A Nonconforming Finite Element Method for the 2D Vector Laplacian

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A Nonconforming Finite Element Method for the 2D Vector Laplacian
by
Mary Barker

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requirements for the degree
of Doctor of Philosophy

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Mary Barker

Washington University in St. Louis

May 2022
Dedicated to my family and friends
ABSTRACT OF THE DISSERTATION
A Nonconforming Finite Element Method for the 2D Vector Laplacian
by
Mary Barker
Doctor of Philosophy in Mathematics
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Professor Ari Stern, Chair

ABSTRACT

The vector Laplacian presents difficulties in finite element approximation. It is well known that for nonconvex domains, $H^1$-conforming approximation spaces form a closed subspace of the solution space $H(div; \Omega) \cap H(curl; \Omega)$. Hence $H^1$-conforming approximations will fail to converge. This is problematic as it is highly difficult to construct more general finite dimensional approximation spaces for this space. We will present an extension of a nonconforming method introduced by Brenner et al. The method was originally given for $P_1$-nonconforming spaces in two dimensions. Our extension is given for degree $r$ polynomials, but which agrees with the preceding method for the lowest degree case. The extended method is a hybridization with equivalent 1-field, 2-field, and 3-field formulations.

The regularity of the solution, and the corresponding convergence estimates are obtained in terms of weighted Sobolev spaces, and numerical results are presented.
Chapter 1

Introduction and background

1.1 Introduction

In two dimensions, the augmented vector Laplacian operator yields the boundary value problem

\[-\nabla \nabla \cdot u + \nabla \times \nabla \times u + \alpha u = f \quad \text{in } \Omega \subset \mathbb{R}^2, \quad \nabla \cdot u = 0 \quad \text{and} \quad u \times n = 0 \quad \text{on } \partial \Omega,
\]

where \( \alpha \) is a real-valued constant. This second order differential equation arises in the context of Maxwell’s equations. With the addition of the term \( \alpha u \), the problem can be used to generalize a family of equations where the parameter \( \alpha \) varies. When \( \alpha \) is chosen to be distinct from the eigenvalues of the corresponding vector Laplacian, the augmented vector Laplacian problem is well-posed.

The aim of this current work is to present a nonconforming method for the augmented vector Laplacian. This method is posed as an extension of the nonconforming one discussed in [15], and in the expanded form is demonstrated to generalize the existing method to higher degree approximation and also admit solution techniques using static condensation. Specifically, it is formulated as the hybridization of a nonconforming method for the Hodge Laplacian with penalty terms.

The remainder of this work is organized as follows:

- In the remainder of this chapter, we will present a brief overview of the historical development of finite element methods. This is followed by a discussion of Sobolev
spaces, which provide the general function-theoretic framework for the mathematical analysis of the differential equations that will be used. In addition to Sobolev spaces, we will give a summary of the Galerkin approach for variational problems in finite element methods and the context for nonconforming methods. Finally, we will introduce weighted Sobolev spaces, which are introduced in the context of possibly nonconvex domains with corners.

- In Chapter 2 the finite element exterior calculus (FEEC) framework is presented, and hybridization of finite element methods after the unified method of Cockburn et al [29, 28]. These two areas are combined in the recent work [6], which is discussed as the setting for the nonconforming method.

- In Chapter 3, we present the main results for this work. The nonconforming method for the augmented vector Laplacian is introduced with a priori estimates and numerical results. Regularity conditions and the problems associated with nonconvex domains are discussed.

### 1.1.1 Historical background

The finite element method is a mathematical framework for numerically approximating solutions to partial differential equations. The finite element approach was developed mainly in engineering applications before the mid 1960s but a rapid expansion of mathematical analyses has been transforming the field, beginning in the 1960s and continuing up to today.

Notable early finite element results are due to Céa [23], and Zlámal [53]. One of the most fundamental objects used in the theory of finite elements is the family of Sobolev spaces defined on a given space. A large body of work in the theory of Sobolev spaces and applications in finite element analysis is due to Bramble and Hilbert [13], Bramble and Zlámal [14], Babuška [7, 8] and Fix and Strang [36]. An early and widely influential mathematical
study of finite element methods was produced by Ciarlet in 1978 [27]. While much of the previous work implemented the theory of Céa for a coercive bilinear form defined on a general Hilbert space, a further important development was the introduction of various finite element spaces for convergence in mixed formulations, which are posed in terms of multiple forms, or of a form that fails the coercivity condition. Notable early results in this context are due to Brezzi, Douglas, and Marini [22], Raviart and Thomas [46], and Nédélec [44]. These results were generalized in a Hilbert complex framework by Arnold, Falk and Winther in their work on finite element exterior calculus, first in [3] and then more fully explored in [4]. In these works, the exterior calculus framework not only posed a generalized setting for families of variational problems, but also unified methods for determining stability and convergence of the problems.

Although the literature is less extensive than in the conforming case, the analysis of finite element applications with nonconforming or other nonstandard methods is not new. Ciarlet and Raviart [26] introduced one of the earliest nonconforming elements, and in 1977, this was connected to the mixed method formulations in [46]. Strang’s lemmas [49] form a crucial step in the analysis of the majority of nonconforming methods.

A nonconforming method and various stability and error estimates for Maxwell’s equations were developed by Brenner, Cui, Li, and Sung in a series of papers [16, 17, 18, 19, 15]. This method is defined using the vector-valued extension of the $P_1$-nonconforming finite elements defined by Crouzeix and Raviart [34]. A quadratic nonconforming element compatible with this method was later introduced by Brenner and Sung [21] with a conjectured extension to higher degree. This conjecture was subsequently proven by Mirebeau [42].
1.2 Background for finite element methods

We will begin with a review of finite element methods in general to provide context for the nonconforming method that is introduced later. This material requires an overview of Sobolev spaces with some standard results that form the basis for mathematical analysis of finite element methods.

1.2.1 Review of Sobolev spaces

The following general results can be found in many introductory works on finite element methods. See, e.g. [20]. Let $\Omega \subset \mathbb{R}^n$. A fundamental object in the definition of Sobolev spaces is the weak derivative, which allows us to treat a wider range of functions than those that are classically differentiable up to a given order.

**Definition 1.2.1 (Weak derivative).** Given a function $f \in L^1_{\text{loc}}(\Omega)$, we say $g \in L^1_{\text{loc}}(\Omega)$ is a weak derivative of $f$ and write

$$\partial^\alpha f = g$$

if

$$\int_\Omega f(x)\partial^\alpha \phi(x)dx = (-1)^{|\alpha|}\int_\Omega g(x)\phi(x)dx \quad \forall \phi \in C^\infty_c(\Omega)$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index, and $\partial^\alpha \phi := \frac{\partial^{\alpha_1} \phi}{dx_1^1 ... dx_n^\alpha_n}$.

**Definition 1.2.2 (Sobolev norm).** Define the Sobolev norm and semi-norm on a space $\Omega$ as:

$$\|f\|_{W^k_p} := \begin{cases} \left( \sum_{|\alpha| \leq k} ||\partial^\alpha f||_p^p \right)^{\frac{1}{p}}, & 1 \leq p < \infty \\ \max_{|\alpha| \leq k} ||\partial^\alpha f||_\infty, & p = \infty \end{cases}$$
Then define the Sobolev space $W^k_p(\Omega)$ as:

$$W^k_p(\Omega) := \left\{ f \in L^1_{\text{loc}}(\Omega) : \partial^\alpha f \in L^p(\Omega) \text{ for all } |\alpha| \leq k \right\}.$$ 

This is a Banach space, and for $p < \infty$, the space $W^k_p(\Omega)$ has a useful property with respect to the density of smooth functions [20, Theorem 1.3.4]:

**Theorem 1.2.3.** If $\Omega$ is any open set, then $C^\infty \cap W^k_p(\Omega)$ is dense in $W^k_p(\Omega)$ for all $k < \infty$.

It follows from the definition that $W^{k+1}_p(\Omega) \subset W^k_p(\Omega)$ for all $k \geq 0$. We will adopt the convention of referring to the space $H^k(\Omega) := W^k_2(\Omega)$ and denote its associated norm and seminorm as $\| \cdot \|_k$, $| \cdot |_k$ respectively.

The differentiability condition inherent in the Sobolev space definition makes these spaces a highly useful choice for finding solutions to differential equations. A fundamental principle underlying the majority of finite element methods is obtained by applying the weak derivative to a given differential equation and thereby shifting the differential to a "test function." Using integration by parts and the definition of the weak derivative, we obtain the variational form of the differential equation:

**Example 1.2.4 (Variational form).** Starting from the Laplace equation

$$-\Delta u = f \quad \text{on } \Omega$$

subject to the boundary conditions $u = 0$ on $\partial \Omega$, we multiply both sides of the equation by a "test function" $v$, and integrate to obtain the equation:

$$-\int_\Omega (\Delta u)v = \int_\Omega f v$$
Using integration by parts on the left hand side, this becomes

\[ \int_{\Omega} (\nabla u) \cdot (\nabla v) - \int_{\partial \Omega} \frac{\partial u}{\partial \nu} v = \int_{\Omega} fv \]

where \( \nu \) is the outward pointing normal to \( \partial \Omega \). Note that, since \( \nabla v \) is present in this form, the test function \( v \) is required to be at least \( H^1(\Omega) \). If we further assume \( v \in H^1(\Omega) \) the space of functions that are in \( H^1(\Omega) \) and which have vanishing boundary trace, then we have the primal variational form:

\[ \int_{\Omega} (\nabla u) \cdot (\nabla v) = \int_{\Omega} fv \quad \forall v \in H^1(\Omega) \]

In this form, the differentiability requirement on \( u \) is weakened. Whereas in the original problem, the definition of the Laplacian as a second order differential operator means that a solution to the equation must be in \( H^2(\Omega) \), the weak variational form only requires that \( u \in H^1(\Omega) \).

An alternative approach is to define the variable \( \sigma \in H(div; \Omega) \) by

\[ \sigma := -\nabla u, \]

and solve the system of first order equations:

\[ \sigma + \nabla u = 0 \]

\[ \nabla \cdot \sigma = f \]
for \((\sigma, u) \in H(\text{div}; \Omega) \times L^2(\Omega)\). Using a test function of \(\tau \in H(\text{div}; \Omega)\) on the first equation, and a test function of \(v \in L^2(\Omega)\) on the second, we obtain the system:

\[
\int_{\Omega} \sigma \tau - \int_{\Omega} u \nabla \cdot \tau = 0 \\
\int_{\Omega} (\nabla \cdot \sigma) v = \int_{\Omega} f v
\]

This is the mixed variational formulation for the Laplace equation.

Given a Hilbert space \(V\), \(f \in V^*\), and a bilinear form \(a\), and the associated problem of finding \(u \in V\) satisfying \(a(u, v) = f(v)\) for all \(v \in V\), the Galerkin approximation to the problem consists of defining a finite dimensional subspace \(V_h \subset V\) and solving the approximate problem

\[
a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h.
\]

**Theorem 1.2.5** (Lax-Milgram). Let \(V\) be a Hilbert space, let \(a(\cdot, \cdot)\) be a continuous, coercive bilinear form, and let \(f \in V^*\) be a continuous linear functional. Then there exists a unique \(u \in V\) such that

\[
a(u, v) = f(v) \quad \forall v \in V.
\]

By contrast, a mixed form contains more than one unknown, and often corresponds to a saddle point problem for two Hilbert spaces \(V, Q\) and continuous linear functionals \(f \in V^*,\) \(g \in Q^*:\)

\[
a(u, v) + b(v, p) = f(v) \quad \forall v \in V \quad (1.2.1a) \\
b(u, q) = g(q) \quad \forall q \in Q. \quad (1.2.1b)
\]

The conditions for existence and uniqueness for a saddle point problem are slightly more complex than for a single bilinear form. Even if the bilinear form \(a\) is symmetric, the saddle
point problem is no longer convex, and so does not correspond to a minimization problem. However, there are standard results due to Babuška and Brezzi in this context that establish conditions that can be satisfied to ensure existence and uniqueness of a solution. First, we will define the inf-sup condition. This condition is related to the notion of coercivity, but generalized to the situation of a map between disparate spaces.

**Definition 1.2.6.** The bilinear form \( b : V \times Q \rightarrow \mathbb{R} \) satisfies the inf-sup condition if there exists \( \beta > 0 \) such that

\[
\inf_{q \neq 0} \sup_{v \neq 0} \frac{|b(v, q)|}{\|v\|_V \|q\|_Q} \geq \beta
\]

When \( b \) is continuous and satisfies the inf-sup condition, then the map \( B^t : Q \rightarrow V^* \) defined by \( q \mapsto b(\cdot, q) \) is injective, since there exists \( \beta > 0 \) such that

\[
\sup_{\substack{v \in V \\text{and } v \neq 0}} \frac{|b(v, q)|}{\|v\|_V} \geq \beta \|q\|_Q \quad \text{for each } q \in Q.
\]

Continuity of \( b \) gives us that the range of \( B^t \) is closed.

In addition to the injectivity result, we see also that the map \( B : V \rightarrow Q^* \) by \( v \mapsto b(v, \cdot) \) is surjective. By the Closed Range Theorem, the range of \( B \) is

\[
\text{range}(B) = \text{nullspace}(B^t)^\perp \subset Q^*
\]

Then using the fact that the inf-sup condition gives that \( B^t \) is injective, the range of \( B \) corresponds to \( \{0\}^\perp \) in \( Q^* \). Hence, \( B \) is surjective.

We define the space:

\[
Z := \{ v \in V : b(v, q) = 0 \ \forall q \in Q \}
\]
Theorem 1.2.7. Let $V$ and $Q$ be Hilbert spaces, and $f \in V^*$ and $g \in Q^*$ be continuous linear functionals. If $a$ is a continuous and coercive bilinear form on $Z$, and $b$ is a continuous bilinear form $V \times Q \to \mathbb{R}$ that satisfies the inf-sup condition, then the mixed problem 1.2.1 has a unique solution.

Proof. This has been shown in several ways. We will follow the argument in [20].

We will first consider the case when $g = 0$. Note that the space $V$ is a Hilbert space with inner product denoted $(\cdot, \cdot)_V$. This space can be decomposed into the sum $Z \oplus Z^\perp$. Since $a$ is coercive on $Z$, by the Lax-Milgram theorem, there exists a unique solution $u \in Z$ to the problem

$$a(u, v) = (f, v) \quad \forall v \in Z \quad (1.2.2)$$

which is the solution to the saddle point problem defined for $a$ restricted to $Z$ and $b$ restricted to $Z \times Q$.

We will solve for $p \in Q$ by solving

$$b(v, p) = -a(u, v) + (f, v) \quad \forall v \in V$$

And since $-a(u, \cdot) + (f, \cdot) \in Z^\perp = \text{range}(B^t)$, this problem is well posed.

Finally, for the general case of $g \in Q^*$, we will add an alternative solution technique to (1.2.2) to define $u$. However, the method used to solve for $p$ with the unknown $u$ remains unchanged. Let $u_0$ be any solution to the problem

$$b(u_0, q) = (g, q) \quad \forall q \in Q$$
The surjectivity of $B$ implies the existence of such a solution. Then set $u = u_0 + u_1$ where $u_1 \in Z$ is the solution to the problem

$$a(u_1, v) = (f, v) - a(u_0, v) \quad \forall v \in Z$$

The uniqueness of the solution $(u, p)$ for the saddle point problem follows from the coercivity of $a$ and the inf-sup condition for $b$: Letting $f = 0$ and $g = 0$, we have that $u \in Z$, and so the saddle point system reduces to:

$$a(u, v) + b(v, p) = 0 \quad \forall v \in V$$

Using a test function of $v = u \in Z$, we have that

$$a(u, u) = 0$$

Since $a$ is coercive, this implies that $u = 0$. Similarly, since $b$ satisfies the inf-sup condition, the map $B^t$ is injective, and hence $p = 0$. Thus the solution to the homogeneous problem is uniquely 0.

Often when a system of equations is not coercive, it can satisfy a weaker condition: coercive “up to a constant” that allows us to obtain similar results. A condition that will be made use of in this context is Gårding’s inequality:

**Definition 1.2.8** (Gårding’s inequality). There exists a finite constant $K > 0$ such that:

$$a(v, v) + K \|v\|^2_{L^2(\Omega)} \gtrsim \|v\|^2_{1, \Omega}$$
Here we use the notation $\gtrsim$ to denote the relation:

$$x \gtrsim y \implies x \geq Cy$$

for some constant $C$.

### 1.2.2 Finite element method

**Definition 1.2.9.** Define a finite element as the triple $(K, \mathcal{N}, \mathcal{P})$ where

- $K$ is a bounded domain $K \subset \mathbb{R}^n$
- $\mathcal{N}$ consists of nodes on $K$.
- $\mathcal{P}$ is a finite-dimensional function space defined on $K$

We take $\mathcal{N}$ as the basis for dual space of $\mathcal{P}$. Thus functions $v \in \mathcal{P}$ can be uniquely determined by its values at the nodes $\mathcal{N}$.

Then the finite element method consists of two steps. First, the domain $\Omega \subset \mathbb{R}^n$ is partitioned to form a set $\mathcal{T}_h$ of elements denoted as $K$ with each $K \subset \mathbb{R}^n$. Second, the solutions to the given differential equation are approximated using a subspace of the finite dimensional approximation space

$$V_h \subset \prod_{K \in \mathcal{T}_h} \mathcal{P}(K)$$

of functions on $\mathcal{T}_h$ that restrict to an element of $\mathcal{P}(K)$ on each $K$. Here the choice of subspace is determined by compatibility conditions for degrees of freedom on the interface between two elements $K \in \mathcal{T}_h$.

There are various well-known theorems that give a priori error estimates for finite element approximations when the variational form of a problem can be posed as a bilinear form, or as a system of bilinear forms as the mixed formulation. If the test functions are chosen in
the appropriate finite element space, then the resulting system is amenable to the results of Céa [23] and Brezzi [10]:

**Theorem 1.2.10 (Céa).** Suppose $V$ is a Hilbert space, $a(\cdot, \cdot)$ a bilinear form on $V$ that is continuous and coercive, and suppose $V_h$ is a finite-dimensional subspace of $V$, $f \in V^*$. Then for $u_h \in V_h$ satisfying

$$a(u_h, v) = f(v)$$

for all $v \in V_h$, we have that

$$||u - u_h||_V \leq \frac{C}{\alpha} \min_{v \in V_h} ||u - v||_V$$

where $C$ is the continuity constant and $\alpha$ the coercivity constant.

**Theorem 1.2.11 (Brezzi).** Let $V, Q$ be Hilbert spaces, and let $(u, p)$ be the solution to the variational problem given by the saddle point problem:

$$a(u, v) + b(v, p) = f(v) \quad \forall v \in V$$

$$b(u, q) = g(q) \quad \forall q \in Q.$$

where $a$ is continuous and coercive, and $b$ satisfies the inf-sup condition, and $f \in V^*$ and $g \in Q^*$. Let $(u_h, p_h)$ be the Galerkin approximation where $V_h, Q_h$ are finite dimensional subspaces of $V$ and $Q$ respectively. Then

$$||u - u_h||_V \leq \left( 1 + \frac{C}{\alpha} \right) \inf_{v_h \in Z_h} ||u - v_h||_V + \frac{C}{\alpha} \inf_{q_h \in Q_h} ||p - q_h||_Q$$

where $\alpha$ is the coercivity constant, $C$ is the continuity constant for $a$, and where $Z_h = \{ v_h \in V_h : b(v_h, q_h) = 0 \ \forall q_h \in Q_h \}.$
One of the most widely used results in polynomial approximation estimates is the Bramble-Hilbert lemma [11]:

**Lemma 1.2.12** (Bramble-Hilbert). Suppose $\Omega \subset \mathbb{R}^n$ satisfies the strong cone condition. Let $F$ be a bounded sublinear linear functional on $W^k_p(\Omega)$ such that

$$F(p) = 0 \quad \text{for polynomials } p \text{ of degree up to } k - 1.$$ 

Then there exists a constant $C > 0$ such that

$$|F(u)| \leq C|u|_{W^k_p(\Omega)} \quad \forall u \in W^k_p(\Omega).$$

This result has been generalized to $\Omega \subset \mathbb{R}^n$ a union of star-shaped domains [20]. An immediate consequence of this lemma is the following result for polynomial approximation in Sobolev spaces:

**Corollary 1.2.13.** Suppose $\Omega \subset \mathbb{R}^n$ satisfies the strong cone property. Then for any integer $m$ and $u \in W^k_p(\Omega)$ with $0 \leq k \leq m$, there exists a polynomial $v$ of degree $m - 1$ such that

$$|u - v|_{W^k_p(\Omega)} \lesssim d^{m-k}|u|_{W^m_p(\Omega)}.$$

where $d$ is the diameter of $\Omega$.

### 1.2.3 Nonconforming finite element methods

There are many types of nonconforming finite element methods. The method we will consider falls into the set of methods where the approximation space does not satisfy the constraints of the differential equation, or is not a subspace of the solution space. These methods are often necessary in contexts where it is difficult to construct finite dimensional subspaces of
the solution space. For example, the space \( V = H(\text{div}; \Omega) \cap \dot{H}(\text{curl}; \Omega) \) in two dimensions, which will be considered in more detail later. This space poses certain difficulties when the domain \( \Omega \) is nonconvex. In particular, when \( \Omega \) contains a re-entrant corner, it is known \([15][32]\) that the space \([H^1(\Omega)]^2 \cap V\) is a proper closed subspace of \( V \). Thus a method using an \( H^1 \)-conforming approximation space will fail to converge if the solution is general enough.

When the approximation space does not conform to the constraints of the original problem, it is necessary to add terms to the variational formulation to ensure that the approximation satisfies the conditions imposed by the differential equation. Such terms can be in the form of a weakened Lagrange multiplier, an additional system of equations, or boundary conditions, etc.

A fundamental result that underpins most analysis for nonconforming methods is the following result of Strang which generalizes Céa’s Lemma in the case that \( V_h \not\subset V \).

**Lemma 1.2.14 (Strang’s Lemma).** Let \( V, V_h \) be subsets of a function space \( S \). Given a coercive bilinear form \( a(\cdot, \cdot) \) on \( V \times V \) such that \( a \) extends to \( V + V_h \), then for \( u \) the solution to

\[
a(u, v) = f(v) \quad \forall v \in V
\]

and \( u_h \) the solution to

\[
a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h
\]

we have

\[
\|u - u_h\|_h \leq c \inf_{v \in V_h} \|u - v\|_h + \sup_{v \in V_h \atop v \neq 0} \frac{|a(u, v) - a_h(u_h, v)|}{\|v\|_h}.
\]

Here the energy norm \( \| \cdot \|_h \) is defined by

\[
\|u\|_h^2 := a(u, u)
\]
1.2.4 Weighted Sobolev spaces

We now introduce a concept of weighted Sobolev spaces for domains with corners. These spaces are used to describe a more general space of functions that meet local regularity conditions. Weights are introduced so that the resulting product has global regularity. These spaces were introduced for domain with re-entrant corners by Kondrat’ev [39]. A rigorous and general setting for these results on polygonal domains can be found in [40]. In addition to early works on elliptic problems in domains with corners and edges, Costabel and Dauge [32] considered weighted spaces in applications to Maxwell’s equations.

We will first define the weighted space for a cone \( V \) emanating from the origin, and then extend this definition to more general domains with cuspidal points.

Let \( V \) be a cone with vertex at the origin, and interior angle \( \omega \). The cone can be expressed in the polar coordinates:

\[
\{(r, \theta) : 0 \leq r, 0 \leq \theta \leq \omega\}
\]

Define the weighted seminorm for integer \( k < \infty \) and real value \( \beta \) as

\[
|v|_{k,\beta}^2 := \sum_{|\alpha|=k} \int_V r^{2\beta} |\partial^\alpha v|^2 dx.
\]

Then the weighted Sobolev space \( H^k_{\beta}(V) \) will be defined as the closure of \( C_0^\infty(V \setminus \{0\}) \) with respect to the norm:

\[
\|v\|_{k,\beta} := \left( \sum_{|\alpha| \leq k} \int_V r^{2(\beta-k+|\alpha|)} |\partial^\alpha v|^2 dx \right)^{1/2}
\]

Let \( \Omega \) be a polyhedral domain with cuspidal points denoted \( (c_\ell), \ell \in \{1, \ldots, L\} \). We will assume that \( \partial \Omega \setminus \cup_{\ell=1}^L (c_\ell) \) is smooth, and that about each point \( c_\ell \) there exists an open neighborhood such that the resulting set of neighborhoods is disjoint. Let \( \omega_\ell \) be the interior angle at \( c_\ell \), and assume that the wedge \( V_\ell = B(c_\ell, \delta) \cap \Omega \) is diffeomorphic to the wedge at
the origin with rays emanating with an angle of $\omega_\ell$. Here $B(c_\ell, \delta)$ is the ball with radius $\delta$ and centered at $c_\ell$. Assume further that at the point $c_\ell$ the diffeomorphism is an isometric transformation, so that the angle $\omega_\ell$ is preserved. Let $\Omega_\delta$ be the deleted domain $\Omega \setminus \bigcup_\ell \mathcal{V}_\ell$. Let $\xi_\ell \in C^\infty(\Omega)$ with $0 \leq \xi_\ell \leq 1$ on $\Omega$ such that $\xi_\ell$ is equal to 1 in a neighborhood of $c_\ell$ and 0 on $\Omega \setminus \mathcal{V}_\ell$, and define $\xi_0 := 1 - \xi_1 - \cdots - \xi_L$. Then $v \in H^k_\beta(\Omega)$ if $\xi_0 v \in H^k(\Omega)$ and for each $\ell = 1, \ldots, L$, $\xi_\ell v \in H^k_\beta(\mathcal{V}_\ell)$. This space is equipped with the norm:

$$\|v\|_{k,\beta} := \left( \|\xi_0 v\|_{k,\Omega}^2 + \sum_{\ell=1}^L \|\xi_\ell v\|_{k,\beta,\mathcal{V}_\ell}^2 \right)^{\frac{1}{2}}$$

To generalize to the non-integer setting $H^s_\beta(\Omega)$, $s \notin \mathbb{N}$, we will utilize the method outlined in [47, Section 2.4]. For $s \notin \mathbb{N}$, let

$$k := \lfloor s \rfloor$$

$$\theta := s - k.$$

Then $H^s_\beta := [H^k_{\beta-s+k}, H^{k+1}_{\beta-s+k+1}]_\theta$ where $[\cdot, \cdot]_\theta$ denotes complex interpolation [9]. Following the argument [47, Remark 2.4], the generalized definition of $H^s_\beta$ by means of interpolation for non-integer values is well-defined for all parameters $s$.

It is known [32, 47] that for $\ell \leq k$

$$H^k_\beta \subset H^{k-\ell}_\beta$$

and $\alpha = (\alpha_\ell)$ satisfying $\alpha_i \leq \beta_i$ for all $i$,

$$H^k_\alpha \subset H^k_\beta.$$  

Finally, [32, Section 3], the inclusion $H^{k+1}_{\beta+1} \subset H^k_\beta$ is compact.
Chapter 2

Finite Element Exterior Calculus and the Hodge Laplacian

We will briefly introduce concepts from finite element exterior calculus (FEEC), including a review of differential forms and Hilbert complexes. These results are then applied to the Hodge Laplacian, which will be used as the main example for exploring the FEEC framework.

2.1 Hilbert complexes

Definition 2.1.1. A cochain complex \((V, d)\) is a sequence

\[
\ldots \to V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \to \ldots
\]

of vector spaces \(V^k\) together with maps \(d^k\) that satisfy \(d \circ d = 0\). Where we use \(d\) to denote the generalized operator that restricts to \(d^k\) when applied to an element of the space \(V^k\).

Definition 2.1.2. A Hilbert complex is a cochain complex \((W, d)\) such that

- \(W^k\) is a Hilbert space for all \(k\), and
- \(d^k : W^k \to W^{k+1}\) is a closed, densely defined linear operator with
- \(\text{range}(d^k) \subset \text{domain}(d^{k+1})\).

A Hilbert complex is closed if the image of \(d^k\) is closed in \(W^{k+1}\) for all \(k\).

Definition 2.1.3. We will now define some basic subspaces associated to a complex:
\[ \mathcal{Z}^k = \text{kernel}(d^k) \text{ elements of } \mathcal{Z}^k \text{ are } k\text{-cocycles} \]

\[ \mathcal{B}^k = \text{Image}(d^{k-1}) \text{ elements of } \mathcal{B}^k \text{ are } k\text{-coboundaries} \]

\[ \mathcal{H}^k = \mathcal{Z}^k / \mathcal{B}^k \] is called the \( k\text{-th cohomology group}. \text{ For Hilbert complexes, this is } \text{isomorphic to } \mathcal{Z}^k \cap \mathcal{B}^{k\perp}, \text{ the space of harmonic forms.} \]

For a Hilbert complex \( \cdots \rightarrow W^{k-1} \xrightarrow{d^{k-1}} W^k \xrightarrow{d^k} W^{k+1} \rightarrow \cdots \), we have the Hodge decomposition:

\[ W^k = \mathcal{B}^k \oplus \mathcal{H}^k \oplus \mathcal{Z}^{k\perp}\]

That is, we can break each of the vector spaces into direct sums of the image of \( d^{k-1} \), which is contained in the nullspace of \( d^k \), its orthogonal complement in the nullspace, and the orthogonal complement of the nullspace.

For any Hilbert complex \((W, d)\), there is an associated domain complex \((V, d)\) where \( d \) is inherited from \((W, d)\), and \( V^k = \text{domain } (d^k) \) with the inner product

\[ (u, v)_{V^k} = (u, v)_{W^k} + (du, dv)_{W^{k+1}}. \]

Thus we have the inclusions:

\[ \cdots \rightarrow W^{k-1} \xrightarrow{d^{k-1}} W^k \xrightarrow{d^k} W^{k+1} \rightarrow \cdots \]

\[ \cup \quad \cup \quad \cup \]

\[ \cdots \rightarrow V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \rightarrow \cdots \]

And define the dual complex \((W, d^*)\) which uses the same spaces \( W^k \), but with the adjoint \( d^* \) of \( d \). The dual complex then is of the form

\[ \cdots \rightarrow W^{k+1} \xrightarrow{d^*_{k+1}} W^k \xrightarrow{d^*_k} W^{k-1} \rightarrow \cdots \]
Or, alternatively:

\[ \cdots \xrightarrow{d_{k-1}} W_{k-1} \xrightarrow{d_k} W_k \xrightarrow{d_k} W_{k+1} \xrightarrow{d_{k+1}} \cdots \]

Note that the complex \((W, d^*)\) is closed or bounded if and only if \((W, d)\) is.

Let

1. \(Z_k^* = B^{k \perp W} = \ker(d_k^*)\) by the Closed Range Theorem
2. \(B_k^* = \text{Im}(d_{k+1}^*)\).

Then \(\mathcal{H}_k = Z_k \cap Z_k^*\) is the space of harmonic forms for both \((W, d)\) and \((W, d^*)\). So

\[ W^k = B^k \oplus \mathcal{H}_k \oplus B_k^*. \]

The following Poincaré inequality given in [4] is a result of Banach’s bounded inverse Theorem:

**Theorem 2.1.4 (Poincaré inequality).** Let \((W, d)\) be a closed Hilbert complex with domain complex \((V, d)\). Then there exists \(C_p\) such that

\[ ||v||_V \leq C_p ||dv||_W \quad \forall v \in Z^{k \perp} \cap V^k \]

The transition from considering results for individual methods to families of methods defined on a complex with a general differential operator is one of the major breakthroughs of FEEC.

### 2.2 Subcomplexes of a Hilbert complex

Given the Hilbert complex \((W, d)\) with domain complex \((V, d)\), choose finite dimensional subspaces \(V_h^k \subset V^k\) such that \((V_h, d)\) is a subcomplex of \((V, d)\). That is, that \(dV_h^k \subset dV_h^{k+1}\). Define
• $W_h^k = V_h^k$ with the $W$-norm rather than the $V$-norm, and

• $d^*_h : V_h^{k+1} \to V_h^k$ by $(d^*_h u, v) = (u, dv)$ for all $u \in V_h^{k+1}, v \in V_h^k$.

Then the following two results [4, Theorem 3.4 and 3.7] of Arnold, Falk, and Winther establish the relation between properties of a subcomplex and the existence of a bounded projection.

**Theorem 2.2.1.** Let $(V, d)$ be a closed Hilbert complex, $(V_h, d_h)$ a subcomplex, and $\pi_h : (V, d) \to (V_h, d_h)$ a bounded cochain projection. Then if for all $k$ it holds that

$$|| q - \pi_h q ||_V \leq ||q||_V \quad \forall 0 \neq q \in \mathcal{H}^k,$$

then the induced map on cohomology is an isomorphism.

Thus if the subcomplex admits a bounded cochain projection, then the approximations are good in the sense that they have the same cohomology as the solution space. Thus the solution to a differential equation and its finite dimensional approximation will satisfy similar properties with respect to the original differential operator. A natural question in this context is when there exists such a bounded projection.

**Theorem 2.2.2.** Let $(V, d)$ be a closed Hilbert complex, $(V_h, d_h)$ subcomplex. Then there exists a bounded cochain projection $\pi_h : (V, d) \to (V_h, d_h)$ if and only if there exists constants $C_1, C_2$ such that

$$||v||_V \leq C_1 ||dv||_V \quad \forall v \in \mathcal{Z}_h^1$$

and

$$||q||_V \leq C_2 ||\mathcal{P}_h q||_V \quad \forall q \in \mathcal{F}_h.$$
2.3 Exterior calculus of differential forms

The machinery associated to Hilbert complexes and cochain complexes in general can be applied to the exterior algebra associated to the space $\Omega$ to generalize results for families of differential equations. First we will define the exterior algebra for a vector space $V$, and then extend these to the general setting for a manifold $\Omega$. Let $V$ be an $n$-dimensional vector space.

**Definition 2.3.1.** Define the alternating $k$-forms on $V$, denoted $\Lambda^k(V)$ as the space of antisymmetric $k$-linear functions:

$$f : V \times \cdots \times V \to \mathbb{R}.$$  

Note that $\Lambda^k(V) = 0$ for $k > n$. The space $\Lambda^0(V)$ is taken to be $\mathbb{R}$, hence $\Lambda^1(V) = V^*$.

**Definition 2.3.2.** Given two forms $f \in \Lambda^k(V)$ and $g \in \Lambda^\ell(V)$, define the wedge product $f \wedge g \in \Lambda^{k+\ell}(V)$ as

$$f \wedge g := \sum_{(\sigma, \tau) \in S_{k,\ell}} \text{sign}(\sigma, \tau) f \circ \sigma \circ g \circ \tau$$

where $S_{k,\ell}$ denotes the set of $(k, \ell)$-shuffles of the set $\{1, \ldots, k + \ell\}$. That is, the set of all permutations of $k + \ell$ elements, where the first $k$ terms are denoted $\sigma$, and the last $\ell$ terms are denoted by $\tau$, and such that $\sigma$, $\tau$ satisfy the ordering $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$, and $\tau(1) < \tau(2) < \cdots < \tau(\ell)$.

Then the elements of $C^\infty\Lambda^k$ are expressed in terms of the wedge product of $k$ basis elements $dx^i$ with coefficients given by smooth functions $a$. Here $x_i$ forms a basis for $V$, and $dx^i$ is the associated dual basis for $V^* = \Lambda^1(V)$.

As a matter of notation, we will define $S\Lambda^k(V)$ to be the set of $k$-forms on $V$ with coefficients consisting of functions in the set $S$. Thus $L^2\Lambda^k(V)$ consists of the sums of terms.
of the form

\[ adx^{i_1} \wedge \cdots \wedge dx^{i_k} \]

where \( a \in L^2(V) \).

The space \( \Lambda^k(V) \) inherits an inner product from the vector space \( V \). Let \( \{x_i\} \) be an orthonormal basis for \( V \). Then for any \( u, v \in \Lambda^k(V) \), define the inner product \( (u, v)_V := \sum_I u(x_I)v(x_I) \) where the sum is over all multi-indices \( I = \{i_1, \ldots, i_k\} \) of length \( k \) satisfying the condition \( i_1 < \cdots < i_k \).

Note that the results are given in terms of a vector space \( V \), but can be generalized to the case of a smooth \( n \)-dimensional manifold with boundary \( \Omega \) by localizing at the a point \( x \in \Omega \) and considering the vector space \( T_x\Omega \). For a detailed description of the mapping, see [50]. For the remainder of the discussion on exterior calculus, we will assume that \( \Omega \) is a smooth orientable \( n \)-dimensional Riemannian manifold, possibly with boundary.

Recall the exterior derivative acts on differential forms. For a single term of a \( k \)-form \( u = adx^{i_1} \wedge \cdots \wedge dx^{i_k} \), the exterior derivative of \( u \) is defined as

\[ du := \sum_j \frac{\partial u}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \]

A choice of orientation on \( V \) is determined by a choice of order on the basis \( dx^i \), which induces the volume form

\[ \text{vol} := dx^1 \wedge \cdots \wedge dx^n \]

which, in turn, yields the Hodge star operator defined by the relation

\[ u \wedge *v = (u, v) \text{vol} \quad \forall u, v \in \Lambda^k(V). \]
The $L^2$-inner product then relates to the Hodge star operator and the wedge product on by
\[(u, v)_\Omega := \int_\Omega u \wedge \star v \quad \forall u, v \in C^\infty \Lambda^k(\Omega).\]

Define the codifferential $d^*$ as:
\[d^* = (-1)^k \star^{-1} d\]

Then $d, d^*$ satisfy the following relation as a result of the Leibniz rule:
\[d(u \wedge \star v) = du \wedge \star v - u \wedge d^* v\]

Now, it follows directly from the definition of $d$ that $dd = d^*d^* = 0$. Therefore we can define the complex
\[0 \rightarrow \Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(\Omega) \xrightarrow{d} 0\]

### 2.4 Finite element families as differential forms

A primary step in interpreting a finite element method in the framework of FEEC consists in identifying the finite element spaces with corresponding finite-dimensional spaces of differential forms that form a subcomplex that satisfies the commuting property with the differential $d$ and the projection. To that end, Arnold, Falk, and Winther identified two families of piecewise polynomial-valued forms that are commonly implemented.

**Notation:**

- $\mathcal{P}_r(\mathbb{R}^n) =$ polynomials of degree $\leq r$
- $\mathcal{P}_r \Lambda^k(\mathbb{R}^n)$ the differential $k$-forms with coefficients in $\mathcal{P}_r$.
- $\mathcal{H}_r(\mathbb{R}^n) =$ homogeneous polynomials of degree $r$,
• $\mathcal{H}_r \Lambda^k(\mathbb{R}^n)$ the corresponding differential forms.

We will then define the space

$$\mathcal{P}_r^\Lambda^k := \mathcal{P}_{r-1} \Lambda^k + \kappa \mathcal{H}_{r-1} \Lambda^{k+1} = \mathcal{P}_{r-1} \Lambda^k + \kappa \mathcal{P}_{r-1} \Lambda^{k+1} = \{ w \in \mathcal{P}_r \Lambda^k : \kappa w \in \mathcal{P}_r \Lambda^{k-1} \}$$

where $\kappa$ is the Koszul differential [4, Section 5.1.2] $\kappa : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k-1}(\mathbb{R}^n)$ defined by

$$(\kappa w)_x(v_1, ..., v_{k-1}) = w_x(X(x), v_1, ..., v_{k-1})$$

where $X(x)$ is the vector field corresponding to $x$.

The following result is given in [4, Theorem 5.4]. This relation between the spaces $\mathcal{P}_r^-$ and $\mathcal{P}_r$ is the key for a stable choice of spaces.

**Theorem 2.4.1.** If $w \in \mathcal{P}_r^\Lambda^k$ and $dw = 0$, then

$$w \in \mathcal{P}_{r-1} \Lambda^k$$

Moreover, $d\mathcal{P}_r \Lambda^k = d\mathcal{P}_r^- \Lambda^k$

Note that for each $n$, we obtain $2^{n-1}$ distinct resolutions of $\mathbb{R}$, since the images $d\mathcal{P}_r$ and $d\mathcal{P}_r^-$ are identical. The possible combinations are enumerated below.

$$\mathbb{R} \leftrightarrow \mathcal{P}_r \Lambda^0 \xrightarrow{d} \begin{cases} \mathcal{P}_{r-1} \Lambda^1 \\ \mathcal{P}_r^- \Lambda^1 \end{cases} \xrightarrow{d} \begin{cases} \mathcal{P}_{r-2} \Lambda^2, \mathcal{P}_{r-1} \Lambda^2 \\ \mathcal{P}_r^- \Lambda^2 \end{cases} \xrightarrow{d} \cdots$$

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**Notation:** For a simplex $T$, let $\Delta_d(T)$ denote the set of sub-simplices of $T$ of dimension $d$. Then for all $f \in \Delta_d(T)$,

\[
\begin{align*}
\mathcal{P}_r\Lambda^k(f) &= tr_{\mathbb{R}^n} f \mathcal{P}_r\Lambda^k(\mathbb{R}^n), \\
\hat{\mathcal{P}}_r\Lambda^k(f) &= \{w \in \mathcal{P}_r\Lambda^k(f) : tr_{f,\partial f} w = 0\}
\end{align*}
\]

### 2.5 Hodge Laplacian

Arnold, Falk, and Winther [4] consider the de Rham complex of differential forms under the Hodge-Laplace operator $L : W^k \to W^k$ defined by $L = dd^* + d^*d$, as it applies to each of the spaces of $k$-forms $\Lambda^k(\Omega)$. Note that if $u$ is a solution to the problem

\[
Lu = f \quad \text{on } \Omega \\
tr u = 0, \ tr du = 0 \quad \text{on } \partial \Omega
\]

then taking a variation of $Lu$, we obtain the problem

\[
(du, dv) + (d^*u, d^*v) = (f, v) \quad \forall v \in V^k \cap V^*_k.
\]

The functions in $\mathcal{H}^k$ measure the well-posedness of the Hodge formulation as follows: if $Lu = 0$, then $u \in \mathcal{H}^k$. Moreover, if there exists a solution $Lu = f$ for $f \neq 0$, then $f \perp \mathcal{H}^k$. Then we can write this in weak form as a system of lower order equations

\[
\begin{align*}
\sigma &= d^*u \in V^{k-1} \\
d\sigma + d^*du &= f \in V^k.
\end{align*}
\]
See [4] for a more detailed discussion of the Hodge Laplace operator and the restriction to the domain complex with spaces

\[ \tilde{H}^k(\Omega) := \{ v \in L^2 \Lambda^k(\Omega) : dv \in L^2 \Lambda^{k+1}(\Omega), trv = 0 \}. \]

The choice of approximation space for a finite dimensional approximation to the problem is viewed as a subspace of the corresponding space of forms, and hence the stability and convergence of the method is examined in terms of the results for general Hilbert complexes with the existence of a commuting cochain projection.

Given a closed Hilbert complex \((W, d)\) with finite-dimensional subspaces \(V_h^k\) of the domain complex \((V, d)\) that give rise to the subcomplex \((V_h, d)\), and given a bounded cochain projection \(\pi_h : (V, d) \to (V_h, d)\), we will examine the abstract Hodge Laplacian in discretized space: for \(f \in W^k\), find \((\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times V_h^{k+1}\) such that

\[
(\sigma_h, \tau) - (d\tau, u_h) = 0 \quad \forall \tau \in V_h^{k-1} \\
(d\sigma_h, v) + (du_h, v) + (v, p_h) = (f, v) \quad \forall v \in V_h^k \\
(u_h, q) = 0 \quad \forall q \in \tilde{H}_h^{k+1}
\]

The choice of finite dimensional subspaces yields a family of finite element methods associated to the differential equations obtained at each step of the complex.

We write the unique solution as [4, Section 3.4]:

\[
u_h = K_h P_h f, \quad \sigma_h = d_h^* u_h, \quad p_h = P_{\delta_h} f
\]

where \(P_h : W^k \to V_h^k\) is \(W^k\)-orthogonal projection. And the stability and consistency for this family of finite element methods are obtained as follows [4, Theorem 3.9].
Theorem 2.5.1. Let \((V_h, d)\) be a family of subcomplexes of \((V, d)\), the domain complex of a closed Hilbert complex such that each \((V_h, d)\) admits uniformly \(V\)-bounded cochain projection. Let \((\sigma, u, p) \in V^{k-1} \times V^k \times \delta^k\) be a solution to the mixed Hodge Laplacian, and \((\sigma_h, u_h, p_h)\) the solution to the discretized version. Then:

\[
||\sigma - \sigma_h||_V + ||u - u_h||_V + ||p - p_h|| \leq C \left( \inf_{\tau \in V^{k-1}_h} ||\sigma - \tau||_V + \inf_{v \in V^k_h} ||u - v||_V \\
+ \inf_{q \in V^k_h} ||p - q||_V + \mu \inf_{v \in V^k_h} ||\mathcal{P}_B u - v||_V \right)
\]

where \(\mu = \sup_{r \in \delta^k} ||(I - \pi^k_h) r||.\)

2.6 The Hodge Laplacian in two dimensions: an overview

We will now apply FEEC to the case of the two dimensional Hodge Laplacian with specific choice of finite element spaces. We obtain three versions of the Laplacian.

2.6.1 \(k = 0\) case

The exterior derivative \(d\) applied to scalars corresponds to the gradient. The \(k = -1\) space in the complex is trivial, and hence the image of the codifferential \(d^*\) applied to scalars is zero. Therefore the Hodge-Laplace operator simplifies to \(d^* d\). Thus the \(k = 0\) version of the Laplacian to finding \(u \in H^1(\Omega)\) that satisfies

\[
-\nabla \cdot \nabla u = f \quad \text{on } \Omega \\
u = 0 \quad \text{on } \partial \Omega
\]
2.6.2 $k = 1$ case

In the two dimensional case, the action of the exterior derivative on a 1-form corresponds to the two dimensional curl operator applied to the associated proxy field. So the vector Laplacian problem is to find $u \in H(\text{div}; \Omega) \cap H(\text{curl}; \Omega)$ that satisfies

$$-\nabla \nabla \cdot u + \nabla \times \nabla \times u = f \quad \text{on } \Omega \quad (2.6.1a)$$

$$u \times n = 0 \quad \text{on } \partial \Omega \quad (2.6.1b)$$

$$\nabla \cdot u = 0 \quad \text{on } \partial \Omega \quad (2.6.1c)$$

2.6.3 $k = 2$ case

In two dimensions, 2-forms are of the form $adx \wedge dy$, and so the corresponding proxy fields are scalar-valued. Similarly to the $k = 0$ case, the exterior derivative in this case has trivial image, and hence the Hodge-Laplace operator simplifies to $dd^*$. Thus the final problem in the Hodge Laplacian is to find $u \in H^1(\Omega)$ satisfying

$$-\nabla \cdot \nabla u = f \quad \text{on } \Omega$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega.$$ 

2.7 Hybridization

Hybridization is a process for finite element methods whereby the chosen approximation spaces are discontinuous at the inter-element boundaries, and continuity is imposed through the introduction of Lagrange multipliers.

Hybridization is closely connected to the solution technique of static condensation. In this way, the resulting problem is posed as a coupled system, where the global terms can be
isolated and solved for independently from the local terms, giving a potentially much smaller system. Due to the locally-defined nature of the remaining terms, the second problem is easily parallelizable, allowing for an increased scaling in problem size. In addition to adding scalability, hybridization can also be viewed as a first step in postprocessing solutions in the manner originally introduced by Arnold and Brezzi [1], used subsequently by Brezzi, Douglas, and Marini [22], and more recently explored by Stenberg [48].

A large body of work by Cockburn et al [30, 29, 28] forms the foundation for much of the analysis related to hybridization. In these works, the authors present a unifying approach to hybridization techniques for a variety of problems. Hybridization in the context of FEEC is a more recent contribution, and is due to (author?) [6].

A key observation in this setting is that the trace of a differential form and its restriction to a lower-dimensional face are not always identical, and can be interpreted differently.

Recall that \( u \in \Lambda^k(\Omega) \) can be expressed uniquely in terms of the basis \( dx^I \) where \( I \) consists of all multi-indices of length \( k \) for the set \( \{dx^i\}_{i=1}^{n} \). Thus there are \( \binom{n}{k} \) components of \( u \).

If \( \Omega \) has a boundary, then under the pullback map induced by the inclusion \( \partial \Omega \hookrightarrow \Omega \), we obtain the trace operator

\[
\text{tr} : \Lambda(\Omega) \rightarrow \Lambda(\partial \Omega).
\]

The boundary is \( n-1 \)-dimensional, and hence \( \text{tr}(u) \in \Lambda^k(\partial \Omega) \) is expressed uniquely in terms of \( \binom{n-1}{k} \) components. The “lost” information from \( u \) can be preserved with the definition of a second trace operator after the manner of [6].

**Definition 2.7.1.** Let \( u \in \Lambda^k(\Omega) \). Define the tangential trace of \( u \) as

\[
u^{\text{tan}} := \text{tr} u \in \Lambda^k(\partial \Omega)\]
and the normal trace of $u$ as

$$u^{\text{nor}} := \hat{\star}^{-1} \text{tr} \star u \in \Lambda^{k-1}(\partial\Omega).$$

Where $\hat{\star}$ is the Hodge star defined on $\partial\Omega$ rather than on $\Omega$. This decomposition of $u$ into tangential and normal traces when restricting to $\partial\Omega$ is chosen to make use of the integration by parts formula [6]

$$(u^{\text{tan}}, v^{\text{nor}})_{\partial\Omega} = (du, v)_{\Omega} - (u, d^{*}v)_{\Omega}$$

Here the subscripts $\partial\Omega$ and $\Omega$ respectively denote the inner product on the appropriate space of forms.

Following the work of Weck [51] and Mitrea, Mitrea, and Shaw [43], Awanou, Fabien, Guzmán, and Stern derive the tangential and normal traces in the abstract setting and in terms of the scalar and vector proxy fields for $\Omega \subset \mathbb{R}^{3}$. In this setting, they re-state the Hodge-Laplace problem as finding $(u, p) \in \mathbb{H}^{k-\perp} \times \mathbb{H}^{k}$ such that

$$dd^{*}u + d^{*}du + p = f \quad \text{on } \Omega$$

$$u^{\text{nor}} = 0, (du)^{\text{nor}} = 0 \quad \text{on } \partial\Omega.$$  

### 2.7.1 Notation and definitions

Let $\mathcal{T}_{h}$ denote the partition of the domain $\Omega$ into $n$-dimensional simplices with codimension 1 subsimplices forming the set $\mathcal{E}_{h}$. In two dimensions, a triangle $K \in \mathcal{T}_{h}$ has edges $e \in \partial K \subset \mathcal{E}_{h}$. We introduce notations and definitions associated to such a discretization. Let $S$ denote a function space of coefficients. Then define the broken space of forms on the discretization as
follows:

\[ S\Lambda^k(T_h) := \prod_{K \in T_h} S\Lambda^k(K), \]
\[ S\Lambda^k(E_h) := \prod_{e \in E_h} S\Lambda^k(e), \]
\[ S\Lambda^k(\partial T_h) := \prod_{K \in T_h} \prod_{e \in \partial K} S\Lambda^k(e). \]

and let \( \tilde{S}\Lambda^k \) denote the space of \( k \)-forms with coefficients in \( S \) that have vanishing tangential trace on \( \partial \Omega \). Note that the spaces \( S\Lambda^k(T_h) \) and \( S\Lambda^k(\partial T_h) \) are defined element-wise, whereas \( S\Lambda^k(E_h) \) is single-valued on the skeleton of the discretization. We will use the notation \( \langle \cdot, \cdot \rangle_K \) and \( \langle \cdot, \cdot \rangle_e \) to denote the \( L^2 \)-inner product over an element \( K \) and an edge \( e \) respectively, and

\[ (u, v)_{T_h} := \sum_{K \in T_h} (u, v)_K \]
\[ \langle u, v \rangle_{\partial T_h} := \sum_{K \in T_h} \langle u, v \rangle_{\partial K} \]
\[ \langle u, v \rangle_{E_h} := \sum_{e \in E_h} \langle u, v \rangle_e \]

for the \( L^2 \) inner products defined on the discretization.

In [6], Awanou, Fabien, Guzmán, and Stern demonstrate that the Hodge Laplacian problem with domain complex \((H\Lambda(\Omega), d)\) is equivalent to the following hybridized problem: find \((\sigma, u, p, \bar{u}, \bar{p}) \in W^{k-1} \times W^k \times \tilde{\mathcal{H}}_k \times \tilde{\mathcal{H}}_k \times \tilde{\mathcal{H}}_k, (\hat{\sigma}_{\text{tan}}, \hat{u}_{\text{tan}}) \in \hat{V}^{k-1, \text{nor}} \times \hat{V}^{k, \text{tan}}, \) and
\((\hat{u}^{\text{nor}}, \hat{\rho}^{\text{nor}}) \in \hat{W}^{k-1,\text{nor}} \times \hat{W}^{k,\text{nor}}\) satisfying the system of equations

\[
\begin{align*}
(\sigma, \tau)_{\mathcal{T}_h} - (u, d\tau)_{\mathcal{T}_h} + \langle \hat{u}^{\text{nor}}, \tau^{\text{tan}} \rangle_{\partial\mathcal{T}_h} &= 0 & \forall \tau \in W^{k-1} \quad (2.7.1a) \\
(d\sigma, v)_{\mathcal{T}_h} + (du, dv)_{\mathcal{T}_h} + (\bar{p} + p, v)_{\mathcal{T}_h} - \langle \hat{\rho}^{\text{nor}}, v^{\text{tan}} \rangle_{\partial\mathcal{T}_h} &= (f, v)_{\mathcal{T}_h} & \forall v \in W^k \quad (2.7.1b) \\
\langle \hat{u} - u, \bar{q} \rangle_{\mathcal{T}_h} &= 0 & \forall \bar{q} \in \tilde{\mathcal{H}}^k \quad (2.7.1c) \\
\langle \hat{\sigma}^{\text{tan}} - \sigma^{\text{tan}}, \hat{\nu}^{\text{nor}} \rangle_{\partial\mathcal{T}_h} &= 0 & \forall \hat{\nu}^{\text{nor}} \in \hat{W}^{k-1,\text{nor}} \quad (2.7.1d) \\
\langle \hat{u}^{\text{tan}} - u^{\text{tan}}, \hat{\eta}^{\text{nor}} \rangle_{\partial\mathcal{T}_h} &= 0 & \forall \hat{\eta}^{\text{nor}} \in \hat{W}^{k,\text{nor}} \quad (2.7.1e) \\
(u, q)_{\mathcal{T}_h} &= 0 & \forall q \in \mathcal{H}^k \quad (2.7.1f) \\
\langle \tilde{p}, \bar{v} \rangle_{\mathcal{T}_h} &= 0 & \forall \bar{v} \in \tilde{\mathcal{H}}^k \quad (2.7.1g) \\
\langle \hat{u}^{\text{nor}}, \hat{\tau}^{\text{tan}} \rangle_{\partial\mathcal{T}_h} &= 0 & \forall \hat{\tau}^{\text{tan}} \in \hat{V}^{k-1,\text{tan}} \quad (2.7.1h) \\
\langle \hat{\rho}^{\text{nor}}, \hat{\nu}^{\text{tan}} \rangle_{\partial\mathcal{T}_h} &= 0 & \forall \hat{\nu}^{\text{nor}} \in \hat{V}^{k,\text{tan}} \quad (2.7.1i)
\end{align*}
\]

where the spaces are defined as follows:

\[
\begin{align*}
W^k &:= H\Lambda^k(\mathcal{T}_h) & \tilde{\mathcal{H}}^k &:= \prod_{K \in \mathcal{T}_h} \tilde{\mathcal{H}}^k(K) \\
\hat{W}^{k,\text{nor}} &:= \{ \eta^{\text{nor}} : \eta \in H^*\Lambda^{k+1}(\mathcal{T}_h) \} & \hat{V}^{k,\text{tan}} &:= \{ v^{\text{tan}} : v \in H\Lambda^k(\Omega) \}.
\end{align*}
\]

Note that the spaces are defined element-wise. Those that are defined on the space \(\mathcal{T}_h\) therefore admit discontinuities across inter-element boundaries. The spaces of Lagrange multipliers \(V^{k,\text{tan}}\) are defined as the spaces of tangential traces \(k\)-forms defined on the space \(\Omega\), and hence correspond to single-valued terms on the edges. The problem (2.7.1) reduces to a system with seven unknowns when \(k < n\) and \(\Omega\) is contractible. In this case, the space of local harmonic \(k\)-forms \(\hat{\mathcal{H}}^k\) is trivial. This highly general formulation can be shown to reduce to various methods for the Hodge Laplacian in \(n\) dimensions and for \(k\)-forms. Note that the formulation is conforming, in the sense that the solution \(u\) resides in the space \(H\Lambda^k(\Omega)\), due to the definition of the trace spaces. When the approximation spaces satisfy
the duality conditions, the continuity constraints yield a solution that satisfies the original Hodge Laplace problem on the entire domain.

An alternative variational principle is introduced in this context [6, Section 8], which forms the basis for extension of the hybridized FEEC framework to various nonconforming and hybridable discontinuous Galerkin methods. The general system for this method is derived as finding

\[
\begin{align*}
\sigma_h & \in W_h^{k-1}, & u_h & \in W_h^k, & \rho_h & \in W_h^{k+1}, & \bar{p}_h & \in \bar{S}_h^k, \\
p_h & \in S_h^k, & \bar{u}_h & \in \bar{S}_h^k, & \hat{\sigma}_h^{\tan} & \in \hat{V}_h^{k-1,tan}, & \hat{u}_h^{\tan} & \in \hat{V}_h^{k,tan},
\end{align*}
\]

satisfying

\[
\begin{align*}
(\sigma_h, \tau_h)_T - (u_h, d\tau_h)_T + (\hat{u}_h^{\nor}, \tau_h^\tan)_{\partial T_h} &= 0 & \forall \tau_h & \in W_h^{k-1}, \\
(\sigma_h, \delta v_h)_T + (\rho_h, d\delta v_h)_T + (\bar{p}_h + p_h, v_h)_T & + (\hat{\sigma}_h^{\tan}, v_h^{\nor})_{\partial T_h} - (\hat{\rho}_h^{\nor}, v_h^{\tan})_{\partial T_h} &= (f, v_h)_T & \forall v_h & \in W_h^k, \\
(\rho_h, \eta_h)_T - (u_h, \delta \eta_h)_T - (\hat{u}_h^{\tan}, \eta_h^{\nor})_{\partial T_h} &= 0 & \forall \eta_h & \in W_h^{k+1}, \\
(\bar{u}_h - u_h, \bar{q}_h)_T &= 0 & \forall \bar{q}_h & \in \bar{S}_h^k, \\
(u_h, q_h)_T & = 0 & \forall q_h & \in S_h^k, \\
(\bar{p}_h, \bar{v}_h)_T & = 0 & \forall \bar{v}_h & \in \bar{S}_h^k, \\
(\hat{u}_h^{\nor}, \hat{z}_h^{\tan})_{\partial T_h} & = 0 & \forall \hat{z}_h^{\tan} & \in \hat{V}_h^{k-1,tan}, \\
(\hat{\rho}_h^{\nor}, \hat{z}_h^{\tan})_{\partial T_h} & = 0 & \forall \hat{z}_h^{\tan} & \in \hat{V}_h^{k,tan}.
\end{align*}
\]

Here the tangential trace spaces \(\hat{V}_h^{k,tan}\) are again taken in some unbroken space, though possibly not corresponding to the space of tangential traces for \(W_h^k\). The choice of definition for the normal traces gives a variety of methods that are explored by Awanou, Fabien, Guzmán, and Stern [6, Section 8]. In particular, replacing \(u_h^{\nor}, \rho_h^{\nor}\) with the terms \(\hat{u}_h^{\nor} \in W_0^k\)
\[ W_h^{k-1,\text{nor}}, \hat{\rho}_h^{\text{nor}} \in \hat{W}_h^{k,\text{nor}}, \]
and adding two additional equations with test functions from the corresponding choice of normal trace spaces to the system (2.7.2):

\[ \langle \sigma_h^{\text{tan}} - \hat{\sigma}_h^{\text{tan}}, \hat{\nu}_h^{\text{nor}} \rangle_{\partial T_h} = 0 \quad \forall \hat{\nu}_h^{\text{nor}} \in \hat{W}_h^{k-1,\text{nor}} \]  
(2.7.3a)

\[ \langle u_h^{\text{tan}} - \hat{u}_h^{\text{tan}}, \hat{\eta}_h^{\text{nor}} \rangle_{\partial T_h} = 0 \quad \forall \hat{\eta}_h^{\text{nor}} \in \hat{W}_h^{k,\text{nor}}. \]  
(2.7.3b)

In the next chapter, we will derive a nonconforming method that weakly enforces continuity through alternative approximation spaces for the tangential and normal trace spaces respectively.
Chapter 3

A nonconforming method for the two dimensional vector Laplacian

The augmented vector Laplacian is obtained by adding a term $\alpha u$ to the vector Laplacian to yield the problem:

$$
-\nabla \nabla \cdot u + \nabla \times \nabla \times u + \alpha u = f \quad \text{in } \Omega \subset \mathbb{R}^2, \quad \nabla \cdot u = 0 \text{ and } u \times n = 0 \quad \text{on } \partial \Omega.
$$

The primal variational form of this is given by

$$
(\nabla \cdot u, \nabla \cdot v)_\Omega + (\nabla \times u, \nabla \times v)_\Omega + \alpha (u, v)_\Omega = (f, v)_\Omega, \quad \forall v \in H(\text{div}; \Omega) \cap \dot{H}(\text{curl}; \Omega). \quad (3.0.1)
$$

A scalar nonconforming function space, the $CR$ space, was introduced by Crouzeix and Raviart in [34]. The lowest-order, or linear, such space, is denoted by the $CR_1$ functions. This is the space of functions that restrict to linear functions on each $K \in \mathcal{T}_h$ and which are continuous at the midpoints of each edge $e \in \mathcal{E}_h$.

In [15], Brenner, Cui, Li, and Sung developed a method for the augmented vector Laplacian based on the space $[CR_1(\mathcal{T}_h)]^2$ of vector fields on $\mathcal{T}_h$ with values in the nonconforming $CR_1$ space. Their method is given by the problem of finding

$$
(\nabla \cdot u_h, \nabla \cdot v_h)_{\mathcal{T}_h} + (\nabla \times u_h, \nabla \times v_h)_{\mathcal{T}_h} + \alpha (u_h, v_h)_{\mathcal{T}_h} + \gamma [u_h \cdot n, v_h \cdot n]_{\mathcal{E}_h} + \gamma [u_h \times n, v_h \times n]_{\mathcal{E}_h} = (f, v_h)_{\mathcal{T}_h}. \quad (3.0.2)
$$
Here $\gamma$ is a penalty parameter defined on $e \in \mathcal{E}_h$. The authors demonstrate that the non-penalized ($\gamma = 0$) primal variational form of the vector Laplacian fails to converge for certain source terms $f$ for which the corresponding exact solution has low regularity, and that the problem is that the vector valued $CR_1$ space does not impose enough continuity on the inter-element jumps. In addition, they give a priori estimates for the method to demonstrate optimal convergence on graded meshes.

We have developed a hybridizable method that extends the approach of Brenner et al to higher degree nonconforming spaces and admits solution using the technique of static condensation. In addition, the results are shown for general meshes.

### 3.1 Notation and definitions

Let $\mathcal{T}_h$ be a triangulation of the domain $\Omega \subset \mathbb{R}^2$, and $\mathcal{E}_h$ be the set of edges associated to the triangulation. We will consider separately the interior edges $\mathcal{E}^0_h = \{ e \in \mathcal{E}_h : e = K^+ \cap K^- \text{ for some } K^\pm \in \mathcal{T}_h \}$ and the boundary edges $\mathcal{E}^\partial_h = \{ e \in \mathcal{E}_h : e = K \cap \partial \Omega \text{ for some } K \in \mathcal{T}_h \}$. We will assume that $\partial \Omega$ is piecewise smooth, and that $\mathcal{T}_h$ satisfies the shape regularity condition:

**Definition 3.1.1 (Shape regularity).** The family of triangulations $\{ \mathcal{T}_h \}$ is shape regular if there exists $c > 0$ independent of $h$ such that

$$\max_{K \in \mathcal{T}_h} \frac{h_K^2}{|K|} \leq c \quad \forall \mathcal{T}_h,$$

where $h_K$ denotes the diameter of the triangle $K$, and $|K|$ its area in $\mathbb{R}^2$.

For any edge $e \in \mathcal{E}_h$, denote $e^\pm = \partial K^\pm \cap e$. The outer unit normal vector associated to $e^\pm$ with respect to $K^\pm$ is denoted $n^\pm$. Without loss of generality, we assume that the globally chosen normal vector is such that $n^+$ coincides with the outer unit normal of $\Omega$ if
For any piecewise continuous function $v_h$ defined on $\mathcal{T}_h$, we denote its restriction to the interface $e$ between two elements $K^+$ and $K^-$ as $v_h^\pm(x) := \lim_{\epsilon \to 0^+} v_h|_{K^\pm}(x - \epsilon n_\pm)$. Then the jump and average respectively of a vector- or scalar-valued function $v_h$ is defined as follows:

$$\{v_h\}^e := \begin{cases}
\frac{1}{2} (v_h^+ + v_h^-) & \text{if } e \in \mathcal{E}^0_h \\
v_h & \text{if } e \in \mathcal{E}^\partial_h
\end{cases}$$

$$\llbracket v_h \rrbracket^e := \begin{cases}
v_h^+ - v_h^- & \text{if } e \in \mathcal{E}^0_h \\
0 & \text{if } e \in \mathcal{E}^\partial_h.
\end{cases}$$

It follows immediately from the definition that a single-valued function $u$ has $\llbracket u \rrbracket^e = 0$ for all $e \in \mathcal{E}^0_h$. The average and jump provide a useful decomposition for a function. Let $u$ be defined on the edges $\mathcal{E}_h$. Then for any $e \in \mathcal{E}^0_h$,

$$u|_e = \{u\}^e \pm \frac{1}{2} \llbracket u \rrbracket^e.$$  \hfill (3.1.1)

The following results are a direct result of this decomposition.

**Lemma 3.1.2.** If $u$ is single-valued on edges $\mathcal{E}_h$, then for all $q$,

$$\langle u, q \rangle_{\partial \mathcal{T}_h} = 2 \langle \{u\}, \{q\} \rangle_{\mathcal{E}_h}.$$

**Lemma 3.1.3.** If the moments up to degree $k - 1$ of $u$ are continuous on each edge $e \in \mathcal{E}_h$, then for any $q \in \prod_{e \in \mathcal{E}_h} [P_{k-1}(e)]^2$,

$$\langle u, q \rangle_{\partial \mathcal{T}_h} = 2 \langle u, q \rangle_{\mathcal{E}_h}.$$  

For simplicity of notation, we will use $H^s(\Omega)$ to denote the vector-valued space $[H^s(\Omega)]^2$.

### 3.2 Nonconforming scalar cases

The $k = 0$ and $k = 2$ problems correspond to the Dirichlet and Neumann versions of the scalar Laplace equation respectively.
Let $V_h$ denote the space of $CR_1$ functions on $T_h$. Then the $P_1$-nonconforming problem corresponding to the scalar Laplacian is given by solving the Galerkin approximation for the problem of finding $u \in \hat{H}^1(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall v \in \hat{H}^1(\Omega)$$

where the bilinear form is given by the sum of $L^2$ inner products:

$$a(u, v) = \sum_{K \in T_h} \int_K (\nabla u) \cdot (\nabla v) dx$$

It is a standard result [20, Theorem 10.3.11] that the $P_1$-nonconforming method for the scalar Laplacian is convergent in the energy norm $\|u\|_h^2 := a(u, u)$, and, for $u_h$ the approximate solution, we have

$$\|u - u_h\|_h \lesssim h|u|_{H^2(\Omega)}.$$ 

### 3.3 Nonconforming vector case

As mentioned in Section 1.2.3, the problem corresponding to the $k = 1$ case poses difficulties for general approximation. It is extremely difficult to develop conforming finite-dimensional approximation spaces for $H(\text{div}; \Omega) \cap H(\text{curl}; \Omega)$ that are not strictly $H^1$-conforming for a general domain $\Omega$. A range of nonconforming methods is therefore introduced, where the space $V_h$ is taken to be a subspace of the space of broken vector-valued polynomials:

$$\prod_{K \in T_h} [P_r(K)]^2$$

where a weaker continuity condition is imposed at the interfaces between elements than strict equality at the degrees of freedom. For the $k = 1$ case, the tangential and normal traces of
a 1-form \( w = adx + bdy \) along an edge \( e \) correspond to the dot and cross products of the vector proxy \( u = (a, b)^t \) with the outward unit normal \( n_e \) respectively:

\[
 w^{tan} \sim n \times (u \times n), \quad w^{nor} \sim u \cdot n
\]

The hybridized conforming method is derived similarly to the hybridization defined in [6, Section 8.1]. Since this is the case \( k = 1 \) in two dimensions, and the space \( \Omega \) is assumed to be polygonal and the triangulation \( \mathcal{T}_h \) is assumed to satisfy shape regularity, the space of harmonic forms is trivial, and so the system obtained from (2.7.2) with the two additional equations (2.7.3) reduces to the following problem: find \((\sigma, \hat{\sigma}^{tan}, \hat{u}^{nor}, u, \hat{\sigma}^{tan}, \rho, \hat{\rho}^{nor}) \in W^0 \times \hat{V}^{0,tan} \times \hat{W}^{0,nor} \times W^1 \times \hat{V}^{1,tan} \times W^2 \times \hat{W}^{1,nor}\) satisfying

\[
\begin{align*}
(\sigma, \tau)_{\mathcal{T}_h} - (u, \nabla \tau)_{\mathcal{T}_h} + \langle \hat{\sigma}^{nor}, \tau^{tan} \rangle_{\partial \mathcal{T}_h} &= 0 & \forall \tau \in W^0 \quad (3.3.1a) \\
(\sigma, \nabla \cdot v)_{\mathcal{T}_h} + (\rho, \nabla \times v)_{\mathcal{T}_h} + \alpha(u, v)_{\mathcal{T}_h} + \langle \hat{\sigma}^{tan}, v^{nor} \rangle_{\partial \mathcal{T}_h} - \langle \hat{\rho}^{nor}, v^{tan} \rangle_{\partial \mathcal{T}_h} &= (f, v)_{\mathcal{T}_h} & \forall v \in W^1 \quad (3.3.1b) \\
(\rho, \eta)_{\mathcal{T}_h} - (u, \nabla \times \eta)_{\mathcal{T}_h} - \langle \hat{u}^{tan}, \eta^{nor} \rangle_{\partial \mathcal{T}_h} &= 0 & \forall \eta \in W^2 \quad (3.3.1c) \\
\langle \hat{u}^{nor}, \tau^{tan} \rangle_{\partial \mathcal{T}_h} &= 0 & \forall \tau^{tan} \in \hat{V}^{0,tan} \quad (3.3.1d) \\
\langle \hat{\rho}^{nor}, v^{tan} \rangle_{\partial \mathcal{T}_h} &= 0 & \forall v^{tan} \in \hat{V}^{1,tan} \quad (3.3.1e) \\
\langle u^{nor} - \hat{u}^{nor}, v^{nor} \rangle_{\partial \mathcal{T}_h} &= 0 & \forall \tilde{v}^{nor} \in \hat{W}^{0,nor} \quad (3.3.1f) \\
\langle u^{tan} - \hat{u}^{tan}, \eta^{nor} \rangle_{\partial \mathcal{T}_h} &= 0 & \forall \tilde{\eta}^{nor} \in \hat{W}^{1,nor} \quad (3.3.1g)
\end{align*}
\]

Given the penalty function \( \gamma \), we will define the penalized terms:

\[
\hat{u}^{nor} := u^{nor} + \gamma^{-1}(\hat{\sigma}^{tan} - \hat{\sigma}^{tan}), \quad \hat{\rho}^{nor} := \rho^{nor} - \gamma(u^{tan} - \hat{u}^{tan}).
\]

The unknowns \( \sigma, \rho \) can be eliminated by using integration by parts on the second terms of (3.3.1a) and (3.3.1c) and substituting these into (3.3.1b) with the equalities (3.3.1d), (3.3.1e)
respectively. We thereby obtain a reduced problem: find
\[
(\hat{\sigma}^\text{tan}, \hat{\sigma}^\text{tan}, u, \hat{u}^\text{tan}, \hat{\rho}^\text{nor}) \in \hat{W}_0^\text{tan} \times \hat{V}_0^\text{nor} \times W^1 \times \hat{V}_1^\text{tan} \times \hat{W}_1^\text{nor}
\]
satisfying:
\[
(\nabla \cdot u, \nabla \cdot v)_{\mathcal{T}_h} + (\nabla \times u, \nabla \times v)_{\mathcal{T}_h} + \alpha(u, v)_{\mathcal{T}_h} + \langle \hat{\sigma}^\text{tan}, v^\text{nor} \rangle_{\partial \mathcal{T}_h} - \langle \hat{\rho}^\text{nor} - \gamma(u^\text{tan} - \hat{u}^\text{tan}), v^\text{tan} \rangle_{\partial \mathcal{T}_h} = (f, v)_{\mathcal{T}_h} \quad \forall v \in W^1
\]
(3.3.2a)
\[
\langle \hat{\sigma}^\text{tan} - \bar{\sigma}^\text{tan}, \bar{\tau}^\text{tan} \rangle_{\partial \mathcal{T}_h} = 0 \quad \forall \bar{\tau}^\text{tan} \in \hat{W}_0^\text{tan}
\]
(3.3.2b)
\[
\langle u^\text{nor} + \gamma^{-1}(\hat{\sigma}^\text{tan} - \bar{\sigma}^\text{tan}), \bar{\tau}^\text{tan} \rangle_{\partial \mathcal{T}_h} = 0 \quad \forall \bar{\tau}^\text{tan} \in \hat{V}_0^\text{tan}
\]
(3.3.2c)
\[
\langle u^\text{tan} - \hat{u}^\text{tan}, \bar{\eta}^\text{nor} \rangle_{\partial \mathcal{T}_h} = 0 \quad \forall \bar{\eta}^\text{nor} \in \hat{W}_1^\text{nor}
\]
(3.3.2d)
\[
\langle \hat{\rho}^\text{nor} - \gamma(u^\text{tan} - \hat{u}^\text{tan}), \hat{v}^\text{tan} \rangle_{\partial \mathcal{T}_h} = 0 \quad \forall \hat{v}^\text{tan} \in \hat{V}_1^\text{tan}
\]
(3.3.2e)

If we instead solve for the unknown \( \hat{u}^\text{nor} \) and define the term \( \hat{\sigma}^\text{tan} \) in terms of \( \hat{\sigma}^\text{tan}, \hat{u}^\text{nor}, \) and \( u \):
\[
\hat{\sigma}^\text{tan} := \hat{\sigma}^\text{tan} - \gamma(u^\text{nor} - \hat{u}^\text{nor})
\]

We obtain a system where the tangential and normal traces for \( u \) are treated symmetrically: find \((\hat{\sigma}^\text{tan}, u, \hat{u}^\text{nor}, \hat{u}^\text{tan}, \bar{\eta}^\text{nor}) \in \hat{W}^\text{tan} \times V \times \hat{V}^\text{nor} \times \hat{V}^\text{tan} \times \hat{W}^\text{nor} \) that satisfy the system of
The solution to (3.3.3) corresponds to the critical points of the functional

\[
J(\tilde{\sigma}^{\text{tan}}, u, \hat{u}^{\text{nor}}, \hat{u}^{\text{tan}}, \bar{\rho}^{\text{nor}}) = \frac{1}{2} \left( \|\nabla \cdot u \|^2_{\mathcal{T}_h} + \|\nabla \times u\|^2_{\mathcal{T}_h} + \alpha \|u\|^2_{\mathcal{T}_h} + \gamma \left( \|\tilde{\sigma}^{\text{tan}} - \hat{u}^{\text{nor}}\|^2_{\partial \mathcal{T}_h} + \|\hat{u}^{\text{tan}} - u^{\text{tan}}\|^2_{\partial \mathcal{T}_h} \right) \right) + \langle \tilde{\sigma}^{\text{tan}}, u^{\text{nor}} - \hat{u}^{\text{nor}} \rangle_{\partial \mathcal{T}_h} + \langle \bar{\rho}^{\text{nor}}, u^{\text{tan}} - \hat{u}^{\text{tan}} \rangle_{\partial \mathcal{T}_h} - (f, u)_{\mathcal{T}_h}
\]

Although this system is derived from the version (3.3.2), it is not the same in general, since for nonconforming methods the choice of spaces for \( \hat{\mathcal{V}}^{0,\text{nor}} \) and \( \hat{\mathcal{W}}^{0,\text{tan}} \) is not necessarily the same, and the continuity constraints imposed on \( \hat{\sigma}^{\text{tan}} \) in Eq. (3.3.2) and \( \hat{u}^{\text{nor}} \) in Eq. (3.3.3) are not necessary results of the alternative formulations. In the original hybridization of FEEC, the finite dimensional approximation spaces are chosen to be Lagrange finite elements of degree \( r \) and \( r - 1 \), since the derivative of a polynomial valued vector field of degree at most \( r \) is contained in the space of polynomials of degree at most \( r - 1 \). For the nonconforming method, we use degree \( r \) Lagrange finite elements to approximate the solution \( u \) and its traces, but degree \( k - 1 \) elements for the auxiliary variables \( \hat{\sigma}^{\text{tan}}_h \) and \( \hat{\rho}^{\text{nor}}_h \), where \( r = 2k - 1 \). In particular, the tangential and normal trace spaces for \( u_h \) are taken in \( \hat{\mathcal{V}}^{\text{tan}}_h \) and \( \hat{\mathcal{V}}^{\text{nor}}_h \).
respectively, which are defined to be single-valued spaces containing the corresponding traces of the vector-valued polynomial spaces \( \prod_{K \in T_h} [P_r(K)]^2 \). However, the spaces \( \tilde{W}_h^{\text{nor}} \), \( \tilde{W}_h^{\text{tan}} \) are taken to be the lower degree “broken” spaces

\[
\tilde{W}_h^{\text{nor}} := \prod_{K \in T_h} \prod_{e \in \partial K} P_{k-1}(e), \quad \tilde{W}_h^{\text{tan}} := \prod_{K \in T_h} \prod_{e \in \partial K} P_{k-1}(e)
\]

This problem can be simplified to a formulation involving only three unknowns, since the restriction of a vector-valued function \( u \) to an edge \( e \) can be expressed as an orthogonal decomposition into the terms \( u^{\text{tan}}, u^{\text{nor}} \). Let \( \tau := (n_y, -n_x) \) denote the unit tangent vector orthogonal to \( n \), and combine the unknowns \( \bar{\sigma}_h^{\text{tan}}, \bar{\eta}_h^{\text{nor}} \) into a single unknown \( p_h \) as

\[
p_h := -\bar{\sigma}_h^{\text{tan}} n + \bar{\eta}_h^{\text{nor}} \tau.
\]

Similarly, we will use a single variable \( \hat{u}_h \) for the normal and tangential trace spaces for \( u_h \).

Define the finite dimensional function spaces for integers \( r, k \) such that \( r = 2k - 1 \):

- \( V_h = \prod_{K \in T_h} [P_r(K)]^2 \),

- \( \hat{V}_h = \left\{ \hat{v}_h \in \prod_{e \in E_h} [P_r(e)]^2 : \hat{v}_h \times n = 0 \text{ on } E^\partial_h \right\}, \quad (3.3.4) \)

- \( Q_h = \prod_{K \in T_h} \prod_{e \in \partial K} [P_{k-1}(e)]^2 \).
Then the finite dimensional system corresponding to (3.3.3) is equivalent to the problem of finding $u_h, \hat{u}_h, p_h \in V_h \times \hat{V}_h \times Q_h$ such that

\[
\begin{align*}
(\nabla \cdot u_h, \nabla \cdot v_h)_T + (\nabla \times u_h, \nabla \times v_h)_T &+ \alpha (u_h, v_h)_T + \langle p_h + \gamma (u_h - \hat{u}_h), v_h \rangle_{\partial T_h} = (f, v_h)_T \quad \forall v_h \in V_h \quad (3.3.5a) \\
\langle u_h - \hat{u}_h, q_h \rangle_{\partial T_h} &= 0 \quad \forall q_h \in Q_h \quad (3.3.5b) \\
\langle p_h + \gamma (u_h - \hat{u}_h), \hat{v}_h \rangle_{\partial T_h} &= 0 \quad \forall \hat{v}_h \in \hat{V}_h. \quad (3.3.5c)
\end{align*}
\]

This will be shown to be equivalent up to constant multiples to the formulation (3.0.2) in the lowest order case.

### 3.3.1 Regularity and weighted Sobolev spaces

For the domain $\Omega$ possibly with corners $\{c_\ell\}_{\ell=1}^L$, we will use the notation

\[\omega_\ell = \text{the interior angle at the corner } c_\ell\]

and let $\mu_\ell = 1$ for $\omega_\ell \leq \frac{\pi}{2}$ and $\mu_\ell < \frac{\pi}{2\omega_\ell}$ whenever $\omega_\ell > \frac{\pi}{2}$. We define the penalty function $\gamma$ on each edge $e \in \mathcal{E}_h$ to be

\[\gamma = \frac{(\Phi_\mu(e))^2}{|e|}\]

Where $\mu = \{\mu_\ell\}$ and the function $\Phi_\mu$ is defined as follows:

\[\Phi_\mu(x) = \prod_\ell |x - c_\ell|^{1-\mu_\ell}\]

And for an edge $e$, define $\Phi_\mu(e)$ as the value of $\Phi_\mu(\cdot)$ at the midpoint of the edge $e$, and similarly, for any $K \in \mathcal{T}_h$, $\Phi_\mu(K)$ as the product of the distances from the centroid of $K$.  

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The term $|\bar{E} - c_\ell| \leq \frac{1}{2} h_K$ on an element $K$ adjoining $c_\ell$. Also, shape regularity of the triangulation $\mathcal{T}_h$ implies that for each $K$ and $e \in \partial K$:

$$\Phi_\mu(e) \sim \Phi_\mu(K) \tag{3.3.6}$$

For each corner $c_\ell$, we also define a $\delta$-neighborhood of $c_\ell$, denoted $N_{\ell,\delta}$, and assume that $h \ll \delta$. Finally, we define $\Omega_\delta := \Omega \setminus \cup_{\ell} N_{\ell,\delta}$.

Now we will present some results concerning the regularity of the solution $u$ to (3.0.1) both globally and in a neighborhood of a re-entrant corner or corner with interior angle $\omega_\ell \geq \frac{\pi}{2}$.

The elliptic regularity of the problem gives that if the source term $f \in L^2(\Omega)$, then $u \in H^2(\Omega_\delta)$. Moreover [15, Section 2], in each neighborhood $N_{\ell,\delta}$ the solution $u$ to (3.0.1) can be decomposed [16] into a regular and a harmonic singular component $u = u_R + u_S$ where the singular part can be expressed in terms of the basis functions

$$\left\{ \psi_{j,\ell} := r_\ell^{j\pi/\omega_\ell-1} \sin \left((j\pi/\omega_\ell - 1)\theta_\ell \right) : j \in \mathbb{N}, j\pi/\omega_\ell \in (0,2) \setminus \{1\} \right\}$$

with $(r_\ell, \theta_\ell)$ polar coordinates centered at the corner $c_\ell$, and the regular part satisfies $u_R \in H^{2-\epsilon}(N_{\ell,\delta})$. In particular, the solution $u$ satisfies

$$u \in H^{2\mu_{\min}}(\Omega) \tag{3.3.7}$$

where $2\mu_{\min} < \min_{\ell} \frac{\pi}{\omega_\ell}$.

**Example 3.3.1.** In the case of the L shaped domain $[-\frac{1}{2}, \frac{1}{2}]^2 \setminus [0, \frac{1}{2}]^2$, there is one re-entrant corner $c$ with $\omega = 3\pi/2$. Given the decomposition of $u = u_S + u_R$ in the corner $c$, We have that $u_R \in H^{2-\epsilon}(N_{\ell,\delta})$, and $u_S \in H^{\pi/\omega_\ell-\epsilon}(N_{\ell,\delta}) = H^{2/3-\epsilon}(N_{\ell,\delta})$
Since outside of the neighborhoods \( N_{\ell,\delta} \), the \((t, \mu)\)-seminorm scales with the unweighted, i.e. Sobolev, \( t \)-seminorm, we have for integer valued \( t \)

\[
|u|_{t,1-\mu}^2 = \sum_{K \in T_h \atop K \subset \Omega_{\delta}} |u|_{t,1-\mu,K}^2 + \sum_{K \in T_h \atop K \subset \cup_j N_{\ell,\delta}} |u|_{t,1-\mu,K}^2 \\
\approx \sum_{K \in T_h \atop K \subset \Omega_{\delta}} |u|_{t,K}^2 + \sum_{K \in T_h \atop K \subset \cup_j N_{\ell,\delta}} \left( \sum_{|\alpha|=t} \int_K \Phi_\mu^2(x) |\partial^\alpha u|^2 dx \right)
\]

and since \( \Phi_\mu \sim 1 \) in \( \Omega_{\delta} \), this can be written:

\[
|u|_{t,1-\mu}^2 \approx \sum_{K \in T_h} \left( \sum_{|\alpha|=t} \int_K \Phi_\mu^2(x) |\partial^\alpha u|^2 dx \right)
\]

The penalty term \( \gamma \) scales in each neighborhood of a corner with the distance from that corner. This penalty acts in polar coordinates as a radial distance raised to the power \( 2(1 - \mu) \). On \( K \subset N_{\ell,\delta} \) not adjacent to \( c_\ell \), mesh regularity implies that \( \Phi_\mu^2(x)|u|_{t,K}^2 \approx \Phi_\mu^2(K)|u|_{t,K}^2 \). However, since \( \Phi_\mu(K) \) is strictly positive, we must treat those elements where \( \Phi_\mu(x) \) goes to zero separately. Define the set

\[
T_\ell^c := \{ K \in T_h : K \subset N_{\ell,\delta} \text{ and } K \text{ is adjacent to } c_\ell \}
\]

Then for \( \mu \leq 1 \), we obtain the approximations on \( K \in T_h \setminus \cup_\ell T_\ell^c \):

\[
|\Phi_\mu(K)|u|_{t,K}^2 \leq \|u\|_{t,1-\mu,K}^2 \tag{3.3.8}
\]

\[
|\Phi_\mu^{-1}(K)|u|_{t,K}^2 \leq \|u\|_{t,\mu-1,K}^2 \tag{3.3.9}
\]
For $K \in \mathcal{T}_h$ adjacent to a corner $c_t$, $\Phi_{\mu}(x) \preceq \Phi_{\mu}(K)$, and so Equation (3.3.9) extends to these terms as

$$|\Phi_{\mu}^{-1}(K)u|_{t,K}^2 \preceq \|u\|_{t,\mu-1,K}^2.$$  \hspace{1cm} (3.3.10)

### 3.3.2 Well-posedness and static condensation

In this section we will demonstrate that the problem (3.3.5) is well-posed and can be solved using static condensation.

Of the finite dimensional function spaces $V_h$, $\hat{V}_h$, and $Q_h$ introduced in Eq. (3.3.4), both $V_h$ and $Q_h$ are defined on the elements $K \in \mathcal{T}_h$ but without inter-element constraints. The space $\hat{V}_h$ is defined on the skeleton of the mesh. Thus the term $\hat{u}$ in (3.3.5) is found using a global solve, while the remaining terms $u_h, p_h$ are found using a local solver on each $K$ that is posed in terms of the source term $f$ and the solution $\hat{u}_h$.

The local problem in (3.3.5) can be written as a saddle point problem on each $K \in \mathcal{T}_h$. Define bilinear forms

$$a_K : V_h \times V_h \to \mathbb{R}, \quad \text{and} \quad b_K : V_h \times Q_h \to \mathbb{R}$$

as

$$a_K(u_h, v_h) := (\nabla \cdot u_h, \nabla \cdot v_h)_K + (\nabla \times u_h, \nabla \times v_h)_K + \alpha(u_h, v_h)_K + \langle \gamma u_h, v_h \rangle_{\partial K}$$

$$b_K(v_h, p_h) := \langle v_h, p_h \rangle_{\partial K}.$$

Then, given $\hat{u}_h$ we can find $(u_h, p_h)$ by solving the local problem:

$$a_K(u_h, v_h) + b_K(v_h, p_h) = (f, v_h)_K + \langle \gamma \hat{u}_h, v_h \rangle_{\partial K} \quad \forall v_h \in V_h$$

$$b_K(u_h, q_h) = \langle \hat{u}_h, q_h \rangle \quad \forall q_h \in Q_h$$

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combined with a global problem defined on the skeleton of the mesh by rearranging (3.3.5c):

\[
\langle \hat{u}_h, \hat{v}_h \rangle_{E_h} = \frac{1}{2} \langle \{\{p_h\} + \gamma \{u_h\}, \hat{v}_h \rangle_{E_h}
\]

Now, \(a_K(\cdot, \cdot)\) is coercive for \(\alpha \geq 0\) and satisfies Gårding’s inequality 1.2.8 for \(\alpha < 0\). Furthermore, by definition of the space \(Q_h\), for any \(q_h \in Q_h\), there is \(v_h \in V_h\) such that the moments of \(v_h|_{\partial K}\) are equal to \(q_h\). Thus the map \(V_h \rightarrow Q_h^*\) given by

\[
v_h \mapsto \langle v_h, \cdot \rangle_{\partial K}
\]

is surjective. And so [10, Section 4] the local problem is well-posed on each \(K \in T_h\).

**Remark 3.3.2.** The solution \((u_h, p_h)\) to the local system can be written as \(u_h = U_f + U_{\tilde{u}_h}\)

\(p_h = P_f + P_{\tilde{u}_h}\) where \(U_f, P_f\) is the solution to

\[
a_K(U_f, v_h) + b_K(v_h, P_f) = (f, v_h) \quad \forall v_h \in V_h
\]

\[
b_K(U_f, q_h) = 0 \quad \forall q_h \in Q_h
\]

and \(U_{\tilde{u}_h}, P_{\tilde{u}_h}\) is the solution to

\[
a_K(U_{\tilde{u}_h}, v_h) + b_K(v_h, P_{\tilde{u}_h}) = \langle \gamma \hat{u}_h, v_h \rangle_{\partial K} \quad \forall v_h \in V_h
\]

\[
b_K(U_{\tilde{u}_h}, q_h) = \langle \hat{u}_h, q_h \rangle_{\partial K} \quad \forall q_h \in Q_h
\]

We will consider an equivalent formulation which eliminates the hybrid variable in \(\hat{V}_h\) to produce an interior penalty formulation. This equivalent version will be used to demonstrate error estimates and to establish the well-posedness of the global system. To this end, the
following “check space” (see e.g., [38]) is introduced:

\[
\tilde{Q}_h = \left\{ q_h \in \prod_{e \in \mathcal{E}_h} [P_{k-1}(e)]^2 : q_h \cdot n = 0 \text{ on } \partial \Omega \right\}.
\] (3.3.11)

Subsequently, \( a_{h,IP}(\cdot, \cdot) : V_h \times V_h \to \mathbb{R} \) and \( b_{h,IP}(\cdot, \cdot) : V_h \times \tilde{Q}_h \to \mathbb{R} \) are defined as

\[
a_{h,IP}(u_h, v_h) := (\nabla \cdot u_h, \nabla \cdot v_h)_\mathcal{T}_h + (\nabla \times u_h, \nabla \times v_h)_\mathcal{T}_h + \alpha (u_h, v_h)_\mathcal{T}_h + \frac{1}{2} \langle \gamma [u_h], [v_h] \rangle_{\mathcal{E}_h^0} + \langle \gamma u_h \times n, v_h \times n \rangle_{\mathcal{E}_h^0},
\]

\[
b_{h,IP}(v_h, \tilde{p}_h) := \langle \tilde{p}_h, [v_h] \rangle_{\mathcal{E}_h^0} + \langle \tilde{p}_h \times n, v_h \times n \rangle_{\mathcal{E}_h^0}.
\] (3.3.12b)

And define the reduced system as follows:

Find \((u_h, \tilde{p}_h) \in V_h \times \tilde{Q}_h\) such that for any \((v_h, \tilde{q}_h) \in V_h \times \tilde{Q}_h\)

\[
a_{h,IP}(u_h, v_h) + b_{h,IP}(v_h, \tilde{p}_h) = (f, v_h)_\mathcal{T}_h, \quad (3.3.13a)
\]

\[
b_{h,IP}(u_h, \tilde{q}_h) = 0. \quad (3.3.13b)
\]

**Lemma 3.3.3.** Let \((u_h, \hat{u}_h, p_h)\) be the solution to (3.3.5). Then \((u_h, \tilde{p}_h)\) satisfies (3.3.13) where

\[
\tilde{p}_h|_e := \begin{cases} 
\frac{1}{2} [p_h]|_e, & e \in \mathcal{E}_h^0 \\
p_h|_e, & e \in \mathcal{E}_h^\partial 
\end{cases}
\]

Conversely, if \((u_h, \tilde{p}_h)\) is the solution to (3.3.13) then \((u_h, \hat{u}_h, p_h)\) satisfies (3.3.5) where \(\hat{u}_h\) and \(p_h\) are defined in terms of \(u_h, \tilde{p}_h\) as follows:

\[
\hat{u}_h|_e := \{u_h\}|_e, \quad p_h|_{e^\pm} := \pm \tilde{p}_h|_e, \quad \text{for } e \in \mathcal{E}_h^0
\]

\[
\hat{u}_h \cdot n|_e := u_h \cdot n|_e, \quad p_h|_e := \tilde{p}_h|_e, \quad \text{for } e \in \mathcal{E}_h^\partial
\]
Proof. We will first show that the solution to (3.3.5) satisfies (3.3.13) by deriving the reduced system from the original. First, we will show that \( \hat{u}_h = \{u_h\} \) on any interior edges. To this end, consider using a test function \( \{q_h\} \in \bar{Q}_h \subset Q_h \) in (3.3.5b). This yields:

\[
\langle u_h - \hat{u}_h, \{q_h\} \rangle_{\partial T_h} = 2\langle \{u_h\} - \hat{u}_h, \{q_h\} \rangle_{E_h} = 0.
\]  

(3.3.14)

Now let \( \hat{v}_h = \{u_h\} - \hat{u}_h \) on \( e \in \mathcal{E}_h^0 \) and zero on boundary edges, (3.3.5c) becomes on \( \mathcal{E}_h^0 \)

\[
\langle p_h + \gamma(u_h - \hat{u}_h), \{u_h\} - \hat{u}_h \rangle_{\partial T_h} = 2\langle \{p_h\} + \gamma(\{u_h\} - \hat{u}_h), \{u_h\} - \hat{u}_h \rangle_{\mathcal{E}_h^0} = 2\langle \gamma(\{u_h\} - \hat{u}_h), \{u_h\} - \hat{u}_h \rangle_{\mathcal{E}_h^0} = 0,
\]  

(3.3.15)

where the last equality is obtained by the orthogonality in (3.3.14). Thus, \( \hat{u}_h = \{u_h\} \) since \( \gamma > 0 \).

Next, for the sum on any \( e \in \mathcal{E}_h^0 \), by a similar argument as the one in (3.3.14), consider the test function being \( Q_h \ni q_h = (p_h \cdot n_e)n_e \) on \( e \) and zero elsewhere, we have

\[
\langle u_h - \hat{u}_h, q_h \rangle_{\partial T_h} = \langle ((u_h - \hat{u}_h) \cdot n)n, (p_h \cdot n)n \rangle_e = 0.
\]

Consequently in (3.3.5c), letting \( \hat{v}_h = ((u_h - \hat{u}_h) \cdot n_e)n_e \) on this given \( e \) and zero elsewhere yields \( \hat{u}_h \cdot n = \{u_h \cdot n\} = u_h \cdot n \) on this \( e \in \mathcal{E}_h^0 \).

Now by \( r > k - 1 \) in (3.3.4) and the result above, let \( \hat{v}_h = \{p_h\} \) on any \( e \in \mathcal{E}_h^0 \) and be zero elsewhere in (3.3.5c), we obtain \( \{p_h\} = 0 \) on \( e \in \mathcal{E}_h^0 \). An analogous argument for any \( e \in \mathcal{E}_h^0 \) with \( \hat{v}_h = (p_h \cdot n_e)n_e \) on \( e \) and zero elsewhere yields \( p_h \cdot n = \{p_h \cdot n\} = 0 \) on \( \mathcal{E}_h^0 \).

To derive the IP-hybrid formulation, it suffices to show that the boundary term in (3.3.5a) can be written as a hybridization of the jump term in (3.0.2). Without loss of generality
consider an edge of interest \( e = \partial K^+ \cap \partial K^- \in \mathcal{E}_h^0 \). By the decomposition (3.1.1), we have

\[
p_h^+ = \{p_h\} + \frac{1}{2}[p_h] = \frac{1}{2}[p_h], \quad \text{and} \quad p_h^- = \{p_h\} - \frac{1}{2}[p_h] = -\frac{1}{2}[p_h],
\]

and since \( \hat{u}_h = \{u_h\} \),

\[
\langle p_h^+ + \gamma(u_h^+ - \hat{u}_h), v_h^+ \rangle_e + \langle p_h^- + \gamma(u_h^- - \hat{u}_h), v_h^- \rangle_e = \frac{1}{2}\langle [p_h] + \gamma[u_h], v_h \rangle_e + \frac{1}{2}\langle [p_h] + \gamma[u_h], v_h \rangle_e = \frac{1}{2}\langle [p_h] + \gamma[u_h], v_h \rangle_e.
\]

If \( e \in \mathcal{E}_h^0 \), by \( \hat{u}_h \times n = 0 \) on \( e \), as well as \( p_h \cdot n = 0 \) and \( (u_h - \hat{u}_h) \cdot n = 0 \) from previous arguments, we have

\[
\langle p_h + \gamma(u_h - \hat{u}_h), v_h \rangle_e
\]

\[
= \langle p_h \cdot n + \gamma(u_h - \hat{u}_h) \cdot n, v_h \cdot n \rangle_e + \langle p_h \times n + \gamma(u_h - \hat{u}_h) \times n, v_h \times n \rangle_e
\]

\[
= \langle p_h \times n + \gamma u_h \times n, v_h \times n \rangle_e
\]

Let \( \tilde{p}_h = \frac{1}{2}[p_h] \) on \( e \in \mathcal{E}_h^0 \) and \( p_h \) on \( e \in \mathcal{E}_h^0 \). Then (3.3.5a) becomes (3.3.13a). For (3.3.13b), it is straightforward to check by exploiting \( q_h = \{q_h\} \pm \frac{1}{2}[q_h] \) on \( e \cap K^\pm \).

And now let \( (u_h, \tilde{p}_h) \) be the solution to (3.3.13). We will show that the system (3.3.5) is satisfied by the triple \( (u_h, \hat{u}_h, p_h) \), where \( \hat{u}_h \) is defined as the average of \( u_h \) on inter-element boundaries, and the normal components \( (\hat{u}_h \cdot n) \) are equal to \( (u_h \cdot n) \) on \( \mathcal{E}_h^0 \), and \( p_h \in Q_h \) is defined on each \( e = K^+ \cap K^- \in \mathcal{E}_h^0 \) by:

\[
p_h|_{e^\pm} = \pm \tilde{p}_h|_e
\]

and \( p_h|_e = \tilde{p}_h|_e \) for \( e \in \mathcal{E}_h^0 \). Note that this implies that \( p_h \) is the signed restriction of \( \tilde{p}_h \) to \( \partial K^\pm \) for an edge \( e = K^+ \cap K^- \).
By (3.3.13b), the moments up to degree \( k - 1 \) of \( u_h \) are continuous on each edge \( e \in \mathcal{E}_h^0 \), and hence we have that
\[
\langle u_h - \hat{u}_h, q_h \rangle_{\partial K} = 0
\]
for all \( q_h \in Q_h \) and all \( K \in \mathcal{T}_h \). Thus \( (u_h, \hat{u}_h, p_h) \) as defined satisfy the system of equations (3.3.5). \( \square \)

We will define the space
\[
\mathcal{N}_h := \{ v_h \in V_h : b_{h,IP}(v_h, q_h) = 0 \ \forall q_h \in Q_h \}
\]
And the energy norm on \( V + V_h \) as
\[
\| u \|^2_h := (\nabla \cdot u, \nabla \cdot u)_{\mathcal{T}_h} + (\nabla \times u, \nabla \times u)_{\mathcal{T}_h} + (u, u)_{\mathcal{T}_h} + \frac{1}{2}(\gamma[u], [u])_{\mathcal{E}_h} \tag{3.3.16}
\]
where \( [u]_e \) is defined as \( n \times (u \times n) \) for \( e \in \mathcal{E}_h^0 \). The condition \( r = 2k - 1 \) and the definition of the spaces \( V_h \) and \( \hat{Q}_h \) imply that by setting \( u_h \in \mathcal{N}_h \) in Eq. (3.3.13b), continuity conditions are imposed on the moments of \( u_h \) only up to degree \( k - 1 \). The solution \( (u_h, p_h) \) satisfies a much weaker continuity condition than with conforming methods.

For a domain \( \Omega \subset \mathbb{R}^2 \), recall the Brenner-Sung-Mirebeau (BSM) space defined for \( k = 1 \) in [21] and extended to higher degrees in [42]. The BSM space of degree \( k \) is defined in terms of the spaces \( (\mathcal{P}_k(K))^2 \) of polynomial-valued vector fields of degree at most \( k \), and \( \mathcal{H}_k(K) \) of homogeneous harmonic polynomials of degree \( k \) as:
\[
(\mathcal{P}_k(K))^2 \oplus [\nabla \mathcal{H}_{k+2}(K) \oplus \cdots \oplus \nabla \mathcal{H}_{2k}(K)]
\]
and the corresponding projection \( \Pi \) is defined in terms of the linear functionals \( \mathcal{N}_k^1 \), where elements of \( \mathcal{N}_k^1 \) determine the \( d \) components of the vector field for \( k - 2 \) interior degrees of
freedom, and elements of $\mathcal{N}_k^2$ the $k-1$ moments on the edges. The projection $\Pi u \in BSM_k(K)$ is defined on each $K \in \mathcal{T}_h$ as

$$
\langle \Pi u, q \rangle_e = \langle u, q \rangle_e \quad \forall e \in \partial K, \forall q \in [P_{k-1}(e)]^2 \quad (3.3.17a)
$$

$$
(\Pi u, w)_K = (u, w)_K \quad \forall w \in [P_{k-2}(K)]^2 \quad (3.3.17b)
$$

This operator is particularly useful as it satisfies the commuting properties, which were proven in [21, Lemma 2]. The proof extends naturally through the definition of the higher-degree space using harmonic polynomials.

$$
\nabla \cdot (\Pi u) = Q_{k-1}(\nabla \cdot u) \quad (3.3.18a)
$$

$$
\nabla \times (\Pi u) = Q_{k-1}(\nabla \times u). \quad (3.3.18b)
$$

Here $Q_{k-1}$ is the $L^2$-orthogonal projection onto the polynomial-valued space:

$$
\prod_{K \in \mathcal{T}_h} P_{k-1}(K)
$$

And further, since $\Pi u$ is continuous at the $k-1$ moments on each edge $e \in \mathcal{E}_h$, and is at most degree $2k-1$ on each $K \in \mathcal{T}_h$, this projection satisfies

$$
\Pi u \in \mathcal{N}_h \quad \forall u \in V + V_h
$$

**Lemma 3.3.4.** The problem (3.3.13) is well posed.

**Proof.** Recall the system of equations (3.3.13) is presented in the form of a saddle point problem with bilinear forms $a_{h,IP}$ and $b_{h,IP}$ given in (3.3.12a) and (3.3.12b) respectively. In order to show that it is well posed, it suffices to demonstrate [10, Theorem 4.2.1] that $a_{h,IP}$
is coercive on the kernel $\mathcal{N}_h$ of the map $V_h \to \tilde{Q}_h^*$ defined by $v_h \mapsto b_{h,IP}(v_h, \cdot)$, and that this map is surjective.

For $\alpha > 0$, the bilinear form $a_{h,IP}$ is coercive on $V_h$ with respect to the energy norm defined in (3.3.16), and if $\alpha = 0$, then since by assumption $\Omega$ is polygonal, the Poincaré inequality applies.

Now, for any $\tilde{p}_h \in \tilde{Q}_h$, define $v_h \in \mathcal{N}_h$ as follows: on each $e \in \mathcal{E}_h^0$, define $v_h$ at the $k - 1$ degrees of freedom on each edge by setting
\[
\langle v_h, q_h \rangle_{e^\pm} = \frac{1}{2} \langle \tilde{p}_h, q_h \rangle_{e^\pm} \quad \forall q_h \in [P_{k-1}(e)]^2.
\]
and setting $v_h|_e = \tilde{p}_h|_e$ on $e \in \mathcal{E}_h^0$. We use the BSM interpolant to extend the definition of $v_h$ to the interior of each $K \in \mathcal{T}_h$. Then, $\{v_h\} = \tilde{p}_h$ on the $k - 1$ moments of the skeleton of the mesh $\mathcal{E}_h^0$, and $v_h \in \mathcal{N}_h \subset V_h$. Thus for all $\tilde{p}_h \in \tilde{Q}_h$ there exists $v_h \in V_h$ satisfying:
\[
\langle v_h, \tilde{q}_h \rangle_{\partial \mathcal{T}_h} = \langle \tilde{q}_h, \tilde{p}_h \rangle_{\mathcal{E}_h}.
\]
Thus the map is surjective, and so the system of equations is well-posed.

\[\square\]

### 3.4 A priori estimates

Recall [32] if $u$ satisfies (3.0.1) on a bounded polygonal domain $\Omega$, then $u$ satisfies the regularity condition on each $K \in \mathcal{T}_h$ for $\mu \leq 1$:
\[
u|_K \in H^{1+\mu_{\min}-\epsilon}_h(K) \subset H^{2\mu_{\min}-\epsilon}_h(K).
\]
3.4.1 Energy estimate

Recall the BSM interpolant $\Pi$ from (3.3.17) is defined in terms of the polynomial spaces of degree at most $k-1$ and $k-2$ on the facets and interior of mesh elements $K \in \mathcal{T}_h$ respectively.

**Remark 3.4.1.** For a function $u \in H^t(K)$ with $t \geq 1$, we have the following interpolation estimate from the Bramble Hilbert Lemma:

$$\| u - \Pi u \|_{L^2(K)} + h_K |u - \Pi u|_{1,K} \lesssim h^t |u|_{t,K} \ \forall \ t \leq k + 1$$

and for $u$ with low regularity such that $u \in H^t(K)$, $t < 1$, we have:

$$\| u - \Pi u \|_{L^2(K)} + h^t_K |u - \Pi u|_{t,K} \lesssim h^t_K |u|_{t,K}$$

Thus in general, we have for all $t \leq k + 1$:

$$\| u - \Pi u \|_{L^2(K)} + h^{\min(t,1)}_K |u - \Pi u|_{\min(t,1),K} \lesssim h^t_K |u|_{t,K}$$

and combining this with the trace theorem with scaling [20, Section 10.3], we have for all $e \in \partial K$:

$$|e| \| u - \Pi u \|_{L^2(e)}^2 \lesssim \| u - \Pi u \|_{L^2(K)}^2 + h^{2\min(t,1)}_K |u - \Pi u|_{\min(t,1),K}^2 \lesssim h^{2t}_K |u|_{1,K}^2$$

**Lemma 3.4.2.** Let $u \in H^{s+1}_1(\Omega) \cap V$ be the solution to (3.0.1). Then for all $s \leq k$,

$$\langle \gamma [u - \Pi u], [u - \Pi u] \rangle_{E_h} \lesssim h^{2s} \| u \|_{s+1,1-\mu, \mathcal{T}_h}^2$$
Proof. For any $e \in \mathcal{E}_h^0$, we have that $\|u \times n\|_{L^2(e)}^2 \leq \|u\|_{L^2(e)}^2$, and so for the remainder of this section, we will use $[u]$, as defined on $\mathcal{E}_h^0$ and $\mathcal{E}_h^0$ without discriminating the choice of $e$.

Then for any $e \in \mathcal{E}_h$, let $\mathcal{T}_e$ denote the set of elements adjoining $e$. Then

$$\langle \gamma [u - \Pi u], [u - \Pi u] \rangle_e \lesssim \sum_{K \in \mathcal{T}_e} \langle \gamma (u - \Pi u), u - \Pi u \rangle_{\partial K \cap e}$$  \hspace{1cm} (3.4.1)

Recall Remark 3.4.1 gives us for $u \in \mathbf{H}^t(K)$:

$$|e|\|u - \Pi u\|_{L^2(e)}^2 \lesssim \|u - \Pi u\|_{L^2(K)}^2 + h_K^{2\min(t,1)}|u - \Pi u|_{\min(t,1),K}^2 \lesssim h_K^2 |u|_{t,K}^2$$  \hspace{1cm} (3.4.2)

Combining (3.3.8), (3.4.1) and (3.4.2), together with the definition $\gamma = \frac{\Phi_{\mu(e)}^2}{|e|}$ and (3.3.6) we obtain:

$$\sum_{e \in \mathcal{E}_h} \langle \gamma [u - \Pi u], [u - \Pi u] \rangle_e \lesssim \sum_{K \in \mathcal{T}_e \setminus \mathcal{T}_0} \frac{1}{h_K^{2(s+1)-1}}|\Phi_{\mu}(K)u|_{s+1,K}^2$$  \hspace{1cm} (3.4.3)

under the assumption of shape regularity, which gives that that $|e| \sim h_K$ for all edges $e \in \partial K$.

In the case of a convex domain with interior angles satisfying $0 \leq \omega_\ell \leq \frac{\pi}{2}$, $\frac{\pi}{2\omega} \geq 1$ and so $\mu = 1$. Hence $\gamma \approx \frac{1}{h}$ and so Eq. (3.4.3) becomes:

$$\sum_{e \in \mathcal{E}_h} \langle \gamma [u - \Pi u], [u - \Pi u] \rangle_e \lesssim h^{2s}|u|_{s+1}^2 \ \forall s \leq k$$

For a domain with corners that have interior angles $\omega_\ell > \frac{\pi}{2}$, recall that outside of the neighborhoods $N_{\ell,\delta}$ the weighted seminorm is equivalent to the unweighted seminorm. Thus on $K \in \mathcal{T}_h$ with $K \subset \Omega_{\delta}$, $u \in \mathbf{H}^{s+1}(K)$. On the other hand for $K \in N_{\ell,\delta}$, if $K \notin \mathcal{T}_h^\ell$, then $\Phi_{\mu}(K) \sim \Phi_{\mu}(x)$, and so $\Phi_{\mu}(\cdot)u(\cdot)|_K \in \mathbf{H}^{s+1}(K)$. And finally we have that $u|_K \in \mathbf{H}^{s+1}_{1-k}(K)$, 55
and so on each $K \in \mathcal{T}_h^\ell$, $u \in H^{s+\mu}(K)$. Thus

$$
\sum_{e \in \mathcal{E}_h} \langle \gamma \{u - \Pi u\}, \{u - \Pi u\} \rangle_e \lesssim \sum_{K \in \mathcal{T}_h \setminus \bigcup N_{\ell,\delta}} h_K^{2s} |u|_{s+1,K}^2 \quad (3.4.4a)
$$

$$
+ \sum_{K \in \mathcal{T}_h \setminus \mathcal{T}_h^\ell} h_K^{2s} |\Phi_\mu(K) u|_{s+1,K}^2 \quad (3.4.4b)
$$

$$
+ \sum_{K \in \mathcal{T}_h^\ell} h_K^{2(s+\mu\ell)-2} |\Phi_\mu(K) u|_{s+\mu\ell,K}^2 \quad (3.4.4c)
$$

The first term can be estimated using the fact that on $K \subset \Omega_\delta$,

$$
|u|_{s+1,K}^2 \sim |u|_{s+1,1-\mu,K}^2
$$

For the second term, note that (3.3.8) applies, and we have

$$
h_K^{2s} |\Phi_\mu(K) u|_{s+1,K}^2 \lesssim h_K^{2s} ||u||_{s+1,1-\mu,K}^2
$$

and finally, for the last term note that for $K \in \mathcal{T}_h^\ell$, $\Phi_\mu(K) \lesssim h_K^{1-\mu\ell}$. Therefore we have

$$
h_K^{2(s+\mu\ell)-2} |\Phi_\mu(K) u|_{s+\mu\ell,K}^2 \lesssim h_K^{2(s+\mu\ell)-2} h_K^{2-2\mu\ell} ||u||_{s+\mu\ell,K}^2
$$

$$
\approx h_K^{2s} ||u||_{s+\mu\ell,K}^2
$$

$$
\lesssim h_K^{2s} ||u||_{s+1,1-\mu,K}^2
$$

\[\square\]

**Lemma 3.4.3.** Let $u \in H^{s+1}_{1-\mu}(\Omega)$ be the solution to (3.0.1), and suppose $\nabla \cdot u, \nabla \times u \in H^s_{\mu-1}(\Omega)$. Then for $s \leq k$

$$
\inf_{v_h \in \mathcal{V}_h} \|u - v_h\|_h^2 \leq \|u - \Pi u\|_h^2 \lesssim h^{2s}(\|u\|_{s+1,1-\mu,T_h}^2 + |\nabla \cdot u|_{s,T_h}^2 + |\nabla \times u|_{s,T_h}^2)
$$
Proof. Since $\Pi u \in \mathcal{N}_h$,

$$\inf_{v_h \in \mathcal{N}_h} \|u - v_h\|_h^2 \leq \|u - \Pi u\|_h^2 \quad (3.4.5a)$$

$$= \|\nabla \cdot (u - \Pi u)\|_{L^2(T_h)}^2 + \|\nabla \times (u - \Pi u)\|_{L^2(T_h)}^2$$

$$+ \|u - \Pi u\|_{L^2(T_h)}^2 + \frac{1}{2} \langle \gamma [u - \Pi u], [u - \Pi u] \rangle_{\mathcal{E}_h} \quad (3.4.5b)$$

Using the commuting property of the interpolant (3.3.18), the terms in Eq. (3.4.5b) become

$$\|u - \Pi u\|_h^2 = \|\nabla \cdot (u - \Pi u)\|_{L^2(T_h)}^2 + \|\nabla \times (u - \Pi u)\|_{L^2(T_h)}^2$$

$$+ \|u - \Pi u\|_{L^2(T_h)}^2 + \frac{1}{2} \langle \gamma [u - \Pi u], [u - \Pi u] \rangle_{\mathcal{E}_h}.$$ 

Then since $\mu \leq 1$, we obtain the inclusion [32, Section 3]

$$H^{s+1}_{1-\mu} \subset H^{s+1-\mu_{\min}}_0 = H^{s+\mu_{\min}}_0 \subset H^{s+\mu_{\min}}$$

So the first three terms can be estimated using standard polynomial interpolation theory, for all $K \in \mathcal{T}_h$

$$\|\nabla \cdot u - Q_{k-1} \nabla \cdot u\|_{L^2(T_h)}^2 \lesssim h^{2s} |\nabla \cdot u|_{s, T_h}^2 \quad \forall s \leq k$$

$$\|\nabla \times u - Q_{k-1} \nabla \times u\|_{L^2(T_h)}^2 \lesssim h^{2s} |\nabla \times u|_{s, T_h}^2 \quad \forall s \leq k$$

$$\|u - \Pi u\|_{L^2(T_h)}^2 \lesssim h^{2(s+\mu_{\min})} |u|_{s+\mu_{\min}, T_h}^2$$

$$\lesssim h^{2(s+\mu_{\min})} \|u\|_{s+\mu_{\min}, T_h}^2$$

$$\lesssim h^{2(s+\mu_{\min})} \|u\|_{s+1,1-\mu, T_h}^2$$

$$\lesssim h^{2s} \|u\|_{s+1,1-\mu, T_h}^2 \quad \forall s \leq k$$
The final term is estimated in Lemma 3.4.2 as

\[ \langle \gamma [u - \Pi u], [u - \Pi u] \rangle_{\mathcal{E}_h} \lesssim h^{2s} \| u \|_{s+1,1,\mu,T_h}^2 \]

Combining these results, we have the estimate. \( \square \)

**Remark 3.4.4.** Note that for \( u \) the solution to (3.0.1), [15, Equations 2.21, 2.22] imply that both \( \nabla \cdot u \) and \( \nabla \times u \) are in \( H^1(\Omega) \), and that

\[ |\nabla \cdot u|_{1,T_h} + |\nabla \times u|_{1,T_h} \lesssim \| f \|_{L^2(T_h)} \quad (3.4.6) \]

And since \( u \in H^1(\Omega_\delta) \), \( u \in H^{2\mu_{\min}}(N_{\ell,}\delta) \) for each \( \ell \), where \( \mu_{\min} \) is the minimum of the multi-index \( \mu = \{ \mu_\ell \}_{\ell=1}^L \) then [15, Equations 2.16,2.21,2.22],

\[ \| u \|_{H^{1+2\mu_{\min}}(T_h)} \lesssim \| f \|_{L^2(T_h)} \]

**Corollary 3.4.5.** Given the assumptions of Lemma 3.4.3, we have that \( u \in H^{1+\mu_{\min}} \), \( \nabla \cdot u, \nabla \times u \in H^1 \), and so

\[ \| u - \Pi u \|_h \lesssim h^{\mu_{\min}} \| f \|_{L^2(T_h)} \quad (3.4.7) \]

**Proof.** The proof for this result is obtained by adapting the arguments given for the estimates in Lemma 3.4.2 and Lemma 3.4.3 to the case of lowest regularity and approximation order. Recall (3.4.5):

\[ \| u - \Pi u \|_h^2 = \| \nabla \cdot (u - \Pi u) \|_{L^2(T_h)}^2 + \| \nabla \times (u - \Pi u) \|_{L^2(T_h)}^2 + \| u - \Pi u \|_{L^2(T_h)}^2 + \frac{1}{2} \langle \gamma [u - \Pi u], [u - \Pi u] \rangle_{\mathcal{E}_h} \quad (3.4.8) \]
And \( \nabla \cdot u, \nabla \times u \) are in \( H^1(\Omega) \), and \( \Pi \) satisfies the commuting property. Then, together with the fact that \( u \in H^{2\mu_{\min}}(\Omega) \), the first three terms of (3.4.8) satisfy:

\[
\| \nabla \cdot (u - \Pi u) \|_{L^2(T_h)}^2 \lesssim h^2 \| \nabla \cdot u \|_{L^2(T_h)}^2 \lesssim h^2 \| f \|_{L^2(T_h)}^2 \\
\| \nabla \times (u - \Pi u) \|_{L^2(T_h)}^2 \lesssim h^2 \| \nabla \times u \|_{L^2(T_h)}^2 \lesssim h^2 \| f \|_{L^2(T_h)}^2 \\
\| u - \Pi u \|_{L^2(T_h)}^2 \lesssim h^{2\mu} \| u \|_{L^2(T_h)}^2 \lesssim h^{2\mu_{\min}} \| f \|_{L^2(T_h)}^2
\]

And the result of Lemma 3.4.2 yields:

\[
\langle \gamma [u - \Pi u], [u - \Pi u] \rangle_{E_h} \lesssim h^{2s} \| u \|_{L^2(T_h)}^2
\]

Note that [16, Equation 4.15] gives that \( u|_K \in H^{2-2\mu}(K) \subseteq H^{1+\mu_{\min}}(K) \) for all \( K \in T_h \). Combining this with [32, Theorem 3.1], and the continuous inclusion (1.2.4) we obtain the result:

\[
h^{2\mu_{\min}} \| u \|_{L^2(T_h)}^2 \lesssim h^{2\mu_{\min}} \| u \|_{L^2(T_h)}^2 \lesssim h^{2\mu_{\min}} \| f \|_{L^2(T_h)}^2
\]

\[\square\]

**Lemma 3.4.6.** Let \( w \in H_s^{\mu-1}(\Omega) \). Suppose \( v_h \in \mathcal{N}_h \). Then for \( s \leq k \) we have:

\[
\langle w, \langle v_h \cdot n \rangle \rangle_{E_h} \lesssim h^s \| w \|_{s, \mu-1, T_h} \| v_h \|_h
\]

and

\[
\langle w, \langle v_h \times n \rangle \rangle_{E_h} \lesssim h^s \| w \|_{s, \mu-1, T_h} \| v_h \|_h.
\]

where the scalar jump is defined on \( e \in E_h^\partial \) by \( \[ \phi \]_e := \phi|_e \)
Proof. Since $v_h \in \mathcal{N}_h$, this implies that

$$\langle Q_{k-1} w, [v_h \cdot n] \rangle_{E_h^0} = 0.$$ 

Therefore

$$\langle w, [v_h \cdot n] \rangle_{E_h^0} = \langle w - Q_{k-1} w, [v_h \cdot n] \rangle_{E_h^0},$$

and so for $\gamma$, as in Lemma 3.4.2,

$$\langle w, [v_h \cdot n] \rangle_{E_h^0} = \langle w - Q_{k-1} w, [v_h \cdot n] \rangle_{E_h^0}$$

$$= \langle \gamma^{1/2}(w - Q_{k-1} w), \gamma^{1/2}[v_h \cdot n] \rangle_{E_h^0}$$

$$\lesssim \left( \| \gamma^{-1/2}(w - Q_{k-1} w) \|_{L^2(E_h^0)} \right)^{1/2} \left( \| \gamma^{1/2}[v_h \cdot n] \|_{L^2(E_h^0)} \right)^{1/2}$$

Cauchy Schwarz inequality

$$\lesssim \| \gamma^{-1/2}(w - Q_{k-1} w) \|_{L^2(E_h^0)} \| [v_h \cdot n] \|_h$$

by definition of energy norm

Then for each $e \in E_h^0$ with $e \in \partial K$ for some $K$, we have, by Remark 3.4.1 and the shape regularity assumption

$$\| w - Q_{k-1} w \|_e^2 \lesssim h^{-1}_K h^{2s}_K |w|_{s,K}^2$$

Recall that $\gamma^{-1/2}$ is constant on each edge. Also, since $\mu - 1 \leq 0$, the assumption $w \in H_{\mu-1}^s(\Omega)$ implies $w \in H^s(\Omega)$. Then, using (3.3.6) and the definition of $\gamma$

$$\| \gamma^{-1/2}(w - Q_{k-1} w) \|_{E_h^0}^2 \lesssim \sum_{K \in T_h} h^{-1}_K h^{2s}_K |\Phi^{-1}(K)w|_{s,K}^2$$

$$\lesssim h^{2s} \| w \|_{s,\mu-1,T_h}^2$$

The argument with $v_h \times n$ is precisely similar. \qed
Theorem 3.4.7. Let \((u, p)\) be the solution to the conforming saddle point problem on the domain \(\Omega \subset \mathbb{R}^2\). Let \((u_h, p_h)\) be the solution to (3.3.13). Then for \(u \in H^s_{\mu-1}(\Omega) \cap V\) satisfying \(\nabla \cdot u \in H^s_{\mu-1}(\Omega)\) and \(\nabla \times u \in H^s_{\mu-1}(\Omega)\), for \(s \leq k\)

\[
\|u - u_h\|_h^2 \lesssim h^{2s}(\|u\|_{s+1,1-\mu,T_h}^2 + \|\nabla \cdot u\|_{s,\mu-1,T_h}^2 + \|\nabla \times u\|_{s,\mu-1,T_h}^2)
\]

Proof. Let \(\pi^\perp\) denote the \(\|\cdot\|_h\)-orthogonal projection onto \(N_h\). Then we have:

\[
\|u - u_h\|_h = \|u - \pi^\perp u\|_h + \|\pi^\perp u - u_h\|_h \\
\leq \inf_{v_h \in N_h} \|u - v_h\|_h + \|\pi^\perp u - u_h\|_h \\
\leq \inf_{v_h \in N_h} \|u - v_h\|_h + C \sup_{w_h \in N_h} |a(u - u_h, w_h)| / \|w_h\|_h \\
= \inf_{v_h \in N_h} \|u - v_h\|_h + C \sup_{w_h \in N_h} |a(u - u_h, w_h)| / \|w_h\|_h
\] (3.4.9)

The last term can be estimated as follows: Since \([u] = 0\), the term \(a(u, w_h)\) simplifies to

\[
a(u, w_h) = (\nabla \cdot u, \nabla \cdot w_h)_{\mathcal{T}_h} + (\nabla \times u, \nabla \times w_h)_{\mathcal{T}_h} + \alpha(u, w_h)_{\mathcal{T}_h}.
\]

Using integration by parts on (3.0.1) with the test function \(w_h \in N_h\), we obtain

\[
(\nabla \cdot u, \nabla \cdot w_h)_{\mathcal{T}_h} - (\nabla \cdot u, w_h \cdot n)_{\partial \mathcal{T}_h} + (\nabla \times u, \nabla \times w_h)_{\mathcal{T}_h} + (\nabla \times u, n \times w_h)_{\partial \mathcal{T}_h} + \alpha(u, w_h)_{\mathcal{T}_h} = (f, w_h)_{\mathcal{T}_h}
\]

and subtracting \(a(u_h, w_h)\) and the fact that \(\nabla \cdot u = 0\) on \(e \in \mathcal{E}_h^0\), we have

\[
\frac{a(u - u_h, w_h)}{\|w_h\|_h} = \frac{\langle \nabla \cdot u, [w_h \cdot n]\rangle_{\mathcal{E}_h^0} + \langle \nabla \times u, [w_h \times n]\rangle_{\mathcal{E}_h}}{\|w_h\|_h} \\
= \frac{\langle \nabla \cdot u, [w_h \cdot n]\rangle_{\mathcal{E}_h^0}}{\|w_h\|_h} + \frac{\langle \nabla \times u, [w_h \times n]\rangle_{\mathcal{E}_h}}{\|w_h\|_h}
\]
Since $w_h \in \mathcal{N}_h$, and so by Lemma 3.4.6 applied to both terms, we have

$$a(u - u_h, w_h) \lesssim h^s(\|\nabla \cdot u|_{s,\mu-1,T_h} + |\nabla \times u|_{s,\mu-1,T_h}) \|w_h\|_h$$

And combining the above results together $H^s \subset H^s_{\mu-1}$, we have:

$$\sup_{w_h \in \mathcal{N}_h, w_h \neq 0} \frac{a(u - u_h, w_h)}{\|w_h\|_h} \lesssim h^s (\|\nabla \cdot u|_{s,\mu-1,T_h} + \|\nabla \times u|_{s,\mu-1,T_h})$$

Finally, the first term in Eq. (3.4.9) is estimated in Lemma 3.4.3. 

3.4.2 $L^2$ estimate

In this section, we will prove the result of the following theorem using a duality argument.

**Theorem 3.4.8.** Suppose that $u \in H^{s+1}_{1-\mu}(\Omega) \cap V$ is the solution to the augmented vector Laplacian, and suppose that $\nabla \cdot u \in H^s_{\mu-1}(\Omega)$, and $\nabla \times u \in H^s_{\mu-1}(\Omega)$. Let $u_h$ be the solution to the nonconforming approximation (3.3.13). Then for $s \leq k$

$$\|u - u_h\|_{L^2(T_h)} \lesssim h^{s+\mu}\text{min}(\|u\|_{s+1,1-\mu,T_h} + \|\nabla \cdot u\|_{s,\mu-1,T_h} + \|\nabla \times u\|_{s,\mu-1,T_h}). \quad (3.4.10)$$

Let $z \in H(\text{div}; \Omega) \cap \dot{H}(\text{curl}; \Omega)$ be the solution to the problem

$$-\nabla \nabla \cdot z + \nabla \times \nabla \times z + \alpha z = u - u_h, \quad \nabla \cdot z = 0 \text{ and } z \times n = 0 \quad \text{on } \partial \Omega.$$ 

Then $z$ satisfies

$$(\nabla \cdot z, \nabla \cdot v)_{T_h} + (\nabla \times z, \nabla \times v)_{T_h} + \alpha (z, v)_{T_h} = (v, u - u_h)_{T_h} \quad \forall v \in V.$$ 

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From the estimates Lemma 3.4.3, Remark 3.4.4 and Corollary 3.4.5 applied to $z$, we obtain the estimate:

$$\|z - \Pi z\|_h \lesssim h^{\mu_{\text{min}}} \|u - u_h\|_{L^2(\partial T_h)} \quad (3.4.11)$$

Since $z$ is the solution to the problem (3.0.1), and hence satisfies the minimal regularity assumptions.

Then, by the definition of $z$, we have

$$\begin{align*}
(u - u_h, u - u_h)_{T_h} &= (u - u_h, -\nabla \nabla \cdot z + \nabla \times \nabla \times z + \alpha z)_{T_h} \\
&= (\nabla \cdot (u - u_h), \nabla \cdot z)_{T_h} - ((u - u_h) \cdot n, \nabla \cdot z)_{\partial T_h} \\
&\quad + (\nabla \times (u - u_h), \nabla \times z)_{T_h} - ((u - u_h) \times n, \nabla \times z)_{\partial T_h} + \alpha (u - u_h, z)_{T_h} \\
&= a(u - u_h, z) - ((u - u_h) \cdot n, \nabla \cdot z)_{\partial T_h} - ((u - u_h) \times n, \nabla \times z)_{\partial T_h} \\
&= a(u - u_h, z - \Pi z) + a(u - u_h, \Pi z) \\
&\quad - ((u - u_h) \cdot n, \nabla \cdot z)_{\partial T_h} - ((u - u_h) \times n, \nabla \times z)_{\partial T_h} \\
&= a(u - u_h, z - \Pi z) + a(u - u_h, \Pi z) - ((u - u_h) \cdot n, \nabla \cdot z)_{\partial T_h} - ((u - u_h) \times n, \nabla \times z)_{\partial T_h} \\
&\quad - ((u - u_h) \times n, \nabla \times z)_{\partial T_h} + \alpha (u - u_h, z)_{T_h} \\
&\quad - ((u - u_h) \cdot n, \nabla \cdot z)_{\partial T_h} - ((u - u_h) \times n, \nabla \times z)_{\partial T_h} + \alpha (u - u_h, z)_{T_h} \\
\end{align*}$$

We will estimate the four terms in the last line separately.

Given continuity of the bilinear form $a(\cdot, \cdot)$ with respect to the energy norm, and using the energy norm estimate Theorem 3.4.7, we have the following estimate for the first term:

$$a(u - u_h, z - \Pi z) \leq \|u - u_h\|_h \|z - \Pi z\|_h$$

$$\lesssim h^{\mu_{\text{min}}} \|u - u_h\|_h \|u - u_h\|_{L^2(\partial T_h)} \quad (3.4.13)$$
We will write the second term \( a(u - u_h, \Pi z) \) of (3.4.12) using integration by parts on (3.0.1) together with the fact that \( \Pi z \in \mathcal{N}_h \) which implies \( a(u_h, \Pi z) = (f, \Pi z) \):

\[
a(u - u_h, \Pi z) = a(u, \Pi z) - a(u_h, \Pi z)
\]

\[
= (f, \Pi z)_{\mathcal{T}_h} + \langle \nabla \cdot u, [\Pi z \cdot n] \rangle_{\mathcal{E}_h^0} + \langle \nabla \times u, [n \times \Pi z] \rangle_{\mathcal{E}_h} - (f, \Pi z)_{\mathcal{T}_h}
\]

\[
= \langle \nabla \cdot u, [\Pi z \cdot n] \rangle_{\mathcal{E}_h^0} + \langle \nabla \times u, [n \times \Pi z] \rangle_{\mathcal{E}_h}
\]

And so

\[
a(u - u_h, \Pi z) = \langle \nabla \cdot u, [\Pi z \cdot n] \rangle_{\mathcal{E}_h^0} + \langle \nabla \times u, [n \times \Pi z] \rangle_{\mathcal{E}_h} \tag{3.4.14a}
\]

\[
\lesssim h^s (\| \nabla \cdot u \|_{s, \mu - 1, \mathcal{T}_h} + \| \nabla \times u \|_{s, \mu - 1, \mathcal{T}_h} \| z - \Pi z \|_h) \tag{3.4.14b}
\]

\[
\lesssim h^{s + \mu_{\min}} (\| \nabla \cdot u \|_{s, \mu - 1, \mathcal{T}_h} + \| \nabla \times u \|_{s, \mu - 1, \mathcal{T}_h} \| u - u_h \|_{L^2(\mathcal{T}_h)} \tag{3.4.14c}
\]

Here we used Lemma 3.4.6 to obtain the third inequality and Corollary 3.4.5 for the final one.

The last two terms of (3.4.12) can be estimated using analogous arguments, and so we will only consider the first of these. Let \( \overline{(\phi)}_e \) denote the average over a triangle incident to the edge \( e \) of the quantity \( \phi \). Then since \( u, u_h \in \mathcal{N}_h \) we have:

\[
\langle [u - u_h] \cdot n, \nabla \cdot z \rangle_{\mathcal{E}_h} = \langle \gamma^{1/2} [u - u_h] \cdot n, \gamma^{-1/2} \nabla \cdot z \rangle_{\mathcal{E}_h} \tag{3.4.15a}
\]

\[
= \langle \gamma^{1/2} [u - u_h] \cdot n, \gamma^{-1/2} (\nabla \cdot z - \overline{(\nabla \cdot z)}_e) \rangle_{\mathcal{E}_h} \tag{3.4.15b}
\]

\[
\lesssim \left( \| \gamma^{1/2} [u - u_h] \cdot n \|_{L^2(\mathcal{E}_h)} \right)^{1/2} \left( \| \gamma^{-1/2} (\nabla \cdot z - \overline{(\nabla \cdot z)}_e) \|_{L^2(\mathcal{E}_h)} \right)^{1/2} \tag{3.4.15c}
\]

\[
\lesssim \| u - u_h \|_h \left( h^{1 - (2 - 2\mu_{\min})} \| \nabla \cdot z - \overline{(\nabla \cdot z)}_e \|_{L^2(\mathcal{E}_h)} \right)^{1/2} \tag{3.4.15d}
\]

\[
\lesssim \| u - u_h \|_h \left( h^{\mu} \| \nabla \cdot z \|_{1, \mathcal{T}_h} \right) \tag{3.4.15e}
\]
Since $\nabla \cdot z, \nabla \times z \in H^1(\mathcal{T}_h)$ together with the inequality

$$h|\nabla \cdot z - (\nabla \cdot z)_{eh}|_{L^2(\mathcal{T}_h)}^2 \lesssim h^2|\nabla \cdot z|_{1,\mathcal{T}_h}^2.$$  

So combining 3.4.13, 3.4.14 and 3.4.15, we obtain the result for $s \leq k$

$$\|u - u_h\|_{L^2(\mathcal{T}_h)}^2 \lesssim h^{s+\mu_{\min}} \|u - u_h\|_{L^2(\mathcal{T}_h)} + h^s \mu_{\min} \left( \|\nabla \cdot u\|_{s,\mu-1,\mathcal{T}_h} + \|\nabla \times u\|_{s,\mu-1,\mathcal{T}_h} \right) \|u - u_h\|_{L^2(\mathcal{T}_h)} + h^{\mu_{\min}} \|u - u_h\|_{h} \left( |\nabla \cdot z|_{1,\mathcal{T}_h} + |\nabla \times z|_{1,\mathcal{T}_h} \right) \quad (3.4.16a)$$

Then, applying Eq. (3.4.6) to $z$ as the solution to the problem with source term $u - u_h$, we have

$$|\nabla \cdot z|_{1,\mathcal{T}_h} + |\nabla \times z|_{1,\mathcal{T}_h} \leq \|u - u_h\|_{L^2}$$

And combining this result with Eq. (3.4.16) and Theorem 3.4.7 we have

$$\|u - u_h\|_{L^2(\mathcal{T}_h)} \lesssim h^{s+\mu_{\min}} \left( \|u\|_{s+1,1-\mu,\mathcal{T}_h} + \|\nabla \cdot u\|_{s,\mu-1,\mathcal{T}_h} + \|\nabla \times u\|_{s,\mu-1,\mathcal{T}_h} \right)$$

3.5 Regularity and optimality

The results of Theorem 3.4.7 and Eq. (3.4.10) are given for a general set of problems. The order of convergence is controlled both by the geometry of the domain $\Omega$, in terms of its interior angles, and also in terms of the regularity of the exact solution $u$ and of $\nabla \cdot u$ and $\nabla \times u$. Since solutions with low regularity have a lower bound in terms of the interior angles Eq. (3.3.7), the convergence results are optimal in the sense that the method and penalty are chosen so that convergence is obtained in the most pathological cases.
This method also converges for solutions that are smooth. However, the domain-dependent penalty and the additional assumptions on $\nabla \cdot u$ and $\nabla \times u$ can lead to slower convergence results than with conforming methods. An interesting consequence of the assumption that $\nabla \cdot u, \nabla \times u \in H^s_{\mu-1}(\Omega)$ is that the value of $s$ may be constrained, even if $u$ itself is nonsingular. An example that illustrates this is given below:

Example 3.5.1. Define $\Omega = [-\frac{1}{2}, \frac{1}{2}]^2 \setminus [0, \frac{1}{2}]^2$ the L shaped domain, and let $r, \theta$ denote polar coordinates for $\Omega$ centered at the origin. In this domain, there is only one re-entrant corner $c_\ell$ at the origin, with interior angle $\omega = \frac{3\pi}{2}$ yielding $\mu < \frac{\pi}{2\omega} = \frac{1}{3}$ and so

$$\gamma = \frac{\pi - \bar{0}}{h^{2(1-\mu)}}.$$ 

where $1 - \mu > 2/3$.

Let $u = \nabla \times r^{m+\epsilon}$ for some $0 < \epsilon < 1$ and $m \geq 2$. Then $u \in H^m(\Omega)$, $\nabla \cdot u = 0$, and $\nabla \times u \approx r^{m-2+\epsilon} \in H^{m-1}(\Omega)$. Therefore the expected order of convergence should be in terms of $m$. However, the $m$-th derivative of $\nabla \times u$ yields a term approximately $r^{-2+\epsilon}$, which is not integrable in a neighborhood of the origin when $\epsilon < 1$.

The assumption $\nabla \times u \in H^s_{\mu-1}(\Omega)$ implies that $r^{2(\mu-1)}|\partial^\alpha (\nabla \times u)|^2$ is integrable whenever $|\alpha| = s$. In particular, we have for $0 < \delta < 1$

$$\int_0^\delta r^{2(\mu-1)} r^{2(m-2+\epsilon-s)} r dr < \infty$$

Thus, the parameter $s$ must satisfy $s \leq m - 2 + \mu$, and so for a solution $u \in H^2(\Omega)$, the expected rate of convergence in $L^2$-norm is $O(h^{\frac{s}{2}-\epsilon})$. 

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3.6 Numerical results

We will present numerical results that demonstrate the convergence estimates of Section 3.4. In each case, we will take the parameter $\alpha = 1$. The code for these experiments was written using the Firedrake library [45] and making use of the Slate extension for a solver that uses static condensation [37].

First, we consider the smooth solution on the domain $\Omega = \left[ -\frac{1}{2}, \frac{1}{2} \right]^2 \subset \mathbb{R}^2$. Let $u_{exact}$ be defined as

$$u_{exact} = \begin{bmatrix} \left( \frac{x^3}{3} - \frac{x^2}{4} \right) \left( y^2 - \frac{1}{2} y \right) \sin(y - 1/2) \\ \left( \frac{y^3}{3} - \frac{y^2}{4} \right) \left( x^2 - \frac{1}{2} x \right) \cos(x - 1/2) \end{bmatrix}. \quad (3.6.1)$$

Figure 3.1: Plot of the exact solution $u = \nabla \times r^{2+\epsilon}$ in the L shaped domain
We present the results from an implementation of the three field formulation (3.3.5) on uniform meshes with $N + 1$ vertices for $N \in \{2^n\}_{n=1}^6$ and the source term $f = -\nabla \nabla \cdot u_{\text{exact}} + \nabla \times \nabla \times u_{\text{exact}} + \alpha u_{\text{exact}}$.

<table>
<thead>
<tr>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>$|u - u_h|_h$ rate</td>
<td>$|u - u_h|_L^2$ rate</td>
</tr>
<tr>
<td>2</td>
<td>$3.92 \cdot 10^{-2}$</td>
<td>$7.14 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>4</td>
<td>$2.03 \cdot 10^{-2}$</td>
<td>$9.51 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>8</td>
<td>$1.05 \cdot 10^{-2}$</td>
<td>$4.79 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>16</td>
<td>$5.36 \cdot 10^{-3}$</td>
<td>$1.18 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>32</td>
<td>$2.71 \cdot 10^{-3}$</td>
<td>$9.81 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>64</td>
<td>$1.36 \cdot 10^{-3}$</td>
<td>$7.24 \cdot 10^{-6}$</td>
</tr>
</tbody>
</table>

On a square domain, the interior angles of the domain corners are all $\leq \frac{\pi}{2}$ and so $\mu_{\min} = 1$, and hence $\gamma = \frac{1}{|e|}$. As the solution $u$ as well as $\nabla \cdot u$, $\nabla \times u$ are smooth, The expected rate
of convergence is controlled by the degree of polynomial approximation. The results of Theorem 3.4.7 and Eq. (3.4.10) give an expected rate of \( k \) and \( k + \mu_{\min} = k + 1 \) for the energy and \( L^2 \) norms, respectively. This is indeed seen numerically in Table 3.1.

To demonstrate the optimal convergence rates for low-regularity solutions, we define a problem on the L shaped domain \( \Omega = \left[ -\frac{1}{2}, \frac{1}{2} \right]^2 \setminus \left[ 0, \frac{1}{2} \right]^2 \). As seen in Example 3.5.1, this has a re-entrant corner \( c \) with interior angle \( \omega = \frac{3\pi}{2} \).

Define the solution in polar coordinates \((r, \theta)\):

\[
 u_{\text{exact}} = \nabla \left[ r^{\frac{2}{3}} \sin \left( \frac{2}{3} \left( \theta - \frac{\pi}{2} \right) \right) \right] 
\]  

(3.6.2)

A plot of the exact solution solution on a mesh with spacing \( h \approx 2^{-4} \) is shown in Fig. 3.3. The singularity at the origin is due to the homogeneous tangential boundary condition.

![Figure 3.3: Plot of the exact solution](image)

The solution \( u \) is in \( H^{2/3-\epsilon}(\Omega) \), and the results of the implementation are given in Table 3.2. In this case, the regularity \( s \) of the solution \( u \) in terms of the weighted Sobolev space is then \( u \in H^{4/3-\epsilon}_{2/3} \), and so according to Eq. (3.4.10), the order of accuracy for the
leading term in the energy estimate is $4/3 - 1 = 1/3$. And in the $L^2$ norm the order should be $4/3 - 1 + 1/3 = 2/3$. This is indeed seen numerically.

Table 3.2: Results for solution with low regularity $s = 2/3$ on the L shaped domain

<table>
<thead>
<tr>
<th>N</th>
<th>$|u - u_h|_h$ rate</th>
<th>$|u - u_h|_{L^2}$ rate</th>
<th>$|u - u_h|_h$ rate</th>
<th>$|u - u_h|_{L^2}$ rate</th>
<th>$|u - u_h|_h$ rate</th>
<th>$|u - u_h|_{L^2}$ rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.3</td>
<td>7.11 $\cdot 10^{-2}$</td>
<td>0.21</td>
<td>5.73 $\cdot 10^{-2}$</td>
<td>0.19</td>
<td>4.42 $\cdot 10^{-2}$</td>
</tr>
<tr>
<td>4</td>
<td>0.31</td>
<td>5.02 $\cdot 10^{-2}$</td>
<td>0.35</td>
<td>3.69 $\cdot 10^{-2}$</td>
<td>0.34</td>
<td>2.82 $\cdot 10^{-2}$</td>
</tr>
<tr>
<td>8</td>
<td>0.33</td>
<td>3.23 $\cdot 10^{-2}$</td>
<td>0.34</td>
<td>2.35 $\cdot 10^{-2}$</td>
<td>0.34</td>
<td>1.79 $\cdot 10^{-2}$</td>
</tr>
<tr>
<td>16</td>
<td>0.33</td>
<td>2.07 $\cdot 10^{-2}$</td>
<td>0.34</td>
<td>1.49 $\cdot 10^{-2}$</td>
<td>0.34</td>
<td>1.13 $\cdot 10^{-2}$</td>
</tr>
<tr>
<td>32</td>
<td>0.33</td>
<td>1.31 $\cdot 10^{-2}$</td>
<td>0.34</td>
<td>9.4 $\cdot 10^{-3}$</td>
<td>0.34</td>
<td>7.13 $\cdot 10^{-3}$</td>
</tr>
<tr>
<td>64</td>
<td>0.33</td>
<td>8.32 $\cdot 10^{-3}$</td>
<td>0.34</td>
<td>5.93 $\cdot 10^{-3}$</td>
<td>0.33</td>
<td>4.5 $\cdot 10^{-3}$</td>
</tr>
</tbody>
</table>

Next, we will consider an exact solution with singularity, but with slightly higher regularity than the $s = 2/3$ case, also on the L shaped domain. Define the solution in polar coordinates $(r, \theta)$:

$$u_{exact} = \nabla \left[ r^{8/3} \sin \left( \frac{8}{3} \left( \theta - \frac{\pi}{2} \right) \right) \right]$$

(3.6.3)

Similarly to the previous example, the solution $u$ is in $H^{8/3-\epsilon}(\Omega)$, and the results of the implementation are given in Table 3.3.

Table 3.3: Results for solution with low regularity $s = 8/3$ on L shaped domain

<table>
<thead>
<tr>
<th>N</th>
<th>$|u - u_h|_h$ rate</th>
<th>$|u - u_h|_{L^2}$ rate</th>
<th>$|u - u_h|_h$ rate</th>
<th>$|u - u_h|_{L^2}$ rate</th>
<th>$|u - u_h|_h$ rate</th>
<th>$|u - u_h|_{L^2}$ rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.14</td>
<td>1.79 $\cdot 10^{-2}$</td>
<td>1.16 $\cdot 10^{-3}$</td>
<td>2.28 $\cdot 10^{-4}$</td>
<td>2.11 $\cdot 10^{-4}$</td>
<td>3.84 $\cdot 10^{-5}$</td>
</tr>
<tr>
<td>4</td>
<td>7.49 $\cdot 10^{-2}$</td>
<td>0.92</td>
<td>2.34 $\cdot 10^{-4}$</td>
<td>2.31</td>
<td>3.63 $\cdot 10^{-5}$</td>
<td>2.65</td>
</tr>
<tr>
<td>8</td>
<td>3.85 $\cdot 10^{-2}$</td>
<td>0.96</td>
<td>4.67 $\cdot 10^{-5}$</td>
<td>2.33</td>
<td>5.73 $\cdot 10^{-6}$</td>
<td>2.66</td>
</tr>
<tr>
<td>16</td>
<td>1.95 $\cdot 10^{-2}$</td>
<td>0.98</td>
<td>9.28 $\cdot 10^{-6}$</td>
<td>2.33</td>
<td>9.65 $\cdot 10^{-7}$</td>
<td>2.66</td>
</tr>
<tr>
<td>32</td>
<td>9.83 $\cdot 10^{-3}$</td>
<td>0.99</td>
<td>1.84 $\cdot 10^{-6}$</td>
<td>2.33</td>
<td>1.43 $\cdot 10^{-7}$</td>
<td>2.67</td>
</tr>
<tr>
<td>64</td>
<td>4.93 $\cdot 10^{-3}$</td>
<td>0.99</td>
<td>3.65 $\cdot 10^{-7}$</td>
<td>2.33</td>
<td>2.25 $\cdot 10^{-8}$</td>
<td>2.67</td>
</tr>
</tbody>
</table>
In the results of Table 3.3, the $k = 2$ case demonstrates a rate of convergence greater than 2 in the energy norm. This is a higher rate than indicated by Theorem 3.4.7, and is due to the fact that the solution $u_{exact}$ is the gradient of a harmonic function. The space $\mathcal{N}_h$ contains the BSM$_k$ space, which in turn contains the gradients of harmonic polynomials up to degree $r = 2k − 1$, and hence the first $r$ terms in the Taylor series expansion of $u_{exact}$ are contained in the approximation space.

And finally, we present results that illustrate the discussion in Section 3.5. The exact solution $u_{exact} = \nabla \times (r^{m+\epsilon})$ explored in Example 3.5.1 is given for a range of values for $m$ in Table 3.4. In each case, we take the approximation degree $k = m + 1$. While the exact solution is in $H^m(\Omega)$, the assumption on $\nabla \cdot u$ in $H^{s-1}_\mu(\Omega)$ implies that $s \leq m + \mu - 2$ for each integer $m$. The change in rate for the case $m = 4$ with the finest mesh spacing is due to the fact that the parameter $\epsilon$ is taken to be $10^{-4}$.

Table 3.4: Results of scheme defined on the L shaped domain with solution $u \in H^m(\Omega)$ and regularity parameter $s = m + \mu - 2$ in weighted Sobolev space

<table>
<thead>
<tr>
<th>N</th>
<th>$|u - u_h|_h$</th>
<th>rate</th>
<th>$|u - u_h|_{L^2}$</th>
<th>rate</th>
<th>$|u - u_h|_h$</th>
<th>rate</th>
<th>$|u - u_h|_{L^2}$</th>
<th>rate</th>
</tr>
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<tbody>
<tr>
<td>2</td>
<td>$3.92 \cdot 10^{-4}$</td>
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<td>$1.03 \cdot 10^{-4}$</td>
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<td>$3.66 \cdot 10^{-3}$</td>
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<td>$3.22 \cdot 10^{-4}$</td>
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<td>$6.8 \cdot 10^{-5}$</td>
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<tr>
<td>16</td>
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<td>$2.84 \cdot 10^{-5}$</td>
<td>0.64</td>
<td>$1.15 \cdot 10^{-3}$</td>
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<td>$1.22 \cdot 10^{-4}$</td>
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<td>$4.35 \cdot 10^{-11}$</td>
<td></td>
<td>$0.45 \cdot 10^{-11}$</td>
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</tr>
</tbody>
</table>
3.7 Conclusion and future work

The nonconforming method introduced in the equivalent two and three field formulations (3.3.13) and (3.3.5) has been shown to generalize the method of Brenner et al [15] to admit polynomial approximation with higher degree, with corresponding convergence rates in the energy norm and $L^2$ norm. The regularity of the solution is discussed in terms of weighted Sobolev spaces, which are used to derive normed estimates for high degree polynomial approximations and for the more general family of shape regular triangulations, rather than triangulations that satisfy a grading condition.

While the solution space is shown to contain the BSM space developed in [21] and [42], the method is defined for standard Lagrange finite elements, which are straightforward to implement numerically, and for which standard polynomial approximation theory results can be utilized.

The hybridization is given in the context of recent work in FEEC [4] [6], and is demonstrated to be solvable using static condensation.

Some future directions for this project include the following problems:

- Extending the current method to 3 dimensions. This involves the choice of weighting on the exponents for the penalty function. A rigorous development of singularities in 3 dimensional domains is made by Costabel, Dauge and Nicaise [32, 33].

- Developing analysis for graded meshes to obtain sharp estimates for higher degree polynomial approximation.
Bibliography


