that the boundary of $cu(\mathbb{D})$ has a component both in the left-half and right-half plane, and so must $cu(\mathbb{T})$. Also notice that

$$\begin{aligned}
\operatorname{Re}(cu(z)) &= \frac{1}{2}\left[\frac{c\overline{(a-b)}z}{(1-\overline{a}z)(1-\overline{b}z)} + \frac{\overline{c}(a-b)\overline{z}}{(1-a\overline{z})(1-b\overline{z})}\right] \\
&= \frac{c\overline{(a-b)}z(1-a\overline{z})(1-b\overline{z}) + \overline{c}(a-b)\overline{z}(1-\overline{a}z)(1-\overline{b}z)}{2\left|(1-\overline{a}z)\right|^2\left|(1-\overline{b}z)\right|^2}.
\end{aligned}$$

On the circle, $\operatorname{Re}(cu(e^{it})) = 0$ is a homogeneous trigonometric polynomial equation of degree one, so $\operatorname{Re}(cu(e^{it}))$ has at most two zeros. Equivalently, $cu(\mathbb{T})$ is purely imaginary at most twice. But since $cu(\mathbb{T})$ has a component in both the left-half and right-half plane, it follows that $\mathrm{Re}(cu(e^{it}))$ has precisely two zeros, and changes sign precisely twice. The result follows. $\square$

We have one more important observation to make about the behavior of the zeros of $\mathrm{Re}(c(k_a - k_b))$ on the circle.

Proposition 3.2.8. For any constants $c, d \in \mathbb{C}$, with c not a non-negative multiple of d, we have on $\mathbb{T}$ that

$$\{d(k_a - k_b) + \overline{d(k_a - k_b)} > 0\} \not\subset \{c(k_a - k_b) + \overline{c(k_a - k_b)} > 0\}.$$

Proof. Begin by letting $u(z) := k_a(z) - k_b(z)$ and observing that $\{du + \overline{du} > 0\} = \{cu + \overline{cu} > 0\}$ if and only if c and d are non-negative real multiples. Suppose for contradiction that

$$\{du + \overline{du} > 0\} \subsetneq \{cu + \overline{cu} > 0\}.$$

Then, letting $t_c = \operatorname{Arg} c$ and $t_d = \operatorname{Arg} d$, we have

$$\{e^{it_d}u + \overline{e^{it_d}u} > 0\} \subsetneq \{e^{it_c}u + \overline{e^{it_c}u} > 0\}.$$

Let $e^{i\theta_{c_1}}$ and $e^{i\theta_{c_2}}$ be the solutions to $cu + \overline{cu} = 0$ on $\mathbb{T}$. Similarly, let $e^{i\theta_{d_1}}$ and $e^{i\theta_{d_2}}$ be the solutions to $du + \overline{du} = 0$ on $\mathbb{T}$. By the containment hypothesis and Lemma 3.2.2, we have $e^{i\theta_{c_1}}, e^{i\theta_{c_2}} \in \{du + \overline{du} \leq 0\}$ (we include equality with zero as two of the roots $e^{i\theta_{c_j}}, e^{i\theta_{d_j}}$, $j = 1, 2$, may be equal). Without loss of generality, this implies that $\mathrm{Re}(e^{it_d}u(e^{i\theta_{c_1}})) \leq 0$ and $\mathrm{Re}(e^{it_d}u(e^{i\theta_{c_2}})) < 0$.

By definition, we also have that $\mathrm{Re}(e^{it_c}u(e^{i\theta_{c_1}})) = \mathrm{Re}(e^{it_c}u(e^{i\theta_{c_2}})) = 0$. Further, since $0 \in e^{it_c}u(\overline{\mathbb{D}})$, we have that $\mathrm{Im}(e^{it_c}u(e^{i\theta_{c_1}}))$ and $\mathrm{Im}(e^{it_c}u(e^{i\theta_{c_2}}))$ have different signs. It now follows that for any $\theta \in \mathbb{R}$ not a multiple of π, the signs of $\mathrm{Re}(e^{i\theta}e^{it_c}u(e^{i\theta_{c_1}}))$ and $\mathrm{Re}(e^{i\theta}e^{it_c}u(e^{i\theta_{c_2}}))$ are different. In particular, taking $\theta = t_d - t_c$ (by hypothesis not a multiple of π), we see that $\mathrm{Re}(e^{i(t_d-t_c)}e^{it_c}u(e^{i\theta_{c_1}}))$ and $\mathrm{Re}(e^{i(t_d-t_c)}e^{it_c}u(e^{i\theta_{c_2}}))$ have different signs. But

$$\mathrm{Re}(e^{i(t_d-t_c)}e^{it_c}u(e^{i\theta_{c_1}})) = \mathrm{Re}(e^{it_d}u(e^{i\theta_{c_1}}))$$

and

$$\mathrm{Re}(e^{i(t_d-t_c)}e^{it_c}u(e^{i\theta_{c_2}})) = \mathrm{Re}(e^{it_d}u(e^{i\theta_{c_2}})).$$

This is a contradiction. $\square$

Again, for $\phi \in L^\infty$, recall that the set of eigenvalues of T^t_ϕ relative to $\mathscr{A}_{a,b}$ is

$$\Lambda^{a,b}_\phi := \bigcup_{t\in\widehat{\mathbb{C}}} \sigma_p(T^t_\phi).$$

We now identify $\Lambda^{a,b}_\phi$ with annihilators of $\mathscr{A}_{a,b} + \mathscr{A}^*_{a,b} := \{h_1 + \overline{h_2} : h_1, h_2 \in \mathscr{A}_{a,b}\}$.

Proposition 3.2.9. Let $\phi \in L^\infty$ be real-valued. There exists a $c \in \mathbb{C}$ such that for every $\lambda \in \Lambda^{a,b}_\phi$, there exists an outer function G_λ such that

$$(\phi - \lambda)|G_\lambda|^2 = c(k_a - k_b) + \overline{c(k_a - k_b)}.$$

Moreover, the constant c is unique up to a non-negative real multiple.

Proof. By definition, if $\lambda \in \Lambda^{a,b}_\phi$, then there exists $t \in \widehat{\mathbb{C}}$ and $g_\lambda \in H^2_t$ such that $T^t_\phi g_\lambda = \lambda g_\lambda$, or, equivalently, $g_\lambda \in \ker(T^t_{\phi-\lambda})$. Further, by Proposition 3.2.2, we know the outer part of g_λ, say G_λ, belongs to $\ker(T^s_{\phi-\lambda})$, where $s = G_\lambda(a)/G_\lambda(b)$. Now, use Proposition 3.2.7 applied to $\phi - \lambda$ to get the desired result.

To see that c is unique up to a non-negative constant, let $\lambda_1, \lambda_2 \in \Lambda_\phi^{a,b}$ be distinct with

$$(\phi - \lambda_j)|G_{\lambda_j}|^2 = c_j(k_a - k_b) + \overline{c_j(k_a - k_b)}, \quad j = 1, 2.$$

Without loss of generality, suppose $\lambda_1 > \lambda_2$. Then $\{\phi > \lambda_1\} \subseteq \{\phi > \lambda_2\}$, or, equivalently, $\{c_1(k_a - k_b) + \overline{c_1(k_a - k_b)} > 0\} \subseteq \{c_2(k_a - k_b) + \overline{c_2(k_a - k_b)} > 0\}$. It now follows from Proposition 3.2.8 that c_1 and c_2 must be non-negative real multiples of each other. □

We now note that the collection of outer functions in the kernel of T_ϕ^t, for fixed real-valued ϕ, is quite small. In fact, all such outer functions are essentially unique.

Proposition 3.2.10. Let $\phi \in L^\infty$ be real-valued and let $\mathcal{N}$ denote the collection of outer functions in H^2. Then there is at most one $t \in \widehat{\mathbb{C}}$ so that $\ker(T_\phi^t) \cap \mathcal{N}$ is non-empty. Moreover, when $\ker(T_\phi^t) \cap \mathcal{N}$ is non-empty, it is equal to the span of a single outer function.

Proof. Suppose $G_1 \in \ker(T_\phi^t)$ and $G_2 \in \ker(T_\phi^s)$ are outer, with $s \neq t$. From Proposition 3.2.7, there exist constants $c_1, c_2 \in \mathbb{C}$ such that

$$\phi|G_j|^2 = c_j(k_a - k_b) + \overline{c_j(k_a - k_b)}, \quad j = 1, 2.$$

Since $|G_1|^2$ and $|G_2|^2$ are non-negative, $c_1(k_a - k_b) + \overline{c_1(k_a - k_b)}$ and $c_2(k_a - k_b) + \overline{c_2(k_a - k_b)}$ must be positive (negative) on the same subsets of $\mathbb{T}$. Proposition 3.2.8 indicates that this can happen if and only if c_1 is a non-negative real multiple of c_2. In turn, there exists a positive constant $r \in \mathbb{R}$ so that $\phi|G_1|^2 = \phi r^2 |G_2|^2$. Since outer functions are determined by their modulus on the unit circle, it follows that G_1 and G_2 are constant multiples of one another and therefore $G_2 \in \operatorname{span}\{G_1\}$. Hence $G_2 \in \ker(T_\phi^t) \cap \mathcal{N}$. □

Marching toward our proof of Theorem 3.2.2, we pause to record a lemma that will be used for a key proposition.

Lemma 3.2.3 ([83, p. 146])**.** Given $w \in L^p(\mathbb{T})$ such that $\log(|w|) \in L^1(\mathbb{T})$, the following function

$$W(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log(|w(e^{i\theta})|)\,d\theta\right),$$

defined for $z \in \mathbb{D}$, is an outer function in H^p. Moreover, $|W| = |w|$ on $\mathbb{T}$.

We now introduce some important notation. For $c \in \mathbb{C}$, let

$$S_c^- := \{z \in \mathbb{T} : c(k_a - k_b) + \overline{c(k_a - k_b)} < 0\}$$

and

$$S_c^+ := \{z \in \mathbb{T} : c(k_a - k_b) + \overline{c(k_a - k_b)} > 0\}.$$

It turns out that these sets are crucial in understanding the relative eigenvalues of T_ϕ^t.

Proposition 3.2.11. Let $\phi \in L^\infty$ be real-valued. If there exist $c \in \mathbb{C}$ and $\beta \in \mathbb{R}$ such that

$$\operatorname{ess\,sup}\{(\phi-\beta)|_{S_c^-}\} = m < 0 < M = \operatorname{ess\,inf}\{(\phi-\beta)|_{S_c^+}\},$$

then

(i) $(m+\beta, M+\beta) \subseteq \Lambda_\phi^{a,b}$.

(ii) For every $\beta+\lambda \in (m+\beta, M+\beta)$, there exists an essentially unique outer function $G_{\beta,\lambda}$ such that

$$(\phi - (\beta+\lambda))|G_{\beta,\lambda}|^2 = c(k_a - k_b) + \overline{c(k_a - k_b)}.$$

(iii) The endpoints $M+\beta$ and $m+\beta$ are elements of $\Lambda_\phi^{a,b}$ if and only if

$$\frac{c(k_a - k_b) + \overline{c(k_a - k_b)}}{\phi - (M+\beta)} \quad \text{and} \quad \frac{c(k_a - k_b) + \overline{c(k_a - k_b)}}{\phi - (m+\beta)}$$

are integrable.

Proof. Let $\lambda \in (m, M)$ and consider the function

$$\psi := \frac{c(k_a - k_b) + \overline{c(k_a - k_b)}}{\phi - (\beta + \lambda)}.$$

By definition of M, on S_c^+ we have, $\lambda < M \leq \phi - \beta$ and hence $0 < \phi - (\beta + \lambda)$. Additionally, $\phi - (\beta + \lambda)$ is uniformly bounded away from 0 on S_c^+. Similarly, by definition of m, the function $\phi - (\beta + \lambda)$ is negative and uniformly bounded away from zero on S_c^-. In turn, $|\phi - (\beta + \lambda)|$ is uniformly bounded away from zero. Further, by definition of S_c^+ and S_c^-, it is clear that ψ is non-negative.

Now consider

$$\log(\psi) = \log(|c(k_a - k_b) + \overline{c(k_a - k_b)}|) - \log(|\phi - (\beta + \lambda)|).$$

Note first that $\int_{\mathbb{T}} \log(|\phi - (\beta + \lambda)|)\, dm < \infty$, so that

$$-\int\limits_{\mathbb{T}} \log(|\phi - (\beta + \lambda)|)\, dm > -\infty.$$

It follows from Lemma 3.2.2 that

$$\int\limits_{\mathbb{T}} \log(|c(k_a - k_b) + \overline{c(k_a - k_b)}|)\, dm > -\infty.$$

Altogether, we find that

$$\int\limits_{\mathbb{T}} \log(\psi)\, dm > -\infty.$$

Thus, since $\mathbb{T}$ is compact, it now follows that $\log(\psi) \in L^1$. Now consider the function $\psi^{1/2}$. Since $\psi \in L^1$ and is non-negative, we have that $\psi^{1/2} \in L^2$. Further, $\log(\psi^{1/2}) = \frac{1}{2}\log(\psi) \in$

L^1.

Now, since $\psi^{1/2} \in L^2$ and $\log(\psi^{1/2}) \in L^1$, it follows from Lemma 3.2.3 that the function

$$G_{\beta,\lambda}(z) := \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(|\psi^{1/2}(e^{i\theta})|)\, d\theta \right)$$

is an outer function in H^2 with $|G_{\beta,\lambda}| = \psi^{1/2}$ on $\mathbb{T}$. Thus, we have an outer function $G_{\beta,\lambda} \in H^2$ with

$$|G_{\beta,\lambda}|^2 = \psi = \frac{c(k_a - k_b) + \overline{c(k_a - k_b)}}{\phi - (\beta + \lambda)},$$

where the last equality follows from ψ being non-negative.

It now follows from Proposition 3.2.7 that $T^t_{\phi-(\beta+\lambda)} G_{\beta,\lambda} = 0$ for $t = \frac{G_{\beta,\lambda}(a)}{G_{\beta,\lambda}(b)}$ (where $G_{\beta,\lambda}(b) \neq 0$ since $G_{\beta,\lambda}$ is outer). In light of Proposition 3.2.10, we have that $G_{\beta,\lambda}$ is unique up to a constant. Rewriting $T^t_{\phi-(\beta+\lambda)} G_{\beta,\lambda} = 0$, we find that $T^t_\phi G_{\beta,\lambda} = (\lambda + \beta) G_{\beta,\lambda}$ and therefore $\lambda + \beta$ is an eigenvalue for T^t_ϕ. This means $\lambda + \beta \in \Lambda^{a,b}_\phi$ and therefore $(m + \beta, M + \beta) \subseteq \Lambda^{a,b}_\phi$.

If we require the integrability of

$$\frac{c(k_a - k_b) + \overline{c(k_a - k_b)}}{\phi - (M + \beta)} \quad \text{and} \quad \frac{c(k_a - k_b) + \overline{c(k_a - k_b)}}{\phi - (m + \beta)},$$

the above argument holds for $\lambda = M$ and $\lambda = m$. □

Finally, we have the machinery necessary to prove Theorem 3.2.2.

Theorem 3.2.2. If $\phi \in L^\infty$ is real-valued, then $\Lambda^{a,b}_\phi$ is either empty, a point, or an interval. In particular, $\Lambda^{a,b}_\phi$ is connected.

Proof. If $\Lambda^{a,b}_\phi$ is empty or a point, then we are done. Suppose, then, $\Lambda^{a,b}_\phi \neq \emptyset$ and that there exist distinct $\lambda_1, \lambda_2 \in \Lambda^{a,b}_\phi$. By Proposition 3.2.9, there exist outer functions $G_1, G_2 \in H^2$

and $c_1, c_2 \in \mathbb{C}$ such that

$$(\phi - \lambda_j)|G_j|^2 = c_j(k_a - k_b) + \overline{c_j(k_a - k_b)}, \quad j = 1, 2,$$

where c_1 and c_2 differ by a non-negative real constant. By absorbing this constant into $|G_j|^2$ and relabeling appropriately, it follows that there exists a single $c \in \mathbb{C}$ such that

$$(\phi - \lambda_j)|G_j|^2 = c(k_a - k_b) + \overline{c(k_a - k_b)}, \quad j = 1, 2.$$

Without loss of generality, assume $\lambda_1 < \lambda_2$ and let $\beta \in (\lambda_1, \lambda_2)$. Then we have

$$\phi - \lambda_2 < \phi - \beta < \phi - \lambda_1.$$

Therefore

$$(\phi - \beta)|_{S_c^-} < (\phi - \lambda_1)|_{S_c^-} < 0$$

and

$$0 < (\phi - \lambda_2)|_{S_c^+} < (\phi - \beta)|_{S_c^+}.$$

Finally, it follows from Lemma 3.2.2 that the sets $S_c^{\pm}$ are of positive measure. As a result, we can define the constants

$$m := \operatorname{ess\,sup}\{(\phi - \beta)|_{S_c^-}\}$$

$$M := \operatorname{ess\,inf}\{(\phi - \beta)|_{S_c^+}\}$$

and conclude that $m < 0 < M$. It follows from Proposition 3.2.11 that $(m+\beta, M+\beta) \subseteq \Lambda_\phi^{a,b}$ (with inclusion of endpoints possible).

We claim that this interval is all of $\Lambda_\phi^{a,b}$ and show that if $\lambda + \beta > M$ (resp. $\lambda + \beta < m$), then $\lambda + \beta \notin \Lambda_\phi^{a,b}$. To this end, suppose $\lambda + \beta > M + \beta$, and, for the sake of contradiction, that $\lambda + \beta \in \Lambda_\phi^{a,b}$. By Proposition 3.2.7, there exists a constant $d \in \mathbb{C}$ and $G_{\beta+\lambda} \in H^2$ outer such that

$$(\phi - (\beta + \lambda))|G_{\beta+\lambda}|^2 = d(k_a - k_b) + \overline{d(k_a - k_b)}.$$

Similarly, since $0 \in (m, M)$, we have $\beta \in (m + \beta, M + \beta) \subseteq \Lambda_\phi^{a,b}$ and therefore, by part ((ii)) of Proposition 3.2.11,

$$(\phi - \beta)|G_\beta|^2 = c(k_a - k_b) + \overline{c(k_a - k_b)}$$

for some outer function $G_\beta \in H^2$. Now, since $\lambda + \beta > M + \beta$, we have

$$\{\phi - (\beta + \lambda) > 0\} \subsetneq \{\phi - \beta > 0\}$$

and therefore

$$\begin{aligned}\{d(k_a - k_b) + \overline{d(k_a - k_b)} > 0\} &= \{\phi - (\beta + \lambda) > 0\} \\ &\subsetneq \{\phi - \beta > 0\} \\ &= \{c(k_a - k_b) + \overline{c(k_a - k_b)} > 0\}.\end{aligned}$$

However, by Proposition 3.2.8, this inclusion cannot occur. Hence we arrive at a contradiction and $\lambda + \beta \notin \Lambda_\phi^{a,b}$. An analogous arguments holds for $\lambda + \beta < m + \beta$. □

3.2.5 Further Remarks and Questions

For real-valued symbols, the proof of Theorem 3.2.2 shows, when expecting a relative point spectrum consisting of an interval, the symbol of the Toeplitz operator is quite confined; namely, the hypotheses of Proposition 3.2.11 must be satisfied. However, there is no mention of explicit hypotheses required on the symbol to guarantee the relative point spectrum to be empty or a point – the former of which we argue is most often the case. In fact, it can be shown, using arguments very similar to the ones used in the above proofs, that $\Lambda_\phi^{a,b}$ is a point if and only if there exist constants $c \in \mathbb{C}$ and $\beta \in \mathbb{R}$ such that

$$\operatorname{ess\,sup}\{(\phi - \beta)|_{S_c^-}\} = 0 = \operatorname{ess\,inf}\{(\phi - \beta)|_{S_c^+}\}.$$

Again, the behavior of ϕ is quite restricted. All told, in order for a constrained Toeplitz operator to have non-empty relative spectrum in the real-valued setting, it must have a symbol (up to a translation) with set of positivity (negativity) coinciding with that of $\mathrm{Re}(c(k_a - k_b))$ for some $c \in \mathbb{C}$.

Reflecting back to Section 3.2.1, Widom's connected-spectrum result held for arbitrary symbols. A result of this generality is still unknown for the two-point and Neil cases:

Question 3.2.1 (Open)**.** For general $\phi \in L^\infty$, are the spectrum, essential spectrum, and relative point spectrum of T_ϕ^t connected? In the Neil algebra setting?

The Neil and two-point constraints are, heuristically speaking, the building blocks for general finite codimensional subalgebras of H^∞. This notion, due to Gamelin, was formalized in [86, Theorem 9.8], and reinterpreted in [130, Theorem 2.1]. A natural generalization of the work here, and in [34], would be to consider spectra of Toeplitz operators associated to these algebras:

Question 3.2.2 (Open)**.** For a general finite codimensional subalgebra of H^∞, do the associated Toeplitz operators have connected relative spectrum?

Here, by 'associated' we mean Toeplitz operators acting on subspaces of H^2 which carry a representation for the sublagebra.

Bibliography

[1] M. B. Abrahamse, *Toeplitz operators in multiply connected regions*, Amer. J. Math. **96** (1974), 261–297.

[2] M. B. Abrahamse and R. G. Douglas, *A class of subnormal operators related to multiply-connected domains*, Advances in Math. **19** (1976), no. 1, 106–148.

[3] Jim Agler and John E. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, vol. 44, American Mathematical Society, Providence, RI, 2002.

[4] Patrick Ahern, *On the range of the Berezin transform*, J. Funct. Anal. **215** (2004), no. 1, 206–216.

[5] Gholamreza Akbari Estahbanati, *On the spectral character of Toeplitz operators on planar regions*, Proc. Amer. Math. Soc. **124** (1996), no. 9, 2737–2744.

[6] A. Aleman, S. Richter, and C. Sundberg, *Beurling's theorem for the Bergman space*, Acta Math. **177** (1996), no. 2, 275–310.

[7] J. Alonso, H. Martini, and S. Wu, *On Birkhoff orthogonality and isosceles orthogonality in normed linear spaces*, Aequationes Math. **83** (2012), no. 1-2, 153–189.

[8] Najla Altwaijry, Mubariz Garayev, Hocine Guediri, and Amani Baazeem, *Skew-symmetric and essentially unitary operators via Berezin symbols*, Open Math. **18** (2020), no. 1, 1760–1770.

[9] J. M. Anderson and R. H. Rochberg, *Toeplitz operators associated with subalgebras of the disk algebra*, Indiana Univ. Math. J. **30** (1981), no. 6, 813–820.

[10] Cyrus P. Aryana and Kevin F. Clancey, *On the existence of eigenvalues of Toeplitz operators on planar regions*, Proc. Amer. Math. Soc. **132** (2004), no. 10, 3007–3018.

[11] Cyrus Parsi Aryana, *Self-adjoint Toeplitz operators associated with representing measures on multiply connected planar regions and their eigenvalues*, Complex Anal. Oper. Theory **7** (2013), no. 5, 1513–1524.

[12] Sheldon Axler, *Toeplitz operators*, A glimpse at Hilbert space operators, 2010, pp. 125–133.

[13] Sheldon Axler and Dechao Zheng, *Compact operators via the Berezin transform*, Indiana Univ. Math. J. **47** (1998), no. 2, 387–400.

[14] Mojtaba Bakherad and Mubariz T. Garayev, *Berezin number inequalities for operators*, Concr. Oper. **6** (2019), no. 1, 33–43.

[15] Sriram Balasubramanian, Scott McCullough, and Udeni Wijesooriya, *Szegő and Widom theorems for the Neil algebra*, Interpolation and realization theory with applications to control theory, 2019, pp. 61–70.

[16] J.A. Ball and V. Bolotnikov, *Weighted Bergman spaces: shift-invariant subspaces and input/state/output linear systems*, Integral Equations Operator Theory **76** (2013), no. 3, 301–356.

[17] ______, *Weighted Hardy spaces: shift invariant and coinvariant subspaces, linear systems and operator model theory*, Acta Sci. Math. (Szeged) **79** (2013), no. 3-4, 623–686.

[18] ______, *On the expansive property of inner functions in weighted Hardy spaces*, Complex analysis and dynamical systems VI. Part 2, 2016, pp. 47–61.

[19] Joseph A. Ball, Vladimir Bolotnikov, and Sanne ter Horst, *A constrained Nevanlinna-Pick interpolation problem for matrix-valued functions*, Indiana Univ. Math. J. **59** (2010), no. 1, 15–51.

[20] Debendra P. Banjade, *Estimates for the corona theorem on $H^\infty_{\mathbb{I}}(\mathbb{D})$*, Oper. Matrices **11** (2017), no. 3, 725–734.

[21] Wolfram Bauer and Joshua Isralowitz, *Compactness characterization of operators in the Toeplitz algebra of the Fock space F^p_α*, J. Funct. Anal. **263** (2012), no. 5, 1323–1355.

[22] Bernard Beauzamy, *Introduction to Banach spaces and their geometry*, North-Holland Mathematics Studies, vol. 68, North-Holland Publishing Co., Amsterdam-New York, 1982. Notas de Matemática [Mathematical Notes], 86.

[23] C. Bénéteau, M. Fleeman, D. Khavinson, D. Seco, and A. Sola, *Remarks on inner functions and optimal approximants*, Canad. Math. Bull. **61** (2018), no. 4, 704–716.

[24] C. Bénéteau, M. Fleeman, D. Khavinson, and A. Sola, *On the concept of inner function in Hardy and Bergman spaces in multiply connected domains*, Anal. Math. Phys. **9** (2019), no. 2, 839–866.

[25] Catherine Bénéteau, Alberto A. Condori, Constanze Liaw, Daniel Seco, and Alan A. Sola, *Cyclicity in Dirichlet-type spaces and extremal polynomials*, J. Anal. Math. **126** (2015), 259–286.

[26] Catherine Bénéteau, Dmitry Khavinson, Constanze Liaw, Daniel Seco, and Brian Simanek, *Zeros of optimal polynomial approximants: Jacobi matrices and Jentzsch-type theorems*, Rev. Mat. Iberoam. **35** (2019), no. 2, 607–642.

[27] Catherine Bénéteau, Dmitry Khavinson, Constanze Liaw, Daniel Seco, and Alan A. Sola, *Orthogonal polynomials, reproducing kernels, and zeros of optimal approximants*, J. Lond. Math. Soc. (2) **94** (2016), no. 3, 726–746.

[28] Catherine Bénéteau, Myrto Manolaki, and Daniel Seco, *Boundary behavior of optimal polynomial approximants*, Constr. Approx. **54** (2021), no. 1, 157–183.

[29] F. A. Berezin, *Covariant and contravariant symbols of operators*, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 1134–1167.

[30] C. A. Berger and L. A. Coburn, *A symbol calculus for Toeplitz operators*, Proc. Nat. Acad. Sci. U.S.A. **83** (1986), no. 10, 3072–3073.

[31] Arne Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math. **81** (1948), 239–255.

[32] Kelly Bickel and Pamela Gorkin, *Compressions of the shift on the bidisk and their numerical ranges*, J. Operator Theory **79** (2018), no. 1, 225–265.

[33] ______, *Numerical range and compressions of the shift*, Complex analysis and spectral theory, [2020] ©2020, pp. 241–261.

[34] Adam Broschinski, *Eigenvalues of Toeplitz operators on the annulus and Neil algebra*, Complex Analysis and Operator Theory **8** (2014), no. 5, 1037–1059.

[35] Arlen Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/64), 89–102.

[36] Leon Brown and Allen L. Shields, *Cyclic vectors in the Dirichlet space*, Trans. Amer. Math. Soc. **285** (1984), no. 1, 269–303.

[37] W. L. Bynum, *Weak parallelogram law for Banach spaces*, Canad. Math. Bull. **19** (1976), no. 3, 269–275.

[38] W. L. Bynum and J. H. Drew, *A weak parallelogram law for ℓ_p*, Amer. Math. Monthly **79** (1972), 1012–1015.

[39] Alberto Calderón, Frank Spitzer, and Harold Widom, *Inversion of Toeplitz matrices*, Illinois J. Math. **3** (1959), 490–498.

[40] N. L. Carothers, *A short course on Banach space theory*, London Mathematical Society Student Texts, vol. 64, Cambridge University Press, Cambridge, 2005.

[41] I. Chalendar, E. Fricain, M. Gürdal, and M. Karaev, *Compactness and Berezin symbols*, Acta Sci. Math. (Szeged) **78** (2012), no. 1-2, 315–329.

[42] I. Chalendar, P. Gorkin, and J.R. Partington, *Inner functions and operator theory*, North-West. Eur. J. Math. **1** (2015), 7–22.

[43] R. Cheng, *The zero sets of ℓ^p_A are nested*, Illinois J. Math. **63** (2019), 601–618.

[44] R. Cheng and J. G. Dragas, *On the failure of canonical factorization in ℓ^p_A*, J. Math. Anal. Appl. **479** (2019), 1939–1955.

[45] R. Cheng and C. Felder, *On the geometry of the multiplier space of ℓ^p_a*, 2021.

[46] R. Cheng and C. Harris, *Duality of the weak parallelogram laws on Banach spaces*, J. Math. Anal. Appl. **404** (2013), no. 1, 64–70.

[47] R. Cheng, J. Mashreghi, and W. T. Ross, *Birkhoff–James Orthogonality and the Zeros of an Analytic Function*, Comput. Methods Funct. Theory **17** (2017), no. 3, 499–523.

[48] ______, *Multipliers of sequence spaces*, Concr. Oper. **4** (2017), 76–108.

[49] ______, *Inner functions and zero sets for ℓ^p_A*, Trans. Amer. Math. Soc. **372** (2019), no. 3, 2045–2072.

[50] ______, *Inner functions in reproducing kernel spaces*, Analysis of operators on function spaces, 2019, pp. 167–211.

[51] ______, *Function theory and ℓ^p spaces*, University Lecture Series, vol. 75, American Mathematical Society, Providence, R.I., 2020.

[52] ______, *Inner vectors for Toeplitz operators*, Complex analysis and spectral theory, 2020, pp. 195–212.

[53] R. Cheng, A. G. Miamee, and M. Pourahmadi, *On the geometry of $L^p(\mu)$ with applications to infinite variance processes*, J. Aust. Math. Soc. **74** (2003), no. 1, 35–42.

[54] R. Cheng and W. T. Ross, *Weak parallelogram laws on Banach spaces and applications to prediction*, Period. Math. Hungar. **71** (2015), no. 1, 45–58.

[55] Wai-Shun Cheung and Chi-Kwong Li, *Elementary proofs for some results on the circular symmetry of the numerical range*, Linear Multilinear Algebra **61** (2013), no. 5, 596–602.

[56] C. Chui and A. Chan, *Application of approximation theory methods to recursive digital filter design*, IEEE Transactions on Acoustics, Speech, and Signal Processing **30** (1982), no. 1, 18–24.

[57] Charles K. Chui, *Approximation by double least-squares inverses*, J. Math. Anal. Appl. **75** (1980), no. 1, 149–163.

[58] Kevin F. Clancey, *On the spectral character of Toeplitz operators on multiply connected domains*, Trans. Amer. Math. Soc. **323** (1991), no. 2, 897–910.

[59] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), no. 3, 396–414.

[60] J. B. Conway, *A course in functional analysis*, Second, Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990.

[61] ———, *The theory of subnormal operators*, Mathematical Surveys and Monographs, vol. 36, American Mathematical Society, Providence, RI, 1991.

[62] C. C. Cowen and C. Felder, *Convexity of the Berezin range*, 2021.

[63] C.C. Cowen and B.D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[64] Felipe Cucker and Steve Smale, *On the mathematical foundations of learning*, Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 1, 1–49.

[65] Ulrich Daepp, Pamela Gorkin, Andrew Shaffer, and Karl Voss, *Finding ellipses*, Carus Mathematical Monographs, vol. 34, MAA Press, Providence, RI, 2018. What Blaschke products, Poncelet's theorem, and the numerical range know about each other.

[66] Kenneth R. Davidson, Vern I. Paulsen, Mrinal Raghupathi, and Dinesh Singh, *A constrained Nevanlinna-Pick interpolation problem*, Indiana Univ. Math. J. **58** (2009), no. 2, 709–732.

[67] Karel de Leeuw, *On L_p multipliers*, Ann. of Math. (2) **81** (1965), 364–379.

[68] Allen Devinatz, *Toeplitz operators on H^2 spaces*, Trans. Amer. Math. Soc. **112** (1964), 304–317.

[69] William F. Donoghue Jr., *On the numerical range of a bounded operator*, Michigan Math. J. **4** (1957), 261–263.

[70] R. G. Douglas, *Banach algebra techniques in the theory of Toeplitz operators*, American Mathematical Society, Providence, R.I., 1973. Expository Lectures from the CBMS Regional Conference held at the University of Georgia, Athens, Ga., June 12–16, 1972, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 15.

[71] Ronald G. Douglas, *Banach algebra techniques in operator theory*, Second, Graduate Texts in Mathematics, vol. 179, Springer-Verlag, New York, 1998.

[72] Michael A. Dritschel and Batzorig Undrakh, *Rational dilation problems associated with constrained algebras*, J. Math. Anal. Appl. **467** (2018), no. 1, 95–131.

[73] P. Duren, D. Khavinson, H. S. Shapiro, and C. Sundberg, *Invariant subspaces in Bergman spaces and the biharmonic equation*, Michigan Math. J. **41** (1994), no. 2, 247–259.

[74] P. Duren, D. Khavinson, and H.S. Shapiro, *Extremal functions in invariant subspaces of Bergman spaces*, Illinois J. Math. **40** (1996), no. 2, 202–210.

[75] P. Duren and A. Schuster, *Bergman spaces*, Mathematical Surveys and Monographs, vol. 100, American Mathematical Society, Providence, RI, 2004.

[76] Peter Duren, Dmitry Khavinson, Harold S. Shapiro, and Carl Sundberg, *Contractive zero-divisors in Bergman spaces*, Pacific J. Math. **157** (1993), no. 1, 37–56.

[77] O. El-Fallah, K. Kellay, J. Mashreghi, and T. Ransford, *A primer on the Dirichlet space*, Cambridge Tracts in Mathematics, vol. 203, Cambridge University Press, Cambridge, 2014.

[78] Miroslav Engliš, *Compact Toeplitz operators via the Berezin transform on bounded symmetric domains*, Integral Equations Operator Theory **33** (1999), no. 4, 426–455.

[79] Martin J. Erickson, *Introduction to combinatorics*, Second, Wiley Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2013.

[80] Christopher Felder, *General optimal polynomial approximants, stabilization, and projections of unity*, 2020.

[81] Christopher Felder and Trieu Le, *Analogues of finite Blaschke products as inner functions*, 2021.

[82] Christopher Felder, Douglas T. Pfeffer, and Benjamin P. Russo, *Spectra for toeplitz operators associated with a constrained subalgebra*, 2021.

[83] Emmanuel Fricain and Javad Mashreghi, *The theory of $\mathcal{H}(b)$ spaces. Vol. 1*, New Mathematical Monographs, vol. 20, Cambridge University Press, Cambridge, 2016.

[84] Emmanuel Fricain, Javad Mashreghi, and Daniel Seco, *Cyclicity in reproducing kernel Hilbert spaces of analytic functions*, Comput. Methods Funct. Theory **14** (2014), no. 4, 665–680.

[85] ———, *Cyclicity in non-extreme de Branges–Rovnyak spaces*, Invariant subspaces of the shift operator, 2015, pp. 131–136.

[86] T. W. Gamelin, *Embedding Riemann surfaces in maximal ideal spaces*, J. Functional Analysis **2** (1968), 123–146.

[87] Mubariz T. Garayev, Hocine Guediri, and Houcine Sadraoui, *Applications of reproducing kernels and Berezin symbols*, New York J. Math. **22** (2016), 583–604.

[88] Mubariz T. Garayev, Mehmet Gürdal, and Arzu Okudan, *Hardy-Hilbert's inequality and power inequalities for Berezin numbers of operators*, Math. Inequal. Appl. **19** (2016), no. 3, 883–891.

[89] S.R. Garcia, J. Mashreghi, and W.T. Ross, *Finite Blaschke products and their connections*, Springer, Cham, 2018.

[90] Stephan Ramon Garcia and William T. Ross, *Model spaces: a survey*, Invariant subspaces of the shift operator, 2015, pp. 197–245.

[91] Hwa-Long Gau and Pei Yuan Wu, *Condition for the numerical range to contain an elliptic disc*, Linear Algebra Appl. **364** (2003), 213–222.

[92] I. C. Gohberg and N. Ja. Krupnik, *The algebra generated by the Toeplitz matrices*, Funkcional. Anal. i Priložen. **3** (1969), no. 2, 46–56.

[93] Karl Gustafson, *The Toeplitz-Hausdorff theorem for linear operators*, Proc. Amer. Math. Soc. **25** (1970), 203–204.

[94] Karl E. Gustafson and Duggirala K. M. Rao, *Numerical range*, Universitext, Springer-Verlag, New York, 1997. The field of values of linear operators and matrices.

[95] P. R. Halmos, *A glimpse into Hilbert space*, Lectures on Modern Mathematics, Vol. I, 1963, pp. 1–22.

[96] J. Hansbo, *Reproducing kernels and contractive divisors in Bergman spaces*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **232** (1996), no. Issled. po Lineĭn. Oper. i Teor. Funktsiĭ. 24, 174–198, 217.

[97] Philip Hartman and Aurel Wintner, *On the spectra of Toeplitz's matrices*, Amer. J. Math. **72** (1950), 359–366.

[98] ———, *The spectra of Toeplitz's matrices*, Amer. J. Math. **76** (1954), 867–882.

[99] Felix Hausdorff, *Der Wertvorrat einer Bilinearform*, Math. Z. **3** (1919), no. 1, 314–316.

[100] H. Hedenmalm, *A factorization theorem for square area-integrable analytic functions*, J. Reine Angew. Math. **422** (1991), 45–68.

[101] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics, vol. 199, Springer-Verlag, New York, 2000.

[102] Lars Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140.

[103] Zhenghui Huo and Brett D. Wick, *Compactness of operators on the Bergman space of the Thullen domain*, J. Operator Theory **83** (2020), no. 2, 391–421.

[104] R. C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. **61** (1947), 265–292.

[105] M. T. Karaev, *Berezin symbols and Schatten-von Neumann classes*, Mat. Zametki **72** (2002), no. 2, 207–215.

[106] ______, *On a Beurling-Arveson type theorem for some functional Hilbert spaces and related questions*, Integral Equations Operator Theory **62** (2008), no. 1, 77–84.

[107] ______, *Reproducing kernels and Berezin symbols techniques in various questions of operator theory*, Complex Anal. Oper. Theory **7** (2013), no. 4, 983–1018.

[108] Mubariz T. Karaev, *Berezin symbol and invertibility of operators on the functional Hilbert spaces*, J. Funct. Anal. **238** (2006), no. 1, 181–192.

[109] ______, *Use of reproducing kernels and Berezin symbols technique in some questions of operator theory*, Forum Math. **24** (2012), no. 3, 553–564.

[110] Mübariz T. Karaev and Mehmet Gürdal, *On the Berezin symbols and Toeplitz operators*, Extracta Math. **25** (2010), no. 1, 83–102.

[111] Mubariz T. Karaev and Nizameddin Sh. Iskenderov, *Berezin number of operators and related questions*, Methods Funct. Anal. Topology **19** (2013), no. 1, 68–72.

[112] Rudolf Kippenhahn, *On the numerical range of a matrix*, Linear Multilinear Algebra **56** (2008), no. 1-2, 185–225. Translated from the German by Paul F. Zachlin and Michiel E. Hochstenbach [MR0059242].

[113] M. G. Kreĭn, *Integral equations on the half-line with a kernel depending on the difference of the arguments*, Uspehi Mat. Nauk **13** (1958), no. 5 (83), 3–120.

[114] T. Le, *Inner functions in weighted Hardy spaces*, Anal. Math. Phys. **10** (2020), no. 2, Paper No. 25, 21.

[115] V. Lebedev and A. Olevskiĭ, *Idempotents of Fourier multiplier algebra*, Geom. Funct. Anal. **4** (1994), no. 5, 539–544.

[116] V. V. Lebedev, *Inner functions and l^p-multipliers*, Funktsional. Anal. i Prilozhen. **32** (1998), no. 4, 10–21, 95.

[117] ______, *Spectra of inner functions and l^p-multipliers*, Complex analysis, operators, and related topics, 2000, pp. 205–212.

[118] V. V. Lebedev and A. M. Olevskiĭ, *Fourier L^p-multipliers with bounded powers*, Izv. Ross. Akad. Nauk Ser. Mat. **70** (2006), no. 3, 129–166.

[119] Vladimir Lebedev and Alexander Olevskiĭ, *Bounded groups of translation invariant operators*, C. R. Acad. Sci. Paris Sér. I Math. **322** (1996), no. 2, 143–147.

[120] Chi-Kwong Li, *A simple proof of the elliptical range theorem*, Proc. Amer. Math. Soc. **124** (1996), no. 7, 1985–1986.

[121] T.H. MacGregor and M.I. Stessin, *Weighted reproducing kernels in Bergman spaces*, Michigan Math. J. **41** (1994), no. 3, 523–533.

[122] J.E. McCarthy and L.M. Yang, *Bounded point evaluations on the boundaries of L regions*, Indiana Univ. Math. J. **43** (1994), no. 3, 857–883.

[123] Mishko Mitkovski, Daniel Suárez, and Brett D. Wick, *The essential norm of operators on $A^p_\alpha(\mathbb{B}_n)$*, Integral Equations Operator Theory **75** (2013), no. 2, 197–233.

[124] Mishko Mitkovski and Brett D. Wick, *A reproducing kernel thesis for operators on Bergman-type function spaces*, J. Funct. Anal. **267** (2014), no. 7, 2028–2055.

[125] N. K. Nikol′skiĭ, *On spaces and algebras of Toeplitz matrices operating in l^p*, Sibirsk. Mat. Ž. **7** (1966), 146–158.

[126] N. K. Nikol'skiĭ, *Treatise on the shift operator*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 273, Springer-Verlag, Berlin, 1986. Spectral function theory, With an appendix by S. V. Hruščev [S. V. Khrushchëv] and V. V. Peller, Translated from the Russian by Jaak Peetre.

[127] Eric Nordgren and Peter Rosenthal, *Boundary values of Berezin symbols*, Nonselfadjoint operators and related topics (Beer Sheva, 1992), 1994, pp. 362–368.

[128] A. Olofsson, *Operator-valued Bergman inner functions as transfer functions*, Algebra i Analiz **19** (2007), no. 4, 146–173.

[129] Vern I. Paulsen and Mrinal Raghupathi, *An introduction to the theory of reproducing kernel Hilbert spaces*, Cambridge Studies in Advanced Mathematics, vol. 152, Cambridge University Press, Cambridge, 2016.

[130] Douglas T. Pfeffer and Michael T. Jury, *Szegő and Widom theorems for finite codimensional subalgebras of a class of uniform algebras*, Complex Anal. Oper. Theory **15** (2021), no. 5, Paper No. 83, 37.

[131] Mrinal Raghupathi, *Nevanlinna-Pick interpolation for $\mathbb{C} + BH^\infty$*, Integral Equations Operator Theory **63** (2009), no. 1, 103–125.

[132] N. V. Rao, *The range of the Berezin transform*, J. Math. Sci. (N.Y.) **228** (2018), no. 6, Problems in mathematical analysis. No. 90 (Russian), 684–694.

[133] Stefan Richter, *Invariant subspaces of the Dirichlet shift*, J. Reine Angew. Math. **386** (1988), 205–220.

[134] Stefan Richter and Allen Shields, *Bounded analytic functions in the Dirichlet space*, Math. Z. **198** (1988), no. 2, 151–159.

[135] J. Ryle and T. T. Trent, *A corona theorem for certain subalgebras of $H^\infty(\mathbb{D})$*, Houston J. Math. **37** (2011), no. 4, 1211–1226.

[136] ———, *A corona theorem for certain subalgebras of $H^\infty(\mathbb{D})$ II*, Houston J. Math. **38** (2012), no. 4, 1277–1295.

[137] D. Seco, *A characterization of Dirichlet-inner functions*, Complex Anal. Oper. Theory **13** (2019), no. 4, 1653–1659.

[138] J. Shanks, S. Treitel, and J. Justice, *Stability and synthesis of two-dimensional recursive filters*, IEEE Transactions on Audio and Electroacoustics **20** (1972), no. 2, 115–128.

[139] H.S. Shapiro and A.L. Shields, *On the zeros of functions with finite Dirichlet integral and some related function spaces*, Math. Z. **80** (1962), 217–229.

[140] Joel H. Shapiro, *Composition operators and classical function theory*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993.

[141] Ingo Steinwart and Andreas Christmann, *Support vector machines*, Information Science and Statistics, Springer, New York, 2008.

[142] Karel Stroethoff, *The Berezin transform and operators on spaces of analytic functions*, Linear operators (Warsaw, 1994), 1997, pp. 361–380.

[143] Daniel Suárez, *The essential norm of operators in the Toeplitz algebra on $A^p(\mathbb{B}_n)$*, Indiana Univ. Math. J. **56** (2007), no. 5, 2185–2232.

[144] Dan Timotin, *A short introduction to de Branges–Rovnyak spaces*, Invariant subspaces of the shift operator, 2015, pp. 21–38.

[145] Otto Toeplitz, *Das algebraische Analogon zu einem Satze von Fejér*, Math. Z. **2** (1918), no. 1-2, 187–197.

[146] V. A. Tolokonnikov, *Estimates in the Carleson corona theorem, ideals of the algebra H^∞, a problem of Sz.-Nagy*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **113** (1981), 178–198, 267. Investigations on linear operators and the theory of functions, XI.

[147] Željko Čučković and Sönmez Şahutoğlu, *Axler-Zheng type theorem on a class of domains in $\mathbb{C}^n$*, Integral Equations Operator Theory **77** (2013), no. 3, 397–405.

[148] Željko Čučković and Bo Li, *Berezin transform, Mellin transform and Toeplitz operators*, Complex Anal. Oper. Theory **6** (2012), no. 1, 189–218.

[149] Željko Čučković, Sönmez Şahutoğlu, and Yunus E. Zeytuncu, *A local weighted Axler-Zheng theorem in $\mathbb{C}^n$*, Pacific J. Math. **294** (2018), no. 1, 89–106.

[150] I. È. Verbickiĭ, *Multipliers of spaces l^p_A*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 3, 67–68.

[151] S. A. Vinogradov, *Multipliers of power series with sequence of coefficients from l^p*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **39** (1974), 30–39. Investigations on linear operators and the theory of functions, IV.

[152] ———, *Multiplicative properties of power series with a sequence of coefficients from l^p*, Dokl. Akad. Nauk SSSR **254** (1980), no. 6, 1301–1306.

[153] Harold Widom, *Inversion of Toeplitz matrices. II*, Illinois J. Math. **4** (1960), 88–99.

[154] ———, *On the spectrum of a Toeplitz operator*, Pacific J. Math. **14** (1964), 365–375.

[155] ———, *Toeplitz operators on H_p*, Pacific J. Math. **19** (1966), 573–582.

[156] Aurel Wintner, *Zur Theorie der beschränkten Bilinearformen*, Math. Z. **30** (1929), no. 1, 228–281.

[157] Thomas H. Wolff, *Counterexamples to two variants of the Helson-Szego theorem*, 2002, pp. 41–62. Dedicated to the memory of Tom Wolff.

[158] Pei Yuan Wu, *Numerical ranges as circular discs*, Appl. Math. Lett. **24** (2011), no. 12, 2115–2117.

[159] Ulaş Yamancı and Mehmet Gürdal, *On numerical radius and Berezin number inequalities for reproducing kernel Hilbert space*, New York J. Math. **23** (2017), 1531–1537.

[160] Ulaş Yamancı, Mehmet Gürdal, and Mubariz T. Garayev, *Berezin number inequality for convex function in reproducing Kernel Hilbert space*, Filomat **31** (2017), no. 18, 5711–5717.

[161] L. Zalcman, *Bounded analytic functions on domains of infinite connectivity*, Trans. Amer. Math. Soc. **144** (1969), 241–269.

[162] Kehe Zhu, *Operator theory in function spaces*, Second, Mathematical Surveys and Monographs, vol. 138, American Mathematical Society, Providence, RI, 2007.

[163] Nina Zorboska, *The Berezin transform and radial operators*, Proc. Amer. Math. Soc. **131** (2003), no. 3, 793–800.

[164] ———, *Closed range type properties of Toeplitz operators on the Bergman space and the Berezin transform*, Complex Anal. Oper. Theory **13** (2019), no. 8, 4027–4044.