Essays on Information and Liquidity

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Olin Business School

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Essays on Information and Liquidity
by
Swaminathan Balasubramaniam

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# Table of Contents

Acknowledgements ...........................................................................................................iii

Abstract ..........................................................................................................................v

1. Information sharing among strategic traders: The role of disagreement ..................1

Bibliography ..................................................................................................................43

Appendix ......................................................................................................................48

2. Uncertainty about uncertainty in coordination games ............................................. 82

Bibliography ..................................................................................................................113

Appendix ......................................................................................................................115

3. Liquidity portfolios ...................................................................................................125

Bibliography ..................................................................................................................141
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ABSTRACT OF THE DISSERTATION

Essays on Information and Liquidity
by
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Doctor of Philosophy in Finance

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Professors Philip H. Dybvig and Jonathan Weinstein, (Co-Chairs)

The 2008 financial crisis has highlighted the challenges faced by financial systems in aggregating information, ensuring coordination among various market participants and providing adequate liquidity. In this backdrop, the three chapters of the dissertation explore (i) the role of disagreement in enabling communication and trade among strategic investors (ii) how uncertainty about what others know could be exploited in preventing coordination failures (ii) pricing liquid and illiquid assets in response to unforeseen liquidity demand.

When do competing traders, endowed with different pieces of information pertaining to a security payoff, exchange information before trading? The first essay shows that competing traders share information when they disagree enough. Traders can lose competitive rents by sharing private information, but with sufficient disagreement, they can engage in profitable belief arbitrage by trading against each other's signal. Traders, however, would gain by over-reporting their signals so that competitors make large opposing trades. When information is verifiable, truthful disclosure emerges due to an “unraveling” argument. Mediators (say, sell-side analysts or brokers) could facilitate partial information sharing by aggregating and distributing
information in an incentive compatible manner. Disagreement makes the market more liquid, but information sharing undermines the liquidity benefits.

IPOs, currency attacks, bank runs and stag hunt games all admit a sub-optimal equilibrium in addition to a superior equilibrium where agents co-ordinate on a Pareto optimal, but risky action. The second essay studies situations where players are uncertain about what others know, and analyzes how this second order uncertainty can be useful for achieving coordination among agents. Holding first order beliefs fixed and modifying second order uncertainty, the policymaker designs simple information structures with contagion to induce his preferred action as often as possible. The policymaker chooses information structures where states in his favor are allowed to infect other states, while states that are not in his favor are quarantined. In a stylized bank run model, a bank manager allows perfect information when bank fundamentals are weak, but allows agents to be uncertain about the other agent's knowledge in all other scenarios. This provides a rationale for accounting conservatism in banking: all probable loan losses are publicly recorded when they are discovered, while knowledge of investment gains are privately dispersed among agents.

The third essay studies a model of cash management using a portfolio of assets with varying degrees of liquidity. The agent faces a probabilistic interim liquidity shock and chooses which assets to liquidate. Assets with high liquidation value are valuable in states when the liquidity shock is high or there is a scarcity of other liquid assets in the economy. The downsides of such assets are protected by their expected terminal value, which delivers an option-like convex pricing rule: the price of a liquid asset is an American put option with strike price process given by the liquidation value in each period. The model in this essay explains the empirically observed convex liquidity premium even with a single representative agent.
Chapter 1: Information Sharing Among Strategic Traders: The Role of Disagreement

1 Introduction

When do competing traders share information with each other? The primary explanations for information sharing in financial markets center around (i) collusion or quid pro (as in Hong et al. [2005], where fund managers based in the same city trade in a correlated manner, pointing towards word of mouth communication), or (ii) exploitation of short term movements in stocks (as in Ljungqvist and Qian [2016], where hedge funds disclose information to make uninformed market participants take similar positions as theirs and drive up the price). This paper describes another natural milieu for information exchange: that is, when traders derive private valuations from an asset and find counterparties to exploit gains from trade. This setup is particularly relevant for equity markets or forex markets the fundamental value of an asset is hard to ascertain with complete objectivity and disagreement.

---

1 In 2018, the United States Department of Justice charged three forex traders, working at competing investment banks, with sharing private information in a chat room and using it to their advantage. The precise charge was: "participating in telephone calls and electronic messages, including engaging in near-daily conversations in a private electronic chat room...and discussing, among other things, past, current, and future customer orders and trades; customer names; and risk positions."
about the value of an asset is central to trading.\textsuperscript{2}

This paper builds a model of incentive compatible information exchange motivated by disagreement and analyzes the implications of information sharing on market liquidity. Traders usually make greater profit when they successfully hide their private information from each other. With disagreement, however, they can bet against each other’s beliefs by sharing information and profit from belief arbitrage. While collusion or short termism makes investors trade in a correlated fashion, information sharing due to disagreement (or heterogeneous valuations) result in anti-correlated trades between the sharing parties.

Despite benefits of sharing, truthful communication is a challenge, as traders could actively mislead their competitors to make larger bets than they should. I show that full disclosure can be enforced if information is verifiable. Otherwise, partial disclosure is possible either through cheap talk or a trustworthy third party (a mediator). When disagreeing investors share information and trade against each other, less information is impounded into prices, thus increasing transaction costs for the market. Similar to Harris and Raviv \cite{harris1993}'s argument that “differences of opinion makes a horse race”, in our context, disagreement (without information sharing) makes the market more liquid. Full information sharing, however, undermines the liquidity benefits.

Consider two large investors who trade as in Kyle \cite{Kyle1985}\textsuperscript{3}, taking into account that their trade could impact the stock price. Each investor receives different pieces of information pertaining to a security’s payoff. The two investors disagree about how the different pieces of information aggregate to determine the overall payoff the security: each investor overweights his own signal and underweights the other agent’s signal when evaluating the aggregate

\textsuperscript{2}For example, in the foreign exchange market, Lyons \cite{lyons1995} points out that information sharing platforms (either telephonic or chatrooms) an essential to trading

\textsuperscript{3}The Kyle model features three types of participants: (i) risk-neutral informed traders (ii) liquidity traders and (iii) competitive and risk-neutral uninformed market-makers who determine the clearing price.
payoff. In this case, an informed agent behaves in an overconfident manner when interpreting his signal, and is somewhat dismissive of the other agent’s information. There could be other valid interpretations of disagreement in our model. Each investors could disagree about the weight of each business to the overall cash flow of the asset. In a rational interpretation of disagreement, the investor has his own private valuation for the asset, and deems a particular source of uncertainty more crucial, either for consumption or hedging purposes. In the forex trading example, a trader’s private valuation of a forex asset would reflect the existing balance sheet exposure of the bank.

Suppose that Investor 1 receives positive news about a stock, based on which he plans to submit a large buy order. This puts upward price pressure on the stock. If Investor 2 deems Investor 1’s information inconsequential for predicting the stock price, he would take a short position and relieve the price pressure on the stock for Investor 1. Essentially, by sharing information, disagreeing traders find a counterparty to take opposite sides in a bet and profit from belief arbitrage. On the other hand, when agents with common priors communicate with each other, they form common posterior beliefs (Geanakoplos and Polemarchakis [1982]; Aumann [1976]) and end up trading in the same direction of each other’s information. Prices reflect the true value of the asset more closely and hence become more informative. This eats into the information rents that traders enjoy from their private information, thus intensifying competition. Under sufficient disagreement, profits from belief arbitrage take precedence and traders find it optimal to share information ex-ante. Our result suggests that different information sets alone is insufficient for information exchange: competing market participants must have sufficiently different models for evaluating asset payoffs.  

\footnote{Goldstein et al. [2020] show that one-sided information sharing is possible with common priors but different information. In their paper, coarsely informed investors share information with well informed}
Although sharing information is ex-ante optimal for disagreeing agents, an investor gains from misreporting his signal to as extreme a value as possible, while keeping the sign intact. This enhances profits from belief arbitrage by making the opponent trade as large (positive or negative) quantities as possible, in the opposite direction. Hence, the initial set of results in this paper require a commitment by agents to truthfully report their signals. In Section 3 of this paper, I relax the ex-ante commitment assumption and study communication protocols that enable truthful information sharing. When information is verifiable, I show that full disclosure ensues due to an unraveling argument. Extreme signal realization types reveal their signal to distinguish themselves from the lower types. The next highest type also reveal themselves to separate from the next lower types and so on.

When information is unverifiable, I first construct an informative cheap talk equilibrium with a unique threshold. Disagreeing traders reveal if their signal is good or bad relative to the threshold. I show that it will not be possible, however, to construct a cheap talk equilibrium which is more informative than the two partition scenario. To improve on the cheap talk equilibrium, I consider the role of a third-party mediator (say for e.g. a sell-side research analyst) who pools information costlessly from informed agents and distributes it back to them in an incentive compatible manner. This is in line with the idea that brokers and sell-side analysts act as valuable middlemen to help traders ascertain the “mood of the market”. In essence, I takes the information sales problem of Admati and Pfleiderer investors so that the latter trades against the former’s information and relieves price pressure. I thank Liyan Yang for pointing this out.

5 This commitment could be enforced due to reputation effects

6 To solve for the communication protocols in a tractable manner, I assume that the market-maker fixes prices according to an exogenously given upward-sloping supply curve, akin to a Cournot duopoly model with an uncertain demand intercept. The “Cournot approximation” of the Kyle model has been used, for example, in Lambert et al. [2018] and Bergemann and Morris [2019].

7 This result is a departure from the non-existence of cheap talk protocols in Cournot duopoly, where the firms are constrained to produce only positive quantities.

8 The value added of sell-side analysts has been questioned in the literature with some evidence (see for e.g.
where a monopolistic intermediary is endowed with information, to a case where the intermediary does not have any information of his own, but merely pools information from his clients. Even in the absence of any informational advantage, I show that intermediaries can add value for their clients. Mediators can implement a communication protocol that further divides the cheap talk thresholds into two partitions each. This is because the mediator can play the role of an information filter between agents: the information that the agent gets about a competitor’s report depends on the agent’s own report to the mediator. In particular, investors receive compensation in the form of extra information for reporting intermediate values of signal realization and receive no additional information when they report high values (either positive or negative). Therefore, mediation complements the established role of intermediaries as producing information to bridge information asymmetries in the market.  

If the investors come to the table to share information, would colluding towards submitting a single trade order become a real possibility? Not surprisingly, under common valuations, agents make greatest profits ex-ante when they both share information and collude towards submitting an order to the market-maker. Analyzing collusion under disagreement is trickier than under common priors. It is imperative that a cartel agreement aggregates heterogeneous beliefs of the agents, in addition to aggregating information. I assume that the cartel appoints a firm authority who weights the belief of each informed agent in a convex manner to arrive at the single demand order of the “firm” or the cartel.  

This assumption is practically relevant for analyzing investment firms where teams of different analysts

\[1986\] Swem [2019]) suggesting that buy-side analysts, who are employed by investment firms, often have earlier access to information that is obtainable by sell-side analysts.

\[9\] See for eg Ramakrishnan and Thakor [1984], Millon and Thakor [1985], Chemmanur and Fulghieri [1994]

\[10\] Brunnermeier et al. [2014] present a criterion for allocations under heterogeneous beliefs: an allocation is efficient if it is efficient under any convex combination of agents’ beliefs. Although we assume that the firm authority uses a simple average of the agent’s beliefs to allocate the cartel output, our results would hold for any convex combination.
(macro, sector, country etc.) gather to share information, and some over-arching authority averages the conditional expectations of each agent to arrive at an appropriate trading strategy. Then, if the agents themselves hold diverging private valuations for different pieces of risk associated with the asset, the single demand order computed by “average valuations” is suboptimal for their purposes. In such a scenario, agents would rather trade on their own than be part of an overarching cartel.\(^\text{11}\) That is, agents prefer to collude only if they do not disagree too much.\(^\text{12}\)

### 1.1 Applications

This paper applies well to settings where traders search for counterparties so that they can exploit the gains from trade, either due to differences of beliefs or private valuations. Analyzing the institutional feature of the foreign exchange market, Lyons [1995] points out “Though differing beliefs are likely in any asset market in which fundamentals are not easily verified, the foreign exchange market provides a particularly fertile ground for investigation”, noting that the empirical specifications of fundamentals accommodates a greater lack of consensus among traders.\(^\text{13}\) Further, forex traders have always sought counterparts for trades, making information sharing platforms (either telephonic or chatrooms) an essential feature of the trading process. In terms of policy implications, this paper suggests that collusion may not be the only motive for competing traders to share information. Sufficient disagree-

\(^{11}\)In our context, Clarke [1983] points out that firms in a Cournot oligopoly with imperfect, non-shared information will not collude: “Lacking a confluence of opinion, firms find it difficult to agree on a cooperative strategy. An industry cartel would find it hard to determine optimal output shares if some members believe demand to be contracting by 5% while others believe it to be rising by 20%.”.

\(^{12}\)This idea is similar to the welfare analysis in Baranchuk and Dybvig [2015] where the agents’ beliefs work against implementing an improvement for themselves.

\(^{13}\)I largely abstract from the detailed microstructure of forex trading in decentralized markets, and approximate the trading at single price as in Kyle (1985). In these markets, brokers play the role of market-makers by clearing the market anonymously (Lyons [1991])
ment motivates traders to share information in search for counterparties, but also makes them trading in opposite directions of each other’s information. The precise methodology to estimate the magnitude of anti-correlated trading on different pieces of information is beyond the scope of this paper and is left as an agenda for future work.

In equity markets, large funds have incorporated “capital markets desks” to execute large orders directly with other funds or corporations taking the opposite view on a stock. These funds are also helped by investment banks who prepare information memorandums or sales pitches to buy or offload large stakes. One likely explanation for this negative correlation could be disagreement arising from differential information. In a public anecdote, hedge fund managers Bill Ackman and Carl Icahn had conversations in live television (CNBC) over Herbalife, taking opposite views over the company’s prospects. Bill Ackman consistently built short positions in the stock while Carl Icahn took the opposite position against Ackman, eventually building up his stake in Herbalife to 26%. While part of the public exchange between Ackman and Icahn was directed towards encouraging other investors to trade in the same direction as their respective positions, this paper points towards the possibility of a disagreement-led channel which instead minimized price impact for their large trade orders.

With the advent of social media, expressing disagreement online in a financial context is now commonplace. Analyzing data from Stocktwits, a popular social media platform, Cookson and Niessner [2018] show that investors who disagree with each other not only have different information sets, but also have different models for predicting security returns. This paper contends that conversations in financial markets are naturally polarized because they occur between disagreeing agents with sufficiently divergent beliefs.

14 “The buy side wakes up” by Institutional Investor, March 31, 2002
1.2 Related literature

This paper is related to several strands of literature. Primarily, it contributes to the link between communication and trade in financial markets. The extant literature has largely studied the role of communication between traders engaging in correlated trading. Hong et al. [2005] find that trades among fund managers based in the same city are positively correlated (likely due to cooperative reasons), pointing to the possibility of communication among them. Ljungqvist and Qian [2016] show that hedge fund managers freely reveal private information about their portfolio firms, despite having spent costly effort to acquire the information. In terms of theory, such communication is shown to be in the interest of sharing funds due to (i) short term orientation of funds who want to speeden the incorporation of information into prices or (ii) collusive behavior where a quid-pro-quo emerges between traders who share ideas with each other.\(^{15}\)

However, evidence of correlated trades due to word-of-mouth communication has been hard to find. Hong et al. [2005] write that prior to their study, “in spite of its familiarity, this hypothesis about word-of-mouth information transmission has received little direct support in stock market data”. Feng and Seasholes [2004] find that in the Chinese stock market, the trades of geographically distant investors are negatively correlated. This paper suggests that communication could make a segment of (disagreeing) investors trade in an anticorrelated manner, likely influencing the data. In a different stream of literature, Lakonishok et al. [1992] show that contrary to popular perception, mutual funds do not engage in correlated trading. They write “institutions are heterogeneous, they use a broad variety of different portfolio strategies which by and large offset each other”. This paper points to the possibility

\(^{15}\)See, for example, Bhattacharyya and Nanda [2013], Pasquariello and Wang [2020], Schmidt [2020] for theories explaining this behaviour
of communication in anti-correlated trading by mutual funds. Sufficient disagreement in the models of investors motivates them to share information with each other, so they can trade with each other successfully in the market.

The underlying force driving information sharing in our model is the fact that disagreeing investors trade against each other’s signal. The idea of investors shorting competitors’ signals finds support in some other contexts. In an asymmetric informations sharing context, Goldstein et al. [2020] show that information flows from less informed to more informed. The more informed trades against the “noise” of the less informed agent’s signal and relieves the price pressure. In Colla and Mele [2010], under exogenous information linkages, agents short each other’s shared signals when they are negatively correlated.

In a broader sense, this paper analyzes interaction between traders with diverse information. Goldstein and Yang [2015] show that greater diversity of information among traders could improve price informativeness due to complementarities in information acquisition. This paper suggests that information sharing, either motivated by disagreement or collusion, could impede diverse information from being incorporated into prices. Boyarchenko et al. [2020] study the welfare implications of information sharing among dealers in Treasury auctions. In their paper, traders share information even with agreement because risk aversion of their agents mitigates the degree of competition in the after-market. In this paper, the after-market is different, but more importantly our agents are large risk-neutral traders, whose trading is constrained by price pressure rather than an aversion to taking on risk.

Section 3 of this paper rigorously analyzes communication protocols to enable truthful exchange of information. Stein [2008] presents a model of incentive compatible information exchange between competitors. However, the “information complementarity” driving information exchange in Stein [2008] is the possibility of innovation, while the main driver in
this paper is belief arbitrage due to disagreement among competitors. Further, I consider the role of an intermediary in mediating communications between traders, without producing any information. In this sense, my study complements the established informational role of intermediaries in serving clients, either by selling information (Admati and Pfleiderer [1986], Garcia and Sangiorgi [2011]) or reducing information asymmetries (Ramakrishnan and Thakor [1984], Millon and Thakor [1985], Chemmanur and Fulghieri [1994]).

This paper is also related to studies analyzing disagreement among informed traders in a strategic trading context. In Kyle and Wang [1997] an investor’s overconfidence about the precision of his signal acts as a commitment device to garner higher profits than a rational competitor. In my model, sufficient overconfidence in one’s own signal enables investors to exchange information with each other. Han and Kyle [2018] find that higher order disagreements may have large effects on liquidity. In a continuous model of trading, Kyle et al. [2018] show that greater disagreement about the precision of signals makes the market more liquid. We have a similar result upto a threshold level of disagreement, after which information sharing caps the liquidity benefits.

Finally, the model presented in this paper might also appeal to the industrial organization literature that studies information sharing and collusion in a Cournot oligopoly (see for eg. Clarke [1983], Gal-Or [1985], Vives [1984]). I contribute to this literature in two ways. First, I consider agents having diverse information about the demand intercept, which is especially plausible for multinational firms. Second, I study the impact of non-common priors, which reverses the result of no information sharing in several of these papers. Since traders can go both long and short on an asset, an informative cheap talk equilibrium exists, which is a departure from the results in the industrial organization literature (see for eg. Goltsman and Pavlov [2014] and Ziv [1993]).
The remainder of this paper is organized as follows. Section 2 presents the model under disagreement. In Section 3, I analyze the communication mechanisms to enable truthtelling. In Section 4, I investigate the possibility of collusion. Finally, Section 5 concludes. The proofs of all propositions are relegated to the Appendix.

2 Model

An asset with payoff \( \tilde{v} \) is traded in a financial market. The asset value depends on two sources of uncertainty \( \tilde{v}_1 \) and \( \tilde{v}_2 \) which cannot be traded individually.

**Priors:** There are two agents who have common prior beliefs about \( \tilde{v}_1 \) and \( \tilde{v}_2 \): they believe that each \( \tilde{v}_k \) is independently and identically distributed with \( v_k \sim N(0, 1/\rho) \) for \( k = 1, 2 \). However, they disagree about how \( \tilde{v}_1 \) and \( \tilde{v}_2 \) aggregate to determine \( \tilde{v} \). Agent 1 believes that the payoff is given by \( \tilde{v} = \alpha_1 \tilde{v}_1 + (1 - \alpha_1)\tilde{v}_2 \) while agent 2 believes that \( \tilde{v} = (1 - \alpha_2)\tilde{v}_1 + \alpha_2\tilde{v}_2 \). Therefore, the agents agree to disagree on the individual weights in the overall payoff. In this setup, there are several valid interpretations for the parameter \( \alpha_i \in [0, 1] \). Building on the disagreement literature (see for eg. Hong and Stein [2007], Harris and Raviv [1993]), investors in this model disagree about the weight of each business to the overall cash flow of the asset. When \( \alpha_i > \frac{1}{2} \), agent \( i \) is overconfident, in that he overweights his own signal and underweights the other agent’s signal. In a rational interpretation of disagreement, an investor could have his own private valuation for the asset as one of these business acts as a hedge for the rest of his portfolio. Throughout this paper, I look for a symmetric equilibrium where \( \alpha_1 = \alpha_2 = \alpha \geq \frac{1}{2} \). In some sense, \(|\alpha - 1/2|\) is a measure of disagreement between the two agents towards the composition of the payoffs.\(^{17}\)

\(^{16}\)The payoff structure mirrors Goldstein and Yang [2015] when \( \alpha = 1/2 \)

\(^{17}\)In an extreme situation, if agents believe that \( \alpha = 1 \), \(|\alpha - 1/2| = 1/2 \) and agents have maximum disagreement. They believe that only the source of uncertainty that they are informed about is relevant.
Information structure: Agents $i$ knows $\tilde{v}_i$ precisely but has no information about $\tilde{v}_{-i}$ except what is learned from communication with the other agent.

Trading mechanism: The trading mechanism between the informed agents is based on Kyle [1985]. The informed agents submit their trade quantity to a competitive risk-neutral market maker who sets prices to clear the market. To prevent security prices from fully revealing the agent’s information, the model also has noise traders who submit a random order $\tilde{x} \sim N(0, 1/\chi)$ which is due to liquidity shocks independent of everything else in the model. The market-maker cannot distinguish what part of the total demand consists of orders made by noise traders and what part consists of the order submitted by the informed trader. The market maker observes the total demand in the market ($D = D_1 + D_2 + x$) and sets the price of the asset as $p = \lambda D$ (where $\lambda$ is the market maker’s price sensitivity to demand).

Information sharing stage: Prior to submitting their orders and observing their signals, the informed agents commit to either (i) trade on their own information without sharing (NS) or, (ii) share their information with the other agent truthfully and then trade individually (IS). I assume that the decision of whether to share information or not is the outcome of an unmodeled bargaining procedure without side payments. Alternatively, one could also consider reputation effects that incentivize agents to commit towards information sharing in a repeated game setting (also unmodeled in this paper). In the next section, I relax the commitment assumption and investigate the possibility of information exchange through verifiable disclosure, cheap talk or a mediated communication mechanism. In this section, the action set of the agent at the first stage of the game consists solely of $A_1 = \{NS, IS\}$. I will show later that this is without any loss of generality. The agents then receive their towards determining the overall payoff. When $\alpha = 1/2$, the agents are in complete agreement.
signal \( \tilde{v}_i \) and share the signal with the other agent if they committed to do so in the first stage.

Let \( I \) be the indicator function which takes a value of 1 if information was exchanged after the signals are observed. Contingent on the information received, the informed agent’s action at the second stage consists of choosing an optimal trade quantity \( D_i(\tilde{v}_i, I\tilde{v}_{-i}) \):

\[
\max_{D_i} \mathbb{E}[D_i(\alpha \tilde{v}_i + (1 - \alpha)\tilde{v}_{-i} - \lambda D)|\tilde{v}_i, I\tilde{v}_{-i}]
\]

The optimal demand for each agent \( i \) is given by:

\[
D_i^* = \frac{\alpha \tilde{v}_i + (1 - \alpha)\mathbb{E}[\tilde{v}_{-i}, I\tilde{v}_{-i}]}{2\lambda} - \frac{1}{2} \mathbb{E}[D_{-i}^*|\tilde{v}_i, I\tilde{v}_{-i}]
\]

Since market-making is a perfectly competitive, the market maker sets a price \( p \) such that, given the total order submitted, his profit on the realization of the payoff \( \tilde{v} \) is expected to be zero:

\[
p = \mathbb{E}_{MM}[\tilde{v}|\tilde{D} = D_1 + D_2 + x]
\]

We assume that the market-maker holds an “average belief” : that is he believes that each source of uncertainty is equally important to the overall payoff \( \tilde{v} = \frac{1}{2} \tilde{v}_1 + \frac{1}{2} \tilde{v}_2 \) with each \( \tilde{v}_i \) having the same distributional properties as those assumed by the agent.\(^{18}\) It is important to note that the market-maker cannot observe whether the agents share information.

In summary, the timeline of events for the agents and the market maker is as follows:

1. Agents decide whether they should commit to share information truthfully or not share at all.

\(^{18}\)While all agents (including the market-maker) hold subjective beliefs, I require that the market maker’s beliefs are in the “middle” of the disagreeing informed traders.
2. Each agent \( i \) observes the payoff component \( \tilde{\nu}_i \) and shares this information with the other agent depending on the decision taken at Step 1.

3. The agents then submit an optimal demand \( D_i(\tilde{\nu}_i, \tilde{\nu}_{-i}) \) as a function of information received.

4. The market-maker observes only the cumulative demand \( D = D_1 + D_2 + x \) and simultaneously sets a price \( p = \lambda D \).

5. The payoff, private valuations and profits are realized.

At this juncture, I do not consider partial information disclosure when agents enter into information sharing agreements. However, in Proposition 2, I consider the possibility of agents sharing noisy information (in terms of additive normally distributed noise to the agent’s precise signals). I show that the optimal information sharing strategy is monotone in the precision of the noise added. Depending on parameters, the agent would find it profitable to either reveal his information perfectly or not reveal it at all. Therefore, the agent’s somewhat restricted strategy set \( A_1 = \{\text{NS, IS}\} \) is without any loss of generality. Furthermore, the market-maker’s belief is consistent with the agent’s strategy in equilibrium. For example, whenever agents find it profitable to share information, the market-maker’s equilibrium beliefs are that the agents share information and this is indeed the case. In this setup, the question I ask is the following: when would competing traders share information and what are its implications for market liquidity?

2.1 Definition and characterization of the equilibrium

Given the trading mechanism in the previous subsection, traders make optimal decisions based on their own information and beliefs, while taking into account other traders’ decision rules. The competitive market maker sets prices, without observing the agents’ action, but
taking into account his beliefs about the informed agents actions. We define an equilibrium as a triple $\{(A_i)_{i=1,2}, (D_i)_{i=1,2}, \lambda\}$ where (i) agents collectively decide whether to share information, (ii) agents individually choose demand (iii) the competitive market-maker sets prices as his conditional expectation of the payoff.

**Proposition 1.** Given disagreement parameter $\alpha$, there exists a unique linear equilibrium such that:

If $\alpha > \frac{2}{3}$, agents commit to share information truthfully, with demand given by:

$$D_i = \frac{3\alpha - 1}{3\lambda_is} \tilde{v}_i - \frac{3\alpha - 2}{3\lambda_is} \tilde{v}_{-i}$$

and the market-maker’s sensitivity to demand given by $\lambda_is = \frac{1}{3} \sqrt{\frac{2}{\rho}}$

If $\alpha \leq \frac{2}{3}$, agents choose not to share information, with demand given by:

$$D_i = \frac{\alpha}{2\lambda_{ns}} \tilde{v}_i$$

and the market-maker’s sensitivity to demand given by $\lambda_{ns} = \sqrt{\frac{1}{2\rho} \alpha (1 - \alpha)}$

The proofs of all propositions are relegated to the Appendix. We begin by analyzing agent 1’s conditional profits on realization of his signal under various scenarios. The detailed workings of the conditional profits is presented in the Appendix. Under no sharing the profits of agent 1 is given by:

$$\mathbb{E}[\pi_{1ns}|\tilde{v}_1] = \frac{1}{\lambda} \left( \frac{\alpha}{2} \right)^2 (\tilde{v}_1)^2$$
The profits of agent 1 when both agent 1 and agent 2 share information is given by:

\[ \mathbb{E}[\pi_{1s}|\bar{v}_1] = \frac{1}{\lambda} [(\alpha - \frac{1}{3})^2(\bar{v}_1)^2 + (\frac{2}{3} - \alpha)^2] \]

When \( \alpha > 2/3 \), then \( \mathbb{E}[\pi_{1s}|\bar{v}_1] - \mathbb{E}[\pi_{1ns}|\bar{v}_1] > 0 \) for any signal realization \( \bar{v}_1 \). Therefore it is the interest of every type of each agent to decide to share information. In fact, we can make a stronger claim. Consider the case when only agent 1 shares his information and agent 2 does not share his information. Then the profit of agent 1 is given by:

\[ \mathbb{E}[\pi_{1as}|\bar{v}_1] = \frac{1}{\lambda} (\alpha - \frac{1}{3})^2(\bar{v}_1)^2 \]

while the profit of agent 2 is given by:

\[ \mathbb{E}[\pi_{2as}|\bar{v}_2] = \frac{1}{\lambda} [(\frac{2}{3} - \alpha)^2 \frac{1}{\rho} + (\frac{\alpha}{2})^2(\bar{v}_2)^2] \]

We note that when \( \alpha > 2/3 \), \( \mathbb{E}[\pi_{1as}|\bar{v}_1] - \mathbb{E}[\pi_{1ns}|\bar{v}_1] > 0 \). Every signal realization type of agent 1 would prefer to share irrespective of whether the competitor shares his information or not. Therefore, it is dominant strategy for each agent to commit to share information truthfully. Agent 1 sharing his information benefits agent 2 also, as \( \mathbb{E}[\pi_{2as}|\bar{v}_2] - \mathbb{E}[\pi_{2ns}|\bar{v}_2] > 0 \). Of course, individual profits when both agents share information are higher than the case when only one of them shares information (which in turn is greater than the case when none of them share information).
2.2 Common valuation benchmark

I first establish a benchmark when agents agree about the relevance of \( \tilde{v}_1 \) and \( \tilde{v}_2 \) to the overall cashflow \( \tilde{v} \). In my model, this pertains to the situation when \( \alpha = 1/2 \) and both agents have common valuations of \( \tilde{v} = \frac{1}{2} \tilde{v}_1 + \frac{1}{2} \tilde{v}_2 \).

Corollary 1. Under common valuations (\( \alpha = 1/2 \)), agents prefer to not share information.

Table 1 presents a comparison of key outcome variables under common valuations. Information sharing can act as a source of profits since it allows each informed trader to trade on a new type of private information. But the traders obtain new information only if they exchange their own information. Conditional on the realization of a signal, the total demand of the two informed traders (\( \frac{2}{3} \tilde{v} \)) is higher than the sum of the individual demands without informed sharing (\( \frac{\tilde{v}}{2} \)). The price becomes more informative and reduces the total trading profits that can be made on the information initially possessed by each informed trader. In short, information sharing intensifies competition between informed traders, and eats into their information rents.

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<tbody>
<tr>
<td>No sharing:</td>
<td>( D_i = \frac{\tilde{v}<em>i}{\lambda</em>{ns}} )</td>
<td>( \lambda_{ns} = \frac{1}{2} \sqrt{\frac{1}{\rho}} )</td>
<td>( \tilde{p} = \frac{\tilde{v}}{2} + \lambda_{ns} \tilde{x} )</td>
</tr>
<tr>
<td>Sharing:</td>
<td>( D_i = \frac{1}{3\lambda_{is}} \tilde{v} )</td>
<td>( \lambda_{is} = \frac{1}{3} \sqrt{\frac{1}{\rho}} )</td>
<td>( \tilde{p} = \frac{2\tilde{v}}{3} + \lambda_{is} \tilde{x} )</td>
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Table 1: Key outcome variables under common priors

While information sharing eats into the profits of informed traders with \( \alpha = 1/2 \) (i.e.
complete agreement), it enhances price informativeness and benefits liquidity traders. The market-maker’s price sensitivity \( \lambda \) measures the illiquidity of an asset in terms of the impact of a unit trade on the price. The smaller is \( \lambda \), lower is the price impact and more liquid is the market (\( \lambda^{-1} \) denoted the “depth” of the market). Comparing information sharing (\( \lambda_{ns} = \frac{1}{2} \sqrt{\frac{x}{\rho}} \)) with no sharing (\( \lambda_{is} = \frac{1}{3} \sqrt{\frac{x}{\rho}} \)), we have that \( \lambda_{is} < \lambda_{ns} \): the per unit price impact of a trade is lower when information is shared rather than when it is not shared. However, this comes at a cost to the informed traders as their profits on sharing information is strictly lower. Therefore, in equilibrium, informed traders with common valuations do not share information and the liquidity traders bear the higher trading costs associated with this strategy.

The results in this subsection under common valuations are similar to ’no sharing’ results in the Cournot oligopoly literature (see, Gal-Or [1985], Vives [1984]). Firms that are endowed with private information about an uncertain linear demand and are constrained to produce positive quantities prefer not to share information. This subsection extends these results to a financial markets setting.

### 2.3 Information sharing under disagreement

Having established the common valuation benchmark, I now set forth the conditions under which informed agents find it optimal to share information ex-ante.

**Corollary 2.** *Agents prefer to share information ex-ante when they disagree enough with \( \alpha > 2/3 \)*
Table 2 presents a comparison of the key variables driving this result. I first focus on how trade quantities depend on individual signals. Demand given no sharing is:

\[ D_i = \frac{\alpha}{2 \lambda} \tilde{v}_i \]

and demand when when agents share information is given by:

\[ D_i = \frac{3\alpha - 1}{3\lambda_{is}} \tilde{v}_i - \frac{3\alpha - 2}{3\lambda_{is}} \tilde{v}_{-i} \]

When disagreement is high enough (\( \alpha > 2/3 \)) and agents share information, the agent’s demand is increasing in their own signal (coefficient \( \frac{3\alpha - 1}{3\lambda_{is}} > 0 \)) and decreasing in the other agent’s signal (\( \frac{3\alpha - 2}{3\lambda_{is}} < 0 \)). The main purpose that the other agent’s signal serves is to inform the first agent of his opponent’s demand. Higher the second agent’s signal, more is his demand and the first agent lowers his overall trade quantity in response. In a sense, information sharing allows an overconfident agent to “short” the other agent’s signal. By sharing information, traders essentially find a counterparty to take opposite sides in a bet: we call this source of profits as belief arbitrage. Whenever the profits from belief arbitrage is greater than the loss in information rents from sharing private information, agents indeed share information with each other and trade against the new piece of information obtained. This is the central mechanism driving several of our results throughout this paper.

When \( \alpha > 2/3 \), the price impact is lower when agents do not share information than

\[ \alpha = 2/3 \] threshold is as follows. An agent’s trade aggressiveness (without taking into account the other agent’s aggressiveness, is proportional to \( \alpha \) for the term \( v_i \). He then scales down his demand to account for agent \( -i \)’s aggressiveness, which is proportional to \( -\frac{1}{2}(1 - \alpha) \). The netting out of the trades gives us the term \( \alpha - \frac{1}{2}(1 - \alpha) \), which changes sign when \( \alpha > 2/3 \).

\[ \text{21} \] While the profits from belief arbitrage are robust to different specifications of disagreement, I emphasize that profits from disagreement in this paper depend on the signal received by the competitor. There are no outright optimists are pessimists (irrespective of signal realization) in the model described.
when they share information \( (\lambda_{ns} = \sqrt{\frac{x}{2p} \alpha (1 - \alpha)} \) is less than \( \lambda_{is} = \frac{1}{3} \sqrt{\frac{x}{p}} \). Overconfident agents trade aggressively on their own information but moderate their demand when they exchange information. As the informational motivation of trades becomes relatively more important, \( \lambda \) goes up. When agents exchange information and trade against each other, they are able to successfully hide their information from the market. The information asymmetry increases in the market and so does the price impact of each unit of trade. On the other hand, when agents trade very aggressively on their own information without sharing, they reveal a significant portion of their information to the market-maker, thereby lowering information asymmetry and associated adverse selection costs. In such situation, the market-maker is willing to offer a lot of depth.

Therefore, prices are more informative when agents trade individually without sharing information. Similar to Harris and Raviv [1993]'s argument that “difference of opinion makes a horse race”, in our context, disagreement makes the market more liquid. However, full information sharing caps the benefits of disagreement by making agents trade in the opposite directions on each other’s information and lowers liquidity (refer to Figure 1). This leads to the netting out of \( \alpha \) in the total demand of informed traders:

\[
D_1 + D_2 = \frac{2}{3\lambda_{is}} \tilde{v}
\]

Consequently, the price impact \( \lambda_{ns} \) when agents share information is also independent of

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<tr>
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<td>( D_i = \frac{3\alpha - 1}{2\lambda_{ns}} \tilde{v}<em>i + \frac{2 - 3\alpha}{2\lambda</em>{ns}} \tilde{v}_{-i} )</td>
<td>( \lambda_{is} = \frac{1}{3} \sqrt{\frac{x}{p}} )</td>
<td>( \tilde{p} = \frac{\tilde{v}_1}{3} + \frac{\tilde{v}<em>2}{3} + \lambda</em>{is} \tilde{x} )</td>
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Table 2: Key outcome variables under private valuations
Hirshleifer [1971] argues that access to early information could have an adverse impact on risk sharing and reduces welfare. Information sharing in this context allows agents to engage in belief arbitrage with each other and precludes liquidity traders to benefit from the aggressive stance of the overconfident traders. In this environment, different information sets alone is insufficient to make agents share information. The agents need to have sufficiently different models. Further, this result suggests that only agents with sufficiently polarized beliefs are motivated to share information in the financial markets.

Aumann [1976] shows that if agents share a common prior and have common knowledge of each other’s posterior beliefs, they cannot agree to disagree. Geanakoplos and Polemarchakis [1982] extend this analysis showing that if agents start with different information sets, then communication results in an equilibrium with common posterior beliefs. This paper examines the motivation for competing traders to communicate with each other in a market order setting and shows that agents with different models, in addition to different information sets, will benefit from communication. Moreover, a small disagreement is insufficient to motivate the traders to share information if it still implies that the opponent will trade in the same direction of the information shared. A threshold level of disagreement is required to motivate traders to share information.

In the limit of extreme disagreement (that is, as $\alpha \to 1$) and no information sharing, the absolute demand of investors (either long or short) $|D_i| \to \infty$ as their own information gets fully incorporated into the price. Consequently, the price impact $\lambda_{ns} \to 0$. The market is extremely liquid, in the sense that liquidity traders can sell their holdings at almost fair value of the asset and negligible price impact. Therefore, when informed traders derive private valuations solely from their own information (or are very overconfident about their signals), they compete very aggressively and the price reveals all the information available to
Figure 1: Price impact (Kyle’s $\lambda$) vs. disagreement. As can be seen from the figure, disagreement lowers the price impact under no information, but then information sharing (when $\alpha > 2/3$) caps out the liquidity benefits of disagreement.

traders. For an outside observer the agents would appear to behave irrationally by putting too much weight on their own information. However, this irrationality benefits the market in terms of low trading costs and higher price informativeness if the agent’s information is correlated with the actual payoff. As we have already established, informed agents avoid this extreme situation by sharing information with each other and moderating their demand.

Next, I ask if there is some way for the traders to do better if they add (unbiased) noise to the information. Assuming the revealed signal $\tilde{s}_i = \tilde{v}_i + \eta_i$ where $\eta_i \sim N(0, 1/\tau)$, I show in Proposition 3 that profits are monotonic in the precision of the signal.

\[22\] In a dynamic trading context, Holden and Subrahmanyan [1992], Foster and Viswanathan [1996] and Back et al. [2000] show that when traders have correlated signals, they try to beat others to the market and reveal their information almost immediately through aggressive trading. Information transmission is slower, however, with uncorrelated signals as traders play the “waiting game”). In our static model, overconfident traders with diverse signals (who do not share information) collectively reveal all their information in prices.
Proposition 2. If $\alpha > 2/3$, profits on sharing information ex-ante will monotonically increase in the precision of the signal shared by the agents ($\tau$) while if $\alpha \in [\frac{1}{2}, \frac{2}{3})$, profits monotonically decrease in the precision of the signal.

Proposition 3 implies that partial information disclosure is ruled out as a strategy for sharing information under commitment. Investors either share their information completely or do not share it at all. Therefore, our assumption of share/no share in the beginning of this section is without loss of generality. However, as can be seen, each type gains from reporting as extreme a signal realization as possible, so that the opponent goes as short as possible and relieves the price pressure. To address this issue, we show in the next section that incentive compatibility encourages partial information sharing.

3 Communication mechanisms

In Section 2, under sufficient disagreement, the agents agree to share information based on their (ex ante) expected profits in the sharing and the non-sharing regime. Some sort of pre-commitment was necessary to for to reveal the information truthfully. In this section, I analyze the traders’ decision to share information with each other after observing their signals but before trading. I relax the commitment assumption and establish information sharing protocols that would make it incentive compatible for agents to report their signals truthfully, after receiving their signals.

When information is verifiable, I show that full disclosure ensues due to an unraveling argument. Extreme signal realization types reveal their signal to distinguish themselves from
the lower types. The next highest type also reveal themselves to separate from the next lower types and so on. When information is unverifiable, I specifically consider the role of a mediator, such as sell-side analysts in enabling truthful information exchange among traders. I show that even without any information of their own, sell-side analysts can act as mediators by extracting information from different traders, aggregating them and then distributing it back to the traders in the form of published reports, notes or emails. In essence, this section takes the information sales problem of Admati and Pfleiderer [1986] where a monopolistic intermediary is endowed with information to a case where the intermediary does not have any information of his own, but merely pools information from his clients. It will be in the trader’s interest to reveal his information to the intermediary, knowing that he will get back some information that helps him assess his counterparties’ trades. We first establish the cheap talk benchmark where agents can exchange the sign of the signal (when \( \mathbb{E}[\tilde{v}_i] = 0 \) as we have assumed) in an incentive compatible manner. Then, I show that mediators can improve upon the cheap talk equilibrium through the exchange of sign as well as the magnitude of the signal: low or high. This is in line with the idea that brokers and sell-side analysts act as valuable middlemen to help traders ascertain the “mood of the market”. The value added of sell-side analysts has been questioned in the literature with Swem [2019] suggesting that buy-side analysts, who are employed by investment firms, often have earlier access to the information that is obtainable by sell-side analysts. Anecdotal evidence suggests that institutional investors are accorded higher priority in corporate meetings as they have greater skin in the game. I show that even in the absence of any informational advantage, intermediaries can add value for their clients. Finally, in Section 3.3, I consider situations when disclosures are verifiable ex-post (either in terms of signals or through trade positions). A fully revealing equilibrium ensues due to an unraveling argument, where extreme signal
realization types reveal their signal to distinguish themselves from the lower types, the next highest type also reveal themselves to separate from the next lower types and so on until all types discloses their signal.

At this juncture, I assume that the market-maker fixes prices according to an exogenously given upward-sloping supply curve. That is, given total demand \( D \), the market-maker sets prices as \( \lambda D \) where \( \lambda \) is the market maker’s exogenous sensitivity to the demand. As can be seen, this formulation is akin to a Cournot duopoly model with an uncertain demand intercept, which corresponds to the asset payoff in the context of speculation.\(^{23}\) This is because the demand under the communication protocols could possibly be non-linear in the signal \( \tilde{v}_i \) (I later show that this is indeed the case) and we lose the benefits of linearity associated with normal-normal conditioning. However, we do analyze the information transmission in prices for a fixed \( \lambda \) by comparing it to the no sharing and full information sharing results in the previous sections. All other distributional assumptions regarding payoffs and exogenous liquidity trades are maintained as in \(^2\).

3.1 Verifiable disclosure

In this subsection, I restrict attention to verifiable disclosures of signal reports, as in Grossman [1981] and Milgrom [1981]. Such analysis is relevant in contexts where agents have hard verifiable information linked to the opponent’s signals. For example, funds report stock holdings that could serve as an indication of the trades orders and signals received by the funds. In such case, the traders can, at zero cost, credibly disclose their signal realization \( \tilde{v}_i \) to their competitor; or stay silent and not disclose any information at all.\(^{24}\)

\(^{23}\)This “Cournot approximation” of the Kyle model has been used, for example, in Lambert et al. [2018] and Bergemann and Morris [2019].

\(^{24}\)An economic reason for this assumption is that an untrue message bears a sufficiently large penalty, possible due to reputational reasons, so that it is never sent.
We say an equilibrium features full disclosure if the probability that the agents disclose their signal is 1. We will show the existence of an equilibrium where each agent reports his signal truthfully. To avoid complications surrounding the true value of the asset ex-post, I interpret the disagreement parameter $\alpha$ as indicating the private valuation that agent $i$ attributes to the source of uncertainty that he is informed about.

Throughout this section, I will refer to an agent with a particular signal realization as a 'type'. Consider a situation when agent 1 receives a signal $v_1$ (that his, his type is $v_1$) but could report it as $v'_1$ without any restriction. If agent 2 takes the report at face value, the expected profit for this signal realization type is given by:

$$E[\pi_{1as}|\tilde{v}_1, v'_1] = \frac{1}{\lambda} \left[ \frac{3\alpha - 1}{3\lambda} v_1 \left( (\alpha - \frac{2}{3}) v'_1 + \frac{v_1}{3} \right) \right]$$

When $\alpha > 2/3$, agent 1 would want to report $v'_1$ as large as possible (in the absolute sense) as his profit is increasing in the report $|v'_1|$. Further, as long as $|v_i| > 0$, agent 1's profit is strictly positive when he discloses his signal. Agent 2's profit, on the other hand, is given by:

$$E[\pi_{2as}|\tilde{v}_2, v'_1] = \left( \frac{2 - 3\alpha}{3\lambda} v'_1 + \frac{\alpha}{2\lambda} v_2 \right) \left( \frac{4}{3} - 2\alpha \right) v_1 + \left( \alpha - \frac{2}{3} \right) v'_1 + \frac{\alpha}{2} v_2$$

which is maximized when $v_1 = v'_1$, that is, when agent 1 discloses the truth. Agent 2's profit is decreasing in the distance of a report from the true signal realization.

I claim that in a simultaneous move game, every signal realization type of each agent will choose to reveal their signals in equilibrium. The intuition used to prove this claim is called the unraveling argument. Since the profits of an agent is increasing in the absolute
value of the signal reports, the highest signal realization types always disclose their signals to distinguish themselves from the remaining types. The remaining types now face a similar game. The next highest types therefore report their signals to distinguish themselves from lower-signal types, and the process repeats itself\textsuperscript{25}.

**Proposition 3.** When $ \alpha > 2/3$, the unique equilibrium of the verifiable disclosure game is full disclosure

Unravelling results have been generalized to wider economic applications by Okuno-Fujiwara et al. [1990], who stress the importance of the monotonicity of the sender’s expected utility in the receiver’s beliefs. The agents utility functions in my setting naturally adhere to such requirements. However, we require that the signals are verifiable by the traders. When this is not the case, full disclosure is not possible and instead, coarse partial information sharing is implemented through cheap talk or mediated communication.

### 3.2 Two sided cheap talk

In this subsection, I allow firms to directly communicate with each other using costless and unverifiable messages before choosing their trade quantities. Before moving on to the specifics of the cheap talk setup, I determine the dependence of an agent’s profit on the opponent’s trade quantity choice. This will give us an idea as to the nature of the information sharing equilibrium. Given an opponent’s quantity choice $D_{-i}$, the best response of agent $i$\textsuperscript{25}.

\textsuperscript{25}While the intuition presented is relevant for finite signal space, the proof of the claim for our case relies on a contradiction argument which is somewhat similar to the unraveling argument.
is given by:

\[ D_{i}^{BR} = \frac{\alpha v_i + (1 - \alpha)v_{-i}}{2\lambda} - \frac{1}{2} D_{-i} \]

The profit function under the best response has a quadratic form given by:

\[ \pi_{i}^{BR}(D_{-i}) = \frac{1}{\lambda} \left( \frac{\alpha v_i + (1 - \alpha)v_{-i}}{2} - \frac{\lambda}{2} D_{-i} \right)^2 \]

Then, agent \( i \)'s “best response” profit depends on the opponent’s quantity choice in the following manner:

\[ \frac{\partial \pi^{BR}}{\partial D_{-i}} = -\lambda D_i \]

When \( D_i > 0 \), the best response profit is decreasing in the opponent’s quantity, whereas when \( D_i < 0 \), the profit is increasing in the opponent’s quantity. This suggests that an informative cheap talk equilibrium may be possible in a financial market setting, which stands in contrast to the impossibility results in the Cournot oligopoly literature (see for eg. Ziv [1993] and Goltsman and Pavlov [2014]). Since firms are constrained to produce positive quantities in Cournot competition models, their profit is monotonically decreasing in the other firm’s production. Consequently, they always send messages that induce the competitor to produce as low as possible. In a financial market setting, sufficiently negative values in the support of \( \tilde{v}_i \) could induce negative trade quantity choices (going “short” on the asset). Then, agents with extreme positive and negative signals have different messaging incentives, suggesting that there may be a threshold cheap talk equilibrium with a type that is indifferent to reporting whether his signal is above or below the particular threshold.
However, we still require sufficient disagreement about the relevance of the payoff components \((\alpha)\) for the existence of an informative cheap talk equilibrium. I make these arguments explicit in the following subsection.

Consider the following game where the informed agents can engage in cheap-talk communication before making their trade decisions. Let \(M\) be the sets of possible costless messages for each agent. Agent \(i\)'s pure strategy is thus a pair of functions \((m_i(v_i), D_i(m_i, m_{-i}, v_i))\), where \(m_i : \mathbb{R} \rightarrow M\) is a message strategy and \(D_i : M \times M \times \mathbb{R} \rightarrow \mathbb{R}\) is the trade strategy in the continuation game\(^{26}\) following a pair of messages \((m_i, m_{-i})\) being observed.

In what follows, I look for an informative cheap talk equilibrium where each agent \(i\) reports whether his signal is below or above a particular threshold \(v^*\). In the former case, he reports a message \(m_iL\) while in the latter case he reports \(m_iH\). The best response of an agent when he send a message \(m_i\) and receives a message \(m_{-i}\) is given by:

\[
D_i^{m_i, m_{-i}}(v_i; v^*) = \frac{\alpha v_i + (1 - \alpha)\mathbb{E}[v_{-i}|m_{-i}]}{2\lambda} - \frac{1}{2}\mathbb{E}[D_{m_{-i}m_{-i}}(v_{-i})]
\]

with the threshold \(v^*\) being implicit in the message choice \(m_i\) and \(m_{-i}\) in the right hand side of the equation. Given the best response \(D_i^{m_i, m_{-i}}(v_i; v^*)\) for each agent \(i\), the equilibrium profit is given by:

\[
\pi_i(v_i, m_i, m_{-i}; v^*) = \mathbb{E}[D_i^{m_i, m_{-i}}(v_i; v^*)]\left(\alpha v_i + (1 - \alpha)v_{-i} - \lambda D_i^{m_i, m_{-i}}(v_i; v^*) - \lambda D_{m_{-i}m_{-i}}(v_{-i})\right)
\]

where the expectation is taken with respect to the distribution of the other agent’s signal \(v_{-i}\) conditional on the message \(m_{-i}\) and threshold \(v^*\). For the existence of a partially revealing

\(^{26}\)I only analyze a simultaneous two-sided cheap talk game. Analyzing a sequential cheap talk with possible multiple rounds of communication is beyond the scope of this paper.
cheap talk, the following incentive constraints must hold for each agent $i$:

$$E_i[\pi_i(v_i, m_{iL}; v^*) | \tilde{v}_i < v^*] \geq E_i[\pi_i(v_i, m_{iH}; v^*) | \tilde{v}_i < v^*]$$

and

$$E_i[\pi_i(v_i, m_{iH}; v^*) | \tilde{v}_i > v^*] \geq E_i[\pi_i(v_i, m_{iL}; v^*) | \tilde{v}_i > v^*]$$

where the expectation is taken with respect to the message (and signal realization) being either high or low with respect to the threshold $v^*$ for the other agent. The first incentive constraint implies that the agent would rather report a low message when his signal is below the threshold $v^*$ than (incorrectly) report a high message. The same logic holds for the second constraint when his signal realization is above the threshold.

Given an agent’s signal $\tilde{v}_i$ and an exogenously given threshold $v^*$, I first define the net benefit of sending a low message over a high message as:

$$\Delta \Pi_{L-H}(v_i; v^*) \equiv E_i[\pi_i(v_i, m_{iL}; v^*)] - E_i[\pi_i(v_i, m_{iH}; v^*)]$$

For an informative cheap talk equilibrium, we need that:

(i) there exists some $v^*$ that solves $\Delta \Pi_{L-H}(v^*; v^*) = E_i[\pi_i(v^*, m_{iL})] - E_i[\pi_i(v^*, m_{iH})] = 0$ where $v^*$ belongs to the support of $\tilde{v}_i$

(ii) $\Delta \Pi_{L-H}(v; v^*) > 0$ for any $v < v^*$ and $\Delta \Pi_{L-H}(v; v^*) < 0$ for any $v > v^*$.

The first condition states that the marginal type $v^*$ must be indifferent between reporting a high and a low message. The second condition requires that the net benefit from sending
a low message instead of a high message should be positive for signal values to the left of the threshold \((v < v^\ast)\) and negative to the right of the threshold \((v > v^\ast)\). This would make it incentive compatible for agents to send high or low messages that are consistent with the signal realizations relative to the threshold. Therefore, we parsimoniously capture both the incentive constraints through this representation of the net benefit.

My proof strategy for condition (i) (i.e. the existence of \(v^\ast\)) uses the Intermediate Value Theorem to show that for extreme negative (positive) values of \(v^\ast\), the net benefit to reporting low over high for the threshold type \(\Delta \Pi_{L-H}(v^\ast; v^\ast)\) is positive (negative). Given that the function is continuous, there must be some interior \(v^\ast\) at which the function crosses the value zero. Carrying this solution of \(v^\ast\) over to the expression for the net benefit, I show that for any arbitrary signal \(v\), the net benefit on truthtelling is given by:

\[
\Delta \Pi_{L-H}(v; v^\ast) = \frac{(2 - 3\alpha)(E[\tilde{v}_2|\tilde{v}_2 > v^\ast] - E[\tilde{v}_2|\tilde{v}_2 < v^\ast])}{6} \left\{ \alpha v + \frac{(2 - 3\alpha)E[\tilde{v}_2]}{2} - \frac{(2 - 3\alpha)(P(\tilde{v}_2 > v^\ast)E[\tilde{v}_2|\tilde{v}_2 < v^\ast] + P(\tilde{v}_2 < v^\ast)E[\tilde{v}_2|\tilde{v}_2 > v^\ast])}{6} \right\}
\]

The slope of the net benefit function (with respect to the signal realization \(v\)) is negative if \(\alpha > 2/3\) and hence condition (ii) is satisfied. I make these arguments formal for a situation where each \(\tilde{v}_i\) has any general distribution with full support.

**Proposition 4.** When the payoff components \(\tilde{v}_i\) have full support and \(\alpha \geq 2/3\), there exists a threshold \(v^\ast \in (-\infty, \infty)\) such that the cheap talk with threshold \(v^\ast\) is incentive compatible. Further, when the distribution of \(\tilde{v}_i\) is symmetric around zero, then \(v^\ast = 0\).

Recall that \(\alpha \geq 2/3\) is the same disagreement threshold that made it optimal for agents to
share information ex-ante. Proposition 4 is a distribution-free result. Given any probability
distribution for $\tilde{v}_i$ with full support, as long as there is sufficient disagreement between
agents, there is always an informative cheap talk equilibrium. In fact, we are also able to
show that this is the only equilibrium in monotone intervals.\(^27\)

The following corollary re-introduces the distributional assumptions regarding $\tilde{v}_i$ as in 2.

**Corollary 3.** When $\tilde{v}_i \sim N(0, 1/\rho)$, a cheap talk equilibrium with threshold $v^* = 0$ is
incentive compatible

Let the standard deviation of the normal distribution be given by $\sigma = \sqrt{1/\rho}$. The net
benefit to reporting low over reporting high for any type $v$ given $v^* = 0$ is\(^28\):

$$\Delta \Pi_{L-H}(v; v^* = 0) = \frac{(2 - 3\alpha)\mathbb{E}[\tilde{v}_2 | \tilde{v}_2 > 0]}{3} \alpha v = \frac{(2 - 3\alpha)\sigma \sqrt{\frac{2}{\pi}}}{3} \alpha v$$

Clearly, for $\alpha > 2/3$, the net benefit is positive for $v < 0$ and negative for $v > 0$ implying
an informative cheap talk equilibrium. This means that when agents care about different
payoff components, they would be willing to reveal if their signal is good or bad. Once
they communicate which side of the threshold they belong to, the agents will not divulge
any more information in a meaningful manner. They would rather report a signal towards
either extreme so as to make the other agent trade as large (positive or negative) a quantity

\(^{27}\)It may, however, be possible to construct cheap talk equilibria with non monotone partition.

\(^{28}\)For a normal distribution, $\mathbb{E}[X | X > 0] = \sigma \sqrt{\frac{2}{\pi}}$ and $\mathbb{E}[X | X < 0] = -\sigma \sqrt{\frac{2}{\pi}}$
as possible. Hence, it will not be possible to have a cheap talk equilibrium which is more informative than the two partition scenario. The demand function of say, agent 1, under different messages sent or received is given by:

\[
\begin{array}{c|c|c}
\text{Signal range} & \tilde{v}_2 < 0 & \tilde{v}_2 > 0 \\
\hline
\tilde{v}_1 < 0 & D^L_L(v_1) = \sigma \sqrt{\frac{1}{2\pi} \left(\frac{3\alpha-2}{3\lambda}\right) + \frac{\alpha}{2\lambda} v_1} & D^L_H(v_1) = -\sigma \sqrt{\frac{1}{2\pi} \left(\frac{3\alpha-2}{3\lambda}\right) + \frac{\alpha}{2\lambda} v_1} \\
\tilde{v}_1 > 0 & D^H_L(v_1) = \sigma \sqrt{\frac{1}{2\pi} \left(\frac{3\alpha-2}{3\lambda}\right) + \frac{\alpha}{2\lambda} v_1} & D^H_H(v_1) = -\sigma \sqrt{\frac{1}{2\pi} \left(\frac{3\alpha-2}{3\lambda}\right) + \frac{\alpha}{2\lambda} v_1} \\
\end{array}
\]

The trading aggressiveness of the agent \((\frac{\alpha}{2\lambda})\) is the same as that under no information sharing, except a constant term that moderates the demand based on the messages sent and received from the other agent. Given that the trade demand is non-linear (or piecewise linear) in \(\tilde{v}_i\), it is difficult to endogenize the price sensitivity \(\lambda\) to take into account the market-makers beliefs. However, we can study some information transmission properties of the cheap talk protocol by comparing the price under various scenarios for an exogenously given \(\lambda\).

Recall that under no sharing, the price is given by \(\tilde{p} = \alpha \tilde{v} + \lambda \tilde{x}\) and the price under full information sharing is given by \(\tilde{p} = \frac{2}{3} \tilde{v} + \lambda \tilde{x}\). In the same light, the price under cheap talk is given by

\[
\begin{align*}
\tilde{p} &= \alpha \tilde{v} + \lambda \tilde{x} \text{ if } v_1 v_2 < 0 \\
&= \alpha \tilde{v} - \sqrt{\frac{2}{\pi \rho} \left(\frac{3\alpha-2}{3}\right)} + \lambda \tilde{x} \text{ if } v_1 > 0 \& v_2 > 0 \\
&= \alpha \tilde{v} + \sqrt{\frac{2}{\pi \rho} \left(\frac{3\alpha-2}{3}\right)} + \lambda \tilde{x} \text{ if } v_1 < 0 \& v_2 < 0
\end{align*}
\]

The prices are less informative than the no sharing case by a constant when the signs of
the signals $\tilde{v}_i$ are the same. This implies that partial information sharing is less informative, but the informativeness increases for large signal realizations of $\tilde{v}_i$.

### 3.3 Mediated communication

In this subsection, we suppose that there is a mediator who has access to a set of messages $m \in M$. He receives reports $\hat{v}_i$ from each agent $i$ and then sends a public message $m_i \in M$ to agent $i$ according to a commonly known probability distribution $F(m, \hat{v}_1, \hat{v}_2)$. Given the mechanism designer’s message $m$ and agent $i$’s own report $\hat{v}_i$, his conjecture about agent $-i$’s report is given by $F(\hat{v}_{-i}|\hat{v}_i, m)$. The profit for each agent is given by:

$$
\pi_i(\tilde{v}_i, D_i, D_{-i}, m) = \mathbb{E}[D_i(\tilde{V}_i - \lambda D_i - \lambda D_{-i} - \lambda \tilde{x})|\tilde{v}_i, m]
$$

The best response for each agent given his own signal and the other agent’s quantity choice is:

$$
D_i^{BR}(\tilde{v}_i, D_{-i}) = \arg \max_{q_i} \pi_i(\tilde{v}_i, D_i, D_{-i}, m)
$$

On receiving a signal $\tilde{v}_i$, the agent reports a message $\hat{v}_i$ to the mediator. If $\hat{v}_i = \tilde{v}_i$, then the reporting strategy is deemed truthful. In what follows, we describe a mediated communications equilibrium as a Bayes Nash Equilibrium, where each agent finds it optimal to adhere to a truth-telling strategy.
\[ \alpha = \frac{4}{5} \]

Figure 2: As a function of signal \( v \), Expected profits (LEFT) on reporting low (green) and high (red) and Net benefit (RIGHT) from reporting low instead of high at \( \alpha = 0.8 \) and \( \lambda = 1 \). Notice the incentive compatibility constraints are met in the top graph (\( \alpha = 0.8 \)): the agent would rather report low (green) when \( v < v^* = 0 \) and high (red) when \( v > v^* = 0 \). The agent would rather misreport in the lowest graph (\( \alpha = 0.5 \)) under common valuation. No informative cheap talk exists under common valuations. In the middle graph (\( \alpha = 2/3 \)), the agent is indifferent between reporting truthfully and misreporting.
Definition 1. A mediated communications equilibrium via public messages consists of optimal quantity choices $D_{i}^{MC}(\tilde{v}_{i}, \hat{v}_{i}, m)$, for each agent $i$, given by:

$$D_{i}^{MC}(\tilde{v}_{i}, \hat{v}_{i}, m) = \arg\max_{q_{i}} \int \pi_{i}(\tilde{v}_{i}, D_{i}, D_{-i}^{MC}(\tilde{v}_{-i}, \tilde{v}_{-i}, m))dF(\tilde{v}_{-i}|m, \hat{v}_{i})$$

The optimal quantity choice is given by:

$$D_{i}^{MC}(v_{1}, \hat{v}_{1}, m) = \frac{\mathbb{E}[\tilde{V}_{1}|\tilde{v}_{1} = v_{1}, m]}{2\lambda} - \frac{1}{2}\mathbb{E}[\bar{D}_{2}|\tilde{v}_{1} = v, m]$$

$$= \alpha v + (1 - \alpha)\mathbb{E}[\tilde{v}_{2}|\tilde{v}_{1} = v, m] - \frac{1}{2}\mathbb{E}[\bar{D}_{2}|\tilde{v}_{1} = v, m]$$

Definition 2. On realization of his signal, the expected optimal profit for each agent under mediated communication is given by:

$$\Pi_{i}^{MC}(\tilde{v}_{i}, \hat{v}_{i}, m) = \int \pi_{i}(\tilde{v}_{i}, D_{i}^{MC}(\tilde{v}_{i}, \hat{v}_{i}, m), D_{-i}^{MC}(\tilde{v}_{-i}, \tilde{v}_{-i}, m))dF(\tilde{v}_{-i}|m, \hat{v}_{i})$$

Truth-telling is incentive compatible for each agent $i$ if for all $\hat{v}_{i}$, we have that:

$$\Pi_{i}^{MC}(\tilde{v}_{i}, \tilde{v}_{i}, m) \geq \Pi_{i}^{MC}(\tilde{v}_{i}, \hat{v}_{i}, m)$$

That is, for each $m$ and each $i$:
\[ \tilde{v}_i \in \arg \max_{\hat{v}_i} \Pi_i^{MC}(\tilde{v}_i, \hat{v}_i, m) \]

I look for an incentive compatible mechanism of the following form. Restricting the signals to positive values only, the mediator’s message distribution is given by a “max” mechanism: if \( \max(\hat{v}_1, \hat{v}_2) \leq v_P \), then the mediator reports a public signal \( m_L \) else he reports \( m_H \). Recall that in the cheap talk section, we established that once the agent is in the positive region of his signal (relative to the mean), his incentives would be to report as high a signal as possible so that his opponent chooses as large a short order as possible. Hence the cheap talk will not be possible beyond a two partition scenario. In the mechanism stated above, agents get extra information when they report that they are of the “low” type, thereby compensating the agent for a potentially higher trade quantity choice by his opponent. Further, it would not matter if an agent reports a false signal realization as long as the following condition holds: whenever the true signal is less than \( v_P \), the reported signal is less than \( v_P \) (and vice versa). Hence, we denote an agent’s signal realization as \( L \) if \( \hat{v}_i < v_P \) and \( H \) otherwise. Our main aim in this section is to couple the cheap talk result in the previous section with a max (min)-mechanism. The detailed mechanism under full support of \( \tilde{v}_i \) is as follows.

1. Mediator first reports sign of the signal (similar to Cheap talk)

2. Then, Max mechanism: if \( \max(|\hat{v}_1|, |\hat{v}_2|) \leq v_P \), then the mediator reports a public signal \( m_L \) else he reports \( m_H \)

Then, the agents report their signal position. The following proposition shows the existence of \( v_P \) and therefore, the validity of the mediated communication protocol.
Proposition 5. For $\alpha \geq 3/4$, there exists $v_P \in (0, \infty)$ so that the agents truthfully report whether their signals lie in one of the following regions: $[-\infty, -v_P]$, $[-v_P, 0]$, $[0, v_P]$ and $[v_P, \infty]$

The above Proposition shows that the mediator can improve upon the cheap talk equilibrium by allowing agents to share information in four regions instead of two. This is because the mediator can play the role of an information filter between agents: the information that the agent gets about a competitor’s report depends on the agent’s own report to the mediator. Early models of intermediation provided theoretical foundations for thinking about various intermediaries as information producers and sellers, mainly to bridge information asymmetries in the market.29 Proposition 5 suggests that even without any information of their own, sell-side analysts can act as mediators by extracting information from different traders, aggregating them and then distributing it back to the traders in the form of published reports, notes or emails. Moreover, the profits for the informed traders in the mediated communication protocol is higher than that of the cheap talk benchmark30. If we consider the information sales problem of Admati and Pfleiderer [1986] (or more recently, Garcia and Sangiorgi [2011]) where a monopolistic intermediary sells information to investors, our intermediary can make profits simply by implementing the strategic correlated equilibrium of the Cournot game.

29See for example Ramakrishnan and Thakor [1984], Millon and Thakor [1985], Chemmanur and Fulghieri [1994]
30Proof is available upon request.
4 Collusion

In this section, I consider collusion as a possible action choice at the ex-ante stage in the model presented in Section 2. This Section reinstates the commitment assumption described in Section 2. Additionally, in Section 2, we assumed that each agent’s information: \( \tilde{v}_i \) was normally distributed with mean zero. The results on sharing vs. no-sharing were independent of the mean or variance of the normal distribution. However, when we analyze collusion, the agent’s action choice (in terms of colluding, sharing information but trading individually or not sharing information at all) depend on the mean and possibly on the variance of the distribution. Therefore, in this section we assume that \( \tilde{v}_i \sim N(\mu, 1/\rho) \).

First, I consider the possibility of collusion between agents who have common valuation for the asset \( (\alpha = 1/2) \). I show that under common valuations, agents prefer to collude as long as \( |\mu| > 0 \), otherwise they are indifferent between colluding and not sharing information. The situation gets a bit trickier when we analyze collusion under disagreement. As is natural to financial markets (and unlike industrial organization), I interpret collusion as agents congregating to form a single firm (say, a mutual fund), sharing information and submitting a single order. While trading collectively gives them both information and strategic advantage (with respect to the broader market), the agents are required to pool their somewhat extreme beliefs (especially when \( \alpha \to 1 \)) and settle for a less aggressive trading strategy in search for a common ground. In Proposition 4, I set forth conditions on the parameter \( \alpha \) which would make it optimal for agents to behave as a single firm instead of trading individually.

4.1 Collusion under common valuations

I first analyze the possibility of collusion under common valuations, that is, \( \alpha = 1/2 \) and
each agent has the same belief that $\bar{v} = \frac{1}{2}\bar{v}_1 + \frac{1}{2}\bar{v}_2$. Prior to submitting the order and observing the signals, the agents decide whether they want to (i) trade on their own information without sharing (NS), (ii) share their information with the other agent and then trade individually (IS) or (iii) collude together as a single firm and submit a monopolistic order under shared information (C).\textsuperscript{31} The agent’s problem under collusion (sharing information and trading as a cartel):

\[
\max_{D_1, D_2} E[(D_1(\frac{1}{2}\bar{v}_1 + \frac{1}{2}\bar{v}_2 - \lambda(D_1 + D_2))|\bar{v}_1, \bar{v}_2) + E_2[(\frac{1}{2}\bar{v}_1 + \frac{1}{2}\bar{v}_2 - \lambda(D_1 + D_2))|\bar{v}_1, \bar{v}_2]]
\]

The total optimal trade quantity is then

\[
D_1 + D_2 = \frac{\frac{1}{2}\bar{v}_1 + \frac{1}{2}\bar{v}_2}{2\lambda}
\]

\textbf{Proposition 6.} Under common valuations ($\alpha = 1/2$), informed agents strictly prefer to collude when $|\mu| > 0$, otherwise they are indifferent between colluding and not sharing information.

This has implications for policy. As can be seen, when agents share information and collude, they trade in the same direction of each other’s information. While sufficient disagreement also motivates traders to share information, it makes them trade in the opposite directions of each other’s information. Therefore, if a regulator has knowledge of the infor-

\textsuperscript{31}I assume that the agents are able to enforce the commitment to trade collusively choice through reputation effects or repeated games with punishment (under verifiable trade quantities and verifiable information) that are unmodeled in this paper.
mation shared by agents and their trade patterns, then it will be possible to differentiate whether agents have collusive motives or are just engaging in belief arbitrage. The precise methodology to estimate the magnitude of anti-correlated trading on different pieces of information is beyond the scope of this paper and is left as an agenda for future work.

4.2 Colluding with disagreement

In this subsection, I consider the possibility of differently informed agents cooperating to form a single firm and submitting a single trade order to the market-maker. The agent’s problem under collusion (sharing information and trading as a cartel):

\[
\max_{D_1, D_2} \mathbb{E}[(D_1(\alpha_1 \tilde{v}_1 + (1-\alpha_1)\tilde{v}_2 - \lambda(D_1+D_2))|\tilde{v}_1, \tilde{v}_2] + \mathbb{E}[(D_2((1-\alpha_2)\tilde{v}_1 + \alpha_2 \tilde{v}_2 - \lambda(D_1+D_2))|\tilde{u}_1, \tilde{u}_2]
\]

The total optimal trade quantity is then

\[
D_1 + D_2 = \frac{\alpha_i \tilde{v}_i + (1 - \alpha_i)\tilde{v}_{-i}}{2\lambda}
\]

Clearly, it is not possible for \( D_1 \) and \( D_2 \) to satisfy the above equation if \( \alpha_i \neq 1/2 \) for each \( i \). To resolve the disagreement in choosing optimal quantities, I assume that an overarching firm authority decides the appropriate composition of payoffs to arrive at the optimal trading quantity. In doing so, he averages both the agents’ individual opinions so that \( \mathbb{E}_F[\tilde{v}|\tilde{v}_1, \tilde{v}_2] = \frac{1}{2}[(\alpha_1 + (1-\alpha_2)]\tilde{v}_1 + \frac{1}{2}[\alpha_2 + (1-\alpha_1)]\tilde{v}_2. \) Since \( \alpha_i = \alpha \), the firm authority assigns a weight of 1/2 to each of their sources of uncertainty in a collusive arrangement. Thus, the firm authority’s behavior is similar to that of a market maker. The firm could
be thought of as a mutual fund with a fund manager taking research inputs from analysts specializing in different domains. Once the agent reveals their signals to the fund manager, he averages their signals to arrive at the total payoff and then submits a single order to the market-maker. The realized profits are then equally divided among the research analysts. The following proposition evaluates the dependence of the optimal strategies of the agent on $\alpha$

**Proposition 7.** There exists some $\alpha_1 \in (\frac{1}{2}, \frac{2}{3})$ such for $\alpha \in [\frac{1}{2}, \alpha_1)$, informed agents prefer to collude, for $\alpha \in (\alpha_1, \frac{2}{3})$ they prefer to trade individually on their own information and for $\alpha \in (\frac{2}{3}, 1]$, agents prefer to share information yet trade individually without colluding.

This Proposition indicates that sharing information (without colluding) is the optimal strategy as long as the agents disagree enough. Since the collusive arrangement only puts a weight of $1/2$ on each agent’s belief, they would rather share information and trade individually to maximize their profit. Unsurprisingly, collusion is an optimal arrangement only when $\alpha \in [\frac{1}{2}, \alpha_1)$. Under mild disagreement the agent does not lose out much in terms of pooling of beliefs. However, if the disagreement is large enough, the agents revert to the action choices in line with those presented in Section 2. Brunnermeier et al. [2014] present a criterion for allocations under heterogeneous beliefs: an allocation is efficient if it is efficient under any convex combination of agents’ beliefs. Although we assume that the firm authority uses a simple average of the agent’s beliefs to allocate the cartel output, our results would hold for any convex combination.

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32Research analysts working for a mutual funds are commonly referred to as “buy-side analysts”
5 Conclusion

This paper studies circumstances under which competing traders, endowed with different pieces of fundamental information pertaining to a security, exchange information before trading? I show that when investors disagree with each other about the relevance of each other’s information towards the overall payoff of a security, they prefer to share information with each other. On the other hand, when investors agree about the relevance of each other’s information, they prefer to trade individually on their own information. When agents have sufficiently different models through which they view the world, they trade against each other’s information and create surpluses for themselves through belief arbitrage. Despite the benefits of information sharing, an investor could deceive other informed agents once he is in possession of the other agent’s information. This paper highlights the role of information mediators (say, sell-side analysts) that enable incentive compatible information transmission protocols by acting as information aggregators and distributors. I also show the existence of an informative cheap talk equilibrium between investors despite competing interests, which is a departure from the somewhat similar Cournot oligopoly models in the industrial organization literature. In terms of policy implications, this paper suggests that collusion may not be the only motive for competing traders to share information. Sufficient disagreement motivates traders to share information, but also makes them trading in opposite directions of each other’s information.

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6 Appendix

6.1 Model: Proof for Proposition 1

*Proof.* For each agent:

\[ \tilde{V}_i = \alpha \tilde{v}_i + (1 - \alpha)\tilde{v}_{-i} \]

I first start by assuming that \( \lambda \) is given exogenously for each agent’s demand choice problem. Then, I endogenize \( \lambda \) to determine the market-maker’s sensitivity to demand.

48
**No information sharing:** The informed trader conjectures that the market maker has a price function \( \hat{p} = \gamma + \lambda \hat{D} \) and submits an order \( D_i \). Further, I conjecture that each agent’s demand is a linear function of the signal: \( D_i = a_i + b_i \hat{v} \). For agent 1, given prices \( \hat{p} \), his profits are

\[
\mathbb{E}[\pi_1|\hat{v}_1] = \mathbb{E}[D_1(\hat{V}_1 - \hat{p}) - \hat{v}_1]
= D_1 \left( \alpha \hat{v}_1 - (\lambda D_1 + \lambda \mathbb{E}[D_2^*|\hat{v}_1]) \right)
\]

Taking FOCs with respect to \( D_1 \) in the above equation, the optimal demand for agent 1 is then given by:

\[
D_1^* = \frac{\alpha \hat{v}_1}{2\lambda} - \frac{1}{2} \mathbb{E}_1 \left[ D_2^*|\hat{v}_1 \right]
\]

Directly substituting \( D_2 = a_2 + b_2 \hat{v}_2 \) gives us:

\[
a_1 + b_1 \hat{v}_1 = \frac{\alpha \hat{v}_1}{2\lambda} - \frac{1}{2} \mathbb{E}_1 \left[ a_2 + b_2 \hat{v}_2|\hat{v}_1 \right]
\]

Given the symmetric nature of the problem, \( b_i = \frac{\alpha}{2\lambda \alpha} \) and \( a_i = 0 \). Therefore the demand for agent \( i \) is

\[
D_i = \frac{\alpha}{2\lambda} \hat{v}_i
\]

The total demand is given by:

\[
D = \frac{\alpha}{\lambda} \hat{v}
\]

From the conjectured demand functions of the agents, we have that \( \hat{p} = \lambda D_1 + \lambda D_2 + \lambda \hat{x} \). Matching the two equations for price, the price equals:

\[
\hat{p} = \frac{\alpha \hat{v}_1}{2} + \frac{\alpha \hat{v}_2}{2} + \lambda \hat{x}
\]

The conditionally expected profit for the agent, given his signal is:
\[ E[\pi_{1ns}|\hat{v}_1] = E\left[D_1(\hat{v} - \hat{p})|\hat{v}_1\right] \]

\[ = E\left[\frac{\alpha\hat{v}_1}{2\lambda}(\hat{v} - \frac{\alpha\hat{v}_1}{2} - \frac{\alpha\hat{v}_2}{2} - \lambda_{ns}\hat{x})|\hat{v}_1\right] \]

\[ = \frac{1}{4\lambda}\left[\alpha^2(\hat{v}_1)^2\right] \]

The ex-ante expected profits for the agent 1, given his own beliefs are:

\[ E[\pi_{1ns}] = E\left[D_1(\hat{v} - \hat{p})\right] \]

\[ = E\left[\frac{\alpha\hat{v}_1}{2\lambda}(\hat{v} - \frac{\alpha\hat{v}_1}{2} - \frac{\alpha\hat{v}_2}{2} - \lambda_{ns}\hat{x})\right] \]

\[ = \frac{1}{4\lambda}\left[\frac{\alpha^2}{\rho}\right] \]

As \( \alpha \to 1 \) and the investors are extremely confident about their own beliefs, trade a large quantity and expect that \( E[\pi_i] \to \infty \).

**Full information sharing:** When the agents share their information completely, both of them know \( \hat{v}_1 \) and \( \hat{v}_2 \). Thus, it is as if they know \( \hat{V}_i \). The marketmaker’s prior variance is given by \( \frac{1}{2\rho} \) but each agent’s prior variance given by:

\[ Var(\hat{V}_i) = \alpha^2\frac{1}{\rho} + (1 - \alpha)^2\frac{1}{\rho} \]

\[ = (2\alpha^2 + 1 - 2\alpha)\frac{1}{\rho} \]

For agent 1, given prices \( \hat{p} \), his profits are

\[ E[D_1(\hat{V}_1 - \hat{p}) - \hat{v}_1, \hat{v}_2] = D_1\left(\hat{V}_1 - (\lambda D_1 + \lambda E[D_2^*|\hat{v}_1, \hat{v}_2]\right) \]
The optimal demand for the informed agent is then given by:

$$D^*_1 = \frac{\tilde{V}_1}{2\lambda} - \frac{1}{2} \mathbb{E}[D^*_2|\tilde{v}_1, \tilde{v}_2]$$

Similarly,

$$D^*_2 = \frac{\tilde{V}_2}{2\lambda} - \frac{1}{2} \mathbb{E}[D^*_1|\tilde{v}_1, \tilde{v}_2]$$

Each agent’s demand function is given by:

$$a + b\tilde{v}_i + c\tilde{v}_i = -\frac{1}{2}a + (\frac{\alpha}{2\lambda_i} - \frac{c}{2})\tilde{v}_1 + (\frac{1 - \alpha}{2\lambda_i} - \frac{b}{2})\tilde{v}_2$$

Matching coefficients, $a = 0$, $b = \frac{3\alpha - 1}{3\lambda}$, and $c = \frac{2 - 3\alpha}{3\lambda}$. Therefore the demand for agent $i$ is

$$D_i = \frac{3\alpha - 1}{3\lambda} \tilde{v}_i + \frac{2 - 3\alpha}{3\lambda} \tilde{v}_{-i}$$

I note that $b + c = \frac{3\alpha - 1}{3\lambda_i} + \frac{2 - 3\alpha}{3\lambda_i} = \frac{1}{3\lambda_i}$. The total demand is given by:

$$D = \frac{2}{3\lambda} \tilde{\nu}$$

From the conjectured demand functions of the agents, I have that $\tilde{p} = \lambda D_1 + \lambda D_2 + \lambda \tilde{x}$. Matching the two equations for price gives us:

$$\tilde{p} = \frac{\tilde{v}_1}{3} + \frac{\tilde{v}_2}{3} + \lambda \tilde{x}$$

$$= \frac{2\tilde{v}}{3} + \lambda \tilde{x}$$

51
The conditionally expected profit for the agent 1 when he receives his own signal is:

$$E[\pi_{1|\hat{v}_1}] = E[D_1(\hat{V}_1 - \hat{\rho})|\hat{v}_1]$$

$$= E[(\frac{3\alpha - 1}{3\lambda}\hat{v}_1 + 2 - 3\alpha\hat{v}_2)(\alpha\hat{v}_1 + (1 - \alpha)\hat{v}_2 - \frac{1}{3}(\hat{v}_1 + \hat{v}_2 + \lambda\hat{x})|\hat{v}_1]$$

$$= \frac{1}{\lambda}(\alpha - \frac{1}{3})^2(\hat{v}_1)^2 + (\frac{2}{3} - \alpha)^2\frac{1}{\rho}$$

The ex-ante expected profit for the agent 1 is given by:

$$E[\pi_{1\alpha}] = E[D_1(\hat{V}_1 - \hat{\rho})]$$

$$= E[(\frac{3\alpha - 1}{3\lambda}\hat{v}_1 + 2 - 3\alpha\hat{v}_2)(\alpha\hat{v}_1 + (1 - \alpha)\hat{v}_2 - \frac{1}{3}(\hat{v}_1 + \hat{v}_2 + \lambda\hat{x})]$$

$$= \frac{2\alpha^2 + \frac{5}{3} - 2\alpha}{\lambda\rho}$$

The difference in the conditionally expected profit under information sharing and no information sharing is given by:

$$E[\pi_{1\alpha}|\hat{v}_1] - E[\pi_{1\alpha|\hat{v}_1}] = \frac{1}{\lambda}\left(\left((\alpha - \frac{1}{3})^2(\hat{v}_1)^2 + (\frac{2}{3} - \alpha)^2\frac{1}{\rho}\right)\right) - \frac{1}{4}[\alpha^2(\hat{v}_1)^2]\right)$$

$$= \frac{1}{\lambda}\left(\left((\alpha - \frac{1}{3})(\frac{3\alpha}{2} - \frac{1}{3})\hat{v}_1)^2 + (\frac{2}{3} - \alpha)^2\frac{1}{\rho}\right)$$

Clearly, when $\alpha > 2/3$, the above expression is positive for every realization of $v_1$.

**Endogenizing $\lambda$**

Next, we endogenize the market-maker’s $\lambda$. I claim that the following strategy profile is a Bayes Nash Equilibrium. Let the market-maker’s strategy be given by $\lambda = \lambda_{is}$ when $\alpha > 2/3$ and $\lambda = \lambda_{ns}$ when $\alpha \leq 2/3$. Then, I claim that the agent’s will share information when $\alpha > 2/3$ and not share information when $\alpha \leq 2/3$. To see that, suppose

$$\frac{1}{\lambda\rho}(2\alpha^2 + \frac{5}{9} - 2\alpha) > \frac{1}{\lambda\rho}(\alpha^2)$$

$$\Rightarrow 7\alpha^2 + \frac{20}{9} - 8\alpha > 0$$

This is true as long as $\alpha > 2/3$. We calculate the price impact under no information sharing and full
information sharing, given the respective demand functions of the agents.

**No information sharing:** The market maker knows the true distribution of the payoffs and then sets the price equal to the conditional expectation of the payoff given his total order flow:

\[
\tilde{p} = E[\tilde{v} | b_1 \tilde{v}_1 + b_2 \tilde{v}_2 + \tilde{x}]
\]

\[
= \frac{Cov(\tilde{v}, b_1 \tilde{v}_1 + b_2 \tilde{v}_2 + \tilde{x})}{Var(b_1 \tilde{v}_1 + b_2 \tilde{v}_2 + \tilde{x})} (b_1 \tilde{v}_1 + b_2 \tilde{v}_2 + \tilde{x})
\]

\[
= \frac{Cov(0.5\tilde{v}_1 + 0.5\tilde{v}_2, b_1 \tilde{v}_1 + b_2 \tilde{v}_2 + \tilde{x})}{Var(b_1 \tilde{v}_1 + b_2 \tilde{v}_2 + \tilde{x})} (b_1 \tilde{v}_1 + b_2 \tilde{v}_2 + \tilde{x})
\]

\[
= \frac{b_1(1/2\rho) + b_2(1/2\rho)}{b_1^2(1/\rho) + b_2^2(1/\rho) + 1/\chi} (b_1 \tilde{v}_1 + b_2 \tilde{v}_2 + \tilde{x})
\]

From the conjectured demand functions of the agents, I have that \(\tilde{p} = \lambda_{ns}D_1 + \lambda_{ns}D_2 + \lambda_{ns}\tilde{x}\). Matching the two equations for price, the price equals:

\[
\tilde{p} = \frac{\alpha \tilde{v}_1}{2} + \frac{\alpha \tilde{v}_2}{2} + \lambda_{ns}\tilde{x}
\]

I note here that as \(\alpha \to 1\), the price is equal to the market-maker’s belief about the payoff \(\tilde{v}\) plus a random error term (\(\lambda_{ns}\tilde{x}\)). Clearly, I have that :

\[
\lambda_{ns} = \frac{\alpha \left(\frac{\alpha}{\lambda_{ns}}(1/2\rho) + \frac{\alpha}{\lambda_{ns}}(1/2\rho)\right)}{(\frac{\alpha}{\lambda_{ns}})^2(1/\rho) + (\frac{\alpha}{\lambda_{ns}})^2(1/\rho) + 1/\chi}
\]

\[
\Rightarrow \lambda_{ns} = \sqrt{\frac{1}{2\rho} \alpha (1 - \alpha)}
\]

Again, as \(\alpha \to 1\), \(\lambda_{ns} \to 0\). The variable \(\lambda_{ns}\) measures the impact in price of a one unit trade. The smaller is \(\lambda_{ns}\), the lower the price impact, or the more liquid a market is. Intuitively, \(\alpha\) also implicitly serves as a measure of disagreement between investors. As \(\alpha \to 1\), the investors put almost full weight on their own source of uncertainty. The demand of investors \(D_i \to \infty\) as their own information gets fully incorporated into the price. The market is extremely liquid, in the sense that liquidity traders can sell their holdings at fair values, at a negligible price impact.

**Full information sharing:** The market maker sets the price equal to the conditional expectation of
the payoff given his total order flow \((b + c = d)\) in the below working)
\[
\tilde{p} = E[\tilde{v}(b + c)\tilde{v}_1 + (b + c)\tilde{v}_2 + \tilde{x}]
\]
\[
= \text{Cov}(\tilde{v}, 2d\tilde{v} + \tilde{x}) \over \text{Var}(2d\tilde{v} + \tilde{x}) (2d\tilde{v} + \tilde{x})
\]
\[
= \frac{2d(1/\rho)}{4d^2(1/\rho) + 1/\chi} (2d\tilde{v} + \tilde{x})
\]
where I note that \(d = b + c = \frac{3a_1}{3\lambda_{x}} + \frac{2a_2}{3\lambda_{x}} = \frac{1}{3\lambda_{x}}\). The total demand is given by:
\[
D = \frac{2}{3\lambda_{x}} \tilde{v}
\]
From the conjectured demand functions of the agents, we have that \(\tilde{p} = \lambda D_1 + \lambda D_2 + \lambda \tilde{x}\). Matching the two equations for price gives us:
\[
\tilde{p} = \frac{\tilde{v}_1}{3} + \frac{\tilde{v}_2}{3} + \lambda_{is} \tilde{x}
\]
\[
= \frac{2\tilde{v}}{3} + \lambda_{is} \tilde{x}
\]
Clearly, we have that :
\[
\lambda_{is} = \frac{2\frac{1}{3\lambda_{x}}(1/\rho)}{4\left(\frac{1}{3\lambda_{x}}\right)^2(1/\rho) + 1/\chi}
\]
\[
\Rightarrow \lambda_{is} = \frac{1}{3} \sqrt{\frac{\chi}{\rho}}
\]

6.2 Workings for one-sided information sharing

Proof. Suppose that only agent 1 has the opportunity to share his information without agent 2 doing so. Will agent 1 find it in his interest to do so?

Agent 2 knows \(\tilde{v}_1\) and \(\tilde{v}_2\). Thus, it is as if they know \(\tilde{V}_i\). The marketmaker’s prior variance is given by
but each agent’s prior variance given by:

\[
\text{Var}(\tilde{V}_i) = \alpha_1^2 \frac{1}{\rho} + (1 - \alpha_1)^2 \frac{1}{\rho} = (2\alpha_1^2 + 1 - 2\alpha_1) \frac{1}{\rho}
\]

The optimal demand for the informed agent is then given by:

\[
D_1^* = \frac{\tilde{V}_1}{2\lambda} - \frac{1}{2} \mathbb{E}[D_2^*|\tilde{v}_1]
\]

Similarly,

\[
D_2^* = \frac{\tilde{V}_2}{2\lambda} - \frac{1}{2} \mathbb{E}[D_1^*|\tilde{v}_1, \tilde{v}_2]
\]

Agent 2’s demand function is given by: \(D_2 = a_2 + b_2 \tilde{v}_2 + c_2 \tilde{v}_1\) and agent 1’s demand is given by \(D_1 = a_1 + b_1 v_1\). Matching coefficients, we have that

\[
b_1 = \frac{\alpha_1}{2\lambda} - \frac{c_2}{2}
\]

and

\[
a_1 + \frac{a_2}{2} = 0
\]

and

\[
a_2 + \frac{a_1}{2} = 0
\]

and

\[
b_2 = \frac{\alpha_2}{2\lambda}
\]

and

\[
c_2 = \frac{1 - \alpha_2}{2\lambda} - \frac{b_1}{2}
\]
which implies $a_1 = a_2 = 0$, $b_1 = \frac{2\alpha_1 + \alpha_2 - 1}{3\lambda}$ and $c_2 = \frac{4 - 4\alpha_2 - 2\alpha_1}{6\lambda}$. Therefore the demand for agent 1 is

$$D_1 = \frac{2\alpha_1 + \alpha_2 - 1}{3\lambda} v_1$$

and agent 2 is

$$D_2 = \frac{4 - 4\alpha_2 - 2\alpha_1}{6\lambda} v_1 + \frac{\alpha_2}{2\lambda} v_2$$

We note that $\alpha - 1/3 > \alpha/2$ when $\alpha > 2/3$. Thus agent 1’s trade aggressiveness on his own information will be greater than under no sharing.

$$D = \frac{1 + \alpha_1 - \alpha_2}{3\lambda} v_1 + \frac{\alpha_2}{2\lambda} v_2$$

From the conjectured demand functions of the agents, I have that $\hat{p} = \lambda D_1 + \lambda D_2 + \lambda \hat{x}$. Matching the two equations for price gives us:

$$\hat{p} = \frac{1 + \alpha_1 - \alpha_2}{3} v_1 + \frac{\alpha_2}{2} v_2 + \lambda \hat{x}$$

The conditionally expected profit for the agent 1 when he receives his own signal is:

$$E[\pi_{1as} | \hat{v}_1] = E[D_1(\hat{V}_1 - \hat{p}) | \hat{v}_1]$$

$$= \frac{1}{\lambda} \left( \frac{2\alpha_1 + \alpha_2 - 1}{3} \right)^2 (v_1)^2$$

Comparing with no sharing:

$$E[\pi_{1as} | \hat{v}_1] - E[\pi_{1is} | \hat{v}_1] = \frac{1}{\lambda} \left( \frac{2\alpha_1 + \alpha_2 - 1}{3} \right)^2 (\hat{v}_1)^2 - \frac{1}{\lambda} \left( \frac{\alpha_1}{2} \right)^2 (\hat{v}_1)^2$$

$$= \frac{1}{\lambda} \left( \frac{7\alpha_1 + 2\alpha_2 - 2}{6} \right) \left( \frac{\alpha_1 + 2\alpha_2 - 2}{6} \right) (\hat{v}_1)^2$$

Asymmetric sharing is preferred to no sharing when:

$$\left( \frac{\alpha}{2} - \frac{1}{3} \right) \left( \frac{3\alpha}{2} - \frac{1}{3} \right) (\hat{v}_1)^2 > 0$$

56
The above is true when

(i) \( \frac{2}{3} - \frac{1}{3} > 0 \) which implies \( \alpha > \frac{2}{3} \) (\( \frac{3\alpha}{2} - \frac{1}{3} > 0 \) in this region)

(ii) \( \frac{3\alpha}{2} - \frac{1}{9} < 0 \) which implies \( \alpha < \frac{2}{9} \) (\( \frac{3\alpha}{2} - \frac{1}{3} > 0 \) in this region)

Therefore, in these situations, irrespective of whether the other agent shares information, agent 1 will want to share info.

**Comparing with full sharing:**

\[
\mathbb{E}[\pi_{1as}|\tilde{v}_1] - \mathbb{E}[\pi_{1as}|\tilde{v}_1] = \frac{1}{\lambda}[(\alpha - \frac{1}{3})^2\frac{1}{\rho}(\tilde{v}_1)^2 - \frac{1}{\lambda}[(\alpha - \frac{1}{3})^2(\tilde{v}_1)^2 + (\frac{2}{3} - \alpha)^2\frac{1}{\rho}(\tilde{v}_1)^2]
= -\frac{1}{\lambda}(\frac{2}{3} - \alpha)^2\frac{1}{\rho}(\tilde{v}_1)^2
\]

**Agent 2’s profit under one-sided sharing:** The conditionally expected profit for the agent 2 when he receives his own signal is:

\[
\mathbb{E}[\pi_{2as}|\tilde{v}_2] = \mathbb{E}[D_2(\tilde{V}_2 - \tilde{p})|\tilde{v}_2]
= \frac{1}{\lambda}[(\frac{2}{3} - \alpha)^2\frac{1}{\rho} + (\frac{\alpha}{2})^2(\tilde{v}_2)^2]
\]

Therefore, agent 2 also benefits from agent 1 sharing his information.

\[\square\]

### 6.3 Model: Proof for Proposition 2

**Proof.** I will show that if \( \alpha \notin [0.4762, 0.6667] \), profits on sharing information will monotonically increase in the precision of the signal shared by the agents (\( \tau \)) while if \( \alpha \in [0.4762, 0.6667] \), profits decrease monotonically with the precision of the signal.

The expected payoff of, say, trader 1 one receiving a signal \( \tilde{s}_2 \) for trader 2 is:

\[
\mathbb{E}_1[\alpha\tilde{v}_1 + (1 - \alpha)\tilde{v}_2|\tilde{v}_1, \tilde{s}_2] = \alpha\tilde{v}_1 + (1 - \alpha)\mathbb{E}[\tilde{v}_2|\tilde{s}_2]
\]
The conditional expectation for trader 1 can be calculated as $E_1[\tilde{v}_2|\tilde{s}_2] = \frac{r}{\tau + \rho} \tilde{s}_2$. Similarly, $E_2[\tilde{v}_1|\tilde{s}_1] = \frac{r}{\tau + \rho} \tilde{s}_1$.

I conjecture that each agent’s demand is a linear function of the two signals he receives. For agent 1, $D_1 = a_1 + b_1 \tilde{v}_1 + c_1 \tilde{s}_2$. Given prices $\tilde{p}$, his profits are

$$E_1[D_1(\tilde{V}_1 - \tilde{p}) - \tilde{v}_1, \tilde{s}_2]$$

Taking FOCs with respect to $D_1$ in the above equation:

$$0 = \alpha \tilde{v}_1 + (1 - \alpha) \frac{r}{\tau + \rho} \tilde{s}_2 - 2\lambda D_1 - \lambda E_1[D_2^*|\tilde{v}_1, \tilde{s}_2]$$

The optimal demand for the agent is then given by:

$$D_1^* = \alpha \tilde{v}_1 + (1 - \alpha) \frac{r}{\tau + \rho} \tilde{s}_2$$

Similarly,

$$D_2^* = \alpha \tilde{v}_2 + (1 - \alpha) \frac{r}{\tau + \rho} \tilde{s}_1$$

Given that the agent’s strategies are symmetric, I set $a_1 = a_2 = a$. Matching coefficients,

$$a + b \tilde{v}_1 + c \tilde{s}_2 = [- a_2] + \frac{\alpha}{2} \tilde{v}_1 + \frac{(1 - \alpha) r}{2\lambda(\tau + \rho)} - \frac{\tau b_2}{2(\tau + \rho)}$$

I have that $a = 0$, and

$$b = \frac{\alpha}{2\lambda} - \frac{c}{2}$$
and

\[ c = \frac{(2 - 3\alpha)\tau}{(3\tau + 4\rho)\lambda} \]

and

\[ b = \frac{3\alpha\tau + 2\alpha\rho - \tau}{\lambda(3\tau + 4\rho)} \]

Therefore the demand for agent \( i \) is:

\[
D_i = \frac{1}{\lambda} \left[ 3\alpha\tau + 2\alpha\rho - \tau \right] \hat{v}_i + \frac{1}{\lambda} \left[ (2 - 3\alpha)\tau \right] \hat{s}_i
\]

Henceforth, I use the notation \( a_1 = a_2 = a \) and so on for \( b_i \) and \( c_i \). The market maker knows the true distribution of the payoffs and then sets the price equal to the conditional expectation of the payoff given his total order flow:

\[
\hat{p} = E[\hat{v}|b\hat{v}_1 + c\hat{s}_2 + b\hat{v}_2 + c\hat{s}_1 + \hat{x}]
= \frac{Cov(\hat{v}, b\hat{v}_1 + c\hat{s}_2 + b\hat{v}_2 + c\hat{s}_1 + \hat{x})}{Var(b\hat{v}_1 + c\hat{s}_2 + b\hat{v}_2 + c\hat{s}_1 + \hat{x})} (b\hat{v}_1 + c\hat{s}_2 + b\hat{v}_2 + c\hat{s}_1 + \hat{x})
= \frac{2(b + c)(1/\rho)}{2(b + c)^2(1/\rho) + 2c^2(1/\tau) + 1/\chi} (b\hat{v}_1 + c\hat{s}_2 + b\hat{v}_2 + c\hat{s}_1 + \hat{x})
\]

From the conjectured demand functions of the agents, I have that \( \hat{p} = \lambda D_1 + \lambda D_2 + \lambda \hat{x} \). Matching the two equations for price gives us:

\[
\hat{p} = \frac{1}{\lambda} \left[ 3\alpha\tau + 2\alpha\rho - \tau \right] \hat{v}_i + \frac{1}{\lambda} \left[ (2 - 3\alpha)\tau \right] \hat{s}_i + \lambda \hat{x}
\]
Clearly, I have that:

\[
\lambda = \frac{2(b + c)(1/\rho)}{2(b + c)^2(1/\rho) + 2c^2(1/\tau) + 1/\chi}
\]

\[\Rightarrow \lambda^2/2\chi = \frac{(2\alpha \rho + \tau)[(3\tau + 4\rho) - (2\alpha \rho + \tau)](1/\rho) - (2 - 3\alpha)^2 \tau}{(3\tau + 4\rho)^2}\]

I note that \(\bar{p} = \lambda(b\bar{v}_1 + c\bar{s}_2 + b\bar{v}_2 + c\bar{s}_1 + \bar{x}) = \lambda(b\bar{v}_1 + c\bar{v}_2 + c\bar{n}_2 + b\bar{v}_2 + c\bar{v}_1 + c\bar{n}_1 + \bar{x})\). The expected profit for the agent 1 is given by:

\[
E[\pi_1] = E[D_1(\alpha\bar{v}_1 + (1 - \alpha)\bar{v}_2 - \bar{p})]
\]

\[
= bE[\bar{v}_1 \left( (\alpha - \lambda b - \lambda c)\bar{v}_1 + ((1 - \alpha) - \lambda b - \lambda c)\bar{v}_2 - \lambda c\bar{n}_2 - c\lambda \bar{n}_1 - \lambda \bar{x} \right)]
\]

\[
+ cE[\bar{v}_2 \left( (\alpha - \lambda b - \lambda c)\bar{v}_1 + ((1 - \alpha) - \lambda b - \lambda c)\bar{v}_2 - \lambda c\bar{n}_2 - \lambda c\bar{n}_1 - \lambda \bar{x} \right)]
\]

\[
+ cE[\bar{n}_2 \left( (\alpha - \lambda b - \lambda c)\bar{v}_1 + ((1 - \alpha) - \lambda b - \lambda c)\bar{v}_2 - \lambda c\bar{n}_2 - \lambda c\bar{n}_1 - \lambda \bar{x} \right)]
\]

Simplifying:

\[
E[\pi_1] = E[D_1(\alpha\bar{v}_1 + (1 - \alpha)\bar{v}_2 - \bar{p})]
\]

\[
= b(\alpha - \lambda b - \lambda c)\frac{1}{\rho}
\]

\[
+ c((1 - \alpha) - \lambda b - \lambda c)\frac{1}{\rho}
\]

\[- \lambda c^2 \frac{1}{\tau}
\]

I can write the expected profits as a function \(\Pi(\tau)\). I note that as long as \(\alpha \notin [10/21, 2/3]\), the function \(\Pi(\tau)\) is increasing and strictly concave in \(\tau\), that is \(\Pi'(\tau) > 0\) and \(\Pi''(\tau) < 0\), and the reverse is true when \(\alpha \in [10/21, 2/3]\)
6.4 Communication mechanisms: Proof for Proposition 3

Proof. Suppose to the contrary, there is some non zero measure of signal realization types $S \subseteq [-V, V]$ of an agent $i$ who do not disclose their signal. Suppose, without any loss of generality, $\mathbb{E}_{-i}[\tilde{v}_i | S] > 0$. Then choose some $x \in S$ such that $x > \mathbb{E}_{-i}[\tilde{v}_i | S]$. Then, the agent with type $x$ would be better off deviating (since $\mathbb{E}[\pi_{1as} | \tilde{v}_1, x] > \mathbb{E}[\pi_{1as} | \tilde{v}_1, \mathbb{E}_{-i}[\tilde{v}_i | S]]$ and disclosing his signal truthfully. A similar logic would prevail when $\mathbb{E}_{-i}[\tilde{v}_i | S] < 0$, in which case we choose $x \in S$ such that $x < \mathbb{E}_{-i}[\tilde{v}_i | S]$. This contradiction completes the proof. Only agents with type $0$ are indifferent between disclosing and not disclosing, hence the receiver can conjecture upon non-disclosure that the agent is of type $0$.

6.5 Communication mechanisms: Proof for Proposition 4

Proof. Throughout the proof, I solve for optimal quantity choices and profits using subscripts for agent 1 (without any loss of generality). I define the quantity choices assuming truthtelling under various sent and received messages as follows:

<table>
<thead>
<tr>
<th>$\tilde{v}_1$</th>
<th>$\tilde{v}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v^*$</td>
<td>$\tilde{v}_2 &gt; v^*$</td>
</tr>
<tr>
<td>$\tilde{v}_1 &lt; v^<em>$ &amp; $\tilde{v}_2 &lt; v^</em>$</td>
<td></td>
</tr>
<tr>
<td>$D_{1L}^{LL}(v_1)$ &amp; $D_{1H}^{LL}(v_1)$</td>
<td></td>
</tr>
<tr>
<td>$D_{1L}^{HL}(v_1)$ &amp; $D_{1H}^{HL}(v_1)$</td>
<td></td>
</tr>
</tbody>
</table>

The optimal quantity choice under each scenario is:

(i) $\tilde{v}_1 < v^*$ and $\tilde{v}_2 < v^*$

$$D_{1L}^{LL}(v) = \frac{\mathbb{E}_1[\tilde{V}_1 | \tilde{v}_1 = v, \tilde{v}_2 < v^*]}{2\lambda} - \frac{1}{2} \mathbb{E}_1[\tilde{D}_2 | \tilde{v}_1 = v, \tilde{v}_2 < v^*]$$

$$= \frac{\alpha v + (1 - \alpha) \mathbb{E}[\tilde{v}_2 | \tilde{v}_2 < v^*]}{2\lambda} - \frac{1}{2} \mathbb{E}[D_{2L}^{LL}(v) | \tilde{v}_2 < v^*]$$

61
Assuming that $D_{1L}(v)$ is linear in $v$:

$$D_{1L}(v) = a + bv$$

Substituting,

$$\frac{3a}{2} + bv = \frac{\alpha v}{2\lambda} + \frac{(1 - \alpha)\mathbb{E}[\tilde{v}|\tilde{v} < v^*]}{2\lambda} - \frac{b}{2} \mathbb{E}[v|v < v^*]$$

Matching co-efficients, we have that $b = \frac{\alpha}{2\lambda}$ and,

$$a = \frac{(2 - 3\alpha)\mathbb{E}[\tilde{v}|\tilde{v} < v^*]}{6\lambda}$$

(ii) $\tilde{v}_1 < v^*$ and $\tilde{v}_2 > v^*$

$$D_{1H}(v) = \frac{\mathbb{E}[\tilde{V}_1|\tilde{v}_1 = v, \tilde{v}_2 > v^*]}{2\lambda} - \frac{1}{2} \mathbb{E}[\tilde{D}_2|\tilde{v}_1 = v, \tilde{v}_2 > v^*]$$

$$= \frac{\alpha v + (1 - \alpha)\mathbb{E}[\tilde{v}_2|\tilde{v}_2 > v^*]}{2\lambda} - \frac{1}{2} \mathbb{E}[D_{2L}(v)|\tilde{v}_2 > v^*]$$

(iii) $\tilde{v}_1 > v^*$ and $\tilde{v}_2 < v^*$

$$D_{1L}(v) = \frac{\mathbb{E}[\tilde{V}_1|\tilde{v}_1 = v, \tilde{v}_2 < v^*]}{2\lambda} - \frac{1}{2} \mathbb{E}[\tilde{D}_2|\tilde{v}_1 = v^*, \tilde{v}_2 < v^*]$$

$$= \frac{\alpha v + (1 - \alpha)\mathbb{E}[\tilde{v}_2|\tilde{v}_2 < v^*]}{2\lambda} - \frac{1}{2} \mathbb{E}[D_{2L}(v)|\tilde{v}_2 < v^*]$$

Suppose $D_{1H}(v) = c + dv$ and $D_{1L}(v) = e + fv$. Now substituting these values in the equation for $D_{1H}(v)$:

$$c + dv = \frac{\alpha v + (1 - \alpha)\mathbb{E}[\tilde{v}|\tilde{v} > v^*]}{2\lambda} - \frac{1}{2} e - \frac{1}{2} f \mathbb{E}[v|v > v^*]$$

Matching co-efficients $d = \frac{\alpha}{2\lambda}$, so

$$c = \frac{(1 - \alpha)\mathbb{E}[v|v > v^*]}{2\lambda} - \frac{1}{2} e - \frac{1}{2} f \mathbb{E}[v|v > v^*]$$

$$\Rightarrow c = \frac{(2 - 3\alpha)\mathbb{E}[v|v > v^*]}{4\lambda} - \frac{1}{2} e$$

62
Similarly, in the equation for $D_{HL}^H(v)$:

$$e + fv = \frac{\alpha v + (1 - \alpha)E[\tilde{v}_2 | \tilde{v}_2 < v^*]}{2\lambda} - \frac{1}{2}E_1[c + dv | \tilde{v}_2 < v^*]$$

Matching co-efficients $f = \frac{\alpha}{2\lambda}$, so

$$e = \frac{(2 - 3\alpha)E[\tilde{v}_2 | \tilde{v}_2 < v^*]}{4\lambda} - \frac{1}{2}c$$

Further substituting,

$$e = \frac{(2 - 3\alpha)}{3\lambda} \left[ E[\tilde{v}_2 | \tilde{v}_2 < v^*] - \frac{E[v | \tilde{v} > v^*]}{2} \right]$$

and

$$c = \frac{(2 - 3\alpha)E[v | \tilde{v} > v^*]}{4\lambda} - \frac{1}{2} \left[ \frac{(2 - 3\alpha)E[\tilde{v}_2 | \tilde{v}_2 < v^*]}{4\lambda} - \frac{1}{2}c \right]$$

$$\Rightarrow c = \frac{(2 - 3\alpha)}{3\lambda} \left[ E[v | \tilde{v} > v^*] - \frac{E[v | \tilde{v} < v^*]}{2} \right]$$

(iv) $\tilde{v}_1 > v^*$ and $\tilde{v}_2 > v^*$

$$D_{HL}^H(v) = \frac{E_1[\tilde{V}_1 | \tilde{v}_1 = v^*, \tilde{v}_2 > v^*]}{2\lambda} - \frac{1}{2}E_1[D_2 | \tilde{v}_1 = v, \tilde{v}_2 > v^*]$$

$$= \frac{\alpha v + (1 - \alpha)E[\tilde{v}_2 | \tilde{v}_2 > v^*]}{2\lambda} - \frac{1}{2}E_1[D_2^H | \tilde{v}_2 > v^*]$$

Let $D_{HL}^H(v) = g + hv$. Solving, we have clearly that $h = \frac{\alpha}{2\lambda}$

$$\frac{3}{2}g = \frac{(1 - \alpha)E[\tilde{v} | \tilde{v} > v^*]}{2\lambda} - \frac{\alpha}{4\lambda}E[v | v > v^*]$$

$$\Rightarrow g = \frac{(2 - 3\alpha)E[\tilde{v} | \tilde{v} > v^*]}{6\lambda}$$

Finally, it must be the case that the agent with signal $v_i = v^*$ is indifferent to reporting a high message.
or a low message. His profit on reporting a low message is:

\[
\mathbb{E}[\pi | v_i = v^*, m_1L; v^*] = \mathbb{P}(\hat{v}_2 < v^*)\mathbb{E}\left[ D_{1L}^{LL}(v^*) (\alpha v^* + (1 - \alpha)v_2 - \lambda D_{1L}^{LL}(v^*))|\hat{v}_2 < v^* \right] \\
+ \mathbb{P}(\hat{v}_2 > v^*)\mathbb{E}\left[ D_{1L}^{HL}(v^*) (\alpha v^* + (1 - \alpha)v_2 - \lambda D_{1L}^{HL}(v^*))|\hat{v}_2 > v^* \right]
\]

Similarly for the high report,

\[
\mathbb{E}[\pi | v_i = v^*, m_1H; v^*] = \mathbb{P}(\hat{v}_2 < v^*)\mathbb{E}\left[ D_{1L}^{HL}(v^*) (\alpha v^* + (1 - \alpha)v_2 - \lambda D_{1L}^{HL}(v^*))|\hat{v}_2 < v^* \right] \\
+ \mathbb{P}(\hat{v}_2 > v^*)\mathbb{E}\left[ D_{1L}^{HL}(v^*) (\alpha v^* + (1 - \alpha)v_2 - \lambda D_{1L}^{HL}(v^*))|\hat{v}_2 > v^* \right]
\]

We first write the net benefit to reporting low over reporting high for the marginal type \(v^*\) as:

\[
\Delta \Pi_{L-H}(v^*; v^*) = \frac{(2 - 3\alpha)(\mathbb{E}[\hat{v}_2 | \hat{v}_2 > v^*] - \mathbb{E}[\hat{v}_2 | \hat{v}_2 < v^*])}{6} \left\{ \alpha v^* + \frac{(2 - 3\alpha)}{2} \mathbb{E}[\hat{v}_2] - \frac{(2 - 3\alpha)(\mathbb{P}(\hat{v}_2 > v^*)\mathbb{E}[\hat{v}_2 | \hat{v}_2 < v^*] + \mathbb{P}(\hat{v}_2 < v^*)\mathbb{E}[\hat{v}_2 | \hat{v}_2 > v^*])}{6} \right\}
\]

Setting \(\Delta \Pi_{L-H}(v^*; v^*) = 0\), and since \(\mathbb{E}[\hat{v}_2 | \hat{v}_2 > v^*] - \mathbb{E}[\hat{v}_2 | \hat{v}_2 < v^*] > 0\)

\[
\alpha v^* + \frac{(2 - 3\alpha)}{2} \mathbb{E}[\hat{v}_2] - \frac{(2 - 3\alpha)(\mathbb{P}(\hat{v}_2 > v^*)\mathbb{E}[\hat{v}_2 | \hat{v}_2 < v^*] + \mathbb{P}(\hat{v}_2 < v^*)\mathbb{E}[\hat{v}_2 | \hat{v}_2 > v^*])}{6} = 0
\]

Simplifying

\[
\alpha v^* + \frac{(2 - 3\alpha)\left(4\mathbb{E}[\hat{v}_2] - \mathbb{E}[\hat{v}_2 | \hat{v}_2 < v^*] - \mathbb{E}[\hat{v}_2 | \hat{v}_2 > v^*]\right)}{6} = 0
\]

First, it can be easily shown that when \(\mathbb{E}[\hat{v}_i] = 0\) and the payoff distribution is symmetric about the mean, then \(v^* = \mathbb{E}[\hat{v}_i] = 0\) solves the above expression.\(^{33}\) Whenever \(\hat{v}_i\) has full support, we note that as

\(^{33}\)This is immediately obtained by substituting \(\mathbb{P}(\hat{v}_2 > v^*) = \mathbb{P}(\hat{v}_2 < v^*)\) in the above expression.
\[ v^* \to -\infty, \ E[\hat{v}_2|\hat{v}_2 < v^*] \to v^* = -\infty \text{ and } E[\hat{v}_2|\hat{v}_2 > v^*] \to E[\hat{v}_2]. \]

\[
\lim_{v^* \to -\infty} \Delta \Pi_{L-H}(v^*; v^*) = \lim_{v^* \to -\infty} \left\{ \frac{(2 - 3\alpha)(E[\hat{v}_2] - v^*)}{6} \left( \alpha v^* + \frac{(2 - 3\alpha)3E[\hat{v}_2] - v^*}{6} \right) \right\} \\
= \lim_{v^* \to -\infty} \left\{ \frac{(2 - 3\alpha)E[\hat{v}_2] - v^*)}{6} \left( \frac{9\alpha - 2}{2} v^* + \frac{(2 - 3\alpha)E[\hat{v}_2]}{2} \right) \right\}
\]

We further that the limit of the first term viz. \( \lim_{v^* \to -\infty} \frac{(2 - 3\alpha)\alpha v^*}{6} = -\infty \) for \( \alpha > 2/3 \) and the limit of the second term \( \lim_{v^* \to -\infty} \frac{(9\alpha - 2)v^*}{6} + \frac{(2 - 3\alpha)E[\hat{v}_2]}{2} = -\infty \) for \( \alpha > 2/9 \). Therefore, the product of the two terms viz. \( \lim_{v^* \to -\infty} \Delta \Pi_{L-H}(v^*; v^*) = \infty \). Similarly \( \hat{v}_i \) has full support, we note that as \( v^* \to +\infty \), \( E[\hat{v}_2|\hat{v}_2 > v^*] \to v^* = \infty \) and \( E[\hat{v}_2|\hat{v}_2 < v^*] \to E[\hat{v}_2]. \)

\[
\lim_{v^* \to +\infty} \Delta \Pi_{L-H}(v^*; v^*) = \lim_{v^* \to +\infty} \left\{ \frac{(2 - 3\alpha)\alpha v^*}{6} \left( \frac{(2 - 3\alpha)E[\hat{v}_2] - v^*}{6} \right) \right\} \\
= \lim_{v^* \to +\infty} \left\{ \frac{(2 - 3\alpha)\alpha v^*}{6} \left( \frac{(2 - 3\alpha)E[\hat{v}_2]}{2} \right) \right\}
\]

The product of the two terms viz. \( \lim_{v^* \to +\infty} \Delta \Pi_{L-H}(v^*; v^*) = -\infty \) for \( \alpha > 2/3 \). Since at the extremes, the value of the function \( \Delta \Pi_{L-H}(v^*; v^*) \) has opposite signs, and the function is continuous in \( v^* \), it must be that the function takes a value of zero for some \( v^* \in (-\infty, \infty) \). For any arbitrary signal \( v \), the net benefit on truthtelling is given by:

\[
\Delta \Pi_{L-H}(v; v^*) = \frac{(2 - 3\alpha)(E[\hat{v}_2|\hat{v}_2 > v^*] - E[\hat{v}_2|\hat{v}_2 < v^*])}{6} \left\{ \alpha v^* + \frac{(2 - 3\alpha)\alpha v^*}{6} \right\} \\
= \frac{(2 - 3\alpha)(E[\hat{v}_2|\hat{v}_2 > v^*] - E[\hat{v}_2|\hat{v}_2 < v^*])}{6} \left\{ \frac{(2 - 3\alpha)E[\hat{v}_2] - v^*)}{6} \right\}
\]

Rearranging terms:

\[
\Delta \Pi_{L-H}(v; v^*) = \frac{(2 - 3\alpha)(E[\hat{v}_2|\hat{v}_2 > v^*] - E[\hat{v}_2|\hat{v}_2 < v^*])}{6} \left\{ \alpha v^* + \frac{(2 - 3\alpha)(4E[\hat{v}_2] - E[\hat{v}_2|\hat{v}_2 < v^*] - E[\hat{v}_2|\hat{v}_2 > v^*])}{6} \right\}
\]

The net benefit to truthtelling \( \Delta \Pi_{L-H}(v; v^*) \) for some exogenously specified \( v^* \) is linear, and hence mono-
tonic, in \( v \). Moreover, since \( \mathbb{E}[\tilde{v} | \tilde{v} > v^*] - \mathbb{E}[\tilde{v} | \tilde{v} < v^*] > 0 \), the slope is negative whenever \( \alpha > 2/3 \). Therefore, \( \lim_{v \to -\infty} \Delta \Pi_{L-H}(v; v^*) = \infty \) and \( \lim_{v \to \infty} \Delta \Pi_{L-H}(v; v^*) = -\infty \). Since \( \Delta \Pi_{L-H}(v; v^*) \) is monotonic, the condition (ii) viz. \( \Delta \Pi_{L-H}(v; v^*) > 0 \) for any \( v < v^* \) and \( \Delta \Pi_{L-H}(v^*; v^*) < 0 \) for any \( v > v^* \) is met.

\[
\text{6.6 Communication mechanism: Proof for Proposition 5}
\]

**Proof:** The mediator’s message distribution is given by a “max” mechanism: if \( \max(|\hat{v}_1|, |\hat{v}_2|) \leq v^* \), then the mediator reports a public signal \( m_L \) else he reports \( m_H \). In the stated mechanism, it would not matter if an agent reports a false signal realization as long as the reported signal is less than \( v^* \) when the true signal is less than \( v^* \) (and vice versa). Hence, we denote an agent’s signal realization as \( L \) if \( |\hat{v}_i| < v^* \) and \( H \) otherwise. We consider two separate cases (1) we assume that the agents signals are of the same sign and (2) they are of opposite signs. We derive conditions for incentive compatible information sharing under each of these cases

6.6.1 Same signs

We assume that both \( v_1 > 0 \) and \( v_2 > 0 \). We write the optimal quantity choice for agent 1 as \( q_1(\hat{v}_1, \hat{v}_1, m) \) where the first argument denotes agent 1’s signal and the second argument denotes agent 1’s report to the mediator. We consider the following cases in order:

(i) truthful agent’s signal \( v_1 < v_P \), mediator reports \( m_L \)

\[
q_1(v_1, L, \hat{L}, m_L) = \frac{\mathbb{E}[\tilde{V}_1 | \tilde{v}_1 = \hat{v}_1, m_L]}{2\beta} - \frac{1}{2} \mathbb{E}[\tilde{Q}_2 | \tilde{v}_1 = \hat{v}_1, m_L]
\]

\[
= \alpha v_1 + (1 - \alpha) \mathbb{E}[\tilde{v}_1 | \tilde{v}_1 = \hat{v}_1, m_L] - \frac{1}{2} \mathbb{E}[\tilde{Q}_2 | \tilde{v}_1 = \hat{v}_1, m_L]
\]

\[
= \alpha v_1 + (1 - \alpha) \mathbb{E}[\tilde{v}_2 | 0 < \tilde{v}_2 < v_P] - \frac{1}{2} \mathbb{E}[\tilde{Q}_2(\hat{v}_1 = \hat{v}_1, m_L) | 0 < \tilde{v}_2 < v_P]
\]

(ii) truthful agent’s signal \( v_1 < v_P \), mediator reports \( m_H \)
\[ q_1(v_1, L, \hat{L}, m_H) = \frac{E_1[\hat{V}_1 | \hat{v}_1 = v_1, m_H]}{2 \beta} - \frac{1}{2} E_1[\hat{Q}_2 | \hat{v}_1 = v_1, m_H] \]

\[ = \frac{\alpha v_1 + (1 - \alpha) E_1[\hat{v}_2 | \hat{v}_2 > v_P]}{2 \beta} - \frac{1}{2} E_1[\hat{Q}_2 | \hat{v}_1 = v_1, m_H] \]

(iii) truthful agent’s signal \( v_1 > v_P \), mediator reports \( m_H \)

\[ q_1(v_1, H, \hat{H}, m_H) = \frac{E_1[\hat{V}_1 | \hat{v}_1 = v_1, m_H]}{2 \beta} - \frac{1}{2} E_1[\hat{Q}_2 | \hat{v}_1 = v_1, m_H] \]

\[ = \frac{\alpha v_1 + (1 - \alpha) E_1[\hat{v}_2]}{2 \beta} - \frac{1}{2} E_1[\hat{Q}_2 | \hat{v}_1 = v_1, m_H] \]

\[ = \frac{\alpha v_1 + (1 - \alpha) E_1[\hat{v}_2]}{2 \beta} - \frac{1}{2} \mathbb{P}[\hat{v}_2 < v^*] E_1[q_2(v_2, L, m_H) | 0 < \hat{v}_2 < v_P] \]

\[ - \frac{1}{2} \mathbb{P}[\hat{v}_2 > v^*] E_1[q_2(v_2, H, m_H) | \hat{v}_2 > v_P] \]

Thus, the chosen quantity as a function of own signal \( v_1 \) depends only on the report \( \hat{v}_1 \). We next assume (and later verify) that the agent’s signals are linear in their signal and are symmetric for both agents.

(i) truthful agent’s signal \( v_1 < v_P \), mediator reports \( m_L \)

\[ q_1(v_1, \hat{L}, m_L) = a + bv_1 \]

\[ = \alpha v_1 + (1 - \alpha) E_1[\hat{v}_2 | 0 < \hat{v}_2 < v_P] - \frac{1}{2} E_1[a + bv_2 | \hat{v}_2 < v_P] \]

Simplifying:

\[ a + bv_1 = \frac{\alpha v_1}{2 \beta} + \frac{(1 - \alpha) E_1[\hat{v}_2 | 0 < \hat{v}_2 < v_P]}{2 \beta} - \frac{a}{2} - \frac{b E_1[\hat{v}_2 | 0 < \hat{v}_2 < v_P]}{2} \]

Clearly, \( b = \frac{\alpha}{2 \beta} \). Substituting:

\[ a = \frac{(2 - 3\alpha) E_1[\hat{v}_2 | 0 < \hat{v}_2 < v_P]}{6 \beta} \]

(ii) truthful agent’s signal \( v_1 < v_P \), mediator reports \( m_H \)
\[ q_1(v_1, \hat{L}, m_H) = c + dv_1 \]
\[ = \frac{\alpha v_1 + (1 - \alpha) E_1[\hat{v}_2 | \hat{v}_2 > v^*]}{2\beta} - \frac{1}{2} E_1[\frac{1}{2} q_2(v_2, H, m_H) | \hat{v}_2 > v_P] \]
\[ = \frac{\alpha v_1 + (1 - \alpha) E_1[\hat{v}_2 | \hat{v}_2 > v^*]}{2\beta} - \frac{1}{2} E_1[c + f v_2 | \hat{v}_2 > v_P] \]

Clearly, \( d = \frac{\alpha}{2\beta} \). Substituting:
\[ c + \frac{e}{2} = \frac{(1 - \alpha) E_1[\hat{v}_2 | \hat{v}_2 > v^*]}{2\beta} - \frac{\alpha E_1[\hat{v}_2 | \hat{v}_2 > v_P]}{4\beta} \]

(iii) truthful agent’s signal \( v_1 > v_P \), mediator reports \( m_H \)

\[ q_1(v_1, \hat{H}, m_H) = \frac{\alpha v_1 + (1 - \alpha) E_1[\hat{v}_2]}{2\beta} - \frac{1}{2} P[\hat{v}_2 < v_P] E_1[q_2(v_2, L, m_H) | \hat{v}_2 < v_P] - \frac{1}{2} P[\hat{v}_2 > v_P] E_1[q_2(v_2, H, m_H) | \hat{v}_2 > v_P] \]
\[ = \frac{\alpha v_1 + (1 - \alpha) E_1[\hat{v}_2]}{2\beta} - \frac{1}{2} P[\hat{v}_2 < v_P] [c + d E_1[\hat{v}_2 | 0 < \hat{v}_2 < v_P]] - \frac{1}{2} P[\hat{v}_2 > v_P] [e + f E_1[\hat{v}_2 | \hat{v}_2 > v_P]] \]

Simplifying:
\[ e + fv_1 = \frac{\alpha v_1 + (1 - \alpha) E_1[\hat{v}_2]}{2\beta} - \frac{1}{2} P[\hat{v}_2 < v_P] [c + d E_1[\hat{v}_2 | 0 < \hat{v}_2 < v_P]] - \frac{1}{2} P[\hat{v}_2 > v_P] [e + f E_1[\hat{v}_2 | \hat{v}_2 > v_P]] \]

Clearly, \( f = \frac{\alpha}{2\beta} \). Substituting:
\[ e = \frac{(1 - \alpha) E_1[\hat{v}_2]}{2\beta} - \frac{1}{2} P[\hat{v}_2 < v_P] [c + \frac{\alpha}{2\beta} E_1[\hat{v}_2 | 0 < \hat{v}_2 < v_P]] - \frac{1}{2} P[\hat{v}_2 > v_P] [e + \frac{\alpha}{2\beta} E_1[\hat{v}_2 | \hat{v}_2 > v_P]] \]

The expected profit on truth telling for agent 1 on receiving signal \( v_1 < v_P \) is given by:

\[ E[\pi | \hat{v}_1 = L] = P[v_2 < v_P] E[\pi | \hat{v}_1 = L, 0 < v_2 < v_P, m_L] + P[v_2 > v_P] E[\pi | \hat{v}_1 = L, v_2 > v_P, m_H] \]
where

\[
E[\pi|\hat{v}_1 = L, v_2 < v_P, m_L] = E \left[ q_1(v_1, L, m_L) (\alpha \hat{v}_1 + (1 - \alpha) \hat{v}_2 - \beta q_1(v_1, L, m_L) - \beta q_2(v_2, L, m_L))|\hat{v}_1, v_2 < v_P \right]
= (a + b \hat{v}_1) (\alpha \hat{v}_1 + (1 - \alpha)E[\hat{v}_2|0 < v_2 < v_P] - \beta(a + b \hat{v}_1) - \beta(a + bE[\hat{v}_2|v_2 < v_P])
\]

and

\[
E[\pi|\hat{v}_1 = L, v_2 > v_P, m_H] = E \left[ q_1(v_1, L, m_H) (\alpha \hat{v}_1 + (1 - \alpha) \hat{v}_2 - \beta q_1(v_1, L, m_H) - \beta q_2(v_2, H, m_H))|\hat{v}_1, v_2 > v_P \right]
= (c + d \hat{v}_1) (\alpha \hat{v}_1 + (1 - \alpha)E[\hat{v}_2|v_2 > v_P] - \beta(c + d \hat{v}_1) - \beta(c + dE[\hat{v}_2|v_2 > v_P])
\]

The expected profit on truth telling for agent 1 on receiving signal \(v_1 > v_P\) is given by:

\[
E[\pi|\hat{v}_1 = H] = P[v_2 < v_P]E[\pi|\hat{v}_1 = H, 0 < v_2 < v_P, m_H] + P[v_2 > v^*]E[\pi|\hat{v}_1 = H, v_2 > v_P, m_H]
\]

where

\[
E[\pi|\hat{v}_1 = H, v_2 < v_P, m_H] = E \left[ q_1(v_1, H, m_H) (\alpha \hat{v}_1 + (1 - \alpha) \hat{v}_2 - \beta q_1(v_1, H, m_H) - \beta q_2(v_2, L, m_H))|\hat{v}_1, v_2 < v_P \right]
= (e + f \hat{v}_1) (\alpha \hat{v}_1 + (1 - \alpha)E[\hat{v}_2|0 < v_2 < v_P] - \beta(e + f \hat{v}_1) - \beta(e + fE[\hat{v}_2|v_2 < v_P])
\]

and

\[
E[\pi|\hat{v}_1 = H, v_2 > v_P, m_H] = E \left[ q_1(v_1, H, m_H) (\alpha \hat{v}_1 + (1 - \alpha) \hat{v}_2 - \beta q_1(v_1, H, m_H) - \beta q_2(v_2, H, m_H))|\hat{v}_1, v_2 > v^* \right]
= (e + f \hat{v}_1) (\alpha \hat{v}_1 + (1 - \alpha)E[\hat{v}_2|v_2 > v_P] - \beta(e + f \hat{v}_1) - \beta(e + fE[\hat{v}_2|v_2 > v_P])
\]

On the other hand when the agent, on receiving signal \(v_1 < v_P\), misreports it as \(H\), his optimal quantity
choice is as if he is of the high type. We have already established that the optimal quantity choice for a lying agent who receives a signal \( v_1 < v_P \) but reports it as \( v_1 > v_P \) is \( q_l(v_1, H, m_H) \).

\[
E[\pi' | \hat{v}_1 = H] = P[v_2 < v_P] E[\pi' | \hat{v}_1 = H, 0 < v_2 < v_P, m_H] + P[v_2 > v_P] E[\pi' | \hat{v}_1 = H, v_2 > v_P, m_H]
\]

where

\[
E[\pi' | \hat{v}_1 = H, v_2 < v_P, m_H] = E\left[ q_l(v_1, H, m_H)(\alpha \hat{v}_1 + (1 - \alpha) \hat{v}_2 - \beta q_l(v_1, H, m_H) - \beta q_2(v_2, L, m_H)) | \hat{v}_1, v_2 < v_P \right]
\]

\[= (e + f \hat{v}_1)(\alpha \hat{v}_1 + (1 - \alpha)E[\hat{v}_2 | 0 < v_2 < v_P] - \beta(e + f \hat{v}_1) - \beta(e + f E[\hat{v}_2 | v_2 < v_P]))\]

and

\[
E[\pi' | \hat{v}_1 = H, v_2 > v_P, m_H] = E\left[ q_l(v_1, H, m_H)(\alpha \hat{v}_1 + (1 - \alpha) \hat{v}_2 - \beta q_l(v_1, H, m_H) - \beta q_2(v_2, H, m_H)) | \hat{v}_1, v_2 > v_P \right]
\]

\[= (e + f \hat{v}_1)(\alpha \hat{v}_1 + (1 - \alpha)E[\hat{v}_2 | v_2 > v_P] - \beta(e + f \hat{v}_1) - \beta(e + f E[\hat{v}_2 | v_2 > v_P]))\]

We define the net benefit of reporting low over high as \( \Delta \Pi_{L-H}(v; v^*) = E[\pi | \hat{v}_1 = L] - E[\pi | \hat{v}_1 = H] \) and write it as a function of \( a, b...f \). Let \( E[\hat{v}_2 | v_2 > v_P] \equiv \mu_H, E[\hat{v}_2 | 0 < v_2 < v_P] \equiv \mu_L, P[\hat{v}_2 | v_2 > v_P] \equiv p_H \) and \( P[\hat{v}_2 | 0 < v_2 < v_P] \equiv p_L \).

We find that the net benefit is linear in \( v \) with the following slope:

\[
\frac{\partial \Delta \Pi_{L-H}(v; v_P)}{\partial v} = -\frac{\alpha (3\alpha - 2)(1 - p_H)(\mu_H^2 + \mu_L^2)(1 - p_H) + \mu_H \mu_L p_H - \mu_H - 3\mu_H \mu_L)}{\mu_H + \mu_H p_H + 2 \mu - 2 \mu_L + \mu_L p_H}
\]

When \( \alpha > 2/3 \), the slope is negative. Therefore, (i) there is a \( v = v_P \) that solves that \( \Delta \Pi_{L-H}(v; v_P) = 0 \) and (ii) the net benefit of reporting low when \( v_1 < v_P \) is positive and negative when \( v_1 > v_P \), hence it is incentive compatible for agents to report their signals truthfully in these regions.
6.6.2 Opposite signs

We assume wlog that \( \hat{v}_1 > 0 \) and \( \hat{v}_2 < 0 \). We write the optimal quantity choice for agent 1 as \( q_1(\hat{v}_1, \hat{v}_1, m) \) where the first argument denotes agent 1’s signal and the second argument denotes agent 1’s report to the mediator. We consider the following cases in order:

(i) truthful agent 1’s signal \( v_1 < v^* \), mediator reports \( m_L \)

\[
q_1(v_1, L, \hat{L}, m_L) = \frac{\mathbb{E}_1[\hat{V}_1|\hat{v}_1 = v_1, m_L]}{2\beta} - \frac{1}{2}\mathbb{E}_1[\hat{Q}_2|\hat{v}_1 = v_1, m_L]
\]

\[
= \frac{\alpha v_1 + (1 - \alpha)\mathbb{E}_1[\hat{v}_2|\hat{v}_2 > -v_P, \hat{v}_2 < 0]}{2\beta} - \frac{1}{2}\mathbb{E}_1[1 - \frac{1}{2}q_2(v_2, L, m_L)|\hat{v}_2 > -v_P, \hat{v}_2 < 0]
\]

(ii) truthful agent 1’s signal \( v_1 < v^* \), mediator reports \( m_H \)

\[
q_1(v_1, L, \hat{L}, m_H) = \frac{\mathbb{E}_1[\hat{V}_1|\hat{v}_1 = v_1, m_H]}{2\beta} - \frac{1}{2}\mathbb{E}_1[\hat{Q}_2|\hat{v}_1 = v_1, m_H]
\]

\[
= \frac{\alpha v_1 - (1 - \alpha)\mathbb{E}_1[\hat{v}_2|\hat{v}_2 > v_P]}{2\beta} - \mathbb{E}_1[1 - \frac{1}{2}q_2(v_2, H, m_H)|\hat{v}_2 > -v_P]
\]

(iii) truthful agent 1’s signal \( v_1 > v^* \), mediator reports \( m_H \)

\[
q_1(v_1, H, \hat{H}, m_H) = \frac{\mathbb{E}_1[\hat{V}_1|\hat{v}_1 = v_1, m_H]}{2\beta} - \frac{1}{2}\mathbb{E}_1[\hat{Q}_2|\hat{v}_1 = v_1, m_H]
\]

\[
= \frac{\alpha v_1 + (1 - \alpha)\mathbb{E}_1[\hat{v}_2|\hat{v}_2 < 0]}{2\beta} - \frac{1}{2}\mathbb{E}_1[\hat{Q}_2|\hat{v}_1 = v_1, m_H]
\]

\[
= \frac{\alpha v_1 + (1 - \alpha)\mathbb{E}_1[\hat{v}_2|\hat{v}_2 < 0]}{2\beta} - \frac{1}{2}\mathbb{E}_1[\hat{Q}_2|\hat{v}_1 = v_1, m_H]
\]

\[
\quad - \frac{1}{2}\mathbb{P}[\hat{v}_2 < v_P]\mathbb{E}_1[|q_2(v_2, H, m_H)|\hat{v}_2 < -v_P]
\]

(iv) truthful agent 2’s signal \( v_2 > -v_P \), mediator reports \( m_L \)

\[
q_2(v_2, L, \hat{L}, m_L) = \frac{\mathbb{E}[\hat{V}_2|\hat{v}_2 = v_2, m_L]}{2\beta} - \frac{1}{2}\mathbb{E}_1[\hat{Q}_1|\hat{v}_2 = v_2, m_L]
\]

\[
= \frac{\alpha v_2 + (1 - \alpha)\mathbb{E}[\hat{v}_1|\hat{v}_1 > 0, \hat{v}_1 < v^*]}{2\beta} - \frac{1}{2}\mathbb{E}_1[\frac{1}{2}q_1(v_1, L, m_L)|\hat{v}_1 > 0, \hat{v}_1 < v^*]
\]
(v) truthful agent 2’s signal $v_2 > -v_P$, mediator reports $m_H$

\[
q_2(v_2, L, \hat{L}, m_H) = \frac{E[\tilde{v}_2 | \tilde{v}_2 = v_2, m_H]}{2\beta} - \frac{1}{2}E[\tilde{Q}_1 | \tilde{v}_2 = v_2, m_H] \\
= \alpha v_2 + (1 - \alpha)E[\tilde{v}_1 | \tilde{v}_1 > v_P] - \frac{1}{2}E[\frac{1}{2}q_1(v_1, H, m_H) | \tilde{v}_2 = v_P, m_H]
\]

(vi) truthful agent 1’s signal $v_2 < -v_P$, mediator reports $m_H$

\[
q_2(v_2, H, \hat{H}, m_H) = \frac{E[\tilde{v}_2 | \tilde{v}_2 = v_2, m_H]}{2\beta} - \frac{1}{2}E[\tilde{Q}_1 | \tilde{v}_2 = v_2, m_H] \\
= \alpha v_2 + (1 - \alpha)E[\tilde{v}_1 | \tilde{v}_1 > 0] - \frac{1}{2}E[\tilde{Q}_1 | \tilde{v}_2 = v_2, m_H] \\
= \frac{\alpha v_2}{2\beta} - \frac{1}{2}P[\tilde{v}_1 > 0, \tilde{v}_1 < v_P]E[q_1(v_1, L, m_H) | \tilde{v}_1 < v_P] \\
- \frac{1}{2}P[\tilde{v}_1 > v_P]E[q_1(v_1, H, m_H) | \tilde{v}_1 > v_P]
\]

Thus, the chosen quantity as a function of own signal $v_1$ depends only on the report $\hat{v}_1$. We next assume (and later verify) that the agent’s signals are linear in their signal and are symmetric for both agents.

1. truthful agent’s signal $v_1 < v_P$, mediator reports $m_L$

\[
q_1(v_1, \hat{L}, m_L) = a_1 + b_1 v_1 \\
= \alpha v_1 - (1 - \alpha)E[v_2 | v_2 < v_P] - \frac{1}{2}E[a_2 + b_2 v_2 | v_2 > -v^*, \tilde{v}_2 < 0]
\]

On the other hand,

\[
q_2(v, \hat{L}, m_L) = a_2 + b_2 v_2 \\
= \alpha v_2 + (1 - \alpha)E[v_1 | v_1 < v_P] - \frac{1}{2}E[\frac{1}{2}q_1(v_1, L, m_L) | v_1 < v_P]
\]
Clearly, \( b_1 = b_2 = \frac{\alpha}{2\beta} \).

\[
a_1 = \frac{-(1 - \alpha)E[v_2|v_2 < v_P]}{2\beta} - \frac{a_2}{2} + \frac{b_2E[v_2|v_2 < v_P]}{2}
\]

and

\[
a_2 = \frac{(1 - \alpha)E[v_1|v_1 < v_P]}{2\beta} - \frac{a_1}{2} - \frac{b_1E[v_2|v_2 < v_P]}{2}
\]

Let \( \mu_L = E[\tilde{u}|\tilde{u} < u^*] \) and \( \mu_H = E[\tilde{u}|\tilde{u} > u^*] \). Adding both equations and solving

\[
a_1 = -a_2
\]

and

\[
a_1 = \frac{(3\alpha/2 - 1)\mu_L}{2\beta}
\]

Compare with \( a = \frac{(1 - 3\alpha/2)v^*}{\beta} \) when both agents had the same sign. Therefore

\[
a_2 = \frac{(3\alpha/2 - 1)\mu_L}{2\beta}
\]

2. truthful agent’s signal \( v_1 < v^* \), mediator reports \( m_H \)

\[
q_1(v_1, \hat{L}, m_H) = c_1 + d_1v_1
\]

\[
= \frac{\alpha u_1 - (1 - \alpha)E[\tilde{v}_2|\tilde{v}_2 > v_P]}{2\beta} - \frac{1}{2}E[q_2(v_2, H, m_H)|\tilde{v}_2 < -v_P]
\]

\[
= \frac{\alpha u_1 - (1 - \alpha)E[\tilde{v}_2|\tilde{v}_2 > v_P]}{2\beta} - \frac{1}{2}E[\epsilon_2 + f_2v_2|\tilde{v}_2 < -v_P]
\]
Simplifying:

\[ c_1 + d_1 v_1 = \frac{\alpha v_1 - (1 - \alpha)\mu_H}{2\beta} - \frac{e_2}{2} + \frac{f_2 \mu_H}{2} \]

Clearly, \( d_1 = \frac{\alpha}{2\beta} \), and

\[ c_1 = \frac{-(1 - \alpha)\mu_H}{2\beta} - \frac{e_2}{2} + \frac{f_2 \mu_H}{2} \]

Also,

\[ q_2(v_2, L, \hat{L}, m_H) = c_2 + d_2 v_2 \]

\[ = \frac{\alpha v_2 + (1 - \alpha)\mathbb{E}[^{\hat{\bar{v}}_1} | \bar{v}_1 > v_P]}{2\beta} - \mathbb{E}[^{\hat{\bar{v}}_1} q_1(v_1, H, m_H) | \bar{v}_1 > v_P] \]

\[ = \frac{\alpha v_2 + (1 - \alpha)\mu_H}{2\beta} - \frac{1}{2} \mathbb{E}[e_1 + f_1 v_1 | \bar{v}_1 > v_P] \]

which implies

\[ c_2 + d_2 v_2 = \frac{\alpha v_2 + (1 - \alpha)\mathbb{E}[^{\hat{\bar{v}}_2} | \bar{v}_2 > v_P]}{2\beta} - \frac{e_1}{2} - \frac{f_1 \mu_H}{2} \]

Clearly, \( d_2 = \frac{\alpha}{2\beta} \), and

\[ c_2 = \frac{(1 - \alpha)\mu_H}{2\beta} - \frac{e_1}{2} - \frac{f_1 \mu_H}{2} \]

and

\[ c_1 = \frac{-(1 - \alpha)\mu_H}{2\beta} - \frac{e_2}{2} + \frac{f_2 \mu_H}{2} \]

3. truthful agent 2’s signal \( v_2 < -v^* \), mediator reports \( m_H \)
\[ q_2(v_2, \hat{H}, m_H) = \alpha v_2 + (1 - \alpha)E[\tilde{v}_1] - \frac{1}{2}P[\tilde{v}_1 < v_P]E[q_1(v_1, L, m_H)|\tilde{v}_1 < v_P] \]

\[ - \frac{1}{2}P[\tilde{v}_1 > v_P]E[q_1(v_1, H, m_H)|\tilde{v}_1 > v_P] \]

\[ = \alpha v_2 + (1 - \alpha)E[\tilde{v}_1] - \frac{1}{2}P[\tilde{v}_1 < v_P]E_1[c_1 + d_1\tilde{v}_1 < v_P] \]

\[ - \frac{1}{2}P[\tilde{v}_1 > v_P]E_1[c_1 + f_1\tilde{v}_1 > v_P] \]

Simplifying:

\[ e_2 + f_2v_2 = \frac{\alpha v_2 + (1 - \alpha)E[\tilde{v}_1]}{2\beta} - \frac{1}{2}p_L[c_1 + d_1\mu_L] - \frac{1}{2}p_H[c_1 + f_1\mu_H] \]

Clearly, \( f = \frac{\alpha}{2\beta} \). Substituting:

\[ e_2 + f_2v_2 = \frac{\alpha v_2 + (1 - \alpha)\mu}{2\beta} - \frac{1}{2}p_L[c_1 + d_1\mu_L] - \frac{1}{2}p_H[c_1 + f_1\mu_H] \]

Then

\[ e_2 = \frac{(1 - \alpha)\mu}{2\beta} - \frac{1}{2}p_L[c_1 + d_1\mu_L] - \frac{1}{2}p_H[c_1 + f_1\mu_H] \]

Next, suppose truthful agent 1’s signal \( v_1 > v^* \), mediator reports \( m_H \)

\[ q_1(v_1, H, \hat{H}, m_H) = \frac{E_1[\tilde{v}_1 = v_1, m_H]}{2\beta} - \frac{1}{2}E_1[\tilde{q}_2|\tilde{v}_1 = v_1, m_H] \]

\[ = \alpha v_1 + (1 - \alpha)E_1[\tilde{v}_2|\tilde{v}_2 < 0] - \frac{1}{2}E_1[\tilde{q}_2|\tilde{v}_1 = v_1, m_H] \]

\[ = \alpha v_1 + (1 - \alpha)E_1[\tilde{v}_2|\tilde{v}_2 < 0] - \frac{1}{2}P[\tilde{v}_2 > -v_P, \tilde{v}_2 < 0]E_1[q_2(v_2, L, m_H)|\tilde{v}_2 > -v_P, \tilde{v}_2 < 0] \]

\[ - \frac{1}{2}P[\tilde{v}_2 < -v^*]E_1[q_2(v_2, H, m_H)|\tilde{v}_2 < -v^*] \]

Simplifying:

\[ e_1 + f_1v_1 = \frac{\alpha v_1 + (1 - \alpha)E_1[\tilde{v}_2|\tilde{v}_2 < 0]}{2\beta} - \frac{1}{2}p_L[c_2 - d_2\mu_L] - \frac{1}{2}p_H[e_2 - f_2\mu_H] \]
Clearly, $f = \frac{\alpha}{2\beta}$. Substituting and solving, we find that $e_1 = -e_2$ and $c_1 = -c_2$. As before, we define the net benefit of reporting low over high as $\Delta \Pi_{L-H}(v; v_P) = \mathbb{E}[\pi|\hat{v}_1 = L] - \mathbb{E}[\pi|\hat{v}_1 = H]$. The expression is linear with slope given by:

$$\frac{\partial \Delta \Pi_{L-H}(v; v_P)}{\partial v} = -\frac{\alpha (3\alpha - 2) (1 - p_H) \left(4\mu_H^2 - 2\mu_L^2 (1 - p_H) + 2\mu_H \mu_L - \mu_H \mu_L - 3\mu_H \mu_L p_H\right)}{8\beta (3\mu_H + \mu_H p_H - 2\mu + 2\mu_L - 2\mu_L p_H)}$$

It is clear that the denominator is greater than zero since $\mu_H > \mu > \mu_L$. Focusing on the numerator, we want to show

$$4\mu_H^2 + 2p_H \mu_L^2 + 2\mu_H \mu_L - 2\mu_L^2 - \mu_H \mu_L - 3\mu_H \mu_L p_H > 0$$

We have that $\mu_H > \mu > \mu_L$, which implies $2\mu_L - 2\mu_L^2 > 0$ and $4\mu_H^2 - \mu_H \mu_L - 3\mu_H \mu_L p_H > 0$. Therefore, the numerator is greater than zero.

### 6.7 Proof of Proposition 6

In this section, we suppose that each $\tilde{v}_i \sim N(\mu, 1/\rho)$, where $\rho = 1/\sigma^2$ is the precision of the random variable $\tilde{v}_i$. We evaluate the ex-ante profits under (i) no information sharing, full information sharing and collusion.

(i) **No information sharing:**

Each agent’s demand is given by:

$$D_i = \frac{\tilde{v}_1}{4\lambda} + \frac{\mathbb{E}[\tilde{v}_1]}{12\lambda}$$

The conditionally expected profit is given by

$$\mathbb{E}[D_1(\tilde{v}_1 + \tilde{v}_2 - \lambda(D_1 + D_2))|\tilde{v}_1] = \mathbb{E}[\left(\frac{\tilde{v}_1}{4\lambda} + \frac{\mathbb{E}[\tilde{v}_1]}{12\lambda}\right)(\frac{\tilde{v}_1 + \tilde{v}_2}{2} - \lambda(\frac{\tilde{v}_1}{4\lambda} + \frac{\mathbb{E}[\tilde{v}_1]}{12\lambda} + \frac{\tilde{v}_2}{4\lambda} + \frac{\mathbb{E}[\tilde{v}_1]}{12\lambda}))|\tilde{v}_1]$$

$$= \frac{1}{4} \left[ \frac{(\tilde{v}_1)^2}{4\lambda} + \tilde{v}_1 \mathbb{E}[\tilde{v}_1] + \frac{\mathbb{E}[\tilde{v}_1]^2}{36\lambda} \right]$$

76
The ex-ante profit under no sharing is given by:

\[
E[\pi_{ns}] = \frac{1}{4} \left[ E\left[ \frac{(\hat{v}_1)^2}{4\lambda} + \frac{\hat{v}_1 E[\hat{v}_1]}{6\lambda} + \frac{(E[\hat{v}_1])^2}{36\lambda} \right] \right]
\]

\[
= \frac{9\sigma^2 + 16\mu^2}{144\lambda}
\]

(ii) Full information sharing (but trading individually):

Each agent’s demand is given by:

\[
D_i = \frac{\hat{v}}{3\lambda}
\]

The price is given by:

\[
p = \lambda(D_1 + D_2) = \frac{2\hat{v}}{3}
\]

The agent’s profits are given by:

\[
E[D_i(\hat{v} - \lambda(D_1 + D_2))|\hat{v}_1, \hat{v}_2] = E[\left( \frac{\hat{v}}{3\lambda} \right)(\hat{v} - \frac{2\hat{v}}{3})]
\]

\[
= \frac{\sigma^2 + 2\mu^2}{18\lambda}
\]

Since \(Cov(\hat{v}_1, \hat{v}_2) = E[\hat{v}_1 \hat{v}_2] - E[\hat{v}_1]E[\hat{v}_2]\), and \(Cov(\hat{v}_1, \hat{v}_2) = 0\), we have that \(E[\hat{v}_1 \hat{v}_2] = E[\hat{v}_1]E[\hat{v}_2] = (E[\hat{v}_1])^2\). Comparing no sharing with sharing:

\[
E[\pi_{ns} - \pi_{is}] = \frac{1}{4} \left[ E\left[ \frac{(\hat{v}_1)^2}{4\lambda} + \frac{7(E[\hat{v}_1])^2}{36\lambda} \right] - \frac{1}{18\lambda} \left[ E[(\hat{v}_1)^2] + (E[\hat{v}_1])^2 \right] \right]
\]

\[
= \left[ \frac{9\sigma^2 + 16\mu^2}{144\lambda} - \frac{8\sigma^2 + 16\mu^2}{144\lambda} \right]
\]

\[
= \frac{\sigma^2}{144\lambda} > 0
\]

(iii) Collusion (sharing information and trading as a cartel):

The total demand under collusion is given by:
\[
D_1 + D_2 = \frac{E[\tilde{v}_1, \tilde{v}_2]}{2\lambda} \\
= \frac{\tilde{v}}{2\lambda}
\]

Each agent’s ex-ante profit on collusion (which is clearly greater than information sharing profits given by \(E[\pi_c] = \frac{1}{8\lambda}E[(\tilde{v})^2])\) is:

\[
E[\pi_c] = \frac{1}{8\lambda}E[(\tilde{v})^2] \\
= \frac{1}{8\lambda}E[(\tilde{v}_1 + \tilde{v}_2)^2] \\
= \frac{\sigma^2 + 2\mu^2}{16\lambda}
\]

Comparing collusion with no sharing:

\[
E[\pi_{ns} - \pi_c] = \left[\frac{9\sigma^2 + 16\mu^2}{144\lambda} - \frac{\sigma^2 + 2\mu^2}{16\lambda}\right] \\
= -\frac{2\mu^2}{144\lambda} < 0
\]

Collusion would always be preferred to no information sharing if \(|\mu| > 0\)

### 6.8 Proof for Proposition 7

In this proof, we assume that the agent disagree about the value of the asset, and evaluate the cases under which collusion emerges as the optimal response.

(i) **No information sharing**:

The optimal demand for each agent 1 is given by:

\[
D_1 = \frac{\alpha\tilde{v}_1}{2\lambda} + \frac{(2 - 3\alpha)E[v_2]}{6\lambda}
\]
The conditionally expected profit is given by

$$
\mathbb{E}[D_1(\bar{v}_1 + \bar{v}_2 - \lambda(D_1 + D_2))|\bar{v}_1] = \mathbb{E}[(\frac{\alpha}{2\lambda} \bar{v}_1 + \frac{(2 - 3\alpha)\mathbb{E}[\bar{v}_2]}{6\lambda})(\alpha\bar{v}_1 + (1 - \alpha)\bar{v}_2 - \lambda(\frac{\alpha}{2\lambda} \bar{v}_1 + \frac{\alpha}{2\lambda} \bar{v}_2 + \frac{(2 - 3\alpha)\mathbb{E}[\bar{v}_1]}{3\lambda})]|\bar{v}_1]
$$

$$
= \frac{1}{4\lambda}[(\alpha \bar{v}_1 + (\frac{2}{3} - \alpha)\mathbb{E}[\bar{v}_2])^2
$$

The ex-ante profit under no sharing is given by:

$$
\mathbb{E}[\pi_{ns}] = \frac{1}{4\lambda}[(\alpha^2\mathbb{E}[(\bar{v}_1)^2] + (\frac{2}{3} - \alpha)^2\mathbb{E}[\bar{v}_2])^2 + 2\alpha(\frac{2}{3} - \alpha)\mathbb{E}[\bar{v}_2]^2]
$$

$$
= \frac{1}{4\lambda}[(\alpha^2\mathbb{E}[(\bar{v}_1)^2] + (\frac{2}{3} - \alpha)(\frac{2}{3} + \alpha)\mu^2]
$$

(ii) Full information sharing (but trading individually):

The optimal demand for each agent 1 is given by:

$$
D_i = \frac{3\alpha - 1}{3\lambda} \bar{v}_i + \frac{2 - 3\alpha}{3\lambda} \bar{v}_{-i}
$$

I note that $d = b + c = \frac{3\alpha - 1}{3\lambda} + \frac{2 - 3\alpha}{3\lambda} = \frac{1}{3\lambda}$. The total demand is given by:

$$
D = \frac{2}{3\lambda} \bar{v}
$$

The conditionally expected profit for the agent 1 when he receives both signals is:

$$
\mathbb{E}[\pi_{is}|\bar{v}_1, \bar{v}_2] = \mathbb{E}[(\frac{3\alpha - 1}{3\lambda} \bar{v}_1 + \frac{2 - 3\alpha}{3\lambda} \bar{v}_2)(\alpha\bar{v}_1 + (1 - \alpha)\bar{v}_2 - \frac{1}{3}(\bar{v}_1 + \bar{v}_2))|\bar{v}_1]
$$

$$
= \frac{1}{\lambda}[(\alpha - \frac{1}{3})\bar{v}_1 + (\frac{2}{3} - \alpha)\bar{v}_2]^2
$$

The agent’s ex-ante profits are given by:

$$
\mathbb{E}[\pi_{is}] = \mathbb{E}[(\alpha - \frac{1}{3})\bar{v}_1 + (\frac{2}{3} - \alpha)\bar{v}_2]^2
$$

$$
= (\frac{2}{3} - \alpha)\frac{1}{\lambda}[(\frac{10}{21} - \alpha)\mathbb{E}[(\bar{v}_1)^2] + 2(\alpha - \frac{1}{3})\mu^2]
$$

79
Comparing with no sharing

\[ \mathbb{E}[\pi_{is} - \pi_{ns}] = \frac{1}{\lambda}[(2\alpha^2 - 2\alpha + \frac{5}{9})\mathbb{E}[(\tilde{v}_1)^2] + 2(\alpha - \frac{1}{3})(\frac{2}{3} - \alpha)p^2] 
- \frac{1}{\lambda} \frac{\alpha^2}{4} \mathbb{E}[(\tilde{v}_1)^2] + \frac{2}{3} - \alpha)^2p^2 + 2\alpha(\frac{2}{3} - \alpha)p^2] 
= (\alpha - \frac{2}{3}) \frac{1}{\lambda}[(\alpha - \frac{10}{21})\mathbb{E}[(\tilde{v}_1)^2] + (1 - \alpha)p^2] \]

Therefore, when \( \alpha \in [2/3, 1] \), \( \mathbb{E}[\pi_{is} - \pi_{ns}] > 0 \) and agents want to share information ex-ante.

(iii) collusion (sharing information and trading as a cartel):

We assume that there is an overarching firm authority that averages the beliefs of each agent when predicting the security’s payoff:

\[ \max_{D_1 + D_2} \mathbb{E}[(D_1 + D_2)(\hat{v} - \lambda(D_1 + D_2))|\tilde{v}_1, \tilde{v}_2] \]

which means

\[ D_1 + D_2 = \frac{\mathbb{E}[\tilde{v}_1, \tilde{v}_2]}{2\lambda} \]
\[ = \frac{\hat{v}}{2\lambda} \]

The conditionally expected profit by each agent, according is given by

\[ \mathbb{E}[(D_1 + D_2)(\frac{\tilde{v}_1 + \tilde{v}_2}{2} - \lambda(D_1 + D_2))|\tilde{v}_1, \tilde{v}_2] = \frac{1}{2} \left[ \frac{\tilde{v}_1 + \tilde{v}_2}{4\lambda (\alpha \tilde{v}_1 + (1 - \alpha)\tilde{v}_2 - \lambda(\tilde{v}_1 + \tilde{v}_2))} \right] \]

The ex-ante profits are given by:

\[ \mathbb{E}[\pi_{c}] = \frac{1}{8\lambda} \mathbb{E} \left[ (\alpha - \frac{1}{4})(\tilde{v}_1)^2 + \left( \frac{3}{4} - \alpha \right) \tilde{v}_1 \tilde{v}_2 + (\alpha - \frac{1}{4})\tilde{v}_1 \tilde{v}_2 + (\frac{3}{4} - \alpha)(\tilde{v}_2)^2 \right] \]
\[ = \frac{\mathbb{E}[(\tilde{v}_1)^2] + p^2}{16\lambda} \]
When $\alpha < 2/3$, comparing collusion with no sharing,

$$E[\pi_{ns} - \pi_c] = \frac{1}{4\lambda} \left[ \alpha^2 E[(\tilde{v}_1)^2] + \left( \frac{2}{3} + \alpha \right) \left( \frac{2}{3} - \alpha \right) \mu^2 \right] - \frac{E[(\tilde{v}_1)^2] + \mu^2}{16\lambda}$$

Clearly the profits from no sharing are increasing in $\alpha$ in the $\alpha^2 E[(\tilde{v}_1)^2]$ term but decreasing in the $(\frac{2}{3} + \alpha)(\frac{2}{3} - \alpha)\mu^2$ term. When $\alpha = 2/3$, the profits from the $(\frac{2}{3} + \alpha)(\frac{2}{3} - \alpha)\mu^2$ term is zero.

$$E[\pi_{ns} - \pi_c]_{\alpha=2/3} = \frac{1}{9\lambda} E[(\tilde{v}_1)^2] - \frac{E[(\tilde{v}_1)^2] + \mu^2}{16\lambda}$$

$$= \frac{7E[(\tilde{v}_1)^2] - \mu^2}{16\lambda}$$

$$= \frac{7\sigma^2 + 6\mu^2}{16\lambda} > 0$$

Therefore for some $\alpha_1 \in (\frac{1}{2}, \frac{2}{3})$, collusion is preferred to no sharing from $[\frac{1}{2}, \alpha_1)$, no sharing is preferred to collusion from $(\alpha_1, \frac{2}{3})$ and then for $(\frac{2}{3}, 1)$, information sharing yet trading individually is preferred.
Chapter 2: Uncertainty About Uncertainty in Coordination Games

1 Introduction

On April 2, 1903, the Chicago bank of Kaspar & Karel nearly faced collapse as almost half of its depositors queued outside the bank to withdraw their money. The cause of the run was traced to an April Fool’s Day prank the day before alleging that the bank was in serious financial difficulty. In such situations, a depositor’s decision of whether or not to withdraw from the bank will depend on whether the run is driven by other knowledgeable depositors who are informed of the fundamentals or by uninformed individuals purely out of panic. Similarly, an investor who is contemplating an investment in an initial public offering would base his decision on the extent to which his co-investors are informed about the fundamentals of the stock.

This paper studies coordination games of incomplete information where agents are uncertain about not only the state but also about what others know about the state. In the financial market examples described above, a market participant could have one of many possible information sets. The case where one of the agents is either informed or uninformed about a fundamental state has been studied extensively in finance and economics. A more subtle example involves situations where an agent, say a stock analyst, is precisely informed about some states pertaining to an asset’s cash flow (say, firm-specific information) but not about others (say, macroeconomic events).

\footnote{Retrieved from http://hoaxes.org/af\_database/permalink/run\_on\_bank on Mar 31st, 2021}
like inflation or monetary policy). We call this uncertainty about what others know as the second order uncertainty of an agent and study how it could be useful in achieving coordination among agents in regime change games. Regime-change games have been studied widely in the contexts of financial crises, bank runs, currency attacks, coordinated investments (e.g., IPOs), political revolutions, and others. In such games, agents need to coordinate an attack against the status quo to successfully change the regime and benefit from their attack; otherwise, the regime stays and agents who were attacking it have to bear a cost.

Consider a stylized bank run model where two large traders speculate over a bank’s fundamentals. A policymaker wishes to avoid sunspot runs (ala Diamond and Dybvig [1983]) in states where multiple actions are rationalizable.

One option available to the policymaker is to disclose fundamental information to the agents challenging a regime, which affects all higher order beliefs of the agents. This option has been studied variously in Goldstein and Huang [2016], Inostroza and Pavan [2020] and Basak and Zhou [2020]. The other option available to the policymaker, which we study in this paper, is to directly influence the second order uncertainty of the agents. In the April Fool’s Day prank, for example, a policymaker could send private messages to an undecided depositor highlighting the possibility that the withdrawing depositor is uninformed and motivated purely by rumors.

The main modelling innovation in this paper is to represent second order uncertainty as an agent’s belief over all possible partitional information structures of the other agent. As an example, suppose there are two states of the world $g$ (good) and $m$ (medium). The partitions possible for each player $i$ in this state space is $F_i = \{\{g\}, \{m\}\}$ and $C_i = \{\{g, m\}\}$. In this example, second order uncertainty for each player can be represented through the probability $(p^{F_i}, 1 - p^{F_i})$ of each information partition $(F_{-i}, C_{-i})$ of his opponent. The designer chooses the partition of the agents

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2 Representing this situation in terms of partitions, we say that a stock analyst has a fine partition with respect to company specific events and a coarse partition with respect to macroeconomic events. A macro fund manager overseeing a global portfolio could have the opposite information structure - coarse with respect to company specific events but fine with respect to macroeconomic events.

3 Following Inostroza and Pavan [2020] we assume that whenever multiple actions are rationalizable, the agent chooses the action that is least preferred to the information designer.
prior to the realization of the payoff relevant state. Since signals cannot be conditioned upon the state, this formulation of the problem is weaker than the conventional paradigm of designing signal structures as in Bayesian persuasion (Kamenica and Gentzkow 2011).

An agent’s information structure can be represented by an augmented state space, taking into account both the payoff relevant state space and the partitional realization. We call this the second-order knowledge space of the agent. In the example described above, the augmented state space for each player is given by:

\[
P_1 = \{\{F_1F_2g, F_1C_2g\}, \{C_1F_2g, C_1C_2g, C_1C_2m, C_1F_2m\}, \{F_1C_2m, F_1F_2m\}\}
\[
P_2 = \{\{F_1F_2g, C_1F_2g\}, \{F_1C_2g, C_1C_2g, C_1C_2m, F_1C_2m\}, \{C_1F_2m, F_1F_2m\}\}
\]

Therefore, by introducing second order uncertainty, we are able to create overlapping partitions of the payoff relevant state space \{l, m\} for both players. Suppose that a certain action is dominant for a player at some payoff relevant state (say, state \(g\)). Then, this knowledge implies a best response by some other player at overlapping information set. The original player responds to that knowledge, at a yet another overlapping information set, and so on.

More generally, using second order uncertainty, we construct a chain of types as a sequence of overlapping information sets in an augmented state space, starting with a seed event where some action is dominant. We adapt Morris et al. (1995)’s idea of belief potential and p-dominance to identify the rationalizable set in a given chain. The belief potential measures the extent to which information sets overlap. On the other hand, an action pair is p-dominant for a particular type if

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4 The information structure in this example could be represented through two rounds of coin tosses. In the first round, a coin tosses with probability \(p_i\) determine whether each player \(i\) is informed or uninformed. The outcome of the first round coin tosses is told to each player privately. A second round coin toss determines the payoff relevant state \(m\) or \(l\). If a player is determined to be of the Fine type in the first round coin toss, then the payoff relevant state of the game is revealed to him. To ensure that the second order uncertainty is consistent with the first order priors, we will assume that there is no other information available to the agents than that described under the partitional information structure.

5 This interpretation of knowledge in terms of partitions has a rich history in economic theory, including Aumann (1976)’s seminal paper on agreement and the study of differential information economies (see Glycopantis and Yannelis 2006 for a textbook introduction on the topic).
each action is a best response to the other player taking the action with probability \( p \). We show that when the belief potential of a chain of types exceeds the \( p \)-dominance threshold, then that action pair is played everywhere in a chain of types. Our aim is to find the largest such chain of types. We show that the optimal policy has a 'quarantine and infect' mechanism. The states in the policymaker’s favor are allowed to infect other states, while states not in his favor are quarantined.

In the bank run application, we show that the designer allows perfect information when bank fundamentals are weak, but allows agents to be uncertain about the other agent’s knowledge in all other scenarios. Partial disclosure by pooling this second order uncertainty can eliminate sunspot runs, even though pooling first order priors about states (as in Kamenica and Gentzkow’s Bayesian persuasion) is unable to do so. This provides a rationale for accounting conservatism, especially in banking: all probable loan losses are recognized when they are discovered, while investment gains can only be publicly registered when they are fully realized, thus dispersing knowledge of gains privately among agents. Put in a different way, a designer relies on public information in bad states but resorts to private information dissemination in all other scenarios.

To put it in a different way, the designer should rely on public information in bad states but resort to private information dissemination in all other scenarios.

We further apply this idea to analyze coordinated investment problems such as Initial Public Offerings (IPOs) or Initial Coin Offerings (ICOs), where an entrepreneur wishes to successfully raise funds to finance a project. We show that the entrepreneur can implement the risky action ("Invest", akin to "Attack") if he is publicly forthcoming about bad states but disseminates news about all other states in a private setting. This is reminiscent of two striking features of an IPO process. First, all materiel risk factors are mandated to be disclosed upfront in the public IPO prospectus by the regulator. Further, any forward looking statements are banned by the regulator as part of the prospectus. Such disclosure is enforced by the underwriter and the associated team

\(^6\)Beatty and Liao [2011] show that loss recognition delays make banks more pro-cyclical due to reduced recessionary lending. We predict that these delays would also lead to greater deposit withdrawal and induce ‘selling’ of the bank stock.
of lawyers. The underwriter often acts as a certifying intermediary and is known to call off the fund raising under difficult circumstances. Second, IPO marketing “roadshows” involve private meetings with large investors, where the entrepreneur has more freedom in discussing his future plans. Moreover, this information is often distributed in an asymmetric manner: large institutional investors are in a position to gain more information than retail investors. Our result suggests that this (asymmetric) bifurcation of public and private information in different states is crucial to the IPO’s success.

We also consider more complicated information structures where the second order uncertainties of both agents could be correlated. We study a variant of Rubinstein’s email game where both players are perfectly informed about the two states in the game, and essentially play a complete information game in two states. We then introduce a small probability of types of a player (say, player 2) that are uninformed about the states. The informed types of player 2 then initiate an email exchange indicating that they are informed, where each email has some probability of getting lost. This email exchange is in lieu of the state-dependent messages as in Rubinstein. In Proposition 5, we show that the risk dominant action is selected even as the probability of uninformed types approaches zero. Thus, a small perturbation in second order uncertainty dramatically alters the equilibrium set of complete information games played over multiple states. In effect, we show that Rubinstein’s result is robust to second order uncertainty. This result is reminiscent of complete information games studied by Carlsson and Van Damme, where adding a small amount of noise to the signal of each player in each state alters the set of equilibrium actions in some states. Our result differs in the sense that most of the types retain their first order beliefs. That is, most types who are perfectly informed about the state prior to the modification, have no change even in the support of their beliefs despite the addition of a few uninformed types. The “noise” for these types is primarily limited to second order beliefs of the players, where a small probability of uninformed types is sufficient to rule out the Pareto optimal equilibrium. In contrast,

\footnote{This intuition is in line with Chiang et al., who find evidence that institutional investors's bidding pattern in IPOs suggests that they are informed while those of individual investors suggests that they are uninformed.}
the Carlsson and Van Damme [1993] construction alters the support of the beliefs of all types to include the entire range of fundamentals. Depending on the situation analyzed, one or the other could be relevant in selecting a risk-dominant equilibrium. In our setting, only a small probability of types have information that is different from types facing a complete information game, whereas in a Carlsson and Van Damme [1993] setting, every type faces incomplete information (albeit, to a very small extent). In this context, our conditions for selecting a risk dominant equilibrium could be interpreted as weaker than that of Carlsson and Van Damme [1993].

Infection arguments of this kind have been used in a number of papers. Rubinstein [1989]’s electronic mail game and Carlsson and Van Damme [1993] & Morris and Shin [2001] global games has shown us how outcomes of a game with common knowledge of payoffs can be very different from outcomes of the game with a "small" departure from common knowledge. All of these papers consider information structures in coordination games with slightly noisy payoff signals and show that as the signal noise vanishes, agents play a unique strategy profile that survives iterative dominance. We have somewhat similar goals, but pursue a distinct approach. Instead of perturbing payoffs, we rely purely on second order uncertainty to achieve the same goal. As has been discussed in both Morris and Shin [2001] and Weinstein and Yildiz [2007], the rationalizable action profile in global games is sensitive to the distribution of the public signal. Our approach is more relevant for simple settings where a bank manager sets the policy of disclosing information to insiders. Further, adding vanishing noise relies on an infinite iterative reasoning in terms of higher order beliefs, while our approach relies on fewer iterations of rational thinking. Nagel [1995] and Camerer et al. [2004] give experimental evidence that player’s equilibrium actions are largely consistent with first and second-order depth of reasoning and very few players show depths of reasoning greater than the second order. In this sense, our approach can be thought of less taxing in the mind of the agents for iterative calculations about what the other agents know, what the other players know about what the agents know and so on.

While there is an extensive literature on higher order beliefs, studies focusing specifically on
second order uncertainty (or, uncertainty about what others know) have been relatively few. Zamir \cite{Zamir1985} studies repeated 2-person zero-sum game with incomplete information where one player is uncertain about the beliefs of his opponent. He shows that such a game will not have a value in general and the Aumann-Maschler results on incomplete information no longer hold. In a bargaining context, Feinberg and Skrzypacz \cite{Feinberg2005} show that second order uncertainty can lead to a delay in reaching agreement in bargaining games.

In terms of applications, the most closely related papers are Goldstein and Huang \cite{Goldstein2016}, Inostroza and Pavan \cite{Inostroza2020} and Basak and Zhou \cite{Basak2020}. In Goldstein and Huang \cite{Goldstein2016} propose a policy where the policymaker can commit to abandon the regime automatically if the bank fundamentals are below a certain threshold. Similarly, Inostroza and Pavan \cite{Inostroza2020} show that under certain conditions, the optimal policy is a Pass/Fail threshold. Our paper shares some commonalities with these papers in that agents are always made publicly aware of a fundamental threshold. However, in our setting, private information is used in good/medium states in a nested manner, after agents have been publicly informed of whether the states are bad or not. All of these papers rely on agents being endowed with a noisy signal, while in our setting, whenever agents are informed of a state, they know it precisely.

The fact that equilibrium outcomes are sensitive to information structures is also emphasized in, among others, Kajii and Morris \cite{Kajii1997} and Weinstein and Yildiz \cite{Weinstein2007}. In Kajii and Morris \cite{Kajii1997}, the main object of study is robustness for complete information games whereas this paper analyzes robustness for incomplete information games played over several states. While Weinstein and Yildiz \cite{Weinstein2007} study the role of interim beliefs in incomplete information games, this paper investigates sensitivity to second order uncertainty from an \textit{ex-ante} perspective.

We introduce our methodology for analyzing second order uncertainty and state the information design problem in Section 2. We apply our findings to applications of coordination games in Section 3, 4 and 5. Section 6 concludes.
2 Model

In this section, we set forth the preliminaries of the model and the information design problem, before solving it for specific applications in the following sections.

**Payoff environment:** \( \Omega \) is a set of finite payoff-relevant states with a generic element given by \( \omega \). There are two players \( I = 1, 2 \), where each player \( i \in I \) has a finite set of actions \( A_i \). The payoff function is given by \( u_i : A \times \Omega \to \mathbb{R} \), where \( A = A_1 \times A_2 \) is the set of action profiles. The players share a common prior over the payoff relevant states given by \( \mu \in \Delta \Omega \), which we denote as the “first order uncertainty”. A basic game is given by \( G = \{ I, \{ A_i \}, \{ u_i \}, \Omega, \mu \} \).

**Knowledge space:** \( \mathcal{P} \) is a set of all possible partitions of \( \Omega \). A generic element of \( \mathcal{P} \) is denoted by \( P = (P_k) \) where \( P_k \) is a subset of \( \Omega \) and \( \bigcup_{k=1}^{K} P_k = \Omega \) and \( P_j \cap P_k = \emptyset \) for \( k \neq j \). A first order knowledge space for a player is a pair \((\Omega, P)\). However, the players are uncertain about the partition of other agents. We call this uncertainty about what others know as the “second-order uncertainty” of the player and represent it in terms of a joint probability distribution \( \phi \in \Delta(\mathcal{P} \times \mathcal{P}) \) over all possible partitions of the payoff relevant state space. We note that the partitional probability measure is independent of the payoff relevant state \( \omega \). This formulation can also be explained in terms of the following extensive form game: Nature moves first to randomly choose the information partition of each player. Nature randomizes again to choose the payoff relevant state and informs the players about the cell in the partition to which the payoff relevant state belongs. At the time of play, each agent’s information comprises one particular partition of the state space, but they are uncertain about the information set of the other agent.

An incomplete information game is defined by \( \mathcal{G} = \{ I, \{ A_i \}, \{ u_i \}, \Omega, \phi, \mu \} \). The structure of this game is common knowledge amongst players. The expanded states of the world, taking into account both the underlying state space and the partitional realization is given by \( W \subset \mathcal{P}^{2} \times \Omega \).

The Harsanyi type of each player is defined by partition elements \( T_i = \pi_i(w) \) where \( \pi_i(w) \) is the

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8 When the partitions of two agents are correlated, their partitional knowledge reveals some information about the knowledge of the other agent with respect to the payoff relevant state.
partition of $W$ to which the state of the world $w$ belongs. The set of such types of each player is given by $\Pi_i = \{\pi_i(w) | w \in W\}$. In its expanded form, a generic type is given by:

$$
\pi_i(P_i, P_{-i}, \omega) = \{(P_i, P'_{-i}, \omega') : P'_{-i} \in P_i, \omega' \in \varphi_i(\omega)\}
$$

The set of types of all players is denoted $\Pi = \Pi_1 \cup \Pi_2$. The probability measure on the states of the world $W$ is described by $\xi = \varphi \times \mu$. A second-order knowledge space for a nonempty set of players $I$, is a pair $(W, \Pi)$. From hereon, we assume that both the information structures and payoffs are symmetric for players 1 & 2.

**Using second order uncertainty to design an information system:** There is an information designer who can influence the second order uncertainty $\varphi$ to implement his preferred action in as many payoff relevant states as possible, through the choice of an information policy. Prior to the realization of the payoff relevant state, the information designer chooses the distribution $\varphi_i(P)$ over all possible partitions of the payoff relevant state-space for each agent $i$. For example, the designer could choose independently that the agents are completely informed (and hence have the finest partition of the state space) with a certain probability $p$ and are completely uninformed (and hence have the coarsest partition of the state space) with probability $(1 - p)$. However, he cannot condition this choice on the occurrence of a particular state of nature. An alternate scheme of representing the signal structure (and the joint probabilities) for a two state example is in terms of private signals received by each player:

<table>
<thead>
<tr>
<th>States\Signals</th>
<th>$s_1^F$</th>
<th>$s_2^F$</th>
<th>$s_3^C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>$p_iq$</td>
<td>0</td>
<td>$(1 - p_i)q$</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>0</td>
<td>$p_i(1 - q)$</td>
<td>$(1 - p_i)(1 - q)$</td>
</tr>
</tbody>
</table>

In this example, the unconditional probability of state $\omega_1$ is given by $q$. $s_3^C$ is a completely uninformative signal whereas $s_1^F$ and $s_2^F$ are completely informative signals. Each private signal
corresponds to a partition in an Aumann’s state of the world representation. In the alternate representation, the signals are received on a one shot basis instead of the two stage “coin-toss” representation described earlier. This formulation would be equivalent to a standard incomplete information game with private signals, except that the designer focuses solely on choosing the probability of “Fine” partitional type, \( p_i \).

The fact that the designer chooses the partition of the agents prior to the realization of the payoff relevant state has important consequences for the choice set of signals. Since signals cannot be conditioned upon the state, this formulation of the problem is also a departure from the conventional paradigm of designing signal structures \((S, \pi)\) and a distribution \(\pi(\cdot|\omega) \in \Delta(S)\) for each \(\theta \in \Omega\), as in Bayesian persuasion (Kamenica and Gentzkow [2011]). In our setting, the partition carries no information about the state, and this what the designer can choose. For example, in a 2 state model, the probability that an agent is of the Fine type is independent of the state. Whatever prior the agent puts on a particular state, he is likely to be informed about both states when he is revealed to be of the Fine type. This makes our information design problem weaker than that of the Bayesian persuasion literature.

In summary, this policy differs from the conventional paradigm of choosing signal structures as in [Kamenica and Gentzkow 2011] in two ways: (i) The designer cannot condition the signals on the state of nature and (ii) the first order priors are fixed. The economic applications of this signal structure is relevant for analyzing problems where a policymaker chooses the long term information paradigm of an institution. In the banking application presented in Section 3, the

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9The difference in the sequence of moves by Nature is subtle. The optimal policy outcome when the designer is allowed to commit to a disclosure after the state realization is different from that when he decides upon it upfront. Bergemann and Morris [2019] state “Whether or not the information designer will observe the payoff relevant state is irrelevant, what is important is whether she can condition the signals she sends on the realization of the state and the players private signals.” To see why, we apply the sequence of moves described in this paper to the judge-prosecutor example in Bayesian persuasion games. Since there are only 2 states, the prosecutor’s choice is either to allow the judge access to information or not. If agents are guilty with probability 0.3, then he should fully reveal information under this assumption, and thus leads to an uninteresting conclusion. The interesting “partial disclosure” solution emerges in dominance solvable games where sender-preferred dominant states could be allowed to infect other states (keeping the other receiver–preferred dominant states quarantined). In Section 6, we solve for optimal second order uncertainty when the designer can condition on the realization of a state.
board of a bank could choose the accounting disclosure system of a bank: whether or not a system should be put in place that is perfectly informative about bad states but is coarse with respect to good states. Alternatively, a bank could decide how many agents have access to inside information at the very outset, and this fact is disclosed to all depositors of the bank. In these settings, the designer may not have the freedom to condition the signals on the realization of a particular state.\footnote{Notwithstanding our description of the somewhat restricted, yet economically meaningful choice set of signals, we relax this restriction in Section 6 and Section 7, and allow the designer to condition information partitions on the state of nature. We then compare the impact of these restrictions on the optimal signal structure.}

Designer’s payoffs: The payoffs for the information designer $U(\cdot)$ is given by $+1$ in states where his preferred action is uniquely implemented for each player and $0$ in states where he is unable to uniquely implement his preferred action for a particular player.

Most Aggressive Rationalizable Property: We take a robust approach to information design as in Inostroza and Pavan\cite{2020}, We assume that whenever multiple actions are rationalizable, the agent chooses the action that is least preferred to the information designer. Finally, we assume that the information designer’s preferred action is dominant for the players in some state.

Assumption 1: For agent $i$, there exists a designer-preferred action $a^*$ and $\omega^* \in \Omega$ such that

$$u_i(a^*, a_{-i}; \omega^*) > u_i(a_i, a_{-i}; \omega^*) \quad (\forall a_i \neq a^*_i, \forall a_{-i})$$

The following section sets forth the technical preliminaries for the general approach we take for solving information design problems with second order uncertainty. The information designer’s problem can be solved through multiple rounds of iterative reasoning, although the calculations get tedious for a large number of states. To maintain simplicity and tractability for a large number of states, we model Nature’s move after the designer’s information choices and the subsequent moves by the agents as a one-shot game. However, our framework is especially relevant for situations when Nature moves repeatedly in choosing different states, while the partitional structure of the agent’s information (for eg. Coarse vs. Fine) remains fixed.
of states, we use an extension of Morris et al. [1995]'s idea of belief potential as a general approach towards uniquely rationalizing a designer’s preferred action. Readers who are more interested in specific applications discussed in the introduction can skip directly to the Section 3.

2.1 Belief potential of an information system

We follow the broad approach in Morris et al. [1995] of separating the properties of the information system denoted by $I = \{I, \Omega, \phi, \mu\}$ from the payoffs $u(a, \omega)$. Starting with the information system, we evaluate the belief potential of the system, which measures how easily a belief in a particular event can spread to different states through overlapping partitions. Since an action $a^*$ is dominant for players in payoff relevant state $\omega^*$, we denote the seed event as $E_i = \{(F_i, P_j, \omega^*)\}$ where $F_i \in \mathcal{P}$ is any partitions which includes the singleton set $\omega^*$ and $P_j \in \mathcal{P}$ is any possible partition with $\phi_j(P_j) > 0$. That is, $E_i$ is the event where player $i$ knows that the state is $\omega^*$. However, he is uncertain about agent $j$’s knowledge of the event, which is indicated by all possible partitional realizations for agent $j$ according to $\phi_j$. We wish to evaluate the susceptibility of the information system towards infection by event $E_i$. To do so, we first adopt the concept of chains introduced by Hellman and Samet [2012], by defining chains starting from some $w^* \in E_i$.

**Definition 1.** A chain of length $n$ for a partition profile $\Pi_i$, from one state to another, is defined by induction on $n$. A chain of length $n + 1$, from a state $w^*_0$ to $w^*_n$, is a sequence $c \rightarrow w_n$, where $c$ is a chain of length $n$ from $w^*_0$ to $w^*_n$ and $w_n \in \Pi_i(w^*_n)$. Thus, a chain of positive length $n$ is a sequence $c = w^*_0 \rightarrow^{i_0} w_1 \rightarrow^{i_1} ... \rightarrow^{i_{n-2}} w_{n-1} \rightarrow^{i_{n-1}} w_n$, such that $w_{s+1} \in \Pi_i(w_s)$ for $s = 0, ... n - 1$. Further, a chain $c$ is alternating if no two consecutive states $w_s$ and $w_{s+1}$ in $c$ are the same, and no two consecutive agents $i_s$ and $i_{s+1}$ in $c$ are the same.

We restrict our analysis to alternating chains henceforth, often suppressing the prefix ‘alternating’ when referring to chains. We wish to relate a chain of states to the spread of infection
in terms of beliefs of a particular event at each state. We first use Monderer and Samet’s 1989’s definition of the belief operator applied to an event $E$:

**Definition 2.** Player $i$’s $r$–belief operator over $E$ is the set of states where player $i$ associates a probability of at least $r$ to an event $E$

$$B^r_j(E) = \{ w \in W : \mathbb{P}(E|\Pi_j(w)) \geq r \}$$

Next, we map a chain of states $c$ to a sequence of beliefs $r(c)$ such that at each $w_{s+1}$, we find the largest $r$ for which the agent believes in the partition containing $w_s$.

**Definition 3.** Given any alternating chain $c$, a sequence of beliefs is $r(c) = (r_1^c, r_2^c, ... r_n^c)$ where an element $r_k$ is the largest probability such that

$$B^r_{i_k}(\Pi_{i_{k-1}}(w_{k-1})) = \{ w \in W : \mathbb{P}(\Pi_{i_{k-1}}(w_{k-1})|\Pi_{i_k}(w_k)) \geq r_k \}$$

The sequence of beliefs measures the intensity of spread of a particular event to some other state $w$ outside the event. The belief potential for a state is defined as the maximum probability with which an entire chain $c$ is covered through sequential application of the belief operator.

**Definition 4.** The belief potential of a chain $r(c)$ is given by:

$$r_{min}(c) = \min(r_1^c, r_2^c, ... r_n^c)$$
The belief potential of an event $E$ measures the potential of the event to “infect” all states in a chain $c$. \[11\]

**Definition 5.** Action pair $(a_1, a_2)$ is $r-$dominant in the state $w$ game if for every probability distribution $\lambda$ on $A_j$ such that $\lambda(a_j) \geq r$ and all $b_i \in A_i$

$$\sum_{b_j \in A_j} \lambda(b_j)u_i(a_i, b_j; w)] > \sum_{b_j \in A_j} \lambda(b_j)u_i(a_i, b_j; w)]$$

If action $(a_1, a_2)$ is $r-$dominant in $w$, then $a_i$ is the unique best response as long as he believes that the other player will play $a_j$ with probability at least $r$. Although $w$ refers to the state in the expanded state space, only the payoff relevant component of $w$ (which is indeed $\omega$) is relevant to the determining the $r-$dominance of an action pair.

**Theorem 1.** Suppose that (1) a chain $c$ has belief potential $r$ (2) $(a^*, a^*)$ is $r$-dominant at every state in $c$; Then playing $(a^*, a^*)$ is the unique rationalizable strategy in $c$

**Proof.** Let $R_i$ be the set of rationalizable strategies of $i$ in the chain $c$. Let $c_i^*$ be the set of states in the chain $c$ where agent $i$ plays action $a_i$ in every rationalizable strategy. Then,

$$c_i^* = \{w \in c | s_i(w) = a^*, \text{for all } s_i \in R_i\}$$

We have that $w_0^* \in c_i^*$. Let $E = \Pi_i(w_0^*)$. By (2), $B_j^r(E) \cap c \subset c_j^*$. Further, by induction,

$$[B_i^r B_j^r ...]_{n\text{ times}} \cap c \subset c_i^*$$

\[11\]Here, we deviate from the notion of the minimum belief potential established by Morris et al. [1995] as “the largest $p$ such that the infection argument works for every nontrivial event measurable with respect to some individual’s partition”
By (1) we have that $c \subseteq [B_i^r B_j^r \ldots]_{\text{times}} E$, which implies $c \subseteq [B_i^r B_j^r \ldots]_{\text{times}} E \subseteq c^*_i$. Therefore $c = c^*_i$ and $a^*$ is uniquely rationalizable in $c$

Given any chain of states $c$, we associate with $c$ a chain of types given by $\tau(c) = \tau_{i-1}(w_0^*) \rightarrow^{i_0} \tau_{i_0}(w_1) \rightarrow^{i_2} \ldots \rightarrow^{i_{n-2}} \tau_{i_{n-1}}(w_{n-1}) \rightarrow^{i_n} \tau_{i_n}(w_n)$ where

$$\tau_{i_k}(w_k) = \{\Pi_{i_k}(w) : w \in B_{i_k}^{r_k}(w_{k-1}) \setminus \tau_{i_k}(w_{k-1})\} \text{ for } k \geq 0$$

The seed of the infection $w_0^*$ has types $\tau_{i-1}(w_0^*) = E$. We note here that for any given $r_k^c$ each $\tau_{i_k}(w_k)$ could comprise types in addition to the type $\Pi_{i_k}(w_k)$ - these are the types that get covered by the belief operator $B_{i_k}^{r_k}(w_{k-1})$ for “free”. We now return to our assumption of two player symmetric games. For each alternating chain of states $c$, there is a mirroring chain of $c'$ with $w_0^* = (P_i, P_j, \omega^*)$ replaced by $w_0'^* = (P_j, P_i, \omega^*)$ and each $i_k$ replaced by $i_{k+1}$.

Consider any chain of types $\tau(c)$. Suppose an action $(a_1, a_2)$ is $r$–dominant in each $w \in \tau(c)$. Then it must be that $(a_1, a_2)$ is $r$–dominant for each type $\tau_{i_k}(w_k)$. Consider a generic element $T_{i_k} \in \tau_{i_k}(w_k)$. Let $G(T_{i_k})$ be the game in partition $T_{i_k}$ with payoffs corresponding to the expected payoffs of the state games in $T_{i_k}$ under the common prior $\mu$. Action pair $(a_1, a_2)$ is $r$–dominant in each game $G(T_{i_k})$ if for every probability distribution $\lambda$ on $A_j$ such that $\lambda(a_j) \geq r$ and all $b_i \in A_i$, we have that

$$\sum_{b_j \in A_j} \lambda(b_j) E[u_i(a_i, b_j)|T_{i_k}] > \sum_{b_j \in A_j} \lambda(b_j) E[u_i(a_i, b_j)|T_{i_k}]$$

Since each agent $i$’s information is measurable with respect to $T_{i_k}$, we use the equivalent formulation above to evaluate the $r$–dominance threshold for action profile $(a^*, a^*)$ in $U_n$.

**Definition 6.** Action pair $(a_1, a_2)$ is $t$–dominant in $U_n$ if for every probability distribution $\lambda$ on $A_j$ such that $\lambda(a_j) \geq r(T_{i_k})$ and all $b_i \in A_i$

$$\sum_{b_j \in A_j} \lambda(b_j) E[u_i(a_i, b_j)|T_{i_k}] > \sum_{b_j \in A_j} \lambda(b_j) E[u_i(a_i, b_j)|T_{i_k}]$$

96
for each type $T_{ik} \in \tau_{ik}(w_k)$

The next theorem establishes the interaction between belief potential and $r-$dominance for a chain of types

**Theorem 2.** Suppose that (1) $\tau(c)$ has belief potential $r(c)$ and (2) $(a^*, a^*)$ is $r$-dominant for each type in $\tau(c)$; Then playing $(a_1, a_2)$ is the unique rationalizable strategy for all types in $\tau(c)$

**Proof.** First, we define $U_c$ as a union of types in a given chain of types

$$U_c = \{ \cup \tau : \tau \in \tau(c) \}.$$

Let $R_i$ be the set of rationalizable strategies of $i$ in $U_n$.

Let $U_n^i$ be the set of types in the expanded state space where $i$ plays action $a_i$ in every rationalizable strategy. Then,

$$U_n^i = \{ T_i \in U_c | s_i(T_i) = a^*, \text{for all } s_i \in R_i \}$$

Clearly $E \subset U_n^i$ for some $E \in \mathcal{E}$. Consider $B_j^* U_n^i = \{ T_j \in U_c | \mathbb{P}(U_n^i | T_j) \geq \sigma \}$. By (2) $B_j^* U_n^i \cap U_c \subset U_j^i$.

By a similar logic, $B_i^* U_c^j \subset U_c^j$. Further, by induction, $[B_i^* B_j^* ...]_{n \times \times \times} E \cap U_c \subset U_c^i$.

By (1) we have that $U_c \subset [B_i^* B_j^* ...]_{n \times \times \times} E$. Therefore $U_c \subset U_c^i$ and $(a^*, a^*)$ is uniquely rationalizable for all types in $U_c$.

Using belief potential of an information system to define the information design problem: Consider an alternating chain of states $c$, and associated chain of types $\tau(c)$. We first note that for a given chain of types, there is:
(i) a belief potential vector given by \( r(T, \phi) \) for each \( T \in \tau(c) \)

(ii) a dominance threshold vector given by \( t(T) \) for each \( T \in \tau(c) \). This threshold will be independent of \( \phi \)

Whenever the former is greater than the latter, then the action \((a^*, a^*)\) is rationalized through the entire chain of types \( \tau(c) \). We wish to find the largest chain, in terms of probability measure, for which this is true.

**Problem 1.** The designer’s problem is then to choose \( \phi \) to

\[
\max_{\phi, c} \sum_i \sum_{w \in \tau(c)} \xi_i(w).
\]

subject to:

\[
r(T, \phi) \geq t(T) \text{ for each } T \in \tau(c)
\]

The designer implements partial disclosure rules (or, private information) in states \( w \in W \) that belong to the largest possible chain (in terms of probability weight) and resorts to full disclosure in all other states that are not in his favor.

In the next sections, we make explicit the payoff functions and hence the \( t \)–dominance threshold for all possible types in an incomplete information game. We then solve for the optimal information policy to rationalize the designer’s preferred action. Specifically, we consider three applications. First, we analyze a stylized bank run where the designer wishes to induce the safe action of No Attack as often as possible. Second, we analyze a co-ordinated investment game (e.g., Initial Public Offering/Initial Coin Offering etc) where the policymaker wishes that the agent coordinate on the risky action of investing as often as possible. Finally, we study an email game and show that even in complete information games, Rubinstein \[1989\]’s results are robust to small perturbations in
second order uncertainty.

3 Application: Stylized bank runs

In this section, we consider a stylized bank run model where two large traders with inside connections speculate over a bank’s fundamentals. They choose between (1) rolling over deposits (“No Attack”) or (2) withdrawing their deposits and shorting the illiquid but possibly solvent bank’s stock (“Attack”). The payoffs from a successful “Attack” are risky: they depend both on uncertain fundamentals and coordination with the other agent. The No Attack action is safe but yields zero profits. Unlike the Diamond-Dybig debt run model, creditors in our model have no preference shocks and the allocation of asset is absent. We focus only on the information aspect of a coordination game, which is similar to that of games studied in the global games literature (Morris and Shin [2001], Carlsson and Van Damme [1993]).

When multiple actions are rationalizable in a game, agents are uncertain about the action choice of the other agent, and choose an action only when the other agent has a sufficiently high chance of choosing the same action. The other agent’s action choice in turn depends on his knowledge about the fundamentals of the bank. A bank manager (“designer”) wishes to safeguard the bank by implementing the “No Attack” action in as many states as possible. The manager’s problem is essentially to avoid sunspot runs (ala Diamond and Dybvig [1983]) in the medium state, which is the state that deserves central attention. We assume that this state occurs with large enough probability so that no disclosure cannot be an optimal solution. It allows multiple actions to remain rationalizable in the “average game”. He preserves the status quo by designing information policies that introduce uncertainty in the minds of traders about what the other trader knows about the fundamentals of the bank.
<table>
<thead>
<tr>
<th></th>
<th>Attack</th>
<th>No Attack</th>
<th>Attack</th>
<th>No Attack</th>
</tr>
</thead>
<tbody>
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<td>$\theta_l - 1, 0$</td>
<td>Attack</td>
<td>$\theta_m, \theta$</td>
</tr>
<tr>
<td>No Attack</td>
<td>$0, \theta_l - 1$</td>
<td>$0, 0$</td>
<td>No Attack</td>
<td>$0, \theta_m - 1$</td>
</tr>
</tbody>
</table>

Table 1: Payoff matrix as in Morris and Shin [2001]

3.1 One sided dominance

Consider the payoff matrix in Table 3 derived from Morris and Shin [2001]. There are two payoff relevant states $\Omega = \{l, m\}$ ('low' and 'medium' respectively) and the probability of the medium state is $q$. There are two actions \{Attack, No Attack\} and the (symmetric) payoff when both players Attack is given by $\theta \in \{\theta_l, \theta_m\}$. If only one player attacks, his payoff is $\theta - 1$. The payoff for not attacking is normalized to zero, regardless of the other player’s action. We assume that $\theta_l < 0$ so that the action No Attack is dominant in state $\{l\}$ and $\theta_m \in (0, 1)$ so that both Attack and No Attack are rationalizable in state $\{m\}$. I also assume that $\mathbb{E}[\theta] > 0$ so that both actions Attack and No Attack are rationalizable under no information (that is, $\{l, m\}$).

We represent uncertainty about what others know through a probability distribution over the partitions possible for this state space: $P_F = \{\{l\}, \{m\}\}$ and $P_C = \{\{l, m\}\}$. We denote the probability of the partition $P_F$ for each player as $p_i$ (independent of the state and the other player). This information structure could be represented in terms of two rounds of coin tosses. In the first round, independent coin tosses with probability $p_i$ determine whether each player $i$ is informed (Fine partition - 'F') or uninformed (Coarse partition - 'C'). The outcome of the first round coin tosses are told to each player privately. In the second round, a coin toss with probability $q$ determines the state $m$ or $l$. If a player is determined to be of the Fine type under the first round coin toss, then the state of the game is revealed to him.

The second order knowledge in terms of the expanded states of the world for each player is represented as “Partitional state of player 1 - Partitional state of player 2 - Payoff relevant state”:

$\mathcal{P}_1 = \{\{FFl, FCl\}, \{CFl, CCl, CCm, CFm\}, \{FCm, FFm\}\}$
The partition state for player 2 in terms of states of the world are:
\[ \mathcal{P}_2 = \{\{FFl, CFl\}, \{FCl, CCl, CCm, FCm\}, \{CFm, FFm\}\} \]

The set of types of agent \( i \) in this game is given by \( \Pi_i = \{F_{il}, F_{im}, C\} \), where, for example, \( F_{il} \) is the type of agent 1 who knows the state of nature, but does not know whether the other agent has a Fine partition or a Coarse partition (that is, \( F_{il} = \{FFl, FCl\} \)). Formulating the problem in terms of partitions helps us relate our information system to those developed in the “infection cascade” literature (Eg. Rubinstein [1989], Carlsson and Van Damme [1993], Morris et al. [1995], etc). As described earlier, we assume that a policy maker wishes the agents to play “No Attack” as often as possible and adopts a robust approach to the design of the optimal information structure. That is, he assumes that the agents will play the worst-case scenario action “Attack” whenever multiple actions are rationalizable. Therefore, the principal’s aim is to uniquely rationalize No Attack for the agents.

**Proposition 1.** No attack emerges as a uniquely rationalizable action under the following probability threshold for each type:

- **Type \( F_{il} \):** No Attack is always uniquely rationalizable
- **Type \( C_i \):** No Attack is the uniquely rationalizable strategy for both players if \( \max(p_1, p_2) > \frac{\mathbb{E}[\theta]}{(1-q)} \)
- **Type \( F_{im} \):** No Attack is uniquely rationalizable for both players if \( \min(p_1, p_2) < 1 - \theta_m \) and \( \max(p_1, p_2) > \frac{(1-q)p_l + q\theta_m}{(1-q)} \)

The Appendix presents a detailed proof for Proposition 1 using the methodology developed in Section 2. A sketch of the proof taking the Coarse types as an example is as follows. The Coarse type evaluates the payoff of an average game where the payoffs from Attacking are \( \mathbb{E}[\theta] \) and the payoff on miscoordination is \( \mathbb{E}[\theta] - 1 \). No Attack yields zero payoffs irrespective of the other types choices. The Coarse type prefers No Attack over Attack only when he believes that the other agent
Figure 1: Rationalizable strategies under the prior $q = 0.5$ and payoffs $\theta_m = 0.4$ and $\theta_l = -0.1$ for the payoff matrix described in Section 3. The x-axis and the y-axis represent the probability $p_i$ that each agent is informed (i.e. of the Fine type). The two charts show rationalizable actions choices for the uninformed (Coarse) and informed (Fine) type respectively. The dark colored region represents combinations of $(p_1, p_2)$ where both actions, Attack and No Attack, are rationalizable while the light colored region represents areas where No Attack is uniquely rationalizable. The 45 degree line in the center is the line of symmetric information where $p_1 = p_2$. That is, the ex-ante probability of each agent being informed is the same. Depending on the choices, of $\theta_m$, $\theta_l$, and $q$, it may or may not be possible to rationalize No Attack uniquely with symmetric information policies. We provide an example in the next section (in the context of IPOs) where one investor receives more information than the other.
will play No Attack with a large enough probability:

$$0 > (1 - \mathbb{P}[\text{No Attack}])\mathbb{E}[\theta] + \mathbb{P}[\text{No Attack}](\mathbb{E}[\theta] - 1)$$

This implies that as long as the probability of the opponent playing No Attack: $\mathbb{P}[\text{No Attack}] > \mathbb{E}[\theta]$, the Coarse type will play No Attack. The opponent type will play No Attack only when he is of the Fine type and the state is $l$. This occurs with probability $p_{-i}(1 - q)$. Therefore, when $p_{-i}(1 - q) > \mathbb{E}[\theta]$, the Coarse type of agent $i$ plays No Attack. A similar reasoning holds for the Fine type of agents.

Thus, an information designer, who wishes to uniquely implement a risk dominant action for all types of a player, must ensure that the informed types are not too high or too low. This condition could prove useful for situations where the risk dominant action is a preferred action for the designer, or the designer could modify payoffs of the games to ensure that an action is dominant in some state with high probability. Moreover, it may not be possible for a designer to implement a symmetric policy with respect to $\phi$ for both players. It may be necessary make one player more informed than the other under different payoff parameters. We delve into this issue further in our next application of coordinated investment.

Restricting ourselves to symmetric policies, we now write the payoffs of the information designer as a function of $p_i = p$ under the Most Aggressive Rationalizable Property. The designer’s payoffs are given by:

$$U = \begin{cases} 
0 & \text{if } p \leq \frac{\mathbb{E}[\theta]}{1-q} \\
2 & \text{if } \frac{\mathbb{E}[\theta]}{1-q} < p < 1 - \theta_m \\
2(1-p) + 2p(1-q) & \text{if } p > 1 - \theta_m 
\end{cases}$$

103
3.2 Two-sided dominance

We consider implications of uncertain information structure when there are three payoff relevant states $\Omega = \{l, m, h\}$. Each state has probability $q_\omega$ for $\omega \in \Omega$. The payoffs are symmetric and there are two actions \{Attack, No Attack\} with \textbf{No Attack} being dominant in state $l$, \textbf{Attack} being dominant in state $h$ and both actions \textbf{Attack} and \textbf{No Attack} being rationalizable in state $m$.

To develop intuition for our results, we will assume the following symmetry assumptions (i) values for the fundamentals: $\theta_l = \theta_m - s$ and $\theta_h = \theta_m + s$, for some $\theta_m \in (0, r)$ and (ii) $q_l = q_h$. We also need that $s > \theta_m$ so that \textbf{No Attack} is dominant in state $s$ and $\theta_m + s - r > 0$ so that \textbf{Attack} is dominant in state $h$. These assumptions ensure that the “average game” in states $\{l, h\}$ have both \textbf{Attack} and \textbf{No Attack} as rationalizable actions.

As in the previous section, we assume that the policy maker adopts a robust approach to the design of the optimal information structure, that is she evaluates the “worst-case” scenario whenever multiple actions are rationalizable. In our game, whenever both \textbf{Attack} and \textbf{No Attack} are rationalizable, the principal’s aim is to uniquely rationalize \textbf{No Attack} for the agents.

The partitional states for each player represented as “Partitional state of player 1 - Partitional state of player 2 - Payoff relevant state” are:

\[
P_1 = \{\{FFl, FCl, FBl, FGl\}, \{CFl, CCl, CBl, CGl, ..\}, \{BFl, BCl, BBl, BGl\}\}.
\]
The partition state for player 2 in terms of states of the world are:

\[ P_2 = \{\{FFl, CFl, BFl, GFl\}, \{FCl, CCl, BCl, GCl, CCm, FCm, ..\}, \{CFm, FFm, \}} \]

As usual, we solve for symmetric information structures where \( \phi_1 = \phi_2 = \phi \).

**Proposition 2.** Let \( q = \frac{q_m}{q_m + q_l} \). Then, the optimal information structure under second order uncertainty consists of

(i) \( \phi(\{\{l\}, \{m\}, \{h\}\}) \in (0,1) \) and \( \phi(\{\{l,m\}, \{h\}\}) \in (0,1) \) when \( q < \frac{(1-\theta_m) - \theta_l}{(1-\theta_l)} \). All other partitions have zero probability mass. This means that agents are always completely informed of the high state (and know that the other agent is also informed) but are unsure about the other agent’s information about the low and medium states.

(ii) \( \phi(\{\{l\}, ..\}) = 1 \) when \( q > \frac{(1-\theta_m) - \theta_l}{(1-\theta_l)} \). The designer is indifferent to information partitions \( \{\{m, h\}\} \) and \( \{\{m\}, \{h\}\} \). The agents are completely informed about the low state.

The proof relies on the fact that there in no pooling combination of the high state with any of the other states results in selection of the No Attack action. Therefore, we show that the optimal policy has a 'quarantine and infect' mechanism. The states in the policymaker’s favor are allowed to infect other states, while states which are not in his favor are quarantined. The policy maker allows agents to be uncertain about the other agent’s knowledge in states when the fundamentals are good but allows perfect information in bad states. By being nebulous about good states, the policy maker leaves open the possibility that some agent could privately be aware of the state and play a dominant action that is preferred by the designer. Knowing this, the ignorant type chooses the policymaker’s preferred action, whenever he expects his opponent to be of the knowledgeable type with large enough probability. This has a cascading impact on the knowledgeable agent in a
state where multiple actions are rationalizable. Even though this agent knows that the preferred action is not dominant, he expects the ignorant type to play the preferred action, thus inducing him to join in as well.

This provides a rationale for accounting conservatism: all probable losses are recorded when they are discovered, while gains can only be registered when they are fully realized, thus dispersing knowledge of gains privately among agents. While several arguments have been presented for accounting conservatism (see for Watts [2003] for an overview), we provide a novel coordination argument for firms being nebulous about gains but precise about losses. A firm’s investors would coordinate on a firm’s preferred action (invest in follow-on offerings, roll over debt etc) more often when they are uncertain about whether their fellow investors are aware of gains, but are sure that they are aware of losses.

**Comparison with Bayesian Persusasion**

Suppose the designer had the flexibility to choose partitions as a function of the state. Then, he would conserve the probability mass on state $l$ as the minimum amount required to just induce choose No Attack in state $m$. The excess probability mass would then be used towards rationalizing No Attack as a dominant action in state $h$. $q'_l$ such that

$$\frac{q_m}{q_m + q'_l} = \bar{q}$$

Then, some probability mass from $l$ types can be pooled with $h$ types. For NA to be dominant,

$$q'_l = \frac{\bar{q}}{q_m} - q_m$$

The excess probability mass is then pooled with $h$ types, such that $q''_l = (q_l - q'_l)$

$$(q_l - q''_l)\theta_l + q''_h\theta_h = 0$$
which implies
\[ q''_h = -\frac{\theta_h}{\theta_i} \]

Therefore, the designer could do better when he is allowed to condition the information structure on the payoff relevant state.

## 4 Application: Coordinated investments

We extend the argument made in the previous application to coordinated investment problems such as Initial Public Offerings (IPOs) or Initial Coin Offerings (ICOs), where an entrepreneur wishes to successfully raise funds from two investors to finance a project. The project is successful only if both investors invest in the project. In the following analysis, we restrict the entrepreneur’s disclosure decision to two states: high (where it is dominant for agents to invest) and medium (where both investing and not investing are rationalizable). As in Section 3.2, we assume that an investor is fully transparent in the state which is not in his favor - the low state (where it is dominant for agents to not invest). However, as we will soon show, the entrepreneur relies on partial disclosure in the high and medium states.

Consider the payoff matrix in Table 3 derived from [Morris and Shin 2001]. The payoff matrix is similar to that in bank run application, except that the designer wishes to implement the Invest action (similar to the Attack action in the bank run model) in as many states as possible. There are two payoff relevant states \( \Omega = \{h,m\} \) (‘high’ and ’medium’ respectively) and the probability of the medium state is \( q \). There are two actions \( \{Invest, NoInvest\} \) and the (symmetric) payoff.
when both players Attack is given by $\theta \in \{\theta_h, \theta_m\}$. If only one player invests, his payoff is $\theta - 1$. The payoff for not investing is normalized to zero, regardless of the other player’s action. We assume that $\theta_h > 0$ so that No Attack is dominant in state $\{h\}$ and $\theta_m \in (0,1)$ so that both Attack and No Attack are rationalizable in state $\{m\}$. I also assume that $\mathbb{E}[\theta] > 0$ so that both actions Attack and No Attack are rationalizable under no information (that is, $\{h,m\}$).

In line with earlier sections, the entrepreneur wishes the agents to invest as often as possible and adopts a robust approach to the design of the optimal information structure. That is, he assumes that the agents will play the worst-case scenario action “No Invest” whenever multiple actions are rationalizable.

**Proposition 3.** Invest emerges as a uniquely rationalizable action under the following probability threshold for each type:

- **Type F1:** Invest is always uniquely rationalizable
- **Type C:** Invest is the uniquely rationalizable strategy for both players if $\max(p_1, p_2) > \frac{1-\mathbb{E}[\theta]}{(1-q)}$
- **Type Fm:** Invest is uniquely rationalizable for both players if $\min(p_1, p_2) < \theta_m$ and $\max(p_1, p_2) > \frac{1-\mathbb{E}[\theta]}{(1-q)}$

The proof of this Proposition in exactly on the same lines as the previous application and we leave it as an exercise for the reader. We show that the entrepreneur can implement the risky action (“Invest”, akin to “Attack”) if he is publicly forthcoming about bad states but disseminates news about all other states in a private setting. This is reminiscent of two striking features of an IPO process. First, all materiel risk factors are mandated to be disclosed upfront in the public IPO prospectus by the regulator. Further, any forward looking statements are banned by the regulator as part of the prospectus. Such disclosure is enforced by the underwriter and the associated team of lawyers. The underwriter often acts as a certifying intermediary and is known to call off the fund raising under difficult circumstances. Second, IPO marketing “roadshows” involve private
Figure 2: Rationalizable strategies under the prior $q = \frac{2}{3}$ and payoffs $\theta_m = 0.5$ and $\theta_h = 1.5$ for the payoff matrix described in Section 4. Rationalizable strategies under the prior $q = 0.5$ and payoffs $\theta_m = 0.4$ and $\theta_l = -0.1$ for the payoff matrix described in Section 2. The x-axis and the y-axis represent the probability $p_i$ that each agent is informed (i.e. of the Fine type). The two charts show rationalizable actions choices for the uninformed (Coarse) and informed (Fine) type respectively. The dark colored region represents combinations of $(p_1, p_2)$ where both actions, “Invest” and “No Invest”, are rationalizable while the light colored region represents areas where “Invest” is uniquely rationalizable. The 45 degree line in the center is the line of symmetric information where $p_1 = p_2$. In this example, it is not possible to rationalize the action “Invest” uniquely using symmetric information policies. One investor (whom we call the institutional investor) receives more information than the other.
meetings with large investors, where the entrepreneur has more freedom in discussing his future plans. Moreover, this information is often distributed in an asymmetric manner: large institutional investors are in a position to gain more information than retail investors. This intuition is in line with Chiang et al. [2010] who find evidence that institutional investors’s bidding pattern in IPOs suggests that they are informed while those of individual investors suggests that they are uninformed. Our result suggests that this (asymmetric) bifurcation of public and private information in different states is crucial to the IPO’s success, as long as relatively uninformed retail investors are aware of the participation of more informed institutional investors in the IPO.

5 Application: The Email game

In this section, we allow for more complicated information structures. Specifically, we start by modifying the information structure in Rubinstein [1989]. In his paper, Rubinstein [1989] assumes that player 1 is fully informed about the two states \{l\}, \{m\} and that player 2 can obtain information about the states via an “email” from player 1. Player 1’s computer automatically sends a message if the state is \(l\), while if the state is \(m\), no message is sent. If player 2’s computer receives a message, then it automatically sends a confirmation; this is so not only for the original message but also for the confirmation, the confirmation of the confirmation, and so on. Moreover, each email has a probability \(\epsilon\) of getting lost. Rubinstein [1989] showed that since the players can never have complete knowledge of the other player’s actions, “No Attack” emerges as the unique (and sub-optimal) equilibrium in state \(m\).

<table>
<thead>
<tr>
<th>Attack</th>
<th>No Attack</th>
<th>Attack</th>
<th>No Attack</th>
</tr>
</thead>
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<td>(0, 0)</td>
<td>(0, \theta_m - r)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

Table 4: Payoff matrix as in Morris and Shin [2001]
Email game Variant #1: We now modify Rubinstein’s original example, by introducing a small probability $p$ with which player 2 is of the Fine type in the Email game i.e. his information partition is $\{\{l\}, \{m\}\}$ and he knows the state (as opposed to Coarse type where his information partition is $\{\Omega\}$and he does not know the state). As usual, player 1 is always of the Fine type, and knows the state.

Proposition 4. (Email game Variant #1) There is an interior threshold probability $p^*$ of Fine types of player 2 such that above the threshold, both actions rationalizable while below the threshold, all types of player 2 play $A$ as a unique rationalizable action. This threshold is given by

$$p > 1 - (2 - \epsilon)\theta = p^*$$

Email game Variant #2: We now show a more noteworthy failure of lower hemi-continuity in a second variant of the email game. As usual, we assume that player 1 always knows the state of the game (i.e. he is always of the Fine type). Player 2 is of the Fine type with probability $p$. In this case, we consider messages about the partitions in lieu of messages about state. If player 2 is of the Fine type, a message (“Hey I’m informed!”) is initiated by player 2. These messages bounce back and forth and as in Rubinstein [1989], each message gets lost with probability $\epsilon'$.

Proposition 5. (Email game Variant #2) Given any probability of Fine types $p$, no matter how large, there exists a loss rate $\epsilon'(p)$ small enough such that action $A$ emerges as the uniquely rationalizable action for the modified email game.

This Proposition shows that for any small probability of Coarse types $\delta \to 0$ in the mind of player 1, an analyst can choose $\epsilon'(\delta)$ small enough so that No Attack is a uniquely rationalizable action, even though both players are playing a complete information game with a high probability.
Thus, a small amount of uncertainty in the mind of player 1 can alter the equilibrium set. It is important to note that we have not allowed the analyst to construct signals that depart significantly from the common first order belief held by both agents (pertaining to the payoff relevant states). The action that remains robust to second order uncertainty when $\theta_m \in (0, 1/2)$ is No Attack. If instead of $\theta = \theta_l < 0$, we have that $\theta = \theta_h > 1$ in some state, we can show that Attack is the robust rationalizable action. In what follows, we summarize our results in this section by considering perturbations in $\phi$ keeping $\mu$ and $u_i$ fixed.

6 Conclusion

We represent “second order uncertainty” as an agent’s belief over all possible information partitions of the opponents and study how it could be useful in achieving coordination among agents. By introducing second order uncertainty, we are able to create overlapping partitions in an augmented state space. We describe simple information policies that use second order uncertainty to rationalize the policymaker’s preferred actions. We show that the optimal policy has a ’quarantine and infect’ mechanism. The states in the policymaker’s favor are allowed to infect other states, while states which are not in his favor are quarantined. For example, in a currency attack game or a stylized bank run model, the policy maker allows agents to be uncertain about the other agent’s knowledge in states when the fundamentals are good but allows perfect information in bad states. By being nebulous about good states, the policy maker leaves open the possibility that some agent could privately be aware of the state and play a dominant action that is preferred by the designer. Knowing this, the ignorant type chooses the policymaker’s preferred action, whenever he expects his opponent to be of the knowledgeable type with large enough probability. This has a cascading impact on the knowledgeable agent in a state where multiple actions are rationalizable. Even though this agent knows that the preferred action is not dominant,
he expects the ignorant type to play the preferred action, thus inducing him to join in as well.

This provides a rationale for accounting conservatism: all probable losses are recorded when they are discovered, while gains can only be registered when they are fully realized, thus dispersing knowledge of gains privately among agents. We show how rationalizable actions are sensitive to this second order uncertainty, especially for small perturbations of the information structure. Complete information games played in many states with dominant actions in some states are most vulnerable.

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114
7 Appendix

7.1 Proof of Proposition 1

Proof. The partition-states in terms of states of the world are given by:

\[ P_1 = \{\{FFl, FCl\}, \{CFl, CCl, CCm, CFm\}, \{FCm, FFm\}\} \]

The partition state for player 2 in terms of states of the world are:

\[ P_2 = \{\{FFl, CFl\}, \{FCl, CCl, CCm, FCm\}, \{CFm, FFm\}\} \]

The probability distribution is as follows
Consider the event $E_1 = \{\text{FFl, FCl}\}$ where it is dominant for player 1 to play action A. We want to find the belief potential for the event $E_1$.

**Step 1:** Consider the beliefs of player 2 about $E_1$: The Fine type of player 2 $\{\text{FFl, CFl}\}$ is infected by $E_1$ whenever $r \leq p_1$, that is, at each state, the Fine type believes with probability $r$ that event $E_1$ has occurred. The Coarse type of player 2 $\{\text{FCl, CCl, CCm, FCm}\}$ $r$—believes in event $E_1$ whenever the probability of the state $\text{FCl}$ exceeds $r$. This would happen as long as $r \leq (1-q)p_1$. Considering the lesser of the two probabilities so far, we arrive at $r = (1-q)p_1$. Then $B_2^r(E_1) = \{\text{FFl, FCl, FCl, CCl, CCm, FCm}\}$ if $r \leq (1-q)p_1$ i.e. both the Fine type $l$ and Coarse type of player 2 are infected. On the other hand, if $r \in ((1-q)p_1, p_1)$, only the Fine type of player 2 is infected. We consider this case separately when we evaluate the infection starting from $E_2 = \{\text{FFl, CFl}\}$.

**Step 2:** Considering the Coarse type of player 1:

\[
B_1^r B_2^r(E_1) = \{\text{FFl, FCl, CFl, CCl, CCm, FCm}\} \text{ if } r \leq 1 - qp_2.
\]

Considering the minimum of all three probabilities, $\sigma_1(E_1; P_1 = \{\text{CFl, CCl, CCm, FCm}\}) = (1-q)p_1$.

Finally, $B_1^r B_2^r(E_1) = \{W\}$ if $r \leq 1 - p_2$. The Fine type of player 1 in state $m$ is also infected and the entire space for player 1 is covered. The Fine type of player 1 $\{\text{FCm, FFm}\}$ is infected by $E_1$ whenever the probability of the state $\text{FCm}$ conditional on his partition (given by $(1-p_2)$) exceeds $r$. Thus, the belief potential for player 1 is $\sigma_1(E_1) = \min((1-q)p_1, 1 - p_2)$.

Now suppose that $r \leq \min((1-q)p_1, (1-p_2))$ so that the event $E_1$ infects every state $w$ of player 1 (in

<table>
<thead>
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<tbody>
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</tr>
<tr>
<td>FCl</td>
<td>$(1-q)p_1(1-p_2)$</td>
</tr>
<tr>
<td>CFl</td>
<td>$(1-q)(1-p_1)p_2$</td>
</tr>
<tr>
<td>CCl</td>
<td>$(1-q)(1-p_1)(1-p_2)$</td>
</tr>
<tr>
<td>CCm</td>
<td>$q(1-p_1)(1-p_2)$</td>
</tr>
<tr>
<td>CFm</td>
<td>$q(1-p_1)p_2$</td>
</tr>
<tr>
<td>FCm</td>
<td>$qp_1(1-p_2)$</td>
</tr>
<tr>
<td>FFm</td>
<td>$qp_1p_2$</td>
</tr>
</tbody>
</table>
the sense of \( r\) –belief on event \( E_1\) at every partition containing \( w\). We have already shown that the event \( E_1\) infects the Coarse type of player 2. Does it infect the Fine type \( \{CFm, FFm\}\) of player 2? The answer is trivially yes! Since both the Coarse and Fine type of player 1 \( r\)-believe in \( E_1\), \( B_2 B_1^r B_2^r (E_1) = W\).

By symmetry, the belief potential of player 2 is \( \sigma_2(E_2) = \min((1 - q)p_2, 1 - p_1)\) which obtains over the event \( E_2 = \{FFl, CFl\}\). So far, we have fixed the proportion of Fine types of each players as the vector \( p = (p_1, p_2)\). Momentarily, we assume that we can fix \( p_i\) such that \( r = (1 - q)p_j\) for each agent \( i\). Then, without any loss in generality, if we choose \( (1 - q)p_1 > (1 - q)p_2\)

The maximum belief potential for the information system is

\[
\sigma = \max \left( \min((1 - q)p_1, 1 - p_2), \min((1 - q)p_2, 1 - p_1) \right)
\]

The action \((NA, NA)\) is 0-dominant is state \( l\). In the “Coarse game” (the average of the two games), the payoffs are:

<table>
<thead>
<tr>
<th>Average Game</th>
<th>A</th>
<th>NA</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>((1 - q)\theta_l + q\theta_m, (1 - q)\theta_l + q\theta_m)</td>
<td>((1 - q)\theta_l + q\theta_m - r, 0)</td>
</tr>
<tr>
<td>NA</td>
<td>(0, (1 - q)\theta_l + q\theta_m - r)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

The t-dominance threshold for this game is:

\[
0 > t_C[(1 - q)\theta_l + q\theta_m - r] + (1 - t_C)[(1 - q)\theta_l + q\theta_m]
\]

which implies:

\[
t_C > \frac{(1 - q)\theta_l + q\theta_m}{r}
\]

\(^{12}\)In the case of symmetric information, or \( p_1 = p_2\), we establish that \( \sigma = \min((1 - q)p, 1 - p)\). We are now interested in the maximum value that the belief potential can take for a given value of \( q\).

\[
\sigma_{max} = \max_p \sigma(p)
\]

For our example, \( \sigma_{max} = \max_p \min((1 - q)p, 1 - p)\). The maximum value obtains when \( p = \frac{1}{(2 - q)}\). At this value \( \sigma_{max} = \frac{1 - q}{2 - q}\).
In terms of Definition 5, \((NA, NA)\) is \(\sigma\)-dominant in \(G(\{l, m\})\) for \(\{CFl, CCl, CCm, CFm\}\) as long as \(\sigma \geq t_C\)

\[
\max((1 - q)p_1, (1 - q)p_2) > \frac{(1 - q)\theta_l + q\theta_m}{r}
\]

Separately, At \(\{CFl, CCl, CCm, CFm\}\), player 1 knows that player 2 chooses No attack at \(CFl\). Assuming in the worst case scenario that player 2 potentially chooses Attack at \(CCl, CCm, CFm\), player 1 will continue to choose No attack (uniquely) if:

\[
(1 - q)p_2u_1(N, N; CFl) + (1 - q)(1 - p_2)u_1(N, A; CCl) + q(1 - p_2)u_1(N, A; CCm) + qp_2u_1(N, A; CFm) > \\
(1 - q)p_2u_1(A, N; l) + (1 - q)(1 - p_2)u_1(A, A; l) + q(1 - p_2)u_1(A, A; CCm) + qp_2u_1(A, A; CFm)
\]

Substituting the payoff parameters, we have that:

\[
0 > (1 - q)p_2(-r) + (1 - q)(\theta - h) + q\theta
\]

In state \(m\), the payoffs are given by:

<table>
<thead>
<tr>
<th>Game (G_m(\text{state } m))</th>
<th>A</th>
<th>NA</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(\theta_m, \theta_m)</td>
<td>(\theta_m - r, 0)</td>
</tr>
<tr>
<td>NA</td>
<td>(0, \theta_m - r)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

Suppose player 1 assumes that player 2 plays No attack with probability \(t_f\), then the minimum threshold for No attack to be \(p\)-dominant in state game \(m\) is

\[
t_f.0 + (1 - t_f).0 \geq t_f.(\theta_m - r) + (1 - t_f).\theta_m
\]

which implies that \(t_f \geq \frac{\theta_m}{r}\). From our belief potential equation, we have that:

\[
\max (1 - p_1, 1 - p_2) \geq \frac{\theta_m}{r}
\]
Combining the two equations on \((p_1, p_2)\) gives us the conditions for agents to play No Attack as a uniquely rationalizable action. Overall, the minimum \(r\)-dominance threshold for (NA,NA) to be uniquely rationalizable in all states is given by: \(\max(\frac{\theta_m}{r}, \frac{(1-q)\theta_l+q\theta_m}{r})\). In the symmetric case, we need that:

\[
\min((1-q)p, 1-p)) > \max(\frac{\theta_m}{r}, \frac{(1-q)\theta_l+q\theta_m}{r})
\]

\[\square\]

### 7.2 Proof of Proposition 2

**Proof.** The designer wishes to induce an infection cascade in state \(m\) or \(h\). Given that \(\theta_m < \theta_h\), his strategy would be use the probability mass on state \(l\) to first induce No Attack on state \(m\). This would be possible only when the probability of state \(m\) is small enough. Let \(q\) be the conditional probability of state \(m\), given that the states are \(\{l, m\}\). For symmetric information design, the maximum conditional probability \(q\) on state \(m\) that would permit No Attack as a uniquely rational action is given by Proposition 1 when:

\[
\frac{(1-\bar{q})\theta_l+q\theta_m}{(1-\bar{q})} = 1 - \theta_m
\]

which implies

\[
\bar{q} = \frac{(1-\theta_m) - \theta_l}{(1-\theta_l)}
\]

Whenever \(q > \bar{q}\), an infection cascade cannot be induced. It would then be optimal for the designer to make agents fully knowledgeable about state \(l\) and induce No Attack in this state. When \(q < \bar{q}\), the designer can ‘quarantine’ state \(h\) and make the agents uncertain about what the other agents know about state \(l\) and state \(m\), in line with Proposition 1.

\[\square\]
7.3 Proof of Proposition 4

Proof. The expanded states of the world can be represented in terms of total number of messages crossed \( \times \) the partitional state \((C,F)\) for player 2.

\[
\Pi_1 = \{\{0F_2l,0C_2l\}, \{1C_2m,1F_2m,2C_2m,2F_2m\}, \{3C_2m,3F_2m,4C_2m,4F_2m\}, \{5C_2m,5F_2m,6C_2m,6F_2m\}...\}
\]

\[
\Pi_2 = \{\{0F_2l\}, \{0C_2l,1C_2m\}, \{1F_2m\}, \{2F_2m,3F_2m\}, \{2C_2m,3C_2m\}, \{4F_2m,5F_2m\}, \{4C_2m,5C_2m\}...\}
\]

The probabilities are given by:

<table>
<thead>
<tr>
<th>States</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0F_2l</td>
<td>((1-q)p)</td>
</tr>
<tr>
<td>0C_2l</td>
<td>((1-q)(1-p))</td>
</tr>
<tr>
<td>1F_2m</td>
<td>(qp\epsilon)</td>
</tr>
<tr>
<td>1C_2m</td>
<td>(q(1-p)\epsilon)</td>
</tr>
<tr>
<td>2F_2m</td>
<td>(qp(1-\epsilon)\epsilon)</td>
</tr>
<tr>
<td>2C_2m</td>
<td>(q(1-p)(1-\epsilon)\epsilon)</td>
</tr>
<tr>
<td>3F_2m</td>
<td>(qp(1-\epsilon)^2\epsilon)</td>
</tr>
<tr>
<td>3C_2m</td>
<td>(q(1-p)(1-\epsilon)^2\epsilon)</td>
</tr>
<tr>
<td>kF_2m</td>
<td>(qp(1-\epsilon)^{k-1}\epsilon)</td>
</tr>
<tr>
<td>kC_2m</td>
<td>(q(1-p)(1-\epsilon)^{k-1}\epsilon)</td>
</tr>
</tbody>
</table>

It is clear that at partitions \(\{0F_2l\}\) (player 2) and \(\{0F_2l,0C_2l\}\) (player 1) action No Attack is dominant. The belief potential for player 2 at \(\{0C_2l,1C_2m\}\) is the largest number \(r\) such that

\[
B^*_i(E_1) = \{w \in W : P(E_1 | \Pi_i(w)) \geq r\}
\]

where \(E_1 = \{0F_2l,0C_2l\}\) and \(\Pi_i(w) = \{0C_2l,1C_2m\}\). The probability of \(0C_2l\) is \((1-q)(1-p)\) and \(1C_2m\) is \(q(1-p)\epsilon\). Therefore, the largest number \(r\) such that the event \(E_1\) covers type \(\{0C_2l,1C_2m\}\) of
player 2 is:

\[
\sigma = \frac{(1-q)(1-p)}{(1-q)(1-p) + q(1-p)e} = \frac{1-q}{(1-q) + qe}
\]

At \(\{0C_2, 1C_2m\}\) the average game for this type is:

<table>
<thead>
<tr>
<th>Average Game</th>
<th>A</th>
<th>NA</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>((1-q)\theta_l + qe\theta_m, (1-q)\theta_l + qe\theta_m)</td>
<td>((1-q)\theta_l + qe\theta_m - ((1-q) + qe)r, 0)</td>
</tr>
<tr>
<td>NA</td>
<td>(0, (1-q)\theta_l + qe\theta_m - ((1-q) + qe)r)</td>
<td>(0, 0)</td>
</tr>
</tbody>
</table>

The t-dominance threshold for this game is:

\[
0 > t_C \cdot [(1-q)\theta_l + qe\theta_m] - ((1-q) + qe)r + (1-t_C) [(1-q)\theta_l + qe\theta_m]
\]

which implies:

\[
t_C > \frac{(1-q)\theta_l + qe\theta_m}{((1-q) + qe)r}
\]

\((NA, NA)\) is \(\sigma\)-dominant in \(G(\{l, m\})\) as long as \(\sigma \geq t_C\). When \(r = 1\)

\[
\frac{(1-q)}{(1-q) + qe} > \frac{(1-q)\theta_l + qe\theta_m}{((1-q) + qe)}
\]

This is equivalent to \(q < \frac{1}{1+\frac{r \theta_m}{(1-\theta_l)}}\), which is what we have assumed. It can be easily verified that the potential breaking point of the infections cascade is player 1’s partition \(\{1C_2m, 1F_2m, 2C_2m, 2F_2m\}\). The maximum belief potential of \(E_1\) is:

\[
\sigma = \frac{q(1-p)e}{q(1-p)e + qpe + qp(1-e)e + q(1-p)(1-e)e} = \frac{1-p}{2-e}
\]

In state \(m\), the payoffs are given by:
Suppose player 1 assumes that player 2 plays No attack with probability $t_f$, then the minimum threshold for No attack to be p-dominant in state game $m$ is

$$t_f.0 + (1 - t_f).0 \geq t_f.(\theta_m - 1) + (1 - t_f).\theta_m$$

which implies that

$$t_f \geq \theta_m$$

To uniquely rationalize NA, it must be that the belief potential for \{1C_2m, 1F_2m, 2C_2m, 2F_2m\} is greater than the $r$–dominance threshold.

$$\frac{1 - p}{2 - \epsilon} > \theta_m$$

Separately, action Attack is rationalizable if

$$(1 - p)\epsilon.u_1(N, N; \{1C_2\}) + \left( \epsilon p + (1 - p)(1 - \epsilon)\epsilon + p(1 - \epsilon)\epsilon \right) u_1(N, A; \{1F_2, 2C_2, 2F_2\}) <$$

$$(1 - p)\epsilon u_1(A, N; \{1C_2\}) + \left( \epsilon p + (1 - p)(1 - \epsilon)\epsilon + p(1 - \epsilon)\epsilon \right) u_1(A, A; \{1F_2, 2C_2, 2F_2\})$$

The minimum Fine types of player 2 required to break the infection is

$$p > 1 - (2 - \epsilon)\theta_m = p^*$$
7.4 Proof of Proposition 4

Proof. First we consider a case with the possible set of parameter \( \theta \in \{ \theta_l, \theta_m \} \): The partitions of each player can first be written as

\[
\mathcal{P}_1' = \{ \{0\text{C}_2, 1F_2l\}, \{0\text{C}_2m, 1F_2m\}, \{2F_2l, 3F_2l\}, \{2F_2m, 3F_2m\}, \ldots \},
\]

\[
\mathcal{P}_2' = \{ \{0\text{C}_2, 0\text{C}_2m\}, \{1F_2l, 2F_2l\}, \{1F_2m, 2F_2m\}, \{3F_2l, 4F_2l\}, \ldots \}
\]

**Step 1:** At \( \{0\text{C}_2, 1F_2l\} \), it is dominant for the Fine type of player 1 to choose No attack.

**Step 2:** At partition \( \{0\text{C}_2, 0\text{C}_2m\} \), it is uniquely rationalizable for the Coarse type player 2 to play No attack if:

\[
(1 - q)(u(N, N; l) - u(A, A; l)) + q(u(N, A; m) - u(A, A; m)) > 0
\]

which implies \( q < q^* = \frac{1}{1 + \frac{\theta_m}{\theta_l - 1}} \). Given that \( \theta_l < 0 \) and \( \theta_m \in (0, 1) \), we can bound \( q^* \) as follows. The largest value that \( q^* \) can take is 1, when \( \theta_m = 0 \). The smallest value is then \( \theta_m = 1/2 \) and \( \theta_l = 0 \) which is \( q^* = 2/3 \). Thus as long as \( q < 2/3 \) for all parameter values, No attack will be uniquely rationalizable.

**Step 3:** At \( \{0\text{C}_2, 1F_2m\} \), knowing that the Coarse type of player 2 will play action **No Attack** at \( 0\text{C}_2b \), No attack will be uniquely rationalizable for (Fine type) player 1 if

\[
(1 - p)((\theta_m - 1)) + p\epsilon(-\theta_m) > 0
\]

which implies \( p < \frac{1}{1 + \frac{\epsilon\theta_m}{\theta_m - 1}} \). Note that we have that as \( \epsilon \to 0 \), \( p \to 1 \).

**Step 4:** At \( \{1F_2m, 2F_2m\} \), the Fine type of both players continue choosing No Attack as

\[
0 > p(\theta_m - 1) + p(1 - \epsilon)(\theta_m)
\]

given that \( \theta_m \in (0, 1) \) which will be true when \( \theta_m < 1/2 \)

**Step k+3:** At \( \{kF_2m, (k + 1)F_2m\} \), the Fine type of both players continue choosing No
Attack as

\[ 0 > p(\theta_m - 1) + p(1 - \epsilon)(\theta_m) \]

which will be true when \( \theta_m < 1/2 \).
Chapter 3: Liquidity portfolios

1 Introduction

Firms face liquidity shocks in various forms. They often fund assets through outside sources which may be callable at the unforeseen instance of the investor. In regular course of business, they may need to make unexpected purchases or meet emergencies. To ensure that the firm has the liquidity and cash to meet its obligations, corporate treasuries play an important role in a firm’s cash management.

An early analysis of cash management was presented in [Baumol 1952], where the forgone interest from holding cash was traded off against the "shoe leather cost" of making additional trips to the bank. In this paper, we study cash management in the spirit of [Baumol 1952], but with stochastic liquidity shocks and assets with varying degrees of liquidity. Our main finding is that an agent’s valuation for buying a little bit more of an illiquid asset is equivalent to the agent’s martingale valuation of an American put option that pays the asset’s liquidation value when exercised. This result explains (i) the convex relationship between asset prices and liquidity, as in [Amihud and Mendelson 1986], even with a single representative agent and (ii) the negative covariance between asset prices and aggregate liquidity in the market, as in [Pastor and Stambaugh 2003].

In our model, a corporate treasury manager plans to hold a dynamic portfolio of liquid assets to random liquidity demands in the future. The available assets have different degrees of liquidity with stochastic liquidation values that may evolve over time. Assets may be liquidated or purchased from the market depending on the size and sign of the random liquidity shock. We first study the agent’s choice problem with an exogenously given shadow

*This chapter is joint work with Philip H. Dybvig
price process. Through a set of recursive equations, we show that the shadow price of a security at each interim date is at least the maximum of the discounted liquidation value in the current period and the expectation of the discounted price in the next period. The equation holds with equality when the agent buys a positive amount of the security. Intuitively, the agent can choose a stopping time at which to sell the incremental purchase, which in effect is the choice of when to exercise the American put option to sell the asset under consideration.

However, we face a challenging situation where the agent holds no assets in the previous period and does not trade in the current period. This leads to a degenerate optimization problem where the shadow prices are determined only up to a difference between a continuum of values. To get a sensible result, we solve for the maximum price that an agent would be willing to pay to buy a small quantity of the asset and denote it as the reservation price of the liquid asset. This reservation price process of a liquid asset is an American put option with strike price process given by the liquidation value in each period.

We then solve for equilibrium prices in a three-date example \((t = 0, 1, 2)\) with a single agent (or many identical agents). The agent liquidates assets to satisfy an interim liquidity shock (at \(t = 1\)). Any shortfall in meeting the liquidity requirement results in a convex failure cost. This cost can be interpreted as an increasingly high rate of borrowing or a reputational penalty for delaying payments. Factoring in these liquidity requirements (and the costs associated with non-adherence thereof), the agent forms an optimal portfolio of assets from an endowment at \(t = 0\). Our model shows that the agent chooses different horses for different races. Illiquid assets provide terminal utility while liquid assets insure against cash flow shocks. Some assets perform a dual role of insurance in some states and continuation value in others. Thus, in contrast to existing studies, we model liquidity as an option that gains value from states when it is exercised. Moreover, liquid assets perform the role of precautionary saving in our model. The agent stores liquid asset at \(t = 0\) anticipating the uncertain liquidity shock at date 1. Our main result is a convex pricing rule under a rich enough state space for liquidity shocks (the number of states is greater than two). Highly liquid assets which can be liquidated in several states of the world (corresponding to various liquidity shock values) command a higher price than illiquid assets that are liquidated only in few states. The optimal liquidation policy is a state-contingent threshold policy where assets with liquidation value greater than the threshold are liquidated at \(t = 1\). Moreover, as
one would expect, this threshold is decreasing in the size of the liquidity shock - the higher the size of the liquidity shock, the lower is the liquidation threshold and more assets are liquidated. Because of the convex nature of the failure cost, the agent will always choose to default on some portion of his liability at \( t = 1 \) and pay the costs associated with such failure at the terminal date \( t = 2 \). Through our pricing rule, we offer a novel explanation for the well established concave liquidity-return relationship in the empirical literature (eg. Amihud and Mendelson [1986]).

Next, we provide intuition for the construction of a liquidity beta which measures the sensitivity of a liquid asset’s price to the shifts in market-wide liquidity. Considering a simple case with linear utility, quadratic shortfall costs and normalizing the terminal value of the asset to a unit constant, we obtain the price of an asset \( i \) as \( P_i = \mathbb{E} \left( \max(1, \frac{L_i}{L^*}) \right) \). The parameter \( L^* \) is a state-contingent threshold liquidation policy: only assets with liquidation value \( L_i \geq L^* \) are liquidated. \( L^* \) serves as a proxy for aggregate market liquidity. The threshold policy \( L^* \) depends both on the size of the random liquidity shock and the realized liquidation value of all assets in the economy. An asset gains value from its liquidity only when \( L_i \geq L^* \), i.e. when the asset is indeed liquidated. In states when the asset is not liquidated, its value is equal to the future liquidation value (which in this case equals one). This gives us an intuition about an asset’s dual sources of value: (i) it should have a higher realized liquidation value in states when the liquidity shock is large and, (ii) it should supply liquidity precisely when all the other assets in the market fail to do so. Like an option, a liquid asset is valuable in states when both the liquidity shock is high and there is a scarcity of other liquid assets in the economy. Moreover, the downsides of the option is protected by the terminal value of such asset). The liquidation option differs from a regular stock option in that its value depends not only on the realization of a primary state variable (the liquidity shock) but also on the realized liquidation value of all other assets in the economy. Our pricing rule also generates rules for ordering random payoffs in a liquidity context. For liquidation purposes, assets are ranked in a descending order in terms of the ratio of the liquidation values to the terminal value, as long as they are liquidated in some state.

Finally, we consider an application where the liquidity shock faced by a risk averse firm is due to its liability structure. The firm is endowed with a portfolio of liquid and illiquid assets and the value of the portfolio, net of liabilities, represents the value of the firm. An agent compares asset prices under two regimes: regime \( f \) and regime \( g \) where \( f(\cdot) \) and \( g(\cdot) \) are the
probability distributions of the liquidity shock faced by the agent at the interim date \( t = 1 \). Further, suppose that \( f(\cdot) \) second order stochastically dominates \( g(\cdot) \) (i.e. \( g(\cdot) \) is a mean preserving spread of \( f(\cdot) \)). We show that risk averse firms would choose a liability structure under regime \( f \) such that the distribution of the liquidity shocks is concentrated around the mean. One method of making this happen is to borrow from different independent sources, so that not all of its liabilities are called at once. This provides an explanation to conventional wisdom suggesting that firms should “broad base” their liabilities and not rely on a single source.

1.1 Related literature

Our paper is related to the literature on dynamic portfolio selection under transaction costs. We adapt this approach to analyze liquidation decisions in a cash management setting. In our model, the transactions cost of an asset at each instant can be described as the difference between (i) the price at which the agent would like to buy a little bit more of the asset and (ii) its liquidation value. In general, dynamic portfolio selection under transaction costs is a complex problem. Most papers adopt specific functional forms for the primitives of the model, e.g., agents’ utility functions and asset payoffs to derive closed-form solutions for portfolio optimization and equilibrium under transaction costs (eg. [Constantinides 1986], [Davis and Norman 1990], [Dybvig and Pezzo 2019]). We take a distinct approach in valuing assets to meet random liquidity demands in the future, without assuming any functional form for our main result.

We show that the valuation of an illiquid asset is equal to an American put option with strike price equal to the liquidation value of the asset. In this respect, our paper is closest to [Jouini and Kallal 1995], who show that absence of arbitrage implies that there exists risk neutral probabilities that transforms some process between the bid and ask price processes to a martingale. If we denote the price of buying the asset as the ask price and its liquidation value as the bid price, then the American put valuation for the price of an asset (where strike price = bid price) is an instance of a martingale process that implies no arbitrage. The idea that liquidity could be valued as a put option also appears in [Holmström and Tirole 2001] in an agency theory setting where a cash constrained firm pays a premium for liquidity. We work in a more general setting, with assets of different degrees of liquidity that
evolve stochastically over time and agent’s dynamically choosing portfolios to satisfy random liquidity demands. Therefore, their findings could be interpreted as a special case of our main result, under standard agency theory assumptions.

Our paper is also related to the literature studying illiquidity premium in the cross section. Amihud and Mendelson [1986] derive a convex relationship between asset prices and liquidation values through clientele effects of investors with heterogeneous holding periods. Investors with long horizon are allocated illiquid assets since they face lower per-period transaction costs. The concavity in our model is generated not through clientele effects (as agents have the same “per-period transaction costs”) but through the option-like feature of liquid assets.

Pástor and Stambaugh [2003] established that stock returns respond cross-sectionally to fluctuations in aggregate market liquidity. We study the sensitivity of assets to aggregate liquidity through a three period example in Section 3 of our paper. In line with their finding, our example shows that a liquid asset is valuable in states when market liquidity is tight: that is, there is high liquidity demand and scarcity of other liquid assets. For individual assets, Acharya and Pedersen [2005] propose a linear factor relationship between liquidation value and asset price. However, under a rich enough state space, we obtain a strictly convex relationship between liquidation value and asset price due to option-like value of the liquidation decision.

The rest of the paper is organized as follows. Section 2 solves for the agent’s choice problem, Section 3 solves for equilibrium prices in a three period example and describes two applications, and Section 4 concludes.

2 Agent’s Choice Problem

An agent lives in a world with $T$ dates, $t = 0, 1, 2, ... T$. The agent derives utility $u(w_T)$ from his final portfolio position at $t = T$. At each date $t$ the agent receives a random liquidity shock $y_t$. Investment options at $t = 0$ comprise $n$ assets with random liquidation values $L_{i,t}$ for the $i$’th asset at time $t$. We solve for the agent’s choice problem assuming that he is endowed with a portfolio of assets $\theta_{i, -1}$ at date 0. Depending on whether the liquidity shock $y_t$ is positive or negative, at each date $t < T$, the agent’s choice consists of either (i) liquidating $l_{i,t}$ units of the $i$’th asset or (ii) buying $b_{i,t}$ units of the asset in the market at a
Problem 1. Given initial portfolio holding \( \theta_{i,-1} \), liquidity shock \( y_t \), asset liquidation values \( L_{i,t} \) and purchase prices \( P_{i,t} \), choose adapted liquidation policy \( l_{i,t} \), security purchases \( b_{i,t} \), and portfolio holding \( \theta_{i,t} \), to:

\[
\max_{l_{i,t}, b_{i,t}, \theta_{i,t}, w_T} \mathbb{E}[u(w_T)] \tag{2.1}
\]

subject to:

\[
w_T = \sum_{i=1}^{n} l_{i,T} L_{i,T} \tag{2.2}
\]

\( \forall t = 0, \ldots, T - 1 \) and each state \( \omega_t \)

\[
y_t \leq \sum_{i=1}^{n} (l_{i,t} L_{i,t} - b_{i,t} P_{i,t}) \tag{2.3}
\]

\( \forall t = 0, \ldots, T \) and \( \forall i = 1, \ldots, n \) and each state \( \omega_t \)

\[
\theta_{i,t} = \theta_{i,t-1} + b_{i,t} - l_{i,t} \tag{2.4}
\]

\[
\theta_{i,t} \geq 0 \tag{2.5}
\]

\[
b_{i,t} \geq 0 \tag{2.6}
\]

\[
l_{i,t} \geq 0 \tag{2.7}
\]

We define the agent’s state price density at each time \( t \) as given by \( \xi_t \). We note that the agent’s state price density is equivalent to the shadow price of constraint (2.3) in the agent’s choice problem.

Theorem 1. The agent’s reservation price of a liquid asset is an American put option with
strike price $L_{i,t}$ and is given by:

$$\phi_{i,t} = \max \left( L_{i,t}, \mathbb{E} \left[ \frac{\phi_{i,t+1}\xi_{t+1}}{\xi_t} \right] \right)$$

$\phi_{i,T} = L_{i,T}$. The reservation price $\phi_{i,t} = P_{i,t}$ whenever $b_{i,t} > 0$

**Proof.** The Lagrangian multipliers for the constraints, deflated by the probability density are given by: $\xi_T$ for equation 2.2, $\xi_t$ for equation 2.3, $\rho_{i,t}$ for equation 2.4, $\delta_{i,t}$ for equation 2.5, $\eta_{i,t}$ for equation 2.6 and $\gamma_{i,t}$ for equation 2.7. The Lagrangian is given by:

$$\mathcal{L}_1 = \max_{l_{i,t}, b_{i,t}} \mathbb{E}[u(w_T)] + \xi_T \mathbb{E} \left[ \left( \sum_{i=1}^{n} l_{i,T} L_{i,T} - w_T \right) \right]$$

+ $\sum_{t=0}^{T-1} \mathbb{E}_t \left[ \xi_t \left( \sum_{i=1}^{n} (l_{i,t} L_{i,t} - b_{i,t} P_{i,t}) - y_t \right) \right]$

+ $\sum_{t=0}^{T} \mathbb{E}_t \sum_{i=1}^{n} \left( \rho_{i,t} (\theta_{i,t-1} + b_{i,t} - l_{i,t} - \theta_{i,t}) \right.$

+ $\delta_{i,t} \theta_{i,t} + \eta_{i,t} b_{i,t} + \gamma_{i,t} l_{i,t} \right)$

First, writing the first order conditions (FOCs) with respect to $w_T$:

$$u'(w_T) = \xi_T$$

Next we consider FOCs with respect to $b_{i,t}(\omega_t)$ for $t \leq T - 1$:

$$\xi_t P_{i,t} = \rho_{i,t} + \eta_{i,t}$$

Writing FOCs with respect to $l_{i,t}(\omega_t)$, for $t \leq T - 1$:

$$\rho_{i,t} = \xi_t L_{i,t} + \gamma_{i,t}$$

Writing FOCs with respect to $\theta_{i,t}(\omega_t)$, for $t \leq T - 1$:

$$\rho_{i,t} = \mathbb{E}_t[\rho_{i,t+1}] + \delta_{i,t}$$

131
We consider three separate cases comparing the FOCs with respect to $l_{i,t}$ and $\theta_{i,t}$.

(i) $l_{i,t} = 0$ and $\theta_{i,t} = 0$ (that is the entire holding in asset $i$ has been liquidated at some $t' \leq t - 1$). Then by CS, $\gamma_{i,t} > 0$ and $\delta_{i,t} > 0$. We will take this case up in greater detail shortly.

(ii) $l_{i,t} > 0$, then since the Lagrangian is linear in $l_{i,t}$, it must be that $\theta_{i,t} = 0$. Therefore $\gamma_{i,t} = 0$ and $\delta_{i,t} > 0$. In such case:

\[
\rho_{i,t} = \xi_t L_{i,t} > \mathbb{E}_t[\rho_{i,t+1}]
\]

(iii) $l_{i,t} = 0$ and $\theta_{i,t} > 0$. Therefore $\gamma_{i,t} > 0$ and $\delta_{i,t} = 0$. In such case:

\[
\rho_{i,t} = \mathbb{E}_t[\rho_{i,t+1}] > \xi_t L_{i,t}
\]

Therefore, combining (i), (ii) and (iii)

\[
\rho_{i,t} \geq \max(\xi_t L_{i,t}, \mathbb{E}_t[\rho_{i,t+1}])
\]

We first note that when $b_{i,t} > 0$, then by complementary slackness (CS), $\eta_{i,0} = 0$.

\[
\xi_t P_{i,t} = \rho_{i,t}
\]

Now, when $b_{i,t} = 0$,

\[
\xi_t P_{i,t} > \rho_{i,t}
\]

Therefore

\[
\xi_t P_{i,t} \geq \max(\xi_t L_{i,t}, \mathbb{E}_t[\rho_{i,t+1}])
\]

Finally, we write FOCs with respect to $l_{i,T}(\omega_T)$, for $t = T$:

\[
\rho_{i,T} = \xi_T L_{i,T} + \gamma_{i,T}
\]

Since $l_{i,T}(\omega_T) > 0$ for any asset that is held in positive quantity at $t = T$, by CS, $\gamma_{i,T} = 0$.  

132
This implies $\rho_{i,T} = \xi_T L_{i,T}$ where $\xi_T = u'(w_T)$.

We now consider the case when $l_{i,t} = 0$ and $\theta_{i,t} = 0$, which implies that $\theta_{i,t-1} = 0$ and $b_{i,t} = 0$. This leads to a degenerate optimization problem where the shadow prices are determined only up to a difference between a continuum of values. To get a sensible result, we find the maximum price that an agent would be willing to pay to buy a small quantity of the asset. We keep all the choice variables, exogenous processes and the Lagrange multiplier $\xi_t$ unchanged in the solution so far.

We define a new Lagrange Multiplier $\rho_{i,t}^*$ such that

$$\rho_{i,t}^* = \max (\xi_t L_{i,t}, \mathbb{E}[\rho_{i,t+1}^*])$$

and $\rho_{i,T}^* = \xi_T L_{i,T}$.

We claim that for all $t$, $\rho_{i,t}^* \leq \rho_{i,t}$ and $\rho_{i,t}^* = \rho_{i,t}$ whenever $\theta_{i,t} > 0$ or $l_{i,t} > 0$. This claim is true by induction since we have already established that $\rho_{i,t} \geq \max (\xi_t L_{i,t}, \mathbb{E}[\rho_{i,t+1}])$ and $\rho_{i,T} = \xi_T L_{i,T}$. We note that the complementary slackness condition for the original problem is satisfied by $\rho_{i,t}^*$. Similarly, consider new Lagrange multipliers $\eta_{i,t}^*$, $\gamma_{i,t}^*$ and $\delta_{i,t}^*$:

(a) $\eta_{i,t}^* = \eta_{i,t} + (\rho_{i,t}^* - \rho_{i,t})$.

(b) $\gamma_{i,t}^* = \gamma_{i,t} - (\rho_{i,t}^* - \rho_{i,t})$.

(c) $\delta_{i,t}^* = \rho_{i,t}^* - \mathbb{E}[\rho_{i,t+1}^*]$.

We note that $\eta_{i,t}^* \geq \eta_{i,t}$, $\gamma_{i,t}^* \geq \gamma_{i,t}$ with equality whenever $\theta_{i,t} > 0$ or $l_{i,t} > 0$. Therefore $\eta_{i,t}^*$, $\gamma_{i,t}^*$ and $\delta_{i,t}^*$ satisfies the complementary slackness with their respective constraints. This implies that new Lagrange multipliers $\rho_{i,t}^*, \eta_{i,t}^*, \gamma_{i,t}^*$ and $\delta_{i,t}^*$ support the original solution to the problem. Let $\phi_{i,t} = \frac{\rho_{i,t}^*}{\xi_t}$, $\rho_{i,t}^* = \max (\xi_t L_{i,t}, \mathbb{E}[\rho_{i,t+1}^*])$ and $\rho_{i,T}^* = \xi_T L_{i,T}$. Substituting $\phi_{i,t}$ into $\rho_{i,t}^* = \max (\xi_t L_{i,t}, \mathbb{E}[\rho_{i,t+1}^*])$, we have that the agent’s reservation price of a liquid asset is given by

$$\phi_{i,t} = \max (L_{i,t}, \mathbb{E}[\phi_{i,t+1}^*])$$

Also, since $\phi_{i,T} = \frac{\rho_{i,T}^*}{\xi_t}$ and $\rho_{i,T}^* = \xi_T L_{i,T}$, we have that $\phi_{i,T} = L_{i,T}$.

Therefore, we find that an agent’s reservation price process of a liquid asset is an American put option with strike price process given by the liquidation value in each period.
3 Equilibrium pricing: A three period example

In this section we consider equilibrium pricing in a model with three dates: initial date \( t = 0 \), interim date \( t = 1 \) and terminal date \( t = 2 \). The agent has an endowment \( w_0 > 0 \) at date 0 and starts with a portfolio position \( \theta_{i,0} = 0 \) for each asset \( i \). The agent derives utility from consumption at \( t = 0 \) and \( t = 2 \) but derives no utility from the interim period \( t = 1 \). Further, the agent expects a random liquidity shock \( y_1 > 0 \) at date 1. To plan for the liquidity event, the agent has access to assets with liquidation value \( L_{i,1} \). If the assets are not liquidated, they pay a terminal value \( L_{i,2} \). The agent spends his endowment at date 0 in building a portfolio of assets \( \theta_{i,0} \) which he can purchase at price \( P_{i,0} \). If the agent is unable to meet the liquidity shock fully, he needs to pay a convex cost \( \kappa(S) \) on any shortfall value \( S \), with \( \kappa'(\cdot) > 0 \) and \( \kappa''(\cdot) > 0 \). The cost of shortfall helps us pin down the state price density endogenously at the interim date (recall that in the previous section, the state price density was given exogenously). To enable us to perform comparative statics and avoid other technical problems, we assume that the agent has access to a continuum of assets \( i \in \mathcal{I} \).

**Definition 1.** The shortfall in liquidation (at the interim time period) is \( S_1 = y_1 - \int l_{i,1} L_{i,1} \)

As long as \( \tilde{c} > 0 \), the shortfall in liquidation will always be greater than zero because of the agent’s concave preferences and the convexity of the failure cost function.

**Definition 2.** The agent’s wealth in the terminal period, as calculated in the interim period, is given by \( w_2 = \int l_{i,2} L_{i,2} - \kappa(S_1) \)

**Problem 2.** Given endowment \( w_0 \), liquidity shock \( c_1 \), asset liquidation values \( L_{i,t} \) and purchase prices \( P_{i,0} \), choose adapted purchases \( b_{i,0} \) and liquidation policy \( l_{i,t} \) (for \( t = 1, 2 \))

\[
\max_{S_i, l_{i,t}} u(c_0) + E u(w_2) \tag{3.1}
\]

subject to:

\[
w_2 = \int l_{i,2} L_{i,2} - \kappa(S_1) \tag{3.2}
\]

\[
S_1 = y_1 - \int l_{i,1} L_{i,1} \tag{3.3}
\]
\[ \int_i b_{i,0} P_{i,0} + c_0 \leq w_0 \]  

(3.4)

**Proposition 1.** (i) The solution to the agent’s Problem 2 yields a convex pricing operator which can be represented as:

\[ P_{i,0} = \mathbb{E}_0 \left( \max(\mathbb{E}_1[u'(w_2) L_{i,2}], \mathbb{E}_1\left[\frac{u'(w_2)}{u'(c_0)}] \kappa'(S_1)L_{i,1}\right) \right) \]

(ii) The optimal liquidation policy is a state contingent threshold \( l^* \equiv \frac{1}{\mathbb{E}_1[u'(w_2)]\kappa'(S_1)} \) which has the property that in every state, whenever \( \frac{L_{i,1}}{\mathbb{E}_1[u'(w_2)]L_{i,2}} \geq l^* \) the agent liquidates the asset \( i \).

(iii) The optimal shortfall \( \tilde{S}_1 \) is the unique solution to:

\[ S_1 = y_1 - \int_i l_{i,1}L_{i,1} \mathbb{I}_{\frac{L_{i,1}}{\mathbb{E}_1[u'(w_2)]L_{i,2}} \geq l^*} \]

**Proof.** Denoting \( \xi_1, \xi_1, \) and \( \xi_0 \) as Lagrange multipliers associated with the final wealth constraint, interim liquidity constraint and the budget constraint respectively.

\[ L = u(c_0) + \mathbb{E} u(w) \]

\[ + \xi_2 \mathbb{E}[w_2 - \int_i (b_{i,0} - l_{i,1})L_{i,2} - \kappa(S_1)] \]

\[ + \xi_1 \mathbb{E}[S_1 - (y_1 - \int_i l_{i,1}L_{i,1})] \]

\[ + \xi_0 \mathbb{E}[\int_i b_{i,0} P_{i,0} + c_0 - w_0] \]

From Theorem 1, we already have that

\[ \xi_0 P_{i,0} = \mathbb{E}_0[\max(\xi_1 L_{i,1}, \mathbb{E}_1[\xi_2 L_{i,2}])] \]

In addition, we can determine the Lagrange multipliers by taking FOCs with respect to

(i) \( \frac{\partial L}{\partial w_2} = 0 \) which implies that \( \xi_2 = u'(w_2) \)

(ii) \( \frac{\partial L}{\partial S_1} = 0 \) which implies that \( \xi_1 = \mathbb{E}_1[u'(w_2)]\kappa'(S_1) \)

(iii) \( \frac{\partial L}{\partial c_0} = 0 \) which implies that \( \xi_0 = u'(c_0) \)

We define a liquidation threshold \( \bar{l}^* \equiv \frac{1}{\mathbb{E}_1[u'(w_2)]\kappa'(S_1)} \) which has the property that in every
state, whenever $\frac{L_{i,1}}{E_1[u'(w_2) L_{i,2}]} \geq l^*$ we liquidate the asset. Thus the optimal liquidation policy can be characterized as follows:

$$ l_{i,1} = \begin{cases} 
0, & \text{if } \frac{L_{i,1}}{E_1[u'(w_2) L_{i,2}]} < l^* \\
1, & \text{if } \frac{L_{i,1}}{E_1[u'(w_2) L_{i,2}]} \geq l^* 
\end{cases} $$

The optimal shortfall is then,

$$ S_1 = y_1 - \int_i l_{i,1} L_{i,1} \mathbb{I} \frac{L_{i,1}}{E_1[u'(w_2) L_{i,2}]} \geq l^* $$

The above pricing rule is convex: Consider three assets $A$, $B$ and $C$ such that for some $a \in (0, 1)$, $L_{C,1} = aL_{A,1} + (1 - a)L_{B,1}$ and $L_{C,2} = aL_{A,2} + (1 - a)L_{B,2}$, clearly then $P_{C,0} \geq aP_{A,0} + (1 - a)P_{B,0}$.

We showcase the convex pricing rule through a simple example.

**Numerical Example:** Suppose wealth $w = 200$, terminal value for each asset $V = 100$ and liquidation value $L_1 = 30$, $L_2 = 50$, $L_3 = 100$. Let the liability (equivalently, the maximum potential liquidity shock) faced by the agent $c_{\text{max}} = 160$ with probability 0.375 and $c = 125$ with probability 0.625. Suppose that the agent faces infinite costs if he is unable to satisfy the liquidity demand (for eg. the agent faces bankruptcy). We find that $L^*(160) = 30$ and $\rho(160) = 3.33$. Additionally, $L^*(125) = 50$ and $\rho(125) = \frac{V}{L^*(125)} = 2$. Therefore, $P_3 = 0.625 \times \frac{2+100}{\gamma} + 0.375 \times \frac{3.33+100}{\gamma} = \frac{250}{\gamma}$. We can also derive $P_2 = \frac{125}{\gamma}$ and $P_1 = \frac{100}{\gamma}$. The most liquid asset is completely liquidated in two states, the intermediate asset is marginal when $c = 160$ while the most illiquid asset is liquidated only in the worst state. The pricing of the assets is piecewise linear and strictly convex in its liquidation value. Note that in a model without any interim liquidity shock, each of these assets would command the same price $P = 66.67$. This means that assets with low liquidation values trade at a discount compared to the ones with high liquidation options.
3.1 Application: Liquidity betas

We are interested in rules for pricing and hence, ordering collection of random variables \((L_{i,1}, L_{i,2})\) with different distributions over the liquidation value. To focus on the liquidation policy, we consider a simple case with linear utility function, a convex failure cost function \(\kappa(S) = S^2/2\) and a fixed terminal value \(L_{i,2} = 1\) for all assets. We also choose \(w_0\) such that the Lagrange multiplier for the budget constraint \(\xi_0 = 1\). This setup yields stochastic discount factors \(\xi_1 = S_1\) and \(\xi_2 = 1\). The pricing equation is:

\[
P_{i,0} = \mathbb{E} \left( \max(S_1 L_{i,1}, 1) \right)
\]

(3.5)

The liquidation policy threshold \(l^*\) for this case is given by \(l^* = \frac{1}{S_1}\). We define the absolute liquidation value threshold \(L^* \equiv \frac{1}{S_1}\) which has the property that in every state, whenever \(L_{i,1} \geq L^*\) we liquidate the asset \(i\). Thus the optimal liquidation policy can be characterized as follows:

\[
l_{i,1} = \begin{cases} 0, & \text{if } L_{i,1} < L^* \\ 1, & \text{if } L_{i,1} \geq L^* \end{cases}
\]

The threshold \(L^*\) is a unique solution to the following equation.

\[
\frac{1}{L^*} = y_1 - \int_i b_{i,0} L_{i,1} \mathbb{I}_{L_{i,1} \geq L^*}
\]

(3.6)

The pricing equation can be re-written as:

\[
P_{i,0} = \mathbb{E} \left( \max \left( \frac{L_{i,1}}{L^*}, 1 \right) \right)
\]

(3.7)

In what follows, we look at two comparative statics for \(L^*\). First, we compute the response of \(L^*\) to an increase in \(y_1\) in any given state:

\[
- \frac{1}{L^*} \frac{\partial L^*}{\partial y_1} = 1 + b_{i,0} L^* \frac{\partial L^*}{\partial y_1}
\]

(3.8)

It is clear from Equation 3.8 that \(\frac{\partial L^*}{\partial y_1} < 0\). That is, larger the liquidity shock in a particular state, lower is the liquidation threshold. Next, we fix the liquidation threshold emerging from Equation 3.8 at \(L^*_0\). We then analyze the impact of increasing liquidation value for all assets \(L_{i,1} \geq L^*_0\) in some state of the world keeping \(y_1\) fixed. As a consequence,
the total liquidation amount in said state $\int_i b_i, 0 L_{i, 1, i} 1_{L_{i, 1} \geq L_0^*}$ increases and the right hand side decreases. To shift the balance towards equality, we need to increase $L_0^*$ to another value $L_1^*$ so that both the left hand side decreases and the right hand side increases. The equilibrating value $L_1^*$ that maintains equality under the new liquidation values for all assets is greater than the original $L_0^*$.

From the above two comparative statics, we can glean that the liquidation value of an asset is valuable only when $L_{i,1} \geq L^*$ i.e. when the asset is indeed liquidated. In states when the asset is not liquidated, its value is equal to the terminal value (which in this case equals one). The ratio of the realized liquidation value in a particular state $L_{i,1}$ to the liquidation threshold $L^*$ serves as a proxy for a state contingent sensitivity parameter to the liquidation value of all assets. Averaging across states, $E(L_{i,1} / L^*)$ is representative of the liquidity beta: it measures the sensitivity of a liquid asset’s price to the shifts in market-wide liquidity (see for example, [Acharya and Pedersen 2005]). The liquidation threshold $L^*$ gives us an intuition about the dual sources of value for any asset $i$:

(i) From the liquidation policy equation, we find that $L^*$ is decreasing in $y_1$ i.e. the threshold liquidation value is lower in states where the liquidity shock is higher. In such states, if $L_{i,1}$ is greater than $L^*$, it enters the pricing equation and contributes a value greater than one. For this to happen, an asset $i$ should have a higher realized value of $L_{i,1}$ in states when the liquidity shock is high. A practical example of such an asset would be gold, which derives value from a high expected payoff in bad states of the world - even though the “intrinsic value” is greatly debated.

(ii) The threshold liquidation value $L^*$ is high in states when there are several assets with high liquidation values (keeping the liquidity shock $y_1$ fixed in such state). If an asset $i$ has a high realized liquidation value $L_{i,1}$ in states where all the other assets also have a high liquidation value, it is likely to be worth less. Conversely, for an asset to be valuable, it should supply liquidity precisely when all the other assets in the market fail to do so.

The above characterization re-iterates our earlier point about the option value aspect of liquidity. Like an option, a liquid asset is valuable in states where both the liquidity shock is high and there is a scarcity of other liquid assets in the economy. Moreover, the downsides of such an option are protected by the terminal value of such asset (no asset has value less than one). The liquidation option is also different from a stock option in the sense that its value depends not only on the realization of a primary state variable (the liquidity shock)
but also on the realized liquidation value of all other assets in the economy.

3.2 Application: Liability structure of a firm

So far, we have abstracted away from the source of the liquidity shock. Suppose the liquidity shock is due to funding structure of the firm, where \( y_{max} = \sup(y_1) \) (assumed finite) is represented as the total external liability of the firm and \( f(y_1) \) is the probability density that a random amount \( y_1 \) of the total liability is recalled at \( t = 1 \). Further, suppose that the firm is endowed with a portfolio of liquid and illiquid assets. The value of the portfolio, net of liabilities, represents the value of the firm. An agent compares asset prices (and hence, the value of the portfolio) under two regimes: Regime \( f \) and regime \( g \) where \( f(\cdot) \) and \( g(\cdot) \) are the probability distributions of the liquidity shock faced by the firm at the interim date \( t = 1 \). Further, suppose that \( f(\cdot) \) second order stochastically dominates \( g(\cdot) \) (i.e. \( g(\cdot) \) is a mean preserving spread of \( f(\cdot) \)). We study the impact of changes in the distribution of the liquidity shock due to the liability structure to the prices of assets held by the firm. To do so, we compare the impact of two distributions of the random liquidity demand: \( f(y_1) \) and \( g(y_1) \) on the agent’s reservation price of the asset. Then, we show that a risk-averse firm would prefer the regime \( f \).

**Corollary 1.** Let \( U'''(x) > 0 \) i.e. the marginal utility is convex. Then,

(i) If \( f(c) \) first order stochastically dominates \( g(c) \). Then

\[
P(L_{i,1}, L_{i,2}; f) \geq p(L_{i,1}, L_{i,2}; g)
\]

(ii) If \( f(c) \) second order stochastically dominates \( g(c) \) i.e. \( g(\cdot) \) is a mean preserving spread of \( f(\cdot) \), then

\[
P(L_{i,1}, L_{i,2}; f) \geq p(L_{i,1}, L_{i,2}; g)
\]

Risk averse firms should choose their liability structure such that the distribution of the liquidity shocks is concentrated around the mean. One method of making this happen is by borrowing from different independent sources, so that not all of its liabilities are called at once. This provides an explanation to conventional wisdom suggesting that firms should “broad base” their liabilities.
4 Conclusion

This paper studies a cash management model to investigate the effect of liquidity on asset prices. In a static model, we price liquidity as an option that is exercised through optimal liquidation in states where the investor suffers a liquidity shock. We derive threshold liquidation policies and convex pricing rules which can rank assets with uncertain payoffs in a liquidity context. Our pricing rule is consistent with empirical findings. An agent in our model is a firm or an asset manager. Adding institutional asset and liability details (for eg. a bank) can generate a richer set of predictions. The maximum size of the liquidity shock, which is treated as exogeneous in our economy, could serve as a proxy for the leverage choice a firm. Our findings can shape firm and regulatory policy for firms to (i) allocate wealth among assets of varying liquidity and (ii) choose a liability structure that best insulates the firm from liquidity shocks.
References


