Spring 5-15-2020

A Study on Lexicographic Shellable Posets

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A Study on Lexicographically Shellable Posets

by

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A dissertation presented to
the Graduate School
of Washington University in
partial fulfillment of the
requirements for the degree
of Doctor of Philosophy

May 2020

St. Louis, Missouri

Table of Contents

List of Figures

Acknowledgments

First and foremost, I would like to thank my thesis advisor, Professor John Shareshian, for his guidance and advice. I deeply appreciate every time he works hard to help me through difficulties and help me with my writings. He is an excellent mentor and great mathematician. I would also like to thank my committee members: Martha Precup, Laura Escobar, Renato Feres, and Sheila Sundaram, for taking their time to participate this thesis defense. Though I only know Sheila for a short time, she has helped me extensively with revising this thesis. I sincerely appreciate her patience and kindness.

I would like to express my sincere gratitude to my friends who have helped me through some very difficult times, especially Qi, Qiyiwen, Wei, and Quancheng. Their programming skills have saved me from more troubles than I can count. I greatly appreciate their time and effort. I also want to thank my dearest brothers Weiliang, Haohua, and Moya, especially Moya, who has been so supportive during my last year before graduation. I am extremely grateful for having all of you in my life.

Last but no least, I would like to thank my parents who have been very supportive for all these years.

Tiansi Li

Washington University in St. Louis

May 2020

Abstract of the Dissertation

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Doctor of Philosophy in Mathematics,

Washington University in St. Louis, 2020.

Professor John Shareshian, Chair

In this thesis, I will discuss the relations and differences between EL-shellable and CL-shellable posets. I will present examples of EL-shellable posets that are previously known to be CL-shellable, including rank-selected subposets of EL-shellable posets such as the Smirnov words posets, and comodernistic lattices. I will show that EL-shellability is equivalent to root-independent recursive atom ordering, and present two examples of CL-shellable posets that are not EL-shellable, one of which is a graded poset and the other is ungraded. In the end, I will briefly discuss how I think one might fully characterize CL-shellable posets that are not EL-shellable.

1. Introduction

1.1 Motivation and Introduction

Lexicographic shellability was introduced in the 1980s. Björner defined EL-shellability in [1], and proved a conjecture of Stanley's that the existence of a labeling, satisfying certain conditions, of the edges of the Hasse diagram of a bounded, graded poset P shows that P is Cohen-Macaulay. Björner and Wachs defined in [2] the equivalent notions of CL-shellability and recursive atom ordering for a bounded, graded poset. They extended these notions to bounded, non-graded posets in [3]. The theory of lexicographic shellability has been developed and applied by many authors, as one will see upon checking a list of papers that refer to [2] and [3]. While every EL-shellable poset is CL-shellable, the converse has remained open until now.

Here is the layout of this thesis.

In Chapter 2, we give the definition of recursive atom ordering, an alternate definition for CL-shellable, and prove that a poset P admits a recursive atom ordering such that the atom ordering of any rooted interval is independent of the root if and only if P is EL-shellable.

In Chapter 3, we present two examples in Section 3.1 and 3.2, where we show both ungraded and graded CL-shellable posets are not necessarily EL-shellable.

In Chapter 4, we show that EL-shellability is preserved under rank-selection. We give a simpler EL-Labeling to the Smirnov words posets.

In Chapter 5, we prove that comodernistic lattices are EL-shellable, and give a simpler EL-labaling to order congruence lattices.

1.2 Preliminaries

Definition 1.2.1 *[4, 3.1] For each face F of a simplicial complex Δ, let $\langle F \rangle$ denote the subcomplex generated by F. A simplicial complex Δ is said to be shellable if its facets can be arranged in linear order $F_1, F_2, \ldots, F_t$ such that the subcomplex $\left(\bigcup_{i=1}^{k-1} \langle F_i \rangle\right) \cap \langle F_k \rangle$ is pure and (dimF_k-1)-dimensional for all $k=2, \ldots, t$. Such an ordering of facets is called a shelling. A poset P is shellable if its order complex $\Delta(P)$ is shellable. That is, the simplicial complex with vertex set P and the finite chains in P as faces.*

Shellability has been a very useful tool in topological combinatorics. Every shellable complex has the homotopy type of a wedge of spheres, and we can find the dimensions of the spheres given a shelling.

For a bounded poset P, an edge-labeling is a map λ from the edge set of the Hasse diagram of P to some poset Λ of labels. Given an edge-labeling, we can associate every saturated chain $C = x_1 \lessdot x_2 \lessdot \cdots \lessdot x_l$ with the word $\lambda(x_1)\lambda(x_2)\ldots\lambda(x_l)$. We say that C is weakly increasing if the associated word is weakly increasing.

Definition 1.2.2 *[4, Definition 3.2.1] Let P be a bounded poset. An edge-lexicographical labeling (EL-labeling, for short) of P is an edge labeling such that in each closed interval*

$[x, y]$ of P, there is a unique weakly increasing maximal chain, which lexicographically precedes all other maximal chains of $[x, y]$.

We call P EL-shellable if there is an EL-labeling of P.

Theorem 1.2.1 *[4, Theorem 3.2.2] Denote by $\overline{P}$ the poset obtained from P by removing $\hat{0}$ and $\hat{1}$. Suppose P is a bounded poset with an EL-labeling. Then the lexicographic order of the maximal chains of P is a shelling of $\Delta(P)$. Moreover, the corresponding order of the maximal chains of $\overline{P}$ is a shelling of $\Delta(\overline{P})$.*

For any closed interval $[x, y]$, we call $[x, y]_r$ a rooted interval if r is a maximal chain in $[\hat{0}, x]$, and we call r the root of the interval $[x, y]_r$. A chain-edge labeling of a bounded poset is a map from the set of all pairs (c, e) to the label poset Λ, where c is a maximal chain of P and e is an edge in c, such that (c, e) and (c', e) get the same label if c and c' coincide from $\hat{0}$ to e. We obtain a label of a rooted edge $e_r = [x, y]_r$, where $x \lessdot y$, from the chain-edge label of (c, e), where c is a maximal chain containing r.

Definition 1.2.3 *[4, Definition 3.3.1] Let P be a bounded poset. A chain-lexicographic labeling (CL-labeling, for short) of P is a chain-edge labeling such that in each closed rooted interval $[x, y]_r$ of P, there is a unique strictly increasing maximal chain, which lexicographically precedes all other maximal chains of $[x, y]_r$. A poset that admits a CL-labeling is said to be CL-shellable.*

Example 1.2.1 *To illustrate the difference, we show in Figure 1 an example of an EL-labeling on a bounded poset and in Figure 2 an example of a CL-labeling on a bounded poset.*

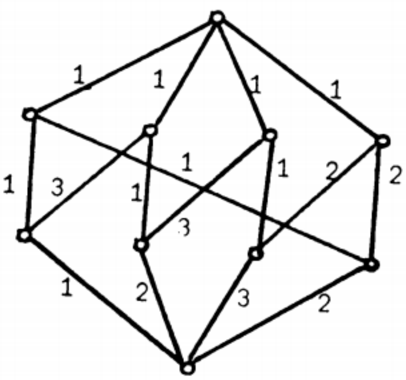

Figure 1.1. An EL-labeling [2, Figure 2.1]

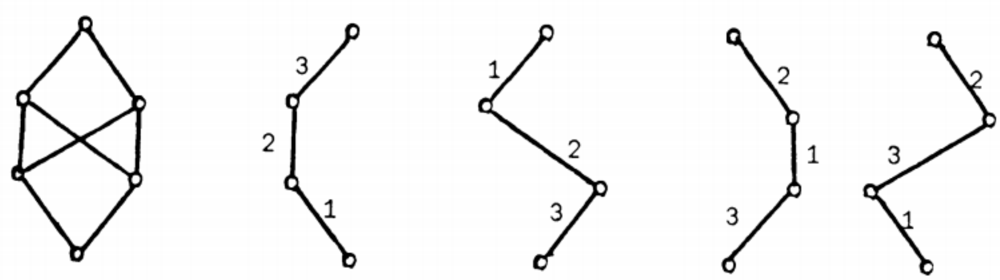

Figure 1.2. A CL-labeling [2, figure 2.2]

Clearly, EL-shellability implies CL-shellability. Both EL-shellability and CL-shellability imply shellability.

Theorem 1.2.2 *[2, Proposition 2.3] EL-shellability* $\Rightarrow$ *CL-shellability* $\Rightarrow$ *Shellability.*

2. Recursive Atom Ordering

Recursive atom ordering is defined in [2], where it is shown that a bounded poset P admits a recursive atom ordering if and only if P is CL-shellable.

Definition 2.0.1 *[4, Definition 4.2.1] A bounded poset P is said to admit a recursive atom ordering if its length $l(P)$ is 1 or if $l(P) > 1$ and there is an ordering $a_1, a_2, \ldots, a_t$ of the atoms of P that satisfies:*

1. *For all $j = 1, 2, \ldots, t$, the interval $[a_j, \hat{1}]$ admits a recursive atom ordering in which the atoms of $[a_j, \hat{1}]$ that belong to $[a_i, \hat{1}]$ for some $i < j$ come first.*

2. *For all $i < j$, if $a_i, a_j < y$ then there is a $k < j$ and an atom z of $[a_j, \hat{1}]$ such that $a_k < z \leq y$.*

A recursive coatom ordering is a recursive atom ordering of the dual poset P^*.

Theorem 2.0.1 *[4, Theorem 4.2.2] A bounded poset P is CL-shellable if and only if P admits a recursive atom ordering.*

In the next section, we derive the connection between EL-shellability and recursive atom ordering.

2.1 Recursive Atom Ordering and EL-Labeling

Proposition 2.1.1 *A bounded poset P admits a recursive atom ordering such that the atom ordering of any rooted interval is independent of the root if and only if P is EL-shellable.*

Proof Let P be a CL-shellable poset with an induced recursive atom ordering such that for every $x \in P\backslash\{\hat{1}\}$, the recursive atom orderings of $[x, \hat{1}]_r$ are the same for all roots r of $[x, \hat{1}]$. We now construct an EL-labeling for P. For every edge $e = [x, y]$ in the Hasse diagram of P, we define a $(\kappa_e + 1)$-tuple, where κ_e is the number of maximal chains in $[\hat{0}, x]$. We assign a 2-tuple to each of the (κ_e+1) coordinates. In the (κ_e+1^{st}) coordinate, we assign the 2-tuple (x, y). In each of the first κ_e coordinates, we assign a 2-tuple whose first entry is the chain obtained by combining a root r of e with e itself, whereas the second entry is the label induced by r in the CL-labeling of P. We then order the κ_e coordinates according to the original CL-labeling of P. That is, if C is lexicographically the k^{th} maximal chain in $[\hat{0}, x]$ according to the CL-labeling, the k^{th} coordinate of e consists of $C \cup e$ and the label induced by C.

Now we define a partial order on the labeling set. Suppose $e = [x, y]$ and $e' = [x', y']$ are two edges in the Hasse diagram labeled as above. Then we say $e < e'$ if $y < x'$, or if $y = x'$ and there exists a coordinate $(C \cup e,\, \alpha)$ of e and a coordinate $(C' \cup e',\, \alpha')$ of e', such that $C \cup e$ is contained in $C' \cup e'$, and $\alpha \leq \alpha'$ in the original labeling poset of P.

We first check that this is a well-defined partial order. Antisymmetry is satisfied because if $e < e'$, x cannot be above or equal y'. Transitivity is satisfied because if $e < e'$ and $e' < e''$, $y < x''$. Hence it is a well-defined partial order.

We then check that this edge-labeling is an EL-labeling of P. For any interval $[x, y]$, let C be the unique weakly-increasing chain in the original CL-labeling (with respect to any roots by assumption). We claim that the new edge-labeling on P makes C the unique weakly increasing and lexicographic first maximal chain in $[x, y]$.

The fact that C is weakly-increasing follows from the consistency of the original CL-labeling. Let C be weakly increasing in $[x, y]_r$ for all roots r. Suppose there exists some $C' = \{c_0 \lessdot c_1 \lessdot \cdots \lessdot c_k\}$ in $[x, y]$ that is also weakly increasing in the new edge-labeling. Then for each $0 < i < k$, there exists a root r_i such that c_i is the first atom in $[c_{i-1}, c_{i+1}]_{r_i}$. Notice that the assumption on the atom orderings implies that whether labels of two consecutive edges in $[c_{i-1}, c_{i+1}]$ are weakly increasing is independent of roots. Hence C' must be weakly increasing in some rooted interval, which would further imply that there are two weakly increasing maximal chains in one rooted interval. This contradicts the original CL-labeling on P. A similar argument shows that C must be the lexicographic first maximal chain of the interval. Hence P is EL-shellable.

Now suppose we have an EL-shellable poset P. First notice that any EL-shelling can be viewed as a CL-shelling where edge labels are independent of roots. By Theorem 7, a CL-shelling induces a recursive atom ordering in which the ordering of atoms above a given element with root r is consistent with those edge labels with root r. That is, if for a fixed linear extension, the label of $[x, y]$ precedes the label of $[x, y']$, where y and y' both cover x, then y precedes y' in the atom ordering of x. Therefore if we start with an EL-shelling, the ordering of atoms above a given element in the induced recursive atom ordering does not depend on how one reached that element from elements below it.

■

3. CL-Shellable Posets with no EL-Shellings

3.1 Ungraded Example

We prove in this section that the Hasse diagram in Figure 3.1 gives an example of an ungraded CL-shellable poset that does not admit any EL-shellings. The difference between CL-shellings and EL-shellings is that in a fixed interval $[x, y]$, CL-shellings allow different weakly increasing maximal chains of $[x, y]_r$ when we consider different roots r, whereas in an EL-shelling, there is a unique weakly increasing maximal chain in $[x, y]$ that does not depend on roots. In terms of recursive atom ordering, the difference is that the atom ordering above every element can be different depending on roots for a CL-shelling, while an EL-shelling induces a recursive atom ordering where the atom ordering above every element is independent of roots (see Proposition 2.1.1).

We claim that the poset as in Figure 3.1 admits a recursive atom ordering where the atom order above each element, except at y, is independent of roots. The recursive atom ordering in $[y, \hat{1}]_r$ relies on which root we choose. That is, the unique weakly increasing chain of $[y, \hat{1}]_r$ must be different when we consider the root through a_1 and the root through a_6. This implies that this poset cannot admit any EL-shelling.

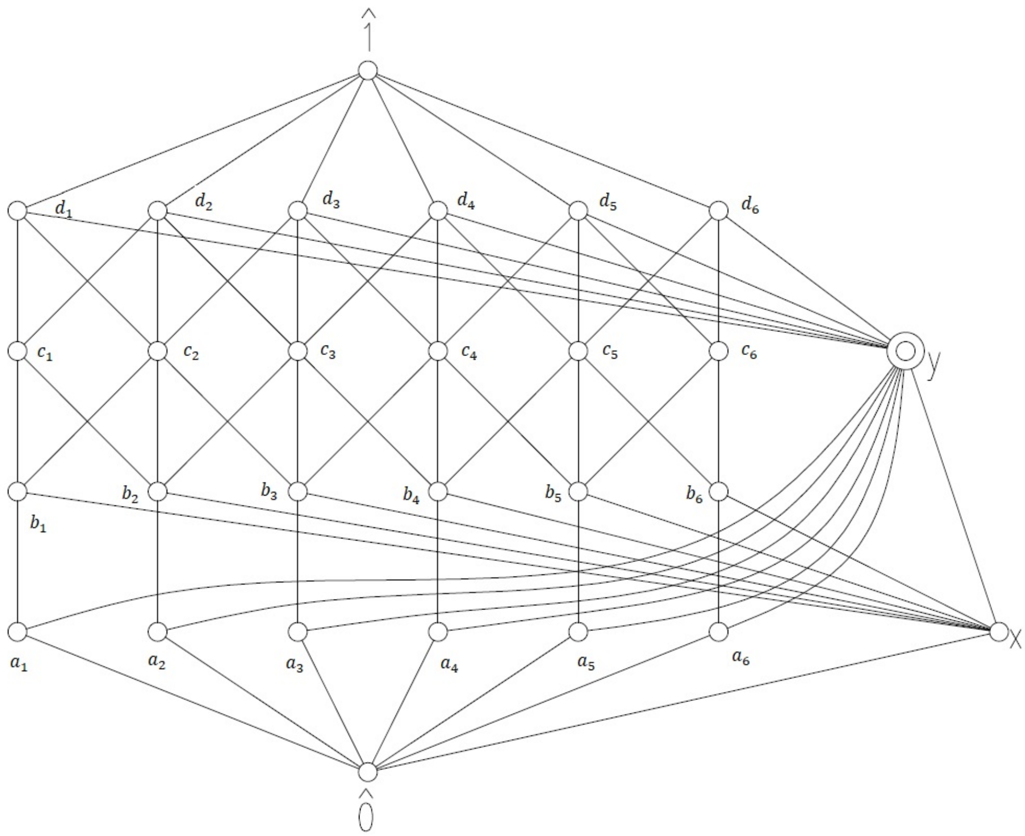

Figure 3.1. An ungraded CL-shellable poset with no EL-shelling

First we show that this poset admits a recursive atom ordering. Notice we only need to consider every element of height at most 2 except y, because for any element z of height larger than 2 (or for y), every atom order of $[z, \hat{1}]$ induces a recursive atom ordering. We claim that listing from left to right, except at $\hat{0}$ where we list x first and then every atom from left to right, gives a recursive atom ordering in every rooted interval.

For elements of height 2, notice from Figure 3.1 that d_i is above all atoms of $[b_i, \hat{1}]$. Therefore, every ordering of the atoms of $[b_i, \hat{1}]$ satisfies the second condition in Definition 6. The recursive atom ordering in $[x, \hat{1}]$ follows similar arguments since for any two atoms b_i and b_j $(i < j)$ of $[x, \hat{1}]$, a height 4 element above both atoms either covers a common atom of $[b_i, \hat{1}]$ and $[b_j, \hat{1}]$, or the leftmost atom of $[b_j, \hat{1}]$, in which case that atom of $[b_j, \hat{1}]$ covers some previous atoms above x. On the other hand, for elements a_i of height 1, $[y, \hat{1}]$ has an atom that is above the other atoms of $[a_i, \hat{1}]$. Finally at $\hat{0}$, both atoms above a_i cover x for $i \in [6]$, and every element of height at least 2 is above x. Hence this poset admits a recursive atom ordering.

Suppose we have an EL-shelling of the poset given in Figure 3.1. In the interval $[y, \hat{1}]$, there is an atom above y that gives the unique increasing chain of that interval. It is independent of roots. Assume this atom is among d_1, d_2 and d_3. Consider the root of y passing through a_6. Notice that y is the second atom in its recursive atom ordering. So the atom above y that comes first along this root must be among d_4, d_5 and d_6. This contradicts the assumption. If we now assume the atom that gives the unique increasing chain of $[y, \hat{1}]$ were among d_4, d_5 and d_6, the root of y through a_1 gives a similar contradiction. Hence this poset cannot be EL-shellable.

3.2 Graded Example

We present in this section a graded poset that is CL-shellable but not EL-shellable. The construction is based on a shellable but not extendably shellable complex exhibited by Hachimori. In [5], Hachimori constructed a shellable simplicial complex where the facet 134 comes last in every shelling of the complex (See Figure 3.2 below). By theorem 4.3 in [2], the dual of the face lattice of Hachimori's complex admits a recursive atom ordering. We will build a CL-shellable complex based on this poset, and we will show that the weakly increasing chain in $[134, \hat{1}]_r$ must be different for r passing through $\hat{0}_a$ and $\hat{0}_d$. Therefore this poset cannot be EL-shellable.

Let us start with four copies of the dual of the face poset of Hachimori's complex. For convenience, we call them posets A, B, C, and D and use y_a or y_b when refering to y in A or B and so on. We build a new poset P by first identifying all four copies of 134 and $\hat{1}$ in A, B, C and D, and then attaching a $\hat{0}$ to the bottom. Next we add a new element, call it x, which sits below all facet elements and above $\hat{0}$. So x has rank 1. Finally we add an edge in the Hasse diagram between every pair of elements y_i and z_j if y and z represent non-empty faces in Hachimori's complex with y a facet of z, and if i and j are consecutive in the lexicographic order. That is to say, for example, y_b covers z_a, z_b and z_c whenever y is a facet of z, but y_b does not cover z_d, whereas y_a covers z_a, z_b, but not z_c or z_d. We claim that P admits a recursive atom ordering, and P admits no EL-shellings. Notice that P is a graded poset of rank 5 with four copies of each element in the Hachimori's lattice except 134 and $\hat{1}$.

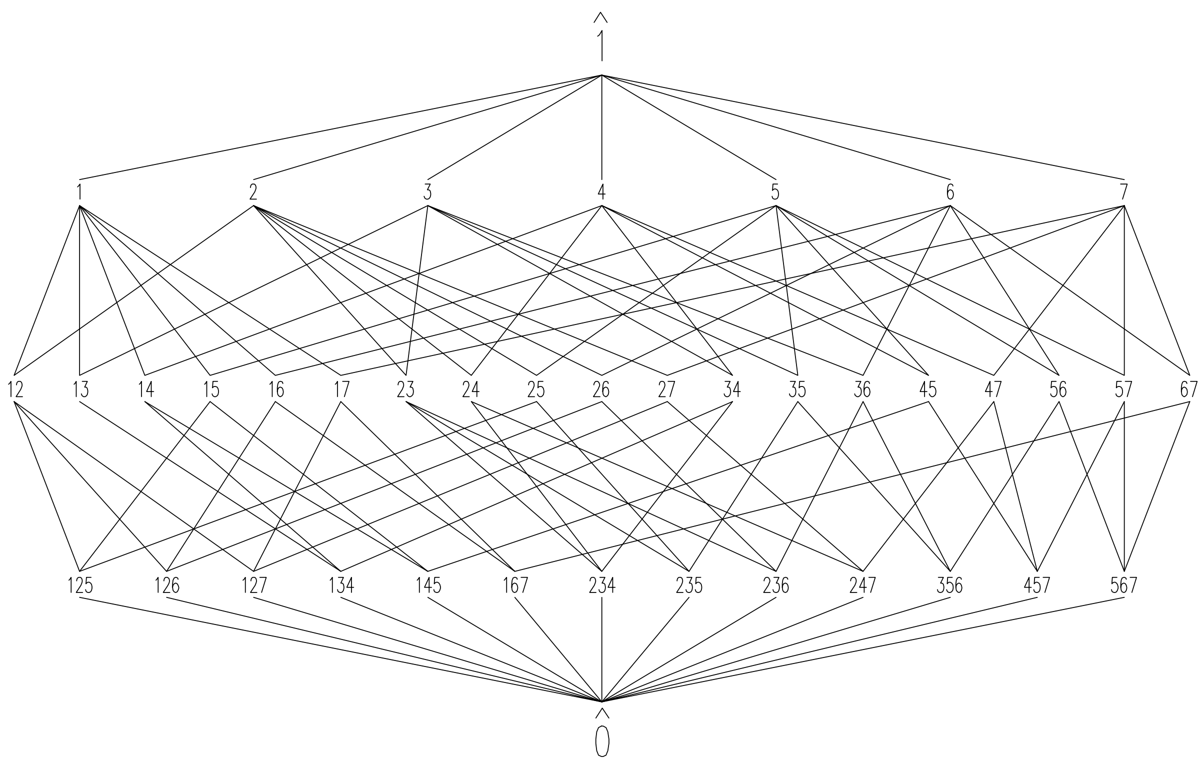

Figure 3.2. Dual of the face lattice of Hachimori's complex

Now we show that P admits a recursive atom ordering. Fix a shelling order of the Hachimori's complex and a recursive coatom ordering of its face lattice induced by the shelling. Then each of A, B, C and D comes with a natural recursive atom ordering, from which we will build the recursive atom ordering of P.

Suppose there is a recursive atom ordering of each $[\hat{0}_i, \hat{1}]$ and $[x, \hat{1}]$. Let the atom order for $[\hat{0}, \hat{1}]$ be $x \rightarrow \hat{0}_a \rightarrow \hat{0}_b \rightarrow \hat{0}_c \rightarrow \hat{0}_d$. Since every atom of each $\hat{0}_i$ covers x, this induces a recursive atom ordering on $[\hat{0}, \hat{1}]$.

Suppose there is a recursive atom ordering in each of $[F_i, \hat{1}]$, where F is a facet. Let the shelling order of Hachimori's complex be the atom order for $[\hat{0}_i.\hat{1}]$. If an element y with index i is above two atoms of $\hat{0}_i$, the recursive atom ordering of Hachimori's complex gives the existence of the element z, with index i, that satisfies the second condition in Definition 2.0.1. If the index of y is not i, z with index $i-1$ or $i+1$ satisfies the second condition in Definition 2.0.1.

For $[x, \hat{1}]$, we consider the following atom ordering:

First facet with index a in the shelling order

$\Downarrow$

First facet with index b in the shelling order

$\Downarrow$

First facet with index c in the shelling order

$\Downarrow$

First facet with index d in the shelling order

$\Downarrow$

Second facet with index a in the shelling order

$$\Downarrow$$

Second facet with index b in the shelling order

$$\Downarrow$$

$$\vdots$$

$$\Downarrow$$

134

For any two atoms y_i, z_j of x, the case where y=z is obvious by construction. If y is prior to z in the shelling of Hachimori's complex, we have the following situations:

1. If i and j are a and d, an element above both y_i and z_j must be rank 4 with index b or c. We can find an atom of z_j with index c below this element such that it covers some previous atoms of x.

2. If i and j are a and c, an element above both y_i and z_j is rank 4 with index a, b or c, or is rank 3 with index b. In both cases we can find an atom of z_j with index b below this element (or it is the element in the rank 3 case), such that it covers some previous atoms of x.

3. If i and j are a and b, or $i = j$, we can find an appropriate atom of z_j with index i similarly. All other cases are equivalent to one of the above.

For rooted intervals $[F_i, \hat{1}]_i$ through $\hat{0}_i$, where F is a facet in Hachimori's complex, the following atom order gives a recursive atom ordering:

Atoms with i-indices which cover facets prior to F

$\Downarrow$

Atoms with j-indices which cover facets prior to F, where j and i are consecutive letters

$\Downarrow$

Atoms with i-indices which do not cover facets prior to F

$\Downarrow$

All other atoms

For rooted intervals $[F_a, \hat{1}]_x$ through x, where F is a facet in Hachimori's complex, the following atom order gives a recursive atom ordering:

Atoms with a-indices which cover facets prior to F

$\Downarrow$

Atoms with b-indices which cover facets prior to F

$\Downarrow$

Atoms with a-indices which do not cover facets prior to F

$\Downarrow$

Atoms with b-indices which do not cover facets prior to F

For rooted intervals $[F_i, \hat{1}]_x$ through x where $i \neq a$ and F is a facet in Hachimori's complex, the following atom order gives a recursive atom ordering:

Atoms with $i-1$-indices which cover facets prior to F

$\Downarrow$

Atoms with i-indices which cover facets prior to F

$$\Downarrow$$

Atoms with $i+1$-indices (if they exist) which cover facets prior to F

$$\Downarrow$$

Atoms with $i+1$-indices (if they exist) which do not cover facets prior to F

$$\Downarrow$$

All other atoms

Notice that we can break ties arbitrarily in the process described above because Hachimori's complex is a simplicial complex.

As for rooted intervals $[134, \hat{1}]_i$, we can still follow those steps except that i stands for the index of $\hat{0}_i$ (root).

For the rooted interval $[134, \hat{1}]_x$ through x, we order the atoms as: $14_a \to 34_a \to 14_b \to 34_b \to \cdots \to 34_d \to 13_a \to \cdots \to 13_d$

For the remaining elements z, the length of $[z, \hat{1}]$ is at most two, hence every atom order induces a recursive atom ordering. We have shown that P admits a recursive atom ordering.

Consider the interval $[134, \hat{1}]$. It is a rank 3 interval with 12 atoms. Suppose P admits an EL-shelling, where the unique increasing chain in this interval goes through one of 13_a, 14_a, 34_a, 13_b, 14_b, or 34_b. Consider the root of 134 through $\hat{0}_d$. None of the six atoms with a or b-indices cover any atoms of $\hat{0}_d$ other than 134, whereas each of the six atoms with c or d-indices cover some atoms of $\hat{0}_d$ other than 134. Since 134 is last in any recursive atom ordering, every atom of 134 with c or d-indices must be prior to every atom with a or b-indices, and we have a contradiction. Similarly, we can get a contradiction by assuming

P admits an EL-shelling where the unique increasing chain in $[134, \hat{1}]$ goes through one of 13_c, 14_c, 34_c, 13_d, 14_d, or 34_d and consider the root through $\hat{0}_a$. Hence P cannot be EL-shellable.

Remark

Notice that both examples in this paper require particular models to begin with, that is, a CL-shellable poset in which there exists two atoms a and b such that a is prior to b in every recursive atom ordering. We used the poset consisting of a 3-chain and a 2-chain (a pentagon in the Hasse diagram) in the ungraded example and the dual of the face lattice of Hachimori's complex in the graded example, both of which have an atom that must come last in any recursive atom ordering. It remains open whether one can construct a CL-shellable but not EL-shellable poset such that for any element e and any two atoms a, b of e, there exists two recursive atom orderings on $[e, \hat{1}]$ where a is prior to b in one and b is prior to a in the other.

4. Rank-Selected Subposets

It is shown in [2] that direct products, ordinal sums, cardinal powers and interval posets preserve both EL-shellability and CL-shellability. Let P be a ranked poset with length n and rank function r. That is, every maximal chain in P has length n. For any subset R of $[n-1]$, the rank-selected subposet P_R is defined to be $P_R = \{x \in P \mid r(x) \in R \cup \{0, n\}\}$. Rank-selection is shown to preserve CL-shellability in Theorem 8.1 of [2]. In this Chapter we will show that rank-selection preserves EL-shellability and give a simpler EL-labeling to Smirnov word posets.

4.1 Rank-Selected Subposets

Theorem 4.1.1 *If P is a EL-shellable poset of length n then P_R is EL-shellable for all $R \subseteq [n-1]$.*

Let P be a graded EL-shellable poset of rank n, and $R = \{r_1 < r_2 < \cdots < r_k\}$ be a subset of $[n-1]$. We introduce an edge labeling on the rank selected subposet P_R using $n+2$ coordinates.

Let a be an atom in P_R, then a comes from an element of rank r_1 in P, which we will also call a. By assumption, there exists a unique weakly increasing maximal chain on the interval $[0, a]$ in P. Call it C_a. Suppose C_a comes with edge labeling $(c_1, c_2, \ldots, c_{r1})$. Then we assign the following edge labeling to $[0, a]$ in P_R:

1. Let coordinate j be c_j for j from 1 to r_1.

2. Let coordinate j be c_{r1} for j from r_1 to n.

3. Let coordinate $n + 1$ be c_1.

4. Let coordinate $n + 2$ be c_{r1}.

For an edge $[x, y]$ in P_R where $rk(x) = r_i$ and $rk(y) = r_{i+1}$, there exists a unique weakly increasing maximal chain C_{xy} of the interval $[x, y]$ in P. Suppose C_{xy} comes with edge labeling $(c_1, c_2, \ldots, c_l)$ where $l = r_{i+1} - r_i$. Then we assign the following edge labeling to $[x, y]$ in P_R:

1. Let coordinate j be c_1 for j from 1 to r_i.

2. Let coordinate j be c_{j-ri} for j from $r_i + 1$ to r_{i+1}.

3. Let coordinate j be c_l for j from $r_{i+1} + 1$ to n.

4. Let coordinate $n + 1$ be c_1 if i is even and c_l if i is odd.

5. Let coordinate $n + 2$ be c_l if i is even and c_1 if i is odd.

(Here we make the convention that $rk(P) + 1 = n$)

Now let us check that this is indeed an EL-labeling on P_R. Suppose for some interval $[x, y]$ in P_R there are two weakly increasing maximal chains C_1 and C_2. Each C_i being weakly increasing means that we have two weakly increasing sequence at coordinate $n+1$

and $n+2$. By the labeling, this implies that there must be two distinct weakly increasing chains in $[x, y]$ in P. This leads to a contradiction since P is EL-shellable by assumption.

Now suppose that for some interval $[x, y]$ in P_R, the unique weakly increasing maximal chain C is not lexicographically first among all maximal chains in $[x, y]$ in P_R. Let D be the maximal chain in $[x, y]$ in P_R that is the first in lexicographic order. Suppose C and D coincides in the lowest m edges. Then the $(m+1)^{st}$ edge of D must be prior to that of C in the linear extension of the label poset. Again, if we trace back to chains in P, we would get a maximal chain in $[x, y]$ that is prior to the unique weakly increasing chain in $[x, y]$. Once again it contradicts P being EL-shellable.

Remark

We can also prove that P_R is EL-shellable with the help of a proof in [2].

Theorem 4.1.2 *[2, Theorem 8.1] If P is a CL-shellable poset of rank n then P_S is CL-shellable for all $S \subset [n-1]$.*

Proof We shall prove the result for $S = [n-1]\backslash\{r\}$ where $r \in [n-1]$. The general result follows by induction. Let λ be a CL-labeling of P with label poset Λ. Define a chain-edge labeling λ_S on P_S with label poset $\Lambda \times \Lambda$ as follows. If $c = (\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{r-1} \lessdot x_{r+1} \lessdot \cdots \lessdot x_n = \hat{1})$ is a maximal chain in P_s and if x_r is the element of rank r on the lexicographically first maximal chain in the rooted interval $[x_{r-1}, x_{r+1}]_{C_R}$, where $C_R = (\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{r-1})$. Let $\lambda_S(c, x_{i-1}, x_i) = (\lambda(c \cup x_r, x_{i-1}, x_i), \lambda(c \cup x_r, x_{i-1}, x_i))$, if $i = 1, 2, \ldots, r-1, r+2, r+3, \ldots, n$, and $\lambda_S(c, x_{r-1}, x_{r+1}) = (\lambda(c \cup x_r, x_{r-1}, x_r), \lambda(c \cup$

$x_r, x_r, x_{r+1})$). Now order $\Lambda \times \Lambda$ lexicographically (this ordering is stronger than direct product order). It is then straightforward to verify that λ_S a CL-labeling of Ps. ■

Notice that in this proof of Björner and Wachs, the recursive atom ordering of the rank-selected subposet is induced from the recursive atom ordering of the original poset. If we start with an EL-shellable poset P with an EL-labeling λ, with this inductive construction, the recursive atom ordering induced by λ_S in each $[x, y]_R$ are the same for all roots R. Combine this observation with Proposition 2.2.1, P_S is EL-shellable. (The above remark is discovered by one of the defense committee members, Sheila Sundaram, while reviewing this thesis.)

4.2 Smirnov Word Posets

A word w over a finite alphabet $[n]$ is called Smirnov (normal in some literature) if no two adjacent letters are the same. That is, $w = w_1 w_2 \dots w_k$ for some $w_1, w_2, \dots, w_k \in [n]$ such that $w_i \neq w_{i+1}$ for $i = 1, 2, \dots, k-1$. A subword of w is of the form $w_{j_1} w_{j_2} \dots w_{j_i}$ where $1 < j_1 < j_2 < \cdots < j_i < k$. Here we adapt notation from [2] and let $N_{n,k}$ denote the poset of Smirnov word with alphabet $[n]$ and length at most k, where the partial order is given by subword inclusion. It is shown in [2] that $N_{n,k}$ is dual CL-shellable. We will show in the theorem below that $N_{n,k}$ is dual EL-shellable.

Theorem 4.2.1 *$N_{n,k}$ is dual EL-shellable, for all $k \geq 1$.*

By Theorem 4.1.1, it suffices to show that any Smirnov word poset can be viewed as a rank-selected subposet of an EL-shellable poset. In this section, we will show that the Smirnov word poset is the rank selected subposet of an interval from the empty word to some Smirnov word. We will show that the dual of this interval is EL-shellable. Therefore, the Smirnov word poset is dual EL-shellable.

Let W be a Smirnov word and let $\emptyset$ denote the empty word. If W_1 covers W_2 in $[W, \emptyset]$, then we assign i to the edge $[W_2, W_1]$ if W_1 is obtained from W_2 by omitting the i^{th} letter. The readers can check that this labeling is indeed well-defined. We now prove that this gives an EL-labeling of the interval.

We claim that there exists a unique maximal chain in $[W_2, W_1]$ whose labeling is weakly increasing and lexicographically prior to other maximal chains in the interval. Notice that there might be multiple identifications of W_2 as a subword of W_1 (e.g. we can see a as the first letter or the last letter in aba). So we need to identify the appropriate copy that gives us the increasing maximal chain. Let x be the last letter in W_2, then we identify that letter as the rightmost x in W_1. Let y be the second letter from the right of W_2, then we identify it as the rightmost y in W_1 that is on the left side of the x in the previous step. We keep doing this procedure until we have identified the word W_2 as a subword in W_1, where each step we always look for the rightmost copy that is on the left of the previous letter. We claim that the maximal chain C obtained from deleting letters from left to right to get the above identification is the desired maximal chain.

First we show lexicographic order. Notice the labeling of C is as many $1'$s as possible followed by as many $2'$s as possible, and as many $3'$s as possible and so on. Suppose there exists another maximal chain C' lexicographically prior to C. Then there exists a smallest positive integer i such that the labeling of C reaches i before C'. Let z be the

$i-1^{st}$ letter of W_2, then C reaches i means that this z is the rightmost z possible for one to identify W_2 as a subword of W_1. Therefore such C' does not exist.

The uniqueness of weakly increasing chain follows similarly. If there exists another weakly increasing chain C'' in $[W_2, W_1]$, by previous argument we know that C is prior to C'' in lexicographic order. Therefore, there exists a smallest positive integer i such that C'' reaches i prior to C. This would result in a decrease in C'' as the i^{th} letter of W_2 identified by C'' is on the left side of that identified by C. We have proved that $[W, \emptyset]$ is EL-shellable.

For a Smirnov word poset P with alphabet size n and maximal length k, there exists a Smirnov word W containing all coatoms of P (e.g. concatenate all coatoms linearly into one word and identify letters next to each other if they are the same). The dual of P can be viewed as a rank selected subposet of $[W, \emptyset]$, hence it is EL-shellable.

Remark

There is a simpler edge labeling on the dual of the Smirnov word poset, where edges among higher ranks are labeled by the order of letter omitted, as described in Theorem 4.2.1, whereas an edge below an atom a is given by the edge label of the last edge in the unique weakly increasing chain of $[W, a]$, where W is a fixed Smirnov word, containing all Smirnov words in $N_{n,k}$, which we use as $\hat{0}$ in the dual of $N_{n,k}$.

Example 4.2.1 *In Figure 4.1, we present an example where we use the edge-labeling as described in the remark as an EL-labeling for the Smirnov word poset* $N_{3,3}$

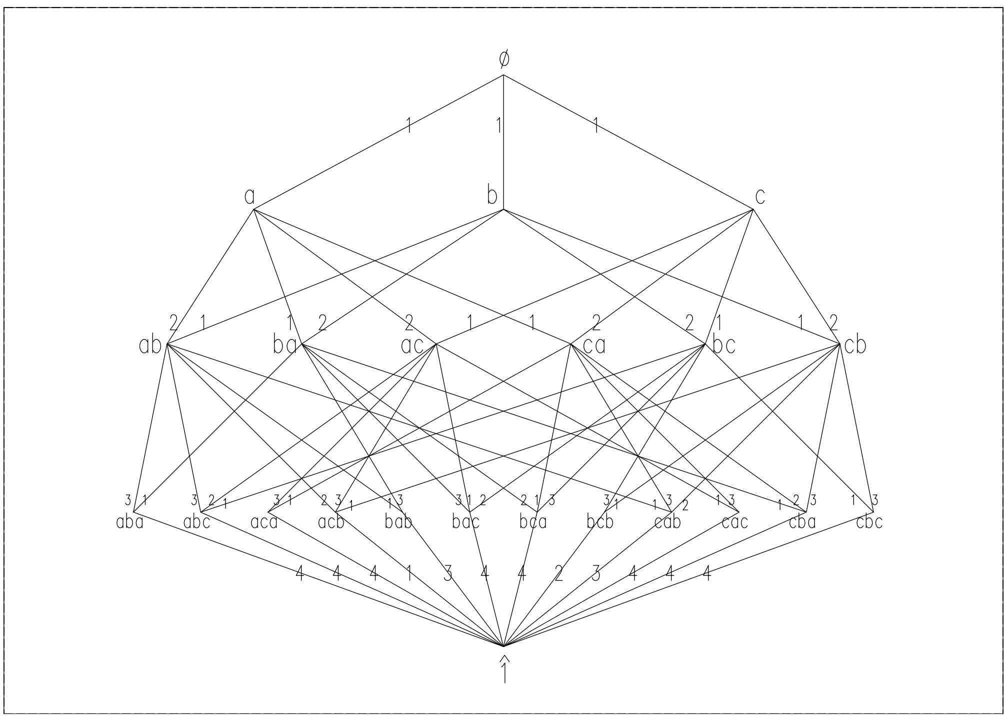

Figure 4.1. EL-shelling on $N_{3,3}$

4.3 Elementary Sequence

In this section, we show that Smirnov word poset provides a class of examples of elementary sequences defined by Stanley [6, Exercise 7.65].

Let φ_n be a character of the symmetric group S_n for each positive integer n. Let us call the sequence $\varphi_1, \varphi_2, \dots$ elementary if for all $\omega \in S_n$ we have that $\varphi_n(\omega)$ is equal either to $\pm$ degφ_m for some $m \leq n$ or to 0. For instance, the characters of the regular representations are elementary.

For every fixed k, we consider the Smirnov word poset $N_{n,k}$. For any element s in the symmetric group S_n, it induces an order-preserving poset map on $N_{n,k}$ by mapping every Smirnov word $w = w_1w_2 \dots w_k$ to $s(w) = s(w_1)s(w_2)\dots s(w_k)$. We consider the fixed point subposet under s. Notice that a word $w = w_1w_2 \dots w_k$ is fixed by s if and only if each w_i is fixed by s, if and only if each w_i is a 1-cycle in s. Therefore, the fixed point subposet under s is isomorphic to $N_{m,k}$ if s has m 1-cycles.

Baclawski and Björner proved in theorem 1.1 of [7] that if we have an order-preserving map on a finite poset, the Euler characteristic of the fixed point subposet under the map equals the Lefschetz number of the map. Since Smirnov word posets are EL-shellable, the Euler characteristic of a Smirnov word poset is the dimension of its top homology group, which means the Euler characteristic of the fixed point poset of a Smirnov word poset under some permutation in S_n is the dimension of its top homology group. On the other hand, the Lefschetz number of a map on the Smirnov word poset is given by the trace of the linear map functorially induced on the top homology group. Therefore, for every fixed k, the dimension of the top homology of the fixed point poset of $N_{n,k}$ under $s \in S_n$

is the character of s of the representation. Since this number is determined by $N_{m,k}$ for some $m < n$, this sequence of characters is an elementary sequence for each $k > 0$.

5. Comodernistic Lattices

For a pair of elements s and t in a poset P, we call $u \in P$ the least upper bound for s and t if $s, t \leq u$, and for every $v \in P$ such that $s, t \leq v$, we have $u \leq v$. We denote it as $u = s \vee t$ (s join t). Similarly, we call $w \in P$ the greatest lower bound for s and t if $s, t \geq u$, and for every $v\ inP$ such that $s, t \geq v$, we have $w \geq v$. We denote it as $w = s \wedge t$ (s meet t). We call a bounded poset L a lattice if every pair of elements has a least upper bound and a greatest lower bound.

Modernistic and Comodernistic lattices are two large classes of finite lattices with shellable order complexes. Schweig and Woodroofe defined and studied these lattices in [8] and showed that a wide range of lattices are either modernistic or comodernistic, including subgroup lattices of finite solvable groups, supersolvable and left-modular lattices, semi-modular lattices, k-equal partition lattices, order congruence lattices, and others. They proved in [8] that comodernistic lattices are CL-shellable, which implies that the order complexes of modernistic and comodernistic lattices are shellable.

In this chapter, we show that comodernistic lattices are EL-shellable, and give an EL-labeling to a special class of comodernistic lattices, namely, order congruence lattices.

5.1 Comodernistic Lattices

Let L denote a lattice. An element m in L is left-modular if for any $x < y$ in L, we have $(x \vee m) \wedge y = x \vee (m \wedge y)$. A lattice L is modernistic if for every interval of L, there exists a left-modular atom in that interval. A lattice is comodernistic if it is the dual of a modernistic lattice. That is, there exists a left-modular coatom in every interval.

In this section, we show that comodernistic lattices are EL-shellable by assigning a recursive atom ordering independent of roots given a sub-M-chain, which can be viewed as an analogy of an M-chain in a left-modular lattice.

Definition 5.1.1 *[8] A maximal chain $\hat{0} = m_0 \lessdot m_1 \lessdot \cdots \lessdot m_n = \hat{1}$ in L is a sub-M-chain if for every i, the element m_i is left-modular in the interval $[\hat{0}, m_{i+1}]$.*

We also list here two lemmas from [8] that will help prove Theorem 5.1.1. We refer readers to [8] for the proofs of the lemmas.

Lemma 5.1.1 *[8, Lemma 3.1] Let L be a lattice with a sub-M-chain $\boldsymbol{m}$ of length n. Then no chain of L has length greater than n.*

Lemma 5.1.2 *[8, Lemma 2.12] Let m be a coatom of the lattice L. Then m is left-modular in L if and only if for every y such that $y \not\leq m$ we have $m \wedge y \lessdot y$.*

Theorem 5.1.1 *Comodernistic lattices are EL-shellable.*

Proof Let L be a comodernistic lattice. Fix a sub-M-chain $\mathbf{m} = \{\hat{0} = m_0 \lessdot m_1 \lessdot \ldots \lessdot m_n = \hat{1}\}$ of L. We assign atom orderings as follows. For atoms a and a' of L, if there exists $m_i \in \mathbf{m}$ such that $a \leq m_i$ and $a' \not\leq m_i$, then a precedes a'. If there exists $m_i \in \mathbf{m}$ such that $a \leq m_i, a' \leq m_i$ and $a \nleq m_{i-1}, a' \nleq m_{i-1}$, then we arbitrarily decide either a before a' or a' before a. For an element $x \neq \hat{0}$, we assign the atom ordering in $[x, \hat{1}]$ according to the atom ordering in $[x \wedge m_i, \hat{1}]$ (given by induction), where $m_i \in \mathbf{m}$, $x \leq m_{i+1}$ and $x \nleq m_i$. For atoms x_a and $x_{a'}$ in $[x, \hat{1}]$, we let $x_{a'}$ precede x_a if either there exist $m_i \in \mathbf{m}$ such that $x_{a'} \leq m_i$ and $x_a \nleq m_i$, or the most previous atom in $[x \wedge m_i, x_{a'} \wedge m_i]$ is prior to the most previous atom in $[x \wedge m_i, x_a \wedge m_i]$ in the atom ordering of $[x \wedge m_i, \hat{1}]$. This atom ordering is well-defined because $x_a \wedge x_{a'} = x$. Hence those two atoms in $[x \wedge m_i, \hat{1}]$ that we used must be distinct. If x_a and $x_{a'}$ are atoms in $[x, \hat{1}]$ such that we can find m_i in $\mathbf{m}$ with $x_a, x_{a'} \leq m_{i+1}$, $x_a, x_{a'} \nleq m_i$, and $x_a \wedge m_i = x_{a'} \wedge m_i = x$, then we can arbitrarily decide whether x_a before $x_{a'}$ or $x_{a'}$ before x_a. Notice that this assignment of atom orderings of L is independent of the choice of roots. Therefore, in order to prove the theorem, all we need to show is that this atom ordering satisfies the definitions of recursive atom ordering.

Suppose x is an element where we have shown that the atom orderings defined above induces recursive atom ordering on every $[x_a, \hat{1}]$, where x_a is an atom in $[x, \hat{1}]$. First, we prove the case where the atom ordering of $[x_a, \hat{1}]$ is obtained from $[x, \hat{1}]$. By construction, the atoms of $[x_a, \hat{1}]$ that belong to $[x_{a'}, \hat{1}]$ for some $x_{a'}$ prior to x_a come first. Suppose the atom ordering of $[x_a, \hat{1}]$ is obtained from $[x', \hat{1}]$, where x' is distinct from x. Then $x' = x_a \wedge m_i$, where $m_i \in \mathbf{m}$, $x_a \leq m_{i+1}$ and $x_a \nleq m_i$. Notice that the atom ordering above x is then obtained from the atom ordering above $x \wedge m_i = x \wedge x_a \wedge m_i = x \wedge x'$. We need to show that if an atom b in $[x_a, \hat{1}]$ is above some atom in $[x, \hat{1}]$ prior to x_a, then b is above some atom in $[x_a \wedge m_i, \hat{1}]$ prior to x_a. Suppose b_0 is an atom in $[x_a, \hat{1}] \cap [x_{a'}, \hat{1}]$, where $x_{a'}$ is an atom in $[x, \hat{1}]$ prior to x_a. Then $b_0 = x_a \vee x_{a'}$, $b \leq m_{i+1}$, and $x_a \neq (b_0 \wedge m_i) \lessdot b$.

This implies that either $b_0 \wedge m_i$ is an atom in $[x', \hat{1}]$ prior to x_a, or $b_0 \wedge m_i$ is above some atom in $[x', \hat{1}]$.

Suppose x_a and $x_{a'}$ are atoms in $[x, \hat{1}]$ with $x_{a'}$ prior to x_a. Let m_i be the element in $\mathbf{m}$ such that $x_a \leq m_{i+1}$ and $x_a \not\leq m_i$. Let b be the first atom in $[x_a, x_a \vee x_{a'}]$. If $x \leq m_i$, then $x = x_a \wedge m_i$. Either $b \wedge m_i$ is an atom in $[x, \hat{1}]$ prior to x_a, or $b \wedge m_i$ is above some atom in $[x, \hat{1}]$, in which case every atom in $[x, b \wedge m_i]$ is prior to x_a. If $x \not\leq m_i$, consider the maximal element m_j in $\mathbf{m}$ such that $x \wedge m_j$ is below some m_k in $\mathbf{m}$, whereas $x_a \wedge m_j$ is not. In this case, the first atom in $[x_a \wedge m_j, (x_a \vee x_{a'} \wedge m_j)]$ is above some atom $z \leq b \wedge m_k$ in $[x \wedge m_j, \hat{1}]$. We prove by induction on $i - j$ that $z \vee x$ is an atom in $[x, x_a \vee x_{a'}]$ that is prior to x_a. Without loss of generality, we assume that $x \wedge m_j \neq x \wedge m_{j+1}$. Notice that $z' = z \vee (x \wedge m_{j+1})$ is an atom in $[x \wedge m_{j+1}, \hat{1}]$. It is prior to $x_a \wedge m_{j+1}$ and below $(b \wedge m_j) \vee (x \wedge m_{j+1}) = b \wedge m_{j+1}$. Now $z' \vee x$ is an atom in $[x, x_a \vee x_{a'}]$ that is prior to x_a.

■

Example 5.1.1 *In the subgroup lattice of S_4, as shown in Figure 5.1, a sub-M-chain is given by the boxed elements. We order the atoms above each element as in Theorem 5,1,1 with a given sub-M-chain (the maximal chain obtained from consecutively taking the first labeled atom). It induces a recursive atom ordering of the lattice, which is independent of roots.*

5.2 Order Congruence Lattices

The order congruence lattice $\mathcal{O}(P)$ of a poset P is the set of all equivalence classes of level set partitions from P to $\mathbb{Z}$. That is, the set of all weakly order preserving maps from

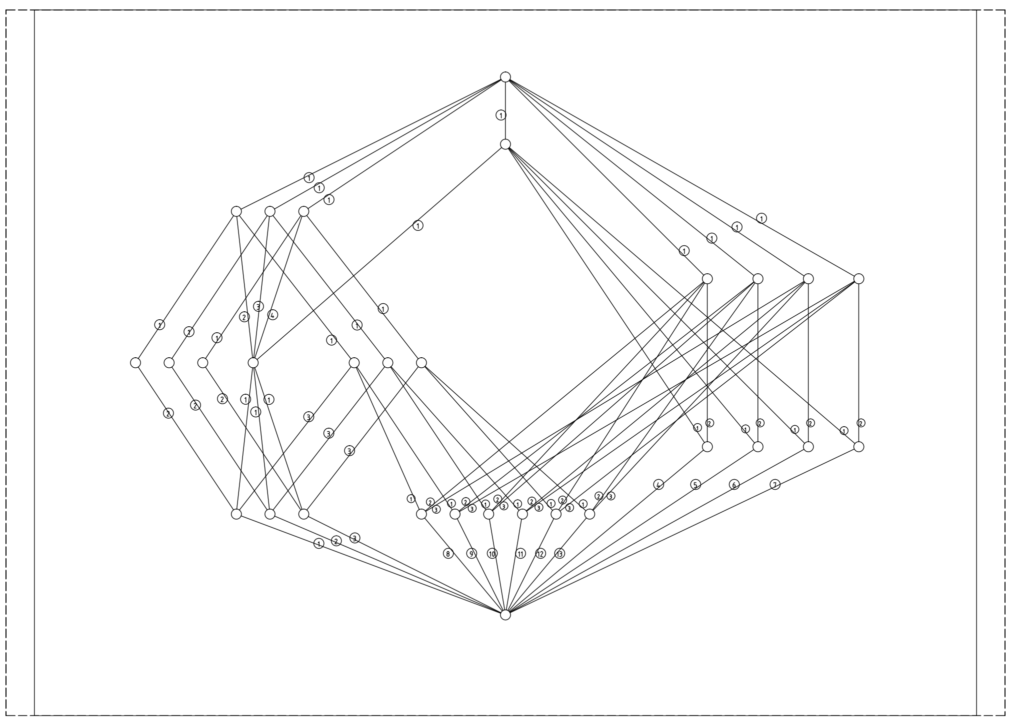

Figure 5.1. Recursive atom ordering independent of roots on the subgroup lattice of S_4

P to $\mathbb{Z}$, where two such maps are considered equivalent if they induce the same partition on P. If $x \lessdot y$ for two elements x and y in $\mathcal{O}(P)$, y is obtained from merging two blocks in x.

For example, the order congruence lattice on a totally ordered set is a boolean lattice. The order congruence lattice on a set of pairwise incomparable elements is isomorphic to a partition lattice. In general, order congruence lattice of any poset can be considered as in between the boolean lattice and the partition lattice.

Schweig and Woodroofe proved in [8] that order congruence lattices are comodernistic, therefore CL-shellable. We here present a different proof where any linear extension of P gives a sub-M-chain and an EL-shelling on $\mathcal{O}(P)$.

Fix a linear extension of $L = \{z_1, z_2, \ldots, z_n\}$. For an element in $\mathcal{O}(P)$ with k blocks, we can index these blocks with $[k]$ as follows. For two blocks B and B', if there exists $x \in B$ and $x' \in B'$ with $x < x'$, then B receives a smaller index than B'. Otherwise, x and x' are incomparable for every pair of $x \in B$ and $x' \in B'$. Let x_0 and x_0' be the smallest elements in B and B' correspondingly given the linear extension L. Then B receives a smaller index than B' if x_0 is prior to x_0' in the linear extension. This indexing is clearly well-defined, and We use these indices to construct a sub-M-chain and EL-shelling as follows.

Theorem 5.2.1 *Let C be the maximal chain $\hat{0} = c_0 \lessdot c_1 \lessdot \cdots \lessdot c_n = \hat{1}$ where c_k is obtained by having the first k elements (z_1 through z_k) in a block, and every other block contains exactly one element. Then C is a sub-M-chain on $\mathcal{O}(P)$. Let Λ be the edge-*

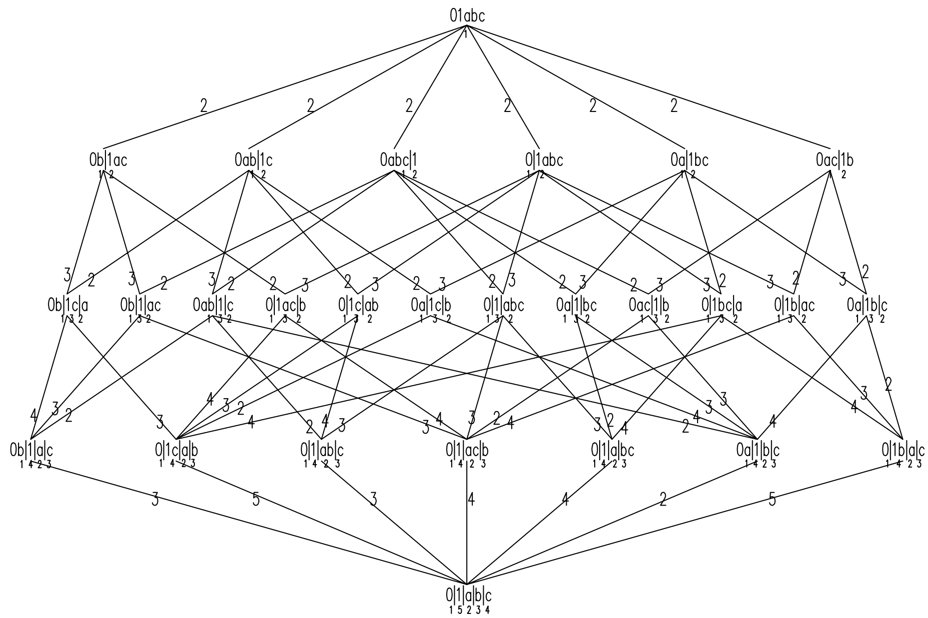

Figure 5.2. EL-shelling on N_5 with linear extension $\hat{0} \to a \to b \to c \to \hat{1}$

labeling $\mathcal{O}(P) \to \mathbb{Z}$ *that assigns* j *to the edge* $x \lessdot y$ *if* y *is obtained from* x *by combining blocks* B_i *and* B_j *with* $i < j$*. Then* Λ *is an EL-labeling.*

If $x \in [\hat{0}, c_k]$, then every z_i is in a single block for $i > k$. We need to show either $x < c_{k-1}$ or $x \wedge c_{k-1} \lessdot x$. If z_k is a single block in x, then $x < c_{k-1}$. Otherwise, $x \wedge c_{k-1}$ is obtained from x by subdividing the block containing z_k into two blocks, one of which contains z_k only. Hence $x \wedge c_{k-1}$ is covered by x and C is a sub-M-chain.

Consider an interval $[x, y]$ in $\mathcal{O}(P)$ with y having k blocks. Notice that any edge in $[x, y]$ can be viewed as combining blocks within one of the k blocks of y. Consider the

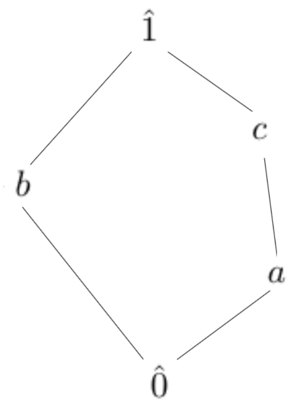

Figure 5.3. N_5

lexicographically first maximal chain $x = x_0 \lessdot x_1 \lessdot x_2 \lessdot \cdots \lessdot x_l = y$ in $[x, y]$. Each x_i is obtained from x_{i-1} by combining the smallest indexed two blocks of x_{i-1} that are contained in the same block of y. It is a weakly increasing chain. We prove that this is the unique weakly increasing chain by induction on the length of the interval. Suppose the first edge in a maximal chain is obtained by merging two blocks of x within a block of y that are not the smallest indexed. Denote this partition by x_1'. By induction, the first edge of the unique weakly increasing chain in $[x_1', y]$ receives a label strictly less than the label of $[x \lessdot x_1']$. Hence this maximal chain cannot be weakly increasing in $[x, y]$.

Example 5.2.1 *In Figure 5.2, we present an EL-labeling on $\mathcal{O}(N_5)$ as described in Theorem 5.2.1, where N_5 is the poset as shown in Figure 5.3. Notice that we ordered the blocks below each partition, as it tells us how to label the edges.*

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