Spring 5-15-2020

Operator Noncommutative Function Theory and Partial Matrix and Operator Convexity

Mark E. Mancuso
Washington University in St. Louis

WASHINGTON UNIVERSITY IN ST. LOUIS
Department of Mathematics and Statistics

Dissertation Examination Committee:
John E. McCarthy, Chair
Mark Alford
Gregory Knese
Xiang Tang
Brett Wick

Operator Noncommutative Function Theory and Partial Matrix and Operator Convexity

by

Mark E. Mancuso

A dissertation presented to
The Graduate School
of Washington University in
partial fulfillment of the
requirements for the degree
of Doctor of Philosophy

May 2020
St. Louis, Missouri

Table of Contents

Acknowledgments

First, I would like to express gratitude to my advisor, John E. McCarthy. He introduced me to the fascinating area of free analysis and provided motivation and support during graduate school. John was also kind enough to support me during my Spring 2019 semester visit at the University of Florida.

I am also thankful for the help and flexibility of my committee members, Mark Alford, Greg Knese, Xiang Tang, and Brett Wick. In addition, I would like to thank Greg, Xiang, Brett, and Victor Wickerhauser for teaching stimulating courses.

Next, I am thankful for my collaborators, Mike Jury, Igor Kelp, Scott McCullough, and James Eldred Pascoe. I especially would like to thank Scott for introducing me to interesting problems in matrix convexity and for being like a second advisor to me.

I appreciate the help of Lisa Kuehne, who gave me several opportunities to teach statistics as the instructor of record. I also want to thank my fellow graduate students for support and good memories.

Next, I of course would like to express immense appreciation for the support of my parents, Michael and Linda Mancuso. Bill and Susie Hicks, my uncle and aunt, have given me similar support and encouragement. I thank the four of you so much.

I wish to thank my friends from Connecticut for the years of encouragement. I especially want to thank Alex Konefal and Brian Paul for their interest in my studies. Finally, I want to thank my loving partner, Kimberly King, for incredible moral and emotional support.

Mark E. Mancuso

Washington University in St. Louis

May 2020

Dedicated to my family and friends.

ABSTRACT OF THE DISSERTATION

Operator Noncommutative Function Theory

and Partial Matrix and Operator Convexity

by

Mark E. Mancuso

Doctor of Philosophy in Mathematics

Washington University in St. Louis, 2020

Professor John E. McCarthy, Chair

This dissertation begins by introducing the foundations of operator noncommutative function theory. That is, the study of noncommutative functions defined on operator domains $\Omega \subset B(\mathcal{H})^d$, where $\mathcal{H}$ is a separable infinite dimensional complex Hilbert space. Inverse and implicit function theorems are established in this setting along with a characterization of global invertibility of an operator noncommutative function's derivative.

Motivated by principles from noncommutative dilation theory, this dissertation then describes a construction that yields a type of (operator) noncommutative Heine-Borel theorem for the strong operator topology. This compactness-like principle is first applied to the study of inversion of operator noncommutative functions that are continuous, in a precise sense, in the strong operator topology.

The dissertation then transitions to the study of noncommutative convexity. A general framework for noncommutative partial convexity, called Γ-convexity, is introduced. This general framework contains several well-studied examples as special cases. Hahan-Banach Effros-Winkler separation theorems are obtained in the matricial and operatorial Γ-convex setting. In the operator setting, we use as a key ingredient the aforementioned noncommutative Heine-Borel theorem. While interesting in their own right, the results obtained in the operator case provide insight into the matrix case as well.

Chapter 1

Introduction

This dissertation is composed of two major pillars; each of these considers problems in the broad area of (free) noncommutative function theory. The first, detailed in Chapters 2-4, introduces the foundations of operator noncommutative function theory and establishes inverse and implicit function theorems in this setting. This material is found in the author's work [14], published in the Journal of Operator Theory. The second pillar, found in Chapters 5 and 6, introduces a general framework for noncommutative partial convexity, called Γ-convexity. Several results along the lines of [11] and [6] in the context of Γ-convexity are established. The material in this second pillar is based on the joint work [12], published in the Journal of Geometric Analysis.

The two pillars of this dissertation are linked by a construction detailed in Chapter 4 and the resulting Theorem 4.1.1 that may be seen as a noncommutative version of the Heien-Borel theorem. This construction and Theorem 4.1.1 are key ingredients to our approach to both operator noncommutative inversion and Γ-convexity.

The theory of noncommutative functions finds its origin in the 1973 work of J.L. Taylor [21], who studied the functional calculus of noncommuting operators. Roughly speaking, noncommutative functions are to polynomials in noncommuting variables as holomorphic functions from complex analysis are to polynomials in commuting variables. Polynomials in d noncommuting indeterminates can naturally be evaluated on d-tuples of square matrices of any size. The resulting function is graded (tuples of $n \times n$ matrices are mapped to $n \times n$ matrices) and preserves direct sums and similarities. Along with polynomials, noncommutative rational functions and power series, the convergence of which has been studied for example in [13], [18], [19], serve as prototypical examples of a more general class of functions called *noncommutative functions.*

Noncommutative functions are classically defined on domains sitting inside of a graded space of d-tuples of square matrices that are closed under direct sums. These matrices are often over the field of complex numbers, but much of the theory works for matrices over a general module over a commutative ring. See the book by D.S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov [13] for a comprehensive, foundational treatment in this generality. In the case of matrices over $\mathbb{C}$, for example, a matricial **(free) noncommutative domain** is a graded set $S = (S(n))_{n=1}^{\infty} \subset \mathcal{M}(\mathbb{C})^d := (\mathcal{M}_n(\mathbb{C})^d)_{n=1}^{\infty}$, where $\mathcal{M}_n(\mathbb{C})$ is the space of $n \times n$ complex matrices and S is closed under direct sums: $x \in S(n)$ and $y \in S(m)$ implies $x \oplus y \in S(n+m)$. Sometimes one requires S to be closed under unitary conjugation. We say S is **open** if each $S(n)$ is open in the Euclidean topology. A **(free) noncommutative function** on S is a graded function $f : S \to \mathcal{M}(\mathbb{C})^r$ which preserves direct sums and similarities. That is, if $x \in S(n)$, $y \in S(m)$, and $s \in \mathcal{M}_n(\mathbb{C})$ is invertible such that $s^{-1}xs \in S$, then $f(x \oplus y) = f(x) \oplus f(y)$ and $f(s^{-1}xs) = s^{-1}f(x)s$.

In Chapter 2 of this dissertation, we provide the foundations of *operator noncommutative function theory.* That is, we consider a notion of a noncommutative function defined on a domain $\Omega \subset B(\mathcal{H})^d$ for an infinite dimensional separable Hilbert space $\mathcal{H}$ over $\mathbb{C}$. This should be seen as a type of *completion* of the classical matricial noncommutative setting. In particular, these functions (NC functions) have the property that they preserve *countable* direct sums and are defined on domains (NC domains) with a "noncommutative exhausting sequence."

With this *purely operatorial* point of view of noncommutative function theory, we are no longer considering a space of infinitely many disjoint levels, but instead are working within the *complete* space $B(\mathcal{H})^d$. This should be seen as a type of completion of the classical matricial noncommutative setting. In this operatorial setting, noncommutative functions are still defined to be direct sum-preserving, but since the domain is no longer graded, we need to make identifications of $\mathcal{H}$ with countable direct sums of $\mathcal{H}$ via unitary equivalence.

The precise definitions and further discussion of operator noncommutative function theory will be given in Section 2.1. Many foundational properties and formulas from the matricial theory, such as those found in the work of Helton, Klep, and McCullough [10], have analogues in this setting. We give their formulations and proofs in Section 2.2, adhering to the formalisms of operator noncommutative function theory as outlined in Section 2.1. In the interest of clarity, when dealing with noncommutative functions defined on operator domains inside of $B(\mathcal{H})^d$, we will use the abbreviation NC; we will use the lower case nc for the matricial noncommutative setting. The main objects of study in operator NC function theory are operator NC domains and operator NC functions. Agler and McCarthy proposed a definition similar to ours for these objects in [3], however our definitions eliminate certain issues and allow for many more examples.

In his resolution of the free Jacobian conjecture, J.E. Pascoe, in [15], proved what is now known as the *free inverse function theorem* in the matricial setting.

Theorem 1.0.1. (Pascoe, 2013) *Let $D \subset \mathcal{M}(\mathbb{C})^d$ be an open nc domain and $f : D \to \mathcal{M}(\mathbb{C})^d$ be a locally bounded nc function. Then the following are equivalent:*

(i) *the derivative Df is injective at all points, i.e. for each n and $X \in D_n$, the linear map $Df(X) : \mathcal{M}_n(\mathbb{C})^d \to \mathcal{M}_n(\mathbb{C})^d$ is injective (non-singular);*

(ii) *f is injective on D;*

(iii) *f is invertible, $f(D)$ is an open nc domain, and $f^{-1} : f(D) \to D$ is an nc function.*

This theorem is striking for several reasons. It asserts that injectivity of an nc function is equivalent to injectivity of its derivative. Examples abound in the commutative case of this failing. Moreover, the result is *global* in nature, a feature in nc function theory that is not found in the commutative setting. This work led to Agler and McCarthy's *free implicit function theorem* in [4]. In that paper, the authors also prove a version of the free implict function theorem in the uniformly-open topology. The notation $Df(X)[H]$ denotes the derivative of the nc function f at the point X in the direction of H and Z_f denotes the zero set of f.

Theorem 1.0.2. (Agler and McCarthy, 2016) *Let $D \subset \mathcal{M}(\mathbb{C})^d$ be an open nc domain and let $f : D \to \mathcal{M}(\mathbb{C})^k$ be a locally bounded nc function, where $1 \leq k \leq d-1$. If for every n, $X \in D_n$, and $H = (H^{d-k+1}, \dots, H^d) \in \mathcal{M}_n(\mathbb{C})^k \setminus \{0\}$,*

$$Df(X)[0, \dots, 0, H^{d-k+1}, \dots, H^d] \neq 0,$$

then there is an open nc domain $V \subset \mathcal{M}(\mathbb{C})^{d-k}$ and a locally bounded nc function $g : V \to \mathcal{M}(\mathbb{C})^k$ such that $Z_f = \{(y, g(y)) : y \in V\}$.

Theorems 1.0.1 and 1.0.2 provided the initial motivation for the author's work on inversion problems in operatorial noncommutative function theory. Our first main results within the context of operator NC function theory are (global) operator NC inverse and implicit function theorems, Theorems 3.0.2 and 3.0.3. In contrast to the finite-dimensional setting, which relied on the invariance of domain theorem from topology to guarantee that the function f in Theorem 1.0.1 is an open mapping, no such tool exists in the infinite-dimensional case. Other noncommutative inverse and implicit function theorems were established in [1] in a quite general matricial setting by making use of what is known as the *uniformly-open* nc topology. As we are not working with graded domains, but instead with domains inside $B(\mathcal{H})^d$, this is another tool that is unavailable to us in the setting of operator NC function theory. In[1], the authors consider nc functions that are locally bounded in the uniformly-open topology which also have a completely bounded and invertible derivative with completely bounded inverse.

In further contrast to the work in [1] and other articles on noncommutative inversion, we give a sufficient condition guaranteeing the invertibility of the derivative map of an operator NC function at all points in a connected operator NC domain. Indeed, Theorem 3.0.1 states that for an NC function f on a connected NC domain in $B(\mathcal{H})^d$, if the derivative Df satisfies a noncommutative bounded below condition (see Definition 2.1.3) and we assume the existence of just *one* point a in the domain such that $Df(a)$ is invertible, then we may conclude the invertibility of $Df(x)$ for *every* x in the domain. This result provides the basis for our inverse and implicit function theorems, Theorems 3.0.2 and 3.0.3.

In Chapter 4, we discuss a construction that was motivated by principles from the noncommutative dilation theory introduced by A. Frazho [7], [8] and G. Popescu [16], [17]. This construction, called the *shift form construction*, underlies a result which may be seen as a (operator) noncommutative version of the Heine-Borel theorem, Theorem 4.1.1. We then consider operator NC functions that are continuous in the *strong operator topology.* It is reasonable to impose these extra assumptions on our NC functions since many examples of interest in applications (such as polynomials and suitable rational functions) are continuous in the strong operator topology on appropriately defined norm-bounded sets. It is proved as a consequence of the shift form construction and Lemma 4.2.2, in Theorem 4.2.1 and Corollary 4.2.1, that injective strongly continuous NC functions, on suitable domains, have everywhere *bounded below* derivative. Therefore, in the operator setting, and especially in the case of strong operator continuity, we are able to obtain global noncommutative inversion-type theorems with only minor hypotheses on the derivative.

We now exposit on some relevant background and history for the material found in the second pillar of this dissertation, noncommutative convexity. We restrict to sets consisting of *self-adjoint* tuples. Let $\mathbb{S}(\mathbb{C})^d = (\mathbb{S}_n(\mathbb{C})^d)_n$ denote the free set of self-adjoint d-tuples. A free set $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^d$ is called **matrix convex** if it is closed under isometric conjugation: for all m and n, if $X \in \mathcal{K}_n$ and $V : \mathbb{C}^m \to \mathbb{C}^n$ is an isometry, then $V^*XV \in \mathcal{K}_m$.

Each level of a matrix convex set is in particular convex in the ordinary sense. Classically, the Hahn-Banach separation theorem asserts that any point outside of a closed convex set can be separated from the set by a continuous linear functional. In particular, a closed convex set is the intersection of all hyperplanes that contain it. In [6], Effros and Winkler proved an analogous result for closed matrix convex sets containing zero. The analogues of linear functionals in the noncommutative setting are **monic linear pencils**: expressions of the form $L = I_\mu + \sum A_j x_j$ where the A_j are self-adjoint matrices of size μ. We say L has size μ. Such a pencil is evaluated at a tuple $X \in \mathbb{S}_n(\mathbb{C})^d$ via the Kronecker tensor product:

$$L(X) = I_\mu \otimes I_n + \sum A_j \otimes X_j.$$

A (matricial) **free spectrahedron** is a set of the form $\widehat{\mathcal{D}}_L = \{X \in \mathbb{S}(\mathbb{C})^d : L(X) \succeq 0\}$ for a monic linear pencil L.

Theorem 1.0.3. (Effros-Winkler, special case, 1997) *Let $\mathcal{K}$ be a closed matrix convex set containing zero. If $Y \notin \mathcal{K}$, then there is a monic linear pencil L such that $L(X) \succeq 0$ for every $X \in \mathcal{K}$, but $L(Y) \not\succeq 0$. If Y has size ℓ, then L may be chosen to have size ℓ. In particular, $\mathcal{K} = \cap \widehat{\mathcal{D}}_L$ where the intersection is taken over the collection of monic linear pencils L that are positive semi-definite on $\mathcal{K}$.*

With motivation from systems engineering and control theory, in [11], Helton and McCullough proved the remarkable fact that every matrix convex free semi-algebraic set (the positivity set of a free polynomial) has a linear matrix inequality (LMI) representation. That is, it can be represented as the positivity set of a monic linear pencil.

Chapter 5 provides the formulation of a general framework of noncommutative *partial* convexity, called Γ-convexity. In many applications of interest, including semi-definite programming and control theory, the situation is not truly convex; one must allow for certain partial notions of convexity. For example, one may consider the notion of convexity in each variable separately; this corresponds closely to the study of bilinear matrix inequalities (BMIs). We will see that a BMI corresponds precisely to the positivity set of what we call an xy-pencil. This particular example of Γ-convexity is explored further in Examples 5.2.2 and 5.2.3.

A primary goal in this setting is to extend classical separation results for matrix convex sets to the general framework of Γ-convexity. This is done in Chapters 5 and 6. We will begin by proving a first version of a Γ-convex generalization of the Effros-Winkler theorem in Theorem 5.2.1. Then, we turn to investigate further the hypotheses in this first version and work toward obtaining several improvements in various directions. This is done in Chapter 6 by appealing to the *operator* setting.

In the operator Γ-convex setting, we prove, in Theorem 6.1.2, that the Hahn-Banach Effros-Winkler theorem holds for strong operator topology closed, bounded, operator Γ-convex sets. It is here that the shift form construction and the noncommutative Heine-Borel principle, Theorem 4.1.1, play a key role. Importantly, we show that even in the operator setting, outlying points can be separated from the convex set by *matrix* pencils. This allows us to effectively use the theory of operator Γ-convexity, while interesting in its own right, as a means to gain insight into the matrix case.

Corollary 6.2.1 and Theorem 6.2.1 analyze the case when we assume the relevant (operator) convex sets contain zero in their *interior*. In particular, Theorem 6.2.1 provides a condition for when an operator Γ-convex set can be realized as the positivity set of a *single* monic operator Γ-pencil.

Chapter 2

Operator Noncommutative Function Theory

2.1 Definitions and Setting

In this section, we elaborate on the general setting of operator noncommutative function theory and provide definitions and examples of the theory's main objects of study: NC operator domains and NC operator functions. Operator noncommutative functions are to be defined on domains sitting inside of $B(\mathcal{H})^d$, where $\mathcal{H}$ is an infinite dimensional separable Hilbert space over $\mathbb{C}$ and $B(\mathcal{H})$ is the Banach space of bounded linear operators on $\mathcal{H}$ equipped with the operator norm. We equip $B(\mathcal{H})^d$ with the maximum norm

$$\|x\| := \max\{\|x^1\|, \dots, \|x^d\|\}.$$

This induces the product topology on $B(\mathcal{H})^d$ with respect to the norm topology on $B(\mathcal{H})$ and turns $B(\mathcal{H})^d$ into a complex Banach space.

The direct sum of l copies of the Hilbert space $\mathcal{H}$, for $l \in \mathbb{N} \cup \{\infty\}$, will be denoted $\mathcal{H}^{(l)}$. Direct sums of operators will often be written as a block diagonal matrix: if $x_1, x_2, \dots$ is a finite or countably infinite bounded sequence of operators in $B(\mathcal{H})$ of length $l \in \mathbb{N} \cup \{\infty\}$, we will write the direct sum operator $\bigoplus_{i=1}^{l} x_i$ as

$$\begin{bmatrix} x_1 & & \\ & x_2 & \\ & & \ddots \end{bmatrix} : \mathcal{H}^{(l)} \to \mathcal{H}^{(l)}.$$

Operations on $B(\mathcal{H})^d$ are defined component-wise: for $L \in B(\mathcal{H})$ and $x \in B(\mathcal{H})^d$, define

$$L(x^1, \ldots, x^d) := (Lx^1, \ldots, Lx^d) \quad \text{and} \quad (x^1, \ldots, x^d)L := (x^1 L, \ldots, x^d L).$$

Similarly, if $s : \mathcal{H} \to \mathcal{H}^{(l)}$ is an invertible linear map and $z \in B(\mathcal{H}^{(l)})^d$, we define

$$s^{-1}zs := (s^{-1}z^1 s, \ldots, s^{-1}z^d s).$$

Direct sums of operator tuples are also defined component-wise. If $x_1, x_2, \ldots$ is a finite or countably infinite bounded sequence of elements of $B(\mathcal{H})^d$ of length $l \in \mathbb{N} \cup \{\infty\}$, we define their direct sum to be the element of $B(\mathcal{H}^{(l)})^d$ given by

$$\begin{bmatrix} x_1 & & \\ & x_2 & \\ & & \ddots \end{bmatrix} := \left(\begin{bmatrix} x_1^1 & & \\ & x_2^1 & \\ & & \ddots \end{bmatrix}, \ldots, \begin{bmatrix} x_1^d & & \\ & x_2^d & \\ & & \ddots \end{bmatrix} \right).$$

Expressions such as

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

for $x, y, z, w \in B(\mathcal{H})^d$ are similarly defined.

We say a subset Ω of $B(\mathcal{H})^d$ is *closed under unitary conjugation* if whenever $x \in \Omega$ and $u \in B(\mathcal{H})$ is a unitary operator, then $u^*xu \in \Omega$. In Definition 2.1.1 below, the interior of a set is with respect to the norm topology on $B(\mathcal{H})^d$.

Definition 2.1.1. A set $\Omega \subset B(\mathcal{H})^d$ is called an **NC domain** if there exists a sequence $(\Omega_k)_{k=1}^{\infty}$ of subsets of Ω with the following properties:

(i) $\Omega_k \subset \operatorname{int} \Omega_{k+1}$ for all $k \geq 1$ and $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$;

(ii) each Ω_k is norm-bounded and closed under unitary conjugation;

(iii) each Ω_k is closed under countable direct sums in the sense that if (x_n) is a sequence in Ω_k of length $l \in \mathbb{N} \cup \{\infty\}$, then there exists a unitary $u : \mathcal{H} \to \mathcal{H}^{(l)}$ such that

$$u^{-1} \begin{bmatrix} x_1 & & \\ & x_2 & \\ & & \ddots \end{bmatrix} u \in \Omega_k. \tag{2.1.1}$$

NC domains are in particular open subsets in the norm topology of $B(\mathcal{H})^d$. Note that by closure under unitary conjugation of each exhaustion level Ω_k, given a finite or countably infinite sequence (x_n) in Ω_k of length l, as soon as (2.1.1) holds for some unitary $u : \mathcal{H} \to \mathcal{H}^{(l)}$, it will in fact hold for *all* unitaries $v : \mathcal{H} \to \mathcal{H}^{(l)}$ by considering $u^{-1}v$.

Let us make a few remarks about Definition 2.1.1. Our notion of operator NC domain using exhausting sequences is a way to reasonably think of the (open) domains as being closed under countably infinite direct sums while still providing a sufficiently large class of examples. Even for bounded domains, it will rarely be the case that one may take an arbitrary sequence in the domain and conclude that its direct sum (conjugated by a unitary) will remain in the domain. Indeed, this fails even for the open unit ball in $d = 1$: consider the sequence $(1 - 1/n)1_{\mathcal{H}}$. When we restrict to sequences contained in a fixed level of an exhaustion as in Definition 2.1.1, however, it is a much less stringent requirement for (2.1.1) to hold. Loosely speaking, considering countable direct sums (rather than just finite) allows one to interact with the completeness of $B(\mathcal{H})$ - one of the inherent advantages of the operatorial theory. This is particularly true in the case that Ω is norm-connected, as will be discussed in Chapters 3 and 4.

Example 2.1.1. A large supply of examples of operator NC domains can be given as follows. Let δ be an $I \times J$ matrix of polynomials in d noncommuting variables (i.e. a matrix whose entries are elements of the free associative algebra $\mathbb{C}\langle x^1, \dots, x^d \rangle$). Define

$$B_\delta := \{x \in B(\mathcal{H})^d : \|\delta(x)\| < 1\},$$

where the norm is taken in $B(\mathcal{H}^{(J)}, \mathcal{H}^{(I)})$. Important concrete examples take this form for particular choices of δ. For example, the noncommutative polydisk $\{x \in B(\mathcal{H})^d : \|x\| < 1\}$ in $B(\mathcal{H})^d$ may be realized as the B_δ for the block diagonal matrix

$$\delta(x^1, \dots, x^d) = \begin{bmatrix} x^1 & & \\ & \ddots & \\ & & x^d \end{bmatrix}.$$

The noncommutative operatorial ball

$$\{x \in B(\mathcal{H})^d : \|x^1(x^1)^* + \cdots + x^d(x^d)^*\|^{1/2} < 1\}$$

is the B_δ corresponding to the row $\delta(x) = [x^1 \cdots x^d]$.

To see that any B_δ is in fact an NC domain according to Definition 2.1.1, one may take the exhausting sequence to be (Ω_k), where

$$\Omega_k = \{x \in B(\mathcal{H})^d : \|\delta(x)\| \leq 1 - 1/k\} \cap \{x \in B(\mathcal{H})^d : \|x\| \leq k\}. \tag{2.1.2}$$

It is immediately checked that (Ω_k) has all of the required properties.

Example 2.1.2. Another example of an NC domain is the set Ω of invertible elements of $B(\mathcal{H})$. In this example, $d = 1$. One may use the exhausting sequence

$$\Omega_k = \{x \in \Omega : \|x\| \leq k, \|x^{-1}\| \leq k\}. \tag{2.1.3}$$

This example illustrates how some care needs to be taken when requiring each level of an exhaustion be closed under countably infinite direct sums. Indeed, the exhausting sequence $W_k = \{x \in \Omega : \|x\| \leq k\}$ satisfies all properties listed in Definition 2.1.1 except condition (iii) for the case $l = \infty$.

We now want to consider functions which act appropriately on NC domains. Namely, we make the following definition of an operator NC function.

Definition 2.1.2. Let $\Omega \subset B(\mathcal{H})^d$ be an NC domain. We say a function $f : \Omega \to B(\mathcal{H})^r$ is an **NC function** if it preserves direct sums in the sense that whenever $x, y \in \Omega$ and $s : \mathcal{H} \to \mathcal{H}^{(2)}$ is a bounded invertible linear map with

$$s^{-1} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} s \in \Omega,$$

then

$$f\left(s^{-1} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} s\right) = s^{-1} \begin{bmatrix} f(x) & 0 \\ 0 & f(y) \end{bmatrix} s.$$

By definition, NC functions are only defined on NC domains. Thus, when we assume that a function $f : \Omega \subset B(\mathcal{H})^d \to B(\mathcal{H})^r$ is an NC function, we are implicitly assuming that Ω is an NC domain. If we write $f : \Omega \to B(\mathcal{H})^r$ as $f = (f^1, \ldots, f^r)$, where each $f^j : \Omega \to B(\mathcal{H})$, then it follows from the definitions that f is an NC function if and only if each f^j is an NC function.

As discussed in Chapter 1, any polynomial in d noncommuting variables is an NC function when defined on any NC domain $\Omega \subset B(\mathcal{H})^d$. Furthermore, rational functions and convergent noncommutative power series, on appropriately defined NC domains, provide us with several classes of examples of NC functions. We provide a simple, explicit example below.

Example 2.1.3. Consider the rational function

$$f(x, y) := (1 - xy)^{-1} = \sum_{n=0}^{\infty} (xy)^n \tag{2.1.4}$$

defined on the operatorial noncommutative unit bidisk $\Omega = \{(x, y) : \|x\| < 1, \|y\| < 1\}$ in $B(\mathcal{H})^2$. We verify here through a direct calculation that this function is in fact NC. Let (x^1, x^2) and (y^1, y^2) be points in Ω and suppose $s : \mathcal{H} \to \mathcal{H}^{(2)}$ is such that

$$s^{-1} \begin{bmatrix} (x^1, x^2) & 0 \\ 0 & (y^1, y^2) \end{bmatrix} s \in \Omega.$$

Then,

$$\begin{aligned} f\left(s^{-1} \begin{bmatrix} (x^1, x^2) & 0 \\ 0 & (y^1, y^2) \end{bmatrix} s \right) &= \sum_{n=0}^{\infty} \left(s^{-1} \begin{bmatrix} x^1 & 0 \\ 0 & y^1 \end{bmatrix} \begin{bmatrix} x^2 & 0 \\ 0 & y^2 \end{bmatrix} s \right)^n \\ &= s^{-1} \sum_{n=0}^{\infty} \begin{bmatrix} (x^1 x^2)^n & 0 \\ 0 & (y^1 y^2)^n \end{bmatrix} s \\ &= s^{-1} \begin{bmatrix} f(x^1, x^2) & 0 \\ 0 & f(y^1, y^2) \end{bmatrix} s, \end{aligned}$$

as claimed.

One may also define this function on the larger NC domain $\Omega = \{(x, y) : \|xy\| < 1\}$. That this is an NC domain follows from noting it is an example of a B_δ. The same calculation as above shows it to be an NC function on this NC domain.

We conclude this section with terminology that will be used in the statements of the operator NC inverse and implicit function theorems, Theorems 3.0.2 and 3.0.3. Recall that an operator $T \in B(X)$, where X is a Banach space, is said to be bounded below if there is a constant $C > 0$ such that $\|Tx\| \geq C\|x\|$ for all $x \in X$.

Definition 2.1.3. Let $\Omega \subset B(\mathcal{H})^d$ be an NC domain. A map $\Psi : \Omega \to B(B(\mathcal{H})^r)$ is said to have the **NC bounded below property** if whenever $(x_n)_{n=1}^{\infty}$ is a bounded sequence in Ω such that $\Psi(x_n)$ is bounded below for every n and $u : \mathcal{H} \to \mathcal{H}^{(\infty)}$ is a unitary such that

$$z := u^{-1} \begin{bmatrix} x_1 & & \\ & x_2 & \\ & & \ddots \end{bmatrix} u \in \Omega,$$

then $\Psi(z)$ is bounded below.

We note that in the notation of Definition 2.1.3, the supposed bound below for $\Psi(x_n)$ is allowed to depend on n. When Ψ arises naturally from an operator NC function, for example when Ψ is the derivative map of an NC function, the argument in the proof of Theorem 3.0.1 shows that Ψ being bounded below when evaluated at the direct sum of such a sequence (x_n) implies a *uniform* bound below for the sequence $(\Psi(x_n))$. As a result, we show that this property characterizes global invertibility of the derivative of an operator NC function on a norm-connected NC domain. That is, the NC bounded below property, when imposed on the derivative of an operator NC function, may be thought of as an operatorial analogue of the assumption of injectivity of the derivative in the matricial nc theory.

2.2 Foundational Properties

The purpose of this section is to collect basic properties and formulas for NC functions defined on operator domains. Our first lemma is an operatorial version of a fundamental formula for noncommutative functions. In [10], Helton, Klep, and McCullough proved a similar formula for matricial nc functions. In this and other related formulas to follow, the presence of unitaries or some invertible linear map s in the statements is necessary as we need a way of identifying $\mathcal{H}$ with some $\mathcal{H}^{(l)}$. Several results in this section have analogues in the classical matricial nc theory. However, we present precise statements and complete proofs here, adhering to the formalisms introduced in Section 2.1.

Lemma 2.2.1. *Let $f : \Omega \subset B(\mathcal{H})^d \to B(\mathcal{H})^r$ be an NC function and let $L \in B(\mathcal{H})$. If $x, y \in \Omega$ and $s : \mathcal{H} \to \mathcal{H}^{(2)}$ is any invertible linear map such that*

$$s^{-1}\begin{bmatrix} x & Ly - xL \\ 0 & y \end{bmatrix} s \in \Omega,$$

then

$$f\left(s^{-1}\begin{bmatrix} x & Ly - xL \\ 0 & y \end{bmatrix} s\right) = s^{-1}\begin{bmatrix} f(x) & Lf(y) - f(x)L \\ 0 & f(y) \end{bmatrix} s.$$

Proof. Define $\sigma : \mathcal{H} \to \mathcal{H}^{(2)}$ to be the invertible map $\sigma := \begin{bmatrix} 1 & -L \\ 0 & 1 \end{bmatrix} s$. Then a computation shows

$$\sigma^{-1}\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\sigma = s^{-1}\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\begin{bmatrix} 1 & -L \\ 0 & 1 \end{bmatrix} s = s^{-1}\begin{bmatrix} x & Ly - xL \\ 0 & y \end{bmatrix} s \in \Omega.$$

Since f is NC, we have

$$f\left(s^{-1}\begin{bmatrix}x & Ly-xL\\ 0 & y\end{bmatrix}s\right)=f\left(\sigma^{-1}\begin{bmatrix}x & 0\\ 0 & y\end{bmatrix}\sigma\right)$$
$$=\sigma^{-1}\begin{bmatrix}f(x) & 0\\ 0 & f(y)\end{bmatrix}\sigma$$
$$=s^{-1}\begin{bmatrix}f(x) & Lf(y)-f(x)L\\ 0 & f(y)\end{bmatrix}s,$$

which completes the proof. □

Corollary 2.2.1. *NC functions preserve intertwinings. That is, if* $f:\Omega\subset B(\mathcal{H})^d\to B(\mathcal{H})^r$ *is an NC function and* $L\in B(\mathcal{H})$ *and* $x,y\in\Omega$ *are such that* $Ly=xL$*, then* $Lf(y)=f(x)L$.

It follows from Corollary 2.2.1 that whenever $f:\Omega\subset B(\mathcal{H})^d\to B(\mathcal{H})^r$ is NC and (Ω_k) is an exhausting sequence of Ω as in Definition 2.1.1, then each $f(\Omega_k)$ is norm-bounded. In particular, NC functions are automatically locally bounded - a property that must be *assumed* throughout the matricial theory. To see this, take a sequence (x_n) in a fixed Ω_k. There is a unitary $u:\mathcal{H}\to\mathcal{H}^{(\infty)}$ such that $x:=u^{-1}[\bigoplus x_n]u\in\Omega_k$. Define $\Gamma_n:\mathcal{H}^{(\infty)}\to\mathcal{H}$ to be projection onto the nth component and let $L_n:=\Gamma_n u$. By definition of L_n, we have $L_nx=x_nL_n$ for all n. Since f preserves intertwinings, we then have

$$L_nf(x)=f(x_n)L_n. \tag{2.2.1}$$

Since $x\in\Omega$, $f(x)$ is an element of $B(\mathcal{H})^r$ and thus has finite norm. Hence, relation (2.2.1) implies $(f(x_n))$ is uniformly bounded.

Similar reasoning lets us conclude the basic fact that operator NC functions preserve *countable* direct sums: if (x_n) is a sequence in Ω of length $l\in\mathbb{N}\cup\{\infty\}$ and $s:\mathcal{H}\to\mathcal{H}^{(l)}$ is linear and invertible with

$$s^{-1}\begin{bmatrix} x_1 & & \\ & x_2 & \\ & & \ddots \end{bmatrix} s \in \Omega,$$

then $(f(x_n))$ is uniformly bounded and we have

$$f\left(s^{-1}\begin{bmatrix} x_1 & & \\ & x_2 & \\ & & \ddots \end{bmatrix} s\right) = s^{-1}\begin{bmatrix} f(x_1) & & \\ & f(x_2) & \\ & & \ddots \end{bmatrix} s.$$

Recall that if X and Y are Banach spaces and $U \subset X$ is open, then a function $g : U \to Y$ is said to be *Gâteaux differentiable* if for all $x \in U$ and all $h \in X$, the limit

$$Dg(x)[h] := \lim_{t\to 0} \frac{g(x+th) - g(x)}{t}$$

exists. It is a well-known general fact (see [20]) that over complex scalars, a norm-continuous and Gâteaux differentiable function is automatically *Fréchet* differentiable, and the two derivatives must then coincide. In particular, $Dg(x) : X \to Y$ is then a bounded linear map for each $x \in U$.

Lemma 2.2.2. *An NC function is norm-continuous and Gâteaux differentiable and therefore is Fréchet differentiable.*

Proof. We begin by showing that if $f : \Omega \subset B(\mathcal{H})^d \to B(\mathcal{H})^r$ is NC, then f is norm-continuous. Fix $x \in \Omega$ and $\varepsilon > 0$, and let (Ω_k) be an exhausting sequence for Ω as in Definition 2.1.1. There is $k \geq 1$ such that $x \in \Omega_k$, so there is $u : \mathcal{H} \to \mathcal{H}^{(2)}$ unitary such that

$$z := u^{-1}\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} u \in \Omega_k.$$

Then there is some $r > 0$ such that the norm balls centered at x and z with radius r are contained in Ω_{k+1}. By the discussion immediately following Corollary 2.2.1, there is $M > 0$ such that $\|f\| < M$ on Ω_{k+1}.

Set $\delta := \min\{\frac{r\varepsilon}{2M}, r/2\}$ and let $\|y - x\| < \delta$. Then

$$\begin{aligned}\left\| u^{-1}\begin{bmatrix} x & \frac{M}{\varepsilon}(y-x) \\ 0 & y \end{bmatrix} u - z \right\| &= \left\| \begin{bmatrix} 0 & \frac{M}{\varepsilon}(y-x) \\ 0 & y-x \end{bmatrix} \right\| \\ &\leq M/\varepsilon\|y-x\| + \|y-x\| \\ &< r,\end{aligned}$$

so we have, by Lemma 2.2.1,

$$\left\| \begin{bmatrix} f(x) & \frac{M}{\varepsilon}(f(y)-f(x)) \\ 0 & f(y) \end{bmatrix} \right\| = \left\| f\left(u^{-1}\begin{bmatrix} x & \frac{M}{\varepsilon}(y-x) \\ 0 & y \end{bmatrix} u \right) \right\| < M.$$

It then follows that $\|f(y) - f(x)\| < \varepsilon$.

Next, we show f is Gâteaux differentiable. Fix $x \in \Omega$ and $h \in B(\mathcal{H})^d$. There is $k \geq 1$, $u : \mathcal{H} \to \mathcal{H}^{(2)}$ unitary, and $\varepsilon > 0$ so that $x \in \Omega_k$ and

$$u^{-1}\begin{bmatrix} x & \varepsilon h \\ 0 & x \end{bmatrix} u \in \Omega_k.$$

Then for all $t \neq 0$ with small enough modulus,

$$\Omega_{k+1} \ni u^{-1}\begin{bmatrix} x+th & \varepsilon h \\ 0 & x \end{bmatrix} u = u^{-1}\begin{bmatrix} x+th & \frac{\varepsilon}{t}(x+th-x) \\ 0 & x \end{bmatrix} u.$$

By Lemma 2.2.1 again,

$$f\left(u^{-1}\begin{bmatrix} x+th & \varepsilon h \\ 0 & x \end{bmatrix} u\right) = u^{-1}\begin{bmatrix} f(x+th) & \frac{\varepsilon}{t}(f(x+th)-f(x)) \\ 0 & f(x) \end{bmatrix} u. \tag{2.2.2}$$

By norm-continuity of f, as $t \to 0$, the limit on the left-hand side of (2.2.2) exists, and therefore so does that of the 1-2 entry of the matrix on the right-hand side of (2.2.2), thus proving f is Gâteaux differentiable. Since f is also norm-continuous, the discussion immediately preceding this proof implies f is Fréchet differentiable. □

Moreover, the second part of the above proof also provides the following derivative formula for operator NC functions. It is reminiscent of a formula obtained in [10], and will be an important tool for us moving forward.

Proposition 2.2.1. *Let $f : \Omega \subset B(\mathcal{H})^d \to B(\mathcal{H})^r$ be an NC function. Suppose $x \in \Omega$, $h \in B(\mathcal{H})^d$, and $s : \mathcal{H} \to \mathcal{H}^{(2)}$ is any invertible linear map such that*

$$s^{-1}\begin{bmatrix} x & h \\ 0 & x \end{bmatrix} s \in \Omega.$$

Then,

$$f\left(s^{-1}\begin{bmatrix} x & h \\ 0 & x \end{bmatrix} s\right) = s^{-1}\begin{bmatrix} f(x) & Df(x)[h] \\ 0 & f(x) \end{bmatrix} s. \tag{2.2.3}$$

One scenario where we can apply Proposition 2.2.1 is as follows. Suppose $x \in \Omega$ and $s = u$ is a given unitary. Then by closure under direct sums and unitary conjugation, $u^{-1}(x \oplus x)u$ is an element of Ω (for the *given* unitary u) and the conclusion of Proposition 2.2.1 holds for all $h \in B(\mathcal{H})^d$ with sufficiently small norm.

The next theorem is an operatorial analogue of J.E. Pascoe's inverse function theorem [15] for locally bounded matricial nc functions. It is a first step towards a bonafide inverse function theorem for operator NC functions. In the operator setting, it is unclear (without an invariance of domain theorem in infinite dimensions) if injective NC functions are open mappings. In Section 4.2, we revisit this idea in the context of the strong operator topology. We remark that, in contrast to the finite dimensional case, it is possible for a linear map $B(\mathcal{H})^d \to B(\mathcal{H})^r$ to be injective even if $d > r$. Therefore, this theorem has content even when $d \neq r$, and so we state it in this generality.

Theorem 2.2.1. *An NC function $f : \Omega \subset B(\mathcal{H})^d \to B(\mathcal{H})^r$ is injective if and only if $Df(x) : B(\mathcal{H})^d \to B(\mathcal{H})^r$ is injective for every $x \in \Omega$.*

Proof. Suppose first f is injective and let $x \in \Omega$. Assume that $Df(x)[h] = 0$. There is u unitary and $\varepsilon > 0$ small enough so that $u^{-1} \begin{bmatrix} x & \varepsilon h \\ 0 & x \end{bmatrix} u \in \Omega$. Formula (2.2.3) then yields

$$f\left(u^{-1} \begin{bmatrix} x & \varepsilon h \\ 0 & x \end{bmatrix} u\right) = u^{-1} \begin{bmatrix} f(x) & Df(x)[\varepsilon h] \\ 0 & f(x) \end{bmatrix} u$$
$$= f\left(u^{-1} \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} u\right).$$

By injectivity of f, it must hold that

$$u^{-1} \begin{bmatrix} x & \varepsilon h \\ 0 & x \end{bmatrix} u = u^{-1} \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} u,$$

which implies $h = 0$. Thus, $Df(x)$ has trivial kernel.

To prove the converse, suppose $x, y \in \Omega$ and $f(x) = f(y)$. There are unitaries $u, v : \mathcal{H} \to$

$\mathcal{H}^{(2)}$ and $\varepsilon > 0$ such that $v^{-1}\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} v \in \Omega$ and

$$z := u^{-1}\begin{bmatrix} v^{-1}\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} v & v^{-1}\begin{bmatrix} 0 & \varepsilon(x-y) \\ 0 & 0 \end{bmatrix} v \\ 0 & v^{-1}\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} v \end{bmatrix} u \in \Omega.$$

First, by Proposition 2.2.1, and because f preserves direct sums, we know

$$f(z) = u^{-1}\begin{bmatrix} v^{-1}\begin{bmatrix} f(x) & 0 \\ 0 & f(y) \end{bmatrix} v & Df\left(v^{-1}\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} v\right)\left[v^{-1}\begin{bmatrix} 0 & \varepsilon(x-y) \\ 0 & 0 \end{bmatrix} v\right] \\ 0 & v^{-1}\begin{bmatrix} f(x) & 0 \\ 0 & f(y) \end{bmatrix} v \end{bmatrix} u. \quad (2.2.4)$$

On the other hand, a calculation shows that if we define $w : \mathcal{H} \to \mathcal{H}^{(4)}$ by $w := (v \oplus v)u$ and $s : \mathcal{H} \to \mathcal{H}^{(4)}$ by

$$s := \begin{bmatrix} 1 & 0 & 0 & \varepsilon 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} w,$$

then z may be rewritten as

$$z = s^{-1}\begin{bmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \end{bmatrix} s.$$

Therefore, as f is NC, we have

$$\begin{aligned} f(z) &= f\left(s^{-1} \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \end{bmatrix} s \right) \\ &= s^{-1} \begin{bmatrix} f(x) & 0 & 0 & 0 \\ 0 & f(y) & 0 & 0 \\ 0 & 0 & f(x) & 0 \\ 0 & 0 & 0 & f(y) \end{bmatrix} s \\ &= w^{-1} \begin{bmatrix} f(x) & 0 & 0 & \varepsilon(f(x) - f(y)) \\ 0 & f(y) & 0 & 0 \\ 0 & 0 & f(x) & 0 \\ 0 & 0 & 0 & f(y) \end{bmatrix} w \\ &= w^{-1} \begin{bmatrix} f(x) & 0 & 0 & 0 \\ 0 & f(y) & 0 & 0 \\ 0 & 0 & f(x) & 0 \\ 0 & 0 & 0 & f(y) \end{bmatrix} w \\ &= u^{-1} \begin{bmatrix} v^{-1} \begin{bmatrix} f(x) & 0 \\ 0 & f(y) \end{bmatrix} v & 0 \\ 0 & v^{-1} \begin{bmatrix} f(x) & 0 \\ 0 & f(y) \end{bmatrix} v \end{bmatrix} u. \end{aligned}$$

Comparing this to equation (2.2.4) implies

$$Df\left(v^{-1}\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} v\right)\left[v^{-1}\begin{bmatrix} 0 & \varepsilon(x-y) \\ 0 & 0 \end{bmatrix} v\right] = 0$$

in $B(\mathcal{H})^r$. By the assumption of the derivative being injective at all points,

$$v^{-1}\begin{bmatrix} 0 & \varepsilon(x-y) \\ 0 & 0 \end{bmatrix} v = 0,$$

and we conclude $x = y$ as desired. □

Other results on "lack of dimensionality" were observed by Cushing, Pascoe, and Tully-Doyle in [5]. Theorem 2.2.1 already provides a stark contrast between classical function theory and the noncommutative theory; examples abound of functions with globally invertible derivative who fail to be injective.

We now recall the definition of the Hessian of a Gâteaux differentiable function and later prove an analogous formula to Proposition 2.2.1 for the Hessian of an operator NC function. The formula is of similar flavor to one derived by Agler and McCarthy in [4] for matricial nc functions.

Definition 2.2.1. Let X and Y be Banach spaces and $U \subset X$ be open. For a Gâteaux differentiable function $g : U \to Y$, we define the *Hessian* of g at the point $x \in U$ to be

$$Hg(x)[h,k] := \lim_{t\to 0} \frac{Dg(x+tk)[h] - Dg(x)[h]}{t}, \tag{2.2.5}$$

whenever the limit exists for all $h, k \in B(\mathcal{H})^d$.

In the next lemma, we show that the derivative of an operator NC function is itself NC, that the Hessian exists for NC functions, and that the Hessian is again NC. As an application of these facts, we give a simple, calculus-based proof using boundedness of the Hessian that an operator NC function must, in particular, be of class C^1. Knowing they are C^1 allows us to use differential geometric tools from Banach space theory such as the classical inverse function theorem for C^1 maps between Banach spaces.

Lemma 2.2.3. *If $f : \Omega \subset B(\mathcal{H})^d \to B(\mathcal{H})^r$ is an NC function, then the following properties must hold:*

(i) *the derivative map $\phi : \Omega \times B(\mathcal{H})^d \to B(\mathcal{H})^r$ given by*

$$\phi(x, h) := Df(x)[h]$$

is an NC function;

(ii) *the Hessian $Hf(x)$ exists at all $x \in \Omega$, the map $\Omega \times B(\mathcal{H})^{2d} \to B(\mathcal{H})^r$ given by $(x, h, k) \mapsto Hf(x)[h, k]$ is an NC function, and $Hf(x)[h, k] = D\phi(x, h)[k, 0]$;*

(iii) *f is C^1.*

Proof. (i) Let (Ω_k) be an exhaustion of Ω as in the definition of NC domain. A natural candidate for an NC exhausting sequence for $\Omega \times B(\mathcal{H})^d$ is

$$W_k := \Omega_k \times \{h \in B(\mathcal{H})^d : \|h\| \leq k\}.$$

Indeed, the requirements of Definition 2.1.1 are readily seen, so $\Omega \times B(\mathcal{H})^d$ is an NC domain.

We now show ϕ is an NC function. This is a simple matter of using the definition of the derivative. Let (x_1, h_1) and (x_2, h_2) be in $\Omega \times B(\mathcal{H})^d$ and let $s : \mathcal{H} \to \mathcal{H}^{(2)}$ be invertible such that

$$(X, H) := s^{-1} \begin{bmatrix} (x_1, h_1) & 0 \\ 0 & (x_2, h_2) \end{bmatrix} s \in \Omega \times B(\mathcal{H})^d.$$

Since f is NC,

$$\begin{aligned}\phi(X,H) &= \lim_{t\to 0}\frac{1}{t}\left\{f\left(s^{-1}\begin{bmatrix}x_1 & 0\\ 0 & x_2\end{bmatrix}s + ts^{-1}\begin{bmatrix}h_1 & 0\\ 0 & h_2\end{bmatrix}s\right) - f\left(s^{-1}\begin{bmatrix}x_1 & 0\\ 0 & x_2\end{bmatrix}s\right)\right\}\\ &= \lim_{t\to 0} s^{-1}\begin{bmatrix}\frac{f(x_1+th_1)-f(x_1)}{t} & 0\\ 0 & \frac{f(x_2+th_2)-f(x_2)}{t}\end{bmatrix}s\\ &= s^{-1}\begin{bmatrix}\phi(x_1,h_1) & 0\\ 0 & \phi(x_2,h_2)\end{bmatrix}s,\end{aligned}$$

which proves part (i).

(ii) Since ϕ is NC on $\Omega \times B(\mathcal{H})^d$, we apply Lemma 2.2.2 to conclude ϕ is Gâteaux differentiable. Unraveling the definitions therefore shows that the Hessian $Hf(x)$ exists for all $x \in \Omega$, and the equality $Hf(x)[h,k] = D\phi(x,h)[k,0]$ must hold. Applying the result in part (i) to the NC function ϕ shows the map $\Omega \times B(\mathcal{H})^{3d} \to B(\mathcal{H})^r$ given by $(x,h,k,k') \mapsto D\phi(x,h)[k,k']$ is NC. Therefore, the Hessian map $(x,h,k) \mapsto Hf(x)[h,k] = D\phi(x,h)[k,0]$ must also be NC on $\Omega \times B(\mathcal{H})^{2d}$.

(iii) Fix $x \in \Omega$. By part (ii), it in particular holds that there is a norm ball B about x and $M > 0$ such that $\|Hf(y)[h,k]\| \leq M\|h\|\|k\|$ for all $y \in B$ and all $h,k \in B(\mathcal{H})^d$. Then for $y \in B$ and $h \in B(\mathcal{H})^d$, the map $t \mapsto Hf(x+t(y-x))[h, y-x]$ is continuous on the interval $[0,1]$ by part (ii). We may then estimate

$$\begin{aligned}\|Df(y)[h]-Df(x)[h]\| &= \left\| \int_0^1 \frac{d}{dt} Df(x+t(y-x))[h]dt \right\| \\ &= \left\| \int_0^1 Hf(x+t(y-x))[h,y-x]dt \right\| \\ &\le \int_0^1 \|Hf(x+t(y-x))[h,y-x]\|dt \\ &\le M\|h\|\|y-x\|.\end{aligned}$$

By definition of the operator norm, it then holds that

$$\|Df(y)-Df(x)\| \le M\|y-x\|$$

for $y \in B$. Therefore, the map $x \mapsto Df(x)$ is continuous on Ω. $\square$

Proposition 2.2.2. *Suppose $f : \Omega \subset B(\mathcal{H})^d \to B(\mathcal{H})^r$ is NC, $x \in \Omega$, and $u, v : \mathcal{H} \to \mathcal{H}^{(2)}$ are unitaries. If we set $w := (u \oplus u)v$, then for all $h, k \in B(\mathcal{H})^d$ of sufficiently small norm,*

$$\begin{aligned} & f\left(v^{-1} \begin{bmatrix} u^{-1} \begin{bmatrix} x & k \\ 0 & x \end{bmatrix} u & u^{-1} \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix} u \\ 0 & u^{-1} \begin{bmatrix} x & k \\ 0 & x \end{bmatrix} u \end{bmatrix} v \right) \\ &= w^{-1} \begin{bmatrix} f(x) & Df(x)[k] & Df(x)[h] & Hf(x)[h,k] \\ 0 & f(x) & 0 & Df(x)[h] \\ 0 & 0 & f(x) & Df(x)[k] \\ 0 & 0 & 0 & f(x) \end{bmatrix} w. \end{aligned} \tag{2.2.6}$$

Proof. For ease of reading, let us write

$$X := u^{-1} \begin{bmatrix} x & k \\ 0 & x \end{bmatrix} u \qquad \text{and} \qquad H := u^{-1} \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix} u.$$

By closure under direct sums and unitary conjugation, $X \in \Omega$ for $\|k\|$ sufficiently small, and

$$v^{-1} \begin{bmatrix} X & H \\ 0 & X \end{bmatrix} v \in \Omega$$

for $\|h\|$ sufficiently small. We may then compute, by letting ϕ be the derivative as in Lemma 2.2.3,

$$\phi(X, H) = u^{-1} \begin{bmatrix} \phi(x,h) & D\phi(x,h)[k,0] \\ 0 & \phi(x,h) \end{bmatrix} u = u^{-1} \begin{bmatrix} Df(x)[h] & Hf(x)[h,k] \\ 0 & Df(x)[h] \end{bmatrix} u.$$

The left-hand side of (2.2.6) is then equal to

$$f\left(v^{-1} \begin{bmatrix} X & H \\ 0 & X \end{bmatrix} v \right) = v^{-1} \begin{bmatrix} f(X) & Df(X)(H) \\ 0 & f(X) \end{bmatrix} v$$

$$= v^{-1} \begin{bmatrix} f(X) & u^{-1} \begin{bmatrix} Df(x)[h] & Hf(x)[h,k] \\ 0 & Df(x)[h] \end{bmatrix} u \\ 0 & f(X) \end{bmatrix} v$$

$$= v^{-1} \begin{bmatrix} u^{-1} \begin{bmatrix} f(x) & Df(x)[k] \\ 0 & f(x) \end{bmatrix} u & u^{-1} \begin{bmatrix} Df(x)[h] & Hf(x)[h,k] \\ 0 & Df(x)[h] \end{bmatrix} u \\ 0 & u^{-1} \begin{bmatrix} f(x) & Df(x)[k] \\ 0 & f(x) \end{bmatrix} u \end{bmatrix} v,$$

which is equal to the right-hand side of (2.2.6). □

Chapter 3

Operator NC Inverse and Implicit Function Theorems

This section contains statements and proofs of the operator NC inverse and implicit function theorems. They are both global results in the sense that global assumptions on the derivative imply global invertibility conclusions. Operator NC functions are, in particular, C^1 in the Fréchet sense by Lemma 2.2.3. The notation $Df(x)$ denotes the derivative mapping $B(\mathcal{H})^d \to B(\mathcal{H})^r$ of f at the point x in the domain of f. We denote by Df the map $x \mapsto Df(x)$. The notation $B(X, Y)$ denotes the space of bounded linear maps between the complex Banach spaces X and Y endowed with the operator norm.

We begin by studying the derivatives of operator NC functions and ask: When are the derivative maps $Df(x)$ invertible for *every* x in the domain of f? Theorem 3.0.1 below provides an answer to this question on norm-connected NC domains. In what follows, what is meant by the phrase "$Df(x)$ is invertible" is "$Df(x)$ is invertible in $B(B(\mathcal{H})^d)$." It is suggested that the reader recall the NC bounded below property defined in Definition 2.1.3.

Theorem 3.0.1. *Let $f : \Omega \subset B(\mathcal{H})^d \to B(\mathcal{H})^d$ be an NC function and suppose Ω is norm-connected. If Df has the NC bounded below property and there exists a point $a \in \Omega$ such that $Df(a)$ is invertible, then $Df(x)$ is invertible for every $x \in \Omega$.*

To prove Theorem 3.0.1, we need a general result on linear maps between Banach spaces. As the author could not find a suitable source in the literature, we include its statement and routine proof below for convenience.

Lemma 3.0.1. *Let X and Y be Banach spaces and fix $\alpha > 0$. The set of maps in $B(X, Y)$ that are surjective and bounded below by α is norm-closed.*

Proof. Let T_n be a sequence of such linear maps converging to T. As $\alpha\|x\| \leq \|T_n x\|$ holds for all n and $x \in X$, we see that $\alpha\|x\| \leq \|Tx\|$ for all $x \in X$, so T is bounded below by α.

Now we show T must also be surjective. The uniform bound below on the T_n implies the sequence of inverses T_n^{-1} is uniformly bounded in operator norm by $1/\alpha$. Thus, the estimate

$$\|T_n^{-1} - T_m^{-1}\| \leq \|T_m^{-1}\|\|T_m - T_n\|\|T_n^{-1}\| \leq 1/\alpha^2\|T_m - T_n\|$$

shows T_n^{-1} is a convergent sequence in $B(Y, X)$. Since $T_n T_n^{-1} = 1_Y$ for all n, we immediately see that T is surjective. □

Proof of Theorem 3.0.1. By hypothesis, the set

$$U := \{x \in \Omega : Df(x) \text{ is invertible in } B(B(\mathcal{H})^d)\}$$

is non-empty. Since invertible maps form a norm-open set in $B(B(\mathcal{H})^d)$, the continuity of the map $x \mapsto Df(x)$ implies U is open in norm.

As Ω is norm-connected, it suffices to show U is also relatively norm-closed in Ω. To that end, take a sequence (x_n) in U converging in norm to $x \in \Omega$. We claim there is a uniform $\alpha > 0$ such that each $Df(x_n)$ is bounded below by α. To see this, take an exhaustion (Ω_k) of Ω as in Definition 2.1.1. Since $x_n \to x \in \Omega$, and since the exhaustion satisfies $\Omega_k \subset \operatorname{int}\Omega_{k+1}$, there is k large enough so that all the x_n lie in Ω_k. Since Ω_k is closed under countably infinite direct sums, there is $u : \mathcal{H} \to \mathcal{H}^{(\infty)}$ unitary such that

$$z := u^{-1} \begin{bmatrix} x_1 & & \\ & x_2 & \\ & & \ddots \end{bmatrix} u \in \Omega_k.$$

By the hypothesis of Df satisfying the NC bounded below property, $Df(z)$ must be bounded below, say by $\alpha > 0$. Now fix n and let $h \in B(\mathcal{H})^d$ be arbitrary. Let $h_n \in B(\mathcal{H}^{(\infty)})^d$ denote the direct sum of h as the nth summand and 0 else. Since

$$Df(z)[u^{-1}h_n u] = u^{-1} \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & Df(x_n)[h] & & \\ & & & 0 & \\ & & & & \ddots \end{bmatrix} u \tag{3.0.1}$$

holds by Lemma 2.2.3 (i), we may take norms in (3.0.1) to get

$$\|Df(x_n)[h]\| = \|Df(z)[u^{-1}h_n u]\| \geq \alpha \|u^{-1}h_n u\| = \alpha \|h\|.$$

This implies each $Df(x_n)$ is bounded below by α. Since f is C^1, we have $Df(x_n) \to Df(x)$ in norm. Lemma 3.0.1 then implies that $Df(x)$ is surjective and bounded below (by α) and therefore is invertible in $B(B(\mathcal{H})^d)$. Thus $x \in U$ and U is relatively closed in Ω. □

With this result giving a sufficient condition for the invertibility of the derivative map of an NC function at *all* points of a connected NC domain, we arrive at an operator NC inverse function theorem. Theorem 3.0.1 justifies the NC bounded below property as a substitute for injectivity in the operatorial setting. The hypotheses for the NC inverse function theorem are the same as in Theorem 3.0.1.

Theorem 3.0.2. (Operator NC Inverse Function Theorem) *Let $f : \Omega \subset B(\mathcal{H})^d \to B(\mathcal{H})^d$ be an NC function and suppose Ω is norm-connected. If Df has the NC bounded below property and there exists a point $a \in \Omega$ such that $Df(a)$ is invertible, then $f(\Omega)$ is an NC domain and $f^{-1} : f(\Omega) \to \Omega$ exists and is an NC function.*

Proof. By Theorem 3.0.1, $Df(x)$ is an invertible linear mapping $B(\mathcal{H})^d \to B(\mathcal{H})^d$ for every $x \in \Omega$. Theorem 2.2.1 tells us that f is then injective on Ω, so f^{-1} exists as a map $f(\Omega) \to \Omega$. We must show $f(\Omega)$ and f^{-1} are both NC. In fact, we claim that if we take an exhaustion (Ω_k) for Ω as in Definition 2.1.1, then the sequence of images $(f(\Omega_k))$ is such an exhaustion for $f(\Omega)$.

First, we show $f(\Omega)$ is an NC domain. All required properties in Definition 2.1.1 of the sequence $(f(\Omega_k))$ are immediate from the corresponding properties of Ω_k and the fact that f is NC, except possibly the containments $f(\Omega_k) \subset \operatorname{int} f(\Omega_{k+1})$. But since f is C^1, the classical inverse function theorem for Banach spaces (see [2] for a reference) implies f is an open map because each $Df(x)$ is invertible. Hence,

$$f(\Omega_k) \subset f(\operatorname{int} \Omega_{k+1}) = \operatorname{int} f(\operatorname{int} \Omega_{k+1}) \subset \operatorname{int} f(\Omega_{k+1}).$$

Finally, we show f^{-1} is an NC function. Let $f(x_1)$ and $f(x_2)$ be in $f(\Omega)$ and let $s : \mathcal{H} \to \mathcal{H}^{(2)}$ be invertible with

$$s^{-1} \begin{bmatrix} f(x_1) & 0 \\ 0 & f(x_2) \end{bmatrix} s \in f(\Omega). \tag{3.0.2}$$

It suffices to show $w := s^{-1} \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} s$ lies in Ω, since we may then apply f and use the fact that f preserves direct sums to get

$$f^{-1}\left(s^{-1}\begin{bmatrix} f(x_1) & 0 \\ 0 & f(x_2) \end{bmatrix} s\right) = s^{-1}\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} s.$$

This then shows f^{-1} preserves direct sums. Note that the membership $w \in \Omega$ does not immediately follow since s is not necessarily unitary.

Call the expression in (3.0.2) $f(z)$ for a unique $z \in \Omega$. We know there is a unitary $u : \mathcal{H} \to \mathcal{H}^{(2)}$ such that

$$x := u^{-1}\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} u \in \Omega.$$

Since f is NC, if we define $L := s^{-1}u \in B(\mathcal{H})$, then

$$f(x) = u^{-1}\begin{bmatrix} f(x_1) & 0 \\ 0 & f(x_2) \end{bmatrix} u = L^{-1}f(z)L.$$

We claim that $z = LxL^{-1}$, which proves $w \in \Omega$, since $LxL^{-1} = w$. There is unitary $v : \mathcal{H} \to \mathcal{H}^{(2)}$ such that $v^{-1}\begin{bmatrix} z & 0 \\ 0 & x \end{bmatrix} v \in \Omega$. For sufficiently small $\varepsilon > 0$, apply Lemma 2.2.1:

$$\begin{aligned} f\left(v^{-1}\begin{bmatrix} z & \varepsilon(Lx - zL) \\ 0 & x \end{bmatrix} v\right) &= v^{-1}\begin{bmatrix} f(z) & \varepsilon(Lf(x) - f(z)L) \\ 0 & f(x) \end{bmatrix} v \\ &= v^{-1}\begin{bmatrix} f(z) & 0 \\ 0 & f(x) \end{bmatrix} v \\ &= f\left(v^{-1}\begin{bmatrix} z & 0 \\ 0 & x \end{bmatrix} v\right). \end{aligned}$$

It now follows from injectivity of f that $Lx = zL$, as desired. □

As one might expect, the operator NC inverse function theorem gives rise to an operator NC implicit function theorem under the hypothesis that an augmented derivative map satisfies the NC bounded below property. The notation Z_f denotes the zero set of the function f. In the implicit function theorem, we write, for notational convenience, $(h^{d-r+1}, \ldots, h^d)$ for elements of $B(\mathcal{H})^r$.

Theorem 3.0.3. (Operator NC Implicit Function Theorem) *Let $f : \Omega \subset B(\mathcal{H})^d \to B(\mathcal{H})^r$ be NC, where $1 \leq r \leq d-1$, and suppose Ω is norm-connected. Suppose the map $\Psi : \Omega \to B(B(\mathcal{H})^r)$ defined by*

$$\Psi(x)(h^{d-r+1}, \ldots, h^d) = Df(x)[0, \ldots, 0, h^{d-r+1}, \ldots, h^d]$$

has the NC bounded below property and there exists a point $a \in \Omega$ such that $\Psi(a)$ is invertible.

Then, there exists $V \subset B(\mathcal{H})^{d-r}$ an NC domain and $\phi : V \to B(\mathcal{H})^r$ an NC function such that

$$Z_f = \{(y, \phi(y)) : y \in V\}.$$

Furthermore, V is given by the projection onto the first $d-r$ coordinates of the zero set Z_f.

Proof. Consider the NC function $F : \Omega \to B(\mathcal{H})^d$ given by the formula $F(x) = (x^1, \ldots, x^{d-r}, f(x))$. We claim that F satisfies the hypotheses of Theorem 3.0.2. The derivative of F is computed as

$$DF(x)[h] = (h^1, \ldots, h^{d-r}, Df(x)[h]). \tag{3.0.3}$$

We first show that $DF(a)$ is invertible in $B(B(\mathcal{H})^d)$, where $a \in \Omega$ is the point such that $\Psi(a)$ is assumed to be invertible. Let $(v, w) \in B(\mathcal{H})^{d-r} \times B(\mathcal{H})^r$ be arbitrary. By hypothesis, there is $(h^{d-r+1}, \ldots, h^d) \in B(\mathcal{H})^r$ such that

$$Df(a)[0, \ldots, 0, h^{d-r+1}, \ldots, h^d] = w - Df(a)[v, 0, \ldots, 0].$$

Linearity of the derivative and (3.0.3) then give

$$\begin{aligned} DF(a)[v, h^{d-r+1}, \ldots, h^d] &= (v, Df(a)[v, h^{d-r+1}, \ldots, h^d]) \\ &= (v, Df(a)[v, 0, \ldots, 0] + Df(a)[0, \ldots, 0, h^{d-r+1}, \ldots, h^d]) \\ &= (v, w), \end{aligned}$$

so $DF(a)$ is surjective. As $DF(a)$ is clearly injective when $\Psi(a)$ is, we conclude that $DF(a)$ is invertible.

We now show that DF has the NC bounded below property by showing, for $x \in \Omega$, that $DF(x)$ is bounded below if and only if $\Psi(x)$ is bounded below. It is immediate to see that $\Psi(x)$ is bounded below if $DF(x)$ is, so we prove the converse. Fix $x \in \Omega$ such that $\Psi(x)$ is bounded below. Then there is $\varepsilon > 0$, depending only on x, such that

$$\|Df(x)[0, \ldots, 0, h^{d-r+1}, \ldots, h^d]\| \geq \varepsilon \max\{\|h^{d-r+1}\|, \ldots, \|h^d\|\}$$

for all $(h^{d-r+1}, \dots, h^d) \in B(\mathcal{H})^r$. Therefore we may estimate

$$\begin{aligned}\|Df(x)[h^1, \dots, h^d]\| &= \|Df(x)[h^1, \dots, h^{d-r}, 0, \dots, 0] \\ &\qquad + Df(x)[0, \dots, 0, h^{d-r+1}, \dots, h^d]\| \\ &\geq \|Df(x)[0, \dots, 0, h^{d-r+1}, \dots, h^d]\| \\ &\qquad - \|Df(x)[h^1, \dots, h^{d-r}, 0, \dots, 0]\| \\ &\geq \varepsilon \max\{\|h^{d-r+1}\|, \dots, \|h^d\|\} \\ &\qquad - \|Df(x)\| \max\{\|h^1\|, \dots, \|h^{d-r}\|\}.\end{aligned}$$

This combined with taking norms in (3.0.3) gives us

$$\begin{aligned}\|DF(x)[h^1, \dots, h^d]\| &= \max\{\|h^1\|, \dots, \|h^{d-r}\|, \|Df(x)[h^1, \dots, h^d]\|\} \\ &\geq \frac{\varepsilon}{\varepsilon + \|Df(x)\| + 1} \max\{\|h^1\|, \dots, \|h^d\|\},\end{aligned}$$

so $DF(x)$ is bounded below.

Therefore, by Theorem 3.0.2, we know $F^{-1} : F(\Omega) \to \Omega$ is NC. We may write F^{-1} it terms of its coordinates, say $F^{-1} = (G^1, \dots, G^d)$. Let V be the projection onto the first $d-r$ coordinates of the zero set Z_f. Thus, V can explicitly be written as the set of $y \in B(\mathcal{H})^{d-r}$ such that there exists $z \in B(\mathcal{H})^r$ with $(y, z) \in \Omega$ and $f(y, z) = 0$. Then V is seen to be an NC domain with exhaustion (V_k), where V_k is defined to be the set of $y \in B(\mathcal{H})^{d-r}$ such that there exists $z \in B(\mathcal{H})^r$ with $(y, z) \in \Omega_k$ and $f(y, z) = 0$. (The containment $V_k \subset \operatorname{int} V_{k+1}$ follows since F is an open map by Theorem 3.0.1 and the classical Banach space inverse function theorem.) Now define $\phi : V \to B(\mathcal{H})^r$ by

$$\phi(y) := (G^{d-r+1}(y, 0), \dots, G^d(y, 0)).$$

It is immediate to check that ϕ is an NC function.

Let $y \in V$. From the definitions,

$$\begin{aligned}(y,0) &= F(G^1(y,0),\dots,G^{d-r}(y,0),\phi(y)) \\ &= (G^1(y,0),\dots,G^{d-r}(y,0),f(F^{-1}(y,0))).\end{aligned}$$

Therefore $y = (G^1(y,0),\dots,G^{d-r}(y,0))$ and $f(F^{-1}(y,0)) = 0$, so $(y,\phi(y)) \in Z_f$. Conversely, let $x = (y,z) \in Z_f$, where $y \in B(\mathcal{H})^{d-r}$ and $z \in B(\mathcal{H})^r$. Then $y \in V$ and $F(x) = (y,f(x)) = (y,0)$. Thus, $(y,z) = F^{-1}(y,0)$, which implies $z = \phi(y)$. This establishes the desired parametrization of Z_f. □

Theorem 3.0.3 is an operatorial analogue of Agler and McCarthy's implicit function theorem (Theorem 6.1, [4]) for the fine matricial nc topology. For further emphasis, the parametrizing function ϕ in Theorem 3.0.3 and the inverse mapping obtained in Theorem 3.0.2, being themselves operator NC, are *countable* direct sum-preserving. It is important to note that the conclusions of Theorems 3.0.2 and 3.0.3 are *global*, a phenomenon that is rare outside of the noncommutative setting. As mentioned in Chapter 1, results similar to Theorems 3.0.2 and 3.0.3 are obtained in [1] for a quite general matricial nc setting. In that paper, the authors obtain local invertibility conclusions with hypotheses of analyticity in the "uniformly-open" topology and a completely bounded and invertible derivative map with completely bounded inverse. In the operator NC setting, we note once more that the notion of a uniformly-open topology is no longer available. We instead made extensive use of the completeness of $B(\mathcal{H})$ and its interaction with functions that preserve countable direct sums. In Chapter 4, we utilize again that we are working in $B(\mathcal{H})^d$ by considering domains and functions with structure in the strong operator topology.

Chapter 4

The Shift Form Construction

4.1 A Noncommutative Heine-Borel Theorem

This chapter concerns the construction mentioned in Chapter 1 that was motivated by noncommutative dilation theory. This construction, the *shift form construction*, has proved remarkably useful in the author's research. After discussing the details of the construction, this section concludes with our first application of shift forms: an operator noncommutative version of the Heine-Borel theorem, Theorem 4.1.1, for the strong operator topology (SOT.) Sections 4.2 and 5.4 include applications of this result in the directions of SOT-continuous NC functions and operator noncommutative partial convexity, respectively.

We now elaborate on the shift form construction. The author would like to thank J.E. Pascoe for many helpful conversations pertaining to this construction. Separability of the underlying Hilbert space will be used extensively. Throughout this chapter, we fix a countable orthonormal basis $\{e_1, e_2, \ldots\}$ for $\mathcal{H}$. Given a d-tuple $X \in B(\mathcal{H})^d$, the idea is to find a unitary operator in $B(\mathcal{H})$ which provides a basis for $\mathcal{H}$ on which the coordinates of X essentially act as shifts.

Let M be the shift operator, $Me_k = e_{k+1}$. For the sake of brevity, we write (X, M) for the $(d+1)$-tuple $(X^1, \ldots, X^d, M)$. We will denote the complex vector space of polynomials in $(d+1)$ noncommuting variables of degree less than or equal to k by $\mathcal{P}(k, d)$ and write

$\alpha(k,d)$ for its dimension. Begin by defining a nested sequence of subspaces of $\mathcal{H}$:

$$V_k^X := \{p(X,M)e_1 : p \in \mathcal{P}(k,d)\},$$

for $k \geq 0$. We record the following properties of the V_k^X :

(i) for each $k \geq 0$, we have $e_1, \ldots, e_{k+1} \in V_k^X$: in particular,

$$\mathcal{H} = \overline{\bigcup_{k=0}^{\infty} V_k^X};$$

(ii) the inclusion $X^i V_k^X \subset V_{k+1}^X$ holds for all $i = 1, \ldots, d$ and $k \geq 0$;

(iii) the V_k^X form a strictly increasing sequence;

(iv) the inequality

$$\dim V_k^X \leq \alpha(k,d)$$

holds for all $k \geq 0$, independent of the choice of d-tuple X.

Properties (i), (iii), and (iv) above imply that there exists a unitary operator $U \in B(\mathcal{H})$, depending on d and X, but not k, such that

$$U\left(\operatorname{span}\{e_1, \ldots, e_k, e_{k+1}\}\right) \subset V_k^X \tag{4.1.1}$$

and

$$U^*(V_k^X) \subset \operatorname{span}\{e_1, \ldots, e_{\alpha(k,d)}\} \tag{4.1.2}$$

hold for every $k \geq 0$.

Definition 4.1.1. A d-tuple $\widetilde{X} \in B(\mathcal{H})^d$ is called a **shift form** of $X \in B(\mathcal{H})^d$ if there is a unitary $U \in B(\mathcal{H})$ satisfying (4.1.1) and (4.1.2) such that $\widetilde{X} = U^*XU$.

This construction allows us to prove an SOT Heine-Borel compactness-like theorem, Theorem 4.1.1 for bounded subsets of $B(\mathcal{H})^d$. By SOT on $B(\mathcal{H})^d$, we mean the product topology induced by endowing $B(\mathcal{H})$ with the strong operator topology. It is well-known that the unit ball of $B(\mathcal{H})$ is not SOT (sequentially) compact when $\mathcal{H}$ is infinite dimensional. Nonetheless, we prove in Theorem 4.1.1 that for any bounded sequence in $B(\mathcal{H})^d$, and given any sequence of unitaries (U_n) such that $\widetilde{X_n} := U_n^* X_n U_n$ is a shift form of X_n for each n, there is a subsequence along which $\widetilde{X_n}$ converges in SOT. This statement lends itself nicely to applications with strong NC functions since they preserve conjugations by unitary operators and are SOT-continuous when restricted to certain sets that are closed under conjugation by unitaries.

Lemma 4.1.1 below is a technical ingredient used only in the proof of Lemma 4.2.2 but is an interesting property of shift forms in its own right.

Lemma 4.1.1. *If $X \in B(\mathcal{H})^d$ and $k \geq 1$, then by letting P_k denote the projection onto the subspace spanned by the first k basis vectors $e_1, \ldots, e_k$, we have*

$$\|P_k X^i P_k\| \leq \|P_{\alpha(k,d)} \widetilde{X}^i P_{\alpha(k,d)}\| \tag{4.1.3}$$

for each $i = 1, \ldots, d$ and choice of shift form $\widetilde{X}$ of X.

Proof. Write $\widetilde{X} = U^*XU$ for a unitary $U \in B(\mathcal{H})$ satisfying (4.1.1) and (4.1.2). Let $y \in \operatorname{span}\{e_1, \ldots, e_k\}$ with $\|y\| \leq 1$. Since $\operatorname{span}\{e_1, \ldots, e_k\} \subset V_{k-1}^X$, we know by (4.1.2) that the containment $U^*y \in \operatorname{span}\{e_1, \ldots, e_{\alpha(k-1,d)}\}$ holds. Furthermore, this implies $X^i y \in V_k^X$, and so $U^*X^iy \in \operatorname{span}\{e_1, \dots, e_{\alpha(k,d)}\}$. Therefore we may estimate

$$\begin{aligned}\|P_kX^iP_ky\| &\leq \|X^iy\| = \|U^*X^iy\| \\ &= \|P_{\alpha(k,d)}U^*X^iy\| = \|P_{\alpha(k,d)}[U^*X^iU]U^*y\| \\ &= \|P_{\alpha(k,d)}[U^*X^iU]P_{\alpha(k,d)}U^*y\| \\ &\leq \|P_{\alpha(k,d)}\widetilde{X}^iP_{\alpha(k,d)}\|.\end{aligned}$$

Taking supremum over such y finishes the proof. □

The proof of Lemma 4.1.1 shows that the norm inequality (4.1.3) can be refined slightly. For example, under the hypotheses of Lemma 4.1.1, it holds that

$$\|X^iP_k\| \leq \|P_{\alpha(k,d)}\widetilde{X}^iP_{\alpha(k-1,d)}\|.$$

Since we do not require this inequality moving forward, we opt for the more visually symmetric (4.1.3).

We now state the main result of this section, Theorem 4.1.1.

Theorem 4.1.1. *Let (X_n) be a bounded sequence in $B(\mathcal{H})^d$. For any sequence $(\widetilde{X_n})$ of shift forms of X_n, there is a subsequence along which $\widetilde{X_n}$ converges in SOT.*

In particular, given a bounded sequence (X_n) in $B(\mathcal{H})^d$, there exists a sequence (U_n) of unitaries in $B(\mathcal{H})$ and a subsequence along which $U_n^*X_nU_n$ and U_n^* both converge in SOT.

Proof. For each n, let $\widetilde{X_n} = U_n^*X_nU_n$ be any shift form of X_n. For every n, $i = 1, \dots, d$, and

$k \geq 1$, properties (4.1.1) and (4.1.2), along with property (ii) above, imply

$$\begin{aligned}
\widetilde{X_n}^i(\operatorname{span}\{e_1, \ldots, e_k\}) &= U_n^* X_n^i U_n(\operatorname{span}\{e_1, \ldots, e_k\}) \\
&\subset U_n^* X_n^i (V_{k-1}^{X_n}) \\
&\subset U_n^*(V_k^{X_n}) \\
&\subset \operatorname{span}\{e_1, \ldots, e_{\alpha(k,d)}\}.
\end{aligned}$$

Therefore, for every $i = 1, \ldots, d$ and $k \geq 1$, the sequence $\{\widetilde{X_n}^i e_k\}_{n=1}^{\infty}$ is bounded and contained in a finite dimensional subspace. By a diagonalization argument, we may then find a subsequence n_j so that $\widetilde{X_{n_j}}^i e_k$ converges for every $i = 1, \ldots, d$ and $k \geq 1$. By boundedness again, this implies $\widetilde{X_{n_j}}^i$ converges in SOT for every $i = 1, \ldots, d$. Hence, $\widetilde{X_{n_j}}$ converges in SOT.

To establish the final part, use property (4.1.2) and apply a similar argument to the sequence $(U_{n_j}^*)$ to get a further subsequence along which both $U_n^* X_n U_n$ and U_n^* converge in SOT. □

4.2 Strong Operator NC Functions

It is reasonable to ask, in view of the operator NC inversion theorems established in Chapter 3, if additional structure imposed on the NC domains and functions in the strong operator topology allows us to weaken the assumptions on their derivatives. Here, we use the shift form construction and Theorem 4.1.1.

For $\varepsilon > 0$ and $U \subset B(\mathcal{H})^d$, let U^ε be the set $U^\varepsilon := \{x \in B(\mathcal{H})^d : \operatorname{dist}(x, U) < \varepsilon\}$ consisting of all points in $B(\mathcal{H})^d$ whose distance (in maximum norm) is less than ε from U.

Definition 4.2.1. We say $\Omega \subset B(\mathcal{H})^d$ is a **strong NC domain** if there exists an exhausting sequence $(\Omega_k)_{k=1}^{\infty}$ of Ω as in Definition 2.1.1, with the additional requirements that:

(i) each Ω_k is closed in the strong operator topology on $B(\mathcal{H})^d$;

(ii) for each k there is $\varepsilon_k > 0$ such that $\Omega_k^{\varepsilon_k} \subset \Omega_{k+1}$.

Definition 4.2.2. Let $\Omega \subset B(\mathcal{H})^d$. A function $f : \Omega \to B(\mathcal{H})^r$ is called a **strong NC function** if:

(i) there exists an exhausting sequence $(\Omega_k)_{k=1}^{\infty}$ of Ω as in Definition 4.2.1 such that each restriction $f|_{\Omega_k}$ is continuous in the strong operator topology;

(ii) f is an NC function.

By definition, strong NC functions are only defined on strong NC domains, and assuming that $f : \Omega \subset B(\mathcal{H})^d \to B(\mathcal{H})^r$ is a strong NC function implicitly implies Ω is a strong NC domain.

Since the strong operator topology is metrizable on norm-bounded subsets of $B(\mathcal{H})^d$ when $\mathcal{H}$ is separable, the continuity condition (i) in Definition 4.2.2 is equivalent to the following sequential criterion: for every k, whenever (x_n) is a sequence in Ω_k with $x_n \to x$ in SOT, we have $f(x_n) \to f(x)$ in SOT. Similarly, condition (i) of each Ω_k being SOT-closed in Definition 4.2.1 is equivalent to the sequential characterization given by: whenever (x_n) is a sequence in Ω_k and $x \in B(\mathcal{H})^d$ with $x_n \to x$ in SOT, then $x \in \Omega_k$.

We remark further about Definitions 4.2.1 and 4.2.2. Any B_δ, as described in Section 2.1, is a strong NC domain since the exhaustion given in (2.1.2) satisfies the additional requirements of Definition 4.2.1. Indeed, such a δ is Lipschitz on bounded sets and multiplication is strongly continuous on bounded sets. Moreover, as noncommutative polynomials and rational functions (such as the example in (2.1.4) on the bidisk) are strongly continuous on appropriate norm-bounded sets, in practice these additional requirements seem rather mild and natural in the setting of operator noncommutative function theory.

Secondly, condition (ii) in Definition 4.2.1 is a technical strengthening of the condition $\Omega_k \subset \operatorname{int} \Omega_{k+1}$ (which we have been using so far), and it is used to ensure that the derivative of a strong NC function is also a strong NC function. Indeed, this is the content of the following lemma.

Lemma 4.2.1. *If f is a strong NC function, then so is its derivative. More precisely, if $f : \Omega \subset B(\mathcal{H})^d \to B(\mathcal{H})^r$ is a strong NC function, then so is the map $\phi : \Omega \times B(\mathcal{H})^d \to B(\mathcal{H})^r$ given by $\phi(x, h) = Df(x)[h]$.*

Proof. Choose an exhausting sequence (Ω_k) of Ω as in Definition 4.2.2. Considering the sequence $(W_k := \Omega_k \times \{h \in B(\mathcal{H})^d : \|h\| \leq k\})$ shows $W := \Omega \times B(\mathcal{H})^d$ to be a strong NC domain.

We now show that the restriction $\phi|_{W_k}$ is SOT-continuous for every k. This is equivalent to showing, for every k, if $((x_n, h_n))$ is a sequence in W_k with $x_n \to x$ in SOT and $h_n \to h$ in SOT, then $Df(x_n)[h_n] \to Df(x)[h]$ in SOT. To see this, fix k and note that by closure under direct sums and unitary conjugation of Ω_k, there is a unitary $u : \mathcal{H} \to \mathcal{H}^{(2)}$ such that

$$u^{-1} \begin{bmatrix} x_n & 0 \\ 0 & x_n \end{bmatrix} u \in \Omega_k$$

for all n. As the sequence (h_n) is bounded, condition (ii) in Definition 4.2.1 implies there is $\varepsilon > 0$ (independent of n) such that

$$u^{-1} \begin{bmatrix} x_n & \varepsilon h_n \\ 0 & x_n \end{bmatrix} u \in \Omega_{k+1}$$

for all n. Therefore, by Proposition 2.2.1 and because $f|_{\Omega_{k+1}}$ is SOT-continuous,

$$\begin{aligned} u^{-1}\begin{bmatrix} f(x) & \varepsilon Df(x)[h] \\ 0 & f(x) \end{bmatrix} u &= f\left(u^{-1}\begin{bmatrix} x & \varepsilon h \\ 0 & x \end{bmatrix} u \right) \\ &= \text{SOT-lim}_n\, f\left(u^{-1}\begin{bmatrix} x_n & \varepsilon h_n \\ 0 & x_n \end{bmatrix} u \right) \\ &= \text{SOT-lim}_n\, u^{-1}\begin{bmatrix} f(x_n) & \varepsilon Df(x_n)[h_n] \\ 0 & f(x_n) \end{bmatrix} u. \end{aligned}$$

We then conclude that $Df(x_n)[h_n] \to Df(x)[h]$ in SOT, as asserted. □

To prove non-trivial inversion properties of strong NC functions, we will need a refinement of a special case of our main result on shift forms, Theorem 4.1.1. In the notation of this lemma, we need $H' \neq 0$ to ensure it is not in the kernel of any injective derivative map of a strong NC function.

Lemma 4.2.2. *Suppose $X \in B(\mathcal{H})^d$ and $H_n \in B(\mathcal{H})^d$ with $\|H_n\| = 1$ for all n. Then there exist unitaries $W_n \in B(\mathcal{H})$, a point $(X', H') \in B(\mathcal{H})^{2d}$ with $H' \neq 0$, and a subsequence along which*

$$W_n^*(X, H_n)W_n \to (X', H')$$

in SOT.

Proof. By passing to a subsequence if necessary, we may assume there is $i \in \{1, \dots, d\}$ such that $\|H_n^i\| = 1$ for all n. We again denote by P_k the projection onto the subspace spanned by the first k basis vectors $e_1, \dots, e_k$. First note that if $T \in B(\mathcal{H})$ has operator norm equal to 1, then for every $0 < \varepsilon < 1$, there exists a unitary $Q \in B(\mathcal{H})$ such that $1 - \varepsilon \leq \|P_2 Q^* T Q P_2\|$. This can be seen by choosing a unit vector v which approximates the norm of T, and then defining a unitary which maps e_1 to v and e_2 to a suitable linear combination of v and Tv.

Applying the above observation to H_n^i for each n, we can find unitaries $Q_n \in B(\mathcal{H})$ so that

$$1 - \frac{1}{n} \leq \|P_2 Q_n^* H_n^i Q_n P_2\|.$$

Since $X_n := Q_n^*(X, H_n)Q_n$ forms a bounded sequence in $B(\mathcal{H})^{2d}$, by Theorem 4.1.1, there exist a sequence of unitaries (U_n) such that $\widetilde{X_n} := U_n^* X_n U_n$ is a shift form of X_n for every n and a subsequence n_j along which $\widetilde{X_n}$ converges in SOT. Call this limit $(X', H') \in B(\mathcal{H})^{2d}$. Define $W_n := Q_n U_n$. Combining this and (4.1.3) with $k = 2$ and $2d$ variables, we estimate

$$\begin{aligned} 1 - \frac{1}{j} &\leq \|P_2 Q_{n_j}^* H_{n_j}^i Q_{n_j} P_2\| \\ &= \|P_2 X_{n_j}^{(d+i)} P_2\| \\ &\leq \|P_{\alpha(2,2d)} \widetilde{X_{n_j}}^{(d+i)} P_{\alpha(2,2d)}\| \\ &= \|P_{\alpha(2,2d)} W_{n_j}^* H_{n_j}^i W_{n_j} P_{\alpha(2,2d)}\|. \end{aligned}$$

Since SOT convergence is equivalent to norm convergence on finite dimensional spaces, taking the limit as $j \to \infty$ in the above estimate implies

$$1 \leq \|P_{\alpha(2,2d)} (H')^i P_{\alpha(2,2d)}\| \leq \|H'\|.$$

Therefore $H' \neq 0$, which concludes the proof. □

Our primary application of the shift form construction to operator noncommutative inversion is the following, rather surprising theorem.

Theorem 4.2.1. *Suppose $f : \Omega \subset B(\mathcal{H})^d \to B(\mathcal{H})^r$ is a strong NC function. If $Df(x)$ is injective for every $x \in \Omega$, then $Df(x)$ is bounded below for every $x \in \Omega$.*

Proof. Suppose there is $x \in \Omega$ such that $Df(x)$ is not bounded below. Then we can find a sequence (h_n) in $B(\mathcal{H})^d$ of unit vectors such that

$$\|Df(x)[h_n]\| \to 0.$$

Let (Ω_k) be an exhaustion for Ω as in Definition 4.2.2 and say the point x lies in Ω_k. By Lemma 4.2.2, there are unitaries v_n and a point $(x', h') \in B(\mathcal{H})^{2d}$ with $h' \neq 0$ such that

$$v_n^*(x, h_n)v_n \to (x', h')$$

in SOT along a subsequence n_j. Since Ω_k is closed under unitary conjugation and is SOT-closed, it follows that $v_{n_j}^* x v_{n_j} \in \Omega_k$ for every j and that $x' \in \Omega_k$. Therefore, by Lemma 4.2.1, we have

$$v_{n_j}^* Df(x)[h_{n_j}]v_{n_j} = Df(v_{n_j}^* x v_{n_j})[v_{n_j}^* h_{n_j} v_{n_j}] \to Df(x')[h']$$

in SOT. But by the choice of h_n and because the v_n are unitary, we also have

$$v_{n_j}^* Df(x)[h_{n_j}]v_{n_j} \to 0$$

in norm. Therefore, $Df(x')[h'] = 0$, contradicting the hypothesis of injectivity of $Df(x')$. □

Example 4.2.1. Let $A \in B(\mathcal{H})$ be an injective operator that is not bounded below. Consider the function $g : B(\mathcal{H}) \to B(\mathcal{H})$ given by $g(X) = AX$. Then g is defined on the strong NC domain $B(\mathcal{H})$ and is SOT-continuous on its domain, but g is not an NC function.

As g is an injective bounded linear operator on $B(\mathcal{H})$ that is not bounded below, g has the property that $Dg(X)$ is injective for every X in its domain, but $Dg(X)$ is bounded below for no X. This example illustrates the extent to which Theorem 4.2.1 fails for functions which are SOT-continuous but not operator NC.

Theorem 4.2.1 does not require the strong NC domain to be connected in any topology.

On the other hand, injective strong NC functions on norm-connected domains are especially nice:

Corollary 4.2.1. *Let $f : \Omega \subset B(\mathcal{H})^d \to B(\mathcal{H})^d$ be an injective strong NC function. If Ω is norm-connected and there exists a point $a \in \Omega$ such that $Df(a)$ is surjective, then $Df(x)$ is invertible for every $x \in \Omega$.*

Proof. Apply Theorem 2.2.1 to conclude that $Df(x)$ is injective for every $x \in \Omega$. In particular, $Df(a)$ is therefore invertible. By Theorem 4.2.1, each $Df(x)$ is bounded below since f is strong NC. Theorem 3.0.1 then implies the desired conclusion. □

Results such as Theorem 4.2.1 and Corollary 4.2.1 suggest it may be natural to have some structure in the strong operator topology built into the definitions of NC domain and function. However, Theorems 3.0.1, 3.0.2, and 3.0.3 along with the foundations found in Section 2.2, require no such hypotheses. As such, there is merit to also studying a more general theory.

Chapter 5

Noncommutative Partial Matrix and Operator Convexity

5.1 Background and Foundations

In this chapter, we introduce the foundations of a general framework for noncommutative partial convexity, called Γ-convexity. We define this notion in both the matricial and operatorial settings. Our results in the operator setting, while interesting in their own right, lend themselves nicely to applications in the matrix setting. In Chapter 1, we outlined the definitions of (noncommutative) matrix convexity. Recall from Chapter 1 that a subset $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^d$ is matrix convex if it is free and closed under isometric conjugation.

However, in many applications of interest, the situation is not truly (matrix) convex. We must allow for certain notions of *partial* convexity. For example, in [9] the authors considered convexity in some of the variables (the "unknown" variables) with the other variables unrestricted (the "known" variables.) Partial convexity has also been widely studied in the context of *biconvexity* (i.e. convex in each variable separately) and the closely related objects of bilinear matrix inequalities (BMIs). These are both special cases of our general framework for noncommutative partial convexity, Γ-convexity. LMIs and BMIs each correspond to the positivity set of a Γ-pencil for some Γ.

In Chapter 6, we prove Hahn-Banach Effros-Winkler separation results in both the matricial and operatorial Γ-convex settings. Of particular importance, we investigate the Γ-convexity of sets of the form $\{X : p(X) \succeq 0\}$ for a noncommutative polynomial $p \in \mathbb{C}\langle x \rangle$. The material found in Chapters 5 and 6 is primarily based on the joint work [12] published in the Journal of Geometric Analysis with M. Jury, I. Klep, S. McCullough, and J.E. Pascoe.

5.2 Γ-Convexity: A Framework for Noncommutative Partial Convexity

In this section, we formally define the notion of Γ-convexity and obtain foundational results along with a first version of the Effros-Winkler theorem for Γ-convexity. In later sections, Sections 5.4 and 6.1, we further investigate the hypotheses found in this first version and obtain several improvements in various directions by appealing to the operator setting.

We say a noncommutative polynomial p in d noncommuting variables is **symmetric** if $p(X)^* = p(X)$ for all $X \in \mathbb{S}(\mathbb{C})^d$. In this case, p determines a mapping $p : \mathbb{S}(\mathbb{C})^d \to \mathbb{S}(\mathbb{C})^1$. If $p_1, \ldots, p_r$ are symmetric polynomials in d noncommuting variables, then $p = (p_1, \ldots, p_r)$ determines a mapping $p : \mathbb{S}(\mathbb{C})^d \to \mathbb{S}(\mathbb{C})^r$.

Let $\Gamma = (\gamma_1, \ldots, \gamma_r)$ denote a tuple of symmetric free polynomials with $\gamma_j = x_j$ for $1 \leq j \leq d \leq r$. Also denote by $\Gamma : \mathbb{S}(\mathbb{C})^d \to \mathbb{S}(\mathbb{C})^r$ the resulting mapping (properly understood when $d = r$)

$$\Gamma(X) = (\gamma_1(X), \ldots, \gamma_r(X)) = (X_1, \ldots, X_d, \gamma_{d+1}(X), \ldots, \gamma_r(X)).$$

We now state the definition of Γ-convexity.

Definition 5.2.1. A pair (X, V), where $X \in \mathbb{S}_n(\mathbb{C})^d$ and $V : \mathbb{C}^m \to \mathbb{C}^n$ is an isometry, is called a **Γ-pair** provided

$$V^*\Gamma(X)V = \Gamma(V^*XV).$$

Let $\mathcal{C}_\Gamma$ denote the collection of Γ-pairs. A subset $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^d$ is a **Γ-convex set** if it is free and if $X \in \mathcal{K}$ and $(X, V) \in \mathcal{C}_\Gamma$ implies $V^*XV \in \mathcal{K}$.

If U is an $n \times n$ unitary matrix and $X \in \mathbb{S}_n(\mathbb{C})^d$, then (X, U) is a Γ-pair. In the special case that $r = d$, equivalently when $\Gamma(x) = x$, the notion of Γ-convexity reduces to ordinary matrix convexity. Our two primary examples of Γ-convexity, y^2-convexity and xy-convexity, are of particular interest in both applications and theory.

Example 5.2.1. For ease of notation, consider the case of two variables for now. We say a set is **y^2-convex** if it is Γ-convex for $\Gamma = \{x, y, y^2\}$. Since our variables are self-adjoint, it can be seen that a pair $((X, Y), V)$ is in $\mathcal{C}_\Gamma$ if and only if the range of the isometry V reduces Y. It is shown in Proposition 5.2.2 that a free set $\mathcal{K}$ that is closed with respect to restrictions to reducing subspaces is y^2-convex if and only if it is convex in X, i.e. $(X_1, Y), (X_2, Y) \in \mathcal{K}(n)$ implies $(\frac{X_1+X_2}{2}, Y) \in \mathcal{K}(n)$.

For a positive integer d, the "TV screen" $\mathrm{TV}^d = (\mathrm{TV}^d(n))_n$ defined by

$$\mathrm{TV}^d(n) = \{(X, Y) : I - X^2 - Y^{2d} \succeq 0\} \subset \mathbb{S}_n(\mathbb{C})^2 \tag{5.2.1}$$

is y^2-convex.

Example 5.2.2. Again, we consider the case of two variables. We say a set is ***xy*-convex** if it is Γ-convex for $\Gamma = \{x, y, xy + yx, i(xy - yx)\}$. The convexity in this example is intimately connected with BMIs. A set that is xy-convex is, in particular, both x^2-convex and y^2-convex, and hence, by Proposition 5.2.2, convex in each variable separately.

We only define Γ-convexity for *free* sets; this is the natural setting to make proper sense of Definition 5.2.1. Our next proposition says that Γ-convex sets are in fact closed under

Γ-convex combinations.

Proposition 5.2.1. *Suppose* $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^d$ *is* Γ*-convex. Let* $k, m, n_1, \ldots, n_k$ *be positive integers and, for* $1 \le j \le k$*, let* $X_j \in \mathcal{K}(n_j)$ *and* $V_j : \mathbb{C}^m \to \mathbb{C}^{n_j}$ *be linear maps. If* $\sum_{j=1}^k V_j^* V_j = I_m$ *and*

$$\sum_{j=1}^{k} V_j^* \Gamma(X_j) V_j = \Gamma\left(\sum_{j=1}^{k} V_j^* X_j V_j\right), \tag{5.2.2}$$

then $\sum_{j=1}^k V_j^* X_j V_j \in \mathcal{K}(m)$.

Proof. Since $\mathcal{K}$ is a free set, we have

$$X := \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_n \end{bmatrix} \in \mathcal{K}(n),$$

where $n = \sum_j n_j$. If we define $V : \mathbb{C}^m \to \mathbb{C}^n$ by $V := (V_1 \cdots V_k)^*$, then V is an isometry by hypothesis. By equation (5.2.2) and because Γ preserves direct sums,

$$V^* \Gamma(X) V = \sum_{j=1}^{k} V_j^* \Gamma(X_j) V_j = \Gamma\left(\sum_{j=1}^{k} V_j^* X_j V_j\right) = \Gamma(V^* X V).$$

Therefore, $(X, V) \in \mathcal{C}_\Gamma$. Since $\mathcal{K}$ is Γ-convex, this implies

$$V^* X V = \sum_{j=1}^{r} V_j^* X_j V_j \in \mathcal{K}(m),$$

which completes the proof. □

We now show that a free set $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^d \times \mathbb{S}(\mathbb{C})^h$, which is also closed with respect to restrictions to reducing subspaces is y^2-convex if and only if, for each n and $Y \in \mathbb{S}_n(\mathbb{C})^h$, the slice $\{X : (X, Y) \in \mathcal{K}(n)\}$ is convex in the ordinary sense. That is, convex in x. A free set $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^d \times \mathbb{S}(\mathbb{C})^h$ is y^2-convex if it is Γ-convex for $\Gamma = (x_1, \ldots, x_d, y_1, \ldots, y_h, y_1^2, \ldots, y_h^2)$.

Let us write elements Z of $\mathcal{K}(n)$ as $Z = (X, Y)$ with $X \in \mathbb{S}_n(\mathbb{C})^d$ and $Y \in \mathbb{S}_n(\mathbb{C})^h$.

Proposition 5.2.2. *A free set $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^d \times \mathbb{S}(\mathbb{C})^h$ that is y^2-convex is convex in x. If, in addition, $\mathcal{K}$ is closed with respect to restrictions to reducing subspaces, then the converse holds as well.*

Proof. First, suppose $\mathcal{K}$ is a y^2-convex set. Fix n and $E \in \mathbb{S}_n(\mathbb{C})^h$. Given $A, B \in \mathbb{S}_n(\mathbb{C})^d$ such that $(A, E), (B, E) \in \mathcal{K}(n)$, note that, since $\mathcal{K}$ is a free set,

$$(X, Y) := (A, E) \oplus (B, E) = \left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \right) \in \mathcal{K}(2n).$$

For $0 \leq t \leq 1$, $V = \begin{pmatrix} \sqrt{t} I_n & \sqrt{1-t} I_n \end{pmatrix}^*$ is an isometry satisfying $V^* Y^2 V = (V^* Y V)^2$. Hence, $((X, Y), V)$ is a y^2-pair. Since $\mathcal{K}$ is y^2-convex,

$$V^*(X, Y)V = (tA + (1-t)B, E) \in \mathcal{K}(n),$$

and $\mathcal{K}$ is convex in x.

Now suppose $\mathcal{K}$ is closed with respect to restrictions to reducing subspaces and is convex in x. To prove $\mathcal{K}$ is y^2-convex, suppose $(X, Y) \in \mathcal{K}(n+m)$ and $V : \mathbb{C}^n \to \mathbb{C}^{n+m}$ is an isometry such that $((X, Y), V)$ is a y^2-pair, so that $V^* Y^2 V = (V^* Y V)^2$. Thus the range of V reduces Y. Hence, with respect to the decomposition $\mathbb{C}^n \oplus \mathbb{C}^m$,

$$(X, Y) = \left(\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}, \begin{bmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{bmatrix} \right)$$

and $V^*(X,Y)V = (X_{11}, Y_{11})$. Letting U denote the unitary matrix,

$$U = \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix},$$

$U^*(X,Y)U \in \mathcal{K}(n+m)$ since free sets are closed under unitary conjugation. Since $\mathcal{K}$ is convex in x,

$$(X', Y') := \frac{1}{2}[U^*(X,Y)U + (X,Y)] = \left(\begin{bmatrix} X_{11} & 0 \\ 0 & X_{22} \end{bmatrix}, \begin{bmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{bmatrix} \right) \in \mathcal{K}(n+m).$$

Finally, since $\mathbb{C}^n \oplus \{0\}$ reduces (X', Y'), we must have $V^*(X,Y)V = (X_{11}, Y_{11}) \in \mathcal{K}(n)$ by hypothesis. Therefore, $\mathcal{K}$ is y^2-convex. $\square$

We turn our attention toward a first version of the Effros-Winkler separation theorem for Γ-convexity. In analogy with the main separating objects of linear pencils in the ordinary theory of matrix convexity, we make the following definition of a Γ-pencil.

Definition 5.2.2. A **Γ-pencil** is a symmetric matrix-valued polynomial of the form

$$L(x) = A_0 + \sum_{j=1}^{r} A_j \gamma_j(x),$$

where $(A_1, \ldots, A_r) \in \mathbb{S}_\mu(\mathbb{C})^r$. We refer to μ as the **size of the pencil** L. In the case $A_0 = I_\mu$, we say that L is **monic**.

Such a pencil L is naturally evaluated at a tuple $X \in \mathbb{S}_n(\mathbb{C})^d$ via the Kronecker tensor product:

$$L(X) = A_0 \otimes I_n + \sum_{j=1}^{r} A_j \otimes \gamma_j(X).$$

Example 5.2.3. In the case of two variables, a monic y^2-pencil is of the form

$$L(x,y) = I + A_x x + A_y y + By^2,$$

where A_x, A_y, and B are self-adjoint.

Similarly, a monic xy-pencil can be expressed in the form

$$L(x,y) = I + A_x x + A_y y + Bxy + B^* yx,$$

where A_x, A_y are self-adjoint.

The main result of this section is Theorem 5.2.1, giving an Effros-Winkler type separation result for Γ-convex sets via monic Γ-pencils. To state it, we briefly discuss convex hulls and establish some basic properties.

Definition 5.2.3. The **Γ-convex hull** of a free set $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^d$ is the intersection of all Γ-convex sets containing $\mathcal{K}$. It is denoted by $\Gamma\text{-matco}(\mathcal{K})$.

In the above definition, containment between two free sets is defined levelwise. When $r = d$, equivalently $\Gamma(x) = x$, $\Gamma\text{-matco}(\mathcal{K})$ is the ordinary matrix convex hull of $\mathcal{K}$, which we denote by $\text{matco}(\mathcal{K})$.

Proposition 5.2.3. *If $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^d$ is a free set, then*

$$\Gamma\text{-matco}(\mathcal{K}) = \{V^* X V : X \in \mathcal{K},\ (X,V) \in \mathcal{C}_\Gamma\}.$$

Proof. For ease of notation, we will let $\mathscr{K} := \{V^* X V : X \in \mathcal{K},\ (X,V) \in \mathcal{C}_\Gamma\}$. First note that $\mathscr{K}$ is a Γ-convex set that contains $\mathcal{K}$. By definition, $\Gamma\text{-matco}(\mathcal{K})$ must then be contained in $\mathscr{K}$. On the other hand, $\Gamma\text{-matco}(\mathcal{K})$ is immediately seen to contain $\mathscr{K}$. $\square$

Proposition 5.2.4. *Suppose $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^d$ is a free set. A tuple $X \in \mathbb{S}(\mathbb{C})^d$ is in* Γ-matco$(\mathcal{K})$ *if and only if* $\Gamma(X)$ *is in* matco$(\Gamma(\mathcal{K}))$. *Equivalently,*

$$\Gamma^{-1}(\text{matco}(\Gamma(\mathcal{K}))) = \Gamma\text{-matco}(\mathcal{K}).$$

Proof. First suppose $X \in \Gamma$-matco$(\mathcal{K})$. By Proposition 5.2.3, there exists a Y in $\mathcal{K}$ and an isometry V such that $V^*\Gamma(Y)V = \Gamma(V^*YV)$ and $X = V^*YV$. Thus, $\Gamma(X) = V^*\Gamma(Y)V$, and therefore $\Gamma(X) \in \text{matco}(\Gamma(\mathcal{K}))$.

Conversely, suppose $\Gamma(X) \in \text{matco}(\Gamma(\mathcal{K}))$. There is a $Y \in \mathcal{K}$ and an isometry V such that $\Gamma(X) = V^*\Gamma(Y)V$. Comparing the first d coordinates gives $X = V^*YV$ and hence (Y, V) is Γ-pair. Since $Y \in \mathcal{K}$ and $(Y, V) \in \mathcal{C}_\Gamma$, Proposition 5.2.3 implies $X \in \Gamma$-matco$(\mathcal{K})$. □

A free set $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^d$ is **closed** if it is closed at each level. That is $\mathcal{K}(n)$ is a closed subset of $\mathbb{S}_n(\mathbb{C})^d$ for all n. Given a free subset $\mathcal{S} \subset \mathbb{S}(\mathbb{C})^d$, its (levelwise) closed matrix convex hull is denoted by $\overline{\text{matco}}(\mathcal{S})$. In other words, $\overline{\text{matco}}(\mathcal{S})(n)$ is the closure of matco$(\mathcal{S})(n)$ in $\mathbb{S}_n(\mathbb{C})^d$ for every n. A routine argument shows $\overline{\text{matco}}(\mathcal{S})$ is matrix convex.

Theorem 5.2.1. *Suppose $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^d$ is Γ-convex and $0 \in \overline{\text{matco}}(\Gamma(\mathcal{K}))$. If $Y \in \mathbb{S}_\ell(\mathbb{C})^d$ and $\Gamma(Y) \notin \overline{\text{matco}}(\Gamma(\mathcal{K}))(\ell)$, then there exists a monic Γ-pencil L of size ℓ such that $L(X) \succ 0$ for all $X \in \mathcal{K}$, but $L(Y) \not\succeq 0$. In particular, if* matco$(\Gamma(\mathcal{K}))$ *is closed, then for each $Y \in \mathbb{S}_\ell(\mathbb{C})^d \setminus \mathcal{K}(\ell)$ there exists a monic Γ-pencil L of size ℓ such that $L(\mathcal{K}) \succ 0$, but $L(Y) \not\succeq 0$.*

Proof. Since $\Gamma(Y) \notin \overline{\text{matco}}(\Gamma(\mathcal{K}))(\ell)$ and $\overline{\text{matco}}(\Gamma(\mathcal{K}))$ is a closed matrix convex subset of $\mathbb{S}(\mathbb{C})^r$ containing 0, Theorem 1.0.3 implies there is a monic linear pencil

$$M = I_\ell + \sum_{j=1}^{r} A_j z_j$$

of size ℓ such that $M(Z) \succeq 0$ for all $Z \in \overline{\text{matco}}(\Gamma(\mathcal{K}))$, but $M(\Gamma(Y)) \not\succeq 0$. Thus $L' := M \circ \Gamma$ is a monic Γ-pencil of size ℓ that is indefinite at Y and positive semidefinite on $\mathcal{K}$. Replacing

L' by

$$L := tI_\ell + (1-t)L' = I_\ell + \sum_{j=1}^{r}(1-t)A_j\gamma_j$$

for some $t \in (0,1)$ produces a monic Γ-pencil of size ℓ that is indefinite at Y such that $L(X) \succ 0$ for every $X \in \mathcal{K}$.

To complete the proof, suppose $\operatorname{matco}(\Gamma(\mathcal{K}))$ is closed and $Y \notin \mathcal{K} = \Gamma\text{-matco}(\mathcal{K})$. Since, by Proposition 5.2.4, $\Gamma(Y) \notin \operatorname{matco}(\Gamma(\mathcal{K})) = \overline{\operatorname{matco}}(\Gamma(\mathcal{K}))$ the existence of L follows from what has already been proved. □

Theorem 5.2.1 is a natural generalization of the Effros-Winkler separation theorem, Theorem 1.0.3, for Γ-convex sets. However, we consider it a first version because, on the level of matrices, we are unable to eliminate the hypothesis that $\operatorname{matco}(\Gamma(\mathcal{K}))$ is closed. That is, on matrices, the geometry is such that it is unclear to the author (and probably false) if $\operatorname{matco}(\Gamma(\mathcal{K}))$ is closed whenever $\mathcal{K}$ is closed. To remedy this difficulty, we appeal to the operatorial setting, as is done in Section 5.4. We obtain several results in the context of operator Γ-convexity. These operatorial results then provide insights in matricial Γ-convexity that circumvent the closedness issue.

We conclude this section with a result describing when 0 lies in the *interior* of the matrix convex hull of $\Gamma(\mathcal{K})$.

Theorem 5.2.2. *Suppose $\mathcal{K} \subset \mathbb{S}(\mathbb{C})^d$ is free and $0 \in \operatorname{matco}(\Gamma(\mathcal{K}))$. If the real span of $\{\gamma_j : 1 \le j \le r\}$ contains a non-zero polynomial $q \in \mathbb{C}\langle x_1, \dots, x_d\rangle$ such that $q(X) \succeq 0$ for every $X \in \mathcal{K}$, then 0 is not in the interior of $\operatorname{matco}(\Gamma(\mathcal{K}))(1) \subset \mathbb{R}^r$. If $\{\gamma_j : 1 \le j \le r\}$ is linearly independent over $\mathbb{R}$, then the converse holds.*

Proof. To prove the first statement, suppose 0 is in the interior of $\operatorname{matco}(\Gamma(\mathcal{K}))(1)$ and $q \neq 0$ is in the real span of $\{\gamma_j : 1 \le j \le r\}$. Thus there is a $\lambda \in \mathbb{R}^r \setminus \{0\}$ such that $q = \sum_{j=1}^r \lambda_j\gamma_j$. View $\lambda : \mathbb{R}^r \to \mathbb{R}$ as the linear map $\lambda(z) = \sum_{j=1}^r \lambda_j z_j$. Since $\lambda \neq 0$ and 0 is in the interior of $\operatorname{matco}(\Gamma(\mathcal{K}))(1)$, there exists $y \in \operatorname{matco}(\Gamma(\mathcal{K}))(1)$ such that $\lambda(y) < 0$. Since

$y \in \mathrm{matco}(\Gamma(\mathcal{K}))(1)$, by Proposition 5.2.3, there exists a positive integer n, a vector $h \in \mathbb{C}^n$, and $Y \in \mathcal{K}(n)$ such that $y = h^*\Gamma(Y)h$. We observe

$$\begin{aligned} \lambda(y) =& \sum_{j=1}^{r} \lambda_j y_j = \sum_{j=1}^{r} \lambda_j h^* \gamma_j(Y) h \\ =& h^* \left[\sum_{j=1}^{r} \lambda_j \gamma_j(Y) \right] h = h^* q(Y) h. \end{aligned} \tag{5.2.3}$$

Therefore, $0 > \lambda(y) = h^*q(Y)h$ and $q(Y) \not\succeq 0$.

Next, suppose $\{\gamma_j : 1 \leq j \leq r\}$ is linearly independent over $\mathbb{R}$ and that 0 is not in the interior of $\mathrm{matco}(\Gamma(\mathcal{K}))(1)$. Then 0 must be in the boundary of $\mathrm{matco}(\Gamma(\mathcal{K}))(1)$, since $0 \in \mathrm{matco}(\Gamma(\mathcal{K}))$. Hence, as $\mathrm{matco}(\Gamma(\mathcal{K}))(1)$ is a convex subset of $\mathbb{R}^r$, there exists a non-zero linear functional $\lambda : \mathbb{R}^r \to \mathbb{R}$ such that λ is nonnegative on $\mathrm{matco}(\Gamma(\mathcal{K}))(1)$. There are $\lambda_j \in \mathbb{R}$, not all zero, so that $\lambda(z) = \sum_{j=1}^{r} \lambda_j z_j$. Let $q \in \mathbb{C}\langle x_1, \ldots, x_d \rangle$ be the polynomial defined by $q := \sum_{j=1}^{r} \lambda_j \gamma_j$. It is a non-zero element of the real span of $\{\gamma_j : 1 \leq j \leq r\}$ by the real linear independence hypothesis.

If $Y \in \mathcal{K}(n)$ and $h \in \mathbb{C}^n$ is any unit vector, then by Proposition 5.2.3, $y := h^*\Gamma(Y)h \in \mathrm{matco}(\Gamma(\mathcal{K}))(1)$. Thus, since λ is nonnegative on $\mathrm{matco}(\Gamma(\mathcal{K}))(1)$, equation (5.2.3) implies

$$0 \leq \lambda(y) = h^* q(Y) h.$$

It follows that $q(Y) \succeq 0$ and the proof is complete. □

5.3 Γ-Convex Polynomials and Free Semialgebraic Sets

Given a symmetric polynomial $p \in \mathbb{C}\langle x \rangle$ with $p(0) \succ 0$ and a positive integer n, we consider the three sets

$$\begin{aligned}
\widehat{\mathcal{D}}_p(n) &= \{X \in \mathbb{S}_n(\mathbb{C})^d : p(X) \succeq 0\}; \\
\mathcal{P}_p(n) &= \{X \in \mathbb{S}_n(\mathbb{C})^d : p(X) \succ 0\}; \\
\mathcal{D}_p(n) &= \overline{\mathcal{P}_p(n)}.
\end{aligned}$$

Form the free sets $\widehat{\mathcal{D}}_p = (\widehat{\mathcal{D}}_p(n))_n$, $\mathcal{P}_p = (\mathcal{P}_p(n))_n$, and $\mathcal{D}_p = (\mathcal{D}_p(n))_n$. The sets $\widehat{\mathcal{D}}_p$, $\mathcal{D}_p$, and $\mathcal{P}_p$ are free analogs of basic semialgebraic sets. We refer to all of these (possibly distinct) sets as **free semialgebraic sets**. As an example, the "TV screen" sets TV^d are free semialgebraic and of the form $\widehat{\mathcal{D}}_p$ for $p = 1 - x^2 - y^{2d}$.

Free semialgebraic sets that have additional geometric properties, such as being star-like, satisfy a cleaner version of Theorem 5.2.1. This is Theorem 6.3.1. One of our goals, along the lines of [11], is to develop constrained simple representations of semialgebraic sets with certain geometric properties. That is, represent them as a positivity set of a Γ-pencil.

We conclude this chapter with a brief discussion of Γ-convexity for noncommutative polynomials. In analogy with Definition 5.2.1 and the classical notion of matrix convex polynomials, we make the following definition.

Definition 5.3.1. A symmetric polynomial $p \in \mathbb{C}\langle x\rangle$ is **Γ-convex** if whenever $X \in \mathbb{S}(\mathbb{C})^d$ and V is an isometry such that $(X, V) \in \mathcal{C}_\Gamma$, then

$$V^* p(X) V - p(V^* X V) \succeq 0.$$

We say p is **Γ-concave** if $-p$ is Γ-convex.

We observe in Proposition 5.3.1 that Γ-concavity of p implies Γ-convexity of its associated free semialgebraic sets.

Proposition 5.3.1. *If $p \in \mathbb{C}\langle x\rangle$ is a Γ-concave polynomial, then $\widehat{\mathcal{D}}_p$ and $\mathcal{P}_p$ are Γ-convex.*

Proof. If $X \in \widehat{\mathcal{D}}_p$ and (X, V) is a Γ-pair, then $V^*p(X)V \succeq 0$. Since p is Γ-concave,

$$p(V^*XV) \succeq V^*p(X)V \succeq 0.$$

Therefore $V^*XV \in \widehat{\mathcal{D}}_p$ and hence $\widehat{\mathcal{D}}_p$ is Γ-convex. A similar argument shows that the strict positivity set $\mathcal{P}_p$ is also Γ-convex. □

5.4 Operator Γ-Convexity and Shift Forms

As discussed after the proof of Theorem 5.2.1, we wish to develop an analogous theory of Γ-convexity in the setting of operator tuples. In the operatorial setting, we will obtain Effros-Winkler separation results, such as Theorems 6.1.2, 6.2.1, and 6.2.1 from Chapter 6, for SOT-closed, bounded, Γ-convex sets $\mathfrak{K} \subset B(\mathcal{H})^d_{sa}$ without requiring a closedness hypothesis on the (operator) convex hull of $\Gamma(\mathfrak{K})$; it will be closed in the strong operator topology automatically by Theorem 5.4.1. The proof of this theorem relies on the shift-form construction from Section 4.1 and its associated operator noncommutative compactness-type principle, Theorem 4.1.1.

Fix a complex infinite dimensional separable Hilbert space $\mathcal{H}$ and let $B(\mathcal{H})^d_{sa}$ denote the d-tuples of self-adjoint bounded operators on $\mathcal{H}$. As we did previously with $B(\mathcal{H})^d$, equip $B(\mathcal{H})^d_{sa}$ with the maximum norm $\|X\| = \max\{\|X_1\|, \ldots, \|X_d\|\}$.

Definition 5.4.1. A subset $\mathfrak{K} \subset B(\mathcal{H})^d$ is a **free set** if it is closed under unitary conjugation and closed under direct sums in the sense that if X and Y are in $\mathfrak{K}$, there is a unitary $U : \mathcal{H} \to \mathcal{H}^{(2)}$ such that

$$U^{-1} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} U \in \mathfrak{K}.$$

We note for emphasis that this definition of a free set of operator tuples differs from the one given in Definition 2.1.1 of an operator NC domain. Indeed, an NC domain is open in norm, while we will primarily be considering free sets that are SOT-closed. Furthermore, for a free set, we do not require the notion of closure under countably infinite direct sums that is built into the definition of NC domain.

In analogy with the matricial theory of Γ-convexity from Section 5.2, let $\Gamma = (\gamma_1, \ldots, \gamma_r)$ denote a tuple of symmetric polynomials in d noncommuting variables with $\gamma_j = x_j$ for $1 \le j \le d \le r$. Also let $\Gamma : B(\mathcal{H})^d_{sa} \to B(\mathcal{H})^r_{sa}$ denote the resulting mapping on self-adjoint operator tuples.

Definition 5.4.2. A pair (X, V) is called a **Γ-pair** provided $X \in B(\mathcal{H})^d_{sa}$, $V : \mathcal{H} \to \mathcal{H}$ is an isometry, and

$$V^*\Gamma(X)V = \Gamma(V^*XV).$$

Let $\mathfrak{C}_\Gamma$ denote the collection of (operator) Γ-pairs.

Definition 5.4.3. A free set $\mathfrak{K} \subset B(\mathcal{H})^d_{sa}$ is called **operator convex** if whenever $X \in \mathfrak{K}$ and $V : \mathcal{H} \to \mathcal{H}$ is an isometry, then $V^*XV \in \mathfrak{K}$. It is called **operator Γ-convex** if $X \in \mathfrak{K}$ and $(X, V) \in \mathfrak{C}_\Gamma$ implies $V^*XV \in \mathfrak{K}$.

The **operator Γ-convex hull** of a free set $\mathfrak{K} \subset B(\mathcal{H})^d_{sa}$ is the intersection of all operator Γ-convex sets containing $\mathfrak{K}$ and is denoted $\Gamma\text{-opco}(\mathfrak{K})$.

In the special case that $r = d$, equivalently $\Gamma(x) = x$, Γ-convexity reduces to ordinary operator convexity. The notions of operator y^2-convexity and operator xy-convexity may be defined in analogy with their matricial definitions. Propositions 5.2.3 and 5.2.4 have analogues in the context of operator Γ-convexity.

Proposition 5.4.1. *If $\mathfrak{K} \subset B(\mathcal{H})^d_{sa}$ is a free set, then*

$$\Gamma\text{-opco}(\mathfrak{K}) = \{V^*XV : X \in \mathfrak{K},\ (X, V) \in \mathfrak{C}_\Gamma\}$$

and

$$\Gamma^{-1}(\operatorname{opco}(\Gamma(\mathfrak{K}))) = \Gamma\text{-opco}(\mathfrak{K}).$$

We now use the shift form construction and Theorem 4.1.1 to show that the operator Γ-convex hull of a bounded SOT-closed free set is again SOT-closed, eliminating the difficulties discussed at the end of Section 5.2 in the matricial setting.

Theorem 5.4.1. *Suppose $\mathfrak{K} \subset B(\mathcal{H})^d_{sa}$ is a free set. If $\mathfrak{K}$ is bounded and SOT-closed, then* $\operatorname{opco}(\Gamma(\mathfrak{K}))$ *and* $\Gamma\text{-opco}(\mathfrak{K})$ *are SOT-closed.*

Proof. By the second set equality in Proposition 5.4.1, it suffices to show $\operatorname{opco}(\Gamma(\mathfrak{K}))$ is SOT-closed, since Γ is SOT-continuous on bounded sets because it is a noncommutative polynomial mapping.

Suppose Y is in the SOT-closure of $\operatorname{opco}(\Gamma(\mathfrak{K}))$. Because $\mathfrak{K}$ is bounded, there exist isometries $V_n : \mathcal{H} \to \mathcal{H}$ and $X_n \in \mathfrak{K}$ such that $V_n^*\Gamma(X_n)V_n \overset{SOT}{\to} Y$. By Theorem 4.1.1 applied to $(V_n, X_n) \in B(\mathcal{H})^{d+1}$, there exist unitaries U_n such that, after passing to a subsequence, $U_n^*V_nU_n \overset{SOT}{\to} V$, $U_n^*X_nU_n \overset{SOT}{\to} X$, and $U_n^* \overset{SOT}{\to} W$, where V, W are isometries, and $X \in \mathfrak{K}$ since $\mathfrak{K}$ is free and SOT-closed.

Since $U_n^*V_n = (U_n^*V_nU_n)U_n^*$, we have $U_n^*V_n \overset{SOT}{\to} VW$ and therefore $V_n^*U_n \overset{WOT}{\to} W^*V^*$. Moreover, $\Gamma(U_n^*X_nU_n)U_n^*V_n \overset{SOT}{\to} \Gamma(X)VW$ since multiplication is SOT-continuous on bounded sets. Note that if $S_n \overset{WOT}{\to} S$ and $T_n \overset{SOT}{\to} T$, then $S_nT_n \overset{WOT}{\to} ST$. Hence,

$$\begin{aligned}
Y &= \text{SOT-lim}_n\, V_n^*\Gamma(X_n)V_n \\
&= \text{SOT-lim}_n\, V_n^*U_n\Gamma(U_n^*X_nU_n)U_n^*V_n \\
&= \text{WOT-lim}_n\, V_n^*U_n[\Gamma(U_n^*X_nU_n)U_n^*V_n] \\
&= W^*V^*[\Gamma(X)VW] = (VW)^*\Gamma(X)VW.
\end{aligned}$$

Therefore, $Y = (VW)^*\Gamma(X)VW \in \operatorname{opco}(\Gamma(\mathfrak{K}))$, since VW is an isometry. □

In the context of Theorem 5.4.1, as $\mathrm{opco}(\Gamma(\mathfrak{K}))$ is convex and SOT-closed, it is also WOT-closed. The proof that $\mathrm{opco}(\Gamma(\mathfrak{K}))$ is SOT-closed shows in fact that if $P : B(\mathcal{H})_{sa}^d \to B(\mathcal{H})_{sa}^r$ is any noncommutative polynomial mapping, then $\mathrm{opco}(P(\mathfrak{K}))$ is SOT-closed. With Theorem 5.4.1 established, the main technical issue in proving separation results for Γ-convexity is resolved.

Chapter 6

Separation via Monic Γ-Pencils

6.1 Hahn-Banach Effros-Winkler Separation Theorem for Operator Γ-Convexity

Our first main goal in this chapter is to establish an Effros-Winkler theorem for SOT-closed, bounded, operator Γ-convex sets. This uses the automatic closedness result in Theorem 5.4.1. We begin by treating the ordinary operator convex case (i.e. $\Gamma(x) = x$) and show that a version of the Effros-Winkler theorem holds for bounded, SOT-closed, operator convex subsets of $B(\mathcal{H})^d_{sa}$.

For this, we need to state what is meant by evaluating a pencil at an operator tuple. Given a monic Γ-pencil $L = I_N + \sum A_j\gamma_j$ of (finite) size N, or an operator Γ-pencil $\widetilde{L} = I_{\widetilde{\mathcal{H}}} + \sum B_j\gamma_j$ for a separable infinite dimensional Hilbert space $\widetilde{\mathcal{H}}$ and $B_j \in B(\widetilde{\mathcal{H}})_{sa}$, its **evaluation on operator tuples** is defined as

$$L(X) = I_N \otimes I_{\mathcal{H}} + \sum A_j \otimes \gamma_j(X)$$

or

$$\widetilde{L}(X) = I_{\widetilde{\mathcal{H}}} \otimes I_{\mathcal{H}} + \sum B_j \otimes \gamma_j(X)$$

respectively for $X \in B(\mathcal{H})^d_{sa}$.

Given a symmetric noncommutative polynomial $p \in \mathbb{C}\langle x\rangle$, in addition to $\widehat{\mathcal{D}}_p$, $\mathcal{P}_p$, and $\mathcal{D}_p$

defined in Section 5.3, we consider the following sets describing its positivity on the operator level:

$$\begin{aligned} \widehat{\mathfrak{D}}_p &= \{X \in B(\mathcal{H})^d_{sa} : p(X) \succeq 0\}; \\ \mathfrak{P}_p &= \{X \in B(\mathcal{H})^d_{sa} : p(X) \succ 0\}; \\ \mathfrak{D}_p &= \overline{\mathfrak{P}_p}^{SOT}. \end{aligned}$$

We also refer to all of these (possibly distinct) sets as **free semialgebraic sets**. Observe that $\mathfrak{D}_p$ and $\widehat{\mathfrak{D}}_p$ are SOT-closed when they are bounded, so Theorem 5.4.1 applies. We may define similar sets associated to a monic Γ-pencil:

$$\begin{aligned} \widehat{\mathfrak{D}}_L &= \{X \in B(\mathcal{H})^d_{sa} : L(X) \succeq 0\}; \\ \mathfrak{P}_L &= \{X \in B(\mathcal{H})^d_{sa} : L(X) \succ 0\}; \\ \mathfrak{D}_L &= \overline{\mathfrak{P}_L}^{SOT}. \end{aligned}$$

We need a lemma which says an operator convex set containing 0 is closed under conjugation by contractions.

Lemma 6.1.1. *If $\mathfrak{K} \subset B(\mathcal{H})^d_{sa}$ is operator convex and contains 0, then $\mathfrak{K}$ is closed under conjugation by contractions: if $X \in \mathfrak{K}$ and $C \in B(\mathcal{H})$ is such that $\|C\| \leq 1$, then $C^*XC \in \mathfrak{K}$.*

Proof. If $X \in \mathfrak{K}$, the hypotheses imply there is a unitary $U : \mathcal{H} \to \mathcal{H}^{(2)}$ such that

$$Z := U^{-1} \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} U$$

is in $\mathfrak{K}$. For a contraction C, define the isometries

$$V := \begin{bmatrix} C \\ (1 - C^*C)^{1/2} \end{bmatrix}$$

and $W := U^{-1}V$. It is readily verified that $C^*XC = W^*ZW$, which is in $\mathfrak{K}$ because $\mathfrak{K}$ is assumed operator convex. $\square$

Theorem 6.1.1 is our main result on operator convex sets.

Theorem 6.1.1. *Suppose $\mathfrak{K} \subset B(\mathcal{H})^d_{sa}$ is operator convex, SOT-closed, bounded, and contains 0. If $Y \notin \mathfrak{K}$, then there is a positive integer N and a monic linear pencil*

$$L = I_N + \sum_{j=1}^{d} A_j x_j,$$

where $A_j \in \mathbb{S}_N(\mathbb{C})$ such that $L(\mathfrak{K}) \succeq 0$, but $L(Y) \not\succeq 0$.

In particular, $\mathfrak{K} = \cap \widehat{\mathfrak{D}}_L$, where the intersection is over (finite) monic linear pencils L such that $\widehat{\mathfrak{D}}_L \supset \mathfrak{K}$.

Finally, if ℓ is a positive integer, $\mathscr{F}$ is an ℓ dimensional subspace of $\mathcal{H}$, and $Y = Y' \oplus 0$ with respect to the orthogonal decomposition $\mathscr{F} \oplus \mathscr{F}^\perp$ of $\mathcal{H}$, then L can be chosen to have size ℓ.

We record for convenience some elementary observations about the preservation of positivity of evaluations of monic linear pencils in Lemma 6.1.2.

Lemma 6.1.2. *Suppose $L = I_N + \sum A_j x_j$ is a finite monic linear pencil, n is a positive integer, $V : \mathbb{C}^n \to \mathcal{H}$ is an isometry, and $Y \in B(\mathcal{H})^d_{sa}$. Then the following hold:*

(i) if $L(VV^*YVV^*) \succeq 0$, then $L(V^*YV) \succeq 0$;

(ii) if $L(V^*YV) \succeq 0$, then $L(VV^*YVV^*) \succeq 0$;

(iii) if $L(Y) \succeq 0$, then $L(V^*YV) \succeq 0$.

Proof. Since V is an isometry, we have

$$(I_N \otimes V)^* L(VV^* Y VV^*)(I_N \otimes V) = L(V^*YV).$$

Item (i) now follows.

To prove item (ii), note that if $L(V^*YV) \succeq 0$, then

$$L(VV^* YVV^*) = (I_N \otimes V)L(V^*YV)(I_N \otimes V)^* + I_N \otimes (I_{\mathcal{H}} - VV^*) \succeq 0.$$

Item (iii) follows from observing that $(I_N \otimes V)^* L(Y)(I_N \otimes V) = L(V^*YV)$. $\square$

The proof of Theorem 6.1.1 uses the following lemma, whose proof in turn relies on Lemma 6.1.2. Given a free set $\mathfrak{K} \subset B(\mathcal{H})^d_{sa}$ and a positive integer n, let $\mathfrak{K}_{mat}(n)$ denote the set of matrix tuples $X \in \mathbb{S}_n(\mathbb{C})^d$ of the form V^*YV, where $V : \mathbb{C}^n \to \mathcal{H}$ is an isometry and $Y \in \mathfrak{K}$.

Lemma 6.1.3. *If $\mathfrak{K} \subset B(\mathcal{H})^d_{sa}$ is operator convex, SOT-closed, bounded, and contains 0, then*

(i) $\mathfrak{K}_{\mathrm{mat}}$ is a (levelwise) closed matrix convex set containing 0;

(ii) if $V : \mathbb{C}^n \to \mathcal{H}$ is an isometry, $Y \in B(\mathcal{H})^d_{sa}$, and $V^*YV \in \mathfrak{K}_{\mathrm{mat}}$, then $VV^* Y VV^* \in \mathfrak{K}$;

(iii) if $Y \notin \mathfrak{K}$, then there is a positive integer N and an isometry $V : \mathbb{C}^N \to \mathcal{H}$ such that $V^*YV \notin \mathfrak{K}_{\mathrm{mat}}(N)$ and for Y in the last statement of Theorem 6.1.1 N can be chosen to be ℓ;

(iv) if L is a (finite) monic linear pencil, then $L(\mathfrak{K}) \succeq 0$ if and only if $L(\mathfrak{K}_{\mathrm{mat}}) \succeq 0$.

Proof. Suppose $X_j \in \mathfrak{K}_{mat}(n_j)$ for $j = 1, 2$. Thus there exist isometries $V_j : \mathbb{C}^{n_j} \to \mathcal{H}$ and $Y_j \in \mathfrak{K}$ such that $X_j = V_j^* Y_j V_j$. Let $n := n_1 + n_2$, define $V : \mathbb{C}^n = \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2} \to \mathcal{H} \oplus \mathcal{H}$ by $V := V_1 \oplus V_2$, and let $Y := Y_1 \oplus Y_2$. Hence V is an isometry and $X := X_1 \oplus X_2 = V^* Y V$. Since $\mathfrak{K}$ is a free set as in Definition 5.4.1, there is a unitary $U : \mathcal{H} \to \mathcal{H}^{(2)}$ such that $Z := U^{-1} Y U \in \mathfrak{K}$. Now $W := U^{-1} V : \mathbb{C}^n \to \mathcal{H}$ is an isometry and $X = W^* Z W$. Hence $\mathfrak{K}_{mat}$ is closed under direct sums. By construction $\mathfrak{K}_{mat}$ is closed under conjugation by isometries. Hence $\mathfrak{K}_{mat}$ is matrix convex, and it contains 0 since $\mathfrak{K}$ does.

Fix a positive integer n. We want to prove that $\mathfrak{K}_{mat}(n) \subset \mathbb{S}_n(\mathbb{C})^d$ is closed. Take a sequence (S_j) in $\mathfrak{K}_{mat}(n)$ converging to S. There exists $T_j \in \mathfrak{K}$ and isometries $V_j : \mathbb{C}^n \to \mathcal{H}$ such that $S_j = V_j^* T_j V_j$. Fix an isometry $V : \mathbb{C}^n \to \mathcal{H}$. For each j, there is a unitary $U_j : \mathcal{H} \to \mathcal{H}$ such that $U_j V_j = V$. Thus

$$S_j = (U_j V_j)^* U_j T_j U_j^* (U_j V_j) = V^* W_j V,$$

where $W_j = U_j T_j U_j^*$. Since $\mathfrak{K}$ is a free set and $T_j \in \mathfrak{K}$, it follows that $W_j \in \mathfrak{K}$. As $\mathfrak{K}$ is bounded, (W_j) converges WOT along a subsequence to $W \in B(\mathcal{H})_{sa}^d$. But SOT-closed, convex sets are WOT-closed, so $W \in \mathfrak{K}$. It follows that $S_j = V^* W_j V$ converges along a subsequence to $V^* W V$ in $\mathbb{S}_n(\mathbb{C})^d$ and thus $S = V^* W V \in \mathfrak{K}_{mat}(n)$. Hence $\mathfrak{K}_{mat}(n)$ is closed and the proof of item (i) is complete.

To prove item (ii), since $V^* Y V \in \mathfrak{K}_{mat}(n)$, there is an isometry $W : \mathbb{C}^n \to \mathcal{H}$ and $Z \in \mathfrak{K}$ such that $V^* Y V = W^* Z W$. Hence $V V^* Y V V^* = (W V^*)^* Z (W V^*)$. As $W V^*$ is a contraction, by Lemma 6.1.1 we conclude $V V^* Y V V^* \in \mathfrak{K}$.

Choose an increasing sequence (P_n) of projections that SOT-converges to the identity on $\mathcal{H}$ such that P_n has rank n. To prove item (iii), suppose $Y \notin \mathfrak{K}$. Since $P_n Y P_n$ converges SOT to Y and $\mathfrak{K}$ is SOT-closed by hypothesis, there is an N such that $P_N Y P_N \notin \mathfrak{K}$. There is an isometry $V : \mathbb{C}^N \to \mathcal{H}$ such that $P_N = V V^*$. By the contrapositive of item (ii),

$V^*YV \notin \mathfrak{K}_{mat}(N)$. If Y is as in the last statement of Theorem 6.1.1, choose the sequence (P_n) such that P_ℓ is the projection onto $\mathscr{F}$. Since $P_nYP_n = Y$ for all $n \geq \ell$, it follows that $P_\ell Y P_\ell \notin \mathfrak{K}$ and hence N can be taken to be ℓ.

To prove item (iv), first suppose L is a monic linear pencil and $L(\mathfrak{K}_{mat}) \succeq 0$. We continue to let (P_n) denote an increasing sequence of projections that SOT-converges to the identity. For each n, there is an isometry $V_n : \mathbb{C}^n \to \mathcal{H}$ such that $V_nV_n^* = P_n$. Further, if $X \in \mathfrak{K}$, then $V_n^*XV_n \in \mathfrak{K}_{mat}$. Hence $L(V_n^*XV_n) \succeq 0$ for all n and therefore, by Lemma 6.1.2(ii), $L(P_nXP_n) \succeq 0$. Since $L(P_nXP_n)$ converges SOT to $L(X)$, it follows that $L(X) \succeq 0$. Thus $L(\mathfrak{K}) \succeq 0$.

Conversely suppose $L(\mathfrak{K}_{mat}) \not\succeq 0$. Thus there is an n an isometry $V : \mathbb{C}^n \to \mathcal{H}$ and $X \in \mathfrak{K}$ such that $L(V^*XV) \not\succeq 0$. By Lemma 6.1.2(i), $L((VV^*)X(VV^*)) \not\succeq 0$ and, by Lemma 6.1.1, $(VV^*)X(VV^*) \in \mathfrak{K}$. Hence $L(\mathfrak{K}) \not\succeq 0$, which concludes the proof. □

Proof of Theorem 6.1.1. By Lemma 6.1.3(iii), there is an N and an isometry $V : \mathbb{C}^N \to \mathcal{H}$ such that $V^*YV \notin \mathfrak{K}_{mat}(N)$. Further, if Y is as in the last part of the theorem, N can be chosen to be ℓ. By Lemma 6.1.3(i), $\mathfrak{K}_{mat}$ is closed, matrix convex, and contains 0. Hence, by the Effros-Winkler theorem, Theorem 1.0.3, there is a monic linear pencil of size N such that $L(\mathfrak{K}_{mat}) \succeq 0$, but $L(V^*YV) \not\succeq 0$. By Lemma 6.1.3(iv), $L(\mathfrak{K}) \succeq 0$, and by the contrapositive of Lemma 6.1.2(iii), $L(Y) \not\succeq 0$. □

Combining Theorems 5.4.1 and 6.1.1 yields Theorem 6.1.2 below - the main result of this section. It may be seen as an improvement of Theorem 5.2.1 for bounded, SOT-closed operator Γ-convex sets since it does not require that the (operator) convex hull of $\Gamma(\mathfrak{K})$ be closed a priori.

Theorem 6.1.2. *Suppose $\mathfrak{K} \subset B(\mathcal{H})^d_{sa}$ is SOT-closed, operator Γ-convex, bounded, and that*

$0 \in \operatorname{opco}(\Gamma(\mathfrak{K}))$. *Then for each* $Y \notin \mathfrak{K}$, *there is a positive integer* N *and a monic linear pencil*

$$M = I_N + \sum_{j=1}^{r} A_j z_j,$$

where $A_j \in \mathbb{S}_N(\mathbb{C})$, *such that* $M(\operatorname{opco}(\Gamma(\mathfrak{K}))) \succeq 0$, *but* $M(\Gamma(Y)) \not\succeq 0$. *Thus the monic* Γ*-pencil*

$$L = M \circ \Gamma = I_N + \sum_{j=1}^{r} A_j \gamma_j$$

is positive semidefinite on $\mathfrak{K}$, *but* $L(Y) \not\succeq 0$.

In particular, $\mathfrak{K} = \cap \widehat{\mathfrak{D}}_L$ *where the intersection is over all (finite) monic* Γ*-pencils* L *such that* $\widehat{\mathfrak{D}}_L \supset \mathfrak{K}$.

Finally, suppose $0 \in \mathfrak{K}$ *and that* $\Gamma(0) = 0$. *If* ℓ *is a positive integer,* $\mathscr{F}$ *is an* ℓ *dimensional subspace of* $\mathcal{H}$, *and* $Y = Y' \oplus 0$ *with respect to the orthogonal decomposition* $\mathscr{F} \oplus \mathscr{F}^{\perp}$ *of* $\mathcal{H}$, *then* L *and* M *can be chosen to have size* ℓ.

Proof. The assumptions imply $\operatorname{opco}(\Gamma(\mathfrak{K}))$ is a bounded, operator convex set containing 0. It is SOT-closed by Theorem 5.4.1. If $Y \notin \mathfrak{K} = \Gamma\text{-opco}(\mathfrak{K})$, then by the set equality in Proposition 5.4.1, $\Gamma(Y) \notin \operatorname{opco}(\Gamma(\mathfrak{K}))$. Therefore, by Theorem 6.1.1, there exists a positive integer N and a monic linear pencil

$$M = I_N + \sum_{j=1}^{r} A_j z_j$$

with self-adjoint coefficients $A_j \in \mathbb{S}_N(\mathbb{C})$ such that $M \succeq 0$ on $\operatorname{opco}(\Gamma(\mathfrak{K}))$, but $M(\Gamma(Y)) \not\succeq 0$.

In the case of the final assertion, we have $\Gamma(Y) = \Gamma(Y') \oplus \Gamma(0) = \Gamma(Y') \oplus 0$, where $\Gamma(Y')$ has size ℓ, and $\Gamma(Y) \notin \operatorname{opco}(\Gamma(\mathfrak{K}))$. Thus, by the corresponding assertion in Theorem 6.1.1, M, and hence L, can be chosen to have size ℓ. □

In Theorems 6.1.1 and 6.1.2, while the convex sets consist of operator tuples, outliers are separated from these sets via a monic pencil of *finite* size; that is, a matrix pencil. In the particular cases described in their final statements, the theorems assert further control over this finite size. That we can arrange the pencils to have finite size in the operator setting allows us to apply these results effectively in the matrix setting. This is done in Section 6.3.

6.2 Further Separation Results for Operator Γ-Convexity

Theorem 6.1.2 establishes separation via monic Γ-pencils when 0 lies in the operator convex hull of $\Gamma(\mathfrak{K})$. That is, within the context of that theorem, we may write $\mathfrak{K}$ as a (possibly infinite) intersection $\cap\widehat{\mathfrak{D}}_L$. If we require that 0 lies in the *interior* of the operator convex hull of $\Gamma(\mathfrak{K})$, Theorem 6.2.1 says we may write $\mathfrak{K}$ as the non-negativity set of a *single* operator Γ-pencil.

In Corollary 6.2.1 below, the topological notions of boundary and interior are with respect to the relative norm topology on $B(\mathcal{H})^d_{sa}$. Note that, in the case $d = r$ (equivalently $\Gamma(x) = x$) the condition of equation (6.2.1) in Corollary 6.2.1 is automatically satisfied.

Corollary 6.2.1. *Suppose $\mathfrak{K} \subset B(\mathcal{H})^d_{sa}$ is SOT-closed, operator Γ-convex, and bounded. Suppose also that* 0 *is in the norm-interior of* $\operatorname{opco}(\Gamma(\mathfrak{K}))$ *and that*

$$\operatorname{int}(\mathfrak{K}) \subset \Gamma^{-1}(\operatorname{int}(\operatorname{opco}(\Gamma(\mathfrak{K})))). \tag{6.2.1}$$

If Y is in the norm-boundary of $\mathfrak{K}$, then there exists a monic operator Γ-pencil L such that $L(X) \succ 0$ for all X in the interior of $\mathfrak{K}$ and such that $L(Y)$ is not bounded below.

Proof. There is a sequence (Y_n) in $B(\mathcal{H})^d_{sa}$ such that each $Y_n \notin \mathfrak{K}$ and $Y_n \to Y$ in norm. By the first statement in Theorem 6.1.2, there are monic linear pencils M_n of finite size such that $M_n(\operatorname{opco}(\Gamma(\mathfrak{K}))) \succeq 0$, but $M_n(\Gamma(Y_n)) \not\succeq 0$. Since 0 is in the norm-interior of $\operatorname{opco}(\Gamma(\mathfrak{K}))$,

$M := \bigoplus_n M_n$ defines a monic linear pencil whose coefficients are bounded operators on a separable Hilbert space. Put $L := M \circ \Gamma$.

By construction, $M \succeq 0$ on $\operatorname{opco}(\Gamma(\mathfrak{K}))$. It follows that $M(Z) \succ 0$ for $Z \in \operatorname{int}(\operatorname{opco}(\Gamma(\mathfrak{K})))$. Indeed, if $Z \in \operatorname{int}(\operatorname{opco}(\Gamma(\mathfrak{K})))$, then there is $\varepsilon > 0$ such that $(1+\varepsilon)Z \in \operatorname{opco}(\Gamma(\mathfrak{K}))$. Therefore, $M(Z) \succeq \varepsilon/(1+\varepsilon)$. By the inclusion in (6.2.1), if $X \in \operatorname{int}(\mathfrak{K})$, then $\Gamma(X) \in \operatorname{int}(\operatorname{opco}(\Gamma(\mathfrak{K})))$ and $M(\Gamma(X)) = L(X) \succ 0$.

Finally, we show $L(Y)$ is not bounded below by a positive multiple of the identity. Since the sequence Y_n tends to Y in norm, $\Gamma(Y_n) \to \Gamma(Y)$ in norm. As $M_n(\Gamma(Y_n))$ is a summand of $M(\Gamma(Y_n))$, it follows that $M(\Gamma(Y_n)) \not\succeq 0$. Since $M(\Gamma(Y_n))$ converges to $M(\Gamma(Y))$ in norm, there does not exist an $\varepsilon > 0$ such that $L(Y) = M(\Gamma(Y)) \succeq \varepsilon$. □

Theorem 6.2.1. *Suppose $\mathfrak{K} \subset B(\mathcal{H})^d_{sa}$ is SOT-closed, operator Γ-convex, and bounded. Suppose also that 0 is in the norm-interior of* $\operatorname{opco}(\Gamma(\mathfrak{K}))$. *Then there exists a monic operator Γ-pencil L such that $\mathfrak{K} = \widehat{\mathfrak{D}}_L$.*

Proof. For each positive integer N, let $\mathcal{L}_N$ denote the collection of monic linear pencils $M = I_N + \sum A_j x_j$ of size N such that the entries of each A_j have rational real and imaginary parts and such that $M(\operatorname{opco}(\Gamma(\mathfrak{K}))) \succeq 0$. In particular, $\cup_N \mathcal{L}_N$ is an (at most) countable set. Apply Theorem 6.1.2 to each $X \in B(\mathcal{H})^d_{sa} \setminus \mathfrak{K}$ to get a finite monic linear pencil M_X such that $M_X \succeq 0$ on $\operatorname{opco}(\Gamma(\mathfrak{K}))$ and $M_X(\Gamma(X)) \not\succeq 0$. Without loss of generality, we may assume each M_X is in $\mathcal{L}_N$ for some N.

Since 0 is in the norm-interior of $\operatorname{opco}(\Gamma(\mathfrak{K}))$ and since $\cup_N \mathcal{L}_N$ is countable,

$$\mathbb{M} := \bigoplus_N \bigoplus_{M \in \mathcal{L}_N} M$$

is a monic operator linear pencil with each coefficient a bounded operator on a separable Hilbert space. The pencil $\mathbb{M}$ has the property that $\mathbb{M}(X) \succeq 0$ for all $X \in \operatorname{opco}(\Gamma(\mathfrak{K}))$ and $\mathbb{M}(\Gamma(X)) \not\succeq 0$ for $X \notin \mathfrak{K}$. Setting $L = \mathbb{M} \circ \Gamma$ completes the proof. □

6.3 An Application to Matricial Γ-Convexity

In this final section, we return to the study of Γ-convexity on the matrix level. In particular, we use the final assertion of Theorem 6.1.2 to get an improvement of Theorem 5.2.1 when the Γ-convex set under consideration is free semialgebraic. The fact that, over operators, we can find a separating Γ-pencil of *finite* size in Theorem 6.1.2 is essential to gaining insight into the matrix case.

Definition 6.3.1. We say a symmetric $p \in \mathbb{C}\langle x \rangle$ is **regular** if $\mathcal{D}_p = \widehat{\mathcal{D}}_p$, the set $\widehat{\mathfrak{D}}_p$ is bounded, and $\mathfrak{D}_p = \widehat{\mathfrak{D}}_p$.

It is clear that the inclusions $\mathcal{D}_p \subset \widehat{\mathcal{D}}_p$ and $\mathfrak{D}_p \subset \widehat{\mathfrak{D}}_p$ always hold when $\widehat{\mathfrak{D}}_p$ is bounded. However equality may not hold: consider the polynomial $q(x) = -(1-x^2)^2$. An example of a regular polynomial is $p(x,y) = 1 - x^2 - y^{2d}$.

A simple and natural geometric sufficient condition for when a symmetric polynomial p is regular may be described as follows. We say a symmetric $p \in \mathbb{C}\langle x \rangle$ is **star-like** if the set $\widehat{\mathfrak{D}}_p$ is bounded and if $X \in B(\mathcal{H})^d_{sa}$ and $p(X) \succeq 0$ implies $p(tX) \succ 0$ for all $0 \leq t < 1$. Note that if p is star-like and $\widehat{\mathfrak{D}}_p \neq \varnothing$, then $p(0) \succ 0$.

Example 6.3.1. For d a positive integer, the polynomial $p(x,y) = 1 - x^2 - y^{2d}$ is star-like since

$$p(t(x,y)) = (1-t^2) + t^2 p(x,y) + t^2(1-t^{2d-2})y^{2d}.$$

Thus, if $p(X,Y) \succeq 0$, then $p(t(X,Y)) \succeq 1 - t^2$ for $0 \leq t < 1$.

Proposition 6.3.1. *If a symmetric $p \in \mathbb{C}\langle x \rangle$ is star-like, then p is a regular polynomial.*

Proof. Let $X \in \widehat{\mathfrak{D}}_p$, so that $p(X) \succeq 0$, and define $X_t = tX$ for $0 \leq t < 1$. Since p is star-like, $p(X_t) \succ 0$. As $X_t \to X$ in SOT as $t \to 1^-$, we conclude that $X \in \mathfrak{D}_p$. Thus, $\widehat{\mathfrak{D}}_p = \mathfrak{D}_p$.

Now suppose $Y \in \widehat{\mathcal{D}}_p$ and let 0 be the zero operator on a separable infinite dimensional Hilbert space. Then, $X := Y \oplus 0 \in \widehat{\mathfrak{D}}_p$ and thus $tX = (tY) \oplus 0 \in \mathfrak{P}_p$ for all $0 \le t < 1$. Hence, $tY \in \mathcal{P}_p$ and since $tY \to Y$ as $t \to 1^-$, $Y \in \mathcal{D}_p$. Therefore $\widehat{\mathcal{D}}_p = \mathcal{D}_p$ and p is regular. $\square$

The main result of this section is Theorem 6.3.1 below. It may be seen as another improvement on our first Γ-separation result, Theorem 5.2.1.

Theorem 6.3.1. *Let $p \in \mathbb{C}\langle x \rangle$ be a regular polynomial and suppose that $\mathfrak{D}_p$ is operator Γ-convex, $0 \in \mathcal{D}_p$, and $\Gamma(0) = 0$. If $Y \in \mathbb{S}_\ell(\mathbb{C})^d$ and $Y \notin \mathcal{D}_p(\ell)$, then there is a finite monic Γ-pencil L of size ℓ such that $L \succeq 0$ on $\mathcal{D}_p$ but $L(Y) \not\succeq 0$. In particular, $\mathcal{D}_p = \cap \widehat{\mathcal{D}}_L$, where the intersection is over all (finite) monic Γ-pencils L such that $L(\mathcal{D}_p) \succeq 0$.*

Proof. Let 0 be the zero operator on a separable infinite dimensional Hilbert space. Since $Y \notin \mathcal{D}_p$, we have $Y \oplus 0 \notin \mathfrak{D}_p$. By the final assertion of Theorem 6.1.2 applied to $Y \oplus 0$ and the SOT-closed $\mathfrak{D}_p$, there is a monic Γ-pencil L of size ℓ such that $L \succeq 0$ on $\mathfrak{D}_p$ and $L(Y \oplus 0) \not\succeq 0$. It then follows that $L(Y) \not\succeq 0$ and $L \succeq 0$ on $\mathcal{D}_p$. $\square$

Though we are able to prove a general separation result for $\mathcal{D}_p$ in Theorem 6.3.1, the statement does not require as an assumption nor conclude that $\operatorname{matco}(\Gamma(\mathcal{D}_p))$ is (levelwise) closed. Instead, it depends only on the SOT-closedness of $\mathfrak{D}_p$, and hence of $\operatorname{opco}(\Gamma(\mathfrak{D}_p))$, by Theorem 5.4.1.

Example 6.3.2. In light of Example 6.3.1 and Theorem 6.3.1, if $p(x, y) = 1 - x^2 - y^{2d}$ and $(X, Y) \notin \mathcal{D}_p$, then there is a monic y^2-pencil L such that $L \succeq 0$ on $\mathcal{D}_p$ and such that $L(X, Y) \not\succeq 0$. Hence $\mathcal{D}_p = \cap \widehat{\mathcal{D}}_L$, where the intersection is over all monic y^2-pencils L that are positive semidefinite on $\mathcal{D}_p$.

Bibliography

[1] G. Abduvalieva and D.S. Kaliuzhnyi-Verbovetskyi, *Implicit/inverse function theorems for free noncommutative functions*, J. Funct. Anal. **269** (2015), no. 9, 2813–2844.

[2] R. Abraham, J.E. Marsden, and R. Ratiu, *Manifolds, tensor analysis, and applications: 2nd edition*, Springer-Verlag, Berlin, Heidelberg, 1988.

[3] J. Agler and J.E. McCarthy, *Non-commutative holomorphic functions on operator domains*, Eur. J. Math. **1** (2015), no. 4, 731–745.

[4] ______, *The implicit function theorem and free algebraic sets*, Trans. Amer. Math. Soc. **368** (2016), no. 5, 3157–3175.

[5] D. Cushing, J.E. Pascoe, and R. Tully-Doyle, *Free functions with symmetry*, Math. Z. **289** (2018), 837—857.

[6] E.G. Effros and S. Winkler, *Matrix convexity: operator analogues of the bipolar and hahn-banach theorems*, J. Funct. Anal. **144** (1997), 117–152.

[7] A.E. Frazho, *Models for noncommuting operators*, J. Funct. Anal. **48** (1982), no. 1, 1 –11.

[8] ______, *Complements to models for noncommuting operators*, J. Funct. Anal. **59** (1984), no. 3, 445 –461.

[9] D.M. Hay, J.W. Helton, A. Lim, and S. McCullough, *Matrix convexity: operator analogues of the bipolar and hahn-banach theorems*, Indiana Univ. Math. J. **57** (2008), 2815–2842.

[10] J.W. Helton, I. Klep, and S. McCullough, *Proper analytic free maps*, J. Funct. Anal. **260** (2011), no. 5, 1476–1490.

[11] J.W. Helton and S. McCullough, *Every convex free basic semi-algebraic set has an lmi representation*, Ann. of Math. **176** (2012), no. 2, 979–1013.

[12] J. Jury, I. Klep, M.E. Mancuso, S. McCullough, and J.E. Pascoe, *Noncommutative partial convexity via γ-convexity*, J. Geom. Anal. (2020).

[13] D.S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, *Foundations of free noncommutative function theory*, Mathematical Surveys and Monographs, vol. 199, American Mathematical Society, Providence, RI, 2014.

[14] M.E. Mancuso, *Inverse and implicit function theorems for noncommutative functions on operator domains*, J. Oper. Theor. **83** (2020), no. 2, 447–473.

[15] J.E. Pascoe, *The inverse function theorem and the Jacobian conjecture for free analysis*, Math. Z. **278** (2014), no. 3-4, 987–994.

[16] G. Popescu, *Isometric dilations for infinite sequences of noncommuting operators*, Trans. Amer. Math. Soc. **316** (1989), no. 2, 523–536.

[17] ———, *Models for infinite sequences of noncommuting operators*, Acta Sci. Math. (Szeged) **53** (1989), no. 3-4, 355–368.

[18] ———, *Free holomorphic automorphisms of the unit ball of $B(H)^n$*, J. Reine Angew. Math. **638** (2010), 119–168.

[19] ———, *Free holomorphic functions on the unit ball of $B(H)^n$, ii*, J. Funct. Anal. **258** (2010), no. 5, 1513 –1578.

[20] A.E. Taylor, *Analytic functions in general analysis*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2) **6** (1937), no. 3-4, 277–292.

[21] J.L. Taylor, *Functions of several noncommuting variables*, Bull. Amer. Math. Soc. **79** (197301), no. 1, 1–34.