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A Different Approach to Endpoint Weak-type Estimates for Calderṛn-Zygmund Operators

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Dedicated to my family and friends.

ABSTRACT OF THE DISSERTATION

A Different Approach to Endpoint Weak-type Estimates

for Calderón-Zygmund Operators

by

Cody B. Stockdale

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The study of Calderón-Zygmund singular integral operators is central in harmonic analysis; a Calderón-Zygmund operator, T, is an $L^2(\mathbb{R}^n)$ bounded linear operator that is given by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy$$

in an appropriate sense, where $K(x,y)$ is a singular kernel possessing certain decay and smoothness properties. The weak-type $(1,1)$ property is a key behavior of Calderón-Zygmund operators and asserts that there exists $C > 0$ such that

$$\|Tf\|_{L^{1,\infty}(\mathbb{R}^n)} := \sup_{\lambda>0} \lambda|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leq C\|f\|_{L^1(\mathbb{R}^n)}$$

for all $f \in L^1(\mathbb{R}^n)$. This inequality was originally proved using the Calderón-Zygmund decomposition. To address more general settings, Nazarov, Treil, and Volberg gave a different proof of the weak-type $(1,1)$ estimate.

The aim of this thesis is to investigate weak-type inequalities for linear and multilinear Calderón-Zygmund operators in Euclidean and weighted settings using the Calderón-Zygmund decomposition and ideas inspired by Nazarov, Treil, and Volberg. In the linear setting, a new simple proof of the classical weak-type $(1,1)$ property is given with motivation

from Nazarov, Treil, and Volberg. This technique is adjusted to provide a new proof of a mixed weighted weak-type inequality.

For multilinear Calderón-Zygmund operators, the Nazarov-Treil-Volberg ideas lead to a new proof of the weak-type $\left(1,\ldots,1;\frac{1}{m}\right)$ estimate. Connecting the weighted and multilinear settings, a weighted weak-type estimate for multilinear Calderón-Zygmund operators is proved. Two proofs for the weighted multilinear inequality are presented – one proof uses the Calderón-Zygmund decomposition, and the other proof uses ideas inspired by Nazarov, Treil, and Volberg.

Additionally, a weak-type (q,q) estimate is proved for Calderón-Zygmund operators whose kernels satisfy an $L^q(\mathbb{R}^n)$-adapted integral smoothness condition, weaker than is typically assumed. Two proofs of the weak-type (q,q) result are presented – one uses the Calderón-Zygmund decomposition and the other is inspired by Nazarov, Treil, and Volberg.

Finally, the Nazarov-Treil-Volberg method is used to investigate the dimensional dependence of the weak-type $(1,1)$ norm of the Riesz transforms. Denoting the j^{th} Riesz transform on $\mathbb{R}^n$ by R_j, we show that the weak-type $(1,1)$ norm of R_j grows at most as a constant times $\log n$.

Chapter 1

Introduction

This thesis focuses on the study of Calderón-Zygmund singular integral operators. Calderón-Zygmund operators are central in harmonic analysis and have far-reaching connections with other fields of analysis such as partial differential equations, operator theory, and complex analysis. Within harmonic analysis, Calderón-Zygmund theory is intimately related to Littlewood-Paley theory and Fourier multiplier theory. More broadly, this theory has close ties with probability, mathematical physics, and ergodic theory.

Let n be a positive integer. We introduce some basics of Calderón-Zygmund theory in the Euclidean setting of $\mathbb{R}^n$ equipped with the standard metric and Lebesgue measure.

Definition 1.0.1. A linear operator T is a *Calderón-Zygmund operator with kernel* K if T is bounded on $L^2(\mathbb{R}^n)$ and is given by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy$$

for smooth compactly supported f and $x \notin \operatorname{supp} f$, where K is a kernel function defined on $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(x,y) : x = y\}$ that, for some $C_K > 0$ and $0 < \delta \leq 1$, satisfies

$$\text{(size)} \qquad |K(x,y)| \leq \frac{C_K}{|x-y|^n} \tag{1.0.1}$$

whenever $x \neq y$ and

$$\text{(smoothness)} \qquad |K(x,y) - K(x',y')| \leq C_K \frac{|x-x'|^\delta + |y-y'|^\delta}{|x-y|^{n+\delta}} \tag{1.0.2}$$

whenever $|x-x'|+|y-y'|\le \frac{1}{2}|x-y|$.

The familiar Hilbert transform and Riesz transforms are included in this general framework.

Example 1.0.1. The Hilbert transform H is given by

$$Hf(x)=\frac{1}{\pi}\,\text{p.v.}\int\limits_{\mathbb{R}}\frac{f(y)}{x-y}\,dy$$

for smooth compactly supported f defined on $\mathbb{R}$ and $x\notin \operatorname{supp} f$. Above, "p.v." stands for principle value, and means that the integral is interpretted as $\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y}\,dy$.

Example 1.0.2. Let $1\le j\le n$. The j^{th} Riesz transform R_j is given by

$$R_jf(x)=\widetilde{C}_n\,\text{p.v.}\int\limits_{\mathbb{R}^n}\frac{x_j-y_j}{|x-y|^{n+1}}f(y)\,dy,$$

for smooth compactly supported f defined on $\mathbb{R}^n$ and $x\notin \operatorname{supp} f$, where $\widetilde{C}_n:=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}$.

A key feature of Calderón-Zygmund operators is that the a priori $L^2(\mathbb{R}^n)$ boundedness and the smoothness assumption (1.0.2) imply that all Calderón-Zygmund operators have bounded extensions on $L^p(\mathbb{R}^n)$ for any $1<p<\infty$. However, Calderón-Zygmund operators generally fail to be bounded on $L^1(\mathbb{R}^n)$. At this $p=1$ endpoint, Calderón-Zygmund operators are instead bounded from $L^1(\mathbb{R}^n)$ to the larger space $L^{1,\infty}(\mathbb{R}^n)$. This is known as the *weak-type* $(1,1)$ *property* and is stated precisely as follows.

Theorem 1.0.1. If T is a Calderón-Zygmund operator, then there exists a constant $C>0$ such that

$$\|Tf\|_{L^{1,\infty}(\mathbb{R}^n)}:=\sup_{\lambda>0}\lambda|\{x\in\mathbb{R}^n:|Tf(x)|>\lambda\}|\le C\|f\|_{L^1(\mathbb{R}^n)}$$

for all $f\in L^1(\mathbb{R}^n)$.

Remark 1.0.1. Notice that $L^1(\mathbb{R}^n) \subsetneq L^{1,\infty}(\mathbb{R}^n)$. Indeed, by Chebyshev's inequality, we have

$$\|f\|_{L^{1,\infty}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \leq \sup_{\lambda>0} \lambda \left(\frac{1}{\lambda} \|f\|_{L^1(\mathbb{R}^n)} \right) = \|f\|_{L^1(\mathbb{R}^n)}$$

for any $f \in L^1(\mathbb{R}^n)$. Also, the containment is strict since $f(x) = \frac{1}{|x|^n} \notin L^1(\mathbb{R}^n)$, but

$$\sup_{\lambda>0} \lambda|\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| = \sup_{\lambda>0} \lambda|B(x, \lambda^{-\frac{1}{n}})| = v_n \sup_{\lambda>0} \lambda(\lambda^{-1}) = v_n,$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$. This implies that the weak-type $(1,1)$ property is a weaker condition than $L^1(\mathbb{R}^n)$ boundedness, and therefore Theorem 1.0.1 should be viewed as a substitute for the failed $L^1(\mathbb{R}^n)$ boundedness of Calderón-Zygmund operators.

Theorem 1.0.1 was originally proved using a standard technique called the *Calderón-Zygmund decomposition*, see the original paper [4] and also [9, 10, 28]. This decomposition relies on the *doubling property*, which for a Borel measure μ on a metric space X states that there exists $C > 0$ such that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \tag{1.0.3}$$

for all $x \in X$ and all $r > 0$, where $B(x, r)$ denotes the ball centered at x with radius r. A metric measure space X is called a *space of homogeneous type* if the underlying measure μ possesses the doubling property (1.0.3).

In [23], Nazarov, Treil, and Volberg recovered the basic theory of Calderón-Zygmund operators in a setting where the doubling property is replaced by the following *polynomial growth condition*: there exist $C, m > 0$ such that

$$\mu(B(x, r)) \leq Cr^m \tag{1.0.4}$$

for all $x \in X$ and all $r > 0$. A metric measure space (X, d, μ) is called a *nonhomogeneous space* if the measure μ satisfies the polynomial growth condition (1.0.4). Calderón-Zygmund operators can be defined on nonhomogeneous spaces by replacing $\mathbb{R}^n$ with X, Lebesgue measure with μ, and $|a - b|$ for $a, b \in \mathbb{R}^n$ with $d(a, b)$ for $a, b \in X$ in Definition 1.0.1.

The following theorem was proved in [23].

Theorem 1.0.2. If T is a Calderón-Zygmund operator defined on a nonhomogeneous space X, then there exists a constant $C > 0$ such that

$$\|Tf\|_{L^{1,\infty}(X,\mu)} \leq C\|f\|_{L^1(X,\mu)}$$

for all $f \in L^1(X, \mu)$.

Since the doubling property is not available in nonhomogeneous settings, the authors of [23] developed a new technique to prove the weak-type $(1, 1)$ estimate. We refer to this technique as the *Nazarov-Treil-Volberg method.* The purpose of this thesis is to investigate weak-type inequalities for various types of singular integral operators using the Calderón-Zygmund decomposition and the Nazarov-Treil-Volberg method.

Recall that Lebesgue measure on $\mathbb{R}^n$ satisfies

$$|B(x, r)| = v_n r^n$$

for all $x \in \mathbb{R}^n$, and all $r > 0$. In particular, Lebesgue measure satisfies both the doubling condition (1.0.3) and the polynomial growth condition (1.0.4), and so the Calderón-Zygmund decomposition method and the Nazarov-Treil-Volberg method both can be understood in the Euclidean setting of Theorem 1.0.1. We next compare the Calderón-Zygmund decomposition method and the Nazarov-Treil-Volberg method for proving the weak-type $(1, 1)$ inequality in the context of Euclidean spaces. Refer to Appendices A and B for full details on the

Calderón-Zygmund decomposition proof and the Nazarov-Treil-Volberg proof.

To prove the weak-type $(1,1)$ property, one shows

$$|\{|Tf| > \lambda\}| \leq \frac{C}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}$$

for all $\lambda > 0$ and all $f \in L^1(\mathbb{R}^n)$. Both techniques involve decomposing f into summands,

$$f = g + b = g + \sum_{j=1}^{\infty} b_j,$$

where g is "good" and b is "bad," and then controlling

$$|\{|Tf| > \lambda\}| \leq \left|\left\{|Tg| > \frac{\lambda}{2}\right\}\right| + \left|\left\{|Tb| > \frac{\lambda}{2}\right\}\right|.$$

In both arguments, the term involving g is handled by using Chebyshev's inequality, the boundedness of T on $L^2(\mathbb{R}^n)$, and the $L^\infty(\mathbb{R}^n)$ control of g. The terms involving b are estimated differently.

Much of the effort in the Calderón-Zygmund decomposition method is spent in carefully decomposing f into its "good" and "bad" parts so that the functions b_j have mean value zero and have useful $L^1(\mathbb{R}^n)$ control. This decomposition involves averages of f, the Lebesgue differentiation theorem, and the doubling property. After defining an exceptional set, Ω^*, in terms of the supports of the b_j, one estimates

$$\left|\left\{|Tb| > \frac{\lambda}{2}\right\}\right| \leq |\Omega^*| + \left|\left\{x \in \mathbb{R}^n \setminus \Omega^* : |Tb(x)| > \frac{\lambda}{2}\right\}\right|.$$

The first term is controlled due to properties of the Calderón-Zygmund decomposition and the doubling property. The final term is controlled using cancellation of the b_j, the smoothness assumption of the kernel of T, and the $L^1(\mathbb{R}^n)$ control of the b_j.

Using ideas from [23], the decomposition of f into its "good" and "bad" parts is more direct. The exceptional set is defined explicitly as

$$\Omega := \{|f| > \lambda\},$$

then g and b are defined by

$$g := f\mathbb{1}_{\mathbb{R}^n\setminus\Omega} \qquad \text{and} \qquad b := f\mathbb{1}_{\Omega}.$$

The b_j are defined by applying a Whitney decomposition to write Ω as a disjoint union of cubes and restricting b to each cube. To introduce cancellation in the b_j, point-mass measures, ν_j, are constructed at points within $\operatorname{supp} b_j$. Adding and subtracting T applied to a linear combination of the ν_j reduces the weak-type $(1, 1)$ estimate to proving

$$\|T\nu\|_{L^{1,\infty}(X,\mu)} \leq C\|\nu\|, \tag{1.0.5}$$

where ν is a finite linear combination of point-mass measures and $\|\nu\|$ denotes the total variation of ν. Inequality 1.0.5 involves approximating ν by appropriately constructed Borel sets, and then it is left to estimate a final term using the size condition of the Calderón-Zygmund kernel, the polynomial growth condition, a duality trick involving the adjoint of T, and control of the maximal truncation operator.

The Nazarov-Treil-Volberg method outlined above was originally written for nonhomogeneous spaces and avoids the doubling property; however, the doubling property can again be used when working in the Euclidean setting. In Section 2.2, the doubling property is used to give a new simple proof of the classical weak-type $(1, 1)$ property in the Euclidean setting inspired by ideas of Nazarov, Treil, and Volberg. In this setting, we no longer need estimate (1.0.5) involving point-masses, the size condition of the kernel, the polynomial growth condition, duality, or the maximal truncation operator control. Instead, we obtain cancellation by directly approximating with explicitly constructed Borel sets and, due to the doubling property of Lebesgue measure, we easily bound the remaining term.

In weighted Lebesgue settings, harmonic analysts work with positive, locally finite measures that are absolutely continuous with respect to Lebesgue measure, called *weights*, rather than Lebesgue measure itself. In [16], Hunt, Muckenhoupt, and Wheeden characterized the boundedness of the Hilbert transform on weighted L^p spaces for $1 < p < \infty$ with the class of weights A_p. Generalizing the classes to $\mathbb{R}^n$, we say a weight w satisfies the A_p *condition* for $1 \leq p < \infty$ if

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\,dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{1-p'}\,dx \right)^{p-1} < \infty,$$

where p' is the Hölder conjugate of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$ and the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$ with sides parallel to the coordinate axes; when $p = 1$, the quantity $\left(\frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p-1}$ is interpreted as $(\inf_Q w)^{-1}$.

The weak-type $(1,1)$ inequality appears in many forms in weighted settings. In [25], Ombrosi, Pérez, and Recchi proved the following quantitative weighted weak-type $(1,1)$ inequality using the Calderón-Zygmund decomposition.

Theorem 1.0.3. If $1 \leq p < \infty$ and $w \in A_p$, then there exists $C > 0$ such that

$$\|T(fw)w^{-1}\|_{L^{1,\infty}(w)} \leq C[w]_{A_p} \max\{p, \log(e + [w]_{A_p})\} \|f\|_{L^1(w)}$$

for all $f \in L^1(w)$.

In Section 2.3, a new proof of Theorem 1.0.3 is given using ideas influenced by Nazarov, Treil, and Volberg.

An extension of the classical Calderón-Zygmund theory that received much attention in recent years involves multilinear Calderón-Zygmund operators.

Definition 1.0.2. Let m be a positive integer. A multilinear operator T is a *multilinear Calderón-Zygmund operator with kernel* K if T is bounded from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for some $1 < q_1, \ldots, q_m < \infty$ and $\frac{1}{m} < q < \infty$ satisfying $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$ and if T is given by

$$T(f_1, \ldots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m$$

for smooth compactly supported f_i and $x \notin \bigcap_{i=1}^m \operatorname{supp} f_i$, where K is a kernel function defined on $(\mathbb{R}^n)^m \setminus \{(x, y_1, \ldots, y_m) : x = y_1 = \cdots = y_m\}$ that, for some $C_K > 0$ and $0 < \delta \leq 1$, satisfies

$$\text{(size)} \qquad |K(x, y_1, \ldots, y_m)| \leq \frac{C_K}{\left(\sum_{i=1}^m |x - y_i|\right)^{nm}} \tag{1.0.6}$$

whenever $x \neq y_i$ for some i and

(smoothness)

$$|K(y_0, y_1, \ldots, y_j, \ldots, y_m) - K(y_0, y_1, \ldots, y_j', \ldots, y_m)| \leq C_K \frac{|y_j - y_j'|^\delta}{\left(\sum_{i=1}^m |y_0 - y_i|\right)^{nm+\delta}} \tag{1.0.7}$$

for each $j \in \{0, \ldots, m\}$ whenever $|y_j - y_j'| \leq \dfrac{1}{2} \max\limits_{1 \leq i \leq m} |y_0 - y_i|$

In [14], Grafakos and Torres laid the groundwork for the study of multilinear Calderón-Zygmund operators. In particular, they proved the following weak-type $\left(1, \ldots, 1; \frac{1}{m}\right)$ property.

Theorem 1.0.4. If T is a multilinear Calderón-Zygmund operator, then there exists $C > 0$

such that

$$\|T(f_1,\ldots,f_m)\|_{L^{\frac{1}{m},\infty}(\mathbb{R}^n)} := \sup_{\lambda>0} \lambda|\{|T(f_1,\ldots,f_m)|>\lambda\}|^m \le C\prod_{i=1}^m \|f_i\|_{L^1(\mathbb{R}^n)}$$

for all $f_1,\ldots,f_m \in L^1(\mathbb{R}^n)$.

Their proof, as well as other proofs containing this result (see [22,26]), relies on the Calderón-Zygmund decomposition. A new proof of Theorem 1.0.4 using a variation of the Nazarov-Treil-Volberg method is given in Chapter 3.

In [19], Lerner, Ombrosi, Pérez, Torres, and Trujillo-González connected the weighted and multilinear settings by introducing the multilinear $A_{\vec{P}}$ classes. We use the following notation for multilinear $A_{\vec{P}}$ weights: $1 \le p_1,\ldots,p_m < \infty$, $\frac{1}{m} \le p < \infty$ satisfies $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\vec{P} = (p_1,\ldots,p_m)$, $\vec{w} = (w_1,\ldots,w_m)$, and $v_{\vec{w}} = \prod_{i=1}^m w_i^{\frac{p}{p_i}}$. We say $\vec{w} \in A_{\vec{P}}$ if

$$[\vec{w}]_{A_{\vec{P}}} := \sup_Q \left(\frac{1}{|Q|}\int_Q v_{\vec{w}}\right)^{\frac{1}{p}} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i^{1-p_i'}\right)^{\frac{1}{p_i'}} < \infty;$$

when $p_i = 1$, the quantity $\left(\frac{1}{|Q|}\int_Q w_i^{1-p_i'}\right)^{\frac{1}{p_i'}}$ is interpreted as $(\inf_Q w_i)^{-1}$. Note that the quantities $[\vec{w}]_{A_{\vec{P}}}^p$ and $[w]_{A_p}$ coincide when $m = 1$.

We give two proofs of the following theorem in Chapter 4.

Theorem 1.0.5. If T is a multilinear Calderón-Zygmund operator and $\vec{w} \in A_{(1,\ldots,1)}$, then there exists $C > 0$ such that

$$\left\|T\left(f_1 w_1 v_{\vec{w}}^{\frac{1-m}{m}},\ldots,f_m w_m v_{\vec{w}}^{\frac{1-m}{m}}\right) v_{\vec{w}}^{-1}\right\|_{L^{\frac{1}{m},\infty}(v_{\vec{w}})} \le C[v_{\vec{w}}]_{A_1}^{2m^2+2m-2} \prod_{i=1}^m \|f_i\|_{L^1(w_i)}$$

for all $f_i \in L^1(w_i)$.

The first proof uses the Calderón-Zygmund decomposition and is a weighted version of the proof in [26]; the second proof uses the Nazarov-Treil-Volberg method and is a weighted version of the proof in [31].

Motivated by the search for sharp sufficient conditions on the kernel K guaranteeing boundedness of the associated Calderón-Zygmund operator on $L^p(\mathbb{R}^n)$, in Chapter 5, we introduce the following L^q-adapted integral smoothness condition: we say that a kernel function K defined on $\mathbb{R}^n \setminus \{0\}$ is in H_q, $1 \leq q < \infty$, if

$$[K]_{H_q} := \sup_{R>0} \left[\frac{1}{v_n R^n} \int_{|y| \leq R} \left(\int_{|x| \geq 2R} |K(x-y) - K(x)| \, dx \right)^q dy \right]^{\frac{1}{q}} < \infty, \tag{1.0.8}$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$. Hypothesis (1.0.8) is implied by the standard smoothness condition (1.0.2). Assuming this weaker smoothness condition, we prove the following theorem.

Theorem 1.0.6. Let $1 \leq q < \infty$ and $K \in H_{q'}$. If the associated singular integral operator T is bounded on $L^s(\mathbb{R}^n)$ for some $s \in (q, \infty]$, then there exists $C > 0$ such that

$$\|Tf\|_{L^{q,\infty}(\mathbb{R}^n)} := \sup_{\lambda > 0} \lambda |\{|Tf| > \lambda\}|^{\frac{1}{q}} \leq C \|f\|_{L^q(\mathbb{R}^n)}$$

for all $f \in L^q(\mathbb{R}^n)$. Moreover, T maps $L^q(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ with bound at most a constant multiple of $\|T\|_{L^s(\mathbb{R}^n) \to L^s(\mathbb{R}^n)} + [K]_{H_{q'}}$.

Two proofs of Theorem 1.0.6 are presented. One proof uses the Calderón-Zygmund decomposition, and the other follows the Nazarov-Treil-Volberg model.

Finally, in Chapter 6, we investigate how the weak-type $(1,1)$ norms of the Riesz transforms on $\mathbb{R}^n$ depend on the dimension n. All of the previously discussed methods show that the weak-type $(1,1)$ norm of R_j is at most exponential in n. In [18], Janakiraman improved the exponential dependence by proving the following theorem.

Theorem 1.0.7. There exists an absolute constant $C > 0$ such that

$$\|R_j f\|_{L^{1,\infty}(\mathbb{R}^n)} \leq C \log(n) \|f\|_{L^1(\mathbb{R}^n)}$$

for all $f \in L^1(\mathbb{R}^n)$.

The proof in [18] uses a variation of the Calderón-Zygmund decomposition involving semi-cubes (instead of ordinary cubes) to eliminate dependence on n in the decomposition. The proof then follows by applying this decomposition and using careful estimates involving the geometry of semi-cubes and the kernel of the operator. We give a new proof of Theorem 1.0.7 in Chapter 6 using methods inspired by [23].

The thesis is organized as follows. Chapter 2 includes new proofs of the classical weak-type $(1,1)$ result of Theorem 1.0.1 and the weighted weak-type inequality of Theorem 1.0.3. The multilinear setting is addressed in Chapter 3 where the Nazarov-Treil-Volberg method is adapted to prove the weak-type $\left(1,\ldots,1;\frac{1}{m}\right)$ estimate of Theorem 1.0.4. Chapter 4 connects the weighted and multilinear results by giving two proofs of the weighted weak-type $\left(1,\ldots,1;\frac{1}{m}\right)$ estimate of Theorem 1.0.5. In Chapter 5, we present two proofs of Theorem 1.0.6: the weak-type (q,q) inequality for singular integral operators whose kernels satisfy the $L^q(\mathbb{R}^n)$-adapted integral smoothness condition (1.0.8). Finally, Chapter 6 includes a new proof of the $\log(n)$ dimensional dependence result for the Riesz transforms' weak-type $(1,1)$ norms, stated in Theorem 1.0.7.

Chapter 2

Linear Calderón-Zygmund Operators

2.1 Preliminaries

The work in this chapter can be found in [29]. Throughout this chapter, T will denote a Calderón-Zygmund operator as defined in 1.0.1 and we use the notation $A \lesssim B$ if there exists $C > 0$, possibly depending on n or T, such that $A \leq CB$. If Q is a cube, then rQ denotes the cube with the same center as Q and side length equal to r times the side length of Q.

It is well-known that Calderón-Zygmund operators are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and are unbounded on $L^1(\mathbb{R}^n)$. For the case $p = 1$, we instead have the following fundamental result known as the *weak-type* $(1,1)$ *property*.

Theorem 2.1.1. Any Calderón-Zygmund operator T satisfies

$$\|Tf\|_{L^{1,\infty}(\mathbb{R}^n)} := \sup_{\lambda>0} \lambda|\{|Tf| > \lambda\}| \lesssim \|f\|_{L^1(\mathbb{R}^n)}$$

for all $f \in L^1(\mathbb{R}^n)$.

Theorem 2.1.1 was originally proved using the Calderón-Zygmund decomposition. This decomposition relies on the doubling property, which for a Borel measure μ on a space X means that

$$\mu(B(x,2r)) \lesssim \mu(B(x,r))$$

for all $x \in X$ and all $r > 0$. In [23], Nazarov, Treil, and Volberg proved the weak-type $(1,1)$

property for Calderón-Zygmund operators in a setting where the doubling property of the underlying measure is replaced by the polynomial growth condition

$$\mu(B(x,r)) \lesssim r^m$$

for some $m > 0$ and all $x \in X$ and all $r > 0$. Since the doubling property is not available in this setting, the proof in [23] avoids the Calderón-Zygmund decomposition.

Unlike the setting of [23], the Euclidean space of Theorem 2.1.1 allows the doubling property. We use the doubling property to obtain the main result of Section 2.2 – a new simple proof of Theorem 2.1.1 motivated by the ideas of Nazarov, Treil, and Volberg.

Our proof has some benefits over the Calderón-Zygmund decomposition technique. For example, the decomposition used to write $f = g + b$ in this argument is direct and does not involve studying averages of f or the doubling property. The doubling property is only used later in the proof to gain control over the measure of the exceptional set, E^*. This proof also shows that $L^1(\mathbb{R}^n)$ control of the b_j is not necessary for the weak-type $(1,1)$ estimate and demonstrates a measure-theoretic method to gain cancellation in the b_j.

An application of our technique is given in Section 2.3, where a weak-type $(1,1)$ inequality involving A_p weights is proved. A locally integrable function w on $\mathbb{R}^n$ is called a *weight* if $w(x) > 0$ for almost every $x \in \mathbb{R}^n$. For $1 \leq p < \infty$, the class A_p consists of all weights w satisfying the A_p condition

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{p-1} < \infty,$$

where p' is the Hölder conjugate of p and the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$; when $p = 1$, the quantity $\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\right)^{p-1}$ is interpreted as $(\inf_Q w)^{-1}$. Notice that if

$w \in A_p$, then $w(x)dx$ is a doubling measure with

$$w(Q(x, ar)) \leq a^{np}[w]_{A_p} w(Q(x, r))$$

for all $x \in \mathbb{R}^n$, $r > 0$, and $a > 1$, where we represent the quantity $\int_A w(x)\,dx$ by $w(A)$; see [9, 10].

The following theorem was proved using the Calderón-Zygmund decomposition by Ombrosi, Pérez, and Recchi in [25].

Theorem 2.1.2. If $1 \leq p < \infty$ and $w \in A_p$, then

$$\|T(fw)w^{-1}\|_{L^{1,\infty}(w)} \lesssim [w]_{A_p} \max\{p, \log(e + [w]_{A_p})\} \|f\|_{L^1(w)}$$

for all $f \in L^1(w)$.

A new proof of Theorem 2 is given in Section 2.3. See [3, 7, 20, 24] for related mixed weak-type inequalities.

We first prove two lemmas.

Lemma 2.1.1. If $f \in L^1(\mathbb{R}^n)$ is supported on $Q(x, r)$ and $\int_{Q(x,r)} f(y)dy = 0$ for some $x \in \mathbb{R}^n$ and $r > 0$, then

$$\int\limits_{\mathbb{R}^n \setminus Q(x, 2\sqrt{n}r)} |Tf(y)|dy \lesssim \|f\|_{L^1(\mathbb{R}^n)}.$$

Proof. First, notice that since $\int_{Q(x,r)} f(y)dy = 0$ and $\operatorname{supp} f \subseteq Q(x, r)$,

$$|Tf(y)| = \left| \int\limits_{Q(x,r)} K(y, z)f(z)dz \right| = \left| \int\limits_{Q(x,r)} (K(y, z) - K(y, x))f(z)\,dz \right|.$$

Therefore, using Fubini's theorem and the smoothness estimate of K, we see

$$\begin{aligned}\int_{\mathbb{R}^n \setminus Q(x,2\sqrt{n}r)} |Tf(y)|\,dy &\leq \int_{\mathbb{R}^n \setminus Q(x,2\sqrt{n}r)} \int_{Q(x,r)} |K(y,z)-K(y,x)||f(z)|\,dzdy \\ &= \int_{Q(x,r)} |f(z)| \int_{\mathbb{R}^n \setminus Q(x,2\sqrt{n}r)} |K(y,z)-K(y,x)|\,dydz \\ &\lesssim \int_{Q(x,r)} |f(z)| \int_{|x-y|\geq 2|x-z|} \frac{|x-z|^\delta}{|x-y|^{n+\delta}}\,dydz \quad \lesssim \|f\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

□

Lemma 2.1.2. Let μ be a doubling measure on $\mathbb{R}^n$ such that

$$\mu(Q(x,ar)) \leq C_{\mu,a}\mu(Q(x,r))$$

for all $x \in \mathbb{R}^n$, $r > 0$, and $a > 1$. If N is a positive integer, then

$$\mu\Big(\bigcup_{j=1}^N Q(x_j,ar_j)\Big) \leq C_{\mu,a}\mu\Big(\bigcup_{j=1}^N Q(x_j,r_j)\Big)$$

for all $x_1,\ldots,x_N \in \mathbb{R}^n$, $r_1,\ldots,r_N > 0$, and $a > 1$.

The inequality still holds if the cubes are replaced by balls induced by any ℓ^p norm on $\mathbb{R}^n$.

Proof. Reorder the r_j to assume that $r_1 \geq r_2 \geq \cdots \geq r_N$. Set

$$F_1 := Q(x_1,r_1), \quad F_j := Q(x_j,r_j) \setminus \bigcup_{k=1}^{j-1} F_k, \quad \text{and} \quad F := \bigcup_{j=1}^N F_j = \bigcup_{j=1}^N Q(x_j,r_j).$$

Similarly, set

$$F_1^* := Q(x_1,ar_1), \quad F_j^* := Q(x_j,ar_j) \setminus \bigcup_{k=1}^{j-1} F_k^*, \quad \text{and} \quad F^* := \bigcup_{j=1}^N F_j^* = \bigcup_{j=1}^N Q(x_j,ar_j).$$

For each $j \in \{1, 2, \ldots, N\}$, we claim that F_j^* is a dilation of F_j in the sense that

$$F_j^* \subseteq \widetilde{F}_j,$$

where $\widetilde{F}_j := \{a(y - x_j) + x_j : y \in F_j\}$. Note that $\mu(\widetilde{F}_j) \leq C_{\mu,a}\mu(F_j)$ since $\widetilde{F}_j$ is obtained from F_j by composing a translation and a dilation by a factor of a. Assuming the claim, we conclude

$$\mu(F^*) \leq \sum_{j=1}^{N} \mu(F_j^*) \leq \sum_{j=1}^{N} \mu(\widetilde{F}_j) \leq C_{\mu,a} \sum_{j=1}^{N} \mu(F_j) = C_{\mu,a}\mu(F).$$

It remains to prove $F_j^* \subseteq \widetilde{F}_j$. Let $x \in F_j^*$. Since $F_j^* \subseteq Q(x_j, ar_j)$, we can write $x = a(y - x_j) + x_j$ for some $y \in Q(x_j, r_j)$. Since $Q(x_j, r_j) \subseteq \bigcup_{k=1}^{j} F_k$ and the F_k are pairwise disjoint, $y \in F_{k_0} \subseteq Q(x_{k_0}, r_{k_0})$ for some distinguished $1 \leq k_0 \leq j$. Suppose that $k_0 < j$. Since $r_{k_0} \geq r_j$, we have

$$|x - x_{k_0}|_\infty = |a(y - x_j) + x_j - x_{k_0}|_\infty \leq (a-1)|y - x_j|_\infty + |y - x_{k_0}|_\infty$$

$$< (a-1)r_j + r_{k_0} \leq ar_{k_0}.$$

This implies $x \in Q(x_{k_0}, ar_{k_0}) \subseteq \bigcup_{k=1}^{j-1} F_K^*$, contradicting the fact that $x \in F_j^*$. Therefore $y \in F_j$, and $x \in \widetilde{F}_j$.

□

Remark 2.1.1. Lemma 2.1.2 implies

$$\left|\bigcup_{j=1}^{N} Q(x_j, ar_j)\right| \leq a^n \left|\bigcup_{j=1}^{N} Q(x_j, r_j)\right|,$$

and for $w \in A_p$,

$$w\left(\bigcup_{j=1}^{N} Q(x_j, ar_j)\right) \leq a^{np}[w]_{A_p} w\left(\bigcup_{j=1}^{N} Q(x_j, r_j)\right).$$

2.2 Unweighted Estimate

Proof of Theorem 2.1.1. Let $\lambda > 0$ be given. We wish to show

$$|\{|Tf| > \lambda\}| \lesssim \frac{1}{\lambda} \|f\|_{L^1(\mathbb{R}^n)}.$$

By density, we may assume f is a nonnegative continuous function with compact support. Set

$$\Omega := \{f > \lambda\}.$$

Apply a Whitney decomposition to write

$$\Omega = \bigcup_{j=1}^{\infty} Q_j,$$

a disjoint union of dyadic cubes where

$$2\operatorname{diam}(Q_j) \leq \operatorname{dist}(Q_j, \mathbb{R}^n \setminus \Omega) \leq 8\operatorname{diam}(Q_j).$$

Put

$$g := f\mathbb{1}_{\mathbb{R}^n \setminus \Omega}, \qquad b := f\mathbb{1}_{\Omega}, \qquad \text{and} \qquad b_j := f\mathbb{1}_{Q_j}.$$

Then

$$f = g + b = g + \sum_{j=1}^{\infty} b_j,$$

where

(1) $\|g\|_{L^\infty(\mathbb{R}^n)} \leq \lambda$ and $\|g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$,

(2) the b_j are supported on pairwise disjoint cubes Q_j with $\sum_{j=1}^{\infty} |Q_j| \leq \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}$, and

(3) $\|b\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$.

Then

$$|\{|Tf| > \lambda\}| \leq \left|\left\{|Tg| > \frac{\lambda}{2}\right\}\right| + \left|\left\{|Tb| > \frac{\lambda}{2}\right\}\right|.$$

To control the first term, use Chebyshev's inequality, the boundedness of T on $L^2(\mathbb{R}^n)$, and property (1) to estimate

$$\begin{aligned}\left|\left\{|Tg| > \frac{\lambda}{2}\right\}\right| &\lesssim \frac{1}{\lambda^2}\int\limits_{\mathbb{R}^n} |Tg(x)|^2 dx \\ &\lesssim \frac{1}{\lambda^2}\int\limits_{\mathbb{R}^n} |g(x)|^2 dx \\ &\leq \frac{1}{\lambda}\int\limits_{\mathbb{R}^n} |g(x)| dx \\ &\leq \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

For positive integers N, set $b^{(N)} := \sum_{j=1}^N b_j$. To control the second term, it suffices to handle $\left|\left\{|Tb^{(N)}| > \frac{\lambda}{2}\right\}\right|$ uniformly in N. Let c_j denote the center of Q_j and let $a_j := \int_{Q_j} b_j(x)dx$. Set

$$E_1 := Q(c_1, r_1),$$

where $r_1 > 0$ is chosen so that $|E_1| = \frac{a_1}{\lambda}$. In general, for $j = 2, 3, \ldots, N$, set

$$E_j := Q(c_j, r_j) \setminus \bigcup_{k=1}^{j-1} E_k,$$

where $r_j > 0$ is chosen so that $|E_j| = \frac{a_j}{\lambda}$. Note that such E_j exist since the function $r \mapsto |Q(x, r)|$ is continuous for each $x \in \mathbb{R}^n$. Define

$$E := \bigcup_{j=1}^N E_j = \bigcup_{j=1}^N Q(c_j, r_j) \qquad \text{and} \qquad E^* := \bigcup_{j=1}^N Q(c_j, 2\sqrt{n}r_j).$$

Then

$$\left|\left\{|Tb^{(N)}| > \frac{\lambda}{2}\right\}\right| \leq \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\begin{aligned}
&\mathrm{I} := |\Omega \cup E^*|,\\
&\mathrm{II} := \left|\left\{x \in \mathbb{R}^n \setminus (\Omega \cup E^*) : |T(b^{(N)} - \lambda \mathbb{1}_E)(x)| > \frac{\lambda}{4}\right\}\right|, \quad \text{and}\\
&\mathrm{III} := \left|\left\{|T(\mathbb{1}_E)| > \frac{1}{4}\right\}\right|.
\end{aligned}$$

The control of I follows from Lemma 2.1.2, Chebyshev's inequality, and property (3)

$$\mathrm{I} \leq |\Omega| + |E^*| \lesssim |\Omega| + |E| \leq \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)} + \frac{1}{\lambda}\|b^{(N)}\|_{L^1(\mathbb{R}^n)} \lesssim \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.$$

For II, use Chebyshev's inequality and Lemma 2.1.1, which applies since

$$\operatorname{supp}(b_j - \lambda \mathbb{1}_{E_j}) \subseteq Q_j \cup Q(c_j, r_j), \quad \int_{\mathbb{R}^n} b_j(x) - \lambda \mathbb{1}_{E_j}(x) dx = 0, \text{ and}$$

$$2\sqrt{n}Q_j \cup Q(c_j, 2\sqrt{n}r_j) \subseteq \Omega \cup E^*,$$

to estimate

$$\begin{aligned}
\mathrm{II} &\lesssim \frac{1}{\lambda} \int_{\mathbb{R}^n \setminus (\Omega \cup E^*)} |T(b^{(N)} - \lambda \mathbb{1}_E)(x)| dx\\
&\leq \frac{1}{\lambda} \sum_{j=1}^N \int_{\mathbb{R}^n \setminus (\Omega \cup E^*)} |T(b_j - \lambda \mathbb{1}_{E_j})(x)| dx\\
&\lesssim \frac{1}{\lambda} \sum_{j=1}^N \|b_j - \lambda \mathbb{1}_{E_j}\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

Using the triangle inequality and property (3), we have

$$\mathrm{II} \lesssim \frac{1}{\lambda}\sum_{j=1}^{N}\left(\|b_j\|_{L^1(\mathbb{R}^n)} + \|\lambda \mathbb{1}_{E_j}\|_{L^1(\mathbb{R}^n)}\right) \lesssim \frac{1}{\lambda}\sum_{j=1}^{N}\|b_j\|_{L^1(\mathbb{R}^n)} \leq \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.$$

To control III, use Chebyshev's inequality, the boundedness of T on $L^2(\mathbb{R}^n)$, and the fact that $|E| \leq \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}$ to estimate

$$\mathrm{III} \lesssim \int_{\mathbb{R}^n} |T(\mathbb{1}_E)(x)|^2\, dx \lesssim |E| \leq \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.$$

Putting all estimates together, we get

$$|\{|Tf| > \lambda\}| \lesssim \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.$$

□

2.3 Weighted Estimate

The main difficulty in adapting the proof of Section 3 to the weighted setting is controlling the term with the "good" function. The following optimal quantitative result is used to handle this term, see [17].

Theorem 2.3.1. If $1 < p < \infty$ and $w \in A_p$, then T is bounded on $L^p(w)$ and

$$\|T\|_{L^p(w)\to L^p(w)} \lesssim pp'[w]_{A_p}^{\max\left\{1, \frac{1}{p-1}\right\}}.$$

Proof of Theorem 2.1.2. Let $\lambda > 0$ be given. We wish to show

$$w(\{|T(fw)|w^{-1} > \lambda\}) \lesssim [w]_{A_p} \max\{p, \log(e + [w]_{A_p})\}\frac{1}{\lambda}\|f\|_{L^1(w)}.$$

Assume that $\int_{\mathbb{R}^n} w(x)dx > \lambda^{-1}\|f\|_{L^1(w)}$ (otherwise there is nothing to prove). By density, we may assume f is a nonnegative continuous function with compact support. Set

$$\Omega := \{f > \lambda\}.$$

Apply a Whitney decomposition to write

$$\Omega = \bigcup_{j=1}^{\infty} Q_j,$$

a disjoint union of dyadic cubes where

$$2\text{diam}(Q_j) \leq \text{dist}(Q_j, \mathbb{R}^n \setminus \Omega) \leq 8\text{diam}(Q_j).$$

Put

$$g := f\mathbb{1}_{\mathbb{R}^n\setminus\Omega}, \qquad b := f\mathbb{1}_{\Omega}, \qquad \text{and} \qquad b_j := f\mathbb{1}_{Q_j}.$$

Then

$$f = g + b = g + \sum_{j=1}^{\infty} b_j,$$

where

(1) $\|g\|_{L^\infty(\mathbb{R}^n)} \leq \lambda$ and $\|g\|_{L^1(w)} \leq \|f\|_{L^1(w)}$,

(2) the b_j are supported on pairwise disjoint cubes Q_j with $\sum_{j=1}^{\infty} w(Q_j) \leq \frac{1}{\lambda}\|f\|_{L^1(w)}$, and

(3) $\|b\|_{L^1(w)} \leq \|f\|_{L^1(w)}$.

Then

$$w(\{|T(fw)|w^{-1} > \lambda\}) \leq w\left(\left\{|T(gw)|w^{-1} > \frac{\lambda}{2}\right\}\right) + w\left(\left\{|T(bw)|w^{-1} > \frac{\lambda}{2}\right\}\right).$$

To control the first term, let $r > p$ be a constant to be chosen later (we will actually choose r so that $r > 2$ as well). Then $w \in A_r$, $[w]_{A_r} \leq [w]_{A_p}$, $w^{1-r'} \in A_{r'}$, and $\left[w^{1-r'}\right]_{A_{r'}} = [w]_{A_r}^{r'-1}$. Use Chebyshev's inequality, Theorem 2.3.1, property (1), and the facts listed above to estimate

$$\begin{aligned} w\left(\left\{|T(gw)|w^{-1} > \frac{\lambda}{2}\right\}\right) &\lesssim \frac{1}{\lambda^{r'}} \int\limits_{\mathbb{R}^n} |T(gw)(x)|^{r'} w(x)^{1-r'} dx \\ &\lesssim \left(rr' \left[w^{1-r'}\right]_{A_{r'}}^{\max\left\{1, \frac{1}{r'-1}\right\}}\right)^{r'} \frac{1}{\lambda^{r'}} \int\limits_{\mathbb{R}^n} |g(x)|^{r'} w(x) dx \\ &\lesssim r^{r'} [w]_{A_r}^{r'} \frac{1}{\lambda} \int\limits_{\mathbb{R}^n} |g(x)| w(x) dx \\ &\leq r^{r'} [w]_{A_p}^{r'} \frac{1}{\lambda} \|f\|_{L^1(w)}. \end{aligned}$$

We next address the factors $r^{r'}$ and $[w]_{A_p}^{r'}$. First consider $r^{r'}$. Let $h(x) = \frac{1}{x}(1+x)^{1+\frac{1}{x}}$. Note that $h(1) = 4$ and that $h'(x) = \frac{-1}{x^3}(1+x)^{1+\frac{1}{x}} \log(1+x) \leq 0$ for all $x \in [1, \infty)$. Thus $h(x) \leq 4$ for all $x \in [1, \infty)$. In particular, letting

$$r = 1 + \max\{p, \log(e + [w]_{A_p})\} > 2$$

and computing

$$r' = 1 + \frac{1}{\max\{p, \log(e + [w]_{A_p})\}} < 2,$$

we have

$$\frac{r^{r'}}{\max\{p, \log(e + [w]_{A_p})\}} = h(\max\{p, \log(e + [w]_{A_p})\}) \leq 4.$$

Thus

$$r^{r'} \leq 4 \max\{p, \log(e + [w]_{A_p})\}.$$

Now consider $[w]_{A_p}^{r'}$. Set $k(x) = x^{\frac{1}{\log(e+x)}}$. Notice that $k(1) = 1$, $\lim_{x\to\infty} k(x) = e$, and $k'(x) = x^{\frac{1}{\log(e+x)}} \left(\frac{1}{x\log(e+x)} - \frac{\log(x)}{(e+x)(\log(e+x))^2} \right) \geq 0$ for all $x \in [1,\infty)$. Thus $1 \leq k(x) \leq e$ for all $x \in [1,\infty)$. In particular,

$$[w]_{A_p}^{r'-1} = [w]_{A_p}^{\frac{1}{\max\{p,\log(e+[w]_{A_p})\}}} \leq [w]_{A_p}^{\frac{1}{\log(e+[w]_{A_p})}} = k([w]_{A_p}) \leq e.$$

Thus

$$[w]_{A_p}^{r'} \leq e[w]_{A_p}.$$

Substituting this into the previous estimate yields

$$w\left(\left\{|T(gw)w^{-1}| > \frac{\lambda}{2}\right\}\right) \lesssim r^{r'}\,[w]_{A_p}^{r'} \frac{1}{\lambda}\|f\|_{L^1(w)} \lesssim [w]_{A_p} \max\{p, \log(e+[w]_{A_p})\}\frac{1}{\lambda}\|f\|_{L^1(w)}.$$

For positive integers N, set $b^{(N)} := \sum_{j=1}^N b_j$. To control the second term, it suffices to handle $w\left(\left\{|T(b^{(N)}w)|w^{-1} > \frac{\lambda}{2}\right\}\right)$ uniformly in N. Let c_j denote the center of Q_j and let $a_j := \int_{Q_j} b_j(x)w(x)dx$. Set

$$E_1 := Q(c_1, r_1),$$

where $r_1 > 0$ is chosen so that $w(E_1) = \frac{a_1}{\lambda}$. In general, for $j = 2, 3, \ldots, N$, set

$$E_j := Q(c_j, r_j) \setminus \bigcup_{k=1}^{j-1} E_k,$$

where $r_j > 0$ is chosen so that $w(E_j) = \frac{a_j}{\lambda}$. Note that such E_j exist since the function $r \mapsto w(Q(x,r))$ increases to $w(\mathbb{R}^n)$ as $r \to \infty$, approaches 0 as $r \to 0$, and is continuous from the right for almost every $x \in \mathbb{R}^n$. Define

$$E := \bigcup_{j=1}^N E_j = \bigcup_{j=1}^N Q(c_j, r_j) \qquad \text{and} \qquad E^* := \bigcup_{j=1}^\infty Q(c_j, 2\sqrt{n}r_j).$$

Then

$$w\left(\left\{|T(b^{(N)}w)|w^{-1} > \frac{\lambda}{2}\right\}\right) \leq \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\begin{aligned}
&\mathrm{I} := w(\Omega \cup E^*),\\
&\mathrm{II} := w\left(\left\{x \in \mathbb{R}^n \setminus (\Omega \cup E^*) : |T(b^{(N)}w - \lambda w \mathbb{1}_E)(x)|w(x)^{-1} > \frac{\lambda}{4}\right\}\right), \quad \text{and}\\
&\mathrm{III} := w\left(\left\{|T(w\mathbb{1}_E)|w^{-1} > \frac{1}{4}\right\}\right).
\end{aligned}$$

The control of I follows from Lemma 2.1.2, Chebyshev's inequality, and property (3)

$$\mathrm{I} \leq w(\Omega) + w(E^*) \lesssim w(\Omega) + [w]_{A_p} w(E) \leq \frac{1}{\lambda}\|f\|_{L^1(w)} + [w]_{A_p}\frac{1}{\lambda}\|b^{(N)}\|_{L^1(w)}$$

$$\lesssim [w]_{A_p} \max\{p, \log(e + [w]_{A_p})\}\frac{1}{\lambda}\|f\|_{L^1(w)}.$$

For II, use Chebyshev's inequality and Lemma 2.1.1, which applies since

$$\operatorname{supp}(b_j w - \lambda w \mathbb{1}_{E_j}) \subseteq Q_j \cup Q(c_j, r_j), \int_{\mathbb{R}^n} b_j(x)w(x) - \lambda w(x)\mathbb{1}_{E_j}(x)dx = 0, \text{ and}$$

$$2\sqrt{n}Q_j \cup Q(c_j, 2\sqrt{n}r_j) \subseteq \Omega \cup E^*,$$

to estimate

$$\begin{aligned}
\mathrm{II} &\lesssim \frac{1}{\lambda}\int_{\mathbb{R}^n\setminus(\Omega\cup E^*)} |T(b^{(N)}w - \lambda w\mathbb{1}_E)(x)|dx\\
&\leq \frac{1}{\lambda}\sum_{j=1}^{N}\int_{\mathbb{R}^n\setminus(\Omega\cup E^*)} |T(b_j w - \lambda w \mathbb{1}_{E_j})(x)|dx\\
&\lesssim \frac{1}{\lambda}\sum_{j=1}^{N}\|b_j - \lambda\mathbb{1}_{E_j}\|_{L^1(w)}.
\end{aligned}$$

Using the triangle inequality and property (3), we have

$$\mathrm{II} \lesssim \frac{1}{\lambda}\sum_{j=1}^{N}\left(\|b_j\|_{L^1(w)} + \|\lambda \mathbb{1}_{E_j}\|_{L^1(w)}\right) \lesssim \frac{1}{\lambda}\sum_{j=1}^{N}\|b_j\|_{L^1(w)}$$

$$\leq [w]_{A_p}\max\{p, \log(e+[w]_{A_p})\}\frac{1}{\lambda}\|f\|_{L^1(w)}.$$

To control III, let

$$r = 1 + \max\{p, \log(e+[w]_{A_p})\}$$

and use Chebyshev's inequality, Theorem 2.3.1, and the properties of w described when bounding $w(\{|T(gw)|w^{-1} > \frac{\lambda}{2}\})$ above to estimate

$$\mathrm{III} \lesssim \int_{\mathbb{R}^n} |T(w(\mathbb{1}_E))|^{r'} w(x)^{1-r'} dx \lesssim \left(rr'\left[w^{1-r'}\right]_{A_{r'}}^{\max\left\{1,\frac{1}{r'-1}\right\}}\right)^{r'} w(E) \lesssim r^{r'}\,[w]_{A_r}^{r'}\,\frac{1}{\lambda}\|f\|_{L^1(w)}.$$

As before, $r^{r'} \lesssim \max\{p, \log(e+[w]_{A_p})\}$ and $[w]_{A_p}^{r'} \lesssim [w]_{A_p}$, so

$$\mathrm{III} \lesssim [w]_{A_p}\max\{p, \log(e+[w]_{A_p})\}\frac{1}{\lambda}\|f\|_{L^1(w)}.$$

Putting all estimates together, we get

$$w(\{|T(fw)|w^{-1} > \lambda\}) \lesssim [w]_{A_p}\max\{p, \log(e+[w]_{A_p})\}\frac{1}{\lambda}\|f\|_{L^1(w)}.$$

□

Chapter 3

Multilinear Calderón-Zygmund Operators

3.1 Preliminaries

The work in this chapter is joint with Brett Wick and can be found in [31]. Throughout this chapter, T will denote a multilinear Calderón-Zygmund operator as defined in 1.0.2 and we will write $A \lesssim B$ if there exists $C > 0$, possibly depending on n, m, or T, such that $A \leq CB$. If Q is a cube, then rQ denotes the cube with the same center as Q and side length equal to r times the side length of Q.

Much attention has been given to the study of multilinear Calderón-Zygmund operators in recent years, see [8, 12, 14, 19, 21, 22, 26]. The following theorem was first proved in [14] using the Calderón-Zygmund decomposition, see also [22, 26].

Theorem 3.1.1. If T is a multilinear Calderón-Zygmund operator, then

$$\|T(f_1, \ldots, f_m)\|_{L^{\frac{1}{m},\infty}(\mathbb{R}^n)} := \sup_{\lambda>0} \lambda |\{|T(f_1, \ldots, f_m)| > \lambda\}|^m \lesssim \prod_{i=1}^{m} \|f_i\|_{L^1(\mathbb{R}^n)}$$

for all $f_1, \ldots, f_m \in L^1(\mathbb{R}^n)$.

The main result of this chapter is a new proof of Theorem 3.1.1 inspired by ideas of Nazarov, Treil, and Volberg.

Instead of obtaining cancellation by means of the Calderón-Zygmund decomposition, we do so by subtracting terms involving point-mass measures. The argument is then completed by establishing a weak-type estimate on a mixture of linear combinations of point-mass measures and of $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ functions with appropriate $L^\infty(\mathbb{R}^n)$ norm.

Let $\mathcal{M}(\mathbb{R}^n)$ denote the space of $\mathbb{R}$-valued Borel measures on $\mathbb{R}^n$. For $\nu_1, \ldots, \nu_m \in \mathcal{M}(\mathbb{R}^n)$, we write

$$T(\nu_1, \ldots, \nu_m)(x) = \int\limits_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) d\nu(y_1) \cdots d\nu(y_m)$$

for $x \notin \bigcap_{i=1}^m \operatorname{supp} \nu_i$. We denote the total variation of $\nu \in \mathcal{M}(\mathbb{R}^n)$ by $\|\nu\|$. Notice that if $\nu_i = \sum_{j=1}^N a_{i,j}\delta_{x_{i,j}}$ for $i \in \{1, \ldots, m\}$, then

$$T(\nu_1, \ldots, \nu_m)(x) = \sum_{j_1, \ldots, j_m = 1}^{N} \left(\prod_{i=1}^m a_{i,j_i} \right) K(x, x_{1,j_1}, \ldots, x_{m,j_m}).$$

Theorem 3.1.2. If T is a multilinear Calderón-Zygmund operator, $\lambda > 0$, and $l \in \{1, \ldots, m\}$, then

$$|\{|T(\nu_1, \ldots, \nu_l, f_{l+1}, \ldots, f_m)| > \lambda\}| \lesssim \lambda^{-\frac{1}{m}} \left(\prod_{i=1}^{l} \|\nu_i\|^{\frac{1}{m}} \right) \left(\prod_{i=l+1}^{m} \|f_i\|_{L^1(\mathbb{R}^n)}^{\frac{1}{m}} \right)$$

for all $\nu_1, \ldots, \nu_l \in \mathcal{M}(\mathbb{R}^n)$ of the form $\nu_i = \sum_{j=1}^N a_{i,j}\delta_{x_{i,j}}$ and all $f_{l+1}, \ldots, f_m \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfying $\|f_i\|_{L^\infty(\mathbb{R}^n)} \leq \lambda^{\frac{1}{m}}$.

It is not important that the ν_i are applied in the first l slots of T – an identical proof yields the result whenever the set of indices of the ν_i is a nonempty subset of $\{1, \ldots, m\}$.

We will use the uncentered Hardy-Littlewood maximal function, M, given by

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int\limits_Q |f(y)|\, dy,$$

where the supremum is taken over cubes Q containing x.

Lemma 3.1.1. The maximal function is weak-type $(1, 1)$. That is,

$$\|Mf\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}$$

for all $f \in L^1(\mathbb{R}^n)$.

The proofs of Theorem 3.1.1 and Theorem 3.1.2 use the multilinear geometric Hörmander condition described below. This regularity was first introduced in the bilinear setting by Pérez and Torres in [26]. Throughout the rest of this chapter, we use the notation $\vec{y}_{i,k} = (y_i, y_{i+1}, \ldots, y_k)$, $\vec{f}_{i,k} = (f_i, f_{i+1}, \ldots, f_k)$, $\vec{\nu}_{i,k} = (\nu_i, \nu_{i+1}, \ldots, \nu_k)$, $\vec{c}_{(i,j_i),(k,j_k)} = (c_{i,j_i}, c_{i+1,j_{i+1}}, \ldots, c_{k,j_k})$, and $\vec{\nu}_{(i,j_i),(k,j_k)} = (\nu_{i,j_i}, \nu_{i+1,j_{i+1}}, \ldots, \nu_{k,j_k})$.

Lemma 3.1.2. If K is a multilinear Calderón-Zygmund kernel, $l \in \{1, \ldots, m\}$, and $\mathcal{S}_1, \ldots, \mathcal{S}_l$ are countable collections of sets satisfying either

1. each $\mathcal{S}_i = \{S_{i,1}, S_{i,2}, \ldots\}$ consists of dyadic cubes with disjoint interiors or
2. each $\mathcal{S}_i = \{S_{i,1}, S_{i,2}, \ldots\}$ consists of sets satisfying:
   - $S_{i,j}$ have disjoint interiors,
   - $S_{i,j} \subseteq B(c_{i,j}, r_{i,j})$, and
   - $\Omega_i := \bigcup_{j=1}^{\infty} S_{i,j} = \bigcup_{j=1}^{\infty} B(c_{i,j}, r_{i,j})$,

then

$$\sum_{j_1,\ldots,j_l=1}^{\infty} \prod_{i=1}^{l} |S_{i,j_i}| \times \int_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\ldots,y_l) \\ \in \prod_{i=1}^{l} S_{i,j_i}}} \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^{l} \Omega_i^*\right)} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})| dx d\vec{y}_{l+1,m} \lesssim \sum_{i=1}^{l} |\Omega_i|$$

where $\Omega_i^* := \bigcup_{j=1}^{\infty} 2S_{i,j} = \bigcup_{j=1}^{\infty} B(c_{i,j}, 2r_{i,j})$.

It is not important that the indices of the $\mathcal{S}_i$ range from 1 to l – an identical proof yields the lemma whenever the set of indices is a nonempty subset of $\{1, \ldots, m\}$. This regularity was considered in [26] when the $\mathcal{S}_i$ are collections of dyadic cubes with disjoint interiors. We will use Lemma 3.1.2 when the collections $\mathcal{S}_i$ consist of dyadic cubes in the proof of Theorem 3.1.1 and when the $\mathcal{S}_i$ are of the second type in the proof of Theorem 3.1.2.

Proof. We only prove the statement when the collections $\mathcal{S}_i$ are of the second type. The proof for collections of dyadic cubes is similar and is addressed in the bilinear setting in [26]. For $i = 1, \ldots, l$, fix $S_{i,j_i} \in \mathcal{S}_i$. Use the smoothness condition of K and the fact that $S_{i,j_i} \subseteq B(c_{i,j_i}, r_{i,j_i}) \subseteq \overline{B(c_{i,j_i}, r_{i,j_i})}$ to see

$$\begin{aligned}
&\sup_{\substack{(y_1,\ldots,y_l)\\ \in\prod_{i=1}^l S_{i,j_i}}} \int_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})| dx \\
&\quad\lesssim \sup_{\substack{(y_1,\ldots,y_l)\\ \in\prod_{i=1}^l S_{i,j_i}}} \int_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l |y_i - c_{i,j_i}|^\delta}{(\sum_{i=1}^m |x-y_i|)^{nm+\delta}} dx \\
&\quad\leq \sup_{\substack{(y_1,\ldots,y_l)\\ \in\prod_{i=1}^l \overline{B(c_{i,j_i},r_{i,j_i})}}} \int_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^m |x-y_i|)^{nm+\delta}} dx.
\end{aligned}$$

Since for fixed $y_i \in \overline{B(c_{i,j_i}, r_{i,j_i})}$, $i = l{+}1, \ldots, m$, the function $\displaystyle\int_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^m |x-y_i|)^{nm+\delta}} dx$ is continuous in the variables $y_i \in \overline{B(c_{i,j_i}, r_{i,j_i})}$, $i = 1, \ldots, l$, and since $\overline{B(c_{i,j_i}, r_{i,j_i})}$ is a compact set, we may write

$$\begin{aligned}
&\sup_{\substack{(y_1,\ldots,y_l)\\ \in\prod_{i=1}^l \overline{B(c_{i,j_i},r_{i,j_i})}}} \int_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^m |x-y_i|)^{nm+\delta}} dx \\
&\quad= \int_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^l |x-y_i^*| + \sum_{i=l+1}^m |x-y_i|)^{nm+\delta}} dx
\end{aligned}$$

and

$$\inf_{\substack{(y_1,\dots,y_l)\\ \in\prod_{i=1}^l \overline{B(c_{i,j_i},r_{i,j_i})}}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^m |x-y_i|)^{nm+\delta}}dx$$
$$= \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^l |x-y_{i_*}|+\sum_{i=l+1}^m |x-y_i|)^{nm+\delta}}dx.$$

Note that for $x\in\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)$, $|x-y_{i_*}|\le 2r_i+|x-y_i^*|$ and $|x-y_i^*|\ge r_i$, so

$$\frac{\sum_{i=1}^l |x-y_{i_*}|+\sum_{i=l+1}^m |x-y_i|}{\sum_{i=1}^l |x-y_i^*|+\sum_{i=l+1}^m |x-y_i|} \le \frac{\sum_{i=1}^l 2r_i}{\sum_{i=1}^l |x-y_i^*|+\sum_{i=l+1}^m |x-y_i|}+1\le 3.$$

Then

$$\sup_{\substack{(y_1,\dots,y_l)\\ \in\prod_{i=1}^l S_{i,j_i}}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} |K(x,\vec{y}_{1,m})-K(x,\vec{c}_{(1,j_1),(l,j_l)},\vec{y}_{l+1,m})|dx$$
$$\lesssim \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^l |x-y_{i_*}|+\sum_{i=l+1}^m |x-y_i|)^{nm+\delta}}$$
$$\times\left(\frac{\sum_{i=1}^l |x-y_{i_*}|+\sum_{i=l+1}^m |x-y_i|}{\sum_{i=1}^l |x-y_i^*|+\sum_{i=l+1}^m |x-y_i|}\right)^{nm+\delta} dx$$
$$\lesssim \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^l |x-y_{i_*}|+\sum_{i=l+1}^m |x-y_i|)^{nm+\delta}}dx$$
$$= \inf_{\substack{(y_1,\dots,y_l)\\ \in\prod_{i=1}^l \overline{B(c_{i,j_i},r_{i,j_i})}}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^m |x-y_i|)^{nm+\delta}}dx$$
$$\le \inf_{\substack{(y_1,\dots,y_l)\\ \in\prod_{i=1}^l S_{i,j_i}}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^m |x-y_i|)^{nm+\delta}}dx.$$

Using the previous estimate, trivial estimates, Fubini's theorem, and integral estimates, we get the bound

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\dots,y_l)\\ \in\prod_{i=1}^l S_{i,j_i}}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} |K(x,\vec{y}_{1,m}) - K(x,\vec{c}_{(1,j_1),(l,j_l)},\vec{y}_{l+1,m})|\,dx\,d\vec{y}_{l+1,m}\\
&\quad\lesssim \int\limits_{\mathbb{R}^{n(m-l)}} \inf_{\substack{(y_1,\dots,y_l)\\ \in\prod_{i=1}^l S_{i,j_i}}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^m |x-y_i|)^{nm+\delta}}\,dx\,d\vec{y}_{l+1,m}\\
&\quad\le \inf_{\substack{(y_1,\dots,y_l)\\ \in\prod_{i=1}^l S_{i,j_i}}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \int\limits_{\mathbb{R}^{n(m-l)}} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^m |x-y_i|)^{nm+\delta}}\,d\vec{y}_{l+1,m}\,dx\\
&\quad\lesssim \inf_{\substack{(y_1,\dots,y_l)\\ \in\prod_{i=1}^l S_{i,j_i}}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^l |x-y_i|)^{nl+\delta}}\,dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{j_1,\dots,j_l=1}^\infty \prod_{i=1}^l |S_{i,j_i}|\\
&\qquad\times \int\limits_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\dots,y_l)\\ \in\prod_{i=1}^l S_{i,j_i}}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} |K(x,\vec{y}_{1,m}) - K(x,\vec{c}_{(1,j_1),(l,j_l)},\vec{y}_{l+1,m})|\,dx\,d\vec{y}_{l+1,m}\\
&\lesssim \sum_{j_1,\dots,j_l=1}^\infty \prod_{i=1}^l |S_{i,j_i}| \inf_{\substack{(y_1,\dots,y_l)\\ \in\prod_{i=1}^l S_{i,j_i}}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^l |x-y_i|)^{nl+\delta}}\,dx\\
&\le \sum_{j_1,\dots,j_l=1}^\infty \int\limits_{S_{l,j_l}} \cdots \int\limits_{S_{1,j_1}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^l |x-y_i|)^{nl+\delta}}\,dx\,d\vec{y}_{1,l}\\
&= \sum_{k=1}^l \Bigg(\sum_{\substack{j_1,\dots,j_l=1\\ r_{k,j_k}\ge r_{i,j_i}\text{ all } i}}^\infty \int\limits_{S_{l,j_l}} \cdots \int\limits_{S_{1,j_1}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^l |x-y_i|)^{nl+\delta}}\,dx\,d\vec{y}_{1,l}\Bigg).
\end{aligned}$$

We will control the term of the summation above with $k = 1$; the other terms are handled identically. Using trivial estimates, Fubini's theorem, the fact that the S_{i,j_i} have disjoint interiors, and integral estimates, we obtain

$$\begin{aligned}
&\sum_{\substack{j_1,\dots,j_l=1\\ r_{1,j_1}\geq r_{i,j_i}\text{ all } i}}^{\infty} \int_{S_{l,j_l}} \cdots \int_{S_{1,j_1}} \int_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^l |x-y_i|)^{nl+\delta}} dx d\overrightarrow{y}_{1,l} \\
&\lesssim \sum_{j_1,\dots,j_l=1}^{\infty} \int_{S_{l,j_l}} \cdots \int_{S_{1,j_1}} \int_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{r_{1,j_1}^\delta}{(\sum_{i=1}^l |x-y_i|)^{nl+\delta}} dx d\overrightarrow{y}_{1,l} \\
&\lesssim \sum_{j_1=1}^{\infty} \int_{S_{1,j_1}} \int_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \sum_{j_2,\dots,j_l=1}^{\infty} \int_{S_{l,j_l}} \cdots \int_{S_{2,j_2}} \frac{1}{(\sum_{i=1}^l |x-y_i|)^{nl}} d\overrightarrow{y}_{2,l} \frac{r_{1,j_1}^\delta}{|x-c_{1,j_1}|^\delta} dx dy_1 \\
&\leq \int_{\Omega_1} \int_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \int_{\mathbb{R}^{n(l-1)}} \frac{1}{(\sum_{i=1}^l |x-y_i|)^{nl}} d\overrightarrow{y}_{2,l} \frac{r_{1,j_1}^\delta}{|x-c_{1,j_1}|^\delta} dx dy_1 \\
&\lesssim \int_{\Omega_1} \int_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{r_{1,j_1}^\delta}{|x-y_1|^n |x-c_{1,j_1}|^\delta} dx dy_1 \\
&\lesssim |\Omega_1| \int_{\mathbb{R}^n\setminus \Omega_1^*} \frac{r_{1,j_1}^\delta}{|x-c_{1,j_1}|^{n+\delta}} dx \\
&\leq |\Omega_1| \int_{|x|>2r_{1,j_1}} \frac{r_{1,j_1}^\delta}{|x|^{n+\delta}} dx \\
&\leq |\Omega_1|.
\end{aligned}$$

Similarly, for $k = 2, \dots, l$,

$$\sum_{\substack{j_1,\dots,j_l=1\\ r_{k,j_k}\geq r_{i,j_i}\text{ all } i}}^{\infty} \int_{S_{l,j_l}} \cdots \int_{S_{1,j_1}} \int_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{(\sum_{i=1}^l |x-y_i|)^{nl+\delta}} dx d\overrightarrow{y}_{1,l} \lesssim |\Omega_k|.$$

This completes the proof. $\square$

3.2 Main Results

We now prove the main results. We will first prove Theorem 3.1.1 assuming Theorem 3.1.2 and then prove Theorem 3.1.2 afterwards.

Proof of Theorem 3.1.1. Let $\lambda > 0$ be given. We will show that

$$|\{|T(f_1, \dots, f_m)| > \lambda\}| \lesssim \lambda^{-\frac{1}{m}} \prod_{i=1}^{m} \|f_i\|_{L^1(\mathbb{R}^n)}^{\frac{1}{m}}.$$

Without loss of generality, assume that $f_1, \dots, f_m$ are continuous functions with compact support and that $\|f_1\|_{L^1(\mathbb{R}^n)} = \cdots = \|f_m\|_{L^1(\mathbb{R}^n)} = 1$. Set

$$\Omega_i := \left\{ Mf_i > \lambda^{\frac{1}{m}} \right\} \quad \text{and} \quad \Omega := \bigcup_{i=1}^{m} \Omega_i.$$

Apply a Whitney decomposition to write

$$\Omega_i = \bigcup_{j=1}^{\infty} Q_{i,j}$$

a disjoint union of dyadic cubes, where

$$2\text{diam}(Q_{i,j}) \leq \text{dist}(Q_{i,j}, \mathbb{R}^n \setminus \Omega_i) \leq 8\text{diam}(Q_{i,j}).$$

Put

$$g_i := f_i \mathbb{1}_{\mathbb{R}^n \setminus \Omega_i}, \qquad b_i := f_i \mathbb{1}_{\Omega_i}, \qquad \text{and} \qquad b_{i,j} := f_i \mathbb{1}_{Q_{i,j}}.$$

Then

$$f_i = g_i + b_i = g_i + \sum_{j=1}^{\infty} b_{i,j},$$

where

(1) $\|g_i\|_{L^\infty(\mathbb{R}^n)} \le \lambda^{\frac{1}{m}}$ and $\|g_i\|_{L^1(\mathbb{R}^n)} \le \|f_i\|_{L^1(\mathbb{R}^n)}$,

(2) the $b_{i,j}$ are supported on pairwise disjoint cubes $Q_{i,j}$ satisfying

$$\sum_{j=1}^{\infty} |Q_{i,j}| \lesssim \lambda^{-\frac{1}{m}} \|f_i\|_{L^1(\mathbb{R}^n)},$$

(3) $\|b_{i,j}\|_{L^1(\mathbb{R}^n)} \lesssim \lambda^{\frac{1}{m}} |Q_{i,j}|$, and

(4) $\|b_i\|_{L^1(\mathbb{R}^n)} \le \|f_i\|_{L^1(\mathbb{R}^n)}$.

To justify the above properties, since

$$f_i(x) \le Mf_i(x) \le \lambda^{\frac{1}{m}}$$

for almost every $x \in \mathbb{R}^n \setminus \Omega_i$, it is true that $\|g_i\|_{L^\infty(\mathbb{R}^n)} \le \lambda^{\frac{1}{m}}$. Noticing that g_i is a restriction of f_i, we have $\|g_i\|_{L^1(\mathbb{R}^n)} \le \|f_i\|_{L^1(\mathbb{R}^n)}$, so property (1) holds. We obtain property (2) using Lemma 3.1.1 as follows

$$\sum_{j=1}^{\infty} |Q_{i,j}| = |\Omega_i| \lesssim \lambda^{-\frac{1}{m}}.$$

Addressing (3), for a fixed $Q_{i,j}$, let $Q_{i,j}^* = 17\sqrt{n} Q_{i,j}$. Then $Q_{i,j}^* \cap (\mathbb{R}^n \setminus \Omega_i) \neq \emptyset$, so there is a point $x \in Q_{i,j}^*$ such that $Mf_i(x) \le \lambda^{\frac{1}{m}}$. In particular, $\int_{Q_{i,j}^*} f_i(y)\,dy \le \lambda^{\frac{1}{m}} |Q_{i,j}^*|$. Since $|Q_{i,j}^*| = (17\sqrt{n})^n |Q_{i,j}|$, we have

$$\|b_{i,j}\|_{L^1(\mathbb{R}^n)} = \int_{Q_{i,j}} f_i(y)\,dy \le \int_{Q_{i,j}^*} f_i(y)\,dy \le \lambda^{\frac{1}{m}} |Q_{i,j}^*| \lesssim \lambda^{\frac{1}{m}} v_{\vec{w}}(Q_{i,j}),$$

proving (3). Property (4) follows since b_i is a restriction of f_i.

Set

$$
\begin{aligned}
S_1 &:= \left\{ |T(g_1, g_2, \ldots, g_m)| > \frac{\lambda}{2^m} \right\}, \\
S_2 &:= \left\{ |T(b_1, g_2, \ldots, g_m)| > \frac{\lambda}{2^m} \right\}, \\
S_3 &:= \left\{ |T(g_1, b_2, \ldots, g_m)| > \frac{\lambda}{2^m} \right\}, \\
&\cdots \\
S_{2^m} &:= \left\{ |T(b_1, b_2, \ldots, b_m)| > \frac{\lambda}{2^m} \right\};
\end{aligned}
$$

where each $S_s = \left\{|T(h_1, \ldots, h_m)| > \frac{\lambda}{2^m}\right\}$ with $h_i \in \{b_i, g_i\}$ and all the sets S_s are distinct. Since

$$|\{|T(f_1, \ldots, f_m)| > \lambda\}| \leq \sum_{s=1}^{2^m} |S_s|,$$

it suffices to control each $|S_s|$ by a constant multiplied by $\lambda^{-\frac{1}{m}}$.

Use Chebyshev's inequality, the boundedness of T from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, and property (1) to see

$$
\begin{aligned}
|S_1| &\lesssim \lambda^{-q} \int_{\mathbb{R}^n} |T(g_1, \ldots, g_m)(x)|^q \, dx \\
&\lesssim \lambda^{-q} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} |g_i(x)|^{q_i} \, dx \right)^{\frac{q}{q_i}} \\
&\leq \lambda^{-q + \sum_{i=1}^m \frac{(q_i - q)q}{m q_i}} \prod_{i=1}^m \|g_i\|_{L^1(\mathbb{R}^n)}^{\frac{q}{q_i}} \\
&\leq \lambda^{-\frac{1}{m}}.
\end{aligned}
$$

Consider the set S_s for a fixed $2 \leq s \leq 2^m$. Suppose that there are l functions of the form b_i and $m - l$ functions of the form g_i appearing as entries in the $T(h_1, \ldots, h_m)$ involved in the definition of S_s. By symmetry, we may assume that the b_i are in the first l entries and the g_i are in the remaining $m-l$ entries. It suffices to control (uniformly in N) the measure of S_s with b_i replaced by $b_i^{(N)} := \sum_{j=1}^N b_{i,j}$. Denote this set by $\widetilde{S}_s$.

Let $c_{i,j}$ denote the center of $Q_{i,j}$ and let

$$a_{i,j} = \|b_{i,j}\|_{L^1(\mathbb{R}^n)}, \qquad \nu_{i,j} = a_{i,j}\delta_{c_{i,j}}, \qquad \text{and} \qquad \nu_i^N = \sum_{j=1}^N \nu_{i,j}.$$

Then, by adding and subtracting $T\left(\overrightarrow{\nu^N}_{1,k}, \overrightarrow{b^{(N)}}_{k+1,l}, \vec{g}_{l+1,m}\right)$ for $1 \le k \le l$, we have

$$\begin{aligned}
\left|\widetilde{S}_s\right| &\le \sum_{k=1}^l \left|\left\{\left|T\left(\overrightarrow{\nu^N}_{1,k-1}, b_k^{(N)} - \nu_k^N, \overrightarrow{b^{(N)}}_{k+1,l}, \vec{g}_{l+1,m}\right)\right| > \frac{\lambda}{(l+1)2^m}\right\}\right| \\
&\quad + \left|\left\{\left|T\left(\overrightarrow{\nu^N}_{1,l}, \vec{g}_{l+1,m}\right)\right| > \frac{\lambda}{(l+1)2^m}\right\}\right| \\
&\lesssim |\Omega| + \sum_{k=1}^l \left|\left\{\mathbb{R}^n \setminus \Omega : \left|T\left(\overrightarrow{\nu^N}_{1,k-1}, b_k^{(N)} - \nu_k^N, \overrightarrow{b^{(N)}}_{k+1,l}, \vec{g}_{l+1,m}\right)\right| > \frac{\lambda}{(l+1)2^m}\right\}\right| \\
&\quad + \left|\left\{\left|T\left(\overrightarrow{\nu^N}_{1,l}, \vec{g}_{l+1,m}\right)\right| > \frac{\lambda}{(l+1)2^m}\right\}\right|.
\end{aligned}$$

Using property (2), we have $|\Omega| \le \sum_{i=1}^m \sum_{j=1}^\infty |Q_{i,j}| \lesssim \lambda^{-\frac{1}{m}}$. Therefore

$$|\widetilde{S}_s| \lesssim \lambda^{-\frac{1}{m}} + \sum_{k=1}^l |P_k| + |P|,$$

where

$$\begin{aligned}
P_k &:= \left\{\mathbb{R}^n \setminus \Omega : \left|T\left(\overrightarrow{\nu^N}_{1,k-1}, b_k^{(N)} - \nu_k^N, \overrightarrow{b^{(N)}}_{k+1,l}, \vec{g}_{l+1,m}\right)\right| > \frac{\lambda}{(l+1)2^m}\right\}, \quad \text{and} \\
P &:= \left\{\left|T\left(\overrightarrow{\nu^N}_{1,l}, \vec{g}_{l+1,m}\right)\right| > \frac{\lambda}{(l+1)2^m}\right\}.
\end{aligned}$$

We will first control $|P_k|$ for $k \in \{1, \ldots, l\}$. Begin by using Chebyshev's inequality, the fact that $(b_{k,j_k}^{(N)}dm - v_{k,j_k}^N)(Q_{k,j_k}) = 0$, and trivial estimates to see

$$
\begin{aligned}
|P_k| &\lesssim \lambda^{-1} \int_{\mathbb{R}^n\setminus\Omega} \left| T\left(\overrightarrow{\nu^N}_{1,k-1}, b_k^{(N)} - \nu_k^N, \overrightarrow{b^{(N)}}_{k+1,l}, \vec{g}_{l+1,m}\right)(x)\right| dx \\
&\leq \lambda^{-1} \sum_{j_1,\ldots,j_l=1}^{N} \int_{\mathbb{R}^n\setminus\Omega} \Bigg| \int_{\mathbb{R}^{n(m-l)}} \int_{Q_{l,j_l}} \cdots \int_{Q_{1,j_1}} K(x, \vec{y}_{1,m}) \\
&\qquad \times \left(\prod_{i=k+1}^{l} b_{i,j_i}(y_i)\right)\left(\prod_{i=l+1}^{m} g_i(y_i)\right) \\
&\qquad d\vec{\nu}_{(1,j_1),(k-1,j_{k-1})}(\vec{y}_{1,k-1}) d(b_{k,j_k}dm - \nu_{k,j_k})(y_k) d\vec{y}_{k+1,m} \Bigg| dx \\
&\leq \lambda^{-1} \sum_{j_1,\ldots,j_l=1}^{N} \int_{\mathbb{R}^{n(m-l)}} \int_{Q_{l,j_l}} \cdots \int_{Q_{1,j_1}} \int_{\mathbb{R}^n\setminus\Omega} \\
&\qquad \times |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})| \left(\prod_{i=k+1}^{l} |b_{i,j_i}(y_i)|\right)\left(\prod_{i=l+1}^{m} |g_i(y_i)|\right) \\
&\qquad dx d|\vec{\nu}|_{(1,j_1),(k-1,j_{k-1})}(\vec{y}_{1,k-1}) d|b_{k,j_k}dm - \nu_{k,j_k}|(y_k) d\vec{y}_{k+1,m} \\
&\leq \lambda^{-1} \sum_{j_1,\ldots,j_l=1}^{N} \left(\prod_{i=1}^{k-1} |a_{i,j_i}|\right) |b_{k,j_k}dm - \nu_{k,j_k}|(Q_{k,j_k}) \\
&\qquad \times \left(\prod_{i=k+1}^{l} \|b_{i,j_i}\|_{L^1(\mathbb{R}^n)}\right)\left(\prod_{i=l+1}^{m} \|g_i\|_{L^\infty(\mathbb{R}^n)}\right) \\
&\qquad \times \int_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\ldots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n\setminus\Omega} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})| dx d\vec{y}_{l+1,m}.
\end{aligned}
$$

Use property (1), property (3), Lemma 3.1.2 (which applies since $2\text{diam}(Q_{i,j}) \leq \text{dist}(Q_{i,j}, \mathbb{R}^n \setminus \Omega_i)$), and property (2) to finish the estimate

$$
\begin{aligned}
|P_k| &\lesssim \lambda^{-1} \sum_{j_1,\ldots,j_l=1}^{N} \left(\prod_{i=1}^{l} \|b_{i,j_i}\|_{L^1(\mathbb{R}^n)}\right)\left(\prod_{i=l+1}^{m} \|g_i\|_{L^\infty(\mathbb{R}^n)}\right) \\
&\qquad \times \int_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\ldots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n\setminus\Omega} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})| dx d\vec{y}_{l+1,m}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sum_{j_1,\ldots,j_l=1}^{N} \left(\prod_{i=1}^{l} |Q_{i,j_i}| \right) \\
&\qquad \times \int\limits_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\ldots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int\limits_{\mathbb{R}^n \setminus \Omega} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})| dx d\vec{y}_{l+1,m} \\
&\lesssim \sum_{i=1}^{l} |\Omega_i| \\
&\lesssim \lambda^{-\frac{1}{m}}.
\end{aligned}
$$

The control of $|P|$ follows from applying Theorem 3.1.2.

$$
\begin{aligned}
|P| &\lesssim \lambda^{-\frac{1}{m}} \left(\prod_{i=1}^{l} \|\nu_i^N\|^{\frac{1}{m}} \right) \left(\prod_{i=l+1}^{m} \|g_i\|_{L^1(\mathbb{R}^n)}^{\frac{1}{m}} \right) \\
&\leq \lambda^{-\frac{1}{m}} \left(\prod_{i=1}^{l} \|b_i^N\|_{L^1(\mathbb{R}^n)}^{\frac{1}{m}} \right) \left(\prod_{i=l+1}^{m} \|g_i\|_{L^1(\mathbb{R}^n)}^{\frac{1}{m}} \right) \\
&\leq \lambda^{-\frac{1}{m}}.
\end{aligned}
$$

Put the estimates of $|P_k|$ and $|P|$ together to get

$$
|\widetilde{S}_s| \lesssim \lambda^{-\frac{1}{m}}.
$$

Since the above estimate is independent of N, we have that $|S_s| \lesssim \lambda^{-\frac{1}{m}}$.

Finally, use the estimates of $|S_s|$, $1 \leq s \leq 2^m$ to complete the proof

$$
|\{|T(f_1, \ldots, f_m)| > \lambda\}| \leq |S_1| + \sum_{s=2}^{2^m} |S_s| \lesssim \lambda^{-\frac{1}{m}}.
$$

□

Proof of Theorem 3.1.2. Without loss of generality, assume that each $a_{i,j} > 0$ and that

$\|\nu_1\| = \cdots = \|\nu_l\| = \|f_{l+1}\|_{L^1(\mathbb{R}^n)} = \cdots = \|f_m\|_{L^1(\mathbb{R}^n)} = 1$. For $i = 1, \ldots, l$, set

$$E_{i,1} := B(x_{i,1}, r_{i,1}),$$

where $r_{i,1} > 0$ is chosen so that $|E_{i,1}| = a_{i,1}\lambda^{-\frac{1}{m}}$. In general, for $j = 2, 3, \ldots, N$, set

$$E_{i,j} := B(x_{i,j}, r_{i,j}) \setminus \bigcup_{k=1}^{j-1} E_{i,k},$$

where $r_{i,j} > 0$ is chosen so that $|E_{i,2}| = a_{i,j}\lambda^{-\frac{1}{m}}$. Note that such $E_{i,j}$ exist since the function $r \mapsto |Q(x,r)|$ is continuous for each $x \in \mathbb{R}^n$. Define

$$E_i := \bigcup_{j=1}^{N} E_{i,j}$$

and notice that

$$|E_i| = \sum_{j=1}^{N} |E_{i,j}| = \sum_{j=1}^{N} a_{i,j}\lambda^{-\frac{1}{m}} = \lambda^{-\frac{1}{m}}.$$

Also define

$$E := \bigcup_{i=1}^{m} E_i = \bigcup_{i=1}^{m}\bigcup_{j=1}^{N} B(x_{i,j}, r_{i,j}) \qquad \text{and} \qquad E^* := \bigcup_{i=1}^{m}\bigcup_{j=1}^{N} B(x_{i,j}, 2r_{i,j}).$$

For $k \in \{0, \ldots, l\}$, set

$$\sigma_k := T\left(\lambda^{\frac{1}{m}}\mathbb{1}_{E_1}, \ldots, \lambda^{\frac{1}{m}}\mathbb{1}_{E_k}, \nu_{k+1} \ldots, \nu_l, f_{l+1}, \ldots, f_m\right),$$

noticing that $\sigma_0 = T(\nu_1, \ldots, \nu_l, f_{l+1}, \ldots, f_m)$. Then, by adding and subtracting σ_k for $1 \leq$

$k \le l$, we have

$$|\{|T(\nu_1,\ldots,\nu_l,f_{l+1},\ldots,f_m)|>\lambda\}| \le \sum_{k=1}^{l}\left|\left\{|\sigma_{k-1}-\sigma_k|>\frac{\lambda}{l+1}\right\}\right| + \left|\left\{|\sigma_l|>\frac{\lambda}{l+1}\right\}\right|$$
$$\lesssim |E^*| + \sum_{k=1}^{l}\sum_{k=1}^{l}\left|\left\{\mathbb{R}^n\setminus E^* : |\sigma_{k-1}-\sigma_k|>\frac{t}{l+1}\right\}\right| + \left|\left\{|\sigma_l|>\frac{t}{l+1}\right\}\right|.$$

Using Lemma 2.1.2, we have

$$|E^*| \lesssim |E| \le \sum_{i=1}^{l}|E_i| \lesssim \lambda^{-\frac{1}{m}}.$$

Therefore

$$|\{|T(\nu_1,\ldots,\nu_l,f_{l+1},\ldots,f_m)|>\lambda\}| \lesssim \lambda^{-\frac{1}{m}} + \sum_{k=1}^{l}|P_k| + |P|,$$

where

$$P_k := \left\{\mathbb{R}^n\setminus E^* : |\sigma_{k-1}-\sigma_k|>\frac{\lambda}{l+1}\right\}, \qquad \text{and}$$
$$P := \left\{|\sigma_l|>\frac{\lambda}{l+1}\right\}.$$

We will bound $|P|$ and each $|P_k|$ by a constant multiplied by $\lambda^{-\frac{1}{m}}$.

To control $|P|$, use Chebyshev's inequality, the boundedness of T from $L^{q_1}(\mathbb{R}^n)\times\cdots\times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, the L^∞ control of the f_i, and the fact $\sum_{i=1}^m \frac{1}{q_i} = \frac{1}{q}$ to observe

$$|P| \lesssim \lambda^{-q+\frac{lq}{m}} \int_{\mathbb{R}^n} |T(\mathbb{1}_{E_1},\ldots,\mathbb{1}_{E_l},f_{l+1},\ldots,f_m)(x)|^q \, dx$$
$$\lesssim \lambda^{-q+\frac{lq}{m}} \left(\prod_{i=1}^{l}\|\mathbb{1}_{E_i}\|_{L^{q_i}(\mathbb{R}^n)}^q\right)\left(\prod_{i=l+1}^{m}\|f_i\|_{L^{q_i}(\mathbb{R}^n)}^q\right)$$
$$\lesssim \lambda^{-q+\frac{lq}{m}-\sum_{i=1}^{l}\frac{q}{mq_i}+\sum_{i=l+1}^{m}\frac{q(q_i-1)}{mq_i}} \prod_{i=l+1}^{m}\|f_i\|_{L^1(\mathbb{R}^n)}^{\frac{q}{q_i}}$$

$$= \lambda^{-q+\frac{lq}{m}+\frac{(m-l)q}{m}-\sum_{i=1}^{m}\frac{q}{mq_i}}$$
$$= \lambda^{-\frac{1}{m}}.$$

We will now control $|P_k|$. Notice

$$\sigma_{k-1} - \sigma_k = T\left(\lambda^{\frac{1}{m}}\mathbb{1}_{E_1}, \ldots, \lambda^{\frac{1}{m}}\mathbb{1}_{E_{k-1}}, \nu_k - \lambda^{\frac{1}{m}}\mathbb{1}_{E_k}, \nu_{k+1}, \ldots, \nu_l, f_{l+1}, \ldots, f_m\right).$$

Use Chebyshev's inequality, the fact that $(\nu_{k,j_k} - \lambda^{\frac{1}{m}}\mathbb{1}_{E_{k,j_k}}dm)(E_k) = 0$, and trivial estimates to see

$$|P_k| \lesssim \lambda^{-\frac{m-k+1}{m}} \int\limits_{\mathbb{R}^n\setminus E^*} \left|T\left(\mathbb{1}_{E_1}, \ldots, \mathbb{1}_{E_{k-1}}, \nu_k - \lambda^{\frac{1}{m}}\mathbb{1}_{E_k}, \nu_{k+1}, \ldots, \nu_l, f_{l+1}, \ldots, f_m\right)(x)\right| dx$$
$$\leq \lambda^{-\frac{m-k+1}{m}} \sum_{j_1,\ldots,j_l=1}^{N} \int\limits_{\mathbb{R}^n\setminus E^*} \Bigg| \int\limits_{\mathbb{R}^{n(m-l)}} \int\limits_{E_{l,j_l}} \cdots \int\limits_{E_{1,j_1}} K(x, \vec{y}_{1,m})$$
$$\times \left(\prod_{i=l+1}^{m} f_i(y_i)\right) d\vec{y}_{1,k-1} d(\nu_{k,j_k} - \lambda^{\frac{1}{m}}\mathbb{1}_{E_{k,j_k}}dm)(y_k) d\vec{\nu}_{(k+1,j_{k+1}),(l,j_l)}(\vec{y}_{k+1,l}) d\vec{y}_{l+1,m}\Bigg| dx$$
$$\leq \lambda^{-\frac{m-k+1}{m}} \sum_{j_1,\ldots,j_l=1}^{N} \int\limits_{\mathbb{R}^{n(m-l)}} \int\limits_{E_{l,j_l}} \cdots \int\limits_{E_{1,j_1}} \int\limits_{\mathbb{R}^n\setminus E^*} |K(x, \vec{y}_{1,m}) - K(x, \vec{x}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})|$$
$$\times \left(\prod_{i=l+1}^{m} |f_i(y_i)|\right) dx d\vec{y}_{1,k-1} d\left|\nu_{k,j_k} - \lambda^{\frac{1}{m}}\mathbb{1}_{E_{k,j_k}}dm\right|(y_k) d|\vec{\nu}|_{(k+1,j_{k+1}),(l,j_l)}(\vec{y}_{k+1,l}) d\vec{y}_{l+1,m}$$
$$\leq \lambda^{-\frac{m-k+1}{m}} \sum_{j_1,\ldots,j_l=1}^{N} \left(\prod_{i=1}^{k-1}|E_{i,j_i}|\right) \left|\nu_{k,j_k} - \lambda^{\frac{1}{m}}\mathbb{1}_{E_{k,j_k}}dm\right|(E_{k,j_k}) \left(\prod_{i=k+1}^{l} a_{i,j_i}\right) \left(\prod_{i=l+1}^{m} \|f_i\|_{L^\infty(\mathbb{R}^n)}\right)$$
$$\times \int\limits_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\ldots,y_l)\\ \in\prod_{i=1}^{l} E_{i,j_i}}} \int\limits_{\mathbb{R}^n\setminus E^*} |K(x, \vec{y}_{1,m}) - K(x, \vec{x}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})| dx d\vec{y}_{l+1,m}.$$

Use the fact that $|\nu_{k,j_k} - \lambda^{\frac{1}{m}}\mathbb{1}_{E_{k,j_k}}dm|(E_k) \leq 2\lambda^{\frac{1}{m}}|E_{k,j_k}|$, the $L^\infty(\mathbb{R}^n)$ control of the f_i, and

Lemma 3.1.2 to continue the estimate

$$\begin{aligned}|P_k| &\lesssim \sum_{j_1,\dots,j_l=1}^{N} \left(\prod_{i=1}^{l} |E_{i,j_i}|\right) \\ &\quad \times \int_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\dots,y_l) \\ \in \prod_{i=1}^{l} E_{i,j_i}}} \int_{\mathbb{R}^n \setminus E^*} |K(x,\vec{y}_{1,m}) - K(x,\vec{x}_{(1,j_1),(l,j_l)},\vec{y}_{l+1,m})| dx d\vec{y}_{l+1,m} \\ &\lesssim \sum_{i=1}^{l} |E_i| \\ &\lesssim \lambda^{-\frac{1}{m}}.\end{aligned}$$

Using these estimates of $|P|$ and $|P_k|$, we complete the proof

$$|\{|T(\nu_1,\dots,\nu_l,f_1,\dots,f_{m-l})| > \lambda\}| \lesssim \lambda^{-\frac{1}{m}} + \sum_{k=1}^{l} |P_k| + |P| \lesssim \lambda^{-\frac{1}{m}}.$$

□

Chapter 4

Weighted Estimate for Multilinear Operators

4.1 Preliminaries

The work in this chapter can be found in [30]. Throughout this chapter, T will denote a multilinear Calderón-Zygmund operator as defined in 1.0.2 and we will write $A \lesssim B$ if there exists $C > 0$, possibly depending on n, m, or T, such that $A \le CB$. If Q is a cube, then rQ denotes the cube with the same center as Q and side length equal to r times the side length of Q.

The focus of this chapter is to connect the previously discussed multilinear and weighted settings of Chapter 3 and Section 2.3 by proving a weighted weak-type $\left(1, \dots, 1; \frac{1}{m}\right)$ estimate for multilinear Calderón-Zygmund operators. Two proofs are given – one uses the Calderón-Zygmund decomposition and the other uses the Nazarov-Treil-Volberg method.

Lerner, Ombrosi, Pérez, Torres, and Trujillo-Gonzáles studied the classes of multilinear weights in [19]. We use the following notation for multilinear $A_{\vec{P}}$ weights: $1 \le p_1, \dots, p_m < \infty$, $\frac{1}{m} \le p < \infty$ satisfies $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, $\vec{P} = (p_1, \dots, p_m)$, $\vec{w} = (w_1, \dots, w_m)$, and $v_{\vec{w}} = \prod_{i=1}^m w_i^{\frac{p}{p_i}}$. We say $\vec{w} \in A_{\vec{P}}$ if

$$[\vec{w}]_{A_{\vec{P}}} := \sup_Q \left(\frac{1}{|Q|} \int\limits_Q v_{\vec{w}} \right)^{\frac{1}{p}} \prod_{i=1}^m \left(\frac{1}{|Q|} \int\limits_Q w_i^{1-p_i'} \right)^{\frac{1}{p_i'}} < \infty;$$

when $p_i = 1$, the factor $\left(\frac{1}{|Q|} \int_Q w_i^{1-p_i'} \right)^{\frac{1}{p_i'}}$ is understood as $(\inf_Q w_i)^{-1}$. Note that the quantities $[\vec{w}]_{A_{\vec{P}}}^p$ and $[w]_{A_p}$ coincide when $m = 1$.

Theorem 4.1.1. If T is a multilinear Calderón-Zygmund operator and $\vec{w} \in A_{(1,\ldots,1)}$, then

$$\left\| T\left(f_1 w_1 v_{\vec{w}}^{\frac{1-m}{m}}, \ldots, f_m w_m v_{\vec{w}}^{\frac{1-m}{m}}\right) v_{\vec{w}}^{-1} \right\|_{L^{\frac{1}{m},\infty}(v_{\vec{w}})} \lesssim [v_{\vec{w}}]_{A_1}^{2m^2+2m-2} \prod_{i=1}^{m} \|f_i\|_{L^1(w_i)}$$

for all $f_i \in L^1(w_i)$.

Theorem 4.1.1 reduces to Theorem 3.1.1 when all the weights are constant and parallels Theorem 2.1.2 when $m = 1$. We give two proofs of Theorem 4.1.1. The first proof uses the Calderón-Zygmund decomposition and is a weighted version of the proof in [26]; the second proof is related to the proof in Chapter 3 using the Nazarov-Treil-Volberg method. See [20] for a related result that is deduced using multilinear extrapolation.

Remark 4.1.1. The second proof is actually a weighted version of a simplification of the proof in Chapter 3. The current proof shows that Theorem 3.1.2 is not necessary for the weak-type estimate and that the regularity of Lemma 3.1.2 is only required for collections of pairwise disjoint cubes.

The following theorem was proved by Grafakos and Torres in [14].

Theorem 4.1.2. If T is a multilinear Calderón-Zygmund operator, $1 < p_1, \ldots, p_m < \infty$, and p satisfies $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, then

$$\|T(f_1, \ldots, f_m)\|_{L^p(\mathbb{R}^n)} \lesssim \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbb{R}^n)}$$

for all $f_i \in L^{p_i}(\mathbb{R}^n)$.

We will use Theorem 4.1.2 in the proofs of Theorem 4.1.1 when $p_1 = \cdots = p_m = m$ and $p = 1$.

A characterization of the multilinear $A_{\vec{P}}$ condition in terms of linear A_q conditions was established in [19].

Theorem 4.1.3. The following conditions are equivalent:

1. $\vec{w} \in A_{\vec{P}}$;

2. $v_{\vec{w}} \in A_{mp}$ and $w_i^{1-p_i'} \in A_{mp_i'}$ for all $i \in \{1, \dots, m\}$.

When $p_i = 1$, the condition $w_i^{1-p_i'} \in A_{mp_i'}$ is understood as $w_i^{\frac{1}{m}} \in A_1$.

Remark 4.1.2. Tracking down the estimates in the proof of Theorem 4.1.3 gives the relationships

$$[v_{\vec{w}}]_{A_{mp}} \leq [\vec{w}]_{A_{\vec{P}}}^p,$$

$$\left[w_i^{1-p_i'}\right]_{A_{mp_i'}} \leq [\vec{w}]_{A_{\vec{P}}}^{p_i'} \text{ when } p_i > 1,$$

$$\left[w_i^{\frac{1}{m}}\right]_{A_1} \leq [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{m}} \text{ when } p_i = 1, \text{ and}$$

$$[\vec{w}]_{A_{\vec{P}}} \leq [v_{\vec{w}}]_{A_{mp}}^{\frac{1}{p}} \prod_{i=1}^m \left[w_i^{1-p_i'}\right]_{A_{mp_i'}}^{\frac{1}{p_i'}},$$

where $\left[w_i^{1-p_i'}\right]_{A_{mp_i'}}^{\frac{1}{p_i'}}$ is interpreted as $\left[w_i^{\frac{1}{m}}\right]_{A_1}^m$ when $p_i = 1$.

We will use the following property of A_1 weights.

Lemma 4.1.1. If $w \in A_1$ and $0 \leq \gamma \leq 1$, then $w^\gamma \in A_1$ with $[w^\gamma]_{A_1} \leq [w]_{A_1}^\gamma$.

Proof. The cases when $\gamma = 0$ and $\gamma = 1$ are clear. If $0 < \gamma < 1$, then $\frac{1}{\gamma} > 1$. Applying Hölder's inequality and the A_1 condition of w gives

$$\frac{1}{|Q|} \int_Q w(y)^\gamma dy \leq \left(\frac{1}{|Q|} \int_Q w(y) dy \right)^\gamma \left(\frac{1}{|Q|} \int_Q 1^{\left(\frac{1}{\gamma}\right)'} dy \right)^{\frac{1}{\left(\frac{1}{\gamma}\right)'}} \leq [w]_{A_1}^\gamma \left(\inf_Q w^\gamma \right).$$

□

Given a weight w, define the *uncentered maximal function associated to* w by

$$M_w f(x) := \sup_{Q \ni x} \frac{1}{w(Q)} \int\limits_Q |f(y)| w(y) dy.$$

Lemma 4.1.2. If $w \in A_p$, then

$$\|M_w f\|_{L^{1,\infty}(w)} \lesssim \|f\|_{L^1(w)}$$

for all $f \in L^1(w)$.

The operator norm of M_w does not depend on the A_p characteristic of w.

The following lemma is well-known and proved in [9, 10].

Lemma 4.1.3. Let $k : [0, \infty) \to [0, \infty)$ be decreasing and continuous except at a finite number of points. If $K(x) = k(|x|)$ is in $L^1(\mathbb{R}^n)$, then for all $f \in L^1_{\text{loc}}(\mathbb{R}^n)$,

$$(|f| * K)(x) \le \|K\|_{L^1(\mathbb{R}^n)} M(f)(x),$$

where M denotes the classical Hardy-Littlewood maximal operator.

The following is a weighted version of the multilinear geometric Hörmander condition described in Chapter 3. We again use the vector notations $\vec{y}_{i,k} = (y_i, y_{i+1}, \dots, y_k)$, $\vec{f}_{i,k} = (f_i, f_{i+1}, \dots, f_k)$, and $\vec{c}_{(i,j_i),(k,j_k)} = (c_{i,j_i}, c_{i+1,j_{i+1}}, \dots, c_{k,j_k})$.

Lemma 4.1.4. If $w \in A_1$, $l \in \{1, \dots, m\}$, and each of $\mathcal{Q}_1, \dots, \mathcal{Q}_l$ consists of pairwise disjoint cubes $\mathcal{Q}_i = \{Q_{i,1}, Q_{i,2}, \ldots\}$ with $Q_{i,j} = Q(c_{i,j}, r_{i,j})$, then

$$\sum_{j_1,\ldots,j_l=1}^{\infty} \prod_{i=1}^{l} w^{\frac{1}{m}}(Q_{i,j_i}) \int_{\mathbb{R}^{n(m-l)}} \prod_{i=l+1}^{m} w(y_i)^{\frac{1}{m}}$$
$$\times \sup_{\substack{(y_1,\ldots,y_l)\\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^{l} \Omega_i^*\right)} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})| dx d\vec{y}_{l+1,m}$$
$$\lesssim [w]_{A_1}^{\frac{2m-2}{m}} \sum_{i=1}^{l} w\left(\Omega_i\right),$$

where $\Omega_i := \bigcup_{j=1}^{\infty} Q_{i,j}$ and $\Omega_i^* := \bigcup_{j=1}^{\infty} 2\sqrt{n} Q_{i,j}$.

It is not important that the indices of the $\mathcal{Q}_i$ range from 1 to l – a symmetric proof yields the lemma whenever the set of indices is a nonempty subset of $\{1, \ldots, m\}$.

Proof. For $i = 1, \ldots, l$, fix $Q_{i,j_i} \in \mathcal{Q}_i$. Use the smoothness condition of K to see

$$\sup_{\substack{(y_1,\ldots,y_l)\\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^{l} \Omega_i^*\right)} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})| dx$$
$$\lesssim \sup_{\substack{(y_1,\ldots,y_l)\\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^{l} \Omega_i^*\right)} \frac{\sum_{i=1}^{l} |y_i - c_{i,j_i}|^{\delta}}{\left(\sum_{i=1}^{m} |x - y_i|\right)^{nm+\delta}} dx$$
$$\lesssim \sup_{\substack{(y_1,\ldots,y_l)\\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^{l} \Omega_i^*\right)} \frac{\sum_{i=1}^{l} r_{i,j_i}^{\delta}}{\left(\sum_{i=1}^{m} |x - y_i|\right)^{nm+\delta}} dx.$$

Since for fixed $y_i \in \overline{Q_{i,j_i}}$, $i = l+1, \ldots, m$, the function

$$\int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^{l} \Omega_i^*\right)} \frac{\sum_{i=1}^{l} r_{i,j_i}^{\delta}}{\left(\sum_{i=1}^{m} |x - y_i|\right)^{nm+\delta}} dx$$

is continuous in the variables $y_i \in \overline{Q_{i,j_i}}$, $i = 1, \ldots, l$, we may write

$$\sup_{\substack{(y_1,\ldots,y_l) \\ \in \prod_{i=1}^l Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{\left(\sum_{i=1}^m |x-y_i|\right)^{nm+\delta}} dx$$
$$= \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{\left(\sum_{i=1}^l |x-y_i^*| + \sum_{i=l+1}^m |x-y_i|\right)^{nm+\delta}} dx,$$

and

$$\inf_{\substack{(y_1,\ldots,y_l) \\ \in \prod_{i=1}^l Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{\left(\sum_{i=1}^m |x-y_i|\right)^{nm+\delta}} dx$$
$$= \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{\left(\sum_{i=1}^l |x-y_{i_*}| + \sum_{i=l+1}^m |x-y_i|\right)^{nm+\delta}} dx.$$

Note that for $x \in \mathbb{R}^n \setminus \left(\bigcup_{i=1}^l \Omega_i^*\right)$, $|x-y_{i_*}| \leq \sqrt{n} r_{i,j_i} + |x-y_i^*|$ and $|x-y_i^*| \geq \frac{1}{2} r_{i,j_i}$, so

$$\frac{\sum_{i=1}^l |x-y_{i_*}| + \sum_{i=l+1}^m |x-y_i|}{\sum_{i=1}^l |x-y_i^*| + \sum_{i=l+1}^m |x-y_i|} \leq \frac{\sum_{i=1}^l \sqrt{n} r_{i,j_i}}{\sum_{i=1}^l |x-y_i^*| + \sum_{i=l+1}^m |x-y_i|} + 1 \leq 2\sqrt{n} + 1.$$

Then

$$\sup_{\substack{(y_1,\ldots,y_l) \\ \in \prod_{i=1}^l Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^l \Omega_i^*\right)} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})| dx$$
$$\lesssim \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^l \Omega_i^*\right)} \frac{\sum_{i=1}^l r_{i,j_i}^\delta}{\left(\sum_{i=1}^l |x-y_{i_*}| + \sum_{i=l+1}^m |x-y_i|\right)^{nm+\delta}}$$
$$\times \left(\frac{\sum_{i=1}^l |x-y_{i_*}| + \sum_{i=l+1}^m |x-y_i|}{\sum_{i=1}^l |x-y_i^*| + \sum_{i=l+1}^m |x-y_i|} \right)^{nm+\delta} dx$$

$$\lesssim \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^{l} \Omega_i^*\right)} \frac{\sum_{i=1}^{l} r_{i,j_i}^{\delta}}{(\sum_{i=1}^{l} |x - y_{i_*}| + \sum_{i=l+1}^{m} |x - y_i|)^{nm+\delta}} dx$$

$$= \inf_{\substack{(y_1,\dots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^{l} \Omega_i^*\right)} \frac{\sum_{i=1}^{l} r_{i,j_i}^{\delta}}{(\sum_{i=1}^{m} |x - y_i|)^{nm+\delta}} dx.$$

Using the previous estimate, Fubini's theorem, and trivial estimates, we get the bound

$$\sum_{j_1,\dots,j_l=1}^{\infty} \prod_{i=1}^{l} w^{\frac{1}{m}}(Q_{i,j_i}) \int_{\mathbb{R}^{n(m-l)}} \prod_{i=l+1}^{m} w(y_i)^{\frac{1}{m}}$$
$$\times \sup_{\substack{(y_1,\dots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^{l} \Omega_i^*\right)} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})| dx d\vec{y}_{l+1,m}$$
$$\lesssim \sum_{j_1,\dots,j_l=1}^{\infty} \prod_{i=1}^{l} w^{\frac{1}{m}}(Q_{i,j_i}) \int_{\mathbb{R}^{n(m-l)}} \prod_{i=l+1}^{m} w(y_i)^{\frac{1}{m}}$$
$$\times \inf_{\substack{(y_1,\dots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^{l} \Omega_i^*\right)} \frac{\sum_{i=1}^{l} r_{i,j_i}^{\delta}}{(\sum_{i=1}^{m} |x - y_i|)^{nm+\delta}} dx d\vec{y}_{l+1,m}$$
$$\leq \sum_{j_1,\dots,j_l=1}^{\infty} \int_{\mathbb{R}^{n(m-l)}} \int_{Q_{l,j_l}} \cdots \int_{Q_{1,j_1}} \prod_{i=1}^{m} w(y_i)^{\frac{1}{m}} \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^{l} \Omega_i^*\right)} \frac{\sum_{i=1}^{l} r_{i,j_i}^{\delta}}{(\sum_{i=1}^{m} |x - y_i|)^{nm+\delta}} dx d\vec{y}_{1,m}$$
$$= \sum_{k=1}^{l} \left(\sum_{\substack{j_1,\dots,j_l=1 \\ r_{k,j_k} \geq r_{i,j_i} \text{ all } i}}^{\infty} \int_{\mathbb{R}^{n(m-l)}} \int_{Q_{l,j_l}} \cdots \int_{Q_{1,j_1}} \prod_{i=1}^{m} w(y_i)^{\frac{1}{m}} \right.$$
$$\left. \times \int_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^{l} \Omega_i^*\right)} \frac{\sum_{i=1}^{l} r_{i,j_i}^{\delta}}{(\sum_{i=1}^{m} |x - y_i|)^{nm+\delta}} dx d\vec{y}_{1,m} \right).$$

We will control the term of the summation above with $k = 1$; the other terms are handled similarly. Using trivial estimates, Fubini's theorem, and the fact that the Q_{i,j_i} have disjoint interiors, we obtain

$$\sum_{\substack{j_1,\ldots,j_l=1\\ r_{1,j_1}\geq r_{i,j_i}\text{ all } i}}^{\infty} \int\limits_{\mathbb{R}^{n(m-l)}} \int\limits_{Q_{l,j_l}} \cdots \int\limits_{Q_{1,j_1}} \prod_{i=1}^{m} w(y_i)^{\frac{1}{m}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^{l}\Omega_i^*\right)} \frac{\sum_{i=1}^{l} r_{i,j_i}^{\delta}}{\left(\sum_{i=1}^{m}|x-y_i|\right)^{nm+\delta}} dx d\vec{y}_{1,m}$$

$$\lesssim \sum_{j_1,\ldots,j_l=1}^{\infty} \int\limits_{\mathbb{R}^{n(m-l)}} \int\limits_{Q_{l,j_l}} \cdots \int\limits_{Q_{1,j_1}} \prod_{i=1}^{m} w(y_i)^{\frac{1}{m}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^{l}\Omega_i^*\right)} \frac{r_{1,j_1}^{\delta}}{\left(\sum_{i=1}^{m}|x-y_i|\right)^{nm+\delta}} dx d\vec{y}_{1,m}$$

$$\leq \sum_{j_1=1}^{\infty} \int\limits_{Q_{1,j_1}} w(y_1)^{\frac{1}{m}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^{l}\Omega_i^*\right)} \frac{r_{1,j_1}^{\delta}}{|x-y_1|^{\delta}} \int\limits_{\mathbb{R}^{n(m-l)}}$$

$$\times \sum_{j_2,\ldots,j_l=1}^{\infty} \int\limits_{Q_{l,j_l}} \cdots \int\limits_{Q_{2,j_2}} \frac{\prod_{i=2}^{m} w(y_i)^{\frac{1}{m}}}{\left(\sum_{i=1}^{m}|x-y_i|\right)^{nm}} d\vec{y}_{2,m} dx dy_1$$

$$\leq \sum_{j_1=1}^{\infty} \int\limits_{Q_{1,j_1}} w(y_1)^{\frac{1}{m}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^{l}\Omega_i^*\right)} \frac{r_{1,j_1}^{\delta}}{|x-y_1|^{\delta}} \int\limits_{\mathbb{R}^{n(m-1)}} \frac{\prod_{i=2}^{m} w(y_i)^{\frac{1}{m}}}{\left(\sum_{i=1}^{m}|x-y_i|\right)^{nm}} d\vec{y}_{2,m} dx dy_1.$$

Repeatedly use Lemma 4.1.3 first with $K(\cdot) = \frac{1}{\left(\sum_{i=1}^{m-1}|x-y_i|+|\cdot|\right)^{nm}}$, second with $K(\cdot) = \frac{1}{\left(\sum_{i=1}^{m-2}|x-y_i|+|\cdot|\right)^{n(m-1)}}$, etcetera, and the fact that $M\left(w^{\frac{1}{m}}\right)(x) \leq \left[w^{\frac{1}{m}}\right]_{A_1} w(x)^{\frac{1}{m}}$ (which is true by Lemma 4.1.1) to control the above expression by

$$\sum_{j_1=1}^{\infty} \int\limits_{Q_{1,j_1}} w(y_1)^{\frac{1}{m}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^{l}\Omega_i^*\right)} \frac{r_{1,j_1}^{\delta} M\left(w^{\frac{1}{m}}\right)(x)}{|x-y_1|^{\delta}} \int\limits_{\mathbb{R}^{n(m-2)}} \frac{\prod_{i=2}^{m-1} w(y_i)^{\frac{1}{m}}}{\left(\sum_{i=1}^{m-1}|x-y_i|\right)^{n(m-1)}} d\vec{y}_{2,m-1} dx dy_1$$

$$\leq \left[w^{\frac{1}{m}}\right]_{A_1} \sum_{j_1=1}^{\infty} \int\limits_{Q_{1,j_1}} w(y_1)^{\frac{1}{m}} \int\limits_{\mathbb{R}^n\setminus\left(\bigcup_{i=1}^{l}\Omega_i^*\right)} \frac{r_{1,j_1}^{\delta} w(x)^{\frac{1}{m}} M\left(w^{\frac{1}{m}}\right)(x)}{|x-y_1|^{\delta}}$$

$$\times \int\limits_{\mathbb{R}^{n(m-3)}} \frac{\prod_{i=2}^{m-2} w(y_i)^{\frac{1}{m}}}{\left(\sum_{i=1}^{m-2}|x-y_i|\right)^{n(m-2)}} d\vec{y}_{2,m-2} dx dy_1$$

$$\leq \cdots$$

$$\leq \left[w^{\frac{1}{m}}\right]_{A_1}^{m-1} \sum_{j_1=1}^{\infty} \int\limits_{Q_{1,j_1}} w(y_1)^{\frac{1}{m}} \int\limits_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^{l} \Omega_i^*\right)} \frac{r_{1,j_1}^{\delta} w(x)^{\frac{m-1}{m}}}{|x-y_1|^{n+\delta}} dx dy_1$$

$$\leq \left[w^{\frac{1}{m}}\right]_{A_1}^{m-1} \sum_{j_1=1}^{\infty} \int\limits_{Q_{1,j_1}} w(y_1)^{\frac{1}{m}} \int\limits_{|x-y_1|>\frac{1}{2}r_{1,j_1}} \frac{r_{1,j_1}^{\delta} w(x)^{\frac{m-1}{m}}}{|x-y_1|^{n+\delta}} dx dy_1.$$

Use Lemma 4.1.3 with $K(\cdot) = \frac{r_{1,j_1}^{\delta}}{|\cdot|^{n+\delta}} \mathbb{1}_{\left\{|\cdot|>\frac{1}{2}r_{1,j_1}\right\}}$, the fact that $\|K\|_{L^1(\mathbb{R}^n)} \lesssim 1$, the fact that $M\left(w^{\frac{m-1}{m}}\right)(y_1) \leq \left[w^{\frac{m-1}{m}}\right]_{A_1} w^{\frac{m-1}{m}}(y_1)$ (which is true by Lemma 4.1.1), the estimates in Remark 4.1.2, and the pairwise disjointness of Q_{1,j_1} to further estimate the previous expression by a constant multiplied by

$$\left[w^{\frac{1}{m}}\right]_{A_1}^{m-1} \sum_{j_1=1}^{\infty} \int\limits_{Q_{1,j_1}} w(y_1)^{\frac{1}{m}} M\left(w^{\frac{m-1}{m}}\right)(y_1) dy_1$$

$$\leq \left[w^{\frac{1}{m}}\right]_{A_1}^{m-1} \left[w^{\frac{m-1}{m}}\right]_{A_1} \sum_{j_1=1}^{\infty} \int\limits_{Q_{1,j_1}} w(y_1) dy_1$$

$$\leq [w]_{A_1}^{\frac{2m-2}{m}} w(\Omega_1).$$

Similarly, for $k = 2, \ldots, l$,

$$\sum_{\substack{j_1,\ldots,j_l=1 \\ r_{k,j_k} \geq r_{i,j_i} \text{ all } i}}^{\infty} \int\limits_{\mathbb{R}^{n(m-l)}} \int\limits_{Q_{l,j_l}} \cdots \int\limits_{Q_{1,j_1}} \prod_{i=1}^{m} w(y_i)^{\frac{1}{m}} \int\limits_{\mathbb{R}^n \setminus \left(\bigcup_{i=1}^{l} \Omega_i^*\right)} \frac{\sum_{i=1}^{l} r_{i,j_i}^{\delta}}{\left(\sum_{i=1}^{m} |x-y_i|\right)^{nm+\delta}} dx d\vec{y}_{1,m}$$

$$\lesssim \left[w^{\frac{1}{m}}\right]_{A_1}^{m-1} \left[w^{\frac{m-1}{m}}\right]_{A_1} w\left(\Omega_k\right)$$

$$\leq [w]_{A_1}^{\frac{2m-2}{m}} w\left(\Omega_k\right),$$

completing the proof. □

4.2 Calderón-Zygmund Decomposition Method

We now prove Theorem 4.1.1.

Proof 1. Let $\lambda > 0$ be given. We will show that

$$v_{\vec{w}}\left(\left\{\left|T\left(f_1 w_1 v_{\vec{w}}^{\frac{1-m}{m}}, \ldots, f_m w_m v_{\vec{w}}^{\frac{1-m}{m}}\right)\right| v_{\vec{w}}^{-1} > \lambda\right\}\right) \lesssim [v_{\vec{w}}]_{A_1}^{2m+\frac{2m-2}{m}} \lambda^{-\frac{1}{m}} \prod_{i=1}^{m} \|f_i\|_{L^1(w_i)}^{\frac{1}{m}}.$$

Without loss of generality, assume that $f_1, \ldots, f_m$ are continuous functions with compact support and that $\|f_1\|_{L^1(w_1)} = \cdots = \|f_m\|_{L^1(w_m)} = 1$. Apply the Calderón-Zygmund decomposition to $f_i w_i v_{\vec{w}}^{-1}$ at height $\lambda^{\frac{1}{m}}$ with respect to $v_{\vec{w}}dx$ (see Appendix A)to write

$$f_i w_i v_{\vec{w}}^{-1} = g_i + b_i = g_i + \sum_{j=1}^{\infty} b_{i,j}$$

where the following properties hold:

(1) $\|g_i\|_{L^\infty(\mathbb{R}^n)} \lesssim [v_{\vec{w}}]_{A_1} \lambda^{\frac{1}{m}}$ and $\|g_i\|_{L^1(v_{\vec{w}})} \leq \|f_i\|_{L^1(w_i)}$,

(2) the $b_{i,j}$ are supported on pairwise disjoint cubes $Q_{i,j}$ satisfying

$$\sum_{j=1}^{\infty} v_{\vec{w}}(Q_{i,j}) \leq \lambda^{-\frac{1}{m}} \|f_i\|_{L^1(w_i)},$$

(3) $\int_{Q_{i,j}} b_{i,j}(x) v_{\vec{w}}(x) dx = 0$,

(4) $\|b_{i,j}\|_{L^1(v_{\vec{w}})} \lesssim [v_{\vec{w}}]_{A_1} \lambda^{\frac{1}{m}} v_{\vec{w}}(Q_{i,j})$, and

(5) $\|b_i\|_{L^1(v_{\vec{w}})} \lesssim \|f_i\|_{L^1(w_i)}$.

Set

$$S_1 := \left\{ \left| T\left(g_1 v_{\vec{w}}^{\frac{1}{m}}, g_2 v_{\vec{w}}^{\frac{1}{m}}, \ldots, g_m v_{\vec{w}}^{\frac{1}{m}}\right) \right| v_{\vec{w}}^{-1} > \frac{\lambda}{2^m} \right\},$$
$$S_2 := \left\{ \left| T\left(b_1 v_{\vec{w}}^{\frac{1}{m}}, g_2 v_{\vec{w}}^{\frac{1}{m}}, \ldots, g_m v_{\vec{w}}^{\frac{1}{m}}\right) \right| v_{\vec{w}}^{-1} > \frac{\lambda}{2^m} \right\},$$
$$S_3 := \left\{ \left| T\left(g_1 v_{\vec{w}}^{\frac{1}{m}}, b_2 v_{\vec{w}}^{\frac{1}{m}}, \ldots, g_m v_{\vec{w}}^{\frac{1}{m}}\right) \right| v_{\vec{w}}^{-1} > \frac{\lambda}{2^m} \right\},$$
$$\ldots$$
$$S_{2^m} := \left\{ \left| T\left(b_1 v_{\vec{w}}^{\frac{1}{m}}, b_2 v_{\vec{w}}^{\frac{1}{m}}, \ldots, b_m v_{\vec{w}}^{\frac{1}{m}}\right) \right| v_{\vec{w}}^{-1} > \frac{\lambda}{2^m} \right\};$$

where each $S_s = \left\{ \left| T\left(h_1 v_{\vec{w}}^{\frac{1}{m}}, \ldots, h_m v_{\vec{w}}^{\frac{1}{m}}\right) \right| v_{\vec{w}}^{-1} > \frac{\lambda}{2^m} \right\}$ with $h_i \in \{b_i, g_i\}$ and all the sets S_s are distinct. Since

$$v_{\vec{w}}\left(\left\{ \left| T\left(f_1 w_1 v_{\vec{w}}^{\frac{1-m}{m}}, \ldots, f_m w_m v_{\vec{w}}^{\frac{1-m}{m}}\right) \right| v_{\vec{w}}^{-1} > \lambda \right\} \right) \leq \sum_{s=1}^{2^m} v_{\vec{w}}(S_s),$$

it suffices to control each $v_{\vec{w}}(S_s)$.

Use Chebyshev's inequality, the boundedness of T from $(L^m(\mathbb{R}^n))^m$ to $L^1(\mathbb{R}^n)$ (which holds by Theorem 4.1.2), and property (1) to see

$$\begin{aligned} v_{\vec{w}}(S_1) &\lesssim \lambda^{-1} \int_{\mathbb{R}^n} \left| T\left(g_1 v_{\vec{w}}^{\frac{1}{m}}, \ldots, g_m v_{\vec{w}}^{\frac{1}{m}}\right)(x) \right| dx \\ &\lesssim \lambda^{-1} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} |g_i(x)|^m v_{\vec{w}}(x) dx \right)^{\frac{1}{m}} \\ &\lesssim [v_{\vec{w}}]_{A_1}^{\frac{m-1}{m}} \lambda^{-\frac{1}{m}} \prod_{i=1}^m \|g_i\|_{L^1(v_{\vec{w}})}^{\frac{1}{m}} \\ &\leq [v_{\vec{w}}]_{A_1}^{\frac{m-1}{m}} \lambda^{-\frac{1}{m}}. \end{aligned}$$

Consider the set S_s for a fixed $2 \leq s \leq 2^m$. Suppose that there are l functions of the form b_i and $m-l$ functions of the form g_i appearing as entries in the $T\left(h_1 v_{\vec{w}}^{\frac{1}{m}}, \ldots, h_m v_{\vec{w}}^{\frac{1}{m}}\right)$ involved in the definition of S_s. By symmetry, we may assume that the b_i are in the first l entries and the g_i are in the remaining $m-l$ entries. Let $Q_{i,j}^* := 2\sqrt{n} Q_{i,j}$, $\Omega_i^* := \bigcup_{j=1}^{\infty} Q_{i,j}^*$, and $\Omega^* := \bigcup_{i=1}^{m} \Omega_i^*$.

By the doubling property of $v_{\vec{w}}dx$, the fact that the $Q_{i,j}$ are pairwise disjoint, and property (3), we have

$$v_{\vec{w}}\left(\Omega^*\right) \leq \sum_{i=1}^{m} \sum_{j=1}^{\infty} v_{\vec{w}}(Q_{i,j}^*) \lesssim [v_{\vec{w}}]_{A_1} \sum_{i=1}^{m} \sum_{j=1}^{\infty} v_{\vec{w}}(Q_{i,j}) \leq [v_{\vec{w}}]_{A_1} \lambda^{-\frac{1}{m}}.$$

Therefore

$$\begin{aligned} v_{\vec{w}}(S_s) &\leq v_{\vec{w}}(\Omega^*) + v_{\vec{w}}\left(\left\{\mathbb{R}^n \setminus \Omega^* : \left|T\left(b_1 v_{\vec{w}}^{\frac{1}{m}}, \ldots, b_l v_{\vec{w}}^{\frac{1}{m}}, g_{l+1} v_{\vec{w}}^{\frac{1}{m}}, \ldots, g_m v_{\vec{w}}^{\frac{1}{m}}\right)\right| v_{\vec{w}}^{-1} > \frac{\lambda}{2^m}\right\}\right) \\ &\lesssim [v_{\vec{w}}]_{A_1} \lambda^{-\frac{1}{m}} + v_{\vec{w}}\left(\left\{\mathbb{R}^n \setminus \Omega^* : \left|T\left(b_1 v_{\vec{w}}^{\frac{1}{m}}, \ldots, b_l v_{\vec{w}}^{\frac{1}{m}}, g_{l+1} v_{\vec{w}}^{\frac{1}{m}}, \ldots, g_m v_{\vec{w}}^{\frac{1}{m}}\right)\right| v_{\vec{w}}^{-1} > \frac{\lambda}{2^m}\right\}\right). \end{aligned}$$

Now use Chebyshev's inequality, the fact that $\int_{Q_{i,j_i}} b_{i,j_i}(x) v_{\vec{w}}(x) dx = 0$, and trivial bounds to estimate

$$\begin{aligned} & v_{\vec{w}}\left(\left\{\mathbb{R}^n \setminus \Omega^* : \left|T\left(b_1 v_{\vec{w}}^{\frac{1}{m}}, \ldots, b_l v_{\vec{w}}^{\frac{1}{m}}, g_{l+1} v_{\vec{w}}^{\frac{1}{m}}, \ldots, g_m v_{\vec{w}}^{\frac{1}{m}}\right)\right| v_{\vec{w}}^{-1} > \frac{\lambda}{2^m}\right\}\right) \\ &\lesssim \lambda^{-1} \int\limits_{\mathbb{R}^n \setminus \Omega^*} \left|T\left(b_1 v_{\vec{w}}^{\frac{1}{m}}, \ldots, b_l v_{\vec{w}}^{\frac{1}{m}}, g_{l+1} v_{\vec{w}}^{\frac{1}{m}}, \ldots, g_m v_{\vec{w}}^{\frac{1}{m}}\right)(x)\right| dx \\ &\leq \lambda^{-1} \sum_{j_1, \ldots, j_l = 1}^{\infty} \int\limits_{\mathbb{R}^n \setminus \Omega^*} \Bigg| \int\limits_{\mathbb{R}^{n(m-l)}} \int\limits_{Q_{1,j_1}} \cdots \int\limits_{Q_{l,j_l}} K(x, \vec{y}_{1,m}) \\ &\qquad \times \left(\prod_{i=1}^{l} b_{i,j_i}(y_i)\right)\left(\prod_{i=l+1}^{m} g_i(y_i)\right)\left(\prod_{i=1}^{m} v_{\vec{w}}(y_i)^{\frac{1}{m}}\right) d\vec{y}_{1,m} \Bigg| dx \end{aligned}$$

$$\leq \lambda^{-1} \sum_{j_1,\dots,j_l=1}^{\infty} \left(\prod_{i=1}^{l} \sup_{Q_{i,j_i}} v_{\vec{w}}^{\frac{1-m}{m}} \right) \int_{\mathbb{R}^{n(m-l)}} \int_{Q_{l,j_l}} \cdots \int_{Q_{1,j_1}} \left(\prod_{i=1}^{l} |b_{i,j_i}(y_i)| v_{\vec{w}}(y_i) \right)$$

$$\times \left(\prod_{i=l+1}^{m} |g_i(y_i)| v_{\vec{w}}(y_i)^{\frac{1}{m}} \right) \int_{\mathbb{R}^n \setminus \Omega^*} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})| dx d\vec{y}_{1,m}$$

$$\leq \lambda^{-1} \sum_{j_1,\dots,j_l=1}^{\infty} \left(\prod_{i=1}^{l} \sup_{Q_{i,j_i}} v_{\vec{w}}^{\frac{1-m}{m}} \right) \left(\prod_{i=1}^{l} \|b_{i,j_i}\|_{L^1(v_{\vec{w}})} \right) \left(\prod_{i=l+1}^{m} \|g_i\|_{L^\infty(\mathbb{R}^n)} \right)$$

$$\times \int_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\dots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \Omega^*} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})|$$

$$\times \left(\prod_{i=l+1}^{m} v_{\vec{w}}(y_i)^{\frac{1}{m}} \right) dx d\vec{y}_{l+1,m}.$$

Apply property (4), property (1), the fact that $\left(\sup_{Q_{i,j_i}} v_{\vec{w}}^{\frac{1-m}{m}}\right) = \left(\inf_{Q_{i,j_i}} v_{\vec{w}}\right)^{\frac{m-1}{m}}$, the A_1 condition of $v_{\vec{w}}$, and trivial estimates to bound the above expression by a constant times

$$[v_{\vec{w}}]_{A_1}^{m} \sum_{j_1,\dots,j_l=1}^{\infty} \left(\prod_{i=1}^{l} \sup_{Q_{i,j_i}} v_{\vec{w}}^{\frac{1-m}{m}} \right) \left(\prod_{i=1}^{l} v_{\vec{w}}(Q_{i,j_i}) \right)$$

$$\times \int_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\dots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \Omega^*} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})|$$

$$\times \left(\prod_{i=l+1}^{m} v_{\vec{w}}(y_i)^{\frac{1}{m}} \right) d\vec{y}_{l+1,m} dx$$

$$\leq [v_{\vec{w}}]_{A_1}^{m+l} \sum_{j_1,\dots,j_l=1}^{\infty} \left(\prod_{i=1}^{l} \inf_{Q_{i,j_i}} v_{\vec{w}} \right)^{\frac{1-m}{m}} \left(\prod_{i=1}^{l} |Q_{i,j_i}| \left(\inf_{Q_{i,j_i}} v_{\vec{w}} \right) \right)$$

$$\times \int_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\dots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \Omega^*} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})|$$

$$\times \left(\prod_{i=l+1}^{m} w_i(y_i)^{\frac{1}{m}} \right) d\vec{y}_{l+1,m} dx$$

$$
\begin{aligned}
&\leq [v_{\vec{w}}]_{A_1}^{2m} \sum_{j_1,\ldots,j_l=1}^{\infty} \left(\prod_{i=1}^{l} |Q_{i,j_i}| \left(\inf_{Q_{i,j_i}} v_{\vec{w}}^{\frac{1}{m}} \right) \right) \\
&\quad \times \int_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\ldots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \Omega^*} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})| \\
&\quad \times \left(\prod_{i=l+1}^{m} w_i(y_i)^{\frac{1}{m}} \right) d\vec{y}_{l+1,m} dx \\
&\leq [v_{\vec{w}}]_{A_1}^{2m} \sum_{j_1,\ldots,j_l=1}^{\infty} \prod_{i=1}^{l} v_{\vec{w}}^{\frac{1}{m}}(Q_{i,j_i}) \\
&\quad \times \int_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\ldots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \Omega^*} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})| \\
&\quad \times \left(\prod_{i=l+1}^{m} w_i(y_i)^{\frac{1}{m}} \right) d\vec{y}_{l+1,m} dx.
\end{aligned}
$$

By Lemma 4.1.4 and property (2), the above expression is controlled by a constant times

$$
[v_{\vec{w}}]_{A_1}^{2m+\frac{2m-2}{m}} \sum_{i=1}^{l} v_{\vec{w}} \left(\bigcup_{j=1}^{\infty} Q_{i,j} \right) \lesssim [v_{\vec{w}}]_{A_1}^{2m+\frac{2m-2}{m}} \lambda^{-\frac{1}{m}}.
$$

Therefore

$$
v_{\vec{w}}(S_s) \lesssim [v_{\vec{w}}]_{A_1} \lambda^{-\frac{1}{m}} + [v_{\vec{w}}]_{A_1}^{2m+\frac{2m-2}{m}} \lambda^{-\frac{1}{m}} \lesssim [v_{\vec{w}}]_{A_1}^{2m+\frac{2m-2}{m}} \lambda^{-\frac{1}{m}}.
$$

Putting the previous estimates together gives

$$
\begin{aligned}
v_{\vec{w}} \left(\left\{ \left| T \left(f_1 w_1^{\frac{m+1}{m}} v_{\vec{w}}^{-1}, \ldots, f_m w_m^{\frac{m+1}{m}} v_{\vec{w}}^{-1} \right) \right| v_{\vec{w}}^{-1} > \lambda \right\} \right) &\lesssim [v_{\vec{w}}]_{A_1}^{\frac{m-1}{m}} \lambda^{-\frac{1}{m}} + \sum_{s=2}^{2^m} [v_{\vec{w}}]_{A_1}^{2m+\frac{2m-2}{m}} \lambda^{-\frac{1}{m}} \\
&\lesssim [v_{\vec{w}}]_{A_1}^{2m+\frac{2m-2}{m}} \lambda^{-\frac{1}{m}}.
\end{aligned}
$$

□

4.3 Nazarov-Treil-Volberg Method

Proof 2. Let $\lambda > 0$ be given. We will show that

$$v_{\vec{w}}\left(\left\{\left|T\left(f_1 w_1 v_{\vec{w}}^{\frac{1-m}{m}}, \ldots, f_m w_m v_{\vec{w}}^{\frac{1-m}{m}}\right)\right| v_{\vec{w}}^{-1} > \lambda\right\}\right) \lesssim [v_{\vec{w}}]_{A_1}^{2m+\frac{2m-2}{m}} \lambda^{-\frac{1}{m}} \prod_{i=1}^{m} \|f_i\|_{L^1(w_i)}^{\frac{1}{m}}.$$

Without loss of generality, assume that $f_1, \ldots, f_m$ are nonnegative, continuous functions with compact support and that $\|f_1\|_{L^1(w_1)} = \cdots = \|f_m\|_{L^1(w_m)} = 1$. Assume that $v_{\vec{w}}(\mathbb{R}^n) > \lambda^{-\frac{1}{m}}$ (otherwise there is nothing to prove). Set

$$\Omega_i := \left\{M_{v_{\vec{w}}}\left(f_i w_i v_{\vec{w}}^{-1}\right) > \lambda^{\frac{1}{m}}\right\} \quad \text{and} \quad \Omega := \bigcup_{i=1}^{m} \Omega_i.$$

Apply a Whitney decomposition to write

$$\Omega_i = \bigcup_{j=1}^{\infty} Q_{i,j},$$

a disjoint union of dyadic cubes where

$$2\mathrm{diam}(Q_{i,j}) \leq \mathrm{dist}(Q_{i,j}, \mathbb{R}^n \setminus \Omega_i) \leq 8\mathrm{diam}(Q_{i,j}).$$

Put

$$g_i := f_i w_i v_{\vec{w}}^{-1} \mathbb{1}_{\mathbb{R}^n \setminus \Omega_i}, \qquad b_i := f_i w_i v_{\vec{w}}^{-1} \mathbb{1}_{\Omega_i}, \qquad \text{and} \qquad b_{i,j} := f_i w_i v_{\vec{w}}^{-1} \mathbb{1}_{Q_{i,j}}.$$

Then

$$f_i w_i v_{\vec{w}}^{-1} = g_i + b_i = g_i + \sum_{j=1}^{\infty} b_{i,j},$$

where

(1) $\|g_i\|_{L^\infty(\mathbb{R}^n)} \leq \lambda^{\frac{1}{m}}$ and $\|g_i\|_{L^1(v_{\vec{w}})} \leq \|f_i\|_{L^1(w_i)}$,

(2) the $b_{i,j}$ are supported on pairwise disjoint cubes $Q_{i,j}$ satisfying

$$\sum_{j=1}^{\infty} v_{\vec{w}}(Q_{i,j}) \lesssim \lambda^{-\frac{1}{m}} \|f_i\|_{L^1(w_i)},$$

(3) $\|b_{i,j}\|_{L^1(v_{\vec{w}})} \leq (17\sqrt{n})^n [v_{\vec{w}}]_{A_1} \lambda^{\frac{1}{m}} v_{\vec{w}}(Q_{i,j})$, and

(4) $\|b_i\|_{L^1(v_{\vec{w}})} \leq \|f_i\|_{L^1(w_i)}$.

To justify the above properties, since

$$f_i(x)w_i(x)v_{\vec{w}}^{-1}(x) \leq M_{v_{\vec{w}}}\left(f_i w_i v_{\vec{w}}^{-1}\right)(x) \leq \lambda^{\frac{1}{m}}$$

for almost every $x \in \mathbb{R}^n \setminus \Omega_i$, it is true that $\|g_i\|_{L^\infty(\mathbb{R}^n)} \leq \lambda^{\frac{1}{m}}$. Noticing that g_i is a restriction of $f_i w_i v_{\vec{w}}^{-1}$, we have $\|g_i\|_{L^1(v_{\vec{w}})} \leq \|f_i\|_{L^1(w_i)}$, so property (1) holds. We obtain property (2) using Lemma 4.1.2 as follows

$$\sum_{j=1}^{\infty} v_{\vec{w}}(Q_{i,j}) = v_{\vec{w}}(\Omega_i) \lesssim \lambda^{-\frac{1}{m}}.$$

Addressing (3), for a fixed $Q_{i,j}$, let $Q_{i,j}^* = 17\sqrt{n}Q_{i,j}$. Then $Q_{i,j}^* \cap (\mathbb{R}^n \setminus \Omega_i) \neq \emptyset$, so there is a point $x \in Q_{i,j}^*$ such that $M_{v_{\vec{w}}}(f_i w_i v_{\vec{w}}^{-1})(x) \leq \lambda^{\frac{1}{m}}$. In particular, $\int_{Q_{i,j}^*} f_i(y)w_i(y)\,dy \leq \lambda^{\frac{1}{m}} v_{\vec{w}}(Q_{i,j}^*)$. Since $v_{\vec{w}}(Q_{i,j}^*) \leq (17\sqrt{n})^n [v_{\vec{w}}]_{A_1} v_{\vec{w}}(Q_{i,j})$, we have

$$\|b_{i,j}\|_{L^1(v_{\vec{w}})} = \int_{Q_{i,j}} f_i(y)w_i(y)\,dy \leq \int_{Q_{i,j}^*} f_i(y)w_i(y)\,dy$$

$$\leq \lambda^{\frac{1}{m}} v_{\vec{w}}(Q_{i,j}^*) \leq (17\sqrt{n})^n [v_{\vec{w}}]_{A_1} \lambda^{\frac{1}{m}} v_{\vec{w}}(Q_{i,j}),$$

proving (3). Property (4) follows since b_i is a restriction of $f_i w_i v_{\vec{w}}^{-1}$.

Set

$$S_1 := \left\{ \left| T\left(g_1 v_{\vec{w}}^{\frac{1}{m}}, g_2 v_{\vec{w}}^{\frac{1}{m}}, \ldots, g_m v_{\vec{w}}^{\frac{1}{m}} \right) \right| v_{\vec{w}}^{-1} > \frac{\lambda}{2^m} \right\},$$
$$S_2 := \left\{ \left| T\left(b_1 v_{\vec{w}}^{\frac{1}{m}}, g_2 v_{\vec{w}}^{\frac{1}{m}}, \ldots, g_m v_{\vec{w}}^{\frac{1}{m}} \right) \right| v_{\vec{w}}^{-1} > \frac{\lambda}{2^m} \right\},$$
$$S_3 := \left\{ \left| T\left(g_1 v_{\vec{w}}^{\frac{1}{m}}, b_2 v_{\vec{w}}^{\frac{1}{m}}, \ldots, g_m v_{\vec{w}}^{\frac{1}{m}} \right) \right| v_{\vec{w}}^{-1} > \frac{\lambda}{2^m} \right\},$$
$$\cdots$$
$$S_{2^m} := \left\{ \left| T\left(b_1 v_{\vec{w}}^{\frac{1}{m}}, b_2 v_{\vec{w}}^{\frac{1}{m}}, \ldots, b_m v_{\vec{w}}^{\frac{1}{m}} \right) \right| v_{\vec{w}}^{-1} > \frac{\lambda}{2^m} \right\};$$

where each $S_s = \left\{ \left| T\left(h_1 v_{\vec{w}}^{\frac{1}{m}}, \ldots, h_m v_{\vec{w}}^{\frac{1}{m}} \right) \right| v_{\vec{w}}^{-1} > \frac{\lambda}{2^m} \right\}$ with $h_i \in \{b_i, g_i\}$ and all the sets S_s are distinct. Since

$$v_{\vec{w}} \left(\left\{ \left| T\left(f_1 w_1 v_{\vec{w}}^{\frac{1-m}{m}}, \ldots, f_m w_m v_{\vec{w}}^{\frac{1-m}{m}} \right) \right| v_{\vec{w}}^{-1} > \lambda \right\} \right) \leq \sum_{s=1}^{2^m} v_{\vec{w}}(S_s),$$

it suffices to control each $v_{\vec{w}}(S_s)$.

Use Chebyshev's inequality, the boundedness of T from $(L^m(\mathbb{R}^n))^m$ to $L^1(\mathbb{R}^n)$ (which holds by Theorem 4.1.2), and property (1) to see

$$\begin{aligned} v_{\vec{w}}(S_1) &\lesssim \lambda^{-1} \int_{\mathbb{R}^n} \left| T\left(g_1 v_{\vec{w}}^{\frac{1}{m}}, \ldots, g_m v_{\vec{w}}^{\frac{1}{m}} \right)(x) \right| dx \\ &\lesssim \lambda^{-1} \prod_{i=1}^{m} \left(\int_{\mathbb{R}^n} g_i(x)^m v_{\vec{w}}(x) dx \right)^{\frac{1}{m}} \\ &\leq \lambda^{-\frac{1}{m}} \prod_{i=1}^{m} \|g_i\|_{L^1(v_{\vec{w}})}^{\frac{1}{m}} \\ &\leq \lambda^{-\frac{1}{m}}. \end{aligned}$$

Consider the set S_s for a fixed $2 \leq s \leq 2^m$. Suppose that there are l functions of the form b_i and $m-l$ functions of the form g_i appearing as entries in the $T\left(h_1 w_1^{\frac{1}{m}}, \ldots, h_m w_m^{\frac{1}{m}}\right)$ involved in the definition of S_s. By symmetry, we may assume that the b_i are in the first l entries and the g_i are in the remaining $m-l$ entries.

Let $c_{i,j}$ denote the center of $Q_{i,j}$ and let $a_{i,j} = \|b_{i,j}\|_{L^1(v_{\vec{w}})}(17\sqrt{n})^{-n}[v_{\vec{w}}]_{A_1}^{-1}$. Set

$$E_{i,j} := Q(c_{i,j}, r_{i,j}),$$

where $r_{i,j} > 0$ is chosen so that $v_{\vec{w}}(E_{i,j}) = a_{i,j}\lambda^{-\frac{1}{m}}$. Note that such $E_{i,j}$ exist since the function $r \mapsto v_{\vec{w}}(Q(x,r))$ increases to $v_{\vec{w}}(\mathbb{R}^n) > \lambda^{-\frac{1}{m}}$ as $r \to \infty$, approaches 0 as $r \to 0$, and is continuous from the right for almost every $x \in \mathbb{R}^n$. Using property (3), we see

$$v_{\vec{w}}(E_{i,j}) = a_{i,j}\lambda^{-\frac{1}{m}} \leq v_{\vec{w}}(Q_{i,j}).$$

Since $E_{i,j}$ is a cube with the same center as $Q_{i,j}$ and since $v_{\vec{w}}(E_{i,j}) \leq v_{\vec{w}}(Q_{i,j})$, it is true that $E_{i,j} \subseteq Q_{i,j}$. Define

$$E_i := \bigcup_{j=1}^{\infty} E_{i,j}.$$

For $k = 0, \ldots, l$, define

$$\sigma_k := T\left((17\sqrt{n})^n [v_{\vec{w}}]_{A_1} \lambda^{\frac{1}{m}} \overrightarrow{\mathbb{1}_{E_{1,k}}} v_{\vec{w}}^{\frac{1}{m}}, \overrightarrow{b_{k+1,l}} v_{\vec{w}}^{\frac{1}{m}}, \overrightarrow{g_{l+1,m}} v_{\vec{w}}^{\frac{1}{m}}\right).$$

Then, by adding and subtracting σ_k for $1 \leq k \leq l$, we have

$$\begin{aligned} v_{\vec{w}}(S_s) &\leq \sum_{k=1}^{l} v_{\vec{w}}\left(\left\{|\sigma_{k-1} - \sigma_k| > \frac{\lambda}{(l+1)2^m}\right\}\right) + v_{\vec{w}}\left(\left\{|\sigma_l| > \frac{\lambda}{(l+1)2^m}\right\}\right) \\ &\lesssim v_{\vec{w}}(\Omega) + \sum_{k=1}^{l} v_{\vec{w}}\left(\left\{\mathbb{R}^n \setminus \Omega : |\sigma_{k-1} - \sigma_k| > \frac{\lambda}{(l+1)2^m}\right\}\right) \\ &\quad + v_{\vec{w}}\left(\left\{|\sigma_l| > \frac{\lambda}{(l+1)2^m}\right\}\right). \end{aligned}$$

Using property (2), we have

$$v_{\vec{w}}(\Omega) \leq \sum_{i=1}^{m} \sum_{j=1}^{\infty} v_{\vec{w}}(Q_{i,j}) \lesssim \lambda^{-\frac{1}{m}},$$

therefore

$$v_{\vec{w}}\left(S_s\right) \lesssim \lambda^{-\frac{1}{m}} + \sum_{k=1}^{l} v_{\vec{w}}(P_k) + v_{\vec{w}}(P),$$

where

$$P_k := \left\{ \mathbb{R}^n \setminus \Omega : |\sigma_{k-1} - \sigma_k| > \frac{\lambda}{(l+1)2^m} \right\}, \quad \text{and}$$
$$P := \left\{ |\sigma_l| > \frac{\lambda}{(l+1)2^m} \right\}.$$

We will first estimate $v_{\vec{w}}(P_k)$ for $k \in \{1, \ldots, l\}$. Notice that

$$\sigma_{k-1}(x) - \sigma_k(x)$$

$$= T\left((17\sqrt{n})^n [v_{\vec{w}}]_{A_1} \lambda^{\frac{1}{m}} \overrightarrow{\mathbb{1}_{E_{i,k-1}}} v_{\vec{w}}^{\frac{1}{m}}, \left(b_k - (17\sqrt{n})^n [v_{\vec{w}}]_{A_1} \lambda^{\frac{1}{m}} \mathbb{1}_{E_k} \right) v_{\vec{w}}^{\frac{1}{m}}, \overrightarrow{b_{k+1,l}} v_{\vec{w}}^{\frac{1}{m}}, \overrightarrow{g_{l+1,m}} v_{\vec{w}}^{\frac{1}{m}} \right)(x).$$

Begin by using Chebyshev's inequality, the fact that

$$\int_{\mathbb{R}^n} \left(b_{k,j_k}(x) - (17\sqrt{n})^n [v_{\vec{w}}]_{A_1} \lambda^{\frac{1}{m}} \mathbb{1}_{E_{k,j_k}}(x) \right) v_{\vec{w}}(x)\, dx = 0,$$

and trivial estimates to see

$$v_{\vec{w}}(P_k) \lesssim \lambda^{-1} \int_{\mathbb{R}^n \setminus \Omega} |\sigma_{k-1}(x) - \sigma_k(x)|\, dx$$

$$
\begin{aligned}
&\leq \lambda^{-1} \sum_{j_1,\ldots,j_l=1}^{\infty} \int_{\mathbb{R}^n\setminus\Omega} \bigg| \int_{\mathbb{R}^{n(m-l)}} \int_{Q_{l,j_l}} \cdots \int_{Q_{1,j_1}} K(x, \vec{y}_{1,m}) \\
&\quad \times \left(\prod_{i=1}^{k-1} (17\sqrt{n})^n [v_{\vec{w}}]_{A_1} \lambda^{\frac{1}{m}} \mathbb{1}_{E_{i,i,j_i}}(y_i) v_{\vec{w}}(y_i)^{\frac{1}{m}} \right) \\
&\quad \times \left(b_{k,j_k}(y_k) - (17\sqrt{n})^n [v_{\vec{w}}]_{A_1} \lambda^{\frac{1}{m}} \mathbb{1}_{E_{k,j_k}}(y_k) \right) v_{\vec{w}}(y_k)^{\frac{1}{m}} \\
&\quad \times \left(\prod_{i=k+1}^{l} b_{i,i,j_i}(y_i) v_{\vec{w}}(y_i)^{\frac{1}{m}} \right) \left(\prod_{i=l+1}^{m} g_i(y_i) v_{\vec{w}}(y_i)^{\frac{1}{m}} \right) d\vec{y}_{1,m} \bigg| dx \\
&\leq \lambda^{\frac{k-m-1}{m}} \sum_{j_1,\ldots,j_l=1}^{\infty} \left(\prod_{i=1}^{l} \sup_{Q_{i,j_i}} v_{\vec{w}}^{\frac{1-m}{m}} \right) \int_{\mathbb{R}^n\setminus\Omega} \int_{\mathbb{R}^{n(m-l)}} \int_{E_{l,j_l}} \cdots \int_{E_{1,j_1}} \\
&\quad \times \left| K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m}) \right| \left(\prod_{i=1}^{k-1} v_{\vec{w}}(y_i) \right) \\
&\quad \times \left| b_{k,j_k}(y_k) - (17\sqrt{n})^n [v_{\vec{w}}]_{A_1} \lambda^{\frac{1}{m}} \mathbb{1}_{E_{k,j_k}}(y_k) \right| v_{\vec{w}}(y_k) \left(\prod_{i=k+1}^{l} b_{i,j_i}(y_i) v_{\vec{w}}(y_i) \right) \\
&\quad \times \left(\prod_{i=l+1}^{m} g_i(y_i) v_{\vec{w}}(y_i)^{\frac{1}{m}} \right) d\vec{y}_{1,m} dx.
\end{aligned}
$$

Next use the fact that $v_{\vec{w}}(E_{i,j_i}) \lesssim \lambda^{-\frac{1}{m}} \|b_{i,j_i}\|_{L^1(v_{\vec{w}})}$, Fubini's theorem, trivial estimates, the fact that

$$
\left\| b_{k,j_k} - (17\sqrt{n})^n [v_{\vec{w}}]_{A_1} \lambda^{\frac{1}{m}} \mathbb{1}_{E_{k,j_k}} \right\|_{L^1(v_{\vec{w}})} \lesssim \|b_{k,j_k}\|_{L^1(v_{\vec{w}})},
$$

and property (1) to control

$$
\begin{aligned}
v_{\vec{w}}(P_k) &\lesssim \lambda^{-1} \sum_{j_1,\ldots,j_l=1}^{\infty} \left(\prod_{i=1}^{l} \sup_{Q_{i,j_i}} v_{\vec{w}}^{\frac{1-m}{m}} \right) \left(\prod_{i=1}^{k-1} \|b_{i,j_i}\|_{L^1(v_{\vec{w}})} \right) \left\| b_{k,j_k} - (17\sqrt{n})^n [v_{\vec{w}}]_{A_1} \lambda^{\frac{1}{m}} \mathbb{1}_{E_{k,j_k}} \right\|_{L^1(v_{\vec{w}})} \\
&\quad \times \left(\prod_{i=k+1}^{l} \|b_{i,j_i}\|_{L^1(v_{\vec{w}})} \right) \left(\prod_{i=l+1}^{m} \|g_i\|_{L^\infty(\mathbb{R}^n)} \right) \int_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\ldots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n\setminus\Omega} \\
&\quad \times |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})| \left(\prod_{i=l+1}^{m} v_{\vec{w}}(y_i)^{\frac{1}{m}} \right) dx d\vec{y}_{l+1,m}
\end{aligned}
$$

$$\lesssim \lambda^{-\frac{l}{m}} \sum_{j_1,\ldots,j_l=1}^{\infty} \left(\prod_{i=1}^{l} \sup_{Q_{i,j_i}} v_{\vec{w}}^{\frac{1-m}{m}} \right) \left(\prod_{i=1}^{l} \|b_{i,j_i}\|_{L^1(v_{\vec{w}})} \right)$$
$$\times \int_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\ldots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \Omega} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})|$$
$$\times \left(\prod_{i=l+1}^{m} v_{\vec{w}}(y_i)^{\frac{1}{m}} \right) dx d\vec{y}_{l+1,m}.$$

Use property (3), the fact that $\left(\sup_{Q_{i,j_i}} v_{\vec{w}}^{\frac{1-m}{m}}\right) = \left(\inf_{Q_{i,j_i}} v_{\vec{w}}\right)^{\frac{m-1}{m}}$, the A_1 condition of $v_{\vec{w}}$, and trivial estimates to estimate $v_{\vec{w}}(P_k)$

$$v_{\vec{w}}(P_k) \lesssim [v_{\vec{w}}]_{A_1}^{l} \sum_{j_1,\ldots,j_l=1}^{\infty} \left(\prod_{i=1}^{l} \sup_{Q_{i,j_i}} v_{\vec{w}}^{\frac{1-m}{m}} \right) \left(\prod_{i=1}^{l} v_{\vec{w}}(Q_{i,j_i}) \right)$$
$$\times \int_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\ldots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \Omega} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})|$$
$$\times \left(\prod_{i=l+1}^{m} v_{\vec{w}}(y_i)^{\frac{1}{m}} \right) dx d\vec{y}_{l+1,m}$$
$$\leq [v_{\vec{w}}]_{A_1}^{l} \sum_{j_1,\ldots,j_l=1}^{\infty} \left(\prod_{i=1}^{l} \inf_{Q_{i,j_i}} v_{\vec{w}} \right)^{\frac{1-m}{m}} \left(\prod_{i=1}^{l} v_{\vec{w}}(Q_{i,j_i}) \right)$$
$$\times \int_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\ldots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \Omega} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})|$$
$$\times \left(\prod_{i=l+1}^{m} v_{\vec{w}}(y_i)^{\frac{1}{m}} \right) dx d\vec{y}_{l+1,m}$$
$$\leq [v_{\vec{w}}]_{A_1}^{2l} \sum_{j_1,\ldots,j_l=1}^{\infty} \left(\prod_{i=1}^{l} |Q_{i,j_i}| \left(\inf_{Q_{i,j_i}} v_{\vec{w}}^{\frac{1}{m}} \right) \right)$$
$$\times \int_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\ldots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int_{\mathbb{R}^n \setminus \Omega} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})|$$
$$\times \left(\prod_{i=l+1}^{m} v_{\vec{w}}(y_i)^{\frac{1}{m}} \right) dx d\vec{y}_{l+1,m}$$

$$\leq [v_{\vec{w}}]_{A_1}^{2m} \sum_{j_1,\ldots,j_l=1}^{\infty} \left(\prod_{i=1}^{l} v_{\vec{w}}^{\frac{1}{m}}(Q_{i,j_i}) \right)$$
$$\times \int\limits_{\mathbb{R}^{n(m-l)}} \sup_{\substack{(y_1,\ldots,y_l) \\ \in \prod_{i=1}^{l} Q_{i,j_i}}} \int\limits_{\mathbb{R}^n \setminus \Omega} |K(x, \vec{y}_{1,m}) - K(x, \vec{c}_{(1,j_1),(l,j_l)}, \vec{y}_{l+1,m})|$$
$$\times \left(\prod_{i=l+1}^{m} v_{\vec{w}}(y_i)^{\frac{1}{m}} \right) dx d\vec{y}_{l+1,m}.$$

Use Lemma 4.1.4 and property (2) to finish the estimate

$$v_{\vec{w}}(P_k) \lesssim [v_{\vec{w}}]_{A_1}^{2m+\frac{2m-2}{m}} \sum_{i=1}^{l} v_{\vec{w}}(\Omega) \lesssim [v_{\vec{w}}]_{A_1}^{2m+\frac{2m-2}{m}+1} \lambda^{-\frac{1}{m}}.$$

The control of $v_{\vec{w}}(P)$ follows from Chebyshev's inequality, Theorem 4.1.2, the construction of the sets E_i, property (1), and property (4):

$$\begin{aligned}
v_{\vec{w}}(P) &\lesssim \lambda^{-1} \int\limits_{\mathbb{R}^n} |\sigma_l(x)|\, dx \\
&\lesssim \lambda^{-1+\frac{l}{m}} \left(\prod_{i=1}^{l} v_{\vec{w}}(E_i)^{\frac{1}{m}} \right) \left(\prod_{i=l+1}^{m} \left(\int\limits_{\mathbb{R}^n} g_i(x)^m v_{\vec{w}}(x)\, dx \right)^{\frac{1}{m}} \right) \\
&\lesssim \lambda^{-\frac{1}{m}} \left(\prod_{i=1}^{l} \|b_i\|_{L^1(v_{\vec{w}})}^{\frac{1}{m}} \right) \left(\prod_{i=l+1}^{m} \|g_i\|_{L^1(v_{\vec{w}})}^{\frac{1}{m}} \right) \\
&\leq \lambda^{-\frac{1}{m}}.
\end{aligned}$$

Put the estimates of $v_{\vec{w}}(P_k)$ and $v_{\vec{w}}(P)$ together to get

$$v_{\vec{w}}(S_s) \lesssim [v_{\vec{w}}]_{A_1} \lambda^{-\frac{1}{m}} + \sum_{k=1}^{l} [v_{\vec{w}}]_{A_1}^{2m+\frac{2m-2}{m}+1} \lambda^{-\frac{1}{m}} + \lambda^{-\frac{1}{m}} \lesssim [v_{\vec{w}}]_{A_1}^{2m+\frac{2m-2}{m}+1} \lambda^{-\frac{1}{m}}.$$

Finally, use the estimates of $v_{\vec{w}}(S_s)$, $1 \leq s \leq 2^m$ to complete the proof

$$v_{\vec{w}}\left(\left\{\left|T\left(f_1 w_1 v_{\vec{w}}^{\frac{1-m}{m}}, \ldots, f_m w_m v_{\vec{w}}^{\frac{1-m}{m}}\right)\right| v_{\vec{w}}^{-1} > \lambda\right\}\right) \leq |S_1| + \sum_{s=2}^{2^m} |S_s|$$

$$\lesssim \lambda^{-\frac{1}{m}} + \sum_{s=2}^{2^m} [v_{\vec{w}}]_{A_1}^{2m+\frac{2m-2}{m}+1} \lambda^{-\frac{1}{m}} \lesssim [v_{\vec{w}}]_{A_1}^{2m+\frac{2m-2}{m}+1} \lambda^{-\frac{1}{m}}.$$

□

Chapter 5

Limited-Range Calderón-Zygmund Theorem

5.1 Preliminaries

The work in this chapter is joint with Loukas Grafakos and can be found in [13]. We work with a class of singular integral operators containing the Calderón-Zygmund operators defined in Definition 1.0.1. If Q is a cube, then rQ denotes the cube with the same center as Q and side length equal to r times the side length of Q.

The classical theory of convolution-type singular integral operators says that for kernels defined on $\mathbb{R}^n \setminus \{0\}$ satisfying the smoothness estimate

$$|K(x-y) - K(x)| \leq C_K \frac{|y|^\delta}{|x|^{n+\delta}}$$

for some $C_K > 0$ and some $0 < \delta \leq 1$ whenever $|x| \geq 2|y|$, the weak-type $(1,1)$ bound holds for the associated singular integral operator, assuming that an $L^s(\mathbb{R}^n)$ bound is known for some $1 < s \leq \infty$. This in turn implies that such singular integral operators are bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ via interpolation.

Hörmander extended this theory in [15] to more general kernels K satisfying the smoothness condition

$$[K]_H := \sup_{y \in \mathbb{R}^n} \int_{|x| \geq 2|y|} |K(x-y) - K(x)| \, dx < \infty.$$

The Hörmander condition is an $L^1(\mathbb{R}^n)$-type smoothness condition and has some variants. For example, Watson introduced the following $L^r(\mathbb{R}^n)$ versions in [33]: for $1 \leq r \leq \infty$, we say a kernel K is in the class H^r if

$$[K]_{H^r} := \sup_{R>0} \sup_{\substack{y\in\mathbb{R}^n \\ |y|\le R}} \sum_{m=1}^{\infty} (2^m R)^{\frac{n}{r'}} \left[\int_{\substack{|x|\ge 2^m R \\ |x|<2^{m+1}R}} |K(x-y)-K(x)|^r dx \right]^{\frac{1}{r}} < \infty,$$

where r' is the Hölder conjugate of r. Observe that Watson's condition coincides with Hörmander's condition when $r=1$, and for $r_1, r_2 \in [1,\infty]$ with $r_1 \le r_2$,

$$H^{r_2} \subseteq H^{r_1} \subseteq H^1 = H.$$

In this chapter, we focus on a different set of $L^r(\mathbb{R}^n)$-adapted conditions defined as follows.

Definition 5.1.1. Let $1 \le r \le \infty$. A kernel K defined on $\mathbb{R}^n \setminus \{0\}$ is in the class H_r if

$$[K]_{H_r} := \sup_{R>0} \left[\frac{1}{v_n R^n} \int_{|y|\le R} \left(\int_{|x|\ge 2R} |K(x-y)-K(x)|\, dx \right)^r dy \right]^{\frac{1}{r}} < \infty,$$

where v_n is the volume of the unit ball $B(0,1)$ in $\mathbb{R}^n$.

Notice that this condition coincides with the Hörmander condition when $r=\infty$. Moreover, for $r_1, r_2 \in [1,\infty]$ with $r_1 \le r_2$,

$$H = H_\infty \subseteq H_{r_2} \subseteq H_{r_1},$$

meaning the H_r conditions are weaker than Hörmander's smoothness condition.

We prove the following variant of the weak-type property for singular integral operators, where we assume the smoothness condition of Definition 5.1.1 and weak-type $(1,1)$ is replaced by weak-type (q,q).

Theorem 5.1.1. Let $1 \le q < \infty$ and $K \in H_{q'}$. If the associated singular integral operator

T is bounded on $L^s(\mathbb{R}^n)$ for some $s \in (q, \infty]$, then

$$\|Tf\|_{L^{q,\infty}(\mathbb{R}^n)} := \sup_{\lambda>0} \lambda |\{|Tf| > \lambda\}|^{\frac{1}{q}} \lesssim \|f\|_{L^q(\mathbb{R}^n)}$$

for all $f \in L^q(\mathbb{R}^n)$. Moreover, T maps $L^q(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ with bound at most a constant multiple of $\|T\|_{L^s(\mathbb{R}^n)\to L^s(\mathbb{R}^n)} + [K]_{H_{q'}}$.

We give two proofs of Theorem 5.1.1. The first proof uses the $L^q(\mathbb{R}^n)$ version of the Calderón-Zygmund decomposition and the second proof is motivated by Nazarov, Treil, and Volberg's proof for the weak-type $(1,1)$ inequality in the nonhomogeneous setting.

By interpolation we obtain the following corollary.

Corollary 5.1.1. Under the hypotheses of Theorem 5.1.1, the operator T is bounded on $L^p(\mathbb{R}^n)$ for p in the interval $(\min(s', q), \max(q', s))$.

If $q > 1$ and $s < \infty$, then the interval $(\min(s', q), \max(q', s))$ is properly contained in $(1, \infty)$. Hence in this case, we obtain $L^p(\mathbb{R}^n)$ estimates for a limited-range of values of p. Prior to this work, other "limited-range" versions of the Calderón-Zygmund theorem appeared in Baernstein and Sawyer [1], Carbery [5], Seeger [27], and Grafakos, Honzík, Ryabogin [11].

Remark 5.1.1. The conclusions of Theorem 5.1.1 and Corollary 5.1.1 also follow under the weaker hypothesis that T is bounded from $L^{s,1}(\mathbb{R}^n)$ to $L^{s,\infty}(\mathbb{R}^n)$. Here $L^{s,r}(\mathbb{R}^n)$ is the usual Lorentz space as defined in [9, 10, 28].

Remark 5.1.2. As in the case $q = 1$, there are natural vector-valued extensions of Theorem 5.1.1 and Corollary 5.1.1, in the spirit of [2].

Remark 5.1.3. Theorem 5.1.1 and Corollary 5.1.1 are also valid if the original kernel is not of convolution type. In this setting, we say a kernel K defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) : x = y\}$

is in H_r if

$$\sup_{y' \in \mathbb{R}^n} \sup_{R>0} \left[\frac{1}{v_n R^n} \int_{|y-y'| \leq R} \left(\int_{|x-y| \geq 2R} |K(x,y) - K(x,y')| \, dx \right)^r dy \right]^{\frac{1}{r}} < \infty,$$

and

$$\sup_{x' \in \mathbb{R}^n} \sup_{R>0} \left[\frac{1}{v_n R^n} \int_{|x-x'| \leq R} \left(\int_{|x-y| \geq 2R} |K(x,y) - K(x',y)| \, dy \right)^r dx \right]^{\frac{1}{r}} < \infty,$$

where v_n is the volume of the unit ball $B(0,1)$ in $\mathbb{R}^n$.

5.2 Calderón-Zygmund Decomposition Method

The first proof of Theorem 5.1.1 relies on the $L^q(\mathbb{R}^n)$ version of the Calderón-Zygmund decomposition. See [9, 10, 28] for details on the decomposition.

Proof 1. Fix $f \in L^q(\mathbb{R}^n)$ and $\lambda > 0$. Write $B = \|T\|_{L^s(\mathbb{R}^n) \to L^s(\mathbb{R}^n)}$. We will show that there exists a constant $C_{n,s,q} > 0$ depending on n, s, and q, such that

$$|\{|Tf| > \lambda\}| \leq C_{n,s,q} (B + [K]_{H^{q'}})^q \lambda^{-q} \|f\|^q_{L^q(\mathbb{R}^n)}.$$

Apply the $L^q(\mathbb{R}^n)$-form of the Calderón-Zygmund decomposition to f at height $\gamma\lambda$ (the constant $\gamma > 0$ will be chosen later), to write

$$f = g + b = g + \sum_{j=1}^{\infty} b_j,$$

where

(1) $\|g\|_{L^\infty(\mathbb{R}^n)} \leq 2^{\frac{n}{q}} \gamma \lambda$ and $\|g\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^q(\mathbb{R}^n)}$,

(2) the b_j are supported on pairwise disjoint cubes Q_j satisfying

$$\sum_{j=1}^{\infty} |Q_j| \leq (\gamma\lambda)^{-q} \|f\|_{L^q(\mathbb{R}^n)}^q,$$

(3) $\|b_j\|_{L^q(\mathbb{R}^n)}^q \leq 2^{n+q}(\gamma\lambda)^q |Q_j|$,

(4) $\int_{Q_j} b_j(x)\,dx = 0$, and

(5) $\|b\|_{L^q(\mathbb{R}^n)} \leq 2^{\frac{n+q}{q}} \|f\|_{L^q(\mathbb{R}^n)}$ and $\|b\|_{L^1(\mathbb{R}^n)} \leq 2(\gamma\lambda)^{1-q} \|f\|_{L^q(\mathbb{R}^n)}^q$.

Now,

$$|\{|Tf| > \lambda\}| \leq \left|\left\{|Tg| > \frac{\lambda}{2}\right\}\right| + \left|\left\{|Tb| > \frac{\lambda}{2}\right\}\right|.$$

Assume first that $s < \infty$. Choose $\gamma = (B + [K]_{H_{q'}})^{-1}$. Using Chebyshev's inequality, the bound of T on $L^s(\mathbb{R}^n)$, property (1), and trivial estimates, we have that

$$\begin{aligned}
\left|\left\{|Tg| > \frac{\lambda}{2}\right\}\right| &\leq 2^s \lambda^{-s} \|Tg\|_{L^s(\mathbb{R}^n)}^s \\
&\leq (2B)^s \lambda^{-s} \|g\|_{L^s(\mathbb{R}^n)}^s \\
&\leq 2^{s-n+\frac{ns}{q}} B^s \lambda^{-s} (\gamma\lambda)^{s-q} \|g\|_{L^q(\mathbb{R}^n)}^q \\
&\leq 2^{s-n+\frac{ns}{q}} (B + [K]_{H_{r'}})^q \lambda^{-q} \|f\|_{L^q(\mathbb{R}^n)}^q.
\end{aligned}$$

We next control the second term. Let c_j denote the center of Q_j, let $Q_j^* := 2\sqrt{n}Q_j$, and set $\Omega^* := \bigcup_{j=1}^{\infty} Q_j^*$. Then

$$\left|\left\{|Tb| > \frac{\lambda}{2}\right\}\right| \leq |\Omega^*| + \left|\left\{x \in \mathbb{R}^n \setminus \Omega^* : |Tb(x)| > \frac{\lambda}{2}\right\}\right|.$$

Notice that since $|Q_j^*| = (2\sqrt{n})^n |Q_j|$ and by property (2), we have

$$|\Omega^*| \leq \sum_{j=1}^{\infty} |Q_j^*| = (2\sqrt{n})^n \sum_{j=1}^{\infty} |Q_j| \leq (2\sqrt{n})^n (B + [K]_{H_q'})^q \lambda^{-q} \|f\|_{L^q(\mathbb{R}^n)}^q.$$

It remains to control the last term. Use Chebyshev's inequality, property (4), Fubini's theorem, Hölder's inequality, property (3), and property (2) to estimate

$$\begin{aligned}
\left|\left\{\mathbb{R}^n \setminus \Omega^* : |Tb| > \frac{\lambda}{2}\right\}\right| &\leq 2\lambda^{-1} \int\limits_{\mathbb{R}^n \setminus \Omega^*} |Tb(x)|\, dx \\
&\leq 2\lambda^{-1} \sum_{j=1}^{\infty} \int\limits_{\mathbb{R}^n \setminus \Omega^*} |Tb_j(x)| dx \\
&\leq 2\lambda^{-1} \sum_{j=1}^{\infty} \int\limits_{Q_j} \left[\int\limits_{\mathbb{R}^n \setminus \Omega^*} |K(x-y) - K(x-c_j)|\, dx\right] |b_j(y)|\, dy \\
&\leq 2\lambda^{-1} \sum_{j=1}^{\infty} \left\| \int\limits_{\mathbb{R}^n \setminus \Omega^*} |K(x-\cdot) - K(x-c_j)|\, dx \right\|_{L^{q'}(Q_j)} \|b_j\|_{L^q} \\
&\leq 2\lambda^{-1} \sup_{j\in\mathbb{N}} \left\| \int\limits_{\mathbb{R}^n \setminus \Omega^*} |K(x-\cdot) - K(x-c_j)|\, dx \right\|_{L^{q'}\left(Q_j, \frac{dy}{|Q_j|}\right)} \sum_{j=1}^{\infty} |Q_j|^{\frac{1}{q'}} \|b_j\|_{L^q} \\
&\leq 2^{\frac{n}{q}+2} \gamma \sup_{j\in\mathbb{N}} \left\| \int\limits_{\mathbb{R}^n \setminus \Omega^*} |K(x-\cdot) - K(x-c_j)|\, dx \right\|_{L^{q'}\left(Q_j, \frac{dy}{|Q_j|}\right)} \sum_{j=1}^{\infty} |Q_j| \\
&\leq 2^{\frac{n}{q}+2} \gamma^{1-q} \lambda^{-q} \|f\|_{L^q(\mathbb{R}^n)}^q \sup_{j\in\mathbb{N}} \left\| \int\limits_{\mathbb{R}^n \setminus \Omega^*} |K(x-\cdot) - K(x-c_j)|\, dx \right\|_{L^{q'}\left(Q_j, \frac{dy}{|Q_j|}\right)}.
\end{aligned}$$

For each j, setting $R_j = \frac{\sqrt{n}}{2} l(Q_j)$, we have

$$Q_j \subseteq B(c_j, R_j) \subseteq B(c_j, 2R_j) \subseteq Q_j^*,$$

where $B(x, r)$ denotes the ball centered at x and with radius r. Then the factor involving the supremum is less than or equal to

$$\sup_{j\in\mathbb{N}}\left[\int\limits_{B(c_j,R_j)}\left(\int\limits_{\mathbb{R}^n\setminus B(c_j,2R_j)}|K(x-y)-K(x-c_j)|dx\right)^{q'}\frac{dy}{|Q_j|}\right]^{\frac{1}{q'}},$$

which is bounded by $\left(\frac{\sqrt{n}}{2}\right)^n v_n[K]_{H_{q'}}$ by changing variables $x'=x-c_j$, $y'=y-c_j$ and by replacing the supremum over R_j by the supremum over all $R>0$.

Putting all of the estimates together, we get

$$|\{|Tf|>\lambda\}|\le\left(2^{s-n+\frac{ns}{q}}+(2\sqrt{n})^n+2^{\frac{n}{q}+2-n}n^{\frac{n}{2}}\right)(B+[K]_{H_{q'}})^q\lambda^{-q}\|f\|^q_{L^q(\mathbb{R}^n)}.$$

When $s=\infty$, set $\gamma=2^{-\frac{n}{q}}(4([K]_{H_{q'}}+B))^{-1}$. Then

$$\|Tg\|_{L^\infty(\mathbb{R}^n)}\le B\|g\|_{L^\infty(\mathbb{R}^n)}\le 2^{\frac{n}{q}}B\gamma\lambda\le\frac{\lambda}{4},$$

so

$$\left|\left\{|Tg|>\frac{\lambda}{2}\right\}\right|=0.$$

The part of the argument involving $\left\{|Tb|>\frac{\lambda}{2}\right\}$ is the same as in the case $s<\infty$. □

5.3 Nazarov-Treil-Volberg Method

Proof 2. Fix $f\in L^q(\mathbb{R}^n)$ and $\lambda>0$. Write $B=\|T\|_{L^s(\mathbb{R}^n)\to L^s(\mathbb{R}^n)}$. We will show that there exists a constant $C_{n,s,q}>0$ depending on n, s, and q, such that

$$|\{|Tf|>\lambda\}|\le C_{n,s,q}(B+[K]_{H_{q'}})^q\lambda^{-q}\|f\|^q_{L^q(\mathbb{R}^n)}.$$

By density, we may assume f is a nonnegative continuous function with compact support. Set

$$\Omega := \{M(f^q) > (\gamma\lambda)^q\},$$

where $\gamma > 0$ is to be chosen later and where M denotes the Hardy-Littlewood maximal operator. Apply a Whitney decomposition to write

$$\Omega = \bigcup_{j=1}^{\infty} Q_j,$$

a disjoint union of dyadic cubes where

$$2\mathrm{diam}(Q_j) \leq \mathrm{dist}(Q_j, \mathbb{R}^n \setminus \Omega) \leq 8\mathrm{diam}(Q_j).$$

Put

$$g := f\mathbb{1}_{\mathbb{R}^n\setminus\Omega}, \qquad b := f\mathbb{1}_{\Omega}, \qquad \text{and} \qquad b_j := f\mathbb{1}_{Q_j}.$$

Then

$$f = g + b = g + \sum_{j=1}^{\infty} b_j,$$

where we claim that

(1) $\|g\|_{L^\infty(\mathbb{R}^n)} \leq \gamma\lambda$ and $\|g\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^q(\mathbb{R}^n)}$,

(2) the b_j are supported on pairwise disjoint cubes Q_j satisfying

$$\sum_{j=1}^{\infty} |Q_j| \leq 3^n(\gamma\lambda)^{-q}\|f\|^q_{L^q(\mathbb{R}^n)},$$

(3) $\|b_j\|^q_{L^q(\mathbb{R}^n)} \leq (17\sqrt{n})^n(\gamma\lambda)^q|Q_j|$, and

(4) $\|b\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^q(\mathbb{R}^n)}$ and $\|b\|_{L^1(\mathbb{R}^n)} \leq (17\sqrt{n})^{\frac{n}{q}}3^n(\gamma\lambda)^{1-q}\|f\|^q_{L^q(\mathbb{R}^n)}$.

Indeed, since for almost every $x \notin \Omega$, we have

$$|g(x)|^q = |f(x)|^q \leq M(f^q)(x) \leq (\gamma\lambda)^q,$$

it follows that $\|g\|_{L^\infty(\mathbb{R}^n)} \leq \gamma\lambda$. Since g is a restriction of f, we have $\|g\|_{L^q(\mathbb{R}^n)} \leq \|f\|_{L^q(\mathbb{R}^n)}$, and so (1) holds. Using the weak-type $(1,1)$ bound for M with $\|M\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)} \leq 3^n$, we obtain property (2) as follows

$$\sum_{j=1}^{\infty} |Q_j| = |\Omega| \leq 3^n(\gamma\lambda)^{-q}\|f\|^q_{L^q(\mathbb{R}^n)}.$$

Addressing (3) and (4), let $Q_j^* := 17\sqrt{n}Q_j$ be the cube with the same center as Q_j but side length $17\sqrt{n}$ times as large. Then $Q_j^* \cap (\mathbb{R}^n \setminus \Omega) \neq \emptyset$, so there is a point $x \in Q_j^*$ such that $M(f^q)(x) \leq (\gamma\lambda)^q$. In particular, $\int_{Q_j^*} |f(y)|^q dy \leq (\gamma\lambda)^q |Q_j^*|$. Since $|Q_j^*| = (17\sqrt{n})^n |Q_j|$, we have

$$\|b_j\|^q_{L^q(\mathbb{R}^n)} = \int\limits_{Q_j} |f(y)|^q dy \leq \int\limits_{Q_j^*} |f(y)|^q dy \leq (\gamma\lambda)^q |Q_j^*| = (17\sqrt{n})^n (\gamma\lambda)^q |Q_j|.$$

This proves (3). We use Hölder's inequality, property (3), and property (2) to justify property (4)

$$\|b\|_{L^1(\mathbb{R}^n)} = \sum_{j=1}^{\infty} \|b_j\|_{L^1(\mathbb{R}^n)} \leq \sum_{j=1}^{\infty} \|b_j\|_{L^q(\mathbb{R}^n)} |Q_j|^{\frac{1}{q'}} \leq (17\sqrt{n})^{\frac{n}{q}} (\gamma\lambda) \sum_{j=1}^{\infty} |Q_j|$$
$$\leq (17\sqrt{n})^{\frac{n}{q}} 3^n (\gamma\lambda)^{1-q} \|f\|^q_{L^q(\mathbb{R}^n)}.$$

Now,

$$|\{|Tf| > \lambda\}| \leq \left|\left\{|Tg| > \frac{\lambda}{2}\right\}\right| + \left|\left\{|Tb| > \frac{\lambda}{2}\right\}\right|.$$

Assume first that $s < \infty$. Choose $\gamma = (B + [K]_{H_{q'}})^{-1}$. Use Chebyshev's inequality, the bound of T on $L^s(\mathbb{R}^n)$, and property (1) to see

$$\begin{aligned}\left|\left\{|Tg| > \frac{\lambda}{2}\right\}\right| &\leq 2^s \lambda^{-s} \|Tg\|_{L^s(\mathbb{R}^n)}^s \\ &\leq (2B)^s \lambda^{-s} \|g\|_{L^s(\mathbb{R}^n)}^s \\ &\leq (2B)^s (\gamma\lambda)^{s-q} \lambda^{-s} \|g\|_{L^q(\mathbb{R}^n)}^q \\ &\leq 2^s (B + [K]_{H_{q'}})^q \lambda^{-q} \|f\|_{L^q(\mathbb{R}^n)}^q.\end{aligned}$$

We will now control the second term. Let E_j be a concentric dilate of Q_j; precisely,

$$E_j := Q(c_j, r_j),$$

where c_j is the center of Q_j and $r_j > 0$ is chosen so that $|E_j| = \frac{1}{(17\sqrt{n})^{\frac{n}{q}}\gamma\lambda} \int_{Q_j} b_j(x)\, dx$. Note that such E_j exist since the function $r \mapsto |Q(x,r)|$ is continuous for each $x \in \mathbb{R}^n$. Applying Hölder's inequality and property (3), we have

$$|E_j| = \frac{1}{(17\sqrt{n})^{\frac{n}{q}}\gamma\lambda} \int\limits_{Q_j} b_j(x)\, dx \leq \frac{1}{(17\sqrt{n})^{\frac{n}{q}}\gamma\lambda} |Q_j|^{\frac{1}{q'}} \|b_j\|_{L^q(\mathbb{R}^n)} \leq |Q_j|.$$

Since E_j is a cube with the same center as Q_j and since $|E_j| \leq |Q_j|$, the containment $E_j \subseteq Q_j$ holds. In particular, the E_j are pairwise disjoint. Set

$$E := \bigcup_{j=1}^{\infty} E_j.$$

Then

$$\left|\left\{|Tb| > \frac{\lambda}{2}\right\}\right| \leq \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$
\begin{aligned}
\mathrm{I} &= |\Omega|, \\
\mathrm{II} &= \left|\left\{x \in \mathbb{R}^n \setminus \Omega : \left|T\left(b - (17\sqrt{n})^{\frac{n}{q}}\gamma\lambda\mathbb{1}_E\right)(x)\right| > \frac{\lambda}{4}\right\}\right|, \text{ and} \\
\mathrm{III} &= \left|\left\{(17\sqrt{n})^{\frac{n}{q}}\gamma\lambda|T(\mathbb{1}_E)| > \frac{\lambda}{4}\right\}\right|.
\end{aligned}
$$

The control of I follows from property (2),

$$
|\Omega| = \sum_{j=1}^{\infty} |Q_j| \le 3^n (B + [K]_{H_{q'}})^q \lambda^{-q} \|f\|_{L^q(\mathbb{R}^n)}^q.
$$

For II, use Chebyshev's inequality, the fact that $\int_{Q_j} b_j(y) - (17\sqrt{n})^{\frac{n}{q}}\gamma\lambda\mathbb{1}_{E_j}(y)\,dy = 0$, Fubini's theorem, and Hölder's inequality to estimate

$$
\begin{aligned}
\mathrm{II} &\le 4\lambda^{-1} \int_{\mathbb{R}^n\setminus\Omega} \left|T\left(b - (17\sqrt{n})^{\frac{n}{q}}\gamma\lambda\mathbb{1}_E\right)(x)\right| dx \\
&\le 4\lambda^{-1} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n\setminus\Omega} \left|T\left(b_j - (17\sqrt{n})^{\frac{n}{q}}\gamma\lambda\mathbb{1}_{E_j}\right)(x)\right| dx \\
&\le 4\lambda^{-1} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n\setminus\Omega} \int_{Q_j} |K(x-y) - K(x-c_j)| \left|b_j(y) - (17\sqrt{n})^{\frac{n}{q}}\gamma\lambda\mathbb{1}_{E_j}(y)\right| dydx \\
&= 4\lambda^{-1} \sum_{j=1}^{\infty} \int_{Q_j} \left(\int_{\mathbb{R}^n\setminus\Omega} |K(x-y) - K(x-c_j)| dx\right) \left|b_j(y) - (17\sqrt{n})^{\frac{n}{q}}\gamma\lambda\mathbb{1}_{E_j}(y)\right| dy \\
&\le 4\lambda^{-1} \sum_{j=1}^{\infty} \left\|\int_{\mathbb{R}^n\setminus\Omega} |K(x-y) - K(x-c_j)| dx\right\|_{L^{q'}(Q_j)} \left\|b_j - (17\sqrt{n})^{\frac{n}{q}}\gamma\lambda\mathbb{1}_{E_j}\right\|_{L^q(\mathbb{R}^n)} \\
&\le 4\lambda^{-1} \sup_{j\in\mathbb{N}} \left\|\int_{\mathbb{R}^n\setminus\Omega} |K(x-y) - K(x-c_j)| dx\right\|_{L^{q'}\left(Q_j, \frac{dy}{|Q_j|}\right)} \\
&\qquad \times \sum_{j=1}^{\infty} |Q_j|^{\frac{1}{q'}} \left\|b_j - (17\sqrt{n})^{\frac{n}{q}}\gamma\lambda\mathbb{1}_{E_j}\right\|_{L^q(\mathbb{R}^n)}.
\end{aligned}
$$

Using the triangle inequality, property (3), and the fact that $|E_j| \leq |Q_j|$, we have

$$\left\| b_j - (17\sqrt{n})^{\frac{n}{q}} \gamma\lambda \mathbb{1}_{E_j} \right\|_{L^q(\mathbb{R}^n)} \leq \|b_j\|_{L^q(\mathbb{R}^n)} + (17\sqrt{n})^{\frac{n}{q}} \gamma\lambda |E_j|^{\frac{1}{q}} \leq 2(17\sqrt{n})^{\frac{n}{q}} \gamma\lambda |Q_j|^{\frac{1}{q}}.$$

Using the above estimate and property (2), we control

$$\begin{aligned} \text{II} &\leq 8(17\sqrt{n})^{\frac{n}{q}} \gamma \sup_{j\in\mathbb{N}} \left\| \int_{\mathbb{R}^n\setminus\Omega} |K(x-y) - K(x-c_j)| dx \right\|_{L^{q'}\left(Q_j, \frac{dy}{|Q_j|}\right)} \sum_{j=1}^{\infty} |Q_j| \\ &\leq 8(17\sqrt{n})^{\frac{n}{q}} 3^n \gamma^{1-q} \lambda^{-q} \|f\|^q_{L^q(\mathbb{R}^n)} \sup_{j\in\mathbb{N}} \left\| \int_{\mathbb{R}^n\setminus\Omega} |K(x-y) - K(x-c_j)| dx \right\|_{L^{q'}\left(Q_j, \frac{dy}{|Q_j|}\right)}. \end{aligned}$$

For each j, setting $R_j = \frac{\sqrt{n}}{2} l(Q_j)$, we have

$$Q_j \subseteq B(c_j, R_j) \subseteq B(c_j, 2R_j) \subseteq \Omega.$$

Then the supremum is bounded by

$$\sup_{j\in\mathbb{N}} \left[\int_{B(c_j,R_j)} \left(\int_{\mathbb{R}^n\setminus B(c_j,2R_j)} |K(x-y) - K(x-c_j)| dx \right)^{q'} \frac{dy}{|Q_j|} \right]^{\frac{1}{q'}},$$

which is bounded by $\left(\frac{\sqrt{n}}{2}\right)^n v_n [K]_{H_{q'}}$ by changing variables $x' = x - c_j$, $y' = y - c_j$ and by replacing the supremum over R_j by the supremum over all $R > 0$. Therefore

$$\text{II} \leq 8(17\sqrt{n})^{\frac{n}{q}} \left(\frac{3\sqrt{n}}{2}\right)^n v_n (B + [K]_{H_{q'}})^q \lambda^{-q} \|f\|^q_{L^q(\mathbb{R}^n)}.$$

To control III, use Chebyshev's inequality, the bound of T on $L^s(\mathbb{R}^n)$, the fact that

$|E| \le |\Omega|$, and property (2) to estimate

$$\begin{aligned}\mathrm{III} &\le 4^s (17\sqrt{n})^{\frac{ns}{q}} \gamma^s \int\limits_{\mathbb{R}^n} |T(\mathbb{1}_E)(x)|^s \, dx \\ &\le 4^s (17\sqrt{n})^{\frac{ns}{q}} \gamma^s B^s |E| \\ &\le 4^s (17\sqrt{n})^{\frac{ns}{q}} |\Omega| \\ &\le 4^s (17\sqrt{n})^{\frac{ns}{q}} 3^n (B + [K]_{H_{q'}})^q \lambda^{-q} \|f\|_{L^q(\mathbb{R}^n)}^q.\end{aligned}$$

Putting the estimates together, we get

$$|\{|Tf| > \lambda\}| \le \left(2^s + 3^n + 8(17\sqrt{n})^{\frac{n}{q}} \left(\frac{3\sqrt{n}}{2}\right)^n v_n + 4^s (17\sqrt{n})^{\frac{ns}{q}} 3^n\right) \frac{(B + [K]_{H_{q'}})^q}{\lambda^q} \|f\|_{L^q(\mathbb{R}^n)}^q.$$

Since we assumed that f was nonnegative, we must double the constant above to prove the statement for general $f \in L^q(\mathbb{R}^n)$.

When $s = \infty$, set $\gamma = (4(B + [K]_{H_{q'}}))^{-1}$. Then

$$\|Tg\|_{L^\infty(\mathbb{R}^n)} \le B\|g\|_{L^\infty(\mathbb{R}^n)} \le B\gamma\lambda \le \frac{\lambda}{4},$$

so $\left|\left\{|Tg| > \frac{\lambda}{2}\right\}\right| = 0$. The part of the argument involving the set $\left\{|Tb| > \frac{\lambda}{2}\right\}$ is the same as in the case $s < \infty$. $\square$

Chapter 6

Riesz Transform Dimensional Dependence

6.1 Preliminaries

Throughout this chapter, we write $A \lesssim B$ if there exists an absolute constant $C > 0$ such that $A \leq CB$. We focus on studying the Riesz transforms on $\mathbb{R}^n$.

Definition 6.1.1. Let $1 \leq j \leq n$. The j^{th} Riesz transform R_j is given by

$$R_j f(x) = \widetilde{C}_n \int\limits_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy,$$

for smooth compactly supported f and $x \notin \operatorname{supp} f$, where $\widetilde{C}_n := \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}$.

Since the Riesz transforms are examples of Calderón-Zygmund operators, they satisfy the weak-type $(1,1)$ property:

$$\|R_j f\|_{L^{1,\infty}(\mathbb{R}^n)} := \sup_{\lambda > 0} \lambda |\{|R_j f| > \lambda\}| \leq C_n \|f\|_{L^1(\mathbb{R}^n)}$$

for all $f \in L^1(\mathbb{R}^n)$, where $C_n > 0$ is a constant that is allowed to depend on n. The smallest $C_n > 0$ where the above inequality holds is denoted by $\|R_j\|_{L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)}$. We are interested in determining how $\|R_j\|_{L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)}$ depends on the dimension n.

Classical proofs of the weak-type $(1,1)$ property, as well as all proofs given earlier in this thesis, imply that $\|R_j\|_{L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)}$ depends at most exponentially on n. In [18], Janakiraman improved the exponential dependence by proving the following theorem.

Theorem 6.1.1. The Riesz tranforms satisfy

$$\|R_j f\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim \log(n)\|f\|_{L^1(\mathbb{R}^n)}$$

for all $f \in L^1(\mathbb{R}^n)$.

Janakiraman's proof of Theorem 6.1.1 follows the Calderón-Zygmund decomposition method for proving the weak-type $(1,1)$ property, with modifications in the decomposition and more careful estimates involving the kernel. We present a different proof of Theorem 6.1.1, relying on ideas inspired by Nazarov, Treil, and Volberg. Our proof closely follows the model of the proof in Chapter 2, together with the kernel estimates found in [18].

As in [18], we prove a weak-type inequality for operators in a certain class of dilation commuting singular integral operators. We note that the Riesz transforms are examples of such singular integral operators, and obtain Theorem 6.1.1 as a consequence.

Definition 6.1.2. Let T be a singular integral operator associated to a kernel function K defined on $\mathbb{R}^n \setminus \{0\}$ and having the form $K(x) = \frac{\Omega(x)}{|x|^n}$, where

(1)
$$\Omega(x) = \Omega\left(\frac{x}{|x|}\right) = \Omega(\delta x),$$

for all $x \neq 0$ and $\delta > 0$,

(2)
$$\int_{S^{n-1}} \Omega(x)\, d\sigma(x) = 0,$$

where σ denotes surface measure on S^{n-1}, and

(3)
$$\int_{S^{n-1}} |\Omega(x - \xi\delta) - \Omega(x)|\, d\sigma(x) \lesssim n\delta \int_{S^{n-1}} |\Omega(x)|\, d\sigma(x)$$

for all $\xi \in S^{n-1}$ and all $0 < \delta < \frac{1}{n}$.

Note that R_j is an example of such a singular integral operator with $\Omega(x) = \widetilde{C}_n \frac{x_j}{|x|}$.

6.2 Main Result

Theorem 6.2.1. If T is a singular integral operator as defined in Definition 6.1.2, then

$$\|Tf\|_{L^{1,\infty}(\mathbb{R}^n)} \lesssim \log(n) \int\limits_{S^{n-1}} |\Omega(x)|\, d\sigma(x) \|f\|_{L^1(\mathbb{R}^n)}$$

for all $f \in L^1(\mathbb{R}^n)$.

Theorem 6.1.1 follows from Theorem 6.2.1 since, taking $T = R_j$, we have

$$\int\limits_{S^{n-1}} |\Omega(x)|\, d\sigma(x) = \widetilde{C}_n \int\limits_{S^{n-1}} |x_j|\, d\sigma(x) = \frac{2}{\pi}.$$

Proof. Let $\lambda > 0$ and $f \in L^1(\mathbb{R}^n)$ be given. We will actually show that

$$|\{|Tf| > \lambda\}| \lesssim \left(\log(n) \int\limits_{S^{n-1}} |\Omega(x)|\, d\sigma(x) + B \right) \frac{1}{\lambda} \|f\|_{L^1(\mathbb{R}^n)},$$

where $B = \|T\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)}$. This suffices since, by [18, Theorem 3.1],

$$B \lesssim \log(n) \int\limits_{S^{n-1}} |\Omega(x)|\, d\sigma(x).$$

By density, we may assume f is a nonnegative continuous function with compact support. Set

$$G := \left\{ f > \frac{\lambda}{B} \right\}.$$

Apply a Whitney decomposition to write

$$G = \bigcup_{i=1}^{\infty} Q_i,$$

a disjoint union of dyadic cubes where

$$2\text{diam}(Q_i) \le \text{dist}(Q_i, \mathbb{R}^n \setminus G) \le 8\text{diam}(Q_i).$$

Put

$$g := f\mathbb{1}_{\mathbb{R}^n\setminus G}, \qquad b := f\mathbb{1}_G, \qquad \text{and} \qquad b_i := f\mathbb{1}_{Q_i}.$$

Then

$$f = g + b = g + \sum_{i=1}^{\infty} b_i,$$

where

(1) $\|g\|_{L^\infty(\mathbb{R}^n)} \le \frac{\lambda}{B}$ and $\|g\|_{L^1(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}$,

(2) the b_i are supported on pairwise disjoint cubes Q_i satisfying

$$\sum_{i=1}^{\infty} |Q_i| \le \frac{B}{\lambda}\|f\|_{L^1(\mathbb{R}^n)},$$

and

(3) $\|b\|_{L^1(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}$.

Then

$$|\{|Tf| > \lambda\}| \le \left|\left\{|Tg| > \frac{\lambda}{2}\right\}\right| + \left|\left\{|Tb| > \frac{\lambda}{2}\right\}\right|.$$

To control the first term, use Chebyshev's inequality, the boundedness of T on $L^2(\mathbb{R}^n)$, and property (1) to estimate

$$\begin{aligned}\left|\left\{|Tg| > \frac{\lambda}{2}\right\}\right| &\lesssim \frac{1}{\lambda^2}\|Tg\|_{L^2(\mathbb{R}^n)}^2 \\ &\le \frac{B^2}{\lambda^2}\|g\|_{L^2(\mathbb{R}^n)}^2\end{aligned}$$

$$\le \frac{B}{\lambda}\|g\|_{L^1(\mathbb{R}^n)}$$
$$\le \frac{B}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.$$

For positive integers N, set $b^{(N)} := \sum_{i=1}^N b_i$. To control the second term, it suffices to handle $\left|\left\{|Tb^{(N)}| > \frac{\lambda}{2}\right\}\right|$ uniformly in N. Let c_i denote the center of Q_i and set

$$E_1 := B(c_1, r_1),$$

where $r_1 > 0$ is chosen so that $|E_1| = \frac{B}{\lambda}\|b_1\|_{L^1(\mathbb{R}^n)}$. In general, for $i = 2, 3, \ldots, N$, set

$$E_i := B(c_i, r_i) \setminus \bigcup_{k=1}^{i-1} E_k,$$

where $r_i > 0$ is chosen so that $|E_i| = \frac{B}{\lambda}\|b_i\|_{L^1(\mathbb{R}^n)}$. Note that such E_i exist since the function $r \mapsto |B(x,r)|$ is continuous for each $x \in \mathbb{R}^n$. Define

$$E := \bigcup_{i=1}^N E_i = \bigcup_{i=1}^N B(c_i, r_i) \quad \text{and} \quad E^* := \bigcup_{i=1}^N B\left(c_i, \left(1 + \frac{1}{n}\right) r_i\right).$$

Then

$$\left|\left\{|Tb^{(N)}| > \frac{\lambda}{2}\right\}\right| \le \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\mathrm{I} := |G \cup E^*|,$$
$$\mathrm{II} := \left|\left\{x \in \mathbb{R}^n \setminus (G \cup E^*) : \left|T\left(b^{(N)} - \frac{\lambda}{B}\mathbb{1}_E\right)(x)\right| > \frac{\lambda}{4}\right\}\right|, \quad \text{and}$$
$$\mathrm{III} := \left|\left\{|T(\mathbb{1}_E)| > \frac{B}{4}\right\}\right|.$$

Using Lemma 2.1.2 and Chebyshev's inequality,

$$\begin{aligned}
\mathrm{I} &\leq |G| + |E^*| \\
&\leq |G| + \left(1 + \frac{1}{n}\right)^n |E| \\
&\leq \frac{B}{\lambda}\|f\|_{L^1(\mathbb{R}^n)} + e\frac{B}{\lambda}\sum_{i=1}^{N}\|b_i\|_{L^1(\mathbb{R}^n)} \\
&\lesssim \frac{B}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

For the second term, we will use the facts that for $x \in \mathbb{R}^n \setminus (G \cup E^*)$ and $y \in Q_i \cup E_i$,

$$|x - y| \geq \frac{|y - c_i|}{n} \tag{6.2.1}$$

and

$$|x - c_i| \geq \frac{n+1}{n}|y - c_i|. \tag{6.2.2}$$

Use Chebyshev's inequality, and the cancellation $\int_{\mathbb{R}^n} b_i(y) - \frac{\lambda}{B}\mathbb{1}_{E_i}(y)\, dy = 0$ to estimate

$$\begin{aligned}
\mathrm{II} &\lesssim \frac{1}{\lambda}\int\limits_{\mathbb{R}^n\setminus(G\cup E^*)} \left|T\left(b^{(N)} - \frac{\lambda}{B}\mathbb{1}_E\right)(x)\right| dx \\
&\leq \frac{1}{\lambda}\sum_{i=1}^{N}\int\limits_{\mathbb{R}^n\setminus(G\cup E^*)} \left|T\left(b_i - \frac{\lambda}{B}\mathbb{1}_{E_i}\right)(x)\right| dx \\
&\leq \frac{1}{\lambda}\sum_{i=1}^{N}\int\limits_{Q_i\cup E_i} (A_i + B_i)\left|b_i(y) - \frac{\lambda}{B}\mathbb{1}_{E_i}(y)\right| dy,
\end{aligned}$$

where

$$A_i := \int\limits_{\{|x-y|\leq n|y-c_i|\}\setminus(G\cup E^*)} |K(x-y)|\, dx$$

and

$$B_i := \int_{\{|x-y|>n|y-c_i|\}\setminus(G\cup E^*)} |K(x-y)-K(x-c_i)|\,dx.$$

We claim that

$$A_i + B_i \lesssim \log(n) \int_{S^{n-1}} |\Omega(x)|\,d\sigma(x).$$

Indeed, conditions (6.2.1) and (6.2.2) allow for the same argument as in [18, pages 549–552] to prove our claim; we include the details for completeness.

We first estimate A_i. Without loss of generality, assume that $c_i = 0$. Use (6.2.1) and polar coordinates to estimate

$$\begin{aligned}
A_i &= \int_{\{|x-y|\le n|y|\}\setminus(G\cup E^*)} |K(x-y)|\,dx \\
&\le \int_{\{|y|/n\le|x-y|\le n|y|\}} \frac{|\Omega(x-y)|}{|x-y|^n}\,dx \\
&= \int_{\{|y|/n\le|x|\le n|y|\}} \frac{|\Omega(x)|}{|x|^n}\,dx \\
&= \int_{|y|/n}^{n|y|} \int_{\partial B(0,r)} \frac{|\Omega(x)|}{r^n}\,d\sigma(x)dr \\
&= \int_{|y|/n}^{n|y|} \frac{1}{r}\,dr \int_{S^{n-1}} |\Omega(x)|\,d\sigma(x) \\
&\lesssim \log(n) \int_{S^{n-1}} |\Omega(x)|\,d\sigma(x).
\end{aligned}$$

We next estimate B_i. Again assume that $c_i = 0$. First,

$$\begin{aligned}
B_i &= \int\limits_{\{|x-y|>n|y|\}\setminus(G\cup E^*)} |K(x-y) - K(x)|\, dx \\
&= \int\limits_{\{|x-y|>n|y|\}\setminus(G\cup E^*)} \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| dx \\
&= \int\limits_{\{|x-y|>n|y|\}\setminus(G\cup E^*)} \left| \frac{\Omega(x-y) - \Omega(x)}{|x-y|^n} + \Omega(x)\left(\frac{1}{|x-y|^n} - \frac{1}{|x|^n} \right) \right| dx \\
&\le B_i' + B_i'',
\end{aligned}$$

where

$$B_i' := \int\limits_{\{|x-y|>n|y|\}\setminus(G\cup E^*)} \frac{|\Omega(x-y) - \Omega(x)|}{|x-y|^n}\, dx$$

and

$$B_i'' := \int\limits_{\{|x-y|>n|y|\}\setminus(G\cup E^*)} |\Omega(x)| \left| \frac{1}{|x-y|^n} - \frac{1}{|x|^n} \right| dx.$$

We control B_i' as follows

$$\begin{aligned}
B_i' &\le \int\limits_{|x-y|>n|y|} \frac{|\Omega(x-y) - \Omega(x)|}{|x-y|^n}\, dx \\
&= \int\limits_{|x|>n|y|} \frac{|\Omega(x+y) - \Omega(x)|}{|x|^n}\, dx \\
&= \int\limits_{n|y|}^{\infty} \int\limits_{B(0,r)} \frac{|\Omega(x+y) - \Omega(x)|}{r^n}\, d\sigma(x) dr \\
&= \int\limits_{n|y|}^{\infty} \int\limits_{S^{n-1}} \frac{|\Omega(x+y/r) - \Omega(x)|}{r}\, d\sigma(x) dr \\
&= \int\limits_{0}^{1/n} \int\limits_{S^{n-1}} \frac{|\Omega(x-\delta\xi) - \Omega(x)|}{\delta}\, d\sigma(x) d\delta
\end{aligned}$$

$$\lesssim \int_0^{1/n} \frac{1}{\delta}\left(n\delta \int_{S^{n-1}} |\Omega(x)|\, d\sigma(x)\right) d\delta$$

$$= \int_{S^{n-1}} |\Omega(x)|\, d\sigma(x).$$

It remains to control B_i''. First

$$B_i'' = \int_{\{|x-y|>n|y|\}\setminus(G\cup E^*)} |\Omega(x)| \left| \frac{|x-y|^n - |x|^n}{|x|^n |x-y|^n} \right| dx = C_1 + C_2,$$

where

$$C_1 := \int_{(\{|x-y|>n|y|\}\cap\{|x-y|\geq|x|\})\setminus(G\cup E^*)} |\Omega(x)| \frac{|x-y|^n - |x|^n}{|x|^n |x-y|^n}\, dx$$

and

$$C_2 := \int_{(\{|x-y|>n|y|\}\cap\{|x-y|\leq|x|\})\setminus(G\cup E^*)} |\Omega(x)| \frac{|x|^n - |x-y|^n}{|x|^n |x-y|^n}\, dx.$$

The change of variables $\xi = x - y$ shows that $C_2 = C_1$, so it is enough to control C_1.

The condition $|x-y| \geq |x|$ and the binomial theorem imply

$$\begin{aligned}
|x-y|^n - |x|^n &\leq (|x| + |y|)^n - |x|^n \\
&= \sum_{k=0}^{n} \binom{n}{k} |y|^k |x|^{n-k} - |x|^n \\
&= |y| \sum_{k=1}^{n} \binom{n}{k} |y|^{k-1} |x|^{n-k} \\
&= |y| \sum_{k-1=0}^{n-1} \frac{n}{k} \binom{n-1}{k-1} |y|^{k-1} |x|^{(n-1)-(k-1)} \\
&\leq n|y|(|x| + |y|)^{n-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
C_1 &\le n|y| \int\limits_{|x-y|>n|y|} |\Omega(x)| \frac{(1+|y|/|x|)^{n-1}}{|x-y|^n|x|}\,dx \\
&\le n|y|\left(1+\frac{1}{n-1}\right)^{n-1} \int\limits_{|x-y|>n|y|} \frac{|\Omega(x)|}{|x-y|^n|x|}\,dx \\
&\le n|y|\left(1+\frac{1}{n-1}\right)^{n-1}\left(1+\frac{1}{n}\right)^{n} \int\limits_{|x-y|>n|y|} \frac{|\Omega(x)|}{|x|^{n+1}}\,dx \\
&\lesssim n|y| \int\limits_{|x|>(n-1)|y|} \frac{|\Omega(x)|}{|x|^{n+1}}\,dx \\
&\le n|y|\frac{1}{(n-1)|y|} \int\limits_{S^{n-1}} |\Omega(x)|\,d\sigma(x) \\
&\lesssim \int\limits_{S^{n-1}} |\Omega(x)|\,d\sigma(x).
\end{aligned}$$

Thus $B_i'' \lesssim \int_{S^{n-1}} |\Omega(x)|\,d\sigma(x)$, and we have proved the claim

$$A_i + B_i \lesssim \log(n) \int\limits_{S^{n-1}} |\Omega(x)|\,d\sigma(x).$$

Using the previous estimates, we control the second term by

$$\begin{aligned}
\mathrm{II} &\lesssim \left(\log(n) \int\limits_{S^{n-1}} |\Omega(x)|\,d\sigma(x)\right)\frac{1}{\lambda}\sum_{i=1}^{N}\left\|b_i - \frac{\lambda}{B}\mathbb{1}_{E_i}\right\|_{L^1(\mathbb{R}^n)} \\
&\lesssim \left(\log(n) \int\limits_{S^{n-1}} |\Omega(x)|\,d\sigma(x)\right)\frac{1}{\lambda}\sum_{i=1}^{N}\|b_i\|_{L^1(\mathbb{R}^n)} \\
&\le \left(\log(n) \int\limits_{S^{n-1}} |\Omega(x)|\,d\sigma(x)\right)\frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

Use Chebyshev's inequality and the boundedness of T on $L^2(\mathbb{R}^n)$ to estimate

$$\begin{aligned}
\mathrm{III} &\lesssim \frac{1}{B^2}\int\limits_{\mathbb{R}^n} |T(\mathbb{1}_E)(x)|^2\,dx \\
&\leq |E| \\
&= \frac{B}{\lambda}\sum_{i=1}^{N} \|b_i\|_{L^1(\mathbb{R}^n)} \\
&\leq \frac{B}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

Putting all of the estimates together gives

$$|\{|Tf| > \lambda\}| \lesssim \left(\log(n)\int\limits_{S^{n-1}} |\Omega(x)|\,d\sigma(x) + B\right)\frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.$$

□

Remark 6.2.1. We remark that inequalities (6.2.1) and (6.2.2) are analagous to (∗) and (∗∗) from Janakiraman's proof in [18]. Our inequalities are better with respect to n because we work with Euclidean balls, whereas in [18], one must consider the geometry of semi-cubes.

Appendices

Appendix A

Calderón-Zygmund Decomposition Method

We describe the classical method for proving the weak-type $(1,1)$ estimate for Calderón-Zygmund operators, see [4, 9, 10, 28]. This method readily extends to handle the theory on spaces of homogeneous type where the underlying measure μ satisfies the doubling property

$$\mu(B(x,2r)) \leq C_\mu \mu(B(x,r))$$

for some $C_\mu > 0$, all $r > 0$, and all x in the space, see [6]. We present the proof in the context of $\mathbb{R}^n$ equipped with a Radon measure, μ, satisfying the doubling property. The Calderón-Zygmund operators in this setting are given by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, d\mu(y)$$

for $x \notin \operatorname{supp} f$. We use the notation $A \lesssim B$ if there exists $C > 0$, possibly depending on n, μ, or T, such that $A \leq CB$

The following lemma is called the Calderón-Zygmund decomposition.

Lemma A.0.1. If $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$ (or $\lambda > \frac{\|f\|_{L^1(\mu)}}{\|\mu\|}$ if μ is a finite measure), then we can write

$$f = g + b = g + \sum_{j=1}^{\infty} b_j,$$

where

(1) $\|g\|_{L^\infty(\mu)} \lesssim \lambda$ and $\|g\|_{L^1(\mu)} \leq \|f\|_{L^1(\mu)}$,

(2) the b_j are supported on pairwise disjoint cubes Q_j satisfying

$$\sum_{j=1}^{\infty} \mu(Q_j) \leq \frac{1}{\lambda} \|f\|_{L^1(\mu)},$$

(3) $\|b_j\|_{L^1(\mu)} \lesssim \lambda \mu(Q_j)$,

(4) $\int_{Q_j} b_j(x)\, d\mu(x) = 0$, and

(5) $\|b\|_{L^1(\mu)} \lesssim \|f\|_{L^1(\mu)}$.

Proof. We only consider the case when $\mu(\mathbb{R}^n_j) = \infty$ for each $j = 1, 2, \ldots, 2^n$, where the $\mathbb{R}^n_j$ denote the 2^n n-dimensional quadrants in $\mathbb{R}^n$. Let $\mathcal{D}$ be the collection of dyadic cubes in $\mathbb{R}^n$ and let $M_{\mathcal{D}}$ denote the dyadic maximal function given by

$$M_{\mathcal{D}} f(x) = \sup_{\substack{Q \in \mathcal{D} \\ Q \ni x}} \frac{1}{\mu(Q)} \int_Q |f(y)|\, d\mu(y).$$

Write

$$\Omega := \{M_{\mathcal{D}} f > \lambda\} = \bigcup_{j=1}^{\infty} Q_j,$$

where the Q_j are maximal dyadic cubes in the sense that

$$\frac{1}{\mu(Q)} \int_Q |f(y)|\, d\mu(y) \leq \lambda < \frac{1}{\mu(Q_j)} \int_{Q_j} |f(y)|\, d\mu(y)$$

whenever $Q \supsetneq Q_j$. Set

$$g := f \mathbb{1}_{\mathbb{R}^n \setminus \Omega} + \sum_{j=1}^{\infty} \left(\frac{1}{\mu(Q_j)} \int_{Q_j} f(y)\, d\mu(y) \right) \mathbb{1}_{Q_j}$$

and

$$b := \sum_{j=1}^{\infty} b_j, \quad \text{where} \quad b_j := f\mathbb{1}_{Q_j} - \left(\frac{1}{\mu(Q_j)} \int\limits_{Q_j} f(y)\, d\mu(y)\right)\mathbb{1}_{Q_j}.$$

Clearly,

$$f = g + b = g + \sum_{j=1}^{\infty} b_j.$$

To prove (1), for almost every $x \notin \Omega$, we have by the Lebesgue differentiation theorem that

$$|g(x)| = |f(x)| \leq M_{\mathcal{D}} f(x) \leq \lambda.$$

For $x \in Q_j$, we use the doubling property of μ and the maximality of Q_j in Ω to obtain

$$|g(x)| \leq \frac{1}{\mu(Q_j)} \int\limits_{Q_j} |f(y)|\, d\mu(y) \leq \frac{\mu(\widehat{Q}_j)}{\mu(Q_j)} \frac{1}{\mu(\widehat{Q}_j)} \int\limits_{\widehat{Q}_j} |f(y)|\, d\mu(y) \lesssim \frac{1}{\mu(\widehat{Q}_j)} \int\limits_{\widehat{Q}_j} |f(y)|\, d\mu(y) \leq \lambda,$$

where for $Q \in \mathcal{D}$, $\widehat{Q}$ denotes the dyadic parent of Q. Therefore $\|g\|_{L^\infty(\mu)} \lesssim \lambda$. Property (1) follows from the above $L^\infty(\mu)$ control and from Fubini's theorem since

$$\|g\|_{L^1(\mu)} \leq \int\limits_{\mathbb{R}^n \setminus \Omega} |f(x)|\, d\mu(x) + \sum_{j=1}^{\infty} \int\limits_{Q_j} \left(\frac{1}{\mu(Q_j)} \int\limits_{Q_j} |f(y)|\, d\mu(y)\right) d\mu(x)$$

$$= \int\limits_{\mathbb{R}^n \setminus \Omega} |f(x)|\, d\mu(x) + \sum_{j=1}^{\infty} \frac{\mu(Q_j)}{\mu(Q_j)} \int\limits_{Q_j} |f(y)|\, d\mu(y) = \|f\|_{L^1(\mu)}.$$

For property (2), notice that $\operatorname{supp} b_j \subseteq Q_j$ by definition of b_j and that the cubes Q_j are pairwise disjoint by maximality. With this and the stopping condition $\lambda < \frac{1}{\mu(Q_j)} \int_{Q_j} |f(y)|\, d\mu(y)$ for each j, we have

$$\sum_{j=1}^{\infty} \mu(Q_j) < \sum_{j=1}^{\infty} \frac{1}{\lambda} \int\limits_{Q_j} |f(y)|\, d\mu(y) \leq \frac{1}{\lambda} \|f\|_{L^1(\mu)}.$$

Property (3) follows Fubini's theorem, the condition $\frac{1}{\mu(\widehat{Q}_j)} \int_{\widehat{Q}_j} |f(y)|\, d\mu(y) \leq \lambda$, and the doubling property of μ

$$\|b_j\|_{L^1(\mu)} \leq \int\limits_{Q_j} |f(x)|\, d\mu(x) + \int\limits_{Q_j} \left(\frac{1}{\mu(Q_j)} \int\limits_{Q_j} |f(y)|\, d\mu(y) \right) d\mu(x) \lesssim \int\limits_{Q_j} |f(x)|\, d\mu(x)$$

$$\leq \int\limits_{\widehat{Q}_j} |f(x)|\, d\mu(x) \leq \lambda\mu(\widehat{Q}_j) \lesssim \lambda\mu(Q_j).$$

Property (4) holds because Fubini's theorem implies that

$$\int\limits_{\mathbb{R}^n} b_j(x)\, d\mu(x) = \int\limits_{Q_j} f(x)\, d\mu(x) - \int\limits_{Q_j} \left(\frac{1}{\mu(Q_j)} \int\limits_{Q_j} f(y)\, d\mu(y) \right) d\mu(x)$$

$$= \int\limits_{Q_j} f(x)\, d\mu(x) - \int\limits_{Q_j} f(x)\, d\mu(x) = 0.$$

Finally, property (5) holds since property (3) and property (2) give

$$\|b\|_{L^1(\mu)} \leq \sum_{j=1}^{\infty} \|b_j\|_{L^1(\mu)} \lesssim \lambda \sum_{j=1}^{\infty} \mu(Q_j) \leq \|f\|_{L^1(\mu)}.$$

□

Theorem A.0.1. If T is a Calderón-Zygmund operator, then

$$\|Tf\|_{L^{1,\infty}(\mu)} := \sup_{\lambda > 0} \lambda\mu(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \lesssim \|f\|_{L^1(\mu)}$$

for all $f \in L^1(\mu)$.

Proof. Fix $f \in L^1(\mu)$ and $\lambda > 0$. We will show that

$$\mu(\{|Tf| > \lambda\}) \lesssim \frac{1}{\lambda} \|f\|_{L^1(\mu)}.$$

Apply Lemma A.0.1, to write

$$f = g + b = g + \sum_{j=1}^{\infty} b_j,$$

where properties (1) – (5) of the lemma hold. Now,

$$\mu(\{|Tf| > \lambda\}) \leq \mu\left(\left\{|Tg| > \frac{\lambda}{2}\right\}\right) + \mu\left(\left\{|Tb| > \frac{\lambda}{2}\right\}\right).$$

Using Chebyshev's inequality, the bound of T on $L^2(\mu)$, property (1), and trivial estimates, we have that

$$\begin{aligned}\mu\left(\left\{|Tg| > \frac{\lambda}{2}\right\}\right) &\lesssim \frac{1}{\lambda^2}\|Tg\|_{L^2(\mu)}^2 \\ &\lesssim \frac{1}{\lambda^2}\|g\|_{L^2(\mu)}^2 \\ &\lesssim \frac{1}{\lambda}\|g\|_{L^1(\mu)} \\ &\leq \frac{1}{\lambda}\|f\|_{L^1(\mu)}.\end{aligned}$$

We next control the second term. Let $Q_j^* := 2\sqrt{n}Q_j$ be the cube with the same center as Q_j and having side length $2\sqrt{n}$ times the side length of Q_j, and set $\Omega^* := \bigcup_{j=1}^{\infty} Q_j^*$. Then

$$\mu\left(\left\{|Tb| > \frac{\lambda}{2}\right\}\right) \leq \mu(\Omega^*) + \mu\left(\left\{x \in \mathbb{R}^n \setminus \Omega^* : |Tb(x)| > \frac{\lambda}{2}\right\}\right).$$

Notice that since $\mu(Q_j^*) \lesssim \mu(Q_j)$ and by property (2), we have

$$\mu(\Omega^*) \leq \sum_{j=1}^{\infty} \mu(Q_j^*) \lesssim \sum_{j=1}^{\infty} \mu(Q_j) \leq \frac{1}{\lambda}\|f\|_{L^1(\mu)}.$$

It remains to control the last term. Use Chebyshev's inequality, Lemma 2.1.1, property

(3), and property (2) to estimate

$$\begin{aligned}
\mu\left(\left\{\mathbb{R}^n \setminus \Omega^* : |Tb| > \frac{\lambda}{2}\right\}\right) &\lesssim \frac{1}{\lambda} \int\limits_{\mathbb{R}^n \setminus \Omega^*} |Tb(x)|\, d\mu(x) \\
&\leq \frac{1}{\lambda} \sum_{j=1}^{\infty} \int\limits_{\mathbb{R}^n \setminus \Omega^*} |Tb_j(x)| d\mu(x) \\
&\lesssim \frac{1}{\lambda} \sum_{j=1}^{\infty} \|b_j\|_{L^1(\mu)} \\
&\lesssim \sum_{j=1}^{\infty} \mu(Q_j) \\
&\leq \frac{1}{\lambda} \|f\|_{L^1(\mu)}.
\end{aligned}$$

Putting all of the estimates together, we get

$$\mu(\{|Tf| > \lambda\}) \lesssim \frac{1}{\lambda} \|f\|_{L^1(\mu)}.$$

□

Appendix B

Nazarov-Treil-Volberg Method

We describe the method for proving the weak-type $(1, 1)$ estimate for Calderón-Zygmund operators given by Nazarov, Treil, and Volberg in [23]. This proof was originally formulated to handle the theory on nonhomogeneous spaces where the underlying measure μ satisfies the polynomial growth condition

$$\mu(B(x, r)) \leq C_\mu r^m$$

for some $C_\mu, m > 0$, all $r > 0$, and all x in the space. See [32] for another approach to Calderón-Zygmund theory on nonhomogeneous spaces. We present the proof in the context of $\mathbb{R}^n$ equipped with a Radon measure, μ, satisfying the above polynomial growth condition. We use the notation $A \lesssim B$ if there exists $C > 0$, possibly depending on n, m, μ, or T, such that $A \leq CB$.

The Calderón-Zygmund operators in this setting are given by

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, d\mu(y)$$

for $x \notin \operatorname{supp} f$. The size and smoothness assumptions of K must be adapted to the measure: there exist $C_K > 0$ and $0 < \delta \leq 1$ satisfying

$$\text{(size)} \qquad |K(x, y)| \leq \frac{C_K}{|x - y|^m} \tag{B.0.1}$$

whenever $x \neq y$ and

$$\text{(smoothness)} \qquad |K(x,y) - K(x',y')| \leq C_K \frac{|x-x'|^\delta + |y-y'|^\delta}{|x-y|^{m+\delta}} \tag{B.0.2}$$

whenever $|x-x'| + |y-y'| \leq \frac{1}{2}|x-y|$.

Let $\mathcal{M}(\mathbb{R}^n)$ denote the space of $\mathbb{C}$-valued Borel measures on $\mathbb{R}^n$ and let $\|\nu\|$ denote the total variation of a measure $\nu \in \mathcal{M}(\mathbb{R}^n)$. For $\nu \in \mathcal{M}(\mathbb{R}^n)$, $T\nu$ is given by

$$T\nu(x) = \int_{\mathbb{R}^n} K(x,y)\, d\nu(y)$$

for $x \notin \operatorname{supp} \nu$. The maximal truncation operator, $T^\#$, is given by

$$T^\# f(x) = \sup_{r>0} |T_r f(x)| \quad \text{and} \quad T^\# \nu(x) := \sup_{r>0} |T_r \nu(x)|,$$

where

$$T_r f(x) := \int_{\mathbb{R}^n \setminus B(x,r)} K(x,y) f(y)\, d\mu(y) \quad \text{and} \quad T_r \nu(x) := \int_{\mathbb{R}^n \setminus B(x,r)} K(x,y)\, d\nu(y).$$

The centered Hardy-Littlewood maximal operator, M, is given for locally integrable functions f and locally finite measures $\nu \in \mathcal{M}(\mathbb{R}^n)$ by

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu(y) \quad \text{and} \quad M\nu(x) = \sup_{r>0} \frac{\nu(B(x,r))}{\mu(B(x,r))}.$$

We also use an auxiliary maximal function, $\widetilde{M}$, defined by

$$\widetilde{M} f(x) = \sup_{r>0} \frac{1}{\mu(B(x,3r))} \int_{B(x,r)} |f(y)|\, d\mu(y) \quad \text{and} \quad \widetilde{M}\nu(x) = \sup_{r>0} \frac{\nu(B(x,r))}{\mu(B(x,3r))}.$$

Note that $\widetilde{M}$ is bounded on $L^p(\mu)$ for $1 < p < \infty$ and weak-type $(1,1)$ with respect to μ since $\widetilde{M}f \leq Mf$ and the $L^p(\mu)$ boundedness and weak-type $(1,1)$ property hold for M.

Lemma B.0.1. If $\nu \in \mathcal{M}(\mathbb{R}^n)$ is supported on $B(x,r)$ and $\nu(B(x,r)) = 0$ for some $x \in \mathbb{R}^n$ and $r > 0$, then

$$\int\limits_{\mathbb{R}^n \setminus B(x,2r)} |T\nu(y)| \, d\mu(y) \lesssim \|\nu\|.$$

Lemma B.0.1 is an immediate extension of Lemma 2.1.1.

Lemma B.0.2. If $F \subseteq \mathbb{R}^n$ is a Borel set with finite μ-measure, then

$$T^{\#}\mathbb{1}_F(x) \lesssim \widetilde{M}T\mathbb{1}_F(x) + 1$$

for almost every $x \in \operatorname{supp} \mu$.

Proof. Fix $r > 0$. Let k be the smallest positive integer such that

$$\mu(B(x, 3^k r)) \leq 4^m \mu(B(x, 3^{k-1} r)).$$

Note that such a k exists because, if not, then $\mu(B(x,3^j)) > 4^m \mu(B(x, 3^{j-1}))$ for each $j \geq 0$. Then using the polynomial growth condition, we have

$$\mu(B(x,r)) \leq 4^{-mj} \mu(B(x, 3^j r)) \lesssim \left(\frac{3}{4}\right)^{mj} r^m.$$

Letting $j \to \infty$ gives that $\mu(B(x,r)) = 0$, which is not possible.

Now,

$$|T_r \mathbb{1}_F(x)| \leq |T_r \mathbb{1}_F(x) - T_{3^k r} \mathbb{1}_F(x)| + |T_{3^k r} \mathbb{1}_F(x)|.$$

To estimate the first term, break up the domain of integration into shells, use the size condition of K, and apply trivial estimates to bound

$$\begin{aligned}
|T_r \mathbb{1}_F(x) - T_{3^k r} \mathbb{1}_F(x)| &= \left| \int\limits_{\mathbb{R}^n} K(x,y) \Big(\mathbb{1}_{F \setminus B(x,r)}(y) - \mathbb{1}_{F \setminus B(x,3^k r)}(y) \Big) d\mu(y) \right| \\
&\leq \int\limits_{B(x,3^k r) \setminus B(x,r)} |K(x,y)| d\mu(y) \\
&= \sum_{j=1}^{k} \int\limits_{B(x,3^j r) \setminus B(x,3^{j-1} r)} |K(x,y)| d\mu(y) \\
&\lesssim \sum_{j=1}^{k} \int\limits_{B(x,3^j r) \setminus B(x,3^{j-1} r)} \frac{1}{|x-y|^m} d\mu(y) \\
&\leq \sum_{j=1}^{k} \frac{\mu(B(x,3^j r))}{(3^{j-1} r)^m}.
\end{aligned}$$

Notice that for each $j \in \{1, \ldots, k\}$, we have

$$\mu(B(x,3^j r)) \leq 4^{m(j+1-k)} \mu(B(x,3^{k-1} r)).$$

Use the above property and the polynomial growth condition of μ to continue estimating $|T_r \mathbb{1}_F(x) - T_{3^k r} \mathbb{1}_F(x)|$ by a constant times

$$\sum_{j=1}^{k} \frac{\mu(B(x,3^j r))}{(3^{j-1} r)^m} \leq \frac{\mu(B(x,3^{k-1} r))}{(3^{k-1} r)^m} \sum_{j=1}^{k} \left(\frac{4}{3} \right)^{m(j-k)} \lesssim 1.$$

It remains to control the second term, $|T_{3^k r} \mathbb{1}_F(x)|$. Let

$$V_k(x) = \frac{1}{\mu(B(x,3^{k-1} r))} \int\limits_{B(x,3^{k-1} r)} T \mathbb{1}_F(y) \, d\mu(y)$$

and further estimate

$$|T_{3^k r}\mathbb{1}_F(x)| \le |T_{3^k r}\mathbb{1}_F(x) - V_k(x)| + |V_k(x)|.$$

Use the adjoint T^* to write

$$V_k(x) = \int\limits_{F\setminus B(x,3^k r)} T^*\left(\frac{\mathbb{1}_{B(x,3^{k-1}r)}}{\mu(B(x,3^{k-1}r))}\right)(y)\,d\mu(y) + \frac{1}{\mu(B(x,3^{k-1}r))}\int\limits_{B(x,3^{k-1}r)} T\mathbb{1}_{F\cap B(x,3^k r)}(y)\,d\mu(y).$$

Write

$$K(x,y) = \int\limits_{\mathbb{R}^n} K(z,y)\,d\delta_x(z) = T^*\delta_x(y),$$

and apply Lemma B.0.1 to the operator T^*, the Cauchy-Schwarz inequality, and the boundedness of T on $L^2(\mu)$ to observe

$$\begin{aligned}
|T_{3^k r}\mathbb{1}_F(x) - V_k(x)| &\le \left|\int\limits_{F\setminus B(x,3^k r)} \left(K(x,y) - T^*\left(\frac{\mathbb{1}_{B(x,3^{k-1}r)}}{\mu(B(x,3^{k-1}r))}\right)(y)\right) d\mu(y)\right| \\
&\quad + \frac{1}{\mu(B(x,3^{k-1}r))}\left|\int\limits_{B(x,3^{k-1}r)} T\mathbb{1}_{F\cap B(x,3^k r)}(y)\,d\mu(y)\right| \\
&\le \int\limits_{\mathbb{R}^n\setminus B(x,2\cdot 3^{k-1}r)} \left|T^*\left(\delta_x - \frac{\mathbb{1}_{B(x,3^{k-1}r)}}{\mu(B(x,3^{k-1}r))}d\mu\right)(y)\right| d\mu(y) \\
&\quad + \frac{1}{\mu(B(x,3^{k-1}r))}\int\limits_{B(x,3^{k-1}r)} |T\mathbb{1}_{F\cap B(x,3^k r)}(y)|d\mu(y) \\
&\lesssim \left\|\delta_x - \frac{\mathbb{1}_{B(x,3^{k-1}r)}}{\mu(B(x,3^{k-1}r))}d\mu\right\| \\
&\quad + \frac{1}{\mu(B(x,3^{k-1}r))}\|\mathbb{1}_{B(x,3^{k-1}r)}\|_{L^2(\mu)}\|T\mathbb{1}_{F\cap B(x,3^k r)}\|_{L^2(\mu)} \\
&\lesssim 1 + \sqrt{\frac{\mu(B(x,3^k r))}{\mu(B(x,3^{k-1}r))}} \\
&\lesssim 1.
\end{aligned}$$

Since

$$|V_k(x)| \le \frac{\mu(B(x,3^k r))}{\mu(B(x,3^{k-1}r))}\widetilde{M}T\mathbb{1}_F(x) \lesssim \widetilde{M}T\mathbb{1}_F(x),$$

and using the previous estimates, we obtain

$$|T_r\mathbb{1}_F(x)| \le |T_r\mathbb{1}_F(x) - T_{3^k r}\mathbb{1}_F(x)| + |T_{3^k r}\mathbb{1}_F(x) - V_k(x)| + |V_k(x)| \lesssim \widetilde{M}T\mathbb{1}_F(x) + 1.$$

Since $r > 0$ was arbitrary, we conclude

$$T^{\#}\mathbb{1}_F(x) \lesssim \widetilde{M}T\mathbb{1}_F(x) + 1.$$

□

The proof of the weak-type $(1,1)$ property relies on the following weak-type estimate on point-mass measures.

Theorem B.0.1. If T is a Calderón-Zygmund operator, then

$$\|T\nu\|_{L^{1,\infty}(\mu)} \lesssim \|\nu\|$$

for all $\nu \in \mathcal{M}(\mathbb{R}^n)$ of the form $\nu = \sum_{j=1}^N a_j\delta_{x_j}$.

Proof. Fix $\nu \in \mathcal{M}(\mathbb{R}^n)$ of the form $\nu = \sum_{j=1}^N a_j\delta_{x_j}$ and $\lambda > 0$. We will show

$$\mu(\{|T\nu| > \lambda\}) \lesssim \frac{1}{\lambda}\|\nu\|.$$

Without loss of generality, assume that $a_j > 0$ for all j.

If $\mu(\mathbb{R}^n) \le \frac{1}{\lambda}\|\nu\|$, the claim is trivial, so we assume $\mu(\mathbb{R}^n) > \frac{1}{\lambda}\|\nu\|$. Select a Borel set E_1 such that

$$B(x_1,r_1) \subseteq E_1 \subseteq \overline{B(x_1,r_1)},$$

where $r_1 > 0$ is chosen so that $\mu(E_1) = \frac{a_1}{\lambda}$. In general, for $j \in \{2, \ldots, N\}$, select a Borel set E_j such that

$$B(x_j, r_j) \setminus \bigcup_{k=1}^{j-1} E_k \subseteq E_j \subseteq \overline{B(x_j, r_j)} \setminus \bigcup_{k=1}^{j-1} E_k,$$

where $r_j > 0$ is chosen so that $\mu(E_j) = \frac{a_j}{\lambda}$. Set

$$\sigma := \sum_{j=1}^{N} \mathbb{1}_{\mathbb{R}^n \setminus \overline{B(x_j, 2r_j)}} T \mathbb{1}_{E_j}.$$

and set

$$E := \bigcup_{j=1}^{N} E_j.$$

Now,

$$\mu(\{|T\nu| > \lambda\}) \leq \mu\left(\left\{|T\nu - \lambda\sigma| > \frac{\lambda}{2}\right\}\right) + \mu\left(\left\{|\sigma| > \frac{1}{2}\right\}\right) \leq \mathrm{I} + \mathrm{II} + \mathrm{III},$$

where

$$\begin{aligned}
&\mathrm{I} := \mu(E), \\
&\mathrm{II} := \mu\left(\left\{x \in \mathbb{R}^n \setminus E : |T\nu(x) - \lambda\sigma(x)| > \frac{\lambda}{2}\right\}\right), \quad \text{and} \\
&\mathrm{III} := \mu\left(\left\{|\sigma| > \frac{1}{2}\right\}\right).
\end{aligned}$$

Since the E_j are pairwise disjoint we have

$$\mathrm{I} = \sum_{j=1}^{N} \mu(E_j) = \frac{1}{\lambda} \sum_{j=1}^{N} a_j = \frac{1}{\lambda} \|\nu\|.$$

To control II, first notice

$$T\nu - \lambda\sigma = \sum_{j=1}^{N} \mathbb{1}_{\mathbb{R}^n \setminus B(x_j, 2r_j)} T(a_j \delta_{x_j} - \lambda \mathbb{1}_{E_j}) + \sum_{j=1}^{N} a_j \mathbb{1}_{B(x_j,2r_j)} T\delta_{x_j}.$$

Use Chebyshev's inequality to estimate

$$\mathrm{II} \lesssim \frac{1}{\lambda} \int\limits_{\mathbb{R}^n \setminus E} |T\nu(x) - \lambda\sigma(x)| \, d\mu(x)$$

$$\leq \frac{1}{\lambda} \sum_{j=1}^{N} \int\limits_{\mathbb{R}^n \setminus B(x_j,2r_j)} |T(a_j \delta_{x_j} - \lambda \mathbb{1}_{E_j})(x)| \, d\mu(x) + \frac{1}{\lambda} \sum_{j=1}^{N} a_j \int\limits_{B(x_j,2r_j) \setminus B(x_j,r_j)} |T\delta_{x_j}(x)| \, d\mu(x).$$

Use Lemma B.0.1 to estimate

$$\frac{1}{\lambda} \sum_{j=1}^{N} \int\limits_{\mathbb{R}^n \setminus B(x_j,2r_j)} |T(a_j \delta_{x_j} - \lambda \mathbb{1}_{E_j})(x)| \, d\mu(x) \lesssim \frac{1}{\lambda} \sum_{j=1}^{N} \|a_j \delta_{x_j} - \lambda \mathbb{1}_{E_j}\| \lesssim \frac{1}{\lambda} \|\nu\|.$$

Using $|T\delta_{x_j}(x)| = |K(x, x_j)|$, the size estimate of K, and the polynomial growth condition, we have

$$\frac{1}{\lambda} \sum_{j=1}^{N} a_j \int\limits_{B(x_j,2r_j) \setminus B(x_j,r_j)} |T\delta_{x_j}(x)| \, d\mu(x) \lesssim \frac{1}{\lambda} \sum_{j=1}^{N} a_j \int\limits_{B(x_j,2r_j) \setminus B(x_j,r_j)} \frac{1}{|x - x_j|^m} \, d\mu(x)$$

$$\leq \frac{1}{\lambda} \sum_{j=1}^{N} a_j \frac{\mu(B(x_j, 2r_j))}{r_j^m} \lesssim \frac{1}{\lambda} \|\nu\|.$$

Therefore

$$\mathrm{II} \lesssim \frac{1}{\lambda} \|\nu\|.$$

We next bound the third term,

$$\text{III} = \mu\left(\left\{|\sigma| > \frac{1}{2}\right\}\right) \leq \frac{C}{\lambda}\|\nu\|,$$

for some constant $C > 0$ independent of λ and ν. Suppose to the contrary that $\mu(\{|\sigma| > \frac{1}{2}\}) > \frac{C}{\lambda}\|\nu\|$ for all such constants C. Then either $\mu(\{\sigma > \frac{1}{2}\}) > \frac{C}{2\lambda}\|\nu\|$ or $\mu(\{\sigma < -\frac{1}{2}\}) > \frac{C}{2\lambda}\|\nu\|$ for all C. Assume that $\mu(\{\sigma > \frac{1}{2}\}) > \frac{C}{2\lambda}\|\nu\|$ for all C, noting that the other case is handled similarly. Choose a Borel set $F \subseteq \{\sigma > \frac{1}{2}\}$ such that $\mu(F) = \frac{C}{2\lambda}\|\nu\|$. Then trivially

$$\int\limits_F \sigma \, d\mu > \frac{1}{2}\mu(F) = \frac{C}{4\lambda}\|\nu\|.$$

We will show that

$$\int\limits_F \sigma(x)\, d\mu(x) \leq \frac{C}{4\lambda}\|\nu\|$$

for a particular C independent of λ and ν, yielding a contradiction.

First, use the adjoint T^* to write

$$\int\limits_F \sigma(x)\, d\mu(x) = \int\limits_{\mathbb{R}^n} \left(\sum_{j=1}^N \mathbb{1}_{F\setminus\overline{B(x_j,2r_j)}} T\mathbb{1}_{E_j}(x)\right) d\mu(x) = \sum_{j=1}^N \int\limits_{E_j} T^*\mathbb{1}_{F\setminus\overline{B(x_j,2r_j)}}(x)\, d\mu(x).$$

Next, use the size estimate of K, Lemma B.0.2 applied to T^*, and the polynomial growth condition to show

$$\begin{aligned}
|T^*\mathbb{1}_{F\setminus\overline{B(x_j,2r_j)}}(x)| &\leq |T^*\mathbb{1}_{F\setminus\overline{B(x_j,2r_j)}} - T^*\mathbb{1}_{F\setminus\overline{B(x,r_j)}}(x)| + |T^*\mathbb{1}_{F\setminus\overline{B(x,r_j)}}(x)| \\
&\leq \int\limits_{\overline{B(x_i,2r_j)}\setminus\overline{B(x,r_j)}} |K(y,x)| d\mu(y) + (T^*)^{\#}\mathbb{1}_F(x) \\
&\lesssim \int\limits_{\overline{B(x_j,2r_j)}\setminus\overline{B(x,r_j)}} \frac{1}{|x-x_j|^m} + \widetilde{M}(T^*\mathbb{1}_F)(x) + 1
\end{aligned}$$

$$\leq \frac{\mu(B(x,2r_j))}{r_j^m} + \widetilde{M}(T^*\mathbb{1}_F)(x) + 1$$
$$\lesssim \widetilde{M}(T^*\mathbb{1}_F)(x) + 1.$$

It follows using the above, the Cauchy-Schwarz inequality, the boundedness of $\widetilde{M}$ and T^* on $L^2(\mu)$, and construction of E and F that

$$\begin{aligned}
\int_F \sigma(x)\,d\mu(x) &= \sum_{j=1}^N \int_{E_j} T^*\mathbb{1}_{F\setminus\overline{B(x_j,2r_j)}}(x)\,d\mu(x) \\
&\lesssim \sum_{j=1}^N \int_{E_j} \left(\widetilde{M}(T^*\mathbb{1}_F)(x) + 1\right) d\mu \\
&= \mu(E) + \int_E \widetilde{M}(T^*\mathbb{1}_F)(x)\,d\mu(x) \\
&\leq \mu(E) + \mu(E)^{\frac{1}{2}} \|\widetilde{M}(T^*\mathbb{1}_F)\|_{L^2(\mu)} \\
&\lesssim \mu(E) + \mu(E)^{\frac{1}{2}}\mu(F)^{\frac{1}{2}} \\
&\lesssim \left(\frac{C}{2}\right)^{\frac{1}{2}} \frac{1}{\lambda}\|\nu\| \\
&= C'\left(\frac{C}{2}\right)^{\frac{1}{2}} \frac{1}{\lambda}\|\nu\|
\end{aligned}$$

for some $C' > 0$ depending on n, μ, and T. Fixing C such that $C'\left(\frac{C}{2}\right)^{\frac{1}{2}} = \frac{C}{4}$, we have $\frac{C}{4\lambda}\|\nu\| < \int_F \sigma\,d\mu \leq \frac{C}{4\lambda}\|\nu\|$, a contradiction. Therefore

$$\mathrm{III} \lesssim \frac{1}{\lambda}\|\nu\|.$$

Collecting the previous estimates, we conclude

$$\mu(\{|T\nu| > \lambda\}) \leq \mathrm{I} + \mathrm{II} + \mathrm{III} \lesssim \frac{1}{\lambda}\|\nu\|.$$

□

Theorem B.0.2. If T is a Calderón-Zygmund operator, then

$$\|Tf\|_{L^{1,\infty}(\mu)} := \sup_{\lambda>0} \lambda\mu(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \lesssim \|f\|_{L^1(\mu)}$$

for all $f \in L^1(\mu)$.

Proof. Let $\lambda > 0$ be given. We wish to show

$$\mu(\{|Tf| > \lambda\}) \lesssim \frac{1}{\lambda}\|f\|_{L^1(\mu)}.$$

By density, we may assume that f is a continuous and nonnegative function with bounded support. Set

$$\Omega := \{f > \lambda\}.$$

Apply a Whitney decomposition to write

$$\Omega = \bigcup_{j=1}^{\infty} Q_j,$$

a disjoint union of dyadic cubes where

$$2\text{diam}(Q_j) \leq \text{dist}(Q_j, \mathbb{R}^n \setminus \Omega) \leq 8\text{diam}(Q_j).$$

Set

$$g := f\mathbb{1}_{\mathbb{R}^n\setminus\Omega}, \quad b := f\mathbb{1}_{\Omega}, \quad \text{and} \quad b_j := f\mathbb{1}_{Q_j}.$$

Then $f = g + b = g + \sum_{j=1}^{\infty} b_j$, where

(1) $\|g\|_{L^\infty(\mu)} \leq \lambda$ and $\|g\|_{L^1(\mu)} \leq \|f\|_{L^1(\mu)}$,

(2) the b_j are supported on pairwise disjoint sets Q_j satisfying

$$\sum_{j=1}^{\infty} \mu(Q_j) \leq \frac{1}{\lambda}\|f\|_{L^1(\mu)},$$

and

(3) $\|b\|_{L^1(\mu)} \leq \|f\|_{L^1(\mu)}$.

Then

$$\mu(\{|Tf| > \lambda\}) \leq \mu\left(\left\{|Tg| > \frac{\lambda}{2}\right\}\right) + \mu\left(\left\{|Tb| > \frac{\lambda}{2}\right\}\right).$$

To control the first term, use Chebyshev's inequality, the boundedness fo T on $L^2(\mu)$, and property (1) to control the first term

$$\begin{aligned}\mu\left(\left\{|Tg| > \frac{\lambda}{2}\right\}\right) &\lesssim \frac{1}{\lambda^2}\|Tg\|_{L^2(\mu)}^2 \\ &\lesssim \frac{1}{\lambda^2}\|g\|_{L^2(\mu)}^2 \\ &\leq \frac{1}{\lambda}\|g\|_{L^1(\mu)} \\ &\leq \frac{1}{\lambda}\|f\|_{L^1(\mu)}.\end{aligned}$$

For positive integers N, set $b^{(N)} := \sum_{j=1}^{N} b_j$. To control the second term, it suffices to handle $\mu(\{|Tb^{(N)}| > \frac{\lambda}{2}\})$ uniformly in N. Let x_j be any point in Q_j, let $a_j := \int_{\mathbb{R}^n} b_j(x)\,d\mu(x)$, and let $\nu_N := \sum_{j=1}^{N} a_i \delta_{x_i}$. Then

$$\mu\left(\left\{|Tb^{(N)}| > \frac{\lambda}{2}\right\}\right) \leq \mu\left(\left\{|T(b^{(N)} - \nu_N)| > \frac{\lambda}{4}\right\}\right) + \mu\left(\left\{|T\nu_N| > \frac{\lambda}{4}\right\}\right)$$

$$\leq \mu(\Omega) + \mu\left(\left\{x \in \mathbb{R}^n \setminus \Omega : |T(b^{(N)} - \nu_N)(x)| > \frac{\lambda}{4}\right\}\right) + \mu\left(\left\{|T\nu_N| > \frac{\lambda}{4}\right\}\right).$$

Using property (2), we have

$$\mu(\Omega) = \sum_{j=1}^{\infty} \mu(Q_j) \leq \frac{1}{\lambda}\|f\|_{L^1(\mu)}.$$

Apply Chebyshev's inequality, Lemma B.0.1, and property (3) to estimate the second term.

$$\begin{aligned}
\mu\left(\left\{x \in \mathbb{R}^n \setminus \Omega : |T(b^{(N)} - \nu_N)(x)| > \frac{\lambda}{4}\right\}\right) &\lesssim \frac{1}{\lambda} \int\limits_{\mathbb{R}^n\setminus\Omega} |T(b^{(N)}(x) - \nu_N)(x)|\, d\mu(x) \\
&\leq \frac{1}{\lambda}\sum_{j=1}^{N} \int\limits_{\mathbb{R}^n\setminus\Omega} |T(b_j - a_j\delta_{x_j})(x)|\, d\mu(x) \\
&\lesssim \frac{1}{\lambda}\sum_{j=1}^{N} \|b_j d\mu - a_j\delta_{x_j}\| \\
&\lesssim \frac{1}{\lambda}\|b\|_{L^1(\mu)} \\
&\leq \frac{1}{\lambda}\|f\|_{L^1(\mu)}.
\end{aligned}$$

We control the last term with Theorem B.0.1

$$\mu\left(\left\{|T\nu_N| > \frac{\lambda}{4}\right\}\right) \lesssim \frac{1}{\lambda}\|\nu_N\| \leq \frac{1}{\lambda}\|f\|_{L^1(\mu)}.$$

Collecting the previous estimates, we conclude

$$\mu(\{|Tf| > \lambda\}) \leq \mu\left(\left\{|Tg| > \frac{\lambda}{2}\right\}\right) + \mu\left(\left\{|Tb| > \frac{\lambda}{2}\right\}\right) \lesssim \frac{1}{\lambda}\|f\|_{L^1(\mu)}.$$

□

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