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Interpolating Matrices

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Chapter 1

Introduction

1.1 Interpolating Matrices

A sequence $Z = (z_n)_{n\in\mathbb{N}}$ of points in the unit disc $\mathbb{D}$ is *interpolating* for the space H^∞ of bounded analytic functions on $\mathbb{D}$ if for any bounded sequence $(w_n)_{n\in\mathbb{N}}$ there exists a function ϕ in H^∞ such that $\phi(z_n) = w_n$, for any n in $\mathbb{N}$. Intuitively, an interpolating sequence is a separated sequence, as we need to be able to specify the values of ϕ *arbitrarily* at the nodes $(z_n)_{n\in\mathbb{N}}$. Since the interpolating functions are holomorphic a natural way to compute distances between points of Z is by using the *pseudo-hyperbolic* distance

$$\rho(z,w) := |b_w(z)| = \left|\frac{w-z}{1-\overline{w}z}\right| \qquad z,w\in\mathbb{D}. \tag{1.1.1}$$

Here b_w will denote the involutive Blaschke factor at a point w in $\mathbb{D}$. We will say that Z is **strongly separated** if

$$\inf_{n\in\mathbb{N}}\prod_{j\neq n}\rho(z_n,z_j) > 0, \tag{1.1.2}$$

while Z is **weakly separated** if

$$\inf_{j\neq n}\rho(z_n,z_j) > 0.$$

Any function f in H^∞ that vanishes with multiplicity m at a point z in $\mathbb{D}$ is divisible in H^∞ by a Blaschke factor:

$$f = b_z^m\, g,$$

where $||g||_\infty = ||f||_\infty$. Therefore, Z is weakly separated if and only if there exists $\delta > 0$ such that, for any $j \neq n$, there exists a bounded analytic function $\phi_{j,n}$ such that $||\phi_{j,n}||_\infty = 1$, $\phi_{j,n}(z_n) = 0$ and $\phi_{j,n}(z_j) = \delta$. In the same way, Z is strongly separated if and only if there exists a positive δ' such that, for any n in $\mathbb{N}$, there exists a function ϕ_n so that $||\phi_n||_\infty = 1$, $\phi_n(z_n) = \delta'$ and ϕ_n vanishes at all the other points of Z.

The celebrated work of Carleson, [10] [9], characterized interpolating sequences for H^∞ in terms of separated sequences:

Theorem 1.1.1 (Carleson)**.** *Let Z be a sequence in $\mathbb{D}$. The following are equivalent:*

(i) *Z is interpolating;*

(ii) *Z is strongly separated;*

(iii) *Z is weakly separated and the measure*

$$\mu_Z := \sum_{n\in\mathbb{N}} (1 - |z_n|^2)\delta_{z_n}$$

on $\mathbb{D}$ satisfies the embedding condition

$$||f||_{L^2(\mathbb{D},\mu_Z)} \leq C_Z \; ||f||_{\mathrm{H}^2} \qquad f \in \mathrm{H}^2. \tag{1.1.3}$$

Here H^2 is the Hardy space on the unit disc $\mathbb{D}$, that is, the reproducing kernel Hilbert space of those power series centered at the origin with square-summable Taylor coefficients,

$$\mathrm{H}^2 := \left\{ f(z) = \sum_{n\geq 0} \hat{f}_n z^n \;\middle|\; \sum_{n\geq 0} |\hat{f}_n|^2 < \infty \right\},$$

with the natural inner product

$$\langle f, g\rangle_{\mathrm{H}^2} := \sum_{n\geq 0} \hat{f}_n \overline{\hat{g}_n}.$$

Its kernel s is the well-studied *Szegö kernel*

$$s_w(z) := \frac{1}{1-\overline{w}z} \qquad w, z \in \mathbb{D},$$

so that for any w in $\mathbb{D}$

$$f(w) = \langle f, s_w \rangle_{\mathrm{H}^2} \qquad f \in \mathrm{H}^2.$$

The *multiplier algebra* of H^2

$$\mathcal{M} := \{\phi \quad \text{holomorphic in } \mathbb{D} \,|\, \phi f \in \mathrm{H}^2,\ f \in \mathrm{H}^2\}$$

can be identified isometrically with H^∞, i.e.,

$$||\phi||_\infty = ||M_\phi||_{\mathcal{B}(\mathrm{H}^2)},$$

where M_ϕ is the linear operator of multiplication by ϕ in H^2.

A measure μ on a domain D of $\mathbb{C}^d$ that embeds continuously $L^2(D, \mu)$ into a reproducing kernel Hilbert space $\mathcal{H}_k$ on D is called a **Carleson measure** for $\mathcal{H}_k$. For a geometric proof of Theorem 1.1.1, see [17, Th. 1.1, Ch. VII].

In [3, Th. 9.42], one can find Theorem 1.1.1 re-phrased in terms of separation conditions on the reproducing kernel functions $(s_{z_n})_{n\in\mathbb{N}}$ in H^2. A sequence of unit vectors $(x_n)_{n\in\mathbb{N}}$ in a Hilbert space is a **Riesz system** if there exists a positve C such that, for any $(a_n)_{n\in\mathbb{N}}$ in l^2,

$$\frac{1}{C^2}\sum_{n\in\mathbb{N}}|a_n|^2 \leq \left|\left|\sum_{n\in\mathbb{N}} a_n x_n\right|\right|^2 \leq C^2 \sum_{n\in\mathbb{N}}|a_n|^2. \tag{1.1.4}$$

The least C for which (1.1.4) holds is the *Riesz bound* of the sequence $(x_n)_{n\in\mathbb{N}}$. If the least positive number for which the right hand side of (1.1.4) is $M < \infty$, we say that $(x_n)_{n\in\mathbb{N}}$ is a **Bessel system** with *Bessel bound* M. If x is an element of a Hilbert space, let $\hat{x}$ denote its normalization

$$\hat{x} := \frac{x}{||x||}.$$

It turns out that (1.1.3) holds if and only if the sequence of normalized kernels $(\hat{s}_{z_n})_{n\in\mathbb{N}}$ is a Bessel system,[3, Prop. 9.5]. This highlights a correspondence between the hyperbolic geometry of the unit disc and the Euclidean geometry of H^2, and allows one to extend Theorem 1.1.1 to more general settings.

Let $\mathcal{H}_k$ be a reproducing kernel Hilbert space of analytic functions on a domain D of $\mathbb{C}^d$, and let $\mathcal{M}_k$ be its multiplier algebra. That is, let $\mathcal{H}_k$ be a Hilbert space of holomorphic functions on D such that point evaluation at any x in D is a continuous linear functional. Thanks to Riesz representation Theorem, for any x in D there exists a **kernel function** k_x in $\mathcal{H}_k$ that *represents* points evaluation at x in $\mathcal{H}$:

$$f(x) = \langle f, k_x \rangle_{\mathcal{H}_k} \qquad f \in \mathcal{H}_k.$$

The collection of such kernel functions define a **kernel** $k\colon D \times D \to \mathbb{C}$

$$k(z, x) := k_x(z) \qquad x, z \in D$$

which is analytic in z and anti-analytic in x. Associated to a kernel k (or, equivalently, to a reproducing kernel Hilbert space $\mathcal{H}_k$) one can define a **multiplier algebra**

$$\mathcal{M}_k := \{\phi \,\text{holomorphic in}\, D \,|\, \phi f \in \mathcal{H}_k,\ f \in \mathcal{H}_k\}$$

as the Banach algebra of those analytic functions on D that multiplies $\mathcal{H}_k$ into itself. Thanks to the closed graph Theorem, each ϕ in $\mathcal{M}_k$ defines then a *bounded* multiplication operator in $\mathcal{B}(\mathcal{H}_k)$

$$M_\phi(f) := \phi\ f \qquad f \in \mathcal{H}_s,$$

and $\mathcal{M}_k$ is normed via $||\phi||_{\mathcal{M}_k} := ||M_\phi||_{\mathcal{B}(\mathcal{H}_k)}$.

A sequence $Z = (z_n)_{n\in\mathbb{N}}$ in D is **interpolating** for $\mathcal{M}_k$ if for any bounded sequence $(w_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ there exists a function ϕ in $\mathcal{M}_k$ such that $\phi(z_n) = w_n$, for any n in $\mathbb{N}$. It is not surprising that the separation conditions we will look at depend on the kernel k: Z is a **weakly separated** sequence if there exists a positive M such that, for any $n \neq j$ in $\mathbb{N}$, there exists a function $\phi_{n,j}$ whose norm in $\mathcal{M}_k$ doesn't exceed M and that separates z_n and z_j, that is,

$$\phi_{n,j}(z_n) = 1 \qquad \phi_{n,j}(z_j) = 0.$$

If there exists a bounded sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_k$ that separates each point of the sequence with the rest of Z

$$\phi_j(z_n) = \delta_{j,n}$$

we will say that Z is **strongly separated**. A class of multiplier algebras for which Theorem 1.1.1 partially extends is the one associated with *complete Pick kernels*. One of the most important properties that connects interpolating sequences to the study of related Hilbert spaces is the fact that, for any multiplier ϕ in $\mathcal{M}_k$, any kernel function k_z in $\mathcal{H}_k$ is an eigenfunction of the adjoint of the multiplication operator M_ϕ:

$$M_\phi^*(k_z) = \overline{\phi(z)}\ k_z \qquad z \in D, \tag{1.1.5}$$

as a straightforward computation using the property of adjoints shows. In particular, if M_ϕ is a contraction and $\phi(z_n) = w_n$ for any n in $\mathbb{N}$, then the linear map T from $S_Z := \overline{\underset{n\in\mathbb{N}}{\text{span}}}\{k_{z_n}\}$ to itself given by

$$T(k_{z_n}) := \overline{w_n}\ k_{z_n} \qquad n \in \mathbb{N} \tag{1.1.6}$$

is a contraction. A reproducing kernel Hilbert space is said to have the **Pick property** if the existence of such a contraction T is also a sufficient condition for the existence of a function ϕ in the unit ball of $\mathcal{M}_k$ such that $\phi(z_n) = w_n$. In particular, this says that M_ϕ^* is an isometric extension of (1.1.6) to $\mathcal{H}_k$. This implies that any two disjoint sets of points Z_1 and Z_2 can be separated by a function in $\mathcal{M}_k$ of norm at most M if and only if the sine of the angle between S_{Z_1} and S_{Z_2} in $\mathcal{H}_k$ is bounded below by $1/M$:

$$\sup\left\{||\phi||_{\mathcal{M}_k}\,\middle|\,\phi_{|Z_1} = 1,\ \phi_{|Z_2} = 0\right\} = \frac{1}{\sin(S_{Z_1}, S_{Z_2})}. \tag{1.1.7}$$

In our discussion, the sine between two closed subspaces H_1 and H_2 of a Hilbert space $\mathcal{H}$ is the least sine of the angle between two vectors chosen from H_1 and H_2 or, equivalently, the inverse of the norm of an operator defined in $\text{span}\{H_1, H_2\}$ that acts like the identity on H_1 and that has H_2 as its kernel.

As a consequence of the Pick property,[3, Th. 9.19], Z is interpolating if and only if the sequence of normalized kernels $(\hat{k}_{z_n})_{n\in\mathbb{N}}$ is a Riesz system. Moreover, weak and strong separation for the sequence Z translates to separation conditions for its kernel functions. Specifically, Z is weakly separated if and only if the sine between any two distinct kernel functions in Z is uniformly bounded below

$$\inf_{j\neq n} \sin(k_{z_j}, k_{z_n}) > 0, \tag{1.1.8}$$

and strong separation translates to a uniform bound from below for the sine of the angle between any kernel function at a point of Z and the closure of the span of all the other kernels:

$$\inf_{n\in\mathbb{N}} \sin\left(k_{z_n}, \overline{\text{span}}_{j\neq n}\{k_{z_j}\}\right) > 0. \tag{1.1.9}$$

A sequence of vectors in a Hilbert space satisfying (1.1.8) is said to be **weakly separated**, whereas a **strongly separated** sequence of vectors is a collection satisfying (1.1.9). Since (1.1.6) being a contraction is equivalent to the infinite matrix

$$(1-\overline{w_n}w_j)k_{z_j}(z_n) \qquad n,j\in\mathbb{N}$$

being positive semi-definite, one can extend the Pick property to the case of *matrix-valued* functions in $\mathcal{H}_k$, by defining $\mathcal{H}_k$ to have the $s\times t$ *Pick property* if whenever $z_1,\dots,z_N$ are points in D and $W_1,\dots,W_N$ are $s\times t$ matrices such that

$$(Id-W_n^*W_j)k_{z_j}(z_n)\geq 0$$

then there exists a multiplier ϕ in the closed unit ball of

$$\mathcal{M}(\mathcal{H}_k\otimes\mathbb{C}^t,\mathcal{H}_k\otimes\mathbb{C}^s):=\left\{\phi=(\phi_{l,r})\,|\,l=1,\dots,s,\ r=1,\dots t,\ \sup_{\mathbf{f}\neq 0}\frac{||\phi\mathbf{f}||_{\mathcal{H}_k\otimes\mathbb{C}^s}}{||\mathbf{f}||_{\mathcal{H}_k\otimes\mathbb{C}^t}}<\infty\right\}$$

such that $\phi(z_i)=W_i$, for $i=1,\dots,N$. We say that $\mathcal{H}_k$ has the **complete Pick property** if it has the $s\times t$ Pick property for any positive integers s and t. The Hardy space H^2 has the complete Pick property, as well as some of its natural generalizations, such as the reproducing kernel Hilbert spaces $\mathcal{H}_s$ on $\mathbb{D}$, $0\leq s\leq 1$, defined by the kernels

$$k_w^s(z):=\sum_{n=0}^{\infty}(n+1)^s(\overline{w}z)^n \qquad z,w\in\mathbb{D}$$

and the *Drury-Arveson space* H^2_d on the d-dimensional unit ball $\mathbb{B}_d$ defined by the kernel

$$b_w(z):=\frac{1}{1-\langle z,w\rangle}, \qquad z,w\in\mathbb{B}_d.$$

For instance, see [3, Ch. 7]. In a recent work, [4], Aleman, Hartz, McCarthy and Richter partially extended Theorem 1.1.1 by showing that any weakly separated sequence Z on a

domain D such that the sequence of normalized kernels $(\hat{k}_{z_n})_{n\in\mathbb{N}}$ is a Bessel system is an interpolating sequence for $\mathcal{M}_k$, provided that $\mathcal{H}_k$ has the complete Pick property. This is done by using the recent positive answer to the *Feichtinger conjecture*, which states that any Bessel system is the disjoint union of finitely many Riesz systems. It has been shown, [11] [24], that the Feichtinger conjecture is equivalent to many other conjectures in operator theory, including the Paving conjecture, who had been proved by the work of Marcus, Spielmann and Srivastava [18].
In conclusion, Theorem 1.1.1 can be re-phrased as follows[3, Th. 9.42]

Theorem 1.1.2. *Let* $Z = (z_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{D}$*, and let* $(\hat{s}_{z_n})_{n\in\mathbb{N}}$ *be the associated sequence of normalized Szegö kernels in* H^2*. The following are equivalent:*

(i) Z *is interpolating for* H^∞*;*

(ii) $(\hat{s}_{z_n})_{n\in\mathbb{N}}$ *is a Riesz system;*

(iii) $(\hat{s}_{z_n})_{n\in\mathbb{N}}$ *is strongly separated;*

(iv) $(\hat{s}_{z_n})_{n\in\mathbb{N}}$ *is a weakly separated Bessel system.*

The aim of Chapter 2 is to extend Theorem 1.1.2 to sequences of square matrices. Throughout the discussion, a holomorphic functions f will be applied to a square matrix M via the Riesz-Dunford functional calculus

$$f(M) := \int_{\partial\mathbb{D}} f(\xi)(\xi\ Id - M)^{-1}\, d\xi, \tag{1.1.10}$$

hence we will assume that the spectra of the matrices in A are all contained in $\mathbb{D}$. In particular, if the power series of f is $\sum_{n\geq 0} c_n z^n$, (1.1.10) coincide with evaluating the power series of f at A:

$$f(M) = \sum_{n\geq 0} c_n A^n.$$

A first attempt to emulate the definition of an interpolating sequence of scalars can then be to define a sequence $(A_n)_{n\in\mathbb{N}}$ of square matrices with spectra in $\mathbb{D}$ to be interpolating if, given a sequence $(W_n)_{n\in\mathbb{N}}$ of square matrices (of the right dimensions) which is bounded in the operator norm, then there exists a function ϕ in H^∞ such that $\phi(A_n) = W_n$.

One reason why this can't be the right approach is given by the following example: let $Z = (z_n)_{n\in\mathbb{N}}$ be a zero sequence in $\mathbb{D}$, that is,

$$\sum_{n\in\mathbb{N}}(1-|z_n|) < \infty,$$

and set

$$A_n = \begin{bmatrix} z_n & 1 \\ 0 & z_n \end{bmatrix}.$$

The least that we can expect from a definition of interpolating sequence of matrices that is consistent with the scalar case is that $(f(A_n))_{n\in\mathbb{N}}$ is a target sequence, for any f in H^∞. But if we set B to be the Blaschke product at the points of Z, then $(B'(z_n))_{n\in\mathbb{N}}$ is unbounded, and so is

$$B(A_n) = \begin{bmatrix} 0 & B'(z_n) \\ 0 & 0 \end{bmatrix}.$$

Therefore, a target sequence for an interpolating sequence of matrices can't simply be a sequence of matrices bounded with respect the operator norm. The problem seems to be that, given a Jordan block

$$J = \begin{bmatrix} z & 1 \\ 0 & z \end{bmatrix} \qquad z \in \mathbb{D}$$

and a bounded holomorphic function f, then

$$f(J) = \begin{bmatrix} f(z) & f'(z) \\ 0 & f(z) \end{bmatrix},$$

and while $f(z)$ can assume only bounded values, $f'(z)$ has bound that depends on the H^∞ norm of f and, most importantly, on z. Another obstacle on the way of identifying a sequence of targets with a bounded sequence in the operator norm comes from the fact that a square matrix can have a non trivial algebraic structure invariant under holomorphic functions (its eigenspaces, for example): given two points z and w in $\mathbb{D}$ there is no function ϕ in H^∞ that maps

$$A = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix}$$

to

$$W = \begin{bmatrix} w & 1 \\ 0 & w \end{bmatrix},$$

although both A and W are bounded in the operator norm and the constant function w is a contraction in H^∞ that sends the spectrum of A to the spectrum of W. Since choosing bounded targets in the operator norm makes even a one-point interpolation problem impossible to solve via bounded analytic functions, in order to define interpolating matrices one has to identify a target with a bounded sequence in H^∞:

Definition 1.1.3 (Interpolating Matrices)**.** *Let $A = (A_n)_{n\in\mathbb{N}}$ be a sequence of complex square matrices (of perhaps different dimensions) with spectra contained in $\mathbb{D}$. Then A is interpolating for H^∞ if, for any bounded sequence $(\phi_n)_{n\in\mathbb{N}}$ in H^∞ there exists a function ϕ in H^∞ such that*

$$\phi(A_n) = \phi_n(A_n) \qquad n \in \mathbb{N}.$$

Observe that if the matrices in A reduce to scalars, then we recover the classic definition of an interpolating sequence in $\mathbb{D}$.

In order to separate the matrices in A one can follow the same outline used for sequences of scalars, and separate matrices via bounded analytic functions. Specifically, we will say that a sequence $A = (A_n)_{n\in\mathbb{N}}$ of matrices with spectra in the unit disc is **weakly separated** if there exists a positive M such that, for any pair of distinct positive integers n and j, there exists a bounded analytic function $\phi_{n,j}$ whose H^∞ norm doesn't exceed M and that separates A_n and A_j, that is,

$$\phi_{n,j}(A_n) = Id \qquad \phi_{n,j}(A_j) = 0.$$

Following the same idea, A is **strongly separated** if there exists a bounded sequence $(\phi_n)_{n\in\mathbb{N}}$ in H^∞ that separates each A_n with the rest of the sequence, i.e.,

$$\phi_n(A_j) = \delta_{n,j}\ Id.$$

In order to extend Theorem 1.1.1 to sequence of matrices, we will in fact extend Theorem 1.1.2, its Euclidean version. To do so, we are required to find a valid replacement for a sequence of normalized kernel functions. With this in mind, let's define, for any positive integer n,

$$H_n := \mathrm{H}^2 \ominus \{f \in \mathrm{H}^2 \,|\, f(A_n) = 0\} \tag{1.1.11}$$

to be the orthogonal complement in H^2 of all functions that vanish at the matrix A_n. If $A_n = z_n$ is a scalar, then H_n is the one dimensional line spanned by the Szegö kernel at z_n. We will see in Section 2.2 how separation conditions on the sequence $(H_n)_{n\in\mathbb{N}}$ corresponds to separation conditions on the matrices in A. At this stage, we only recall the notion of Riesz system and Bessel system for a sequence $X = (X_n)_{n\in\mathbb{N}}$ of closed sub-spaces (eventually multi-dimensional) of a Hilbert space $\mathcal{H}$, by saying that X is a **Riesz system** if there exists a $C \geq 1$ such that, for any sequence $(x_n)_{n\in\mathbb{N}}$ of unit vectors in $\mathcal{H}$ such that x_n belongs to X_n for any n in $\mathbb{N}$, and for any $(a_n)_{n\in\mathbb{N}}$ in l^2,

$$\frac{1}{C^2} \sum_{n\in\mathbb{N}} |a_n|^2 \leq \left\| \sum_{n\in\mathbb{N}} a_n x_n \right\|^2 \leq C^2 \sum_{n\in\mathbb{N}} |a_n|^2. \tag{1.1.12}$$

If the right hand side of (1.1.12) holds we will say that X is a Bessel system. As for the case of one dimensional subspaces, the optimal constants in (1.1.12) are said to be, respectively, the *Riesz bound* and the *Bessel bound* of X.

Our main result extends Theorem 1.1.2, and can be stated as follows:

Theorem 1.1.4. *Let A be a sequence of square matrices with spectra in the unit disc, and let $H = (H_n)_{n\in\mathbb{N}}$ be the sequence of subspaces of* H^2 *defined in* (1.1.11)*. The following are equivalent:*

(i) *A is interpolating;*

(ii) *H is a Riesz system;*

(iii) *A is strongly separated;*

(iv) *A is weakly separated and H is a Bessel system.*

1.2 Interpolating d-tuples of Commuting Matrices

In Chapter 3 we will extend some well-known results on interpolating sequences in the polydisc to sequences of *pairs of commuting matrices*.

Let $\mathrm{H}^\infty(\mathbb{D}^d)$ be the Banach algebra of bounded analytic functions on the polydisc $\mathbb{D}^d$. A sequence $Z = (z_n)_{n\in\mathbb{N}}$ in $\mathbb{D}^d$ is interpolating for $\mathrm{H}^\infty(\mathbb{D}^d)$ if, given any bounded $(w_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ there exists a bounded holomorphic function f on $\mathbb{D}^d$ so that $f(z_n) = w_n$, for all n. Berndtsson, Chang and Lin proved in [7] the following analogous of Theorem 1.1.1:

Theorem 1.2.1 (Berndtsson, Chang and Lin)**.** *Let* $Z = (z_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{D}^d$*, and let* (a)*,* (b) *and* (c) *denote the following statements:*

(a)

$$\inf_{n\in\mathbb{N}} \prod_{j\neq n} \rho_G(z_n, z_j) > 0; \tag{1.2.1}$$

(b) Z *is interpolating for* $\mathrm{H}^\infty(\mathbb{D}^d)$*;*

(c) The measure

$$\mu_Z := \sum_{n\in\mathbb{N}} \left(\prod_{i=1}^d (1 - |z_n^i|^2) \right) \delta_{z_n}$$

is a Carleson measure for $\mathrm{H}^2(\mathbb{D}^d)$ *and*

$$\inf_{n\neq j} \rho_G(z_n, z_j) > 0. \tag{1.2.2}$$

Then (a) $\implies$ (b) $\implies$ (c)*, and none of the converse implications hold.*

$\mathrm{H}^2(\mathbb{D}^d)$ will be here the *Hardy space* on the polydisc, that is, the reproducing kernel Hilbert space on $\mathbb{D}^d$ with kernel

$$s_d(z, w) := \prod_{i=1}^d \frac{1}{1 - \overline{w^i} z^i} \qquad z, w \in \mathbb{D}^d. \tag{1.2.3}$$

Conditions (1.2.1) and (1.2.2) are separation conditions, both stated in terms of the so called *Gleason distance* on the polydisc:

$$\rho_G(w, z) := \max_{i=1,\dots d} \rho(z^i, w^i) \qquad z, w \in \mathbb{D}^d.$$

It turns out that ρ_G corresponds to separating points via bounded analytic function, since

$$\rho_G(z, w) = \sup\{|\phi(z)| \mid ||\phi||_\infty \leq 1, \phi(w) = 0\}. \tag{1.2.4}$$

Throughout this work, (1.2.1) will refer to **strong separation** on the polydisc, while (1.2.2) defines a **weakly separated** sequence on the polydisc.

Even if Theorem 1.2.1 does not characterize interpolating sequences in the polydisc, the case $d = 2$ provides a characterization of such interpolating sequences by looking at the Euclidean geometry of those reproducing kernels Hilbert spaces whose multiplier algebra is $\mathrm{H}^\infty(\mathbb{D}^2)$. The main difference with the one-variable case is the fact that, rather than working with just the Szegö kernel, in order to obtain interpolating properties one has to consider a whole class of different kernels on the polydisc at the same time. In [2], Agler and McCarthy characterized interpolating sequences in the bi-disc in terms of separation conditions on the class of so called **admissible kernels**. A kernel k on $\mathbb{D}^d$ is admissibile if the multiplications by the coordinates

$$M_{z^1}, \ldots, M_{z^d}$$

is a set of commuting contractions on $\mathcal{H}_k$. Let $\mathcal{A}_d$ be the set of all admissible kernels on $\mathbb{D}^d$, and let $\mathcal{B}_d$ be the set of all kernels k on the d-dimensional polydisc whose multiplier algebra coincide with $\mathrm{H}^\infty(\mathbb{D}^d)$. Since the coordinate functions are clearly in the unit ball of H^∞, we have that $\mathcal{B}_d \subseteq \mathcal{A}_d$. Conversely, thanks to Ando's inequality [5], for any kernel in $\mathcal{B}_2$

$$||\phi(T_1, T_2)||_{\mathcal{B}(\mathcal{H}_k)} \leq ||\phi||_\infty$$

for any ϕ in $\mathrm{H}^\infty(\mathbb{D}^2)$ and for any pair (T_1, T_2) of commuting contractions on $\mathcal{H}_k$. Since

$$M_\phi = \phi(M_{z^1}, M_{z^2}) \qquad \phi \in \mathrm{H}^\infty(\mathbb{D}^2)$$

we have that

$$\mathcal{A}_2 = \mathcal{B}_2.$$

Namely, the class of admissible kernels coincides with the class of kernels on $\mathbb{D}^2$ whose multiplier algebra is $\mathrm{H}^\infty(\mathbb{D}^2)$.

Example 1.2.2. *Let $\alpha := (\alpha_1, \dots, \alpha_d)$ be a d-tuple of positive integers. Then the kernel*

$$s^\alpha_w(z) := \prod_{i=1}^{d} \frac{1}{(1-\overline{w^i}z^i)^{\alpha_i}} \qquad z, w \in \mathbb{D}^d$$

is admissible on $\mathbb{D}^d$. Indeed, multiplication by the coordinates being a set of contractions is equivalent to assert that, for any $z_1, \dots, z_N$ in $\mathbb{D}^d$, the matrix

$$[(1-\overline{z^i_n}z^i_j)k_{z_j}(z_n)]_{n,j=1}^N \qquad i = 1, \dots, d$$

is positive semi-definite. Therefore, since the Szegö kernel is admissible and the Schur (point-wise) product of positive semi-definite matrices is positive semi-definite, s^α belongs to $\mathcal{A}_d$ for any α.

Although the Szegö kernel on the polydisc (1.2.3) has not the Pick property, one can prove an analogous extension property provided that the bounds are uniform in $\mathcal{A}_2$. More precisely, let, for any kernel k in $\mathcal{A}_2$

$$S^k_Z := \overline{\operatorname*{span}_{n\in\mathbb{N}}}\{k_{z_n}\} \subseteq \mathcal{H}_k.$$

Given a bounded sequence $(w_n)_{n\in\mathbb{N}}$, if there exits a function ϕ in the unit ball $\mathrm{H}^\infty(\mathbb{D}^2)$ such that

$$\phi(z_n) = w_n \qquad n \in \mathbb{N},$$

then, for any k in $\mathcal{A}_2$, the operator

$$T^k \colon S^k_Z \to S^k_Z$$

such that

$$T^k(k_{z_n}) = \overline{w_n} k_{z_n}$$

is a contraction in $\mathcal{H}_k$. Conversely, [1], if

$$\sup_{k\in\mathcal{A}_2} ||T^k||_{\mathcal{B}(\mathcal{H}_k)} \leq 1,$$

then *each* T^k is in fact the restriction of the adjoint of the multiplication by a function ϕ in the unit ball of $\mathrm{H}^\infty(\mathbb{D}^2)$, and $\phi(z_n) = w_n$. This, together with Theroem 1.2.4 below, gives the following characterization for interpolating sequences for the bidisk, which is part of a result in [2]:

Theorem 1.2.3. *Let $Z = (z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}^2$. Then Z is interpolating for $\mathrm{H}^\infty(\mathbb{D}^2)$ if and only if there exists a constant $C \geq 1$ such that, for any k in $\mathcal{A}_2$, the sequence of normalized kernels $(\hat{k}_{z_n})_{n\in\mathbb{N}}$ is a Reisz sequence in $\mathcal{H}_k$ with Riesz bound C.*

As we will see, the key for the identification between Riesz systems of kernels functions and interpolating sequences for those reproducing kernel Hilbert spaces with a Pick-like property is the following characterization of the Riesz system condition [19, Th. 3.1.4]:

Theorem 1.2.4. *Let $H = (H_n)_{n\in\mathbb{N}}$ be a sequence of closed sub-spaces of a Hilbert space $\mathcal{H}$. The following are equivalent:*

(i) *H is a Riesz system with Riesz bound C;*

(ii) *For any sequence of linear functions $(\chi_n)_{n\in\mathbb{N}}$ such that*

$$\chi_n\colon H_n \to H_n \qquad n \in \mathbb{N}$$

and $\sup_{n\in\mathbb{N}} ||\chi_n|| \leq 1$*, then*

$$\chi\colon \overline{\operatorname{span}}_{n\in\mathbb{N}}\{H_n\} \to \overline{\operatorname{span}}_{n\in\mathbb{N}}\{H_n\}$$

such that

$$\chi_{|H_n} = \chi_n$$

is bounded by C*.*

(iii) *For any sequence* $(w_n)_{n\in\mathbb{N}}$ *in the unit ball of* l^∞ *the linear function*

$$\mu\colon \overline{\operatorname{span}}_{n\in\mathbb{N}}\{H_n\} \to \overline{\operatorname{span}}_{n\in\mathbb{N}}\{H_n\}$$

such that $\mu_{|H_n} = w_n\ Id_{H_n}$ *is bounded by* C^2*.*

(iv) *For any finite subset* σ *of* $\mathbb{N}$*, the linear function* $P_\sigma\colon \overline{\operatorname{span}}_{n\in\mathbb{N}}\{H_n\} \to \overline{\operatorname{span}}_{n\in\mathbb{N}}\{H_n\}$ *such that*

$$P_\sigma(x) := \begin{cases} x & \textit{if} \quad x \in H_j, j \in \sigma \\ 0 & \textit{if} \quad x \in H_j, j \notin \sigma \end{cases}$$

is bounded by $\frac{1}{6}\ C^2$*.*

Let $M = (M^1, \ldots, M^d)$ be a d-tuple of matrices of size m, and let ϕ be a function in $\mathrm{H}^\infty(\mathbb{D}^d)$. In order for

$$\phi(M) = \phi(M^1, \ldots, M^d)$$

to makes sense the matrices in M must be **commuting**, so that the value of a bounded analytic function at M can be expressed via the matrix version of Cauchy integral formula:

$$\phi(M) := \frac{1}{(2\pi i)^d} \int_{\mathbb{T}^d} f(\xi^1, \ldots, \xi^d)(\xi^1\ Id - M^1)^{-1} \ldots (\xi^d\ Id - M^d)^{-1}\, d\xi^1 \ldots d\xi^d, \qquad (1.2.5)$$

where $\mathbb{T}^d$ is the d-dimensional torus. As for the one dimensional case, (1.2.5) makes sense provided that the **joint spectrum** of M belongs to $\mathbb{D}^d$. In order to define the joint spectrum of M, recall that commuting matrices preserve each others invariant sub-spaces: consequently, M can be *jointly upper-triangularizable*, that is, there exits an $m \times m$ non singular matrix P such that

$$N^i = P^{-1} M^i P \tag{1.2.6}$$

is upper triangular, for any $i = 1, \dots, m$. The joint spectrum of M is the (multi)-set

$$\sigma(M) = \{(N^1_{j,j}, \dots, N^d_{j,j}) \mid j = 1, \dots, d\}.$$

In particular, $\lambda = (\lambda^1, \dots, \lambda^d)$ belongs to $\sigma(M)$ if and only if there exists a vector x in $\mathbb{C}^m$ that is a joint eigenvector for the matrices in M:

$$M^i x = \lambda^i x \qquad i = 1, \dots, d.$$

The definition of interpolating d-tuples of commuting matrices for bounded analytic functions in the polydisc follows the outline of Definition 1.1.3:

Definition 1.2.5. *A sequence $A = (A_n)_{n \in \mathbb{N}}$ of d-tuples of commuting matrices with joint spectra in $\mathbb{D}^d$ is* **interpolating** *for $\mathrm{H}^\infty(\mathbb{D}^d)$ if for any bounded target sequence $(\phi_n)_{n \in \mathbb{N}}$ in $\mathrm{H}^\infty(\mathbb{D}^d)$ there exists a bounded analytic function ϕ on the polydisc such that*

$$\phi(A_n) = \phi_n(A_n) \qquad n \in \mathbb{N}.$$

In order to separate the matrices in A we will extend (1.2.4) to the matrix setting by defining the Gleason distance ρ_G between two d-tuples of commuting matrices M and N (of eventual different dimension) to be the supremum of all $|r| > 0$ such that there exists a function ϕ in the unit ball of H^∞ so that

$$\phi(N) = 0, \qquad \phi(M) = r\ Id.$$

We will say then that the sequence A is **weakly separated** if

$$\inf_{n \neq j} \rho_G(A_n, A_j) > 0,$$

whereas (1.2.1) extend to define **strongly separated** sequences of commuting d-tuples by

$$\inf_{n \in \mathbb{N}} \prod_{j \neq n} \rho_G(A_n, A_j) > 0.$$

If τ is any subset of $\mathbb{N}$ not containing n, then the distance between A_n and $A_\tau := (A_j)_{j \in \tau}$ is the inverse of the least norm of a function in $\mathrm{H}^\infty(\mathbb{D}^d)$ that vanishes on $(A_j)_{j \in \mathbb{N}}$ and it's the identity on A_n:

$$\rho_G(A_n, A_\tau) := \frac{1}{\inf\{||\phi||_\infty \mid \phi(A_n) = Id, \phi(A_j) = 0, j \in \tau\}}. \tag{1.2.7}$$

As for the one dimensional case, one can relate interpolation conditions for the sequence A to separation condition on well chosen sub-spaces of some reproducing kernel Hilbert spaces, and the main difference with the theory in one variable is that one has to consider *all* the kernels in $\mathcal{B}_d$, rather than only the Szegö kernel. Let then k be a kernel in $\mathcal{B}_d$, and let M be a d-tuple of commuting matrices with joint spectra in $\mathbb{D}^d$. Assume that a function f in $\mathcal{H}_k$ vanishes at M: without loss of generality, we can assume thanks to (1.2.6) that each M^i is upper triangular. Let $\{z_1, \ldots, z_d\}$ be the joint spectrum of M. For any ξ in $\mathbb{T}^d$,

$$(\xi^i\ Id - M^i)^{-1} \tag{1.2.8}$$

has on its main diagonal multiples of

$$\frac{1}{(\xi^i - z^i_j)} \qquad j = 1, \ldots, d$$

while on its second to main diagonal has a linear combination of factors of the form

$$\frac{1}{(\xi^i - z^i_j)^2} \qquad j = 1, \ldots, d,$$

where such linear combination depends exclusively on the algebraic structure of M, and not on the kernel k. In this fashion, the l-th-to-main diagonal of (1.2.8) will contain linear combinations of terms of the form

$$\frac{1}{(\xi^i - z^i_j)^l} \qquad j = 1, \ldots, d.$$

Hence, thanks to (1.2.5), $f(M) = 0$ if and only if f vanishes to its joint spectrum, and so do some linear combinations of its partial derivatives at the points of the joint spectrum.

Example 1.2.6. *Let λ and γ be in $\mathbb{D}$, and set*

$$M = \left(\begin{bmatrix} \lambda & 2 \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} \gamma & 1 \\ 0 & \gamma \end{bmatrix} \right).$$

Then the joint spectrum of M consists in just the point $z = (\lambda, \gamma)$ in $\mathbb{D}^2$ and, for any f holomorphic in $\mathbb{D}^2$,

$$f(M) = \begin{bmatrix} f(z) & 2\partial_1 f(z) + \partial_2 f(z) \\ 0 & f(z) \end{bmatrix}.$$

Thus $f(M) = 0$ if and only both $f(z) = 0$ and $2\frac{\partial f}{\partial z^1}(z) + \frac{\partial f}{\partial z^2}(z) = 0$.

Let $A = (A_n)_{n\in\mathbb{N}}$ be a sequence of commuting d-tuples with joint spectra in $\mathbb{D}^d$. For any positive integer n and for any kernel k in $\mathcal{B}_d$, define

$$H_n^k := \mathcal{H}_k \ominus \{f \in \mathcal{H}_k | f(A_n) = 0\} \subseteq \mathcal{H}_k. \tag{1.2.9}$$

By the above discussion H_n^k is a finite dimensional sub-space of $\mathcal{H}_k$ spanned by the kernels at the points of the joint spectrum of A_n, together with some linear combinations of kernel functions that represent partial derivatives at the joint spectra of A. Moreover, if the matrices in A_n have size s_n, then

$$H_n^k = \{K_{A_n}(u, v) \,|\, u, v \in \mathbb{C}^{s_n}\},$$

where, if $(e_j)_{j\in\mathbb{N}}$ is any orthonormal basis of $\mathcal{H}_k$, $K_{A_n}(u, v)$ is defined by

$$K_{A_n}(u, v) := \sum_{j\in\mathbb{N}} \langle v, e_j(A_n)u\rangle_{\mathcal{H}_k} e_j. \tag{1.2.10}$$

In particular, the collection $K_{A_n}(u, v)$ satisfies

$$\begin{aligned} \langle f, K_{A_n}(u, v)\rangle_{\mathcal{H}_k} &= \sum_{j\in\mathbb{N}} \langle e_j(A_n)u, v\rangle_{\mathbb{C}^{s_n}} \langle f, e_j\rangle_{\mathcal{H}_k} \\ &= \sum_{j\in\mathbb{N}} \left\langle \langle f, e_j\rangle_{\mathcal{H}_k} e_j(A_n)u, v\right\rangle_{\mathbb{C}^{s_n}} \\ &= \left\langle \left(\sum_{j\in\mathbb{N}} \langle f, e_j\rangle_{\mathcal{H}_k} e_j(A_n)\right) u, v \right\rangle_{\mathbb{C}^{s_n}} \\ &= \langle f(A_n)u, v\rangle_{\mathbb{C}^{s_n}} \qquad f \in \mathcal{H}_k, \end{aligned} \tag{1.2.11}$$

which says that (1.2.10) does not depend on the choice of the basis $(e_n)_{n\in\mathbb{N}}$. Equation (1.2.11) works as a *reproducing property* for $K_{A_n}(u, v)$ and implies that $K_{A_n}(u, v)$ is linear in v and conjugate-linear in u. Most importantly,

$$M_\phi^*(K_{A_n}(u, v)) = K_{A_n}(u, \phi(A_n)^* v) \qquad \phi \in \mathrm{H}^\infty(\mathbb{D}^d), \tag{1.2.12}$$

extending (1.1.5) to the multi-dimensional matrix case. It is worth noticing that, for any positive integer n, some of the $K_{A_n}(u,v)$ might repeat while varying u and v. The following Lemma describes the relation on the pairs (u_1,v_1) and (u_2,v_2) in order for this to happen. If l is a multi-index in $\mathbb{N}^d$, and N is a d-tuple of commuting matrices, we will write

$$N^l := (N^1)^{l_1}\dots(N^d)^{l_d}.$$

Lemma 1.2.7. *Let k be a kernel in $\mathcal{B}_d$, and let N be a d-tuple of $m\times m$ commuting matrices with spectra in $\mathbb{D}^d$. Let u_1,v_1,u_2 and v_2 be in $\mathbb{C}^m$. Then $K_N(u_1,v_1)=K_N(u_2,v_2)$ if and only if*

$$\langle N^l u_1, v_1\rangle_{\mathbb{C}^m} = \langle N^l u_2, v_2\rangle_{\mathbb{C}^m} \qquad l\in\mathbb{N}^d. \tag{1.2.13}$$

Proof. Let us first assume that $K_N(u_1,v_1)=K_N(u_2,v_2)$. Thus, since each monic monomial z^l belongs to $\mathrm{H}^\infty(\mathbb{D}^d)$ and thanks to (1.2.12)

$$K_N(u_1,(N^l)^*v_1) = M^*_{z^l}(K_N(u_1,v_1)) = M^*_{z^l}(K_N(u_2,v_2)) = K_N(u_2,(N^l)^*v_2).$$

Therefore, if ϕ is a polynomial such that $\phi(N)=Id$, the reproducing property in (1.2.11) implies that

$$\langle u_1,(N^l)^*v_1\rangle_{\mathbb{C}^m} = \langle \phi, K_N(u_1,(N^l)^*v_1)\rangle_{\mathcal{H}_k} = \langle \phi, K_N(u_2,(N^l)^*v_2)\rangle_{\mathcal{H}_k} = \langle u_2,(N^l)^*v_2\rangle_{\mathbb{C}^m}.$$

Conversely, assume that (1.2.13) holds. We have that $K_N(u_1,v_1)=K_N(u_2,v_2)$ if and only if their inner product with any f in $\mathcal{H}_k$ coincide:

$$\langle u_1, f(N)^*v_1\rangle_{\mathbb{C}^m} = \langle f, K_N(u_1,v_1)\rangle_{\mathcal{H}_k} = \langle f, K_N(u_2,v_2)\rangle_{\mathcal{H}_k} = \langle u_1, f(N)^*v_1\rangle_{\mathbb{C}^m} \qquad f\in\mathcal{H}_k.$$

The rightmost and the leftmost hand side of the above equation coincide, thanks to (1.2.13) and modulo writing f in its power series. □

Observe that (1.2.13) does **not** depend on the kernel k that we chose in $\mathcal{B}_d$, but it depends exclusively on the algebraic structure of M.

Example 1.2.8. *Let M be the pair of commuting matrices in Example 1.2.6. If e_1 and e_2 are the elements of the canonical basis of $\mathbb{C}^2$, then*

$$K_M(e_1, e_1) = K_M(e_2, e_2) = k_z,$$
$$K_M(e_1, e_2) = 2k_z^{(1,0)} + k_z^{(0,1)}$$
$$K_M(e_2, e_1) = 0,$$

where

$$k_z^{(1,0)} := \frac{\partial k_z}{\partial \overline{z}^1}$$

and

$$k_z^{(0,1)} := \frac{\partial k_z}{\partial \overline{z}^2}$$

are the kernels in $\mathcal{H}_k$ that represent, respectively, one derivative in the first variable and one derivative in the second variable.

In Chapter 3, we will extend Theorem 1.2.3 to pairs of commuting matrices by replacing admissible kernels with the collections $(H_n^k)_{n\in\mathbb{N}}$:

Theorem 1.2.9. *A sequence of pairs of commuting matrices is interpolating if and only if there exists a positive C such that, for any admissible kernel k on the bi-disc, the sequence*

$$H^k = (H_n^k)_{n\in\mathbb{N}}$$

is a Riesz system in $\mathcal{H}_k$ with Riesz bound C.

Since the approach we use to study interpolating sequences of matrices is mainly defined by looking at the Euclidean geometry of reproducing kernel Hilbert spaces, and since we do not know whether $\mathcal{A}_d = \mathcal{B}_d$ for $d \geq 3$, we can only give a (stronger) sufficient condition to replace (1.2.1) in the case of the bi-disc:

Theorem 1.2.10. *Let $A = (A_n)_{n\in\mathbb{N}}$ be a sequence of pairs of commuting matrices with spectra in the bi-disc such that*

$$\prod_{n\in\mathbb{N}} \rho_G(A_n, A_{\mathbb{N}\setminus n}) > 0. \tag{1.2.14}$$

Then A is interpolating.

Let us observe that (1.2.14) is a rather strong separation condition on $A = (A_n)_{n\in N}$. In order to extend the sufficient condition in Theorem 1.2.1, one has to show that if

$$\inf_{n\in\mathbb{N}} \prod_{j\neq n} \rho_G(A_n, A_j) > 0,$$

then A is interpolating. This remain, for us, an open question.

1.3 Interpolating d-tuples of Non-Commuting Matrices

In order to study interpolating properties of eventually non-commuting d-tuples of matrices, one has to change the class of functions to apply to such a sequence. Specifically, the robust and highly active field of **non-commutative (NC) functions** is the environment in which the discussion of Chapter 4 will take place.

Fix d in $\mathbb{N} \cup \{\infty\}$. For any positive integer n, let $\mathbb{M}_n^1$ be the set of all $n \times n$ matrices with coefficient in $\mathbb{C}$ and, more generally, let $\mathbb{M}_n^d$ be the set of all d-tuples of square matrices of size n. If $d = \infty$, we require that the row norm

$$||X|| := ||\sum_i X^i (X^i)^*|| \tag{1.3.1}$$

is bounded for any X in $\mathbb{M}_n^\infty$. Each $\mathbb{M}_n^d$ is normed with (1.3.1) and endowed with the induced topology. Let us consider now arbitrarily large sizes by defining

$$\mathbb{M}_d := \bigcup_{n=1}^{\infty} \mathbb{M}_n^d$$

as the disjoint union in n of d-tuples of $n \times n$ matrices. A topology of interest for us is the so-called *disjoint union (DU) topology*: a subset Ω of $\mathbb{M}_d$ is open if and only if all of its n components

$$\Omega(n) := \Omega \cap \mathbb{M}_n^d \qquad n \in \mathbb{N}$$

are open in $\mathbb{M}_n^d$. Moreover, Ω is a **non-commutative (NC) set** if it is closed under direct sums:

$$Z, W \in \Omega \implies Z \oplus W \in \Omega.$$

A class of functions that can be defined on such an Ω is $\mathbb{C}\langle z_1, \dots, z_d \rangle$, the set of all *free polynomials* in d non-commuting variables. More generally, a function $f \colon \Omega \to \mathbb{M}_1$ is a **non-commutative (NC) function** if

- f is *graded*: if Z is in $\Omega(n)$, then $f(Z)$ belongs to $\mathbb{M}_n^1$;
- f *respects direct sums*: for any Z and W in Ω, then $f(Z \oplus W) = f(Z) \oplus f(W)$;
- f *respects similarities*: for any Z in $\Omega(n)$ and for any invertible P in $\mathbb{M}_n^1$, then $f(P^{-1}ZP) = P^{-1}f(Z)P$, provided that $P^{-1}ZP$ belongs to Ω.

One can easily check that any NC polynomial is an NC function on Ω. An NC function f is *holomorphic* on Ω if it is locally bounded.

The domain of interest for us in order to extend interpolation results to this NC setting is the **NC unit ball**

$$\mathfrak{B}_d := \left\{ Z \in \mathbb{M}_d \,\middle|\, ||Z|| < 1 \right\},$$

and the interpolating functions will be chosen from the algebra of **bounded NC analytic functions**

$$\mathcal{H}^\infty_d := \{ f \quad \text{NC holomorphic on}\, \mathfrak{B}_d \quad | \quad ||f||_\infty < \infty \},$$

where

$$||f||_\infty := \sup_{Z \in \mathfrak{B}_d} ||f(Z)||.$$

Definition 1.1.3 adapts also to this non-commutative setting: a sequence $(Z_n)_{n\in\mathbb{N}}$ in $\mathfrak{B}_d$ is **interpolating** for $\mathcal{H}^\infty_d$ if for any bounded sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{H}^\infty_d$ there exists a function ϕ in $\mathcal{H}^\infty_d$ such that

$$\phi(Z_n) = \phi_n(Z_n) \qquad n \in \mathbb{N}.$$

As for the well-known scalar case, the understanding of NC interpolating sequences goes through the study of related Hilbert spaces. Let $\mathbb{W}_d$ be the set of words with d generators, and let

$$\mathcal{H}^2_d := \left\{ f = \sum_{l \in \mathbb{W}_d} a_l Z^l \quad \middle| \quad ||f||_2^2 := \sum_{l \in \mathbb{W}_d} |a_l|^2 < \infty \right\}$$

be the so-called *NC Drury Arveson space* of those formal NC power series with squared-summable coefficients. It turns out that $\mathcal{H}^2_d$ can be seen as an **NC reproducing kernel Hilbert space**. More specifically, for any W in $\mathfrak{B}_d(n)$ and for any pairs of vectors u and v in $\mathbb{C}^n$, let

$$K_W(u,v)(Z) := \sum_{l \in \mathbb{W}_d} \langle v, W^l u \rangle \, Z^l \qquad Z \in \mathfrak{B}. \tag{1.3.2}$$

The function $K_W(u,v)$ is linear in v and conjugate-linear in v. Salomon, Shalit and Shamovich showed in [22, Prop. 3.2] that $\mathcal{H}^2_d$ is generated by the NC functions in (1.3.2), and that for any f in $\mathcal{H}^2_d$ and for any W in $\mathfrak{B}_d(n)$

$$\langle f, K_W(u,z)\rangle_{\mathcal{H}^2_d} = \langle f(W)u, v\rangle_{\mathbb{C}^n}\,. \tag{1.3.3}$$

Moreover, the multiplier algebra of $\mathcal{H}^2_d$

$$\mathcal{M}_{\mathcal{H}^2_d} := \{\phi \,\text{NC holomorphic function on}\, \mathfrak{B}_d \,|\, \phi f \in \mathcal{H}^2_d, f \in \mathcal{H}^2_d\}$$

is isometrically identifiable with $\mathcal{H}^\infty_d$, [22, Coro. 3.6], and the reproducing property in (1.3.3) implies that

$$M^*_\phi(K_W(u,v)) = K_W(u, \phi(W)^*v) \qquad \phi \in \mathcal{H}^\infty_d.$$

Similarly to Chapter 2 and Chapter 3, in Chapter 4 we will describe interpolating non-commutative sequences in terms of *separated sequences*. Moreover, we will see how to separated elements in $\mathfrak{B}_d$ correspond separated sub-spaces of $\mathcal{H}^2_d$. Specifically, the *NC Gleason distance* between two elements Z and W of $\mathfrak{B}_d$ is the inverse of the least norm of an $\mathcal{H}^\infty_d$ NC function that vanishes at W and that is the identity at Z:

$$\rho_{NC}(Z,W) := \frac{1}{\inf\{||\phi||_\infty \,|\, \phi(Z) = Id, \phi(W) = 0\}}.$$

As in (1.2.7), we might also want to separate Z from a whole sequence $(W_n)_{n\in\mathbb{N}}$ via

$$\rho_{NC}(Z,(W_n)_{n\in\mathbb{N}}) := \frac{1}{\inf\{||\phi||_\infty \,|\, \phi(Z) = Id, \phi(W_n) = 0, n \in \mathbb{N}\}}.$$

We will see in Chapter 4 how an analogous of the Pick property for the NC Drury-Arveson space implies that separation conditions on a sequence $(Z_n)_{n\in\mathbb{N}}$ in $\mathcal{B}_d$ correspond to separated sequences of sub-spaces of $\mathcal{H}^2_d$ defined as

$$\mathcal{H}_n := \mathcal{H}^2_d \ominus \{f \in \mathcal{H}^2_d \,|\, f(Z_n) = 0\}. \tag{1.3.4}$$

Since the right hand side of the reproducing property (1.3.3) vanishes for any u and v if and only if $f(W) = 0$, we have that,

$$\mathcal{H}_n = \{K_{Z_n}(u, v) \,|\, u, v \in \mathbb{C}^{m_n}\}, \tag{1.3.5}$$

where Z_n belongs to $\mathfrak{B}_d(m_n)$ for any n.

Interpolating sequences can be then characterized via separation conditions in $\mathcal{H}^2_d$:

Theorem 1.3.1. *Z is interpolating if and only if the sequence $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is a Riesz system in $\mathcal{H}^2_d$.*

As a consequence, we will show that a separation condition on the sequence Z is sufficient for it to be interpolating:

Theorem 1.3.2. *If*

$$\prod_{n\in\mathbb{N}} \rho_{NC}(Z_n, (Z_j)_{j\neq n}) > 0 \tag{1.3.6}$$

then Z is interpolating.

Condition (1.3.6) is a rather strong separation condition on the sequence Z. It would be interesting for us to know whether such condition can be relaxed:

Question 1. *Is any sequence Z in $\mathfrak{B}_d$ satisfying*

$$\inf_{n\in\mathbb{N}} \prod_{j\neq n} \rho_{NC}(Z_n, Z_j) > 0$$

an interpolating sequence?

A positive answer to Question 1 would extend a result from Berndtsson in [6] to this non commutative setting.

An even more ambitious goal would be to prove an analogous of Carleson interpolation Theorem for the noncommutative setting. In particular, we ask whether strongly separated sequences are interpolating, and whether weakly separated sequences whose sequence $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is a Bessel system are interpolating:

Question 2. *Is Z interpolating, provided that*

$$\inf_{n\in\mathbb{N}} \rho_{NC}(Z_n, (Z_j)_{j\neq n}) > 0?$$

Question 3. *Is Z interpolating, provided that*

$$\inf_{n\neq j} \rho_{NC}(Z_n, Z_j) > 0$$

and $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is a Bessel system?

Chapter 2

Interpolating Matrices for Bounded Analytic Functions on the Unit Disc

The main goal of this Chapter is to prove Theorem 1.1.4.

Section 2.1 will give a definition for interpolating sequences of matrices that uses only diagonal targets (namely, constant sequences $(\phi_n)_{n\in\mathbb{N}}$) which is equivalent to Definition 1.1.3, and it will show that any interpolating sequence of matrices has a so-called sequence of *P. Beurling* functions, extending a result in [23] to the matrix setting.

Section 2.2 will realize the sequence $(H_n)_{n\in\mathbb{N}}$ in (1.1.11) as a sequence of *model spaces* associated to Blaschke products, and it will translate separation conditions on the sequence A as separation conditions on the sine of the angles between the sub-spaces in H.

Section 2.3 will show that, given any sequence of positive integers $(m_n)_{n\in\mathbb{N}}$, there exists a sequence of interpolating matrices with dimensions in $(m_n)_{n\in\mathbb{N}}$.

Finally, Section 2.4 will take care of the proof of Theorem 1.1.4, and Section 2.5.1 will give an example of an interpolating sequence of matrices with equi-distributed eigenvalues. We will also show in Section 2.5.2 that there exists a sequence of matrices such that the associated sequence $(H_n)_{n\in\mathbb{N}}$ of subspaces in H^2 is a Bessel system which can not be partitioned into finitely many Riesz systems, answering a question that the author posed in [13] about extending the positive answer to the Feichtinger conjecture to multi-dimensional closed subspaces of H^2 that arise from sequences of matrices.

The content of this Chapter is based on [13] and [14].

2.1 P. Beurling Functions and Diagonal Targets

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$. An elegant way to solve any interpolation problem with bounded targets $(w_n)_{n\in\mathbb{N}}$ is to construct a sequence of functions $(f_n)_{n\in\mathbb{N}}$ in H^∞ such that $f_n(z_j) = \delta_{n,j}$, and so that

$$\sup_{z\in\mathbb{D}} \sum_{n\in\mathbb{N}} |f_n(z)| < \infty. \tag{2.1.1}$$

Given such a sequence $(f_n)_{n\in\mathbb{N}}$, one can indeed observe that $f := \sum_{n\in\mathbb{N}} w_n f_n$ is a bounded analytic function that interpolates $(z_n)_{n\in\mathbb{N}}$ with $(w_n)_{n\in\mathbb{N}}$.

The functions $(f_n)_{n\in\mathbb{N}}$ are usually called **P. Beurling functions** associated to $(z_n)_{n\in\mathbb{N}}$. In [17, Ch. VII, Th. 2.2], one can find a proof of the existence of the P. Beurling functions for any interpolating sequence on a uniform algebra of holomorphic function on a compact set. It turns out that the same proof adapts to the case of a sequence $(A_n)_{n\in\mathbb{N}}$ of interpolating matrices. This implies that definition 1.1.3 can be re-stated in terms of only *diagonal targets*:

Theorem 2.1.1. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of matrices with spectra in the unit disc. The following are equivalent:*

(i) *A is interpolating;*

(ii) *For any bounded sequence $(w_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ there exists a function ϕ in H^∞ such that*

$$\phi(A_n) = w_n\ Id, \qquad n \in \mathbb{N};$$

(iii) *There exists a sequence $(f_n)_{n\in\mathbb{N}}$ in H^∞ so that (2.1.1) holds and $f_n(A_j) = \delta_{n,j}\ Id$.*

The proof of (i) $\Longrightarrow$ (ii) is trivial, as condition (ii) corresponds to the case in which the sequence $(\phi_n)_{n\in\mathbb{N}}$ in Definition 1.1.3 is constant, that is,

$$\phi_n(z) = w_n \qquad z \in \mathbb{D}.$$

We will say that a sequence A satisfying (ii) is **weakly interpolating**. Moreover, condition (iii) allows one to interpolate any bounded target sequence $(\phi_n)_{n\in\mathbb{N}}$ in H^∞: it suffices to set

$$f := \sum_{n\in\mathbb{N}} f_n\phi_n \in \mathrm{H}^\infty,$$

and $f(A_n) = \phi_n(A_n)$, for any n. Therefore (iii) $\Longrightarrow$ (i), and Theorem 2.1.1 will be proved once we show (ii) $\Longrightarrow$ (iii).

Before doing so, we define the *constant of interpolation* for a sequence $A = (A_n)_{n\in\mathbb{N}}$ of interpolating matrices. Consider the Banach space

$$\mathcal{L}^\infty := \left\{(\phi_n)_{n\in\mathbb{N}} \subset \mathrm{H}^\infty \mid \sup_{n\in\mathbb{N}} ||\phi_n||_\infty < \infty\right\},$$

and define the equivalence relation $\sim_A$ on $\mathcal{L}^\infty$ so that two sequences are equivalent if and only if they agree on A, that is,

$$(\phi_n)_{n\in\mathbb{N}} \sim_A (g_n)_{n\in\mathbb{N}} \qquad \Longleftrightarrow \qquad \phi_n(A_n) = g_n(A_n), \qquad n \in \mathbb{N}.$$

Set $\mathcal{L}^\infty_A := \mathcal{L}^\infty / \sim_A$, and observe that the linear contraction

$$E_A : h \in \mathrm{H}^\infty \mapsto [(h)_{n\in\mathbb{N}}]_{\sim_A} \in \mathcal{L}^\infty_A$$

is surjective, since A is interpolating. The open map Theorem implies that there exists a constant C so that, if $\sup_{n\in\mathbb{N}} ||\phi_n||_\infty \leq 1$, and $\phi(A_n) = \phi_n(A_n)$ for any n, one has

$$||\phi||_\infty \leq C.$$

The least C that works is denoted by *interpolation constant* of the sequence A.

In the same fashion, the *weak interpolation constant* is the least M given by the open map

Theorem in case A is weakly interpolating, being in that case

$$E'_A \colon h \in \mathrm{H}^\infty \mapsto (w_n \ Id)_{n\in\mathbb{N}} \in l^\infty$$

surjective (modulo the obvious identification of such diagonal operators with l^∞).

Theorem 2.1.1 now follows from a normal family argument and the following Proposition:

Proposition 2.1.2. *Let A be a weakly interpolating sequence, and M be its weak interpolation constant. Then, for any n in $\mathbb{N}$ and for any $\varepsilon > 0$, there exist f_1, f_2, ..., f_n in H^∞ so that*

$$f_i(A_j) = \delta_{i,j} \cdot \mathrm{Id} \qquad i, j = 1, \dots, n$$

and

$$\sup_{z\in\mathbb{D}} \sum_{j=1}^{n} |f_j(z)| \le M^2 + \varepsilon.$$

The proof is a straightforward adaptation of the elegant argument in [17, h. VII, Th. 2.2]:

Proof. Fix a positive integer n, a positive ε, and let ω be a primitive n-th root of unity in $\mathbb{C}$. For any $j = 1, \dots, n$, solve the the interpolating problem with constant data

$$\phi_{j,k} = \begin{cases} \omega^{jk} & \text{if } 1 \le k \le n \\ 0 & \text{if } k > n \end{cases},$$

and find then, for any positive γ, a function g_j so that $||g_j||_\infty \le M + \gamma$ and

$$g_j(A_k) = \omega^{jk} \ \mathrm{Id}, \qquad j, k = 1, \dots, n.$$

Define

$$f_j(z) := \left(\frac{1}{n} \sum_{k=1}^{n} \omega^{-jk} g_k(z) \right)^2,$$

and since ω is a primitive n-th root of unity one can check that $f_j(A_k) = \delta_{j,k}$ Id. Moreover, for any z in $\mathbb{D}$,

$$\begin{aligned}
\sum_{j=1}^{n} |f_j(z)| &= \frac{1}{n^2} \sum_{j=1}^{n} \left(\sum_{l,k=1}^{n} \omega^{(l-k)j} g_k(z) \overline{g_l(z)} \right) \\
&= \frac{1}{n^2} \sum_{k,l=1}^{n} g_k(z) \overline{g_l(z)} \sum_{j=1}^{n} \omega^{(l-k)j} \\
&= \frac{1}{n^2} \sum_{k=1}^{n} n |g_k(z)|^2 \leq (M+\gamma)^2 \leq M^2 + \varepsilon,
\end{aligned}$$

if γ is chosen small enough. □

Remark 2.1.3. *Although weak interpolating sequences and interpolating sequences of matrices coincides, the constants of interpolation might not. Nevertheless, the argument of Proposition 2.1.2 shows that the weak interpolation constant is at most the square of the interpolation constant.*

In [23], Vinogradov, Gorin and Khruschen deduced the existence of the P. Beurling functions for a sequence $Z = (z_n)_{n \in \mathbb{N}}$ in $\mathbb{D}$ by only assuming that Z is strongly separated. It would be interesting for us to find a similar argument for a matrix interpolation problem.

2.2 Separated Model Spaces

A key role for the study of the function theory and the hyperbolic geometry of the unit disc is played by *inner functions*, which are those bounded analytic functions on the unit disc with an unimodular radial limit almost everywhere on the unit circle. Given an inner function Θ, one can define the associated *model space*

$$H_\Theta := \mathrm{H}^2 \ominus \Theta\mathrm{H}^2$$

as the orthogonal complement in the Hardy space of all multiples of Θ in H^2. A great treatment of the main properties of model spaces, together with their interactions with operator theory on spaces of analytic functions, can be found in [16].

Example 2.2.1. *Any Blaschke factor is an inner function, and since any function in* H^2 *that vanishes with multiplicity* m *at a point* z *in* $\mathbb{D}$ *is divisible in* H^2 *by the Blaschke factor* b_z^m*, the model space associated to* b_z^m *is* m*-dimensional and it is spanned by the kernels at* z *that represent up to* $m-1$ *derivatives of any function of* H^2 *at* z*, that is,*

$$H_{b_z^m} = \operatorname{span}\left\{ s_z, \frac{\partial\ s_z}{\partial \overline{z}}, \dots, \frac{\partial^{m-1}\ s_z}{\partial \overline{z}^{m-1}} \right\},$$

where

$$\left\langle f, \frac{\partial^m\ s_w}{\partial \overline{w}^m} \right\rangle = f^{(m)}(w) \qquad f \in \mathrm{H}^2.$$

In order to characterize interpolating sequences of matrices, a rather trivial yet important observation is that, for any pair of similar matrices M and N with spectra in $\mathbb{D}$ and for any holomorphic function f on the unit disc then $f(M)$ and $f(N)$ are similar as well, and the matrix that performs both similarities is the same,

$$M = P^{-1}NP \implies f(M) = P^{-1}f(N)P,$$

as an elementary computation using the power series of f shows. As a consequence, we can assume without loss of generality that each matrix of the sequence A is in its *Jordan* *canonical form*: if, for any positive integer n, $z_{n,1}, \ldots, z_{n,k_n}$ are the eigenvalues of A_n, then

$$A_n = \operatorname{diag}(J_{n,1}, \ldots, J_{n,k_n}),$$

where $J_{n,j}$ is a Jordan block of size $m_{n,j}$

$$J_{n,j} = \begin{bmatrix} z_{n,j} & 1 & 0 & \ldots & 0 \\ 0 & z_{n,j} & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & z_{n,j} & 1 \\ 0 & 0 & 0 & 0 & z_{n,j} \end{bmatrix}.$$

Since, for any function f holomorphic in $\mathbb{D}$,

$$f(A_n) = \operatorname{diag}(f(J_{n,1}), \ldots, f(J_{n,k_n})),$$

where

$$f(J_{n,j}) = \begin{bmatrix} f(z_{n,j}) & f'(z_{n,j}) & \frac{f''(z_{n,j})}{2} & \ldots & \frac{f^{(m_{n,j}-1)}(z_{n,j})}{(m_{n,j}-1)!} \\ 0 & f(z_{n,j}) & f'(z_{n,j}) & \ldots & \frac{f^{(m_{n,j}-2)}(z_{n,j})}{(m_{n,j}-2)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & f(z_{n,j}) & f'(z_{n,j}) \\ 0 & 0 & 0 & 0 & f(z_{n,j}) \end{bmatrix},$$

one realizes that a holomorphic function vanishes at A_n if and only if it vanishes at its eigenvalues with the right multiplicity. Specifically, the multiplicity of $z_{n,j}$ as a zero of f must be the maximal size $m_{n,j}$ of a Jordan block of A_n associated to $z_{n,j}$. We will call such $m_{n,j}$ the **order** of $\lambda_{n,j}$ as an eigenvalue of A_n. In particular, any function in H^2 that vanishes at A_n is a multiple of the Blaschke product

$$B_n = \prod_{j=1}^{k_n} b_{z_{n,j}}^{m_{n,j}}. \tag{2.2.1}$$

Observe also that B_n can be constructed from the minimal polynomial P_n of A_n, since

$$B_n(z) = \frac{P_n(z)}{\overline{P}\left(\frac{1}{\overline{z}}\right)}.$$

Hence, for any n in $\mathbb{N}$, the subspace

$$\begin{aligned} H_n :=& \mathrm{H}^2 \ominus \{f \in \mathrm{H}^2 \mid f(A_n) = 0\} \\ =& \mathrm{span}\left\{ s_{z_{n,j}}, \frac{\partial\ s_{z_{n,j}}}{\partial \overline{w}}, \ldots, \frac{\partial^{m_{n,j}-1}\ s_{z_{n,j}}}{\partial \overline{w}^{m_{n,j}-1}} \,\middle|\, j = 1, \ldots, k_n \right\} \end{aligned} \tag{2.2.2}$$

containing all the interpolation information of the matrix A_n is in fact a model space:

$$H_n = H_{B_n} \qquad n \in \mathbb{N}.$$

Each H_n can be seen also as a *kernel at the matrix* A_n. More precisely, let M be a $m \times m$ square matrix with eigenvalues in the unit disc, and let H_M be the associated model space in H^2. Let us define, for any u and v in $\mathbb{C}^m$, the H^2 function

$$K_M(u, v)(z) := \sum_{n \in \mathbb{N}} \langle v, M^n u \rangle_{\mathbb{C}^m} z^n \qquad z \in \mathbb{D}.$$

Then, thanks to the definition of the inner product in H^2,

$$\begin{aligned}\langle f, K_M(u,v)\rangle_{\mathrm{H}^2} &= \sum_{n\in\mathbb{N}} \langle M^n u, v\rangle_{\mathbb{C}^m} \langle f, z^n\rangle_{\mathrm{H}^2} \\ &= \sum_{n\in\mathbb{N}} \langle \langle f, z^n\rangle_{\mathrm{H}^2} M^n u, v\rangle_{\mathbb{C}^m} \\ &= \left\langle \left(\sum_{n\in\mathbb{N}} \langle f, z^n\rangle_{\mathrm{H}^2} M^n \right) u, v \right\rangle_{\mathbb{C}^m} \\ &= \langle f(M)u, v\rangle_{\mathbb{C}^m} \qquad f \in \mathrm{H}^2.\end{aligned} \tag{2.2.3}$$

Equation (2.2.3) works as a *reproducing property* for the collection

$$X_M := \{K_M(u,v) \,|\, u, v \in \mathbb{C}^m\}.$$

In particular, since $f(A) = 0$ if and only if the right hand side of (2.2.3) vanishes for any u and v, we have that X_M coincides with the model space H_M. Moreover, (2.2.3) implies that the collection of functions in X_M is linear in v, conjugate-linear in u, and that for any ϕ in H^∞

$$M_\phi^*(K_M(u,v)) = K_M(u, \phi(M)^* v) \qquad u, v \in \mathbb{C}^m, \tag{2.2.4}$$

extending (1.1.5) to the matrix setting. In particular, if $M = w$ is a point in $\mathbb{D}$, then u and v are scalars and $K_M(u,v) = \overline{u}v\ s_w$ is a multiple of the Szegö kernel at w.

Another analogy with the scalar case comes from separation, since the Pick property of the Szegö kernel extends to this matrix-nodes setting. One way to see that is by applying the commutant lifting Theorem [3, Th. 10.29]: suppose that two collections of matrices $A = (A_i)_{i\in\mathbb{N}}$ and $B = (B_j)_{j\in\mathbb{N}}$ can be separated by a multiplier ϕ in H^∞:

$$\phi(A_i) = Id \quad i \in \mathbb{N} \qquad \phi(B_j) = 0 \quad j \in \mathbb{N},$$

and suppose that $||\phi||_\infty$ does not exceed $1/\delta$. Then M_ϕ^* is an operator of norm less than or equal to $1/\delta$ that restricts, thanks to (2.2.4), to the identity on $H_A := \overline{\text{span}}_{i\in\mathbb{N}}\{H_{A_i}\}$ and to the zero operator on $H_B := \overline{\text{span}}_{j\in\mathbb{N}}\{H_{B_j}\}$. Therefore, the sine of the angle between H_A and H_B is bounded below by δ. Conversely, if $\sin(H_A, H_B) \geq \delta$, then the operator $R\colon \text{span}\{H_A, H_B\} \to \text{span}\{H_A, H_B\}$ such that

$$R_{|H_A} = Id_{H_A} \qquad R_{|H_B} = 0$$

satisfies $||R|| \leq 1/\delta$ and, most importantly, *it commutes with the adjoint of multiplication by* z

$$M_z^*\colon K_M(u,v) \mapsto K_M(u, M^*v).$$

By the commutant lifting Theorem, R extends to an operator $\tilde{R}$ on H^2 bounded by $1/\delta$ and that commutes with M_z^*. Such an operator has all the Szegö kernels as its eigenvectors, and in particular it is the adjoint of multiplication by a multiplier ϕ. This gives a bounded analytic function ϕ of norm not exceeding $1/\delta$ that separates the sequences A and B. Therefore, (1.1.8) extends by saying that $H = (H_n)_{n\in\mathbb{N}}$ is **weakly separated** if

$$\inf_{j\neq n} \sin(H_j, H_n) > 0 \tag{2.2.5}$$

or, equivalently, if A is weakly separated. Following the same idea, $(H_n)_{n\in\mathbb{N}}$ is **strongly separated** if

$$\inf_{n\in\mathbb{N}} \sin(H_n, \overline{\text{span}}_{j\neq n}\{H_j\}) > 0,$$

which is equivalent to asserting that A is strongly separated.

The scalar case has an even more geometric viewpoint on separation via bounded analytic functions: given two points z and w in the unit disc there exists a function ϕ whose H^∞ norm doesn't exceed M that separates z and w (and hence the sine of the angle between s_z and s_w is bounded below by $1/M$) if and only if their pseudo-hyperbolic distance $\rho(z,w)$

is bounded below by $1/M$. This extends to the matrix case by looking at the action of the adjoint of the multiplication by a Blaschke product on different model spaces:

Lemma 2.2.2. *Let A_1 and A_2 be two square matrices corresponding to the Blaschke products B_1 and B_2, and let $H_1 := H_{B_1}$ and $H_2 := H_{B_2}$ be the associated model spaces. Then the sine of the angle between H_1 and H_2 is equal to $\delta > 0$ if and only if the restriction of $M^*_{B_1}$ to H_2 is bounded below by δ, that is,*

$$\inf_{x \in H_2} ||M^*_{B_1}(\hat{x})|| = \sin(H_1, H_2).$$

Proof. The sine of the angle between H_1 and H_2 is equal to δ if and only if the least H^∞ norm of a function ϕ such that

$$(M^*_\phi)_{|H_1} = 0$$

and

$$(M^*_\phi)_{|H_2} = Id_{H_2}$$

is $1/\delta$. Any such ϕ vanishes on A_1, which is equivalent to asserting that there exists a function g in H^∞ such that $||g||_\infty = 1/\delta$ and

$$\phi = B_1 \; g. \tag{2.2.6}$$

Since

$$Id_{H_2} = (M^*_\phi)_{|H_2} = (M^*_{B_1})_{|H_2} \; (M^*_g)_{|H_2},$$

equation (2.2.6) says that $(M^*_{B_1})_{|H_2}$ has an inverse bounded by $1/\delta$, as $||M^*_g|| = ||g||_\infty = 1/\delta$. Conversely, if $M^*_{B_1}$ admits an inverse T bounded by $1/\delta$ on H_2, the Pick property of the

Szegö kernel says that T extends isometrically to some M_g^*, and thanks to (2.2.4)

$$B_1(A_2)\ g(A_2) = Id,$$

as $M_{B_1 g}^*$ acts like the identity on H_2. □

2.3 Existence of Interpolating Matrices

Since a model space is a subspace of H^2 generated by an inner function, it comes natural to ask whether function theoretical properties of a sequence of inner functions $(\Theta_n)_{n\in\mathbb{N}}$ translate to Euclidean properties for the sequence $(H_{\Theta_n})_{n\in\mathbb{N}}$. Out of the many results that constitute such a valuable dictionary between operator theory and function theory, one of the most significant for the purpose of this work can be found in [19, Th. 3.2.14]:

Theorem 2.3.1. *Let $(\Theta_n)_{n\in\mathbb{N}}$ be a sequence of inner functions such that $\prod_{n=1}^{\infty}\Theta_n$ converges uniformly on any compact subset of $\mathbb{D}$. The following are equivalent:*

(i) *For any bounded sequence $(\phi_n)_{n\in\mathbb{N}}$ in H^∞ there exists a function ϕ in H^∞ such that*

$$\phi - \phi_n \in \Theta_n \mathrm{H}^2, \qquad n \in \mathbb{N}; \tag{2.3.1}$$

(ii) *$(H_{\Theta_n})_{n\in\mathbb{N}}$ is a Riesz system;*

(iii) *There exists a positive δ such that, for any z in $\mathbb{D}$,*

$$\sup_{n\in\mathbb{N}} \prod_{j\neq n} |\Theta_j(z)| \geq \delta. \tag{2.3.2}$$

Condition (iii) is a function theoretical property on the inner functions $(\Theta_n)_{n\in\mathbb{N}}$ whose importance can not be overstated, being equivalent, thanks to Carleson's corona Theorem,

[9], to the existence of two bounded sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in H^∞ such that

$$f_n\Theta_n + g_n \cdot \prod_{j\neq n} \Theta_j = 1 \qquad n \in \mathbb{N}.$$

Moreover, condition (iii) is related in [20, Lec. IX] to separation conditions on the subspaces in $(H_{\Theta_n})_{n\in\mathbb{N}}$:

Theorem 2.3.2. *There exists a constant $c \geq 1$ such that, for any Θ_1 and Θ_2 inner functions on $\mathbb{D}$ satisfying*

$$\inf_{z\in\mathbb{D}} \max\{|\Theta_1(z)|, |\Theta_2(z)|\} = \delta \geq 0,$$

then

$$\frac{\delta^3}{c} \leq \sin(H_{\Theta_1}, H_{\Theta_2}) \leq c\delta.$$

In particular, (iii) says that $(H_{\Theta_n})_{n\in\mathbb{N}}$ is strongly separated.

Suppose now that $\Theta_n = b_{z_n}^{m_n}$, for some sequence $(z_n)_{n\in\mathbb{N}}$ in the unit disc and some sequence $(m_n)_{n\in\mathbb{N}}$ of positive integers. Condition (i) of Theorem 2.3.1 becomes then an *interpolation property*: for any bounded sequence $(\phi_n)_{n\in\mathbb{N}}$ in H^∞ there exists a bounded analytic function ϕ that, for any n in $\mathbb{N}$, agrees with ϕ_n at z_n up to its $m_n - 1^{\text{st}}$ derivative. Theorem 2.3.1 is therefore a great example of how the deep interconnection between operator theory and function theory greatly helps the studying and the understanding of interpolating sequences. Let A be a sequence of matrices with spectra in the unit disc, and let H be its associated sequence of model spaces. If, for any positive integer n, Θ_n is the Blaschke factor B_n defined in (2.2.1), then Theorem 2.3.1, together with the discussion in Section 2.2 on the correspondence between strongly separated matrices and strongly separated model spaces, proves the following

Theorem 2.3.3. *The following are equivalent:*

(i) *A is interpolating;*

(ii) *H is a Riesz system;*

(iii) *A is strongly separated.*

(iv) *There exists a positive δ such that, for any z in $\mathbb{D}$,*

$$\sup_{n\in\mathbb{N}} \prod_{j\neq n} |B_j(z)| \geq \delta. \tag{2.3.3}$$

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$. Note that no such sequence is strongly (or weakly) separated if some terms are repeated. If we want to take into account multiplicities, a natural generalization for strong separation is

$$\inf_{n\in\mathbb{N}} \prod_{j\neq n} \rho(z_n, z_j)^{m_j} > 0, \tag{2.3.4}$$

where m_j is the multiplicity of z_j, and ρ is the pseudo-hyperbolic distance in (1.1.1). A more demanding generalization of (1.1.2) is given by the case $\Theta_n = b_{z_n}^{m_n}$ of (2.3.2), :

Definition 2.3.4 (Uniform Strong Separation)**.** *Let $Z = (z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$, with a given sequence $(m_n)_{n\in\mathbb{N}}$ of multiplicities. Then Z is uniformly strongly separated if*

$$\inf_{z\in\mathbb{D}} \sup_{n\in\mathbb{N}} \prod_{j\neq n} \rho(z, z_j)^{m_j} > 0. \tag{2.3.5}$$

It can be easily checked that inequality (2.3.4) is inequality (2.3.5) for z in Z. Therefore, a uniformly strongly separated sequence is strongly separated. Before asking ourselves if, or under what conditions, a strongly separated sequence is uniformly strongly separated, we should check that such sequences exist:

Theorem 2.3.5. *For any sequence of multiplicities $(m_n)_{n\in\mathbb{N}}$ and for any $0 < \delta < 1$, there* *exists a sequence* $(z_n)_{n\in\mathbb{N}}$ *such that*

$$\inf_{z\in\mathbb{D}} \sup_{n\in\mathbb{N}} \prod_{j\neq n} \rho(z,z_j)^{m_j} \geq \delta.$$

Proof. Fix $\delta < 1$ and $1 < \nu < 1/\delta$. Set $B_j := b_{z_j}^{m_j}$. Observe that

$$\rho(z,z_j)^{m_j} = |B_j(z)|, \qquad j \in \mathbb{N}.$$

The proof now proceeds inductively:

- *There exist* z_1 *and* z_2 *in* $\mathbb{D}$ *so that*

$$\inf_{z\in\mathbb{D}} \max_{j=1,2} |B_j(z)| \geq \delta\nu.$$

To see that it is true, choose any z_1 in $\mathbb{D}$, and then a radius $r < 1$ such that

$$|B_1(z)| \geq \delta\nu, \qquad z \in \{r \leq |z| \leq 1\}.$$

It suffices then to choose z_2 close enough to $\partial\mathbb{D}$ so that

$$|B_2(z)| \geq \delta\nu \qquad |z| \leq r,$$

by observing that the family $(b_\tau)_{\tau\in\mathbb{D}}$ is locally uniformly bounded, and therefore by Montel's Theorem

$$\lim_{n\to\infty} |b_{\tau_n}| = 1$$

uniformly on compact subsets of $\mathbb{D}$, for some sequence $(\tau_n)_{n\in\mathbb{N}}$ approaching the unit circle.

- *If $z_1, \ldots, z_n$ are points in $\mathbb{D}$ so that*

$$\inf_{z\in\mathbb{D}} \max_{l=1,\ldots,n} \prod_{j\neq l} |B_j(z)| \geq \gamma > 0,$$

then there exists a point z_{n+1} such that

$$\inf_{z\in\mathbb{D}} \max_{l=1,\ldots,n+1} \prod_{j\neq l} |B_j(z)| \geq \frac{\gamma}{\nu^{2^{-n}}}.$$

To prove it, observe that

$$\inf_{z\in\mathbb{D}} \max_{l=1,\ldots,n+1} \prod_{j\neq l} |B_j(z)| = \inf_{z\in\mathbb{D}} \max \left\{ \prod_{j=1}^{n} |B_j(z)| \, ; \, |B_{n+1}(z)| \max_{l=1,\ldots,n} \prod_{j\neq l} |B_j(z)| \right\}.$$

Then it suffices to find a radius r_n so that

$$\prod_{j=1}^{n} |B_j(z)| \geq \gamma \qquad z \in \{r_n \leq |z| < 1\}$$

and z_{n+1} so that

$$|B_{n+1}(z)| \geq \nu^{-2^{-n}} \qquad |z| \leq r_n.$$

This inductive construction shows that the sequence $(z_n)_{n\in\mathbb{N}}$ that one can build in this fashion satisfies

$$\inf_{z\in\mathbb{D}} \max_{l=1,\ldots,n} \prod_{j\leq n, j\neq l} |B_j(z)| \geq \delta \nu^{1-1/2-\cdots-1/2^n}, \qquad n \in \mathbb{N}.$$

To conclude, let n going to ∞. □

Remark 2.3.6. *One can note that in the above proof we used only the fact that each B_j is a finite Blaschke product. Therefore the same argument can be applied to the sequence of* *inner functions*

$$B_n := \prod_{j=1}^{k_n} b_{z_{n,j}}^{m_{n,j}},$$

for any sequence $(k_n)_{n\in\mathbb{N}}$ *and* $\{m_{n,j} \mid j = 1, \ldots, k_n\}_{n\in\mathbb{N}}$. *This gives that a sequence of matrices* $(A_n)_{n\in\mathbb{N}}$ *whose Blaschke factors* $(B_n)_{n\in\mathbb{N}}$ *satisfy* (2.3.2) *exists, for any choice that we can make of their dimensions. Such a sequence, thanks to Theorem 2.3.3, is interpolating.*

It is not difficult to show that, if the sequence of multiplicities $(m_n)_{n\in\mathbb{N}}$ is bounded, then any strongly separated sequence is actually uniformly strongly separated: a proof can be found in [19, Lemma 3.2.18.]. Therefore, Theorem 2.3.1 generalizes Carleson's interpolation Theorem for those interpolation problems in H^∞ that aim to specify m_n derivatives at each z_n or, equivalently, a matrix interpolating problem with $m_n \times m_n$ nodes

$$A_n = \begin{bmatrix} z_n & 1 & 0 & \ldots & 0 \\ 0 & z_n & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & z_n & 1 \\ 0 & 0 & 0 & 0 & z_n \end{bmatrix}.$$

On the other hand, if $(m_n)_{n\in\mathbb{N}}$ is unbounded, a strongly separated sequence is not, in general, uniformly strongly separated:

Theorem 2.3.7. *Let* $(m_n)_{n\in\mathbb{N}}$ *be an increasing and unbounded sequence. Then there exists a sequence* $Z = (z_n)_{n\in\mathbb{N}}$ *in* $\mathbb{D}$ *whose sequence of multiplicities is* $(m_n)_{n\in\mathbb{N}}$ *and so that* Z *is strongly separated but not uniformly strongly separated.*

The proof has a strong geometric flavor: the first tool that is going to be used is the following property of the pseudo-hyperbolic metric:

Lemma 2.3.8. *Let* $0 < z_1 < \gamma < z_2 < 1$*, and suppose that* $\rho(\gamma, z_1) = t\rho(z_1, z_2)$*, for some* $0 < t < 1$*. Then*

$$\rho(\gamma, z_2) = \frac{1-t}{1-\rho^2(z_1, z_2)t}\rho(z_1, z_2).$$

Proof. Let $s_i = \rho(z_i, \gamma)$, $i = 1, 2$. Then $s_1 = -b_{z_1}(\gamma)$, and therefore

$$\gamma = b_{z_1}(-s_1) = \frac{z_1 + s_1}{1 + z_1 s_1},$$

since Blaschke factors are their own inverses. Therefore

$$s_2 = b_{z_2}(\gamma) = b_{z_2}(b_{z_1}(-s_1)) = \frac{\rho(z_1, z_2) - s_1}{1 - \rho(z_1, z_2)s_1}.$$

This, together with

$$s_1 = t\rho(z_1, z_2),$$

concludes the proof. □

Of course, both (2.3.4) and (2.3.5) are separation conditions for the sequence $Z = (z_n)_{n\in\mathbb{N}}$. Rather than proving that a uniform strongly separated sequence is more separated than a strongly separated one, the construction that will prove Theorem 2.3.7 is given by a sequence Z that lacks regularity:

Lemma 2.3.9. *Let* $(m_n)_{n\in\mathbb{N}}$ *be a sequence of multiplicities, and let* $0 < \nu < 1$*. Then there exists a strongly separated sequence* $(z_n)_{n\in\mathbb{N}}$ *in* $\mathbb{D}$ *such that*

$$\rho(z_n, z_{n+1})^{m_{n+1}} = \nu \tag{2.3.6}$$

for infinitely many n *in* $\mathbb{N}$.

Proof. Choose a positive z_1, and then $z_2 > z_1$ so that

$$\rho(z_1, z_2)^{m_2} = \nu.$$

Then pick $z_3 > z_2$ which is close enough to 1 so that

$$\begin{cases} \rho(z_3, z_1)^{m_1}\rho(z_3, z_2)^{m_2} \geq \frac{1}{2} \\ \rho(z_2, z_3)^{m_3}\rho(z_2, z_1)^{m_1} \geq \nu\rho(z_2, z_3)^{m_3} \geq \nu\ 2^{-\frac{1}{2^3}}, \\ \rho(z_1, z_2)^{m_2}\rho(z_1, z_3)^{m_3} = \nu\rho(z_1, z_3)^{m_3} \geq \nu\ 2^{-\frac{1}{2^3}} \end{cases}$$

and $z_4 > z_3$ so that $\rho(z_3, z_4) = \nu$. In this fashion, the choices of the even terms are going to be at distance ν from the previous odd, while an odd term z_{2k+1} will be chosen so that

$$\begin{cases} \prod_{j=1}^{2k} \rho(z_{2k+1}, z_j)^{m_j} \geq \frac{1}{2} \\ \rho(z_{2k+1}, z_l)^{m_{2k+1}} \prod_{j \leq 2k, j \neq l} \rho(z_l, z_j)^{m_j} \geq \nu\ 2^{\sum_{j=1}^{k} 2^{-(2j+1)}} \qquad l = 1, \ldots, 2k. \end{cases} \tag{2.3.7}$$

We conclude the proof by letting k going to ∞. □

We are now ready to prove Theorem 2.3.7:

Proof of Theorem 2.3.7. For any n in $\mathbb{N}$, choose $z_{2n-1} < \xi_n < z_{2n}$ so that $\rho(\xi_n, z_{2n-1}) = t_n\rho(z_{2n-1}, z_{2n})$, where

$$\lim_{n\to\infty} t_n = 1 \tag{2.3.8}$$

but

$$\lim_{n\to\infty} t_n^{m_{2n-1}} = 0. \tag{2.3.9}$$

Observe that we can choose such a $(t_n)_{n\in\mathbb{N}}$ since m_n goes to infinity. To prove the Theorem, it suffices to show that, if $\rho(\xi_n, z_{2n}) = s_n \rho(z_{2n-1}, z_{2n})$, then

$$\lim_{n\to\infty} s_n^{m_{2n}} = 0 \tag{2.3.10}$$

too. Indeed, if both equation (2.3.9) and equation (2.3.10) hold, ξ_n is close enough to both z_{2n-1} and z_{2n} so that

$$\rho(\xi_n, z_{2n-i})^{m_{2n-i}} \underset{n\to\infty}{\to} 0 \qquad i = 0, 1$$

and therefore (2.3.5) fails.

Thanks to Lemma 2.3.8, this is equivalent to showing that

$$\lim_{n\to\infty} \left(\frac{1 - t_n}{1 - \rho^2(z_{2n-1}, z_{2n}) t_n} \right)^{m_{2n}} = \lim_{n\to\infty} \left(1 - \frac{t_n(1 - \rho^2(z_{2n-1}, z_{2n}))}{1 - \rho^2(z_{2n-1}, z_{2n}) t_n} \right)^{m_{2n}} = 0,$$

which is true if and only if

$$\lim_{n\to\infty} \frac{m_{2n} t_n (1 - \rho^2(z_{2n-1}, z_{2n}))}{1 - \rho^2(z_{2n-1}, z_{2n}) t_n} = \infty.$$

Set $\gamma_n := 1 - t_n$ and $\varepsilon_n := 1 - \rho^2(z_{2n-1}, z_{2n})$. Recall that by construction $\rho(z_{2n-1}, z_{2n})^{m_{2n-1}} \geq \nu > 0$, and therefore by (2.3.9) we have

$$\gamma_n + \varepsilon_n \underset{n\to\infty}{\sim} \gamma_n,$$

and so

$$\frac{m_{2n} t_n (1 - \rho^2(z_{2n-1}, z_{2n}))}{1 - \rho^2(z_{2n-1}, z_{2n}) t_n} = \frac{m_{2n}(1 - \gamma_n)\varepsilon_n}{1 - (1 - \gamma_n)(1 - \varepsilon_n)} \underset{n\to\infty}{\sim} m_{2n} \frac{\varepsilon_n}{\gamma_n}.$$

To conclude observe that, thanks to (2.3.6) and (2.3.8),

$$m_{2n} \frac{\varepsilon_n}{\gamma_n} = m_{2n} \frac{1 - \nu^{\frac{2}{m_{2n}}}}{\gamma_n} \underset{n\to\infty}{\sim} \frac{-2 \log \nu}{\gamma_n} \to \infty.$$

□

Theorem 2.3.7 exhibits a reason why the notion of strongly separated matrices in (2.3.2) involves a uniform lower bound on all the points on $\mathbb{D}$, rather than a condition that has to be checked only at the eigenvalues of the matrices composing the sequence. Even though the latter condition would be more similar to (1.1.2), Theorem 2.3.7 shows that it doesn't extend to the matrix case in case the sequence of sizes of their Jordan blocks is unbounded.

Remark 2.3.10. *A characterization of uniform strong separation given only in terms of the mutual distance of the points of a the sequence* $(z_n)_{n\in\mathbb{N}}$ *can be found in [20, Cor. 5, Lec. X]. Indeed, for any sequence of multiplicities* $(m_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$ *is uniformly strongly separated if and only if*

$$\inf_{n\in\mathbb{N}} \prod_{j\neq n} \rho(z_n, z_j)^{m_n m_j} > 0,$$

provided that $(z_n)_{n\in\mathbb{N}}$ *approaches the unit circle non tangentially.*

2.4 Carleson's Interpolation Theorem for Matrices

Thanks to Theorem 2.3.3, in order to prove Theorem 1.1.4 it suffices to show that A is interpolating if and only if it is weakly separated and H is a Bessel system. According to Theorem 2.3.3, if A is interpolating then H is a Riesz system, hence a Bessel system, and A is strongly separated, hence weakly separated. Therefore, Theorem 1.1.4 will be proved once we'll show that any weakly separated Bessel system of model spaces in H^2 is a Riesz system. This will be the aim os Section 2.4.2. Section 2.4.1 will give another Euclidean viewpoint to strong separation, and in particular to (2.3.3). This will lead to a proof of the equivalence between (2.3.3) and strong separation alternative to the one given by Theorem 2.3.2.

2.4.1 Strong Separation

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in the unit disc, and $(m_n)_{n\in\mathbb{N}}$ be a sequence of positive integers. As we saw in Theorem 2.3.1, to study separation properties of the sequence $(z_n)_{n\in\mathbb{N}}$ we need to form the Blaschke product

$$B := \prod_{n\in\mathbb{N}} b_{z_n}^{m_n}.$$

Observe that a Szegö kernel s_w belongs to the model space H_B if and only if $B(w) = 0$. Indeed, the only zeros of B are in the sequence $(z_n)_{n\in\mathbb{N}}$, and therefore if $B(w) = 0$ then w must actually be a point in the sequence. Conversely, if s_w belongs to the subspace of H^2 spanned by

$$\left\{ s_{z_n}, \frac{\partial\ s_{z_n}}{\partial \overline{w}}, \dots, \frac{\partial^{m_n-1}\ s_{z_n}}{\partial \overline{w}^{m-1}}, n = 1, 2, \dots \right\},$$

then we can write

$$s_w = \sum_{i=1}^{l} \sum_{j=0}^{m_{n_i}-1} a_{i,j} \frac{\partial^j\ s_{z_{n_i}}}{\partial \overline{w}^j}$$

and

$$B(w) = \langle B, s_w \rangle_{\mathrm{H}^2} = \sum_{i=1}^{l} \sum_{j=0}^{m_{n_i}-1} a_{i,j} \left\langle B, \frac{\partial^j\ s_{z_{n_i}}}{\partial \overline{w}^j} \right\rangle_{\mathrm{H}^2} = 0,$$

since B has a zero of multiplicity $m_n - 1$ at each z_n.

More generally, we have the following result:

Theorem 2.4.1. *For any Θ inner function in H^2*

$$\sin(s_w, H_\Theta) = |\Theta(w)| \qquad w \in \mathbb{D}. \tag{2.4.1}$$

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of matrices with spectra in the unit disk and let $(H_n)_{n\in\mathbb{N}}$ be its associated sequence of model spaces. Theorem 1.1.4 and Theorem 2.4.1 imply immediately that the sequence $(A_n)_{n\in\mathbb{N}}$ is interpolating if and only if $(H_n)_{n\in\mathbb{N}}$ is, in a certain sense, well separated:

Theorem 2.4.2. *A sequence of matrices* $(A_n)_{n\in\mathbb{N}}$ *is interpolating if and only if*

$$\inf_{z\in\mathbb{D}} \sup_{n\in\mathbb{N}} \sin\left(s_z, \overline{\operatorname{span}}_{j\neq n}\{H_j\}\right) > 0. \tag{2.4.2}$$

Proof. Thanks to Theorem 1.1.4, $(A_n)_{n\in\mathbb{N}}$ is interpolating if and only if

$$\inf_{z\in\mathbb{D}} \sup_{n\in\mathbb{N}} \prod_{j\neq n} |B_j(z)| > 0.$$

Since, for any n,

$$\overline{\operatorname{span}}_{j\neq n}\{H_j\} = H_{\prod_{j\neq n} B_j}$$

equation (2.4.1) concludes the proof. □

The proof of Theorem 2.4.1 follows from two lemmas. The first one states a geometric property of any Hilbert space:

Lemma 2.4.3. *Let* K *be a closed subspace of a Hilbert space* $\mathcal{H}$*, and* x *be a point of* $\mathcal{H}\setminus K$*. Then*

$$\frac{1}{\sin(x,K)} = \inf\{||y|| \mid y\in K^{\perp}\ ,\ \langle y,\hat{x}\rangle = 1\}.$$

For a proof, see [16, Prop. 12.6]. The second Lemma is the solution of a very natural extremal problem for reproducing kernel Hilbert spaces:

Lemma 2.4.4. *Let* $\mathcal{H}_k$ *be a reproducing kernel Hilbert space on a domain* D *of* $\mathbb{C}^d$*. Let* a *be a point of* $\mathbb{C}$*, and* z *in* D*. Then*

$$\inf\{||g|| \mid g\in\mathcal{H}_k,\ g(z)=a\} = \frac{|a|}{||k_z||}.$$

Proof. Let g be any function in $\mathcal{H}_k$ such that $g(z) = a$. Then by Cauchy-Schwartz's inequality

$$|a| = |\langle g, k_z \rangle| \leq ||g|| \cdot ||k_z||,$$

and therefore

$$\inf\{||g|| \mid g \in \mathcal{H}_k,\, g(z) = a\} \geq \frac{|a|}{||k_z||},$$

since g was arbitrary chosen. To prove the other inequality, it suffices to find one function g in $\mathcal{H}_k$ such that $g(z) = a$ and $||g|| = |a|/||k_z||$. The function

$$g = \frac{a}{||k_z||^2} k_z$$

has the requested properties. □

We can prove now Theorem 2.4.1:

Proof of Theorem 2.4.1. By Lemma 2.4.3,

$$\begin{aligned} \sin(s_z, H_\Theta) =& \frac{1}{||s_z|| \inf\{||h|| \mid h(z) = 1\,,\, h = \Theta g\,,\, g \in \mathrm{H}^2\}} \\ =& \frac{1}{||s_z|| \inf\{||g|| \mid g(z) = 1/\Theta(z)\}}. \end{aligned} \tag{2.4.3}$$

By applying Lemma 2.4.4 to the rightmost side of (2.4.3), we get (2.4.1) and we conclude the proof. □

Equation (2.4.2) says, intuitively, the following: the angle between any model space and the span of all the others is large enough so that, fixed a Szegö kernel s_w in H^2, the span of all the model spaces except the one which is the closest from s_w is at a uniform positive distance from s_w. To give a formal justification to our intuition, we are going now to quantify the relation between condition (2.4.2) and the actual sines between the model spaces involved.

A fundamental tool for what follows is Carleson's corona Theorem: a complete treatment of such a classic result can be found in [17, Ch. VIII, Th. 2.1].

Theorem 2.4.5 (Carleson's corona Theorem)**.** *Let Θ_1 and Θ_2 be two inner functions in* H^2 *so that*

$$\inf_{z\in\mathbb{D}} \max\{|\Theta_1(z)|, |\Theta_2(z)|\} \geq \delta > 0. \tag{2.4.4}$$

Then there exists a constant $C_\delta > 0$ (depending only on δ) and two functions g_1 and g_2 in H^∞ *of norm less than C_δ such that*

$$\Theta_1 g_1 + \Theta_2 g_2 = 1.$$

In Theorem 2.3.2, condition (2.4.4) is related to the sine between the model spaces H_{Θ_1} and H_{Θ_2}. Theorem 2.4.6 below gives another quantitative version of such a relation.
Let Θ_1 and Θ_2 be two inner functions in H^2, and let H_1 and H_2 be the respective model spaces. Suppose that Θ_1 and Θ_2 have a common zero z_0 whose kernel function is s_0. Then clearly s_0 belongs to both H_1 and H_2, and therefore $\sin(H_1, H_2) = 0$. We can quantify more precisely the relation between the angle between their model spaces and the separation of their zeros:

Theorem 2.4.6. *Let Θ_1 and Θ_2 be two inner functions, H_1 and H_2 their model spaces, respectively.*

(i) *If*

$$\inf_{z\in\mathbb{D}} \max\{|\Theta_1(z)|, |\Theta_2(z)|\} \leq \varepsilon < 1,$$

then for $i = 1, 2$ there exists x_i in H_i so that $||x_i|| \geq \sqrt{1-\epsilon^2}$ and $||x_1 - x_2|| \leq 2\varepsilon$;

(ii) *If*

$$\inf_{z\in\mathbb{D}} \max\{|\Theta_1(z)|, |\Theta_2(z)|\} \geq \delta > 0,$$

then there exists a constant $c_\delta > 0$ *(depending only on* δ*) such that* $\sin(H_1, H_2) > c_\delta$.

Proof. Let P_i be the orthogonal projection onto H_i, $i = 1, 2$, and P be the orthogonal projection onto H, the subspace spanned by H_1 and H_2. Observe that $H = \mathrm{H}^2 \ominus \Theta_1\Theta_2\mathrm{H}^2$.

(i) Let z be a point of $\mathbb{D}$ such that $|\Theta_i(z)| \le \varepsilon$, $i = 1, 2$, and let

$$x_i := P_i\left(\hat{s}_z\right) \qquad i = 1, 2$$
$$y = P\left(\hat{s}_z\right).$$

By Theorem 2.4.1, $||x_i|| \ge \sqrt{1 - \varepsilon^2}$ and

$$||x_i - y||^2 = |\Theta_i(z)|^2(1 - |\Theta_j(z)|)^2 \le |\Theta_i(z)|^2 \le \varepsilon^2, \qquad \{i, j\} = \{1, 2\}$$

since $x_i = P_i(y)$, $i = 1, 2$. The triangle inequality concludes the proof.

(ii) Let f_1 in H_1 have $||f_1|| = 1$, and $f_2 = P_2(f_1)$. Therefore $f_1 = f_2 + g$, where g belongs to $\Theta_2\mathrm{H}^2$ and $\mathrm{dist}(f_1, H_2) = ||g||$. Thanks to Theorem 2.4.5 there exist h_1 and h_2, $||h_i||_\infty \le C_\delta$, so that $h_1\Theta_1 + h_2\Theta_2 = 1$. Therefore

$$f_1 = h_1\Theta_1 f_1 + h_2\Theta_2 f_1,$$

and by orthogonality and the Cauchy-Schwartz inequality

$$1 = \langle f_1, f_1\rangle = \langle h_2\Theta_2 f_1, f_1\rangle = \langle h_2\Theta_2 f_1, g\rangle \le ||h_2|| \cdot ||g|| \le C_\delta \cdot \mathrm{dist}(f_1, H_2).$$

To conclude the proof, it suffices to choose $c_\delta = 1/C_\delta$ and observe that f_1 was chosen arbitrarily among all the unit vectors in H_1.

□

As a consequence, A (and hence H) is strongly separated if and only if (2.3.3) holds, and A and H are weakly separated if and only if there exists a positive δ such that, for any $n \neq j$,

$$\inf_{z\in\mathbb{D}} \max\{|B_n(z)|, |B_j(z)|\} \geq \delta. \tag{2.4.5}$$

2.4.2 Weakly Separated Bessel Systems of Model Spaces

In [13], the author asked whether the positive answer to the Feichtinger conjecture can be extended to multi-dimensional model spaces of H^2, that is, if any Bessel system of model spaces is the disjoint union of finitely many Riesz systems. We show in Section 2.5.2 that this is not the case, though any weakly separated Bessel systems of model spaces is in fact a Riesz system:

Theorem 2.4.7. *Any weakly separated Bessel system of model spaces in* H^2 *is a Riesz system.*

The proof of Theorem 2.4.7 uses the following re-statement of the Bessel system condition:

Proposition 2.4.8. *A sequence* $(H_n)_{n\in\mathbb{N}}$ *of closed sub-spaces of a Hilbert space* $\mathcal{H}$ *is a Bessel system with Bessel bound* M *if and only if, for any sequence* $(h_n)_{n\in\mathbb{N}}$ *of unit vectors such that* h_n *belongs to* H_n *for any* n *in* $\mathbb{N}$*,*

$$\sup_{||x||=1} \sum_{n\in\mathbb{N}} |\langle x, h_n\rangle|^2 = M^2. \tag{2.4.6}$$

Proof. The idea of the proof comes from [3, Prop. 9.5]. Choose for any n in $\mathbb{N}$ a unit vector

h_n in H_n, and suppose first that (2.4.6) holds. Then, for any $(a_n)_{n\in\mathbb{N}}$ in l^2,

$$\begin{aligned}\left|\left|\sum_{n\in\mathbb{N}} a_n h_n\right|\right|^2 &= \sup_{||x||=1}\left|\langle x, \sum_{n\in\mathbb{N}} a_n h_n\rangle\right|^2\\ &= \sup_{||x||=1}\left|\sum_{n\in\mathbb{N}}\langle x, h_n\rangle\ \overline{a_n}\right|^2\\ &\le M^2\sum_{n\in\mathbb{N}}|a_n|^2,\end{aligned}$$

thanks to (2.4.6) and Cauchy-Schwartz's inequality.

Conversely, let M be the Bessel bound for the sequence $(H_n)_{n\in\mathbb{N}}$, and fix a unit vector x in $\mathcal{H}$. Then set $a_n = \langle x, h_n\rangle$, and observe that

$$\begin{aligned}\sum_{n\in\mathbb{N}}|\langle x, h_n\rangle|^2 &= \langle x, \sum_{n\in\mathbb{N}} a_n h_n\rangle\\ &\le \left|\left|\sum_{n\in\mathbb{N}} a_n h_n\right|\right|\\ &\le M\left(\sum_{n\in\mathbb{N}}|a_n|^2\right)^{\frac{1}{2}}\\ &= M\left(\sum_{n\in\mathbb{N}}|\langle x, h_n\rangle|^2\right)^{\frac{1}{2}}.\end{aligned}$$

This conclude the proof. □

Remark 2.4.9. *Fixed x in $\mathcal{H}$, we can choose $(h_n)_{n\in\mathbb{N}}$ so that the sum in (2.4.6) attains its maximum. We can actually maximizes each term of the sum, by setting h_n to be the orthogonal projection onto H_n of x, divided by its norm:*

$$f_n(x) := \frac{P_{H_n}(x)}{||P_{H_n}(x)||}.$$

Proposition 2.4.8 then says that $(H_n)_{n\in\mathbb{N}}$ has a finite Bessel bound M if and only if

$$\sup_{||x||=1}\sum_{n\in\mathbb{N}}(1-\operatorname{dist}^2_{\mathrm{H}^2}(x,H_n)) = \sup_{||x||=1}\sum_{n\in\mathbb{N}}|\langle x, f_n(x)\rangle|^2 = M^2 \tag{2.4.7}$$

We are now ready to prove Theorem 2.4.7:

Proof of Theorem 2.4.7. Let $(H_{\Theta_n})_{n\in\mathbb{N}}$ be a weakly separated Bessel system. We will show that (2.3.2) holds, and Theorem 2.3.1 will conclude the proof. Thanks to (2.4.7), for any fixed z in $\mathbb{D}$ there exists a positive integer n_z that minimizes the distance between $\hat{s_z}$ and H_n:

$$\operatorname{dist}_{\mathrm{H}^2}(\hat{s_z}, H_{n_z}) = \min_{n\in\mathbb{N}}\operatorname{dist}_{\mathrm{H}^2}(\hat{s_z}, H_n),$$

which thanks to Theorem 2.4.1 becomes

$$|\Theta_{n_z}(z)| = \min_{n\in\mathbb{N}}|\Theta_n(z)|.$$

Therefore, by Theorem 2.3.2 and weak separation we have that

$$\inf_{n\neq n_z}|\Theta_n(z)| > 0,$$

which implies that

$$\prod_{n\neq n_z}|\Theta_n(z)| = \sup_{n\in\mathbb{N}}\prod_{j\neq n}|\Theta_j(z)|$$

is bounded below uniformly on z if and only if

$$\sum_{n\neq n_z}(1-|\Theta_n(z)|^2) = \sum_{n\neq n_z}(1-\operatorname{dist}^2_{\mathrm{H}^2}(\hat{s_z}, H_{\Theta_n}))$$

is uniformly bounded on z, which is true thanks to Remark 2.4.9. □

Observe that the proof of Theorem 2.4.7 used a weaker version of the Bessel system condition, in which the sup in (2.4.6) is taken only on normalized kernel functions, rather than on all unit vectors in H^2. It remains open for us whether such a weaker condition is enough to characterize Bessel systems of model spaces, and a positive answer for the special case in which each Θ_n is a Blaschke product would be of great interest for us.

Question 4. *Is any sequence of model spaces $(H_{\Theta_n})_{n\in\mathbb{N}}$ in H^2 satisfying*

$$\sup_{z\in\mathbb{D}} \sum_{n\in\mathbb{N}} |\langle \hat{s_z}, f_n(\hat{s_z})\rangle|^2 < \infty \tag{2.4.8}$$

a Bessel system? Is it true if Θ_n is, for any positive integer n, a Blaschke product?

Remark 2.4.10. *Question 4 has a positive answer whenever $\Theta_n = b_{z_n}$ is, for any positive integer n, a degree-one Blaschke factor at a point z_n, and therefore whenever H_{Θ_n} is the line spanned by the Szegö kernel at z_n, [17, Ch. VI, Lemma 3.3]. Moreover, a positive answer to Question 4 would give a function theoretical characterization for the Bessel system condition for $(H_{\Theta_n})_{n\in\mathbb{N}}$, as* (2.4.8) *is equivalent to*

$$\sup_{z\in\mathbb{D}} \sum_{n\in\mathbb{N}} 1 - |\Theta_n(z)|^2 < \infty.$$

2.5 Examples

We present below two examples of sequences of matrices having dyadic roots of unity as their eigenvalues. Section 2.5.1 will give an example of an interpolating sequence of matrices, together with some useful tool to estimate the angle between model spaces arising from Blaschke products. Section 2.5.2 will use a similar construction in order to exhibit a Bessel system of model spaces which is not the disjoint union of finitely many weakly separated sequences, and hence not the union of finitely many Riesz systems. This will give a negative answer to a question posed by the author in [13], where it was asked whether the positive answer to the Feichtinger conjecture can be extended to multi-dimensional model sub-spaces of the Hardy space.

2.5.1 Equidistributed Eigenvalues

Let, for any positive integer n,

$$\omega_n := e^{\frac{2\pi i}{2^n}}$$

be a primitive 2^n-root of unity, and let

$$W_n := \operatorname{diag}(1, \omega_n, \ldots, \omega_n^{2^n-1})$$

be a $2^n \times 2^n$ diagonal matrix having 2^n equi-distributed points on the unit circle as its eigenvalues. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence in $(0,1)$ that re-scales the sequence $(W_n)_{n\in\mathbb{N}}$ so that its spectra belong to $\mathbb{D}$:

$$A_n := r_n\, W_n, \qquad n \in \mathbb{N}. \tag{2.5.1}$$

We will discuss here how fast must $(r_n)_{n\in\mathbb{N}}$ go to 1 in order for $A := (A_n)_{n\in\mathbb{N}}$ to be interpolating. In particular, we will show that A is interpolating if and only if it is a *zero sequence*, that is, if and only if there exists a bounded analytic function on $\mathbb{D}$ that vanishes on A and that doesn't vanish outside the spectra of the matrices in A:

Theorem 2.5.1. *The sequence of matrices defined in* (2.5.1) *is interpolating if and only if it is a zero sequence.*

The proof of Theorem 2.5.1 requires that we are able to estimate (from below) the angle between two model spaces arising from Blaschke products. Such a tool is the content of Lemma 2.5.2 below. Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of Blaschke products such that $B = \prod_{n\in\mathbb{N}} B_n$ converges uniformly on any compact subset of $\mathbb{D}$ to a non zero inner function, and let $(H_n)_{n\in\mathbb{N}}$ be the associated sequence of model spaces in H^2. For any subset σ of $\mathbb{N}$ we will define, for the sake of brevity,

$$H_\sigma := \overline{\operatorname{span}}_{i\in\sigma}\{H_i\}$$

and

$$B_\sigma = \prod_{i\in\sigma} B_i.$$

Lemma 2.5.2. *Let σ and τ be two disjoint subsets of $\mathbb{N}$, and suppose that $(H_i)_{i\in\sigma}$ is a Riesz system with Riesz bound γ. Then*

$$\sin(H_\sigma, H_\tau) \geq \frac{1}{\gamma^2} \inf_{i\in\sigma} \sin(H_i, H_\tau).$$

Proof. For any i in σ let $\delta_i := \sin(H_i, H_\tau)$, and let $\delta := \inf_{i\in\sigma}\delta_i$. It suffices to show that

$$T\colon H_{\sigma\cup\tau} \to H_{\sigma\cup\tau}$$

such that

$$T_{|H_\sigma} = \delta \; Id_{|H_\sigma} \qquad T_{|H_\tau} = 0$$

is bounded by γ^2. Let T_i be, for any i in σ, the restriction to H_i of $M^*_{B_\tau}$. Thanks to Lemma 2.2.2, each T_i is bounded below by δ_i. Fix then a vector $x = u + v$ in $H_{\sigma\cup\tau}$, where u is in H_σ and v is in H_τ. There exists a sequence $(h_i)_{i\in\sigma}$ of unit vectors such that h_i is in H_i for any i in σ so that u can be written as a linear combination of $(h_i)_{i\in\sigma}$:

$$u = \sum_{i\in\sigma} \alpha_i h_i.$$

Since each T_i is a contraction and it is bounded below by δ, the sequence $(T_i(h_i))_{i\in\mathbb{N}}$ is bounded above and below and therefore

$$\begin{aligned}
||T(x)||^2 = ||T(u)||^2 &= \left|\left|\sum_{i\in\sigma} \delta\alpha_i h_i\right|\right|^2 \\
&\leq \gamma^2\delta^2 \sum_{i\in\sigma} \alpha_i^2 \leq \gamma^2 \sum_{i\in\sigma} \delta_i^2 \alpha_i^2 \\
&\leq \gamma^2 \sum_{i\in\sigma} \alpha_i^2 ||T_i(h_i)||^2 \leq \gamma^4 \left|\left|\sum_{i\in\sigma} \alpha_i T_i(h_i)\right|\right|^2 \\
&= \gamma^4 ||M^*_{B_\tau}(u)||^2 = \gamma^4 ||M^*_{B_\tau}(x)||^2.
\end{aligned}$$

Since $M^*_{B_\tau}$ is a contraction, this shows that the norm of T doesn't exceed γ^2, as we claimed.

□

Remark 2.5.3. *Suppose that also $(H_j)_{j\in\tau}$ is a Riesz system. Then a double application of* Lemma 2.5.2 *implies that the distance between H_σ and H_τ is comparable with the minimal distance attained by a model space labeled by an index in σ and one with a label in τ. If the sequence $(H_n)_{n\in\mathbb{N}}$ is weakly separated, this says roughly speaking that the set of* sparse subsequences *of $(H_n)_{n\in\mathbb{N}}$ is a separated set as well.*

In particular, if $(H_n)_{n\in\mathbb{N}}$ is weakly separated and it is disjoint union of finitely many Riesz systems, then it is a Riesz system itself. Which, in terms of interpolating sequences of matrices, gives the following Corollary:

Corollary 2.5.4. *Let A be a weakly separated sequence of matrices that can be partitioned into finitely many interpolating sequences. Then A is interpolating.*

We are now ready to prove Theorem 2.5.1. Here $(B_n)_{n\in\mathbb{N}}$ and $(H_n)_{n\in\mathbb{N}}$ arise from the sequence of matrices A defined in (2.5.1). Observe that if A is not a zero sequence then each H_n is contained in the closure of the linear span of all other model spaces, and hence $(H_n)_{n\in\mathbb{N}}$ is not strongly separated. What is left to show then is that A is strongly separated (and hence interpolating) provided that it is a zero sequence:

Proof of Theorem 2.5.1. Let, for any positive integer n,

$$r_n = 1 - \alpha_n 2^{-n}. \tag{2.5.2}$$

Since A is a zero sequence, then the spectra of the matrices in A form a zero sequence and therefore

$$\sum_{n\in\mathbb{N}} \alpha_n < \infty.$$

Let γ_n be the Riesz bound of the basis $\{\hat{s}_{r_n}, \dots, \hat{s}_{r_n\omega_n^{2^n-1}}\}$ of H_n. Then $(\gamma_n)_{n\in\mathbb{N}}$ is uniformly bounded if and only if the strong separation constants

$$\prod_{l=1}^{2^n-1} |b_{r_n\omega_n^l}(r_n)|^2$$

are uniformly bounded below. Let, for any j and n in $\mathbb{N}$,

$$\begin{aligned} M_j(n) :=& \sum_{l=1}^{2^j} |\langle \hat{s}_{r_j\omega_j^l}, \hat{s}_{r_n}\rangle|^2 \\ =& \sum_{l=1}^{2^j} \frac{(1-r_n^2)(1-r_j^2)}{|1-r_nr_j\omega_j^l|^2} \end{aligned} \tag{2.5.3}$$

Since

$$|b_w(z)|^2 = 1 - |\langle \hat{s}_w, \hat{s}_z\rangle|^2 \qquad w, z \in \mathbb{D},$$

then $(\gamma_n)_{n\in\mathbb{N}}$ is bounded if and only if $(M_n(n))_{n\in\mathbb{N}}$ is bounded, which is the case thanks to Lemma 2.5.5 below and since $(\alpha_n)_{n\in\mathbb{N}}$ is bounded. Therefore, by Lemma 2.5.2, in order to show that A is strongly separated it suffices to show that

$$\inf_{n\in\mathbb{N}} \prod_{j\neq n} |B_j(r_n)| > 0,$$

which is true if and only if each term on the product is uniformly bounded below and

$$\sup_{n\in\mathbb{N}} \sum_{j\in\mathbb{N}} M_j(n) < \infty.$$

Thanks to Lemma 2.5.5, this is true since $(\alpha_n)_{n\in\mathbb{N}}$ is summable and the sequence $(r_n)_{n\in\mathbb{N}}$ is weakly separated, together with the fact that thanks to Remark 2.5.3 the sine of the angle between H_n and H_j is, for any n and j, comparable with the pseudo-hyperbolic distance between r_n and r_j. □

A technical tool for the proof of Theorem 2.5.1 is the following computation, which relates the quantity $M_j(n)$ to the parameters $(\alpha_n)_{n\in\mathbb{N}}$ defined in (2.5.2):

Lemma 2.5.5. *Let*

$$r_n := 1 - \alpha_n 2^{-n} \qquad n \in \mathbb{N}$$

be a sequence in $(0, 1)$ *and let* $M_j(n)$ *be defined as in* (2.5.3)*. Then, for any* j *and* n *positive integers,*

$$M_j(n) \simeq \frac{\alpha_n \alpha_j}{\alpha_n + \alpha_j 2^{n-j} - \alpha_n \alpha_j 2^{-j}}.$$

Proof. Let, for any j in $\mathbb{N}$ and for any $l = 1, \ldots, 2^j$,

$$\theta_l^j := \arg \omega_j^l = \frac{2\pi l}{2^j}.$$

Then

$$1 - \cos(\theta_l^j) \simeq \frac{(\theta_l^j)^2}{2} \simeq \frac{l^2}{2^{2j+1}}$$

and therefore

$$
\begin{aligned}
M_j(n) &\simeq (1-r_j)(1-r_n)\sum_{l=1}^{2^j}\frac{1}{|1-r_nr_j\omega_j^l|^2}\\
&=(1-r_j)(1-r_n)\sum_{l=1}^{2^j}\frac{1}{(1-r_nr_j)^2+2((1-\cos(\theta_l^j)))r_nr_j}\\
&\simeq\frac{(1-r_j)(1-r_n)}{(1-r_jr_n)^2}\sum_{l=1}^{2^j}\frac{1}{1+\left(\frac{\sqrt{r_nr_j}}{2^j(1-r_nr_j)}\ l\right)^2}\\
&\simeq\frac{(1-r_j)(1-r_n)}{(1-r_jr_n)^2}\int_1^{2^j}\frac{1}{1+\left(\frac{\sqrt{r_nr_j}}{2^j(1-r_nr_j)}\ x\right)^2}\,dx\\
&=\frac{2^j(1-r_j)(1-r_n)}{(1-r_jr_n)\sqrt{r_jr_n}}\int_{\frac{\sqrt{r_jr_n}}{2^j(1-r_jr_n)}}^{\frac{\sqrt{r_jr_n}}{(1-r_jr_n)}}\frac{1}{1+x^2}\,dx\\
&\simeq\frac{2^j(1-r_j)(1-r_n)}{1-r_jr_n}\\
&=\frac{\alpha_n\alpha_j}{\alpha_n+\alpha_j2^{n-j}-\alpha_n\alpha_j2^{-j}}.
\end{aligned}
$$

□

2.5.2 Finite Unions of Riesz Systems

Thanks to the positive answer to the Feichtinger conjecture, any Bessel system of lines in a Hilbert space is the disjoint union of finitely many Riesz systems. We show here that this is not the case for multi-dimensional model spaces in H^2, as we will construct a sequence of matrices A which can not be written as the disjoint union of finitely many weakly separated sequences and whose associated sequence of model spaces is a Bessel system. This implies that [3, Th. 9.11] doesn't extend to multi-dimensional model spaces, and since any Riesz system is weakly separated it will show that the positive answer to the Feichtinger conjecture doesn't extend to multi-dimensional model spaces whose dimensions are not uniformly bounded.

The starting point of our construction is the sequence A defined in (2.5.1). Let, for any n in $\mathbb{N}$, t_n be the positive number greater than r_n such that

$$|b_{r_n}(t_n)| = \frac{1}{2^n} \qquad n \in \mathbb{N}, \tag{2.5.4}$$

and let $Z = (z_{n,l})$ be the sequence in $\mathbb{D}$ consisting of the union of all 2^n equi-distributed points at distance $1 - t_n$ from the unit circle:

$$z_{n,l} := t_n \omega_n^l, \qquad n \in \mathbb{N},\, l = 1, \ldots, 2^n.$$

The reader can think of Z as a sequence whose elements tends to approach each eigenvalue of the matrices in A. In particular, thanks to (2.5.4), the sequence of matrices

$$B := A \cup Z$$

can not be written as the finite union of weakly separated sequences, as for any positive ε and for any n in $\mathbb{N}$ each of the 2^n eigenvalues of A_n has a point of Z at pseudo-hyperbolic distance less than ε. Therefore what is left to show is that, if $(r_n)_{n\in\mathbb{N}}$ is chosen to be converging to 1 adequately fast, the sequence of model spaces associated with the sequence B is a Bessel system. Let $A' = (r_n \omega_n^l)$ be the (scalar) sequence of all the eigenvalues of the matrices in A, and let

$$B' := A' \cup Z = (z_n)_{n\in\mathbb{N}}.$$

We can then recursively choose the sequence $(r_n)_{n\in\mathbb{N}}$ to approach 1 fast enough so that

$$\sup_{n\in\mathbb{N}} \sum_{n\in\mathbb{N}} 1 - |b_{z_n}(z)|^2 < \infty,$$

and therefore thanks to Remark 2.4.10 the sequence of lines spanned by the Szegö kernels at the point of B' is a Bessel system. This, together with an extra separation condition on the eigenvalues of the matrices in A, implies that the model spaces associated with the sequence B forms a Bessel system:

Lemma 2.5.6. *Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of closed sub-spaces of a Hilbert space $\mathcal{H}$, and let, for any n in $\mathbb{N}$, $\{x_n^1, \ldots, x_n^{m_n}\}$ be a basis of H_n of unit vectors such that*

$$\sum_{l=1}^{m_n} |c_l|^2 \leq C_n^2 \left\| \sum_{l=1}^{m_n} c_l x_n^l \right\|^2, \qquad c_1, \ldots, c_{m_n} \in \mathbb{C}. \tag{2.5.5}$$

If

$$C := \sup_{n\in\mathbb{N}} C_n < \infty \tag{2.5.6}$$

and (x_n^l) is a Bessel system with bound M, then $(H_n)_{n\in\mathbb{N}}$ is a Bessel system with bound CM.

Proof. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of unit vectors in $\mathcal{H}$ such that h_n belongs to H_n for any n in $\mathbb{N}$, and let $(a_n)_{n\in\mathbb{N}}$ be an l^2 sequence. Write $h_n = \sum_{l=1}^{m_n} b_n^l x_n^l$, and observe that

$$\begin{aligned}\left\| \sum_{n\in\mathbb{N}} a_n h_n \right\|^2 &= \left\| \sum_{n\in\mathbb{N}} \sum_{l=1}^{m_n} a_n b_n^l x_n^l \right\|^2 \\ &\leq M^2 \sum_{n\in\mathbb{N}} |a_n|^2 \sum_{l=1}^{m_n} |b_n^l|^2 \\ &\leq C^2 M^2 \sum_{n\in\mathbb{N}} |a_n|^2,\end{aligned}$$

thanks to (2.5.5) and (2.5.6). □

As we showed during the proof of Theorem 2.5.1, by eventually increasing the rate of convergence of (r_n) to 1 we can assume that the Reisz bound of the basis $\{\hat{s}_{r_n}, \ldots, \hat{s}_{r_n \omega_n^{2^n-1}}\}$ is uniformly bounded in n, thus in particular condition (2.5.5) holds. Therefore thanks to Lemma 2.5.6 the sequence of model spaces associated to the sequence B is a Bessel system.

Chapter 3

Pairs of Commuting Matrices

The main goal of this Chapter is to prove Theorem 1.2.9 and Theorem 1.2.10.

Section 3.1 is based on [15], and tries to understand *how far* is Theorem 1.2.1 from being a characterization for interpolating sequences in the polydisc. This will be done by studying sequences with deterministic radii and random (independent and identically distributed) arguments, and by asking whether condition (i) and (iii) of Theorem 1.2.1 coincide at least *almost surely*. This will lead to a quantitative sufficient separation condition for the joint spectra of a sequence of d-tuples of diagonal matrices in order for it to be interpolating.

Section 3.2 will extend Agler's argument [1] on the interaction between admissible kernels and interpolating sequences to sequences of pairs of commuting matrices. As a Corollary, we will prove Theorem 1.2.9 in Section 3.2.1, and we will also point out how this implies that Definition 1.2.5 is equivalent to a weaker notion of interpolating sequences of pairs of commuting matrices which relies only on diagonal target, as for the one variable case. Finally, Section 3.2.2 will use an operator theoretical argument to prove Theorem 1.2.10.

3.1 Random Interpolating Sequences on The Polydisc

In certain instances, the randomization of the conditions studied by Carleson in [9] and [10] become more tractable and provide insight into the structure of interpolating sequences. Cochran studied in [12] separation properties of random sequences. A random sequence in the unit disc is defined as follows: let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables, all distributed uniformly in $(0, 2\pi)$ and defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then, for any choice of a deterministic sequence of radii $(r_n)_{n\in\mathbb{N}}$ approaching 1 define

$$\lambda_n(\omega) := r_n e^{i\theta_n(\omega)}, \qquad \omega \in \Omega.$$

Considering the random sequence $\Lambda(\omega) = (\lambda_n(\omega))_{n\in\mathbb{N}}$, the 0-1 Kolmogorov law yields that events such as

$$\mathcal{W} := \{\Lambda \text{ is weakly separated}\}$$

$$\mathcal{S} := \{\Lambda \text{ is strongly separated}\}$$

$$\mathcal{C} := \{\mu_\Lambda \text{ is a Carleson measure for } \mathrm{H}^2(\mathbb{D})\}$$

$$\mathcal{I} := \{\Lambda \text{ is an interpolating sequence}\}$$

have probability zero or one, thanks to the independence of the arguments of the points in Λ. Let

$$I_j := \{z \in \mathbb{D} : 1 - 2^{-j} \le |z| < 1 - 2^{-(j+1)}\} \qquad j \in \mathbb{N} \tag{3.1.1}$$

be the jth dyadic annulus of $\mathbb{D}$, and let

$$N_j := \#\Lambda \cap I_j. \tag{3.1.2}$$

All the randomness of the sequence is on the arguments of the points in Λ, and therefore $(N_j)_{j\in\mathbb{N}}$ is a deterministic sequence. Cochran proved in [12, Th. 2] that $\mathbb{P}(\mathcal{W}) = 1$ provided that

$$\sum_{j\in\mathbb{N}} N_j^2 2^{-j} < \infty, \tag{3.1.3}$$

and that $\mathbb{P}(\mathcal{W}) = 0$ whenever the sum in (3.1.3) diverges. Later on, Rudowicz showed in [21] that (3.1.3) is a sufficient condition for μ_Λ to be a Carleson measure for $\mathrm{H}^2(\mathbb{D})$ almost surely, and concluded, thanks to Theorem 1.1.1, that $\mathbb{P}(\mathcal{I}) = 1$ if and only if (3.1.3) holds.

In particular, condition (3.1.3) encodes all those random sequences so that $\mathcal{W}$, $\mathcal{S}$ and $\mathcal{I}$ have all probability one.

Theorem 1.2.1 represents one of the best known attempts to characterize interpolating sequences on the polydisc in terms of its hyperbolic geometry. The motivation of this section is to find out whether condition (a), and (c) of Theorem 1.2.1 are equivalent at least *almost surely*. A negative answer would imply that Theorem 1.2.1 is far from being a characterization. A positive answer would give the 0-1 Kolmogorov law for interpolating sequences in the polydisc with random arguments. The construction of a random sequence Λ on the polydisc follows the same outline as for the case of the unit disc. Let $\mathbb{T}^d$ be the d-dimensional torus in $\mathbb{C}^d$, and let $(\theta_n^1, \ldots, \theta_n^d)_{n\in\mathbb{N}}$ be a sequence of independent and indentically distributed random variables taking values on $\mathbb{T}^d$, all distributed uniformly and defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence in $[0,1)^d$, and define a random sequence $\Lambda = (\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{D}^d$ as

$$\lambda_n(\omega) = \left(r_n^1 e^{i\theta_n^1(\omega)}, \ldots, r_n^d e^{i\theta_n^d(\omega)}\right), \qquad \omega \in \Omega.$$

The events of interest are going to be

$$\begin{aligned}
&\mathcal{W}(\mathbb{D}^d) := \{\Lambda \text{ is weakly separated in } \mathbb{D}^d\}\\
&\mathcal{S}(\mathbb{D}^d) := \{\Lambda \text{ is strongly separated in } \mathbb{D}^d\}\\
&\mathcal{C}(\mathrm{H}^2(\mathbb{D}^d)) := \{\mu_\Lambda \text{ is a Carleson measure for } \mathrm{H}^2(\mathbb{D}^d)\}\\
&\mathcal{I}(\mathbb{D}^d) := \{\Lambda \text{ is an interpolating sequence}\}.
\end{aligned}$$

Our aim is to give necessary conditions and sufficient conditions for Λ to be interpolating almost surely. This will be achieved by studying separately the probability of the events $\mathcal{W}(\mathbb{D}^d)$, $\mathcal{S}(\mathbb{D}^d)$ and $\mathcal{C}(\mathrm{H}^2(\mathbb{D}^d))$, and by applying Theorem 1.2.1. Looking for separation conditions on $(r_n)_{n\in\mathbb{N}}$ that yield almost sure separation properties for Λ, (3.1.1) and (3.1.2) are extended to the d dimensional case by considering

$$I_m := \{z \in \mathbb{D}^d : 1 - 2^{-m_i} \leq |z^i| < 1 - 2^{-(m_i+1)}, i = 1, \dots d\} \tag{3.1.4}$$

and

$$N_m = \#\Lambda \cap I_m,$$

for any multi-index $m = (m_1, \dots, m_d)$ in $\mathbb{N}^d$. Here $|m| = m_1 + \cdots + m_d$ will denote the length of m.

We partially extend Cochran's and Rudowicz's works to the polydisc:

Theorem 3.1.1. *Let Λ be a random sequence in $\mathbb{D}^d$. Then*

(i) If

$$\sum_{m\in\mathbb{N}^d} N_m^2 2^{-|m|} < \infty \tag{3.1.5}$$

then $\mathbb{P}(\mathcal{W}(\mathbb{D}^d)) = 1$. If the sum in (3.1.5) *diverges, then $\mathbb{P}(\mathcal{W}(\mathbb{D}^d)) = 0$.*

(ii) If

$$\sum_{m\in\mathbb{N}^d} N_m^{1+\frac{1}{d}} 2^{-\frac{|m|}{d}} < \infty \tag{3.1.6}$$

then $\mathbb{P}(\mathcal{S}(\mathbb{D}^d)) = 1$.

(iii) If (3.1.5) *holds, then $\mathbb{P}(\mathcal{C}(\mathrm{H}^2(\mathbb{D}^d))) = 1$. If the sum in* (3.1.5) *diverges, then $\mathbb{P}(\mathcal{C}(\mathrm{H}^2(\mathbb{D}^d))) = 0$.*

Observe that the case $d = 1$ yields Rudowicz's and Cochran's characterization of random interpolating sequences on the unit disc. In general, part (i) and (iii) of the above Theorem gives the 0-1 Komolgorov law for a sequence to be weakly separated and to generate a Carleson measure for the Hardy space on the polydisc. For the case of strong separation, the result gives a sufficient condition for a sequence to be almost surely strongly separated. In particular, thanks to Theorem 1.2.1, it is the case that the 0-1 Kolmogorov law for almost surely interpolating sequences in the polydisc lies somewhere in between (3.1.6) and (3.1.5):

Corollary 3.1.2. *Let Λ be a random sequence on $\mathbb{D}^d$. Then*

(i) If (3.1.6) *holds, then $\mathbb{P}(\mathcal{I}) = 1$;*

(ii) If the sum in (3.1.5) *diverges, then $\mathbb{P}(\mathcal{I}) = 0$.*

Proposition 3.1.12 will give an example of a class of random sequences for which the 0-1 Kolmogorov law for almost surely interpolating sequences coincides with the one for almost sure weak separation.

3.1.1 Preliminary Tools

Double sums are extensively used throughout this Section. In particular the fact that, for a certain class of double sums involving exponential decay, the terms of the sums on their diagonals contain all the necessary informations to bound the whole sums:

Lemma 3.1.3. *Let $s \geq 1$, and let $(A_m)_{m\in\mathbb{N}}$ and $(B_k)_{k\in\mathbb{N}}$ be two sequences of positive numbers. Then there exists some constant $C = C_s > 0$ such that*

$$\sum_{m\in\mathbb{N}} A_m \left(\sum_{k\in\mathbb{N}} \frac{B_k}{2^m + 2^k} \right)^{\frac{1}{s}} \leq C_s \left(\max\left\{ \sum_{m\in\mathbb{N}} A_m^{1+\frac{1}{s}} 2^{-\frac{m}{s}}, \sum_{k\in\mathbb{N}} B_k^{1+\frac{1}{s}} 2^{-\frac{k}{s}} \right\} + \sum_{m\in\mathbb{N}} A_m B_m^{\frac{1}{s}} 2^{-\frac{m}{s}} \right).$$

Proof. First observe that

$$\sum_{m\in\mathbb{N}} A_m \left(\sum_{k\in\mathbb{N}} \frac{B_k}{2^m+2^k}\right)^{\frac{1}{s}} \leq \sum_{m,k\in\mathbb{N}} \frac{A_m B_k^{\frac{1}{s}}}{(2^m+2^k)^{\frac{1}{s}}}$$
$$\lesssim \sum_{k>m} \frac{A_m B_k^{\frac{1}{s}}}{(2^m+2^k)^{\frac{1}{s}}} + \sum_{k<m} \frac{A_m B_k^{\frac{1}{s}}}{(2^m+2^k)^{\frac{1}{s}}} + \sum_{m\in\mathbb{N}} A_m B_m^{\frac{1}{s}} 2^{-\frac{m}{s}}.$$

Let's first estimate the sum in $k>m$:

$$\sum_{k>m} \frac{A_m B_k^{\frac{1}{s}}}{(2^m+2^k)^{\frac{1}{s}}} \leq C_s \sum_{m=1}^{\infty} A_m 2^{-\frac{m}{s}} \sum_{k=1}^{\infty} B_{m+k}^{\frac{1}{s}} 2^{-\frac{k}{s}}$$
$$= C_s \sum_{k=1}^{\infty} 2^{-\frac{k}{s+1}} \sum_{m=1}^{\infty} A_m 2^{-\frac{m}{s+1}} B_{m+k}^{\frac{1}{s}} 2^{-\frac{m+k}{s(s+1)}}$$
$$\leq C_s \sum_{k=1}^{\infty} 2^{-\frac{k}{s+1}} \left(\sum_{m=1}^{\infty} A_m^{1+\frac{1}{s}} 2^{-\frac{m}{s}}\right)^{\frac{s}{s+1}} \left(\sum_{m=1}^{\infty} B_{m+k}^{1+\frac{1}{s}} 2^{-\frac{m+k}{s}}\right)^{\frac{1}{s+1}}$$
$$\leq C_s \max\left\{\sum_{m\in\mathbb{N}} A_m^{1+\frac{1}{s}} 2^{-\frac{m}{s}}, \sum_{k\in\mathbb{N}} B_k^{1+\frac{1}{s}} 2^{-\frac{k}{s}}\right\},$$

thanks to Holder's inequality with dual exponents $1+1/s$ and $s+1$. The sum in $m>k$ is estimated analogously. This concludes the proof. □

Our take away from Lemma 3.1.3 is the following

Corollary 3.1.4. *Let $s \geq 1$, $d \geq 1$ and $(N_m)_{m\in\mathbb{N}^d}$ be a sequence of positive numbers so that*

$$\sum_{m\in\mathbb{N}^d} N_m^{1+\frac{1}{s}} 2^{-\frac{|m|}{s}} < \infty.$$

Then there exists a positive $C = C_{s,d} > 0$ so that

$$\sum_{m\in\mathbb{N}^d} N_m \left(\sum_{k\in\mathbb{N}^d} N_k \prod_{i=1}^{d} \frac{1}{2^{m_i}+2^{k_i}}\right)^{\frac{1}{s}} \leq C \sum_{m\in\mathbb{N}^d} N_m^{1+\frac{1}{s}} 2^{-\frac{|m|}{s}}. \tag{3.1.7}$$

Proof. The proof is by induction on d:

$d = 1$ apply Lemma 3.1.3 to $A_m = B_m = N_m$;

$d \geq 2$ suppose that (3.1.7) is true for $d-1$, and let $(N_m)_{n\in\mathbb{N}^d}$ be a sequence of positive numbers. Then

$$\begin{aligned}
&\sum_{m\in\mathbb{N}^d} N_m \left(\sum_{k\in\mathbb{N}^d} N_k \prod_{i=1}^{d} \frac{1}{2^{m_i}+2^{k_i}}\right)^{\frac{1}{s}} \\
&= \sum_{m_1\in\mathbb{N}} \sum_{\tilde{m}\in\mathbb{N}^{d-1}} N_{(m_1,\tilde{m})} \left(\sum_{\tilde{k}\in\mathbb{N}^{d-1}} \prod_{i=1}^{d-1} \frac{1}{2^{\tilde{k}_i}+2^{\tilde{m}_i}} \sum_{k_1\in\mathbb{N}} N_{(k_1,\tilde{k})} \frac{1}{2^{k_1}+2^{m_1}}\right)^{\frac{1}{s}} \\
&\leq C_s \sum_{m_1\in\mathbb{N}} \sum_{\tilde{m}\in\mathbb{N}^{d-1}} N_{(m_1,\tilde{m})} \left(\sum_{k_1\in\mathbb{N}} N_{(k_1,\tilde{m})} \frac{1}{2^{k_1}+2^{m_1}}\right)^{\frac{1}{s}} 2^{-\frac{|\tilde{m}|}{s}} \\
&= C_s \sum_{\tilde{m}\in\mathbb{N}^{d-1}} 2^{-\frac{|\tilde{m}|}{s}} \sum_{m_1\in\mathbb{N}} N_{(m_1,\tilde{m})} \left(\sum_{k_1\in\mathbb{N}} N_{(k_1,\tilde{m})} \frac{1}{2^{k_1}+2^{m_1}}\right)^{\frac{1}{s}}.
\end{aligned}$$

To conclude it suffices to apply the case $d = 1$ to the two inner sums.

□

Fairly elementary facts from probability theory are exploited in the proofs. All the events and the random variables that are considered will be defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. For a comprehensive treatment of the probabilistic results used, see [8].

The first tool is the Borel-Cantelli Lemma. Recall that, given a sequence $(A_n)_{n\in\mathbb{N}}$ of events in $\mathcal{A}$, then

$$\limsup_{n\in\mathbb{N}} A_n := \bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} A_n$$

denotes the event made of those ω in Ω that belong to infinitely many of the events in $(A_n)_{n\in\mathbb{N}}$.

Theorem 3.1.5 (Borel-Cantelli Lemma)**.** *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of events in $\mathcal{A}$. Then*

(i) If $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) < \infty$, *then* $\mathbb{P}\left(\limsup_{n\in\mathbb{N}} A_n\right) = 0$;

(ii) If $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) = \infty$ *and the events in* $(A_n)_{n\in\mathbb{N}}$ *are independent, then* $\mathbb{P}\left(\limsup_{n\in\mathbb{N}} A_n\right) = 1$.

Given a random variable X on Ω, its mean value (or expectation) will be denoted by

$$\mathbb{E}(X) := \int_\Omega X \, d\mathbb{P}.$$

In particular, if $\mathbb{E}(X) < \infty$, then $\mathbb{P}\{X = \infty\} = 0$. This leads to a tool for the proofs of Theorem 3.1.1:

Lemma 3.1.6. *Let $(X^i_{n,j})_{n,j\in\mathbb{N}}$ be a sequence of positive random variables, for any $i = 1, \ldots, d$. Set*

$$m(n,j) := \min_{i=1,\ldots,d} X^i_{n,j}, \qquad p(n,j) := \prod_{i=1}^{d} X^i_{n,j}.$$

Assume that

$$\sum_{j\in\mathbb{N}} \left(\sum_{k\in\mathbb{N}} \mathbb{E}(p(k,j)) \right)^{\frac{1}{d}} < \infty.$$

Then

$$\sup_{n\in\mathbb{N}} \sum_{j\neq n} m(n,j)$$

is bounded almost surely.

Proof. Since, for any $n \neq j$ in $\mathbb{N}$,

$$m(n,j) \le p(n,j)^{\frac{1}{d}} \le \left(\sum_{k\neq j} p(k,j) \right)^{\frac{1}{d}},$$

we have

$$\sup_{n\in\mathbb{N}} \sum_{j\neq n} m(n,j) \le \sum_{j\in\mathbb{N}} \left(\sum_{k\neq j} p(k,j) \right)^{\frac{1}{d}}.$$

Thus

$$\mathbb{E}\left(\sup_{n\in\mathbb{N}}\sum_{j\neq n} m(n,j)\right) \leq \sum_{j\in\mathbb{N}} \mathbb{E}\left(\left(\sum_{k\in\mathbb{N}} p(k,j)\right)^{\frac{1}{d}}\right) \leq \sum_{j\in\mathbb{N}}\left(\sum_{k\in\mathbb{N}} \mathbb{E}(p(k,j))\right)^{\frac{1}{d}}. \tag{3.1.8}$$

□

Inequality (3.1.8) uses Jensen's Inequality:

Theorem 3.1.7 (Jensen's Inequality)**.** *Let X be a real-valued random variable on Ω, and let $\phi\colon \mathbb{R}\to\mathbb{R}$ be a convex function. Then*

$$\mathbb{E}(\phi(X)) \geq \phi(\mathbb{E}(X)).$$

In particular, since

$$t \in (0,\infty) \mapsto t^{\frac{1}{s}}$$

is concave, for any $s \geq 1$, this gives

$$\mathbb{E}\left(X^{\frac{1}{s}}\right) \leq \mathbb{E}(X)^{\frac{1}{s}}, \tag{3.1.9}$$

for any positive random variable X on Ω, by applying Jensen's inequality to $\phi(t) = -t^{\frac{1}{s}}$.

3.1.2 The Proof of Theorem 3.1.1

This section is devoted to the proof of Theorem 3.1.1. The events $\mathcal{S}(\mathbb{D}^d)$, $\mathcal{W}(\mathbb{D}^d)$ and $\mathcal{C}(\mathrm{H}^2(\mathbb{D}^d))$ will be analyzed separately.

Weak Separation

For weak separation in the polydisc, it turns out that Cochran's argument in [12, Th. 2] extends to the higher dimensional case. In particular, such an argument can be modified to get a slightly stronger conclusion, at least in one direction:

Proposition 3.1.8. *Let Λ be a random sequence in $\mathbb{D}^d$ so that*

$$\sum_{m\in\mathbb{N}^d} N_m^2 2^{-|m|} = \infty.$$

Then Λ is almost surely not the union of finitely many weakly separated sequences.

Proof. For the sake of readability, Cochran's proof will be adapted only to the case $d = 2$; the proof will lift appropriately to any $d > 1$.

Let l and k be in $\mathbb{N}$. Define

$$A_{l,k} := \bigcup_{r_1,\ldots,r_k\neq n} \{\rho_G(\lambda_{r_i}, \lambda_n) \leq 5\cdot 2^{-l}, i = 1, \ldots, k\}.$$

If ω is in $A_{l,k}$, then $\Lambda(\omega)$ contains k points clustered in a Gleason ball centered at some λ_n of radius $5\cdot 2^{-l}$. Therefore it suffices to show that $\mathbb{P}\left(\bigcap_{l,k} A_{l,k}\right) = 1$. To do so, it suffices to fix l and k and to show that $A_{l,k}$ has probability 1.

For any m in $\mathbb{N}^2$, partition I_m into 2^{2l} "rectangles" of the form

$$\left\{(z^1, z^2) \in \mathbb{D}^2 : 1 - 2^{-(m_1+l-1)} \leq |z^1| < 1 - 2^{-(m_1+l)}, 1 - 2^{-(m_2+l-1)} \leq |z^2| < 1 - 2^{-(m_2+l)}\right\},$$

and observe that at least one of these rectangles, say R_m, must contain at least $M_m := N_m/2^{2l}$ points of Λ. Let B_m be the event

$$\bigcup_{r_1,\ldots,r_k} \bigcap_{i=1}^{k} \left\{\lambda_{r_i} \in R_m, \lambda_n \in R_m, |\theta^1_{r_i} - \theta^1_n| \leq \pi\cdot 2^{-(m_1+l)}, |\theta^2_n - \theta^2_{r_i}| \leq \pi\cdot 2^{-(m_2+l)}\right\}. \quad (3.1.10)$$

Since

$$\limsup_m B_m \subseteq A_{l,k}$$

and the events B_m are independent, by the Borel-Cantelli Lemma, Theorem 3.1.5, it suffices to show that $\sum_{m\in\mathbb{N}^2} \mathbb{P}(B_m) = \infty$.

In order to estimate the probability of each B_m from below, an upper bound for $\mathbb{P}(B_m^c)$ is provided. If τ is in $\mathbb{T}^2$, let $S_m(\tau)$ be a "rectangle" in $\mathbb{T}^2$ centered at τ with base $2^{-(m_1+l)+1}$ and height $2^{-(m_2+l)+1}$. If $\tau_n = (e^{i\theta_n^1}, e^{i\theta_n^2})$, then thanks to the independence of $(\tau_n)_{n\in\mathbb{N}}$ we have

$$\begin{aligned}
\mathbb{P}(B_m^c) &= \mathbb{P}\left(\left\{\tau_1 \in \mathbb{T}^2, \tau_2 \in \mathbb{T}^2 \setminus S_m(\tau_1), \ldots, \tau_{M_m-k} \in S \setminus \bigcup_{j=1}^{M_m-k-1} S_m(\tau_j)\right\}\right) \\
&\leq \left(1 - 2^{-(|m|+2l)}\right)\left(1 - \frac{3}{2}\cdot 2^{-(|m|+2l)}\right)\ldots\left(1 - \frac{M_m-k}{2}\cdot 2^{-(|m|+2l)}\right) \\
&= \prod_{j=2}^{M_m-k}\left(1 - j\cdot 2^{-(|m|+2l+1)}\right).
\end{aligned}$$

If $\liminf_m \mathbb{P}(B_m^c) < 1$, then $\mathbb{P}(B_m)$ is uniformly bounded away from 0 infinitely many times, and therefore $\sum_{m\in\mathbb{N}^2} \mathbb{P}(B_m) = \infty$ trivially.

On the other hand, if $\lim_{|m|\to\infty} \mathbb{P}(B_m^c) = 1$, then

$$\begin{aligned}
\mathbb{P}(B_m) &\geq 1 - \prod_{j=2}^{M_m - k} \left(1 - j \cdot 2^{-(|m|+2l+1)}\right) \\
&\underset{|m|\to\infty}{\sim} -\log \prod_{j=2}^{M_m - k} \left(1 - j \cdot 2^{-(|m|+2l+1)}\right) \\
&= -\sum_{j=2}^{M_m - k} \log\left(1 - j \cdot 2^{-(|m|+2l+1)}\right) \\
&\geq \sum_{j=2}^{M_m - k} j \cdot 2^{-(|m|+2l+1)} \\
&\underset{|m|\to\infty}{\sim} \frac{(M_m - k)^2 2^{-|m|}}{2^{2l+2}} \geq \frac{(N_m - k)^2 2^{-|m|}}{2^{6l+2}},
\end{aligned}$$

which is the general term of a divergent series. □

In particular, if the sum in (3.1.5) diverges, then $\mathbb{P}(\mathcal{W}(\mathbb{D}^d)) = 0$.

Remark 3.1.9. *The Gleason distance is not the only metric that one can choose to work with in the polydisc. Another choice can be made from the Szegö kernel on $\mathbb{D}^d$:*

$$s_d(z, w) := \prod_{i=1}^{d} \frac{1}{1 - z^i \overline{w^i}} \qquad z, w \in \mathbb{D}^d.$$

For any positive kernel k on $\mathbb{D}^d$ one can define the associated metric

$$\rho_k(z, w) := \sqrt{1 - \left|\frac{k(z, w)}{k(z, z)k(w, w)}\right|^2}, \qquad z, w \in \mathbb{D}^d.$$

A sequence $Z = (z_n)_{n\in\mathbb{N}}$ in $\mathbb{D}^d$, is weakly separated with respect the metric ρ_k if

$$\inf_{n\neq j} \rho_k(z_n, z_j) > 0.$$

If B_m is defined as in (3.1.10)*, then*

$$\limsup_m B_m \subset \bigcup_{r_1,\dots,r_k\neq n} \{\rho_{s_d}(\lambda_{r_i},\lambda_n) \leq 11\cdot 2^{-l}, i=1,\dots,k\},$$

and therefore the same argument above yields that Λ is almost surely not the union of finitely many weakly separated sequences with respect ρ_{s_d}, provided that the sum in (3.1.5) *diverges.*

To conclude the proof of Theorem 3.1.1, part (i), it suffices to show that a random sequence Λ in $\mathbb{D}^d$ is almost surely weakly separated whenever (3.1.5) holds. To do so, let

$$\Omega_m := \bigcup_{r\neq n} \left\{\lambda_r \in I_m, \lambda_n \in I_m, |\theta_r^1 - \theta_n^1| \leq \pi\cdot 2^{-m_1}, |\theta_n^2 - \theta_r^2| \leq \pi \cdot 2^{-m_2}\right\}.$$

Then

$$\mathbb{P}(\Omega_m) \leq \binom{N_m}{2} 2^{-|m|} \leq \frac{1}{2} N_m^2 2^{-|m|},$$

and the Borel-Cantelli Lemma provides that, almost surely, any pair (λ_n,λ_r) in all but finitely many rectangles I_m satisfies

$$|\theta_n^1 - \theta_r^1| > \pi 2^{-m_1} \quad \text{or} \quad |\theta_n^2 - \theta_r^2| > \pi 2^{-m_2}. \tag{3.1.11}$$

The same argument applies for the right-shifted "rectangles" I'_m of the form

$$\left\{1 - \frac{3\cdot 2^{-m_1}}{4} \leq |z_1| < 1 - \frac{3\cdot 2^{-(m_1+1)}}{4}, 1 - 2^{-m_2} \leq |z_2| < 1 - 2^{-(m_2+1)}\right\}$$

and the up-shifted "rectangles" I''_m of the form

$$\left\{1 - 2^{-m_1} \leq |z_1| < 1 - 2^{-(m_1+1)}, 1 - \frac{3\cdot 2^{-m_2}}{4} \leq |z_2| < 1 - \frac{3\cdot 2^{-(m_2+1)}}{4}\right\}.$$

This ensures that all but finitely many pairs (λ_n, λ_r) in Λ so that both

$$|\lambda_n^1 - \lambda_r^1| \simeq 2^{-m_1}$$

and

$$|\lambda_n^2 - \lambda_r^2| \simeq 2^{-m_2}$$

have property (3.1.11). Therefore, see [12, Claim, p. 741] Λ is almost surely weakly separated.

Strong Separation

While weak separation behaves essentially in the same way as the dimension d grows, the sufficient condition in (3.1.6) for almost sure strong separation picks up a dependence on d. As will be shown, this is due to some estimates on the expected value of quantities related to the (random) Gleason distances between the points in Λ.

It will also be explained how (3.1.6) can be improved for some choices of $(r_n)_{n\in\mathbb{N}}$. As a corollary, a cut off condition for Λ to be almost surely interpolating for some types of random sequences in the polydisc will be given.

Let s_d be the Szegö kernel on $\mathbb{D}^d$. Then the Hardy space $\mathrm{H}^2(\mathbb{D}^d)$ is the reproducing kernel Hilbert space $\mathcal{H}_{s_d}$. Denote the *normalized* Szegö kernel by

$$S_d(z, w) := \prod_{i=1}^{d} \frac{\sqrt{(1 - |z^i|^2)(1 - |w^i|^2)}}{1 - z^i\overline{w^i}},$$

and observe that, for any z and w in $\mathbb{D}^d$,

$$\rho_G(z, w)^2 = 1 - \min_{i=1,\dots,d} |S_1(z^i, w^i)|^2. \tag{3.1.12}$$

Given a random sequence Λ in $\mathbb{D}^d$ denote, for the sake of readability,

$$S^i(n,j) := S_1(\lambda_n^i, \lambda_j^i).$$

Thanks to (3.1.12), strong separation can be achieved from weak separation and a uniform bound on sums depending on the random sequence $(S_1(n,j))_{n,j\in\mathbb{N}}$:

$$\mathcal{S}(\mathbb{D}^d) = \mathcal{W}(\mathbb{D}^d) \cap \left\{ \sup_{n\in\mathbb{N}} \sum_{j\neq n} \min_{i=1,\dots,d} |S^i(n,j)|^2 < \infty \right\}. \tag{3.1.13}$$

Observe that each $(S^i(n,j))_{n,j\in\mathbb{N}}$ is a sequence of random variables on Ω which is determined, together with Λ, by $(r_n)_{n\in\mathbb{N}}$. It is not surprising then that the expectation of $|S^i(n,j)|^2$ depends, for any i, n and j, only on r_n^i and r_j^i:

Lemma 3.1.10. *Let Λ be a random sequence in $\mathbb{D}^d$. Then, for any $n \neq j$ in $\mathbb{N}$ and for any $i = 1, \dots, d$,*

$$\mathbb{E}(|S^i(n,j)|^2) = \frac{\left(1-(r_n^i)^2\right)\left(1-(r_j^i)^2\right)}{1-\left(r_n^i r_j^i\right)^2}.$$

Proof. Observe that[1]

$$\begin{aligned}
|S^i(n,j)|^2 =& \left(1-(r_n^i)^2\right)\left(1-(r_j^i)^2\right) \left| \sum_{k=0}^{\infty} (r_n^i r_j^i)^k e^{-ik(\theta_n^i - \theta_j^i)} \right|^2 \\
=& \left(1-(r_n^i)^2\right)\left(1-(r_j^i)^2\right) \left(\sum_{k=0}^{\infty} (r_n^i r_j^i)^k e^{-ik(\theta_n^i - \theta_j^i)} \right) \left(\sum_{k=0}^{\infty} (r_n^i r_j^i)^k e^{-ik(\theta_j^i - \theta_n^i)} \right) \\
=& \left(1-(r_n^i)^2\right)\left(1-(r_j^i)^2\right) \sum_{k=0}^{\infty} (r_n^i r_j^i)^k \sum_{l=0}^{k} e^{i(2l-k)(\theta_n^i - \theta_j^i)}.
\end{aligned}$$

[1] We trust the reader to distinguish between the index $i = 0, \dots, d$ and $i = \sqrt{-1}$

Therefore, by making use of the independence of θ_n^i and θ_j^i,

$$\begin{aligned}
\mathbb{E}(|S^i(n,j)|^2) =& \left(1-(r_n^i)^2\right)\left(1-(r_j^i)^2\right)\sum_{k=0}^{\infty}(r_n^i r_j^i)^k \sum_{l=0}^{k} \mathbb{E}\left(e^{i(2j-k)(\theta_n^i-\theta_j^i)}\right)\\
=& \left(1-(r_n^i)^2\right)\left(1-(r_j^i)^2\right)\sum_{k=0}^{\infty}(r_n^i r_j^i)^k \sum_{l=0}^{k} \mathbb{E}\left(e^{i(2l-k)\theta_n^i}\right)\mathbb{E}\left(e^{i(k-2l)\theta_2^i}\right)\\
=& \left(1-(r_n^i)^2\right)\left(1-(r_j^i)^2\right)\sum_{k=0}^{\infty}(r_n^i r_j^i)^k \sum_{l=0}^{k} \delta_{2l,k}\\
=& \left(1-(r_n^i)^2\right)\left(1-(r_j^i)^2\right)\sum_{k=0}^{\infty}(r_n^i r_j^i)^{2k}\\
=& \frac{\left(1-(r_n^i)^2\right)\left(1-(r_j^i)^2\right)}{1-(r_n^i r_j^i)^2}.
\end{aligned}$$

□

Remark 3.1.11. *Let m and k be two multi-indices in $\mathbb{N}^d$, and suppose that λ_n and λ_j belong to I_m and I_k, respectively. Then, thanks to Lemma 3.1.10 and* (3.1.4),

$$\mathbb{E}(|S^i(n,j)|^2) \simeq \frac{2^{-(m_i+k_i)}}{2^{-m_i}+2^{-j_i}-2^{-(m_i+k_i)}} = \frac{1}{2^{k_i}+2^{m_i}-1} \simeq \frac{1}{2^{k_i}+2^{m_i}}.$$

In particular, since $S^i(n,j)$ and $S^r(n,j)$ are independent for any $i \neq r$, we have

$$\mathbb{E}(|S_d(n,j)|^2) \simeq \prod_{i=1}^{d} \frac{1}{2^{k_i}+2^{m_i}}.$$

Part (ii) of Theorem 3.1.1 can now be proved:

Proof of Theorem 3.1.1, (ii). Observe that

$$\sum_{m\in\mathbb{N}^d} N_m^2 2^{-|m|} \leq \sum_{m\in\mathbb{N}^d} N_m^{1+\frac{1}{d}} 2^{-\frac{|m|}{d}},$$

whenever $N_m \le 2^{|m|}$, and so under our assumption Λ is weakly separated, thanks to Theorem 3.1.1, part (i). Therefore, thanks to (3.1.13), it suffices to show that the random sequence $(S_n)_{n\in\mathbb{N}}$ given by

$$S_n := \sum_{j\neq n} \min_{i=1,\dots,d} |S^i(n,j)|^2$$

is bounded almost surely. Thanks to Lemma 3.1.6, one must show that

$$\sum_{j\in\mathbb{N}} \left(\sum_{n\in\mathbb{N}} \mathbb{E}\left(|S_d(n,j)|^2\right)\right)^{\frac{1}{d}} < \infty. \tag{3.1.14}$$

By regrouping the terms of the double sum in (3.1.14) with respect the partition $(I_m)_{m\in\mathbb{N}^d}$ of $\mathbb{D}^d$ and thanks to Remark 3.1.11 and (3.1.9) we get

$$\begin{aligned}
&\sum_{j\in\mathbb{N}} \left(\sum_{n\in\mathbb{N}} \mathbb{E}\left(|S_d(n,j)|^2\right)\right)^{\frac{1}{d}} \\
&= \sum_{m\in\mathbb{N}^d} \sum_{\lambda_n\in I_m} \left(\sum_{k\in\mathbb{N}^d} \sum_{\lambda_j\in I_k} \mathbb{E}(|S_d(n,j)|^2)\right)^{\frac{1}{d}} \\
&\simeq \sum_{m\in\mathbb{N}^d} N_m \left(\sum_{k\in\mathbb{N}^d} N_k \prod_{i=1}^{d} \frac{1}{2^{m_i}+2^{k_i}}\right)^{\frac{1}{d}}.
\end{aligned}$$

Corollary 3.1.4, $d = s$, concludes the proof. □

Condition (3.1.6) is not sharp. Indeed, for some choices of $(r_n)_{n\in\mathbb{N}}$, we can show that the $0-1$ Kolmogorov law for interpolating sequences coincide with the one for weak separation:

Proposition 3.1.12. *Let $d = 2$ and $(t_n)_{n\in\mathbb{N}}$ be a sequence in $(0,1)$. Let*

$$r_n := (t_{n_1}, t_{n_2}) \qquad n = (n_1, n_2) \in \mathbb{N}^2$$

Then the random sequence Λ associated with $(r_n)_{n\in\mathbb{N}^2}$ is interpolating almost surely if an *only if* (3.1.5) *holds.*

Proof. If $\sum_{m\in\mathbb{N}^2} N_m^2 2^{-|m|} = \infty$, then Λ is not weakly separated almost surely, and in particular it is almost surely not interpolating. Thus it suffices to show that Λ is interpolating provided that $\sum_{m\in\mathbb{N}^2} N_m^2 2^{-|m|} < \infty$, which, by construction of $(r_n)_{n\in\mathbb{N}^2}$, it is equivalent to

$$\sum_{n\in\mathbb{N}} T_n^2 2^{-n} < \infty,$$

where $T_n := \#\{n \in \mathbb{N} \quad | \quad 1 - 2^{-n} \leq t_n < 1 - 2^{-(n+1)}\}$. By Rudowicz's Theorem, [21], the random sequence T on $\mathbb{D}$ given by

$$\tau_n := t_n e^{i\theta_n} \qquad n \in \mathbb{N}$$

is almost surely interpolating in $\mathbb{D}$, where $(\theta_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and distributed uniformly on the unit circle. In particular, T has almost surely a sequence of P. Beurling functions, that is, there exists an event Ω' so that $\mathbb{P}(\Omega') = 1$ and, for any ω in Ω', there exists a sequence of $\mathrm{H}^\infty(\mathbb{D})$ functions $(F_{\omega,n})_{n\in\mathbb{N}}$ such that

$$\begin{cases} F_{\omega,n}(\tau_j(\omega)) = \delta_{n,j} \\ \sup_{z\in\mathbb{D}} \sum_{n\in\mathbb{N}} |F_{\omega,n}(z)| < \infty. \end{cases}$$

Let us consider now the product probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$, where $\tilde{\Omega} := \Omega \times \Omega$, $\tilde{\mathcal{A}}$ is the product σ-algebra of $\mathcal{A}$ with itself, and

$$\tilde{\mathbb{P}}(A \times B) = \mathbb{P}(A)\mathbb{P}(B), \qquad A, B \in \mathcal{A}.$$

Then the random variables

$$\theta_{n_1,n_2} \colon \tilde{\Omega} \to \mathbb{T}^2$$

given by

$$\theta_{n_1,n_2}(\omega_1,\omega_2) := (\theta_{n_1}(\omega_1), \theta_{n_2}(\omega_2))$$

are uniformly distributed in $\mathbb{T}^2$ and independent. Thus we can think of the random sequence Λ as

$$\lambda_{n_1,n_2}(\omega_1,\omega_2) := (r_{n_1}e^{i\theta_{n_1}(\omega_1)}, r_{n_2}e^{i\theta_{n_2}(\omega_2)}) \qquad (\omega_1,\omega_2) \in \tilde{\Omega}.$$

Let $\Omega'' := \Omega' \times \Omega'$ and define, for any $n = (n_1, n_2)$ in $\mathbb{N}^2$ and $\tilde{\omega} = (\omega_1, \omega_2)$ in Ω'' the $\mathrm{H}^\infty(\mathbb{D}^2)$ function

$$G_{\tilde{\omega},n}(z_1,z_2) = F_{\omega_1,n_1}(z_1)\ F_{\omega_2,n_2}(z_2) \qquad (z_1,z_2) \in \mathbb{D}^2.$$

Then $(G_{\tilde{\omega},n})_{n\in\mathbb{N}^2}$ is a set of P. Beurling functions for $\Lambda(\tilde{\omega})$, and in particular $\Lambda(\tilde{\omega})$ is interpolating for any $\tilde{\omega}$ in Ω''. Since $\tilde{\mathbb{P}}(\Omega'') = \mathbb{P}(\Omega')^2 = 1$, Λ is interpolating almost surely. □

The argument in Proposition 3.1.12 can be easily extended to any $d > 1$ to show that, whenever the sequence of radii $(r_n)_{n\in\mathbb{N}}$ is the Cartesian product of d sequences in $[0, 1)$, then (3.1.5) encodes all random sequences that are almost surely interpolating. For a general choice of $(r_n)_{n\in\mathbb{N}}$ the following question remains open:

Question 5. *Is any random sequence Λ in $\mathbb{D}^d$ satisfying* (3.1.5) *strongly separated? Or else, does there exist a choice of $(r_n)_{n\in\mathbb{N}}$ so that the random sequence Λ obtained is almost surely weakly separated but not strongly separated?*

Carleson Measures

The same idea that was used for random strong separation works for the proof of Theorem 3.1.1, part (iii), modulo some adaptations. Let $Z = (z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}^d$ and consider the Szegö Grammian

$$G := (S_d(z_n, z_j))_{n,j\in\mathbb{N}}$$

associated with the sequence Z. Therefore,

Theorem 3.1.13. *The following are equivalent:*

(i) μ_Z *is a Carleson measure for* $\mathrm{H}^2(\mathbb{D}^d)$*;*

(ii) $G\colon l^2 \to l^2$ *is bounded.*

A proof of Theorem 3.1.13 can be found in [3, Th. 9.5]. Moreover, any sufficiently strong decay of the coefficients of G outside its diagonal implies that G is bounded:

Lemma 3.1.14. *Let* $A = (a_{n,j})_{n,j\in\mathbb{N}}\colon l^2 \to l^2$ *such that* $a_{i,i} = 1$ *for any* i *in* $\mathbb{N}$*, and suppose that*

$$\sum_{j\in\mathbb{N}} \sum_{n\neq j} |a_{n,j}|^2 = M^2 < \infty.$$

Then $||G|| \leq 1 + M$*.*

Proof. Let $e_i = (0, \ldots, 0, 1, 0 \ldots)$ be the ith element of the canonical basis of l^2, and pick any $v := \sum_{i\in\mathbb{N}} c_i e_i$ in l^2. By assumption,

$$A(e_i) = e_i + \alpha_i,$$

where α_i is the element of l^2 that is 0 in the i-th slot and equal to the i-th column of A elsewhere. By assumption,

$$\sum_{i\in\mathbb{N}} ||\alpha_i||^2_{l^2} = M^2,$$

and therefore

$$\begin{aligned}||A(v)||_{l^2} &= \left|\left|\sum_{i\in\mathbb{N}}(c_ie_i + c_i\alpha_i)\right|\right|_{l^2} \\ &\leq \left|\left|\sum_{i\in\mathbb{N}} c_ie_i\right|\right|_{l^2} + \sum_{i\in\mathbb{N}} |c_i|\ ||\alpha_i||_{l^2} \\ &\leq ||v||_{l^2} + \left(\sum_{i\in\mathbb{N}} |c_i|^2\right)^{\frac{1}{2}} \left(\sum_{i\in\mathbb{N}} ||\alpha_i||_{l^2}^2\right)^{\frac{1}{2}} \\ &\leq ||v||_{l^2}(1+M),\end{aligned}$$

ending the proof. □

Let Λ be a random sequence in $\mathbb{D}^d$. Thanks to Lemma 3.1.14, to show that $\mathbb{P}(\mathcal{C}(\mathrm{H}^2(\mathbb{D}^d))) = 1$ it is enough to show that the random Grammian associated to Λ has a strong decay outside its diagonal almost surely:

Theorem 3.1.15. *Let Λ be a random sequence in $\mathbb{D}^d$ such that* (3.1.5) *holds. Then* $\mathbb{P}(\mathcal{C}(\mathrm{H}^2(\mathbb{D}^d))) = 1$.

Proof. It suffices to show that

$$\sum_{j\in\mathbb{N}}\sum_{n\neq j}\mathbb{E}(|S_d(n,j)|^2) < \infty. \tag{3.1.15}$$

Indeed, if (3.1.15) holds, then

$$\sum_{j\in\mathbb{N}}\sum_{n\neq j}|S_d(n,j)|^2 < \infty \tag{3.1.16}$$

almost surely, and Lemma 3.1.14 would conclude the proof. By Remark 3.1.11 and by regrouping the sum in (3.1.15) with respect the partition $(I_m)_{m\in\mathbb{N}^d}$ of $\mathbb{D}^d$, one obtains

$$\sum_{j\in\mathbb{N}}\sum_{n\neq j}\mathbb{E}(|S_d(n,j)|^2)\leq C\sum_{m,k\in\mathbb{N}^d}N_mN_k\left(\prod_{i=1}^d\frac{1}{2^{m_i}+2^{k_i}}\right).$$

Corollary 3.1.4, $s=1$, concludes the proof. □

As for a necessary condition for $\mathbb{P}(\mathcal{C}(\mathrm{H}^2(\mathbb{D}^d)))=1$, the following result from [3, Prop. 9.11] will be used:

Proposition 3.1.16. *Let Z be sequence in a subset X of $\mathbb{C}^d$, and let $\mathcal{H}_k$ be reproducing kernel Hilbert space of analytic functions on X. Suppose that the Grammian*

$$G_k:=(k(z_n,z_j))_{n,j\in\mathbb{N}}$$

is bounded. Then Z is the union of finitely many weakly separated sequences with respect ρ_k.

Here $X=\mathbb{D}^d$, $k=s_d$, and therefore G_k being bounded is equivalent to μ_Z being a Carleson measure for $\mathrm{H}^2(\mathbb{D}^d)$. Thanks to Remark 3.1.9 and Proposition 3.1.16, $\mathbb{P}(\mathcal{C}(\mathrm{H}^2(\mathbb{D}^d)))=0$ whenever the sum in (3.1.5) diverges. This concludes the proof of Theorem 3.1.1, part (iii).

As we saw in Theorem 1.2.3, the reproducing kernel Hilbert space approach to interpolating sequences uses a whole class of admissible kernels, as the Szegö kernel does not have the Pick property on the polydisc. Nevertheless, (3.1.16) is a rather strong separation condition on the d-dimensional Szegö kernel, so that one can ask wheter it is sufficient for a sequence of points in the polydisc to be interpolating:

Question 6. *Is any sequence in the polydisc whose Szegö Grammian satisfies* (3.1.16) *an interpolating sequence?*

We do not know the answer of Question 6. One of the reasons that makes such a question interesting for us is that (3.1.16) is satisfied almost surely for a random sequence in the polydisc whenever (3.1.15) holds, and therefore a positive answer to Question 6 would imply that the 0-1 Kolmogorov law for almost sure random interpolating sequences coincides with the sum

$$\sum_{m\in\mathbb{N}^d} N_m^2 2^{-|m|}$$

being finite.

3.1.3 An Application to Separated Joint Spectra

Random interpolating sequences help to construct examples of interpolating sequences of d-tuples of commuting matrices of any dimensions. Let indeed $(m_n)_{n\in\mathbb{N}}$ be a sequence of positive integeres, and choose, for any n in $\mathbb{N}$, m_n points on the d-torus $\tau_{n,1},\dots,\tau_{n,m_n}$. A sequence of d-tuples of commuting matrices having those points as their joint spectra is $W=(W_n)_{n\in\mathbb{N}}$, where

$$W_n^i := \operatorname{diag}(\tau_{n,1}^i,\dots,\tau_{n,m_n}^i).$$

In order for the joint spectra to belong to $\mathbb{D}^d$, let us re-scale the matrices in W via a sequence $(r_n)_{n\in\mathbb{N}}$ in $[0,1)^d$:

$$A_n := (r_n^1 W_n^1,\dots,r_n^d W_n^d) \qquad n\in\mathbb{N}.$$

The more sparse the sequence $(r_n)_{n\in\mathbb{N}}$ is the more separated the joint spectra of the matrices in $A:=(A_n)_{n\in\mathbb{N}}$ are. It is natural then to ask if there is a choice of the radii $(r_n)_{n\in\mathbb{N}}$ that makes the sequence A interpolating, for some choice of the sequence $T:=(\tau_{n,j})$. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}^d$ and let

$$r_n^i = 1-2^{-\alpha_n^i}.$$

If the point in T are uniformly randomly chosen $\mathbb{T}^d$, independently one from the other, then Corollary 3.1.2 says that if

$$\sum_{n\in\mathbb{N}} m_n^{1+\frac{1}{d}}\ 2^{-\frac{(\alpha_n^i+\cdots+\alpha_n^d)}{d}} < \infty$$

then the collection Z of all joint eigenvalues of A is interpolating almost surely. Fixed $(m_n)_{n\in\mathbb{N}}$, it will suffice then to choose the sequence $(\alpha_n)_{n\in\mathbb{N}}$ diverging fast enough in order to have a whole family of possible choices of the parameters in T for which Z is interpolating. Thanks to Definition 1.2.5, A is interpolating as well.

3.2 Admissible Kernels and Matrix Nodes

The key step for the proof of Theorem 1.2.9 is to extend Agler's argument in [1] to the matrix case. Let $A_1, \dots, A_N$ be finitely many d-tuples of commuting matrices with spectra in $\mathbb{D}^d$, of eventually different sizes $s_1, \dots, s_N$. Given the formal sets

$$K_j := \{K_j(u,v) | u, v \in \mathbb{C}^{s_j}\} \qquad j = 1, \dots, N$$

define an equivalence relation $\sim_j$ on K_j that makes the elements in K_j linear in v, conjugate linear in u, and that sets $K_j(u_1, v_1) \sim_j K_j(u_2, v_2)$ if and only if

$$\langle A_j^l u_1, v_1 \rangle_{\mathbb{C}^{s_j}} = \langle A_j^l u_2, v_2 \rangle_{\mathbb{C}^{s_j}} \qquad l \in \mathbb{N}^d.$$

Let

$$F_j := K_j / \sim_j \qquad j = 1, \dots, N$$

and let F be the set of formal linear combinations of the elements of $\cup_{j=1}^N F_j$. Since each F_j is a finite dimensional vector space, so is F. A *kernel* k on F is a choice of a strictly positive scalar product on F. Let F_k be the Hilbert space (F, k). In this finite-nodes setting, k is

admissible if, for any $i = 1, \dots, d$, the linear map $T_i^k \colon F_k \to F_k$ defined by

$$T_i^k(K_j(u,v)) := K_j(u, (A_j^i)^* v) \tag{3.2.1}$$

is a contraction. Observe that T is well defined thanks to Lemma 1.2.7. In particular, if k is an admissible kernel on $\mathbb{D}^d$ then F_k is isometric to

$$\text{span}\{H_{A_j}^k \,|\, j = 1 \dots, N\} \subset \mathcal{H}_k.$$

In other words, F_k works as a *restriction* of an admissible kernel to the finite nodes $A_1, \dots, A_N$. Let $\phi_1, \dots, \phi_N$ be targets in $\mathrm{H}^\infty(\mathbb{D}^2)$, and suppose that there exists a contraction ϕ in $\mathrm{H}^\infty(\mathbb{D}^2)$ such that

$$\phi(A_j) = \phi_j(A_j) \qquad j = 1, \dots, N.$$

Then, thanks to (1.2.12), for any kernel in k in $\mathcal{A}_2$ the map $R^k \colon \text{span}\{H_{A_j}^k \,|\, j = 1, \dots, N\} \to \text{span}\{H_{A_j}^k \,|\, j = 1, \dots, N\}$ given by

$$R^k(K_{A_j}(u,v)) = K_{A_j}(u, \phi_j(A_j)^* v) \qquad j = 1, \dots, N \tag{3.2.2}$$

is a restriction of M_ϕ^*, and therefore is a contraction. Conversely,

Theorem 3.2.1. *Let $\phi_1, \dots, \phi_N$ be in $\mathrm{H}^\infty(\mathbb{D}^2)$ such that, if R^k is defined as in (3.2.2),*

$$\sup_{k \in \mathcal{A}_2} ||R^k|| \leq 1. \tag{3.2.3}$$

Then there exists a function ϕ in $\mathrm{H}^\infty(\mathbb{D}^2)$ such that $||\phi||_\infty \leq 1$ and

$$\phi(A_j) = \phi_j(A_j) \qquad j = 1, \dots, N.$$

Theorem 3.2.1 works as a Pick property for all admissible kernels, provided that the necessary bounds are uniform in $\mathcal{A}_2$.

The proof of Theorem 3.2.1 has a strong operator theory flavour: the first main tool is the following Lemma, due to Parrot:

Lemma 3.2.2 (Parrot's Lemma). *Let $\mathcal{H}_i$ and $\mathcal{K}_i$, $i = 1, 2$, be Hilbert spaces, Let $A\colon \mathcal{H}_1 \to \mathcal{K}_1$, $B\colon \mathcal{H}_2 \to \mathcal{K}_1$ and $C\colon \mathcal{H}_1 \to \mathcal{K}_2$ be linear operators. For any $D\colon \mathcal{H}_2 \to \mathcal{K}_2$, let*

$$W_D := \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{K}_1 \oplus \mathcal{K}_2.$$

Then

$$\sup_{D \in \mathcal{B}(\mathcal{H}_2, \mathcal{K}_2)} ||W_D|| = \max \left\{ \left|\left| \begin{bmatrix} A \\ C \end{bmatrix} \right|\right|_{\mathcal{B}(\mathcal{H}_1, \mathcal{K}_1 \oplus \mathcal{K}_2)}, \left|\left| \begin{bmatrix} A & B \end{bmatrix} \right|\right|_{\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{K}_1)} \right\}.$$

For a proof, see [3, Lemma B.1].

The second technical tool will allow us to go from one admissible kernel to another, making a full use of the uniform bound in (3.2.3). Let k be an admissibile kernel on $\mathbb{D}^d$, and let z be a point in $\mathbb{D}^d$. Let

$$H_0 : \operatorname{span}\{H^k_{A_j} \,|\, j = 1, \ldots, N\} \subset \mathcal{H}_k$$

and $H_1 := \operatorname{span}\{H_0, k_z\}$. Assume that k_z is not in H_0, so that H_0 is strictly contained in H_1. Define $\mathcal{G}_0 := H_1 \ominus \operatorname{span}\{k_z\}$, and let $L_0\colon H_0 \to \mathcal{G}_0$ be the restriction on H_0 of the orthogonal projection P_0 onto $\mathcal{G}_0$:

$$L_0 := (P_0)_{|H_0}.$$

As we already pointed out, H_0 is isometric to a finite admissible kernel structure F_k on $A_1, \ldots, A_N$. It turns out that $L(H_0)$ corresponds to a finite admissible kernel structure as well:

Lemma 3.2.3. *Define, for any $j = 1, \dots, N$ and for any u and v in $\mathbb{C}^{s_j}$*

$$G_j(u, v) := L_0(K_{A_j}(u, v)) \in \mathcal{G}_0.$$

Then the vector space

$$\{G_j(u, v) \,|\, j = 1, \dots, N, u, v \in \mathbb{C}^{s_j}\} \subset \mathcal{H}_k$$

together with the inner product

$$g(G_j(u_1, v_1), G_l(u_2, v_2)) := \langle G_j(u_1, v_1), G_l(u_2, v_2) \rangle_{\mathcal{H}_k}$$

is a finite admissible kernel structure on $A_1, \dots, A_N$.

Proof. Observe that H_1 is an admissible kernel structure on the $N+1$ nodes, $A_1, \dots, A_N, z$. Hence the map $\tilde{T}_i^k$ such that

$$\tilde{T}_i^k(x) = \begin{cases} T_i^k(x) \quad \text{if} \qquad x \in H_0 \\ \overline{z^i}\ k_z \quad \text{if} \qquad x = k_z \end{cases} \qquad i = 1, \dots, d$$

extends T_i^k defined in (3.2.1) to a contraction on H_1. Since the maps

$$S_i := P_0(\tilde{T}_i^k)_{|\mathcal{G}_0} \qquad i = 1, \dots, d$$

are contractions, it suffices to show that

$$T_i^g = S_i \qquad i = 1, \dots, d. \tag{3.2.4}$$

Since span$\{k_z\}$ is invariant under each $\tilde{T}_i^k$, we have

$$P_0\tilde{T}_i^k(Id - P_0) = 0 \qquad i = 1, \dots, d,$$

and therefore, for any f in H_0

$$S_iL(f) = P_0\tilde{T}_i^k P_0(f) = P_0\tilde{T}_i^k(f) = P_0T_i^k(f) = LT_i^k(f) \qquad i = 1, \dots, d.$$

Thus $S_i = LT_i^k L^{-1}$ is similar to T_i^k and therefore

$$S_i(G_j(u,v)) = L(K_j(u,(A^i)^*v)) = G_j(u,(A^i)^*v) = T_i^g(G_j(u,v)),$$

proving (3.2.4). □

We are now ready to prove Theorem 3.2.1. In order to do so, it suffices to show that, for any z in $\mathbb{D}^d$, one can choose an optimal w in $\mathbb{C}$ such that, for any admissible kernel, $\tilde{R}^k\colon H_1 \to H_1$ given by

$$\tilde{R}_w^k(x) := \begin{cases} R^k(x) & \text{if} \quad x \in H_0 \\ \overline{w}k_z & \text{if} \quad x = k_z \end{cases} \tag{3.2.5}$$

extends R^k defined in (3.2.2) without changing its norm:

Theorem 3.2.4. *For any z in $\mathbb{D}^d$, there exists a w in $\mathbb{C}$ such that*

$$\sup_{k \in \mathcal{A}_d} ||\tilde{R}_w^k|| = \sup_{k \in \mathcal{A}_d} ||R^k||.$$

Theorem 3.2.1 follows then from Theorem 3.2.4: let $Z = (z_n)_{n \in \mathbb{N}}$ be a *sequence of uniqueness* for $\mathbb{D}^2$, i. e., a sequence such that any holomorphic function is uniquely determined by its values at Z (for example, any sequence converging to 0). By iterating Theorem 3.2.4 we extend R^k isometrically to the nested sub-spaces

$$H_n^k := \text{span}\{H_{n-1}^k, k_{z_n}\} \qquad n \in \mathbb{N}$$

by choosing an optimal value w_i at each step, independently of the admissible kernel k. This leads to construction of a map on the Hardy space $H^2(\mathbb{D}^2)$

$$R\colon \mathrm{H}^2(\mathbb{D}^2) \to \mathrm{H}^2(\mathbb{D}^2)$$

that has all the Szegö kernels as its eigenvetors, and which hence commutes with any $M_{z^i}^*$. Such a map will have then to be the adjoint of multiplication by

$$\phi := R^*(1),$$

and since $M_\phi^* = R$ is a contraction, then $||\phi||_\infty = 1$. Moreover, since M_ϕ^* conicide with $M_{\phi_j}^*$ on each H_j^k, then ϕ agrees with ϕ_j on each pair A_j. This is the only part of the proof that requires d to be equal to 2, since this is the only case in which we know that $\mathcal{A}_2 = \mathcal{B}_2$, and hence the multiplier algebra of each $\mathcal{H}_k$ is $\mathrm{H}^\infty(\mathbb{D}^2)$.

of Theorem 3.2.4. Let k be an admissible kernel on $\mathbb{D}^d$, and let R^k be defined as in (3.2.2). Let $\tilde{R}_w^k$ be the extension in (3.2.5). Define

$$\mathcal{G} := H_1 \ominus \text{span}\{k_z\}$$

$$\mathcal{F} := H_1 \ominus H_0.$$

Split then $\tilde{R}_w^k$ into

$$\tilde{R}_w^k = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{3.2.6}$$

where

$$A\colon H_0 \to \mathcal{G}$$
$$B\colon \mathcal{F} \to \mathcal{G}$$
$$C\colon H_0 \to \operatorname{span}\{k_z\}$$
$$D\colon \mathcal{F} \to \operatorname{span}\{k_z\}.$$

The column $\begin{bmatrix} A \\ C \end{bmatrix}$ is then R^k, while the top row is $\tilde{R}^k_w$ pre-composed with the orthogonal projection P onto $\mathcal{G}$:

$$\begin{bmatrix} A & B \end{bmatrix} = P\tilde{R}^k_w.$$

In particular, $\begin{bmatrix} A \\ C \end{bmatrix}$ does not depend on w. Most importantly, B does not depend on w either, since $\tilde{R}^k_w$ has k_z as one of its eigevenctors and $\mathcal{G}$ is orthogonal to k_z, which in particular implies

$$B = (P\tilde{R}^k_w)_{|\mathcal{F}} = (PR^k)_{|\mathcal{F}}.$$

Therefore *the whole dependence on w in* (3.2.6) *is carried by D.* In particular,

$$D\colon f \in \mathcal{F} \mapsto \frac{\overline{w}||f||^2}{\langle k_z, f\rangle_{\mathcal{H}_k}} k_z$$

ranges among any possible complex number, as w ranges in $\mathbb{D}$. Moreover, observe that if

$$T^k := (P\tilde{R}^k_w)_{|\mathcal{G}},$$

then $||P\tilde{R}^k_w|| \leq ||T^k||$ (and hence the two norms are indeed equal), since for any $v = u + w$

in H_1, where u is in $\mathcal{G}$ and w is in span$\{k_z\}$, one has

$$||P\tilde{R}^k_w(v)|| = ||T^k(u)|| \leq ||T^k||||u|| \leq ||T^k||||v||,$$

thanks to orthogonality. Thus Lemma 3.2.2 implies that there exists a choice of w such that

$$||\tilde{R}^k_w|| = \max\{||R^k||, ||T^k||\}.$$

To conclude, it suffices to observe that thanks to Lemma 3.2.3 $\mathcal{G}$ carries an admissible structure g, and that therefore

$$T^k = p(T^k_1, \ldots, T^k_d)$$

is unitarily equivalent to

$$R^g = p(T^g_1, \ldots, T^g_d),$$

for any polynomial p such that $p(A_j) = \phi_j(A_j)$ for any $j = 1, \ldots, N$. Thus

$$||\tilde{R}^k_w|| = \max\{||R^k||, ||R^g||\} \leq \sup_{g \in \mathcal{A}_d} ||R^g||.$$

Since k was chosen arbitrarily in $\mathcal{A}_d$, this concludes the proof. □

Remark 3.2.5. *As a main consequence of Theorem 3.2.1, the Gleason distance $\rho_G(M, N)$ between two pairs of commuting matrices corresponds to the sine of the least angle between H^k_M and H^k_N, where k ranges among all admissible kernels in $\mathbb{D}^2$. Indeed, if ϕ in $\mathrm{H}^\infty(\mathbb{D}^2)$ has norm C and separates M and N, i.e.*

$$\phi(M) = Id \qquad \phi(N) = 0,$$

then for any admissible kernel k the operator M^*_ϕ acts like the identity on H^k_M and like the *zero operator on* H_N^k*. Since* $||M_\phi^*|| = C$*, the angle between* H_M^k *and* H_N^k *is greater than* $1/C$*. Conversely, if*

$$\inf_{k \in \mathcal{A}_2} \sin(H_M^k, H_N^k) \geq \frac{1}{C},$$

then Theorem 3.2.1 ($A_1 = M, A_2 = N, \phi_1 = 1, \phi_2 = 0$*) implies that there exists a function* ϕ *whose* H^∞ *norm does not exceed* C *that separates* M *and* N*, hence* $\rho_G(M, N) \geq 1/C$*.*

3.2.1 Diagonal Targets

Theorem 3.2.1 allows us to extend Theorem 1.2.3 to pairs of commuting matrices of any sizes:

Theorem 3.2.6. *Let* $A = (A_n)_{n \in \mathbb{N}}$ *be a sequence of pairs of commuting matrices with spectra in the bi-disc, and let* $H^k = (H_n^k)_{n \in \mathbb{N}}$ *be, for any kernel in* $\mathcal{A}_2$*, the associated sequence of closed sub-spaces of* $\mathcal{H}_k$ *defined in* (1.2.9)*. The following are equivalent:*

(i) *A is interpolating;*

(ii) *For any bounded sequence* $(w_n)_{n \in \mathbb{N}}$ *in* $\mathbb{C}$ *there exists a function* ϕ *in* $\mathrm{H}^\infty(\mathbb{D}^2)$ *such that*

$$\phi(A_n) = w_n \; Id \qquad n \in \mathbb{N};$$

(iii) *There exists a* $C > 0$ *such that, for any* k *in* $\mathcal{A}_2$*,* H^k *is a Riesz system in* $\mathcal{H}_k$ *with Riesz bound* C*;*

(iv) *There exists a sequence* $(f_n)_{n \in \mathbb{N}}$ *such that* $f_n(A_j) = \delta_{n,j}$ *and*

$$\sup_{z \in \mathbb{D}^2} \sum_{n \in \mathbb{N}} |f_n(z)| < \infty.$$

The proof of the equivalence between (ii) and (iv) follows the same outline of Theorem

2.1.1: such an argument is indeed valid since Montel's Theorem extends to $\mathrm{H}^\infty(\mathbb{D}^2)$. Therefore, it suffices to show (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i). Moreover, the equivalence between (i) and (ii) says that, as for the one-variable case, the notion of interpolating sequences for pairs of commuting matrices is indeed equivalent to one that a priori is weaker, asking to find interpolating functions for only diagonal targets.

Proof. Trivially, (i) implies (ii), by considering a constant target sequence

$$\phi_n(z) = w_n \qquad z \in \mathbb{D}^2.$$

The implication (ii) $\implies$ (iii) follows from Theorem 1.2.4, by observing that, thanks to (ii), for any admissible kernel k on $\mathbb{D}^2$

$$\mu^k = M_\phi^* \colon \operatorname*{span}_{n\in\mathbb{N}}\{H_n^k\} \to \operatorname*{span}_{n\in\mathbb{N}}\{H_n^k\}$$

whenever

$$(\mu^k)_{H_n^k} = w_n \; Id_{|H_n^k}.$$

Finally, (iii) $\implies$ (i) follows from Theorem 3.2.1, Theorem 1.2.4 and a normal family argument. Indeed, let $(\phi_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathrm{H}^\infty(\mathbb{D}^2)$, and fix a positive integer N. By H^k having a Riesz bound C for any admissible k, one has that the operator R^k defined in (3.2.2) is bounded by C for any k in $\mathcal{A}_2$, thanks to Theorem 1.2.4. Thus by Theorem 3.2.1 there exists a function f_N whose H^∞ norm doesn't exceed C and

$$f_N(A_1) = \phi_1(A_1), \ldots, f_N(A_N) = \phi_N(A_N).$$

Since the sequence $(f_N)_{N\in\mathbb{N}}$ is bounded in H^∞, it has a sub-sequence that converges in $\mathrm{H}^\infty(\mathbb{D}^2)$ to ϕ, which agrees with ϕ_n at A_n, for any n in $\mathbb{N}$. $\square$

Remarkably, both this Euclidean approach and the function theoretical approach in Theorem 2.1.1 for the proof of (i) $\Longleftrightarrow$ (ii) squares the respective constant of interpolation, as one can observe by looking at the above proof and the statement of Theorem 1.2.4, (ii) $\Longleftrightarrow$ (iii).

3.2.2 A More Explicit Sufficient Condition

In this Secion we will prove Theorem 1.2.10. Thanks to Remark 3.2.5, condition (1.2.14) implies that

$$\prod_{n\in\mathbb{N}} \sin(H_n^k, \overline{\operatorname{span}}_{j\neq n}\{H_j^k\}) \tag{3.2.7}$$

is uniformly bounded below in $k \in \mathcal{A}_2$. Thanks to Theorem 1.2.9, it suffices then to show that any sequence $H = (H_n)_{n\in\mathbb{N}}$ of closed subspaces of a Hilbert space $\mathcal{H}$ satisfying (3.2.7) is a Riesz system. In order to do so, we will use Theorem 1.2.4, (i) $\Longleftrightarrow$ (iv). Fix a finite subset σ of $\mathbb{N}$, and enumerate the elements in σ^c by $\{j_1, j_2, \dots\}$. For the sake of brevity, let $H_\sigma := \overline{\operatorname{span}}_{j\in\sigma}\{H_j\}$ and

$$S_i := \operatorname{span}\{H_\sigma, H_{j_1}, \dots, H_{j_i}\} \qquad i \in \mathbb{N}.$$

Namely, if $S_0 = H_\sigma$, then each S_i is obtained by adding a subspace not labeled in σ to the linear span of S_{i-1}. Let P_0 be the identity on H_σ, and define $P_i\colon S_i \to S_i$ by

$$P_i(x) := \begin{cases} P_{i-1}(x) & \text{if} \quad x \in S_{i-1} \\ 0 & \text{if} \quad x \in H_{j_i} \end{cases}.$$

Thanks to Theorem 1.2.4, (i) $\Longleftrightarrow$ (iv), we need to prove the following

Proposition 3.2.7. *Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of closed sub-spaces of a Hilbert space $\mathcal{H}$ such that*

$$\prod_{n\in\mathbb{N}} \sin(H_n, H_{\mathbb{N}\setminus\{n\}}) > 0 \tag{3.2.8}$$

Then

$$\sup_{\sigma\ finite}\ \sup_{i\in\mathbb{N}} ||P_i|| < \infty. \tag{3.2.9}$$

This will be done by analyzing how much does the operator norm increase when extending P_i to P_{i+1}:

Lemma 3.2.8. *Let H and F be two closed subspaces of a Hilbert space $\mathcal{H}$ that intersect trivially. Let d be the distance (equivalently, the sine of the angle) between H and K, and let T be any bounded operator from H to itself. Then the operator $\tilde{T}\colon \operatorname{span}\{H, F\} \to \operatorname{span}\{H, F\}$ such that*

$$\tilde{T}(x) := \begin{cases} T(x) \quad if \qquad x \in H \\ 0 \quad if \qquad\quad x \in F \end{cases}$$

extends T and has norm

$$||\tilde{T}|| \leq \frac{1}{d}||T||.$$

Proof. Let $G := \operatorname{span}\{H, F\} \ominus H$ be the orthogonal complement of H in $\operatorname{span}\{H, F\}$, and fix a unit vector x in $\operatorname{span}\{H, F\}$. Then x can be written uniquely as

$$x = \alpha y + \beta z \qquad y \in G \quad z \in H,$$

where y and z are unit vectors and $|\alpha|^2 + |\beta|^2 = 1$, thanks to orthogonality. Moreover, y can be written as

$$y = h + f \qquad h \in H \quad f \in F,$$

where

$$||h||^2 = \frac{1 - \sin^2(f, H)}{\sin^2(f, H)} \leq \frac{1 - d^2}{d^2}$$

and $\tilde{T}(y) = \tilde{T}(h) = T(h)$. Therefore, if $r := \sqrt{1-d^2}/d$, we have

$$\begin{aligned}||\tilde{T}(x)||^2 =& ||T(\alpha h + \beta z)||^2 \\ \leq& ||T||^2 \ ||\alpha h + \beta z||^2 \\ =& ||T||^2 \ (\alpha^2 r^2 + \beta^2 + 2\alpha\overline{\beta}\mathrm{Re}\,\langle h, z\rangle) \\ \leq& ||T||^2 \ (\alpha^2 r^2 + \beta^2 + 2|\alpha||\beta| r) \\ =& ||T||^2 (r|\alpha| + |\beta|)^2.\end{aligned}$$

Thus

$$||\tilde{T}|| \leq ||T|| \sup_{|\alpha|^2+|\beta|^2=1} (r|\alpha| + |\beta|) = ||T|| \ \sqrt{1+r^2} = \frac{||T||}{d}.$$

□

In particular,

$$||P_n||^2 \leq \prod_{i=1}^{n} \frac{1}{\sin^2(H_{j_i}, S_i)} \leq \prod_{i=1}^{n} \frac{1}{\sin^2(H_i, H_{\mathbb{N}\setminus\{i\}})}$$

is uniformly bounded in n (and σ) if

$$\prod_{i=1}^{n} \sin(H_i, H_{\mathbb{N}\setminus\{i\}})$$

is bounded below, and Proposition 3.2.7 follows. This concludes the proof of Theorem 1.2.10.

Chapter 4

NC-Interpolating Sequences

4.1 The NC Pick Property

The main goal of this final Chapter is to prove Theorem 1.3.1 and Theorem 1.3.2. The key tool for doing so can be found in [22, Th. 4.7], where Salomon, Shalit and Shamovich proved that $\mathcal{H}^2_d$ has the following non-commutative version of the Pick property: let Ω be a subset of $\mathfrak{B}_d$, and let $\overline{\Omega}^{nc}$ be its NC-envelop, that is, the smallest NC set containing Ω that is also closed under left intertwiners:

$$W \in \overline{\Omega}^{nc} \implies P^{-1}WP \in \overline{\Omega}^{nc}.$$

Suppose $f_0\colon \Omega \to \mathbb{M}_1$ is an NC function that extends to an NC function on $\overline{\Omega}^{nc}$ and suppose that the map

$$R_0\colon \operatorname{span}\{K_W(u,v)\,|\,W \in \Omega\} \to \operatorname{span}\{K_W(u,v)\,|\,W \in \Omega\}$$

such that, for any W in $\Omega(n)$,

$$R_0(K_W(u,v)) = K_W(u, f_0(W)^* v) \qquad u, v \in \mathbb{C}^n$$

is a contraction. Then f_0 extends to a contractive multiplier on $\mathfrak{B}_d$, that is, there exists a function f in $\mathcal{H}^\infty_d$ whose norm doesn't exceed 1 such that $f_{|\Omega} = f_0$.

With this in mind we can partially extend Theorem 3.2.6 to this non-commutative setting, and prove Theorem 1.3.1:

Theorem 4.1.1. *Let $Z = (Z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{B}_d$. The following are equivalent:*

(i) *Z is interpolating;*

(ii) *For any bounded $(w_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ there exists a function ϕ in $\mathcal{H}_d^\infty$ such that*

$$\phi(Z_n) = w_n \ Id \qquad n \in \mathbb{N}; \tag{4.1.1}$$

(iii) *The sequence $(\mathcal{H}_n)_{n\in\mathbb{N}}$ defined in (1.3.4) is a Riesz System.*

Proof. The implication (i) $\implies$ (ii) is trivial, as constant functions belong to $\mathcal{H}_d^\infty$. Moreover, (ii) $\implies$ (iii) follows from Theorem 1.2.4, since for any ϕ in $\mathcal{H}_d^\infty$ satisfying (4.1.1) the restriction of M_ϕ^* to $\operatorname{span}\{\mathcal{H}_n \,|\, n \in \mathbb{N}\}$ is a bounded linear operator that acts like $w_n \ Id$ on each $\mathcal{H}_n$. To conclude, it suffices to show then that (iii) $\implies$ (i). To do so, let $(\phi_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{H}_d^\infty$. There exists an function $\phi_0 \colon \overline{Z}^{nc} \to \mathbb{M}_1$ such that

$$\phi_0(Z_n) = \phi_n(Z_n) \qquad n \in \mathbb{N},$$

since each ϕ_n respects direct sums and left intertwiners. Since $(\phi_n)_{n\in\mathbb{N}}$ is bounded, so is the sequence of operator norms $(||R_n||)_{n\in\mathbb{N}}$, where

$$R_n := (M_{\phi_n}^*)_{|\mathcal{H}_n}$$

is the linear map from $\mathcal{H}_n$ to itself such that

$$R_n(K_{Z_n}(u,v)) = K_{Z_n}(u, \phi_n(Z_n)^* v) = K_{Z_n}(u, \phi_0^*(Z_n) v).$$

Thanks to Theorem 1.2.4, the map R_0 from $\text{span}\{\mathcal{H}_n \,|\, n \in \mathbb{N}\}$ to itself such that

$$R_0(K_{Z_n}(u,v)) = K_{Z_n}(u, \phi_0^*(Z_n)v)$$

is bounded, and by the NC Pick property there exists a multiplier ϕ in $\mathcal{H}_d^\infty$ such that

$$\phi(Z_n) = \phi_0(Z_n) = \phi_n(Z_n) \qquad n \in \mathbb{N},$$

concluding the proof. $\square$

Remark 4.1.2. *As in Remark 2.1.3, we would like to point out how the a-priori weaker interpolation property in (ii) is in fact equivalent for Z to be interpolating. Moreover, thanks to Theorem 1.2.4, the respective interpolating constants are one the square of the other, as we saw for the commutative and the one-variable case.*

Theorem 1.3.1 is not the only main consequence of the NC Pick property of the NC Drury-Arveson space. Indeed, since the existence of a function ϕ in $\mathcal{H}_d^\infty$ of norm M separating two points in $\mathfrak{B}_d$

$$\phi(Z) = Id \qquad \phi(W) = 0$$

is equivalent to the operator R: $\text{span}\{\mathcal{H}_W, \mathcal{H}_Z\} \to \text{span}\{\mathcal{H}_W, \mathcal{H}_Z\}$ such that

$$R_{|\mathcal{H}_Z} = Id \qquad R_{|\mathcal{H}_W} = 0$$

being bounded by M, one has that the Gleason NC distance between Z and W coincide with the sine of the angle between $\mathcal{H}_Z$ and $\mathcal{H}_W$ in the Hilbert space $\mathcal{H}_d^2$:

$$\rho_{NC}(Z,W) = \sin(\mathcal{H}_Z, \mathcal{H}_W).$$

As a consequence, (1.3.6) can be re-written as

$$\prod_{n\in\mathbb{N}} \sin(\mathcal{H}_n, \overline{\operatorname{span}}_{j\neq n}\{\mathcal{H}_j\}) > 0,$$

and Theorem 1.3.2 follows from Theorem 1.3.1 and Proposition 3.2.7.

4.2 An Example

We give here an example of an interpolating sequence of pairs of non-commuting matrices. Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ be two sequences of complex numbers, and define the sequence $(Z_n)_{n\in\mathbb{N}}$ in $\mathbb{M}_2$ as

$$Z_n^1 := \begin{bmatrix} 0 & \alpha_n \\ 0 & 0 \end{bmatrix} \quad Z_n^2 := \begin{bmatrix} 0 & 0 \\ \beta_n & 0 \end{bmatrix} \qquad n \in \mathbb{N}.$$

Then Z_n^1 and Z_n^2 do not commute for any n in $\mathbb{N}$, and since

$$Z_n^1(Z_n^1)^* + Z_n^2(Z_n^2)^* = \begin{bmatrix} |\alpha_n|^2 & 0 \\ 0 & |\beta_n|^2 \end{bmatrix}$$

we have that Z belongs to $\mathfrak{B}_2$ if and only if any (α_n, β_n) belongs to the bi-disc $\mathbb{D}^2$. We claim that a separation condition on the sequence $(\alpha_n, \beta_n)_{n\in\mathbb{N}}$ encodes all the cases in which Z is an NC interpolating sequence:

Theorem 4.2.1. *Z is interpolating if and only if $(\alpha_n\beta_n)_{n\in\mathbb{N}}$ is interpolating in $\mathbb{D}$.*

Thanks to Theorem 4.1.1, we need to study separation conditions of the subspaces $(\mathcal{H}_n)_{n\in\mathbb{N}}$ of $\mathcal{H}_2^2$, and according to (1.3.5) this is equivalent to studying the NC kernel functions of the form $(K_{Z_n}(u_n, v_n))_{n\in\mathbb{N}}$, for any sequence $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ in $\mathbb{C}^2$. Observe that

$$(Z_n^i)^2 = 0 \qquad n \in \mathbb{N}, i = 1, 2$$

and therefore, out of all the coefficients of the NC function

$$K_{Z_n}(u_n, v_n)(Z) = \sum_{l \in \mathbb{W}_2} \langle v_n, Z_n^l u_n \rangle \, Z^l \qquad Z \in \mathfrak{B}_2, \tag{4.2.1}$$

the only non zero ones are the ones associated to a word which alternates the two letters a and b. Such words can be written in four different ways:

$$\omega = (ab)^l \qquad l \in \mathbb{N}$$
$$\omega = (ba)^l \qquad l \in \mathbb{N}$$
$$\omega = a(ba)^l \qquad l \in \mathbb{N}$$
$$\omega = b(ab)^l \qquad l \in \mathbb{N},$$

depending on its lenght and its first and last letter. Since

$$Z_n^1 Z_n^2 = \begin{bmatrix} \alpha_n \beta_n & 0 \\ 0 & 0 \end{bmatrix} \quad Z_n^2 Z_n^1 = \begin{bmatrix} 0 & 0 \\ 0 & \alpha_n \beta_n \end{bmatrix}$$
$$Z_n^2 Z_n^1 Z_n^2 = \begin{bmatrix} 0 & 0 \\ \beta_n \alpha_n \beta_n & 0 \end{bmatrix} \quad Z_n^1 Z_n^2 Z_n^1 = \begin{bmatrix} 0 & 0 \\ 0 & \alpha_n \beta_n \alpha_n \end{bmatrix},$$

we can split the NC power series in (4.2.1) into four *orthogonal* pieces

$$\begin{aligned}
K_{Z_n}(u_n,v_n)(Z) = & \langle v_n, u_n\rangle + v_n^1\overline{u_n^1}\ \sum_{l=1}^{\infty}(\overline{\alpha_n\beta_n})^l(Z^1Z^2)^l + v_n^2\overline{u_n^2}\ \sum_{l=1}^{\infty}(\overline{\alpha_n\beta_n})^l(Z^2Z^1)^l + \\
& + \overline{\alpha_n}v_n^1\overline{u_n^2}\ \sum_{l=1}^{\infty}(\overline{\alpha_n\beta_n})^l Z^1(Z^2Z^1)^l + \overline{\beta_n}v_n^2\overline{u_n^1}\ \sum_{l=1}^{\infty}(\overline{\alpha_n\beta_n})^l Z^2(Z^1Z^2)^l \\
= & v_n^1\overline{u_n^1}\ \sum_{l=0}^{\infty}(\overline{\alpha_n\beta_n})^l(Z^1Z^2)^l + v_n^2\overline{u_n^2}\ \sum_{l=0}^{\infty}(\overline{\alpha_n\beta_n})^l(Z^2Z^1)^l + \qquad (4.2.2) \\
& + \overline{\alpha_n}v_n^1\overline{u_n^2}\ \sum_{l=1}^{\infty}(\overline{\alpha_n\beta_n})^l Z^1(Z^2Z^1)^l + \overline{\beta_n}v_n^2\overline{u_n^1}\ \sum_{l=1}^{\infty}(\overline{\alpha_n\beta_n})^l Z^2(Z^1Z^2)^l. \\
= & \sum_{m,j=1}^{2} v_n^m\overline{u_n^j}K_{Z_n}(e^m,e^j)(Z),
\end{aligned}$$

where $\{e^1, e^2\}$ is the standard basis of $\mathbb{C}^2$.

of Theorem 4.2.1. The sequence $(\hat{K}_n(e^1,e^1))_{n\in\mathbb{N}}$ of normalized NC kernels can be unitarily identified with a multiple of the sequence $(s_{\alpha_n\beta_n})_{n\in\mathbb{N}}$ of Szegö kernels at the points of $(\alpha_n\beta_n)_{n\in\mathbb{N}}$ in $\mathbb{D}$. Thus if $(\alpha_n\beta_n)_{n\in\mathbb{N}}$ is not interpolating, the sequence $(\hat{s}_{\alpha_n\beta_n})_{n\in\mathbb{N}}$ is not a Reisz system, and therefore $(\hat{K}_n(e^1,e^1))_{n\in\mathbb{N}}$ is not a Riesz system either. Hence, thanks to Theorem 4.1.1, Z is not interpolating.

Conversely, assume that $(\alpha_n\beta_n)_{n\in\mathbb{N}}$ is an interpolating sequences in $\mathbb{D}$ or, equivalently, that the associated sequence $(\hat{s}_{\alpha_n\beta_n})_{n\in\mathbb{N}}$ of Szegö kernels in H^2 is a Riesz system. Thanks to (4.2.2), both the sequences $(\hat{K}_{Z_n}(e^1,e^1))_{n\in\mathbb{N}}$ and $(K_{Z_n}(e^2,e^2))$ are untarly equivalent to $(\hat{s}_{\alpha_n\beta_n})_{n\in\mathbb{N}}$ in H^2, and hence are Riesz systems. The same holds for $(\hat{K}_{Z_n}(e^1,e^2))_{n\in\mathbb{N}}$ and $(\hat{K}_{z_n}(e^2,e^1))_{n\in\mathbb{N}}$, since they are both unitarly equivalent to a multiple of the sequence of shifted Szegö kernels $(zs_{\alpha_n\beta_n})_{n\in\mathbb{N}}$. Since the four sequences

$$(K_{Z_n}(u^m,u^j))_{n\in\mathbb{N}} \qquad m,j=1,2$$

are pairwise orthogonal in $\mathcal{H}^2_2$, we can conclude that the sequence of subspaces

$$\mathcal{H}_n = \{K_{Z_n}(u_n, v_n) \,|\, u_n, v_n \in \mathbb{C}^2\} \qquad n \in \mathbb{N}$$

is a Riesz system in $\mathcal{H}^2_2$, thanks to Lemma 4.2.2 below. $\square$

Lemma 4.2.2. *Let m be a finite positive integer, and let $X^i := (x^i_n)_{n\in\mathbb{N}}$, $i = 1, \ldots, m$ be m pairwise orthogonal sequences in a Hilbert space $\mathcal{H}$. If each X^i is a Riesz system, then so is the sequence $X = (x_n)_{n\in\mathbb{N}}$ defined as*

$$x_n := x^1_n + \cdots + x^m_n \qquad n \in \mathbb{N}.$$

Proof. Without loss of generality, assume that $||x_n|| = 1$ for any n in $\mathbb{N}$. Thus

$$x_n = t^1_n \hat{x}^1_n + \cdots + t^m_n \hat{x}^m_n,$$

where thanks to orthogonality

$$\sum_{i=1}^{m} |t^i_n|^2 = 1 \qquad n \in \mathbb{N}. \tag{4.2.3}$$

Fix then an arbitrary $(a_n)_{n\in\mathbb{N}}$ in l^2, and observe that thanks to orthogonality

$$\left|\left|\sum_{n\in\mathbb{N}} a_n x_n\right|\right|^2 = \sum_{i=1}^{m} \left|\left|\sum_{n\in\mathbb{N}} a_n t^i_n \hat{x}^i_n\right|\right|^2.$$

Therefore, if C_i is the Riesz bound of each X^i,

$$\left|\left|\sum_{n\in\mathbb{N}} a_n x_n\right|\right|^2 \le \sum_{i=1}^{m} C_i^2 \sum_{n\in\mathbb{N}} |t^i_n|^2 |a_n|^2 \le \max_{i=1,\ldots,m} C_i^2 \sum_{n\in\mathbb{N}} |a_n|^2$$

and

$$\frac{1}{\max\limits_{i=1,\ldots,m} C_i^2} \sum_{n\in\mathbb{N}} |a_n|^2 \leq \sum_{i=1}^{m} \frac{1}{C_i^2} \sum_{n\in\mathbb{N}} |t_n^i|^2 |a_n|^2 \leq \left\| \sum_{n\in\mathbb{N}} a_n x_n \right\|^2,$$

thanks to (4.2.3). □

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