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Adaptive Optimal Market Making Strategies with Inventory Liquidation Cost

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by

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Table of Contents

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Dedicated to My Family.

ABSTRACT OF THE THESIS

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Professor José E. Figueroa-López, Research Advisor

Along the lines of the paper [3], we find a general form of the optimal market making strategy for a high-frequency market maker (HFM) in a discrete-time Limit Order Book (LOB) model. Unlike [3], the optimal market making strategy is adaptive depending on the arrival of Market Order (MO) in the previous time intervals. We provide a method to make each placement of Limit Orders (LO) dependent on previous information in the same trading day and prove the admissibility of the optimal market making strategy under some general assumptions. Empirical study shows the adaptive optimal strategies outperform the non-adaptive strategy and those which place LOs at fixed distance from the midprice.

1. Introduction and Background

In this chapter, we introduce some basic concepts related to the problem we are investigating, such as the Limit Order Book (LOB), the Discrete-Time Optimal Control Problem, and the Dynamic Programming Theorem.

1.1 Limit Order Book

To trade the stock of a publicly listed company, individual investors, investment companies and other market participants will submit buy or sell orders into the electronic system of a stock exchange, such as the Nasdaq, indicating the volume of stock they want to buy or sell. There are two types of orders: **Limit Orders (LO)** and **Market Orders (MO)**. LOs come with a specified price level and the order will only be executed at that price or a better price level (for example, a buy LO can only be executed at a price less than or equal to the price specified in the LO). According to [1], Section 1.4, the **Limit Order Book (LOB)** at a given time is a record of the unexecuted LOs of a stock at that time. In a LOB, buy and sell LOs are recorded according to their price levels, and the price level for buy (sell) LOs are called bid (ask) price. Typically at a given time t, the ask prices are higher than the bid prices, and if we denote the lowest (highest) ask (bid) price as a_t (b_t), then the **Quoted Spread** and the **midprice** (S_t) are defined as:

$$\text{Quoted Spread}_t = a_t - b_t, \qquad S_t = \frac{1}{2}(a_t + b_t) \tag{1.1}$$

respectively. For intensively traded stocks, the Quoted Spread tends to be small and most LOs are concentrated near the best bid or ask price. Unlike LOs, MOs are orders that indicate the market participant wants to buy or sell certain amount of stocks at the best available prices. For example when a buy MO is entered into the electronic system, it will be matched with the sell LOs in the LOB starting from the best ask price, and for LOs at the same price level, the ones which are entered the earliest will be matched to the MO first. If the total volume of LOs at the best ask price is not enough to fulfill the MO, then the electronic system will start to match the remaining volume of the buy MO with the LOs at the second best ask price. The whole process (often called 'walking the book') will come to an end when the total volume of the MO is fulfilled. When a buy (sell) MO walks deeper into the book, which means the MO has exhausted the volume of LOs at a_t (b_t) and is matched to the sell (buy) LOs at the second or third best ask (bid) price, the midprice S_t will increase (decrease).

1.2 Discrete-Time Stochastic Optimal Control Problem

The following definition of a stochastic optimal control model comes from [2]. We changed the definition of $c(x, a)$ from one-stage cost function to one-stage revenue function since the optimization problem in this thesis seeks to maximize the revenue.

Definition 1.2.1 *A stochastic optimal control model is a five-tuple,*

$$(X, A, \{A(x) \subset A | x \in X\}, Q, c(x, a)) \tag{1.2}$$

where

- *X, State Space, is a nonempty Borel space (an element x in X is referred to as a state);*
- *A, Control Space, is a nonempty Borel space (an element a in A is referred to as a control or action);*

- *$\{A(x) \subset A | x \in X\}$ is a collection of nonempty measurable subsets of A, where $A(x)$ denotes the set of feasible controls or actions given state x. It has the property that the set*

$$\mathbb{K} = \{(x, a) | x \in X,\, a \in A(x)\} \tag{1.3}$$

is measurable in $X \times A$;

- *$Q = Q(\cdot|x, a)$ is a probability measure on X for each fixed $(x, a) \in \mathbb{K}$ and $Q(B|\cdot, \cdot)$ is a measurable function on $X \times A$ for each fixed Borel set $B \subset X$.*

- *$c(x, a) : \mathbb{K} \to \mathbb{R}$ is a measurable function called the one-stage revenue function.*

Let $\mathbb{K}^t$ denote the t-time Cartesian Product of the set $\mathbb{K}$ defined in (1.3) and for $t = 0,\, 1,\, 2,\, \ldots$, we define the space H_t as $H_0 = X$, and

$$H_t = \mathbb{K}^t \times X \quad \text{for} \quad t = 1,\, 2,\, \ldots \tag{1.4}$$

An element h_t of H_t is a vector of the form

$$h_t = (x_0, a_0, \ldots, x_{t-1}, a_{t-1}, x_t), \tag{1.5}$$

with $(x_i, a_i) \in \mathbb{K}$ for $i = 0, \ldots, t-1$ and $x_t \in X$. The following definition of randomized control policy comes from [2].

Definition 1.2.2 *A randomized control policy or, more briefly, a control policy, is a sequence $\pi = \{\pi_t,\, t = 0, 1, \ldots\}$ of stochastic kernels π_t, such that $\pi(\cdot|h_t)$ is a probability measure on A for any fixed $h_t \in H_t$ and $\pi(B|\cdot)$ is a measurable function on H_t given any Borel set $B \subset A$. Also, π_t satisfies the constraint*

$$\pi_t(A(x_t)|h_t) = 1 \quad \forall h_t \in H_t, \quad t = 1, 2, \ldots \tag{1.6}$$

For some positive integer N, a **Finite Horizon Discrete-Time Stochastic Optimal Control Problem** is to find a control policy $\pi = (\pi_0, \pi_1, \ldots, \pi_{N-1})$ such that

$$J(\pi, x) = \mathbb{E}\Big(\sum_{t=0}^{N-1} c(x_t, a_t) + c_N(x_N)|x_0 = x\Big) \tag{1.7}$$

attains its maximum for every $x \in X$. Here, c_N is a measurable function on X. Denote this optimal control policy as $\pi^* = (\pi_0^*, \pi_1^*, \ldots, \pi_{N-1}^*)$.

The finite horizon discrete-time stochastic optimal control problem can be solved using the Dynamic Programming Theorem, which gives a recursive algorithm for finding π^*. It turns out that the optimal control policy π^* is actually deterministic in the sense that for every $t \in 0, 1, \ldots, N-1$, there exists a measurable function $f_t(x)$ on X such that

$$f_t(x_t) \in A(x_t), \quad \pi_t(B|h_t) = I_B(f_t(x_t)) \tag{1.8}$$

for any $x_t \in X$ and Borel set $B \subset A$.

1.3 Dynamic Programming Theorem

We now give the statement of the **Dynamic Programming Theorem** (the proof of this theorem is similar to those given in [2], p24).

Theorem 1.3.1 *Define functions $J_N, J_{N-1}, \ldots, J_1, J_0$ on X inductively from $t = N$ to $t = 0$ by*

$$J_N(x) = c_N(x) \tag{1.9}$$

$$J_t(x) = \max_{a \in A(x)} [c(x, a) + \int_X J_{t+1}(y) Q(dy|x, a)] \quad t = N-1, N-2, \ldots, 0 \tag{1.10}$$

Suppose that these functions are measurable and that, for each $t = 0, \ldots, N-1$, there is a measurable function $f_t : X \to A$ such that $f_t(x) \in A(x)$ attains the maximum in (1.10) for all $x \in X$. Then the control policy $\pi^* = (f_0, f_1, \ldots, f_{N-1})$ is optimal, and

$$J(\pi^*, x) = J_0(x) \quad \forall x \in X. \tag{1.11}$$

If we define the expected revenue from time t to terminal time N as

$$C_t(\pi, x_t) = \mathbb{E}\Big(\sum_{n=t}^{N-1} c(x_n, a_n) + c_N(x_N)|h_t\Big), \quad t = 0, 1, \dots, N-1, \tag{1.12}$$

$$C_N(\pi, x) = c_N(x) \tag{1.13}$$

from previous theorem, it can be shown that

$$\begin{aligned} J_t(x_t) &= \max_{\pi} C_t(\pi, x_t) \\ &= \max_{(\pi_t, \dots, \pi_{N-1})} C_t(\pi, x_t) \\ &= \max_{a \in A(x_t)} [c(x_t, a) + \int_X \max_{(\pi_{t+1}, \dots, \pi_{N-1})} C_{t+1}(\pi, x_{t+1}) Q(dx_{t+1}|x_t, a)] \end{aligned} \tag{1.14}$$

From equation (1.14), it can be shown that

$$\max_{(\pi_t, \dots, \pi_{N-1})} \mathbb{E}(\sum_{n=0}^{N-1} c(x_n, a_n) + c_N(x_N)|h_t) = \max_{\pi_t} \mathbb{E}[\max_{(\pi_{t+1}, \dots, \pi_{N-1})} \mathbb{E}(\sum_{n=0}^{N-1} c(x_n, a_n) + c_N(x_N)|h_{t+1})|h_t] \tag{1.15}$$

this equation can be used to find the optimal control policy inductively.

2. Main Result

The model used in this thesis is a generalized version of the model used in [3], and the approach in this thesis is developed based on their original work. First, we describe the specific optimization problem, then we introduce the model and show how it can be fitted into the general framework of discrete-time stochastic optimal control problem in Chaper 1. At last, we find the optimal control policy using the Dynamic Programming Theorem and check its admissibility.

2.1 The Optimization Problem

In the stock market, a high-frequency market maker (HFM) is usually some financial company or fund who constantly submit buy and sell LOs into the trading system and cancel them if they can't be executed in a short period of time. The time gap between two consecutive inputs of LOs can be at the order of a few milliseconds.

Assume that in one trading day, the HFM only put LOs at finite predetermined times. A good strategy for placing these LOs will be to maximize the cash holdings and the value of stock holdings, as well as minimize the risk of holding too much stocks at the end of the day. The problem of finding the best strategy for a HFM falls into the realm of stochastic optimal control problem and in the next section, we setup the finite-horizon stochastic optimal control model corresponding to this problem.

2.2 The Finite-Horizon Stochastic Optimal Control Model

We modify the model in [3] to make the optimal control policy "adaptive" to the arrival of MOs in the previous time intervals. Specifically, instead of assuming the conditional probability of the arriving of MOs being fixed, we assume that the conditional probability will change according to the arrival of MOs in the preceding time intervals.

Assume the HFM places buy and sell LOs at times $0 = t_0 < t_1 < \cdots < t_N$ and assume $T = t_{N+1} > t_N$. The random variables used in the model are all defined on the same probability space $(\Omega, \mathbb{P}, \mathcal{F})$ with a filtration $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$, where $\mathcal{T} = \{t_0, t_1, \ldots, t_{N+1}\}$.

Let $\mathbb{1}^+_{t_{k+1}}$ ($\mathbb{1}^-_{t_{k+1}}$) be a random variable indicating whether there is at least one buy (sell) MO arriving during the time period $[t_k, t_{k+1})$:

$$\begin{aligned} \mathbb{1}^+_{t_{k+1}} &= \mathbb{1}_{\{\text{At least one buy MO arrives during } [t_k,t_{k+1})\}}, \\ \mathbb{1}^-_{t_{k+1}} &= \mathbb{1}_{\{\text{At least one sell MO arrives during } [t_k,t_{k+1})\}}. \end{aligned} \tag{2.1}$$

We assume that $\mathbb{1}^+_{t_{k+1}}, \mathbb{1}^-_{t_{k+1}} \in \mathcal{F}_{t_{k+1}}$. For some positive integer A, B, define

$$\begin{aligned} e^\pm_{t_k} &= (\mathbb{1}^\pm_{t_k}, \mathbb{1}^\pm_{t_{k-1}}, \ldots, \mathbb{1}^\pm_{t_{k-A+1}}) \in \{0,1\}^A, \\ e_{t_k} &= (e^+_{t_k}, e^-_{t_k}) \in \{0,1\}^{2A}. \end{aligned} \tag{2.2}$$

Let $g : \{0,1\}^{2A} \to \mathbb{R}^B$ be a measurable function, so we have $g(e_{t_k}) \in \mathcal{F}_{t_k}$.

If there's a buy (sell) MO arriving during $[t_k, t_{k+1})$, and if the price of sell (buy) LO that the HFM placed at time t_k is $L^+_{t_k}$ ($L^-_{t_k}$) higher (lower) than the midprice S_{t_k} at time t_k, we assume the executed volume of sell LO, denoted as $Q^+_{t_{k+1}}$, is given by

$$Q^+_{t_{k+1}} = \mathbb{1}^+_{t_{k+1}} c^+_{t_{k+1}} \left[(S_{t_k} + p^+_{t_{k+1}}) - (S_{t_k} + L^+_{t_k})\right] = \mathbb{1}^+_{t_{k+1}} c^+_{t_{k+1}} (p^+_{t_{k+1}} - L^+_{t_k}), \tag{2.3}$$

where $p^+_{t_{k+1}} \in \mathcal{F}_{t_{k+1}}$ is a random variable indicating the maximum depth buy MOs can walk into the sell LOB, and $c^+_{t_{k+1}} \in \mathcal{F}_{t_{k+1}}$ is a random variable such that $c^+_{t_{k+1}} p^+_{t_{k+1}}$ indicate the executed volume of sell LO if it is placed at the midprice ($L^+_{t_k} = 0$). From equation

(2.3), we can see that if $c^+_{t_{k+1}}, p^+_{t_{k+1}}$ are held fixed and $\mathbb{1}^+_{t_{k+1}} = 1$, $Q^+_{t_{k+1}}$ is a linear function of $L^+_{t_k}$. Similarly we can assume the executed volume of buy LO, denoted as $Q^-_{t_{k+1}}$, is given by

$$Q^-_{t_{k+1}} = \mathbb{1}^-_{t_{k+1}} c^-_{t_{k+1}} \left[(S_{t_k} - L^-_{t_k}) - (S_{t_k} - p^-_{t_{k+1}}) \right] = \mathbb{1}^-_{t_{k+1}} c^-_{t_{k+1}} (p^-_{t_{k+1}} - L^-_{t_k}), \tag{2.4}$$

where $p^-_{t_{k+1}} \in \mathcal{F}_{t_{k+1}}$ is a random variable indicating the maximum depth sell MOs can walk into the buy LOB, and $c^-_{t_{k+1}} \in \mathcal{F}_{t_{k+1}}$ is a random variable such that $c^-_{t_{k+1}} p^-_{t_{k+1}}$ indicate the executed volume of buy LO if it is placed at the midprice ($L^-_{t_k} = 0$). Next we introduce our main assumptions on $(S_{t_k}, \mathbb{1}^+_{t_{k+1}}, \mathbb{1}^-_{t_{k+1}}, c^+_{t_{k+1}}, p^+_{t_{k+1}}, c^-_{t_{k+1}}, p^-_{t_{k+1}})$, and some notation of their conditional expectation given $(\mathcal{F}_{t_k}, \mathbb{1}^+_{t_{k+1}}, \mathbb{1}^-_{t_{k+1}})$.

Assumption 1 *For $k = 0, \dots, N$, we have:*

1. $\mathbb{E}(S_{t_{k+1}} | \mathcal{F}_{t_k}) = S_{t_k}$.
2. $S_{t_{k+1}} - S_{t_k}$ *and* $(\mathbb{1}^+_{t_{k+1}}, \mathbb{1}^-_{t_{k+1}}, c^+_{t_{k+1}}, p^+_{t_{k+1}}, c^-_{t_{k+1}}, p^-_{t_{k+}})$ *are conditionally independent given* $\mathcal{F}_{t_k}$.
3. *The conditional distribution of* $(c^+_{t_{k+1}}, p^+_{t_{k+1}}, c^-_{t_{k+1}}, p^-_{t_{k+}})$ *given* $(\mathcal{F}_{t_k}, \mathbb{1}^+_{t_{k+1}}, \mathbb{1}^-_{t_{k+1}})$ *does not depend on k and is deterministic.*
4. $(c^+_{t_{k+1}}, p^+_{t_{k+1}})$ *and* $(c^-_{t_{k+1}}, p^-_{t_{k+1}})$ *are independent given* $(\mathcal{F}_{t_k}, \mathbb{1}^+_{t_{k+1}}, \mathbb{1}^-_{t_{k+1}})$.
5. *The conditional probabilities of the arrival of MO depend on $g(e_{t_k})$; i.e.,*

$$\pi^\pm_{t_{k+1}} := \mathbb{P}(\mathbb{1}^\pm_{t_{k+1}} = 1 | \mathcal{F}_{t_k}) = \mathbb{P}(\mathbb{1}^\pm_{t_{k+1}} = 1 | g(e_{t_k})) = f^\pm(g(e_{t_k})),$$
$$\pi_{t_{k+1}}(1,1) := \mathbb{P}(\mathbb{1}^+_{t_{k+1}} = 1, \mathbb{1}^-_{t_{k+1}} = 1 | \mathcal{F}_{t_k}) = \mathbb{P}(\mathbb{1}^+_{t_{k+1}} = 1, \mathbb{1}^-_{t_{k+1}} = 1 | g(e_{t_k})) = f(g(e_{t_k})), \tag{2.5}$$

where f, $f^\pm : \mathbb{R}^B \to [0,1]$ are measurable functions. Instead of being deterministic, which is the case in [3], $\pi^\pm_{t_{k+1}}, \pi_{t_{k+1}}(1,1) \in \mathcal{F}_{t_k}$ are random variables depending on $g(e_{t_k})$.

We use the same notation as [3] for the conditional expectations of $(c^+_{t_{k+1}}, p^+_{t_{k+1}}, c^-_{t_{k+1}}, p^-_{t_{k+1}})$:

$$
\begin{aligned}
\mu^\pm_c &:= \mathbb{E}(c^\pm_{t_{k+1}} | \mathcal{F}_{t_k}, \mathbb{1}^\pm_{t_{k+1}} = 1), \\
\mu^\pm_{c^2} &:= \mathbb{E}((c^\pm_{t_{k+1}})^2 | \mathcal{F}_{t_k}, \mathbb{1}^\pm_{t_{k+1}} = 1), \\
\mu^\pm_{cp} &:= \mathbb{E}(c^\pm_{t_{k+1}} p^\pm_{t_{k+1}} | \mathcal{F}_{t_k}, \mathbb{1}^\pm_{t_{k+1}} = 1), \\
\mu^\pm_{c^2p} &:= \mathbb{E}((c^\pm_{t_{k+1}})^2 p^\pm_{t_{k+1}} | \mathcal{F}_{t_k}, \mathbb{1}^\pm_{t_{k+1}} = 1), \\
\mu^\pm_{c^2p^2} &:= \mathbb{E}((c^\pm_{t_{k+1}} p^\pm_{t_{k+1}})^2 | \mathcal{F}_{t_k}, \mathbb{1}^\pm_{t_{k+1}} = 1).
\end{aligned}
\tag{2.6}
$$

To avoid the risk of holding too much stocks overnight, we use $W_T + S_T I_T - \lambda I_T^2$ as the performance criterion for our control policy, the strategy of placing LOs at different times. Here W_T and I_T respectively represent the cash holding and stock inventory at time T (see below for precise definition). λI_T^2, with λ being some positive number, is a penalizing term for holding too much inventory. So our optimal control problem is to find $L^\pm = (L^\pm_{t_0}, L^\pm_{t_1}, \dots, L^\pm_{t_N})$ that maximizes

$$
\mathbb{E}[W_T + S_T I_T - \lambda I_T^2]. \tag{2.7}
$$

We now show how our optimal control problem can be fitted into the general framework of the discrete-time stochastic optimal control problem in Chaper 1. For $k = 0, 1, \dots, N$, assume the $\sigma-$algebra $\mathcal{F}_{t_k}$ is generated by a random variable x_k and denote $a_k = (L^+_{t_k}, L^-_{t_k})$. Since

$$
W_{t_{k+1}} = W_{t_k} + (S_{t_k} + L^+_{t_k}) \mathbb{1}^+_{t_{k+1}} c^+_{t_{k+1}} (p^+_{t_{k+1}} - L^+_{t_k}) - (S_{t_k} - L^-_{t_k}) \mathbb{1}^-_{t_{k+1}} c^-_{t_{k+1}} (p^-_{t_{k+1}} - L^-_{t_k}), \tag{2.8}
$$

and

$$
I_{t_{k+1}} = I_{t_k} - \mathbb{1}^+_{t_{k+1}} c^+_{t_{k+1}} (p^+_{t_{k+1}} - L^+_{t_k}) + \mathbb{1}^-_{t_{k+1}} c^-_{t_{k+1}} (p^-_{t_{k+1}} - L^-_{t_k}), \tag{2.9}
$$

we can define the one-stage revenue function for $k = 1, 2, \dots, N$ as

$$
\begin{aligned}
c(x_k, a_k) &= \mathbb{E}\Big[W_{t_{k+1}} + S_{t_{k+1}} I_{t_{k+1}} - \lambda I^2_{t_{k+1}} - (W_{t_k} + S_{t_k} I_{t_k} - \lambda I^2_{t_k}) \,\Big|\, \mathcal{F}_{t_k}, L^+_{t_k}, L^-_{t_k} \Big] \\
&= \mathbb{E}\Big[W_{t_{k+1}} + S_{t_{k+1}} I_{t_{k+1}} - \lambda I^2_{t_{k+1}} - (W_{t_k} + S_{t_k} I_{t_k} - \lambda I^2_{t_k}) \,\Big|\, x_k, a_k \Big],
\end{aligned}
\tag{2.10}
$$

and define $c(x_0, a_0) = \mathbb{E}[W_{t_1} + S_{t_1} I_{t_1} - \lambda I_{t_1}^2 \,|\, x_0, a_0]$, $c_T(x_T) = c_{N+1}(x_{N+1}) = 0$. By Theorem 1.3.1, the optimal control problem defined in (1.7) is to find $(a_0, a_1, \ldots, a_N)$ that maximize

$$\begin{aligned}
&\mathbb{E}\Big(\sum_{k=0}^{N} c(x_k, a_k) + c_T(x_T) \Big| x_0 \Big) \\
=&\mathbb{E}\Big\{ \sum_{k=1}^{N} \mathbb{E}[W_{t_{k+1}} + S_{t_{k+1}} I_{t_{k+1}} - \lambda I_{t_{k+1}}^2 - (W_{t_k} + S_{t_k} I_{t_k} - \lambda I_{t_k}^2) \,|\, x_k, a_k] \\
&+ \mathbb{E}[W_{t_1} + S_{t_1} I_{t_1} - \lambda I_{t_1}^2 \,|\, x_0, a_0] \Big| x_0 \Big\} \\
=&\mathbb{E}[W_T + S_T I_T - \lambda I_T^2 \,|\, x_0],
\end{aligned} \tag{2.11}$$

which is the same as (2.7).

2.3 Finding the Optimal Control Policy

To use Theorem 1.3.1, we first define, as in equation (1.15), the value function

$$V_{t_k} = \sup_{(L_{t_k}^{\pm}, \ldots, L_{t_N}^{\pm}) \in \mathcal{A}} \mathbb{E}[W_T + S_T I_T - \lambda I_T^2 | \mathcal{F}_{t_k}], \qquad k = 0, 1, \ldots, N \tag{2.12}$$

where $\mathcal{A}$ is the set of all admissible control policies. As in [3], Theorem 1, it is natural to assume that V_{t_k} takes the following form

$$V_{t_k} = W_{t_k} + \alpha_{t_k} I_{t_k}^2 + S_{t_k} I_{t_k} + h_{t_k} I_{t_k} + g_{t_k}, \tag{2.13}$$

where α_{t_k}, h_{t_k}, $g_{t_k} \in \mathcal{F}_{t_k}$ are real valued random variables. We will determine these functions and show that our ansatz is correct. We use the following notation for the conditional expectations of $\alpha_{t_{k+1}}$ and $h_{t_{k+1}}$

$$\begin{aligned}
\alpha^0_{t_{k+1}} &:= \mathbb{E}[\alpha_{t_{k+1}} \,|\, \mathcal{F}_{t_k}], \\
\alpha^{1\pm}_{t_{k+1}} &:= \mathbb{E}[\alpha_{t_{k+1}} \,|\, \mathcal{F}_{t_k}, \mathbb{1}^{\pm}_{t_{k+1}} = 1], \\
\alpha^2_{t_{k+1}} &:= \mathbb{E}[\alpha_{t_{k+1}} \,|\, \mathcal{F}_{t_k}, \mathbb{1}^{+}_{t_{k+1}} = 1, \mathbb{1}^{-}_{t_{k+1}} = 1], \\
h^0_{t_{k+1}} &:= \mathbb{E}[h_{t_{k+1}} \,|\, \mathcal{F}_{t_k}], \\
h^{1\pm}_{t_{k+1}} &:= \mathbb{E}[h_{t_{k+1}} \,|\, \mathcal{F}_{t_k}, \mathbb{1}^{\pm}_{t_{k+1}} = 1], \\
h^2_{t_{k+1}} &:= \mathbb{E}[h_{t_{k+1}} \,|\, \mathcal{F}_{t_k}, \mathbb{1}^{+}_{t_{k+1}} = 1, \mathbb{1}^{-}_{t_{k+1}} = 1].
\end{aligned} \tag{2.14}$$

According to (1.15)

$$V_{t_k} = \sup_{L^{\pm}_{t_k} \in \mathcal{A}} \mathbb{E}[V_{t_{k+1}} \,|\, \mathcal{F}_{t_k}],$$

or equivalently,

$$\begin{aligned}
W_{t_k} + \alpha_{t_k} I^2_{t_k} + S_{t_k} I_{t_k} + h_{t_k} I_{t_k} + g_{t_k} = \sup_{L^{\pm}_{t_k} \in \mathcal{A}} \mathbb{E}[& W_{t_{k+1}} + \alpha_{t_{k+1}} I^2_{t_{k+1}} + S_{t_{k+1}} I_{t_{k+1}} \\
& + h_{t_{k+1}} I_{t_{k+1}} + g_{t_{k+1}} \,|\, \mathcal{F}_{t_k}]
\end{aligned} \tag{2.15}$$

Replacing $W_{t_{k+1}}$ and $I_{t_{k+1}}$ by (2.8) and (2.9), the conditional expectation on the right hand side of (2.15) is a quadratic function of $(L^{+}_{t_k}, L^{-}_{t_k})$. By taking the first and second order partial derivative, we can find the $(L^{+,*}_{t_{k+1}}, L^{-,*}_{t_{k+1}})$ that maximizes the right hand side of (2.15). We state the conclusion in the following proposition and give the proof in Appendix A.2.

Propisition 1 *The optimal control policy that solves the optimization problem (2.7) are given, for $k = 0, \ldots, N$, by*

$$\begin{aligned}
L^{+,*}_{t_k} &= {}^{(1)}A^{+}_{t_k} I_{t_k} + {}^{(2)}A^{+}_{t_k} + {}^{(3)}A^{+}_{t_k}, \\
L^{-,*}_{t_k} &= -{}^{(1)}A^{-}_{t_k} I_{t_k} - {}^{(2)}A^{-}_{t_k} + {}^{(3)}A^{-}_{t_k},
\end{aligned} \tag{2.16}$$

where the coefficients above are specified as

$$
\begin{aligned}
{}^{(1)}A^{\pm}_{t_k} &= \frac{\beta^{\pm}_{t_k}}{\gamma_{t_k}}, \quad {}^{(2)}A^{\pm}_{t_k} = \frac{\eta^{\pm}_{t_k}}{2\gamma_{t_k}}, \\
{}^{(3)}A^{\pm}_{t_k} &= \frac{\pi^{\mp}_{t_{k+1}}}{2\gamma_{t_k}}(\alpha^{1\mp}_{t_{k+1}}\mu^{\mp}_{c^2} - \mu^{\mp}_c)\left[\pi^{\pm}_{t_{k+1}}(\mu^{\pm}_{cp} - 2\alpha^{1\pm}_{t_{k+1}}\mu^{\pm}_{c^2p}) + 2\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1)\mu^{\pm}_c\mu^{\mp}_{cp}\right] \\
&\quad + \pi_{t_{k+1}}(1,1)\frac{\alpha^2_{t_{k+1}}}{2\gamma_{t_k}}\mu^+_c\mu^-_c\left[\pi^{\mp}_{t_{k+1}}(\mu^{\mp}_{cp} - 2\alpha^{1\mp}_{t_{k+1}}\mu^{\mp}_{c^2p}) + 2\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1)\mu^{\mp}_c\mu^{\pm}_{cp}\right],
\end{aligned}
\tag{2.17}
$$

and

$$
\begin{aligned}
\gamma_{t_k} &:= \left[\pi_{t_{k+1}}(1,1)\alpha^2_{t_{k+1}}\mu^+_c\mu^-_c\right]^2 - \pi^+_{t_{k+1}}\pi^-_{t_{k+1}}(\alpha^{1+}_{t_{k+1}}\mu^+_{c^2} - \mu^+_c)(\alpha^{1-}_{t_{k+1}}\mu^-_{c^2} - \mu^-_c), \\
\beta^{\pm}_{t_k} &:= \alpha^{1\pm}_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}}\mu^{\pm}_c(\alpha^{1\mp}_{t_{k+1}}\mu^{\mp}_{c^2} - \mu^{\mp}_c) - \alpha^{1\mp}_{t_{k+1}}\pi^{\mp}_{t_{k+1}}\pi_{t_{k+1}}(1,1)\alpha^2_{t_{k+1}}\mu^{\pm}_c(\mu^{\mp}_c)^2, \\
\eta^{\pm}_{t_k} &:= h^{1\pm}_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}}\mu^{\pm}_c(\alpha^{1\mp}_{t_{k+1}}\mu^{\mp}_{c^2} - \mu^{\mp}_c) - h^{1\mp}_{t_{k+1}}\pi^{\mp}_{t_{k+1}}\pi_{t_{k+1}}(1,1)\alpha^2_{t_{k+1}}\mu^{\pm}_c(\mu^{\mp}_c)^2.
\end{aligned}
$$

In the expressions above, α_{t_k} and h_{t_k} are specified using the following backward equations: $\alpha_T = -\lambda$, $h_T = 0$ at $T = t_{N+1}$ and, for $k = 0, \dots, N$:

$$
\begin{aligned}
\alpha_{t_k} = \alpha^0_{t_{k+1}} &+ \sum_{\delta=\pm}\pi^{\delta}_{t_{k+1}}\left[(\alpha^{1\delta}_{t_{k+1}}\mu^{\delta}_{c^2} - \mu^{\delta}_c)({}^{(1)}A^{\delta}_{t_k})^2 + 2\alpha^{1\delta}_{t_{k+1}}\mu^{\delta}_c({}^{(1)}A^{\delta}_{t_k})\right] \\
&+ 2\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1)\mu^+_c\mu^-_c({}^{(1)}A^+_{t_k}\,{}^{(1)}A^-_{t_k}),
\end{aligned}
\tag{2.18}
$$

and

$$
\begin{aligned}
h_{t_k} = h^0_{t_{k+1}} &+ \sum_{\delta=\pm}\pi^{\delta}_{t_{k+1}}\Big\{2(\alpha^{1\delta}_{t_{k+1}}\mu^{\delta}_{c^2} - \mu^{\delta}_c)\left[{}^{(1)}A^{\delta}_{t_k}(\delta\,{}^{(3)}A^{\delta}_{t_k} + {}^{(2)}A^{\delta}_{t_k})\right] \\
&+ 2\alpha^{1\delta}_{t_{k+1}}\mu^{\delta}_c(\delta\,{}^{(3)}A^{\delta}_{t_k} + {}^{(2)}A^{\delta}_{t_k}) - 2\alpha^{1\delta}_{t_{k+1}}(\delta\mu^{\delta}_{cp}) + (\delta\,{}^{(1)}A^{\delta}_{t_k})(\mu^{\delta}_{cp} + \delta h^{1\delta}_{t_{k+1}}\mu^{\delta}_c - 2\alpha^{1\delta}_{t_{k+1}}\mu^{\delta}_{c^2p})\Big\} \\
&- 2\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1)\mu^+_c\mu^-_c\left[{}^{(1)}A^+_{t_k}({}^{(3)}A^-_{t_k} - {}^{(2)}A^-_{t_k}) - {}^{(1)}A^-_{t_k}({}^{(2)}A^+_{t_k} + {}^{(3)}A^+_{t_k}) + \frac{\mu^+_{cp}}{\mu^+_c}({}^{(1)}A^-_{t_k}) - \frac{\mu^-_{cp}}{\mu^-_c}({}^{(1)}A^+_{t_k})\right].
\end{aligned}
\tag{2.19}
$$

The following lemma will be needed to show that the critical point $(L^{+,*}_{t_k}, L^{-,*}_{t_k})$ of Proposition 1 is indeed a maximum point. Its proof is deferred to Appendix A.1.

Lemma 1 *The quantity α_{t_k} defined in equation (2.18) is negative and $\alpha_{t_k} > \alpha^0_{t_{k+1}}$ for every $k = 0, \dots, N$.*

Denote a_{t_k} (b_{t_k}) as the price of sell (buy) LOs placed at time t_k under the optimal control policy. So we have

$$\begin{aligned} a_{t_k} &= S_{t_k} + L^{+,*}_{t_k} = S_{t_k} + {}^{(1)}A^{+}_{t_k} I_{t_k} + {}^{(2)}A^{+}_{t_k} + {}^{(3)}A^{+}_{t_k}, \\ b_{t_k} &= S_{t_k} - L^{-,*}_{t_k} = S_{t_k} + {}^{(1)}A^{-}_{t_k} I_{t_k} + {}^{(2)}A^{-}_{t_k} - {}^{(3)}A^{-}_{t_k}. \end{aligned} \tag{2.20}$$

In the next proposition, we provide conditions under which the bid-ask spread is guaranteed to be positive (i.e., $a_{t_k} > b_{t_k}$). We defer the proof to Appendix A.3.

Propisition 2 *Under Assumption 1, the optimal control policy yields positive spreads at all times (i.e., $a_{t_k} > b_{t_k}$, for all $k = 0, \ldots, N$), provided that the following three conditions hold:*

(1) The first and second conditional moments of $c^{\pm}$ defined in (2.6) satisfy

$$\mu_c := \mu_c^+ = \mu_c^-, \quad \mu_{c^2} := \mu_{c^2}^+ = \mu_{c^2}^-. \tag{2.21}$$

(2) Buy and sell market orders arrive with the same probability:

$$\pi^+_{t_{k+1}} = \pi^-_{t_{k+1}}, \tag{2.22}$$

For any $a, b \in \{0,1\}^A$, exchanging finite many coordinates of a with the corresponding coordinates of b to get a' and b', we have

$$\begin{aligned} \mathbb{P}(\mathbb{1}^{\pm}_{t_{k+1}} = 1 | e^+_{t_k} = a, e^-_{t_k} = b) &= \mathbb{P}(\mathbb{1}^{\pm}_{t_{k+1}} = 1 | e^+_{t_k} = a', e^-_{t_k} = b'), \\ \mathbb{P}(\mathbb{1}^{+}_{t_{k+1}} = 1 \mathbb{1}^{-}_{t_{k+1}} = 1 | e^+_{t_k} = a, e^-_{t_k} = b) &= \mathbb{P}(\mathbb{1}^{+}_{t_{k+1}} = 1 \mathbb{1}^{-}_{t_{k+1}} = 1 | e^+_{t_k} = a', e^-_{t_k} = b'). \end{aligned} \tag{2.23}$$

(3) The conditional expectations of $(cp)^{\pm}$ and $(c^2p)^{\pm}$ defined in (2.6) satisfy

$$\mu^{\pm}_{cp} = \mu^{\pm}_c \mu^{\pm}_p, \quad \mu^{\pm}_{c^2p} = \mu^{\pm}_{c^2} \mu^{\pm}_p, \tag{2.24}$$

where $\mu^{\pm}_p := \mathbb{E}(p^{\pm}_{t_{k+1}} | \mathcal{F}_{t_k}, \mathbb{1}^{\pm}_{t_{k+1}} = 1)$.

3. Empirical Study on Actual LOB Data

In this chapter, we test the optimal control policy obtained in (2.16) of Proposition 1 with real LOB data of Microsoft, whose stock is listed as MSFT on Nasdaq. We test 3 different optimal control policies, which adapt to different random variables $g(e_{t_k})$, where e_{t_k} is defined in (2.2) with $A = 3, 4$. Specifically, $g(e_{t_k})$ is defined as follows:

$$\begin{aligned}
&g_1 : \{0,1\}^6 \to \{0,1\}^6, \quad g_1(e_{t_k}) = e_{t_k} \\
&g_2 : \{0,1\}^6 \to \{0,1,2\}^3, \quad g_2(e_{t_k}) = (\mathbb{1}^+_{t_k} + \mathbb{1}^-_{t_k}, \mathbb{1}^+_{t_{k-1}} + \mathbb{1}^-_{t_{k-1}}, \mathbb{1}^+_{t_{k-2}} + \mathbb{1}^-_{t_{k-2}}) \\
&g_3 : \{0,1\}^8 \to \{0,1,2\}^4, \quad g_3(e_{t_k}) = (\mathbb{1}^+_{t_k} + \mathbb{1}^-_{t_k}, \mathbb{1}^+_{t_{k-1}} + \mathbb{1}^-_{t_{k-1}}, \mathbb{1}^+_{t_{k-2}} + \mathbb{1}^-_{t_{k-2}}, \mathbb{1}^+_{t_{k-3}} + \mathbb{1}^-_{t_{k-3}})
\end{aligned} \tag{3.1}$$

In the first section we explain how the data is preprocessed and set up the test environment. In the second section we state the parameter estimation procedure and check if Assumption 1 and the conditions in Proposition 2 are satisfied. In the last section, we present the result of the policy and compare it with "benchmark" control policies that places LOs at fixed ask (bid) price levels.

3.1 Data Preprocessing and environment setup

The data we use, which was obtained from [3], is the daily LOB data of MSFT for every trading day (252 days in total) of year 2019. This is the same dataset as in the paper [3]. For each day there are two datasets, the "book" dataset record the state of the LOB at different times of a trading day. Whenever a change occurs in the LOB, a new row of data is added to record the new volume of LOs at the 20 best ask and bid price levels. The "message" dataset record the reason of every change of the LOB. Whenever a new row is added to the book dataset, correspondingly, a new row of data with the same time index will be added to the message dataset, recording the direction (buy or sell), price, executed volume and type (MO, LO or canceling of outstanding LO) of the order. Since we only use the LOB changes caused by execution of buy and sell MOs, we first divide the type of message dataset into three parts: buy MO, sell MO and others, then we combine consecutive buy (sell) MOs at the same price level into one entry with volume being the sum of volumes of the combined MOs.

The HFM places LOs every second from 10:00 am to 3:30 pm, so the HFM will place LOs 19800 times every trading day. We assume each LO placed by the HFM has a fixed volume of 500 shares and it will be executed ahead of other LOs at the same price level in the LOB. On Nasdaq, the minimal price difference (tick size) is one cent, so the prices of sell (buy) LOs placed under the optimal control policy will be rounded up (down) to nearest available price level.

On each testing day, We place the LOs according to the optimal control policy and using the message dataset to determine how many shares of the LOs will be executed. the parameters in the optimal control policy, namely $\pi^{\pm}_{t_{k+1}}, \pi_{t_{k+1}}(1, 1)$ and the conditional expectation related to $(c^{\pm}, p^{\pm})$ defined in (2.6), will be estimated using the data from previous 20 days. So the first testing day will start from the 21st trading day of 2019.

3.2 Parameter Estimation

Estimation of $\pi^{\pm}_{t_{k+1}}, \pi_{t_{k+1}}(1, 1)$. Since the functions f, $f^{\pm}$, g in (2.5) don't change with time, we can use simple sample proportions to estimate $\pi^{\pm}_{t_{k+1}}$ and $\pi_{t_{k+1}}(1, 1)$. To estimate $\pi^{\pm}_{t_{k+1}}$ and $\pi_{t_{k+1}}(1, 1)$ for the jth trading day, we define I_j to be the collection of all indicators $\mathbb{1}^{\pm}_{t_{k+1}}$ in the previous 20 days. For any $x \in \mathbb{R}^B$, define $I^{\pm}_x = \{\mathbb{1}^{\pm}_{t_{k+1}} \in I_j : g(e_{t_k}) = x, k = 0, 1, \ldots, N\}$ and $I_x = \{\mathbb{1}^{+}_{t_{k+1}}\mathbb{1}^{-}_{t_{k+1}} \in I_j : g(e_{t_k}) = x, k = 0, 1, \ldots, N\}$, then $\pi^{\pm}_{t_{k+1}}$ and $\pi_{t_{k+1}}(1, 1)$ can be estimated by

$$
\begin{aligned}
\bar{\pi}^{\pm}_{t_{k+1}} &= f^{\pm}(g(e_{t_k}) = x) = \frac{\sum_{z \in I^{\pm}_x} z}{\# I^{\pm}_x} \\
\bar{\pi}_{t_{k+1}}(1, 1) &= f(g(e_{t_k}) = x) = \frac{\sum_{z \in I_x} z}{\# I_x}
\end{aligned}
\tag{3.2}
$$

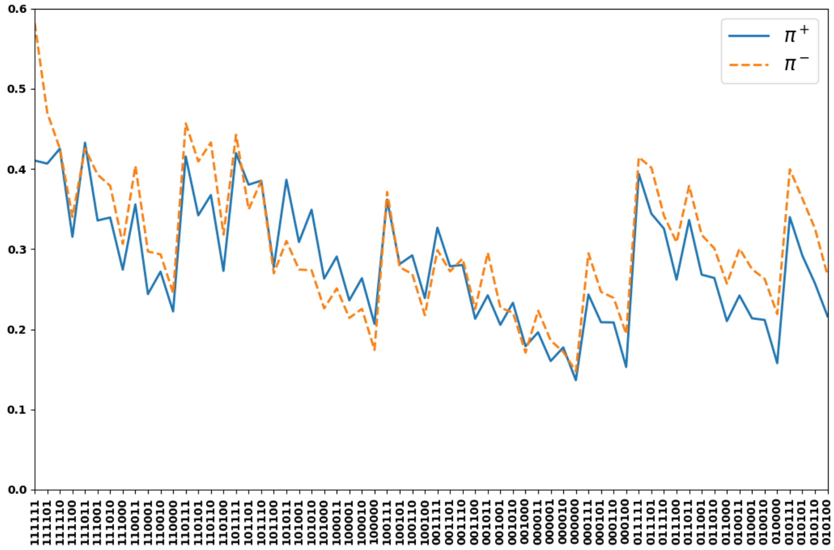
where $\# I^{\pm}_x$, $\# I_x$ are the number of elements in $I^{\pm}_x$, I_x.

To check if Condition (2) in Proposition 2 is satisfied, we first check if $\pi^{+}_{t_{k+1}} = \pi^{-}_{t_{k+1}}$. For different control policy, We plot the value of $\pi^{\pm}_{t_{k+1}}$ for different value of $g(e_{t_k})$ and compare if they are significantly different. As shown in Figure 3.1, for most values of $g(e_{t_k})$, the values of $\pi^{+}_{t_{k+1}}$ and $\pi^{-}_{t_{k+1}}$ are similar.

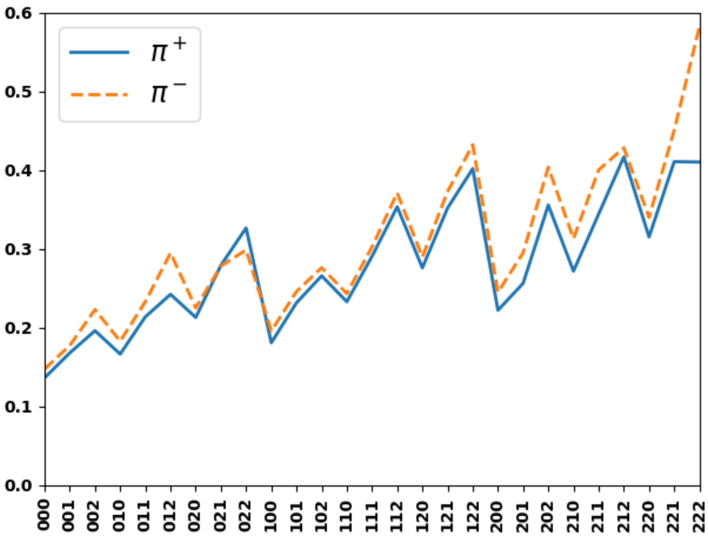
We now check if the second part (Equation (2.23)) of Condition (2) in Proposition 2 is satisfied. Since for the control policies conditioning on g_2 and g_3, this condition holds trivially, we only need to check it for the control policy conditioning on g_1. For any $a, a', b, b' \in \{0, 1\}^3$ and $x = (a, b), y = (a', b') \in \{0, 1\}^6$, denote $x \sim y$ if we can exchange finite many coordinates of a with the corresponding coordinates of b to get a' and b'. Clearly $\sim$ is a equivalence relation on $\{0, 1\}^6$ and (2.23) in Condition (2) of Proposition 2 is satisfied if and only if $\pi^{\pm}_{t_{k+1}}$ and $\pi_{t_{k+1}}(1, 1)$, as functions defined on $\{0, 1\}^6$, take the same value on a equivalence class. For any $x = (x_1, x_2, \ldots, x_6) \in \{0, 1\}^6$, define

$$
s(x) = (x_1 + x_4, x_2 + x_5, x_3 + x_6) = (s_1, s_2, s_3) \in \{0, 1, 2\}^3, \tag{3.3}
$$

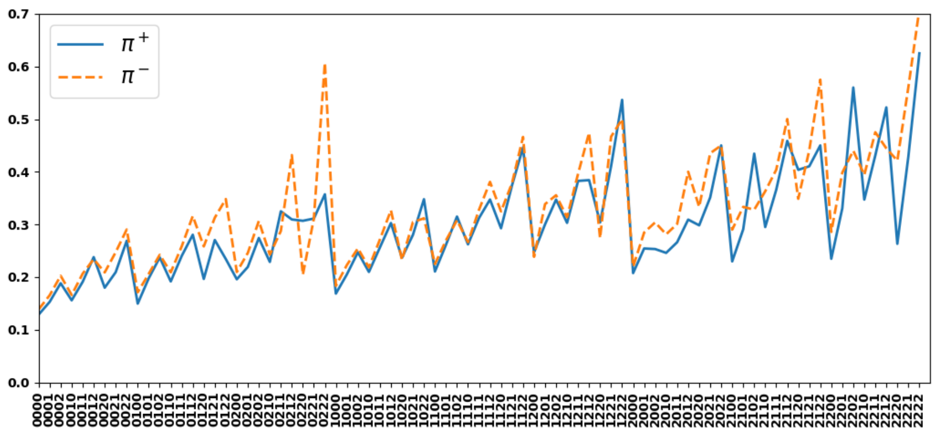
then $x \sim y$ if and only if $s(x) = s(y)$, so there are 27 equivalence classes. On each equivalence class, we calculate the difference between the maximum and minimum values of $\pi^{\pm}_{t_{k+1}}, \pi_{t_{k+1}}(1, 1)$ and as shown in Figure 3.2, the difference is smaller than 0.05 in most equivalence classes, so it's reasonable to say the second part (Equation (2.23)) of Condition (2) in Proposition 2 is satisfied.



(a) $\pi^{\pm}$ as functions of g_1.



(b) $\pi^{\pm}$ as functions of g_2.



(c) $\pi^{\pm}$ as functions of g_3.

Figure 3.1.

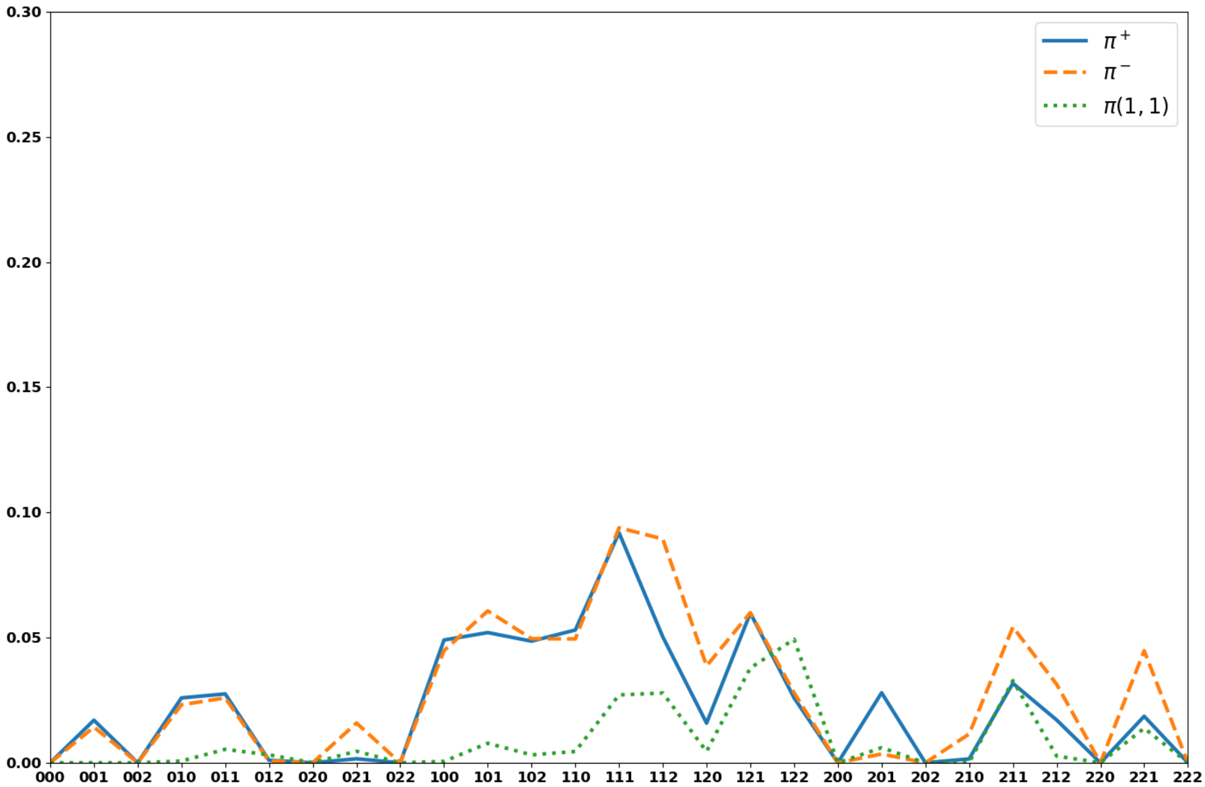


Figure 3.2.: The difference between the maximum and minimum values of $\pi^{\pm}_{t_{k+1}}, \pi_{t_{k+1}}(1,1)$ on 27 equivalence classes

Demands Function. To estimate the conditional expectations defined in (2.6), we first estimate $c^{\pm}, p^{\pm}$ in each 1-second time interval using weighted linear regression. By [3], p28, all $(c^{\pm}_{\cdot}, p^{\pm}_{\cdot})$-related time series defined in (2.6) are reasonably stationary (which means the second assumption in Assumption 1 are satisfied), so we can take the average of all estimated values in one day to get a daily estimate. The conditional expectations related to $c^{\pm}, p^{\pm}$ in the optimal control policy is the 20-day average of the daily estimate.

We now give an example on how to estimate c^+, p^+ in the 1-second time interval $[t, t+1)$. Assume there are k buy MOs executed during this time interval at price $p_1 \leq p_2 \leq, \cdots, \leq p_k$ and with volume v_i, $i = 1, 2, \ldots, k$. We only consider buy MOs executed at a price level higher than or equal to the midprice, and since the midprice is always integer times half tick size, we assume all p_i are integers indicating the number of half tick sizes higher than the midprice. For $j = 0, 1, \ldots, p_k$, a sell LO placed at j half tick size higher than the midprice will be executed by

$$y_j = \sum_{i:p_i \geq j} v_i \tag{3.4}$$

We now do weighted linear regression of $(y_0, y_1, \ldots, y_{p_k}, 0)$ on $(0, \frac{1}{2}, 1, \ldots, p_k/2, \frac{1}{2}(p_k + 1))$ to get the slope and intercept. From (2.3), we can estimate c^+ by $-$(slope) and p^+ by $-$(intercept/slope).

To see if the linear model in (2.3) and (2.4) is a good approximation of the actual volume of shares executed, we plot the average of $(y_0, y_1, \ldots, y_{p_k}, 0)$ and the regression line with slope and intercept as the average of all slopes and intercepts estimated in each 1 second time interval for one trading day. As shown in Figure 3.3, the linear model is a good approximation of the actual volume for prices close to the midprice, where most of the MOs are executed.

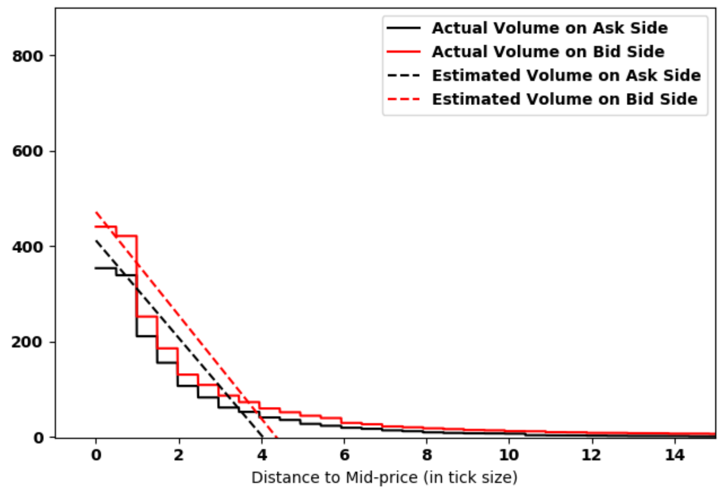


Figure 3.3.: Plot of Actual Demand on October 3rd vs. Estimated Linear Demand over a 1-Second Trading Interval

To check if (2.21) and (2.24) in Proposition 2 holds, we calculate the average of all daily estimates of $c^{\pm}, p^{\pm}$. As shown in Table 3.1, (2.21) and (2.24) in Proposition 2 are largely satisfied.

$\bar{\mu}_c^+ = 125.512$	$\bar{\mu}_c^- = 130.622$
$\bar{\mu}_p^+ = 3.287$	$\bar{\mu}_p^- = 3.292$
$\bar{\mu}_{cp}^+ = 451.263$	$\bar{\mu}_{cp}^- = 471.685$
$\bar{\mu}_{c^2}^+ = 8.47 \times 10^4$	$\bar{\mu}_{c^2}^- = 5.45 \times 10^4$
$\bar{\mu}_{c^2p}^+ = 3.53 \times 10^5$	$\bar{\mu}_{c^2p}^- = 2.25 \times 10^5$

Table 3.1: Average Values of $c^{\pm}, p^{\pm}$ related mean over 252 Trading Days in 2019.

3.3 Results for 3 Optimal Control Policies

In this section we calculate the performance criterion $W_T + S_T I_T - \lambda I_T^2$ for 3 optimal control policies based on the actual LOB data of MSFT in year 2019. As suggested by [3], Section 5.2, we set $\lambda = 0.0005$. On each testing day, we place LOs according to the optimal control policies, and the cash holding W_{t_k}, stock inventory I_{t_k} are given by

$$W_{t_{k+1}} - W_{t_k} = a_{t_k} \widetilde{Q}^+_{t_{k+1}} - b_{t_k} \widetilde{Q}^-_{t_{k+1}},$$

$$I_{t_{k+1}} - I_{t_k} = -\widetilde{Q}^+_{t_{k+1}} + \widetilde{Q}^-_{t_{k+1}},$$

where a_{t_k}, b_{t_k} (defined in (2.20)) are the price of sell (buy) LO placed at time t_k, and $\widetilde{Q}^+_{t_{k+1}}$ ($\widetilde{Q}^-_{t_{k+1}}$) are the executed volume of sell (buy) LO calculated using the message dataset.

Distribution of Performance Criterion Table 3.2 reports the means and standard deviations of the performance criterion $W_T + S_T I_T - \lambda I_T^2$ under different strategies. 'Level 1'- 'Level 6' represent the benchmark policies that place LOs at a fixed level (ith level being the prices in LOB ith closest to the midprice) in the LOB. Figure 3.4 shows the

	Optimal Control Policy Conditioning on $g_1(e_{t_k})$		Optimal Control Policy Conditioning on $g_2(e_{t_k})$		Optimal Control Policy Conditioning on $g_3(e_{t_k})$	
Mean	8.36×10^4		8.50×10^4		8.57×10^4	
Std.	1.63×10^6		1.63×10^6		1.62×10^6	
	Level 1	Level 2	Level 3	Level 4	Level 5	Level 6
Mean	-7.78×10^6	-9.99×10^5	-1.14×10^5	-3.64×10^4	-5.16×10^4	-3.69×10^4
Std.	1.52×10^7	4.49×10^6	2.01×10^6	1.07×10^6	7.16×10^5	4.81×10^5

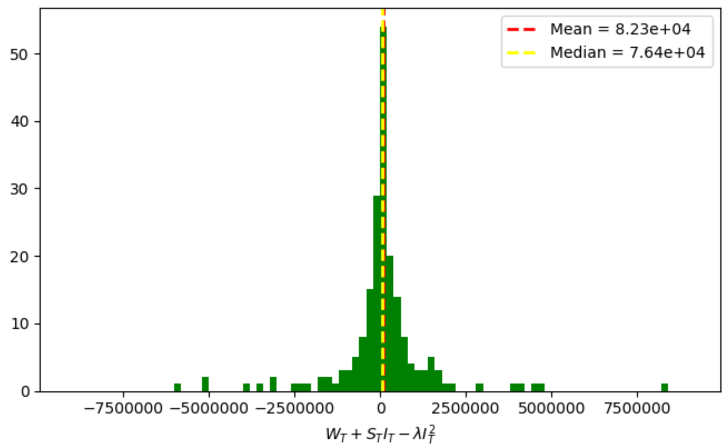
Table 3.2: Mean and Std. of the performance criterion $W_T + S_T I_T - \lambda I_T^2$ over 232 days.

histogram of the performance criterion $W_T + S_T I_T - \lambda I_T^2$ for 232 trading days of year 2019.

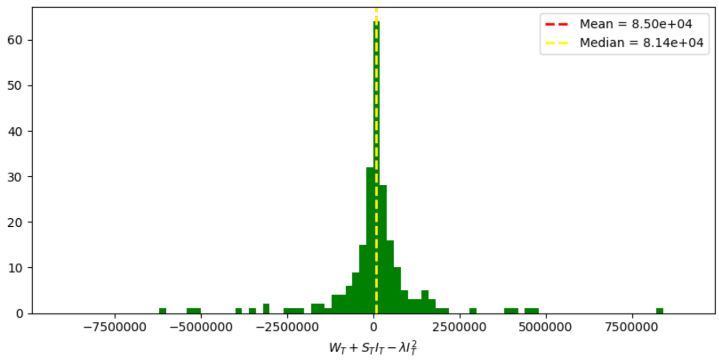
Based on the results of Table 3.2 and Figure 3.4, we can conclude that the optimal control policy outperform the fixed level 1-level 6 policies. From Table 3.2 we can see the mean and standard deviation for all three policies are close and the policy conditioning on $g_3(e_{t_k})$ has the highest mean and lowest standard deviation. As shown in Figure 3.4, the values of the performance criterion for all three policies concentrates around zero.

Control on Terminal Inventory. Figure 3.5 and Figure 3.6 shows the intraday price and inventory paths of the optimal control policies conditioning on $g_1(e_{t_k})$ and $g_3(e_{t_k})$ compared with the 'Level 1'- 'Level 6' policies for August 7th. The price paths are for three 1-minute time intervals at the beginning of the trading day 10:00−10:01, in the middle of the trading day 12:45− 12:46, and at the end of the trading day 15:29−15:30, respectively.

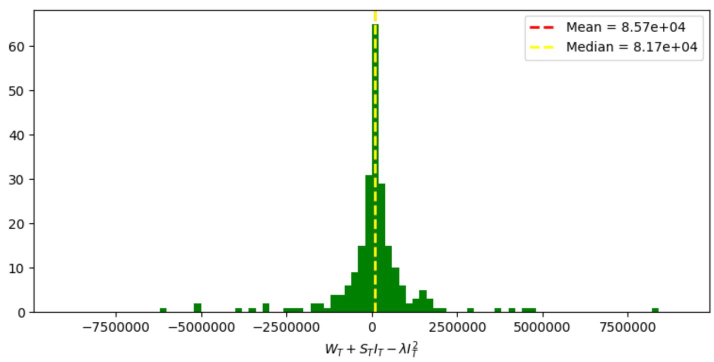
Both control policies have similar price and inventory paths. As we can see from Figure 3.5a, the optimal price for policy conditioning on $g_1(e_{t_k})$ swings between prices



(a) Optimal Control Policy Conditioning on $g_1(e_{t_k})$



(b) Optimal Control Policy Conditioning on $g_2(e_{t_k})$

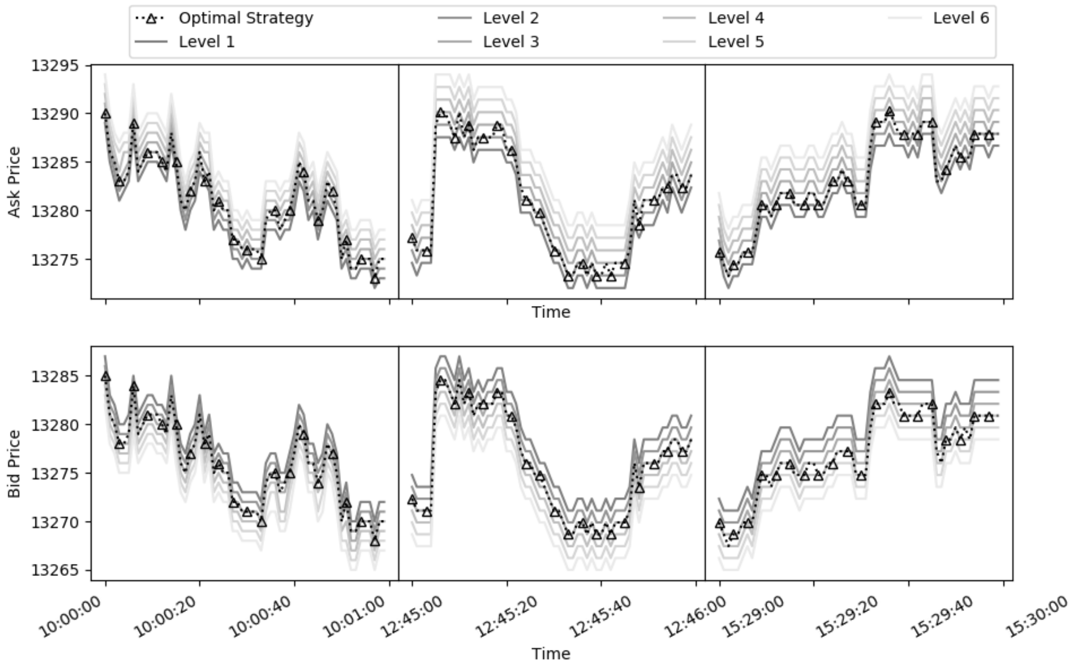


(c) Optimal Control Policy Conditioning on $g_3(e_{t_k})$

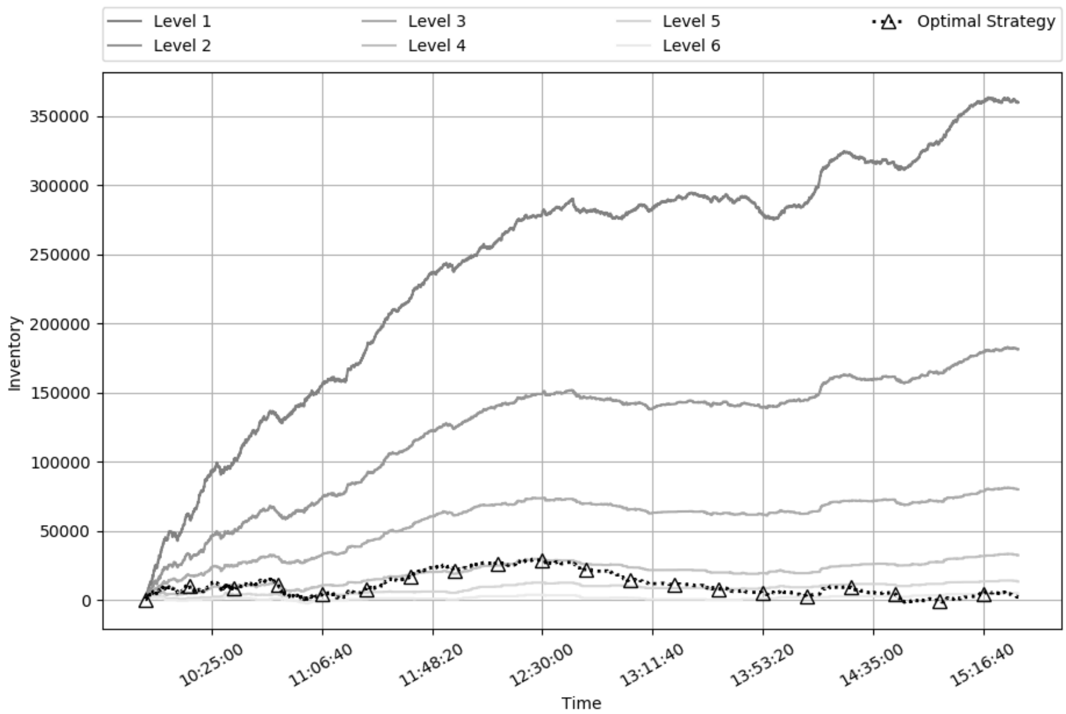
Figure 3.4.: Histogram of the performance criterion obtained from the optimal control policies in year 2019 (232 trading days included). We compute the performance criterion for each trading day starting from the 21st trading day. In each day, we use the prior 20 days to estimate parameters.

for level 1 and level 6. At the end of the trading day, the optimal ask price is more close to level 1 price and the optimal bid price is more close to the level 6 price, which leads to a decrease in the stock inventory.

For both optimal control policies, the end-of-day inventory is lower than the ‘Level 1’- ‘Level 6’ policies. This shows the effectiveness of the liquidation penalty $-\lambda I_T^2$ in controlling inventory and avoiding large end of the day costs.

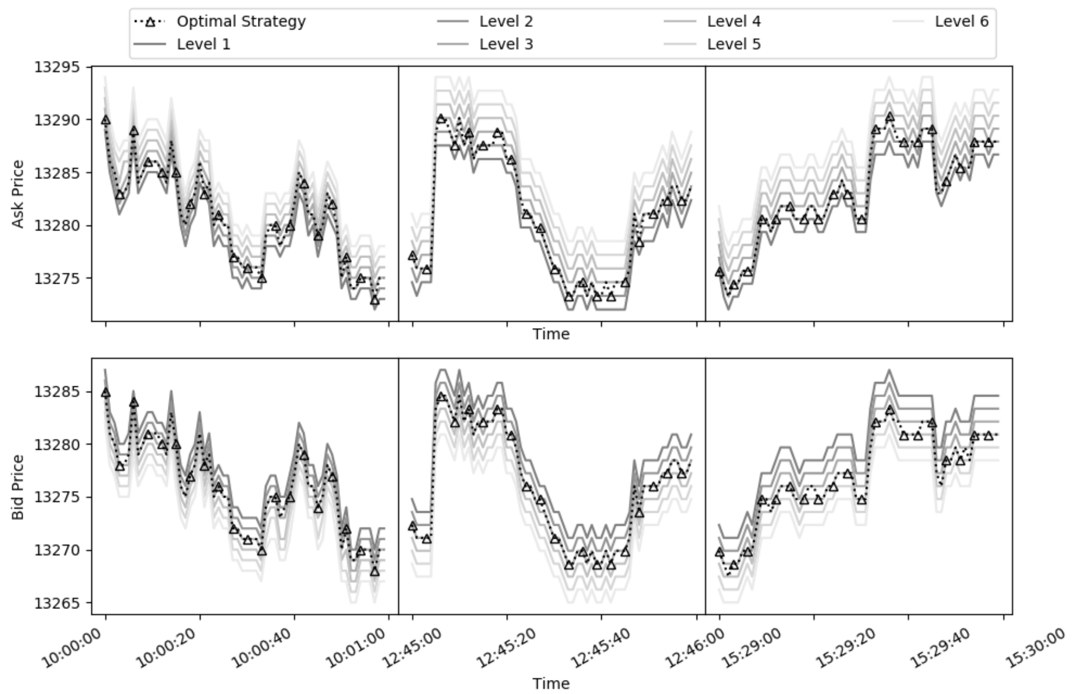


(a) The Intraday Prices Paths.

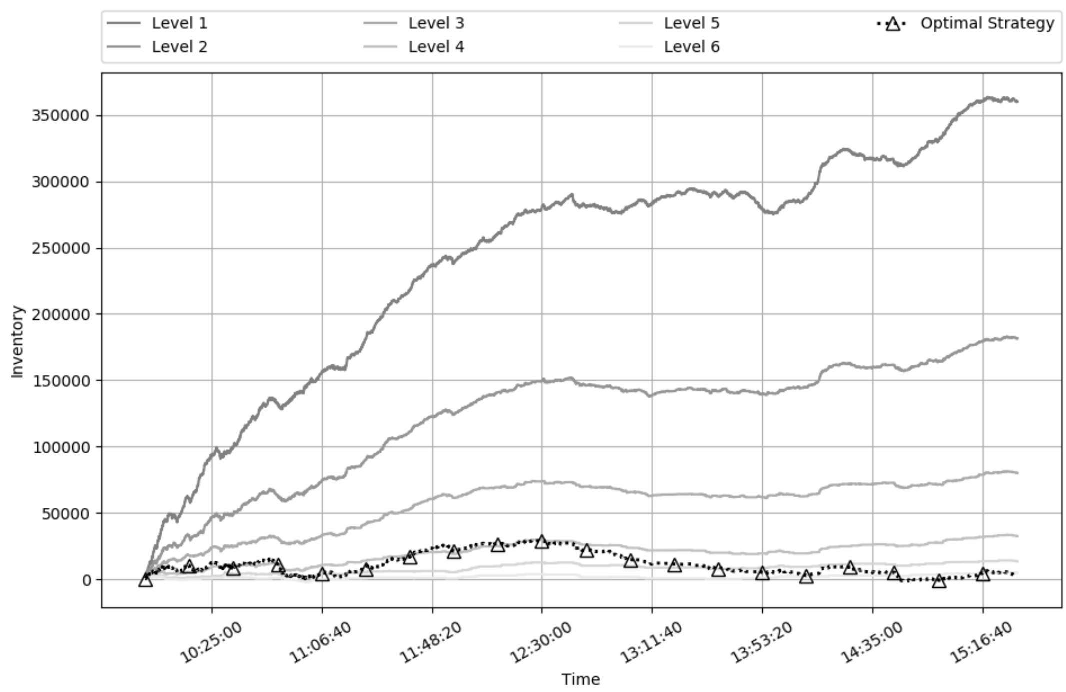


(b) The Intraday Inventory Paths.

Figure 3.5.: The intraday Price and inventory paths of the optimal control policy conditioning on $g_1(e_{t_k})$ compared with the benchmark policies on August 7th.



(a) The Intraday Prices Paths.



(b) The Intraday Inventory Paths.

Figure 3.6.: The intraday Price and inventory paths of the optimal control policy conditioning on $g_2(e_{t_k})$ compared with the benchmark policies on August 7th.

APPENDIX

A. Appendix

A.1 Proof for Lemma 1

Plug in ${}^{(1)}A^{\pm}_{t_k}$ defined in (2.17) into (2.18), we can get $\alpha_{t_k} = \alpha^0_{t_{k+1}} + N_k/D_k$, where

$$
\begin{aligned}
N_k &= (\alpha^{1+}_{t_{k+1}}\mu^+_c\pi^+_{t_{k+1}})^2\pi^-_{t_{k+1}}(\alpha^{1-}_{t_{k+1}}\mu^-_{c^2} - \mu^-_c) + (\alpha^{1-}_{t_{k+1}}\mu^-_c\pi^-_{t_{k+1}})^2\pi^+_{t_{k+1}}(\alpha^{1+}_{t_{k+1}}\mu^+_{c^2} - \mu^+_c) \\
&\quad - 2\alpha^{1+}_{t_{k+1}}\alpha^{1-}_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}}\pi_{t_{k+1}}(1,1)\alpha^2_{t_{k+1}}(\mu^+_c\mu^-_c)^2, \qquad (A.1) \\
D_k &= \left[\pi_{t_{k+1}}(1,1)\alpha^2_{t_{k+1}}\mu^+_c\mu^-_c\right]^2 - \pi^+_{t_{k+1}}\pi^-_{t_{k+1}}(\alpha^{1+}_{t_{k+1}}\mu^+_{c^2} - \mu^+_c)(\alpha^{1-}_{t_{k+1}}\mu^-_{c^2} - \mu^-_c).
\end{aligned}
$$

Therefor, it suffice to show that $N_k/D_k \in (0, -\alpha^0_{t_{k+1}})$ whenever $\alpha_{t_{k+1}} < 0$. First we prove that $N_k < 0$ and $D_k < 0$. Observe that

$$
\begin{aligned}
\alpha^{1+}_{t_{k+1}} &= \mathbb{E}\big[\mathbb{E}(\alpha_{t_{k+1}}|\mathcal{F}_{t_k}, \mathbb{1}^+_{t_{k+1}} = 1, \mathbb{1}^-_{t_{k+1}})\,|\mathcal{F}_{t_k}, \mathbb{1}^+_{t_{k+1}} = 1\big] \\
&= \alpha^2_{t_{k+1}}\mathbb{P}(\mathbb{1}^-_{t_{k+1}} = 1|\mathcal{F}_{t_k}, \mathbb{1}^+_{t_{k+1}} = 1) \\
&\quad + \mathbb{E}(\alpha_{t_{k+1}}|\mathcal{F}_{t_k}, \mathbb{1}^+_{t_{k+1}} = 1, \mathbb{1}^-_{t_{k+1}} = 0)(1 - \mathbb{P}(\mathbb{1}^-_{t_{k+1}} = 1|\mathcal{F}_{t_k}, \mathbb{1}^+_{t_{k+1}} = 1)) \qquad (A.2) \\
&= \alpha^2_{t_{k+1}}\frac{\pi_{t_{k+1}}(1,1)}{\pi^+_{t_{k+1}}} + \mathbb{E}(\alpha_{t_{k+1}}|\mathcal{F}_{t_k}, \mathbb{1}^+_{t_{k+1}} = 1, \mathbb{1}^-_{t_{k+1}} = 0)(1 - \frac{\pi_{t_{k+1}}(1,1)}{\pi^+_{t_{k+1}}})
\end{aligned}
$$

similarly for $\alpha^{1-}_{t_{k+1}}$:

$$
\alpha^{1-}_{t_{k+1}} = \alpha^2_{t_{k+1}}\frac{\pi_{t_{k+1}}(1,1)}{\pi^-_{t_{k+1}}} + \mathbb{E}(\alpha_{t_{k+1}}|\mathcal{F}_{t_k}, \mathbb{1}^+_{t_{k+1}} = 0, \mathbb{1}^-_{t_{k+1}} = 1)(1 - \frac{\pi_{t_{k+1}}(1,1)}{\pi^-_{t_{k+1}}}). \qquad (A.3)
$$

Since $\pi_{t_{k+1}}(1,1) \le \pi^+_{t_{k+1}} \wedge \pi^-_{t_{k+1}}$, we have

$$
\begin{aligned}
\alpha^{1+}_{t_{k+1}}\pi^+_{t_{k+1}} &\le \alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1) \le 0 \\
\alpha^{1-}_{t_{k+1}}\pi^-_{t_{k+1}} &\le \alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1) \le 0
\end{aligned} \qquad (A.4)
$$

Using (A.4) and the fact that $\mu^{\pm}_{c^2} \ge (\mu^{\pm}_c)^2$, we can get

$$
D_k \le \pi^+_{t_{k+1}}\pi^-_{t_{k+1}}\left[\alpha^{1+}_{t_{k+1}}\mu^+_{c^2}\mu^-_c + \alpha^{1-}_{t_{k+1}}\mu^+_{c^2}\mu^-_c - \mu^+_c\mu^-_c\right] < 0, \qquad (A.5)
$$

To prove $N_k < 0$, we rearrange the terms in (A.1)

$$
\begin{aligned}
N_k &= (\alpha^{1+}_{t_{k+1}}\mu^+_c\pi^+_{t_{k+1}})^2\pi^-_{t_{k+1}}(\alpha^{1-}_{t_{k+1}}\mu^-_{c^2} - \mu^-_c) + (\alpha^{1-}_{t_{k+1}}\mu^-_c\pi^-_{t_{k+1}})^2\pi^+_{t_{k+1}}(\alpha^{1+}_{t_{k+1}}\mu^+_{c^2} - \mu^+_c)\\
&\quad - 2\alpha^{1+}_{t_{k+1}}\alpha^{1-}_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}}\pi_{t_{k+1}}(1,1)\alpha^2_{t_{k+1}}(\mu^+_c\mu^-_c)^2\\
&= \alpha^{1+}_{t_{k+1}}\alpha^{1-}_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}}(\mu^+_c)^2\big[\alpha^{1+}_{t_{k+1}}\pi^+_{t_{k+1}}\mu^-_{c^2} - \alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1)(\mu^-_c)^2\big]\\
&\quad + \alpha^{1+}_{t_{k+1}}\alpha^{1-}_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}}(\mu^-_c)^2\big[\alpha^{1-}_{t_{k+1}}\pi^-_{t_{k+1}}\mu^+_{c^2} - \alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1)(\mu^+_c)^2\big]\\
&\quad - (\alpha^{1+}_{t_{k+1}}\mu^+_c\pi^+_{t_{k+1}})^2\mu^-_c - (\alpha^{1-}_{t_{k+1}}\mu^-_c\pi^-_{t_{k+1}})^2\mu^+_c\\
&\le \alpha^{1+}_{t_{k+1}}\alpha^{1-}_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}}(\mu^+_c)^2\pi_{t_{k+1}}(1,1)\alpha^2_{t_{k+1}}[\mu^-_{c^2} - (\mu^-_c)^2]\\
&\quad + \alpha^{1+}_{t_{k+1}}\alpha^{1-}_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}}(\mu^+_c)^2\pi_{t_{k+1}}(1,1)\alpha^2_{t_{k+1}}[\mu^+_{c^2} - (\mu^+_c)^2]\\
&\quad - (\alpha^{1+}_{t_{k+1}}\mu^+_c\pi^+_{t_{k+1}})^2\mu^-_c - (\alpha^{1-}_{t_{k+1}}\mu^-_c\pi^-_{t_{k+1}})^2\mu^+_c\\
&\le -(\alpha^{1+}_{t_{k+1}}\mu^+_c\pi^+_{t_{k+1}})^2\mu^-_c - (\alpha^{1-}_{t_{k+1}}\mu^-_c\pi^-_{t_{k+1}})^2\mu^+_c\\
&< 0
\end{aligned}
\tag{A.6}
$$

Next we prove that $N_k/D_k < -\alpha^0_{t_{k+1}}$ whenever $\alpha_{t_{k+1}} < 0$, which is equivalent to prove $\alpha^0_{t_{k+1}}D_k + N_k > 0$. We can define $f(\mu^+_{c^2}) := \alpha^0_{t_{k+1}}D_k + N_k$ which is a linear function of $\mu^+_{c^2}$, and observe that:

$$
\begin{aligned}
\frac{df}{d\mu^+_{c^2}} &= -\alpha^0_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}}[\alpha^{1+}_{t_{k+1}}\alpha^{1-}_{t_{k+1}}\mu^-_{c^2} - \alpha^{1+}_{t_{k+1}}\mu^-_c] + (\alpha^{1-}_{t_{k+1}}\mu^-_c\pi^-_{t_{k+1}})^2\pi^+_{t_{k+1}}\alpha^{1+}_{t_{k+1}}\\
&= \alpha^{1+}_{t_{k+1}}\alpha^{1-}_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}}[\alpha^{1-}_{t_{k+1}}\pi^-_{t_{k+1}}(\mu^-_c)^2 - \alpha^0_{t_{k+1}}\mu^-_{c^2}] + \alpha^0_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}}\alpha^{1+}_{t_{k+1}}\mu^-_c.
\end{aligned}
\tag{A.7}
$$

Since

$$
\begin{aligned}
\alpha^0_{t_{k+1}} &= \mathbb{E}(\alpha_{t_{k+1}}|\mathcal{F}_{t_k})\\
&= \mathbb{E}\big[\mathbb{E}(\alpha_{t_{k+1}}|\mathcal{F}_{t_k}, \mathbb{1}^{\pm}_{t_{k+1}})|\mathcal{F}_{t_k}\big]\\
&= \alpha^{1\pm}_{t_{k+1}}\pi^{\pm}_{t_{k+1}} + \mathbb{E}(\alpha_{t_{k+1}}|\mathcal{F}_{t_k}, \mathbb{1}^{\pm}_{t_{k+1}} = 0)(1 - \pi^{\pm}_{t_{k+1}}),
\end{aligned}
\tag{A.8}
$$

we have

$$
\alpha^0_{t_{k+1}} \le \alpha^{1\pm}_{t_{k+1}}\pi^{\pm}_{t_{k+1}}. \tag{A.9}
$$

Plug (A.9) into (A.7), we can get

$$\frac{df}{d\mu_{c^2}^+} \geq \alpha_{t_{k+1}}^{1+}\alpha_{t_{k+1}}^{1-}\pi_{t_{k+1}}^{+}\pi_{t_{k+1}}^{-}\alpha_{t_{k+1}}^{0}[(\mu_c^-)^2 - \mu_{c^2}^-] + \alpha_{t_{k+1}}^{0}\pi_{t_{k+1}}^{+}\pi_{t_{k+1}}^{-}\alpha_{t_{k+1}}^{1+}\mu_c^-$$
$$\geq \alpha_{t_{k+1}}^{0}\pi_{t_{k+1}}^{+}\pi_{t_{k+1}}^{-}\alpha_{t_{k+1}}^{1+}\mu_c^- \tag{A.10}$$
$$\geq 0$$

So we have $g(\mu_{c^2}^-) := f((\mu_c^+)^2) \leq f(\mu_{c^2}^+)$, where $g(\mu_{c^2}^-)$ is a linear function in $\mu_{c^2}^-$. Observe that

$$\frac{dg}{d\mu_{c^2}^-} = -\alpha_{t_{k+1}}^{0}\pi_{t_{k+1}}^{+}\pi_{t_{k+1}}^{-}[\alpha_{t_{k+1}}^{1+}\alpha_{t_{k+1}}^{1-}(\mu_c^+)^2 - \alpha_{t_{k+1}}^{1-}\mu_c^+] + (\alpha_{t_{k+1}}^{1+}\mu_c^+\pi_{t_{k+1}}^{+})^2\pi_{t_{k+1}}^{-}\alpha_{t_{k+1}}^{1-}$$
$$= \alpha_{t_{k+1}}^{1+}\alpha_{t_{k+1}}^{1-}\pi_{t_{k+1}}^{+}\pi_{t_{k+1}}^{-}[\alpha_{t_{k+1}}^{1+}\pi_{t_{k+1}}^{+}(\mu_c^+)^2 - \alpha_{t_{k+1}}^{0}(\mu_c^+)^2] + \alpha_{t_{k+1}}^{0}\pi_{t_{k+1}}^{+}\pi_{t_{k+1}}^{-}\alpha_{t_{k+1}}^{1-}\mu_c^+$$
$$\geq \alpha_{t_{k+1}}^{0}\pi_{t_{k+1}}^{+}\pi_{t_{k+1}}^{-}\alpha_{t_{k+1}}^{1-}\mu_c^+$$
$$\geq 0,$$

so we have $\alpha_{t_{k+1}}^0 D_k + N_k \geq g(\mu_{c^2}^-) \geq g((\mu_c^-)^2)$, where

$$g((\mu_c^-)^2) = \mu_c^+\mu_c^-\Big\{\alpha_{t_{k+1}}^{0}\big[(\pi_{t_{k+1}}(1,1)\alpha_{t_{k+1}}^{2})^2\mu_c^+\mu_c^- - \pi_{t_{k+1}}^{+}\pi_{t_{k+1}}^{-}(\alpha_{t_{k+1}}^{1+}\mu_c^+ - 1)(\alpha_{t_{k+1}}^{1-}\mu_c^- - 1)\big]$$
$$+ \mu_c^+(\alpha_{t_{k+1}}^{1+}\pi_{t_{k+1}}^{+})^2\pi_{t_{k+1}}^{-}(\alpha_{t_{k+1}}^{1-}\mu_c^- - 1) + \mu_c^-(\alpha_{t_{k+1}}^{1-}\pi_{t_{k+1}}^{-})^2\pi_{t_{k+1}}^{+}(\alpha_{t_{k+1}}^{1+}\mu_c^+ - 1)$$
$$- 2\alpha_{t_{k+1}}^{1+}\alpha_{t_{k+1}}^{1-}\pi_{t_{k+1}}^{+}\pi_{t_{k+1}}^{-}\pi_{t_{k+1}}(1,1)\alpha_{t_{k+1}}^{2}\mu_c^+\mu_c^-\Big\}$$
$$= \mu_c^+\mu_c^-\ell(\mu_c^+),$$

$\ell(\mu_c^+)$ is a linear function of μ_c^+. Take the derivative of ℓ with respect to μ_c^+ and we get

$$\frac{d\ell}{d\mu_c^+} = \alpha_{t_{k+1}}^{0}\big[(\pi_{t_{k+1}}(1,1)\alpha_{t_{k+1}}^{2})^2\mu_c^- - \pi_{t_{k+1}}^{+}\pi_{t_{k+1}}^{-}(\alpha_{t_{k+1}}^{1+}\alpha_{t_{k+1}}^{1-}\mu_c^- - \alpha_{t_{k+1}}^{1+})\big]$$
$$+ (\alpha_{t_{k+1}}^{1+}\pi_{t_{k+1}}^{+})^2\pi_{t_{k+1}}^{-}(\alpha_{t_{k+1}}^{1-}\mu_c^- - 1) + \mu_c^-(\alpha_{t_{k+1}}^{1-}\pi_{t_{k+1}}^{-})^2\pi_{t_{k+1}}^{+}\alpha_{t_{k+1}}^{1+}$$
$$- 2\alpha_{t_{k+1}}^{1+}\alpha_{t_{k+1}}^{1-}\pi_{t_{k+1}}^{+}\pi_{t_{k+1}}^{-}\pi_{t_{k+1}}(1,1)\alpha_{t_{k+1}}^{2}\mu_c^-$$
$$= \pi_{t_{k+1}}^{+}\pi_{t_{k+1}}^{-}\alpha_{t_{k+1}}^{1+}(\alpha_{t_{k+1}}^{0} - \alpha_{t_{k+1}}^{1+}\pi_{t_{k+1}}^{+}) + \mu_c^- m(\alpha_{t_{k+1}}^{2}\pi_{t_{k+1}}(1,1)),$$

where

$$m(\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1)) = \alpha^0_{t_{k+1}}(\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1))^2 - 2\alpha^{1+}_{t_{k+1}}\alpha^{1-}_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}}(\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1))$$
$$+ (\alpha^{1+}_{t_{k+1}}\pi^+_{t_{k+1}})^2\alpha^{1-}_{t_{k+1}}\pi^-_{t_{k+1}} + (\alpha^{1-}_{t_{k+1}}\pi^-_{t_{k+1}})^2\alpha^{1+}_{t_{k+1}}\pi^+_{t_{k+1}}$$
$$- \alpha^0_{t_{k+1}}\alpha^{1+}_{t_{k+1}}\alpha^{1-}_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}}. \tag{A.11}$$

Observe that

$$\alpha_{t_{k+1}}(\mathbb{1}^+_{t_{k+1}} + \mathbb{1}^-_{t_{k+1}} - 1) \geq \alpha_{t_{k+1}}\mathbb{1}^+_{t_{k+1}}\mathbb{1}^-_{t_{k+1}},$$

so, by taking conditional expectation, we have

$$\alpha^{1+}_{t_{k+1}}\pi^+_{t_{k+1}} + \alpha^{1-}_{t_{k+1}}\pi^-_{t_{k+1}} - \alpha^0_{t_{k+1}} \geq \alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1). \tag{A.12}$$

Combine (A.12) with (A.4), we get the upper and lower bound for $\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1)$:

$$\alpha^{1+}_{t_{k+1}}\pi^+_{t_{k+1}} \vee \alpha^{1-}_{t_{k+1}}\pi^-_{t_{k+1}} \leq \alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1) \leq (\alpha^{1+}_{t_{k+1}}\pi^+_{t_{k+1}} + \alpha^{1-}_{t_{k+1}}\pi^-_{t_{k+1}} - \alpha^0_{t_{k+1}}) \wedge 0, \tag{A.13}$$

If we can show that for any value of $\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1)$, in the range specified by (A.13), $m(\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1)) \geq 0$, then we have $d\ell/d\mu^+_c \geq \pi^+_{t_{k+1}}\pi^-_{t_{k+1}}\alpha^{1+}_{t_{k+1}}(\alpha^0_{t_{k+1}} - \alpha^{1+}_{t_{k+1}}\pi^+_{t_{k+1}}) \geq 0$, then

$$\begin{aligned}
\ell(\mu^+_c) \geq \ell(0) &= \alpha^0_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}}(\alpha^{1-}_{t_{k+1}}\mu^-_c - 1) - \mu^-_c(\alpha^{1-}_{t_{k+1}}\pi^-_{t_{k+1}})^2\pi^+_{t_{k+1}} \\
&= \pi^+_{t_{k+1}}\pi^-_{t_{k+1}}\alpha^{1-}_{t_{k+1}}\mu^-_c(\alpha^0_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^-_{t_{k+1}}) - \alpha^0_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}} \\
&\geq -\alpha^0_{t_{k+1}}\pi^+_{t_{k+1}}\pi^-_{t_{k+1}} \\
&> 0,
\end{aligned}$$

so we have $\alpha^0_{t_{k+1}}D_k + N_k > 0$ and the lemma is proved.

Since $m(\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1))$ is a quadratic function of $\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1)$ opening downwards, we only need to check $m(\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1))$ is non-negative at the boundary points

$\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}} \vee \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}}$ and $(\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}} + \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}} - \alpha^{0}_{t_{k+1}}) \wedge 0$. Without loss of generality, we assume $\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}} \geq \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}}$, then

$$\begin{aligned}
m(\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}}) &= \alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}}\big[(\alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})^2 - (\alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})(\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}}) \\
&\qquad - \alpha^{0}_{t_{k+1}}(\alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}}) + \alpha^{0}_{t_{k+1}}(\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}})\big] \\
&= \alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}}(\alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}} - \alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}})(\alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}} - \alpha^{0}_{t_{k+1}}) \\
&\geq 0
\end{aligned}$$

Now we check that $m((\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}} + \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}} - \alpha^{0}_{t_{k+1}}) \wedge 0) \geq 0$, if $\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}} + \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}} - \alpha^{0}_{t_{k+1}} \geq 0$

$$m(0) = \alpha^{1+}_{t_{k+1}}\alpha^{1-}_{t_{k+1}}\pi^{+}_{t_{k+1}}\pi^{-}_{t_{k+1}}(\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}} + \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}} - \alpha^{0}_{t_{k+1}}) \geq 0.$$

If $\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}} + \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}} - \alpha^{0}_{t_{k+1}} \leq 0$, then

$$\begin{aligned}
m(\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}} + \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}} - \alpha^{0}_{t_{k+1}}) &= \alpha^{0}_{t_{k+1}}(\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}} + \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}} - \alpha^{0}_{t_{k+1}})^2 \\
&\quad - 2\alpha^{1+}_{t_{k+1}}\alpha^{1-}_{t_{k+1}}\pi^{+}_{t_{k+1}}\pi^{-}_{t_{k+1}}(\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}} + \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}} - \alpha^{0}_{t_{k+1}}) \\
&\quad + (\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}})^2\alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}} + (\alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})^2\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}} \\
&\quad - \alpha^{0}_{t_{k+1}}\alpha^{1+}_{t_{k+1}}\alpha^{1-}_{t_{k+1}}\pi^{+}_{t_{k+1}}\pi^{-}_{t_{k+1}} \\
&= (\alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})(\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}})^2 \\
&\quad - (2\alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})(\alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})(\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}}) \\
&\quad + \alpha^{0}_{t_{k+1}}(\alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})^2 \\
&:= n(\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}}),
\end{aligned}$$

where $n(\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}})$ is a quadratic function, opening downward, of $\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}}$. Since $\alpha^{0}_{t_{k+1}} \leq \alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}} \leq \alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}}$ and

$$\begin{aligned}
n(\alpha^{0}_{t_{k+1}}) &= (\alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})(\alpha^{0}_{t_{k+1}})^2 - (2\alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})(\alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})(\alpha^{0}_{t_{k+1}}) \\
&\quad + \alpha^{0}_{t_{k+1}}(\alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})^2 = 0 \\
n(\alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}}) &= (\alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})(\alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})^2 \\
&\quad - (2\alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})(\alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})^2 \\
&\quad + \alpha^{0}_{t_{k+1}}(\alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})^2 = 0,
\end{aligned}$$

we have $n(\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}}) \geq 0$ for all $\alpha^{1+}_{t_{k+1}}\pi^{+}_{t_{k+1}} \in [\alpha^{0}_{t_{k+1}}, (\alpha^{0}_{t_{k+1}} - \alpha^{1-}_{t_{k+1}}\pi^{-}_{t_{k+1}})]$. So we've proved that for any value of $\alpha^{2}_{t_{k+1}}\pi_{t_{k+1}}(1,1)$ in the range specified by (A.13), $m(\alpha^{2}_{t_{k+1}}\pi_{t_{k+1}}(1,1)) \geq 0$.

A.2 Proof for Proposition 1

In equation (2.15), replace $W_{t_{k+1}}$ and $I_{t_{k+1}}$ by (2.8) and (2.9), then we get

$$\begin{aligned}
V_{t_k} = \sup_{L^{\pm}_{t_k}} \mathbb{E}\Bigg\{ &\sum_{\delta=\pm}(S_{t_k} + \delta L^{\delta}_{t_k})\delta \mathbb{1}^{\delta}_{t_{k+1}} c^{\delta}_{t_{k+1}}(p^{\delta}_{t_{k+1}} - L^{\delta}_{t_k}) \\
&+ \alpha_{t_{k+1}}\big[I_{t_k} - \sum_{\delta=\pm}\delta \mathbb{1}^{\delta}_{t_{k+1}} c^{\delta}_{t_{k+1}}(p^{\delta}_{t_{k+1}} - L^{\delta}_{t_k})\big]^2 \\
&+ S_{t_{k+1}}\big[I_{t_k} - \sum_{\delta=\pm}\delta \mathbb{1}^{\delta}_{t_{k+1}} c^{\delta}_{t_{k+1}}(p^{\delta}_{t_{k+1}} - L^{\delta}_{t_k})\big] \\
&+ h_{t_{k+1}}\big[I_{t_k} - \sum_{\delta=\pm}\delta \mathbb{1}^{\delta}_{t_{k+1}} c^{\delta}_{t_{k+1}}(p^{\delta}_{t_{k+1}} - L^{\delta}_{t_k})\big] \\
&+ g_{t_{k+1}} \Big| \mathcal{F}_{t_k}\Bigg\}.
\end{aligned} \tag{A.14}$$

We expand the squares inside the expectation above and arrange the terms as follows:

$$\sum_{\delta=\pm} \mathbb{1}^{\delta}_{t_{k+1}} \big[-c^{\delta}_{t_{k+1}} (L^{\delta}_{t_k})^2 + (c^{\delta}_{t_{k+1}} p^{\delta}_{t_{k+1}} - \delta c^{\delta}_{t_{k+1}} S_{t_k}) L^{\delta}_{t_k} + \delta c^{\delta}_{t_{k+1}} p^{\delta}_{t_{k+1}} S_{t_k} \big] \tag{A.15}$$

$$+ \alpha_{t_{k+1}} \Bigg\{ I^2_{t_k} + \sum_{\delta=\pm} \mathbb{1}^{\delta}_{t_{k+1}} \Big\{ (c^{\delta}_{t_{k+1}})^2 (L^{\delta}_{t_k})^2 + \big[2\delta I_{t_k} c^{\delta}_{t_{k+1}} - 2(c^{\delta}_{t_{k+1}})^2 p^{\delta}_{t_{k+1}} \big] L^{\delta}_{t_k}$$

$$+ (c^{\delta}_{t_{k+1}} p^{\delta}_{t_{k+1}})^2 - 2\delta I_{t_k} c^{\delta}_{t_{k+1}} p^{\delta}_{t_{k+1}} \Big\} \tag{A.16}$$

$$+ 2 \mathbb{1}^{+}_{t_{k+1}} \mathbb{1}^{-}_{t_{k+1}} c^{+}_{t_{k+1}} c^{-}_{t_{k+1}} (-L^{+}_{t_k} L^{-}_{t_k} + p^{+}_{t_{k+1}} L^{-}_{t_k} + p^{-}_{t_{k+1}} L^{+}_{t_k} - p^{+}_{t_{k+1}} p^{-}_{t_{k+1}}) \Bigg\}$$

$$+ S_{t_{k+1}} \Bigg[I_{t_k} + \sum_{\delta=\pm} \mathbb{1}^{\delta}_{t_{k+1}} (-\delta c^{\delta}_{t_{k+1}} p^{\delta}_{t_{k+1}} + \delta c^{\delta}_{t_{k+1}} L^{\delta}_{t_k}) \Bigg] \tag{A.17}$$

$$+ h_{t_{k+1}} I_{t_k} + \sum_{\delta=\pm} \mathbb{1}^{\delta}_{t_{k+1}} (-\delta h_{t_{k+1}} c^{\delta}_{t_{k+1}} p^{\delta}_{t_{k+1}} + \delta h_{t_{k+1}} c^{\delta}_{t_{k+1}} L^{\delta}_{t_k}) + g_{t_{k+1}}. \tag{A.18}$$

Under Assumption 1, the conditional expectation of terms in (A.15) through (A.18) is easy to calculate. We plug in these conditional expectations given $\mathcal{F}_{t_k}$ into the right hand side of (A.14) to get:

$$\alpha_{t_k} I^2_{t_k} + S_{t_k} I_{t_k} + h_{t_k} I_{t_k} + g_{t_k}$$

$$= \sup_{L^{\pm}_{t_k}} \sum_{\delta=\pm} \pi^{\delta}_{t_{k+1}} \Big\{ (\alpha^{1\delta}_{t_{k+1}} \mu^{\delta}_{c^2} - \mu^{\delta}_{c}) (L^{\delta}_{t_k})^2 + \big[\mu^{\delta}_{cp} + \delta h^{1\delta}_{t_{k+1}} \mu^{\delta}_{c} + \alpha^{1\delta}_{t_{k+1}} (2\delta \mu^{\delta}_{c} I_{t_k} - 2\mu^{\delta}_{c^2 p}) \big] L^{\delta}_{t_k}$$

$$+ \alpha^{1\delta}_{t_{k+1}} (\mu^{\delta}_{c^2 p^2} - 2\delta \mu^{\delta}_{cp} I_{t_k}) - \delta h^{1\delta}_{t_{k+1}} \mu^{\delta}_{cp} \Big\}$$

$$+ \alpha^{0}_{t_{k+1}} I^2_{t_k} + 2\alpha^{2}_{t_{k+1}} \pi_{t_{k+1}}(1,1) (-\mu^{+}_{c} \mu^{-}_{c} L^{+}_{t_k} L^{-}_{t_k} + \mu^{+}_{cp} \mu^{-}_{c} L^{-}_{t_k} + \mu^{+}_{c} \mu^{-}_{cp} L^{+}_{t_k} - \mu^{+}_{cp} \mu^{-}_{cp})$$

$$+ I_{t_k} S_{t_k} + h^{0}_{t_{k+1}} I_{t_k} + \mathbb{E}(g_{t_{k+1}} | \mathcal{F}_{t_k}) \tag{A.19}$$

Denote the right hand side of above equation as $\sup_{L^{\pm}_{t_k}} f(L^+_{t_k}, L^-_{t_k})$, where $f(L^+_{t_k}, L^-_{t_k})$ is a quadratic function of $L^+_{t_k}$ and $L^-_{t_k}$. Setting the partial derivatives with respect to $L^+_{t_k}$ and $L^-_{t_k}$, respectively, equal to 0, we have

$$\begin{aligned}
\partial_{L^+_{t_k}} f &= 2\pi^+_{t_{k+1}}(\alpha^{1+}_{t_{k+1}}\mu^+_{c^2} - \mu^+_c)L^+_{t_k} + \pi^+_{t_{k+1}}\left[\mu^+_{cp} + h^{1+}_{t_{k+1}}\mu^+_c + \alpha^{1+}_{t_{k+1}}(2\mu^+_c I_{t_k} - 2\mu^+_{c^2p})\right] \\
&\quad - 2\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1)\mu^+_c\mu^-_c L^-_{t_k} + 2\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1)\mu^+_c\mu^-_{cp} = 0, \\
\partial_{L^-_{t_k}} f &= 2\pi^-_{t_{k+1}}(\alpha^{1-}_{t_{k+1}}\mu^-_{c^2} - \mu^-_c)L^-_{t_k} + \pi^-_{t_{k+1}}\left[\mu^-_{cp} - h^{1-}_{t_{k+1}}\mu^-_c + \alpha^{1-}_{t_{k+1}}(-2\mu^-_c I_{t_k} - 2\mu^-_{c^2p})\right] \\
&\quad - 2\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1)\mu^+_c\mu^-_c L^+_{t_k} + 2\alpha^2_{t_{k+1}}\pi_{t_{k+1}}(1,1)\mu^-_c\mu^+_{cp} = 0.
\end{aligned}$$

. Solving for $L^+_{t_k}$ and $L^-_{t_k}$, we get the solution $L^{+,*}_{t_k}$ and $L^{-,*}_{t_k}$ given in Proposition 1.

To prove that $L^{+,*}_{t_k}$ and $L^{-,*}_{t_k}$ are the maximum point of $f(L^+_{t_k}, L^-_{t_k})$, we show that

$$\begin{aligned}
D &= (\partial^2_{L^+_{t_k}} f)(\partial^2_{L^-_{t_k}} f) - (\partial_{L^+_{t_k}L^-_{t_k}} f)^2 \\
&= 4\pi^+_{t_{k+1}}\pi^-_{t_{k+1}}(\alpha^{1+}_{t_{k+1}}\mu^+_{c^2} - \mu^+_c)(\alpha^{1-}_{t_{k+1}}\mu^-_{c^2} - \mu^-_c) - 4\left[\pi_{t_{k+1}}(1,1)\alpha^2_{t_{k+1}}\mu^+_c\mu^-_c\right]^2 \\
&= -4\gamma_{t_k} > 0
\end{aligned}$$

$$\partial^2_{L^+_{t_k}} f = 2\pi^+_{t_{k+1}}(\alpha^{1+}_{t_{k+1}}\mu^+_{c^2} - \mu^+_c) < 0.$$

so by the second derivative test, $f(L^+_{t_k}, L^-_{t_k})$ takes its maximum value at $L^{\pm,*}_{t_k}$.

A.3 Proof for Proposition 2

By (2.20), we only need to prove that

$$L^{+,*}_{t_k} + L^{-,*}_{t_k} = ({}^{(1)}A^+_{t_k} - {}^{(1)}A^-_{t_k})I_{t_k} + ({}^{(2)}A^+_{t_k} - {}^{(2)}A^-_{t_k}) + ({}^{(3)}A^+_{t_k} + {}^{(3)}A^-_{t_k}) > 0.$$

First we prove that under condition (2.22) and (2.23), $\alpha^{1+}_{t_{k+1}} = \alpha^{1-}_{t_{k+1}}$ and $h^{1+}_{t_{k+1}} = h^{1-}_{t_{k+1}}$. Since the σ-algebra generated by $g(e_{t_k})$ is contained in the σ-algebra generated by $\{e^+_{t_k}, e^-_{t_k}\}$, we can treat α_{t_k} as a function of $(e^+_{t_k}, e^-_{t_k})$ and denote it as $\alpha_{t_k}(e^{\pm}_{t_k})$. By (A.2) and (A.3), we have

$$\alpha^{1+}_{t_{k+1}} = \alpha_{t_{k+1}}(e^{\pm}_{t_k}, \mathbb{1}^+_{t_{k+1}} = 1, \mathbb{1}^-_{t_{k+1}} = 1)\frac{\pi_{t_{k+1}}(1,1)}{\pi^+_{t_{k+1}}} + \alpha_{t_{k+1}}(e^{\pm}_{t_k}, \mathbb{1}^+_{t_{k+1}} = 1, \mathbb{1}^-_{t_{k+1}} = 0)(1 - \frac{\pi_{t_{k+1}}(1,1)}{\pi^+_{t_{k+1}}})$$
$$\alpha^{1-}_{t_{k+1}} = \alpha_{t_{k+1}}(e^{\pm}_{t_k}, \mathbb{1}^+_{t_{k+1}} = 1, \mathbb{1}^-_{t_{k+1}} = 1)\frac{\pi_{t_{k+1}}(1,1)}{\pi^-_{t_{k+1}}} + \alpha_{t_{k+1}}(e^{\pm}_{t_k}, \mathbb{1}^+_{t_{k+1}} = 0, \mathbb{1}^-_{t_{k+1}} = 1)(1 - \frac{\pi_{t_{k+1}}(1,1)}{\pi^-_{t_{k+1}}})$$

For $t_{k+1} = T$, we have $\alpha_T(e^{\pm}_T) \equiv -\lambda$, so by condition (2.22) and (2.23) we have

$$\alpha^{1+}_T(e^{\pm}_{t_N}) = \alpha^{1-}_T(e^{\pm}_{t_N})$$

$$\alpha^{1\pm}_T(e^+_{t_N} = a, e^-_{t_N} = b) = \alpha^{1\pm}_T(e^+_{t_N} = a', e^-_{t_N} = b')$$

and by (A.1) we have

$$\alpha_{t_N}(e^+_{t_N} = a, e^-_{t_N} = b) = \alpha_{t_N}(e^+_{t_N} = a', e^-_{t_N} = b'). \tag{A.20}$$

So, by induction, we have proved that $\alpha^{1+}_{t_{k+1}} = \alpha^{1-}_{t_{k+1}}$ and similarly $h^{1+}_{t_{k+1}} = h^{1-}_{t_{k+1}}$. Denote

$$\alpha^{1\pm}_{t_{k+1}} = \alpha^1_{t_{k+1}},\ h^{1\pm}_{t_{k+1}} = h^1_{t_{k+1}}, \tag{A.21}$$

it is easy to see that

$$\beta^+_{t_k} - \beta^-_{t_k} = \pi^+_{t_{k+1}}\pi^-_{t_{k+1}}(\alpha^1_{t_{k+1}})^2(\mu^+_c\mu^-_{c^2} - \mu^-_c\mu^+_{c^2}) - \alpha^1_{t_{k+1}}\pi_{t_{k+1}}(1,1)\alpha_{t_{k+1}}\mu^-_c\mu^+_c(\pi^-_{t_{k+1}}\mu^-_c - \pi^+_{t_{k+1}}\mu^+_c) = 0.$$

$$\eta^+_{t_k} - \eta^-_{t_k} = \pi^+_{t_{k+1}}\pi^-_{t_{k+1}}\alpha^1_{t_{k+1}}h^1_{t_{k+1}}(\mu^+_c\mu^-_{c^2} - \mu^-_c\mu^+_{c^2}) - h^1_{t_{k+1}}\pi_{t_{k+1}}(1,1)\alpha_{t_{k+1}}\mu^-_c\mu^+_c(\pi^-_{t_{k+1}}\mu^-_c - \pi^+_{t_{k+1}}\mu^+_c) = 0.$$

This directly implies that ${}^{(1)}A^+_{t_k} - {}^{(1)}A^-_{t_k} = 0$ and ${}^{(2)}A^+_{t_k} - {}^{(2)}A^-_{t_k} = 0$. So it suffice to show that ${}^{(3)}A^+_{t_k} + {}^{(3)}A^-_{t_k} > 0$. Notice that, as shown in (A.5) ($D_k = \gamma_{t_k}$), the denominator γ_{t_k} of ${}^{(3)}A^+_{t_k} + {}^{(3)}A^-_{t_k}$ is negative. So, it suffice to show the numerator of ${}^{(3)}A^+_{t_k} + {}^{(3)}A^-_{t_k}$ is also negative.

By Condition (2.24) in Proposition **??** ($\mu^{\pm}_{cp} = \mu^{\pm}_{c}\mu^{\pm}_{p}$ and $\mu^{\pm}_{c^2p} = \mu^{\pm}_{c^2}\mu^{\pm}_{p}$), the numerator of ${}^{(3)}\!A^{+}_{t_k} + {}^{(3)}\!A^{-}_{t_k}$ can be written as

$$
\begin{aligned}
N({}^{(3)}\!A^{+}_{t_k} + {}^{(3)}\!A^{-}_{t_k}) = \Big\{ &\pi^{+}_{t_{k+1}}\pi^{-}_{t_{k+1}}(\alpha^{1}_{t_{k+1}}\mu^{-}_{c^2} - \mu^{-}_{c})(\mu^{+}_{c} - 2\alpha^{1}_{t_{k+1}}\mu^{+}_{c^2}) + 2\big[\alpha^{2}_{t_{k+1}}\pi_{t_{k+1}}(1,1)\mu^{+}_{c}\mu^{-}_{c}\big]^2 \\
&- \pi^{+}_{t_{k+1}}\pi_{t_{k+1}}(1,1)\alpha^{2}_{t_{k+1}}(\mu^{+}_{c})^2\mu^{-}_{c}\Big\}\mu^{+}_{p} \\
+ \Big\{ &\pi^{+}_{t_{k+1}}\pi^{-}_{t_{k+1}}(\alpha^{1}_{t_{k+1}}\mu^{+}_{c^2} - \mu^{+}_{c})(\mu^{-}_{c} - 2\alpha^{1}_{t_{k+1}}\mu^{-}_{c^2}) + 2\big[\alpha_{t_{k+1}}\pi_{t_{k+1}}(1,1)\mu^{+}_{c}\mu^{-}_{c}\big]^2 \\
&- \pi^{-}_{t_{k+1}}\pi_{t_{k+1}}(1,1)\alpha^{2}_{t_{k+1}}(\mu^{-}_{c})^2\mu^{+}_{c}\Big\}\mu^{-}_{p}.
\end{aligned}
$$

We can then show the coefficients of μ^{+}_{p} is negative. Denote the coefficient of μ^{+}_{p} as $r(\mu^{+}_{c^2}, \mu^{-}_{c^2})$, which is a linear function of $\mu^{-}_{c^2}$ with coefficient $\pi^{+}_{t_{k+1}}\pi^{-}_{t_{k+1}}\alpha^{1}_{t_{k+1}}(\mu^{+}_{c} - 2\alpha^{1}_{t_{k+1}}\mu^{+}_{c^2}) < 0$. Since $\mu^{-}_{c^2} \geq (\mu^{-}_{c})^2$, we have that

$$
\begin{aligned}
r(\mu^{+}_{c^2}, \mu^{-}_{c^2}) &\leq r(\mu^{+}_{c^2}, (\mu^{-}_{c})^2) \\
&= \pi^{+}_{t_{k+1}}\pi^{-}_{t_{k+1}}[\alpha^{1}_{t_{k+1}}(\mu^{-}_{c})^2 - \mu^{-}_{c}](\mu^{+}_{c} - 2\alpha^{1}_{t_{k+1}}\mu^{+}_{c^2}) + 2\big[\alpha^{2}_{t_{k+1}}\pi_{t_{k+1}}(1,1)\mu^{+}_{c}\mu^{-}_{c}\big]^2 \\
&\quad - \pi^{+}_{t_{k+1}}\pi_{t_{k+1}}(1,1)\alpha^{2}_{t_{k+1}}(\mu^{+}_{c})^2\mu^{-}_{c}.
\end{aligned}
$$

Similarly, $r(\mu^{+}_{c^2}, (\mu^{-}_{c})^2)$ is linear in $\mu^{+}_{c^2}$ with coefficient $-2\alpha^{1}_{t_{k+1}}\pi^{+}_{t_{k+1}}\pi^{-}_{t_{k+1}}[\alpha^{1}_{t_{k+1}}(\mu^{-}_{c})^2 - \mu^{-}_{c}] < 0$. Therefore,

$$
\begin{aligned}
r(\mu^{+}_{c^2}, (\mu^{-}_{c})^2) &\leq r((\mu^{+}_{c})^2, (\mu^{-}_{c})^2) \\
&= \pi^{+}_{t_{k+1}}\pi^{-}_{t_{k+1}}[\alpha^{1}_{t_{k+1}}(\mu^{-}_{c})^2 - \mu^{-}_{c}][\mu^{+}_{c} - 2\alpha^{1}_{t_{k+1}}(\mu^{+}_{c})^2] + 2\big[\alpha^{2}_{t_{k+1}}\pi_{t_{k+1}}(1,1)\mu^{+}_{c}\mu^{-}_{c}\big]^2 \\
&\quad - \pi^{+}_{t_{k+1}}\pi_{t_{k+1}}(1,1)\alpha^{2}_{t_{k+1}}(\mu^{+}_{c})^2\mu^{-}_{c} \\
&= \mu^{+}_{c}\mu^{-}_{c}\{[2(\alpha^{2}_{t_{k+1}}\pi_{t_{k+1}}(1,1))^2 - 2(\alpha^{1}_{t_{k+1}})^2\pi^{+}_{t_{k+1}}\pi^{-}_{t_{k+1}}]\mu^{+}_{c}\mu^{-}_{c} + 2\alpha^{1}_{t_{k+1}}\pi^{+}_{t_{k+1}}\pi^{-}_{t_{k+1}}\mu^{+}_{c} \\
&\quad + \pi^{+}_{t_{k+1}}[\alpha^{1}_{t_{k+1}}\pi^{-}_{t_{k+1}}\mu^{-}_{c} - \alpha^{2}_{t_{k+1}}\pi_{t_{k+1}}(1,1)\mu^{+}_{c}] - \pi^{+}_{t_{k+1}}\pi^{-}_{t_{k+1}}\}.
\end{aligned}
$$

By (A.4) we have $(\alpha^{2}_{t_{k+1}}\pi_{t_{k+1}}(1,1))^2 \leq (\alpha^{1}_{t_{k+1}})^2\pi^{+}_{t_{k+1}}\pi^{-}_{t_{k+1}}$ and by Lemma 1 we have $\alpha_{t_{k+1}} < 0$, thus the summation of the first two terms in the brackets above is negative. Under the Condition (2.21) and by (A.4), the third term in the brackets is also negative. Thus the coefficients of μ_p^+ in $N({}^{(3)}A_{t_k}^+ + {}^{(3)}A_{t_k}^-)$ is negative. Similarly the coefficients of μ_p^- is also negative. Therefore $N({}^{(3)}A_{t_k}^+ + {}^{(3)}A_{t_k}^-) < 0$ and ${}^{(3)}A_{t_k}^+ + {}^{(3)}A_{t_k}^- > 0$.

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