Spring 5-15-2020

Dynamic Portfolio Optimization with a Noisy Observation of the Hidden Economic Regime

Siqi Peng

WASHINGTON UNIVERSITY IN ST. LOUIS

Department of Mathematics

Dynamic Portfolio Optimization with a Noisy Observation of the Hidden Economic Regime

by

Siqi Peng

A thesis presented to
The Graduate School
of Washington University in
partial fulfillment of the
requirements for the degree
of Master of Arts

May 2020
St. Louis, Missouri

Table of Contents

Acknowledgments

Foremost, I would like to express my sincere gratitude to my advisor Professor José E. Figueroa-López. He gave me the opportunity to work on such an exciting topic and guided me with patience throughout the entire process. I would also like to thank my committee members, Professor Nan Lin and Professor Likai Chen, for providing their valuable time. Last but not least, I would like to thank my parents for their support.

Siqi Peng

Washington University in St. Louis

May 2020

Dedicated to My Family.

ABSTRACT OF THE THESIS

Dynamic Portfolio Optimization with a Noisy Observation of the Hidden Economic Regime

by

Peng, Siqi

Master of Arts in Statistics,

Washington University in St. Louis, 2020.

Professor José E. Figueroa-López, Research Advisor

In this thesis, we aim to maximize the expected utility of a risk-averse investor by allocating her wealth in a risk-free bank account, a stock, and a defaultable security. The securities' rates of return depend on a hidden continuous-time finite-state Markov chain representing the economic regime. The securities' price volatilities are determined by an observable process, which can be considered as a noisy observation of the hidden economic regime. We use a two dimensional Markov chain, with a special construction of its generator matrix, to model the joint dynamics of the hidden process and the observable process. Our model is a generalized version of the original model introduced by Capponi et al. [1]. For our generalized model, we find the dynamics of the filtered economic regime probabilities. Then, using it, we reduce the partially observed optimal control problem to a control problem with complete observation.

1. Introduction and Background

In this chapter, we introduce the mathematical construction of our financial problem. It is common to use stochastic processes to model the dynamics of financial securities' prices, $S_t = (S_t^1, \cdots, S_t^N)$, where S_t^i is the price of the i^{th} security at time t. On the other hand, the investor's holding positions, $\nu_t = (\nu_t^1, \cdots, \nu_t^N)$, on the securities over time can also be viewed as a stochastic process which is under the investor's control. Consequently, the value process V^ν corresponding to the portfolio ν is given by (see Chapter 6 in [2])

$$V_t^\nu = \sum_{i=1}^{N} \nu_t^i S_t^i, \tag{1.1}$$

which represents the wealth of the investor.

In real life, the investor's control over ν is subject to restrictions. If we assume that there is no additional capital invested into or consumption from the portfolio, in order to long (buy) a security, the investor will have to raise the fund by shorting (selling) another security. Namely, the action of making transactions itself will not affect the wealth, V_t, of the investor. We further assume that there is no cash dividend payout (or 100% reinvestment rate), and then money can only be earned from price fluctuations. This leads to the concept of a self-financing portfolio defined as follows.

Definition 1.0.1 *A portfolio ν is called **self-financing** if the value process V^ν satisfies the condition*

$$dV_t^\nu = \sum_{i=1}^{N} \nu_t^i dS_t^i. \tag{1.2}$$

To apply the self-financing constraint to the investor's control, we define the relative portfolio as follows.

Definition 1.0.2 *For a given portfolio* ν *the corresponding* ***relative portfolio*** $\pi_t = [\pi_t^1, \cdots, \pi_t^N]$ *is given by*

$$\pi_t^i = \frac{\nu_t^i S_t^i}{V_t^\nu},\ i = 1, ..., N, \tag{1.3}$$

where we have

$$\sum_{i=1}^{N} \pi_t^i = 1.$$

The definition above indicates that with N different securities, we only have free control over $N-1$ of them. To incorporate such a constraint, we can just set $\pi_N = 1 - \sum_{i=1}^{N-1} \pi_t^i$, and realize that the investor can only freely determine the value of $(\pi^1, \cdots, \pi^{N-1})$. In the following proposition, we provide the dynamics of a self-financing portfolio in terms of π.

Proposition 1.0.1 *A relative portfolio* π *is self-financing if and only if*

$$dV_t^\pi = V_t^\pi \sum_{i=1}^{N} \pi_t^i \frac{dS_t^i}{S_t^i}. \tag{1.4}$$

The fundamental goal of managing a portfolio is to maximize the return by controlling the holding positions. This goal leads us to an optimal stochastic control problem, which is to maximize an specified function, by controlling the dynamics of the value process, V_t, through the control process, π_t. In this thesis, the objective function is the expected utility of the final portfolio value at a fixed terminal time T.

We should notice that the self-financing constraint is not enough to make our problem have a practical meaning. When the investor decides the relative portfolio π_t, she can only use the information available up to time t and cannot trade on future information. To describe this constraint mathematically, we define the investor's filtration, $\mathbb{G}^I = \{\mathcal{G}_t^I\}_{0 \le t}$, as the filtration generated by all observable processes. We require π_t to be $\mathbb{G}^I - adapted$. Therefore, π_t can be expressed as a function of the observable processes up to time t.

Regarding the price processes, we further assume that an unobservable process X_t, representing the economic regime, can affect the rates of return on the securities. A model of this type has been proposed in Capponi et al.'s earlier work [1], which serves as inspiration of this work. In their work. In their work, the volatility of the stock is assumed to be constant. In this thesis, however, we propose a new model that allows the price processes to have time-varying volatilities related to the hidden process in some noisy sense. Our new model is a generalized version of the original model. We present the market model and the optimization problem in Chapter 2, with the part similar to [1] emphasized in the first section, and the new model introduced in the second section.

With X_t assumed to be unobservable, we do not observe the entire system that determines the dynamics of the value process, V_t. Thus, we need to solve an optimal control problem with partial knowledge (see chapter 11 of [3]). In chapter 3, we first solve a filtering problem, which is to find the dynamics of the filtered probability $p_t^i = \mathbb{P}(X_t = e_i | \mathcal{G}_t^I)$ (see chapter 9 of [4]). Then, we will develop an equivalent optimal control problem under a new measure, where the hidden process is independent of the system. Therefore, we reduce the problem to a control problem with complete observation.

2. The Optimal Control Problem

In this thesis, we provide a generalized version of the model introduced by Capponi et al. [1], and the approach in this thesis is developed based on their original work. In the first section, we will have a brief review of the original paper, with the aim of introducing some needed notations. Then, we present our model and new results in the second section.

2.1 The Original Problem

2.1.1 The Market Model

Under a complete filtered probability space, $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$, we define a hidden continuous-time finitie-state Markov chain, X_t, which has N different states, $\{e_1, e_2, \cdots, e_N\}$, where $e_i = (0, \cdots, 1, \cdots, 0)^\top \in \mathbb{R}^N$ with only the i^{th} dimension equals to 1. Let $p_{i,j}(s,t) = \mathbb{P}(X_t = e_j | X_s = e_i)$, $\forall 0 \leq s \leq t$ and $i, j = 1, \cdots, N$, denote transition probabilities and $p^{\circ,i} = \mathbb{P}(X_0 = e_i)$ denote initial probabilities. Assume there exists

$$\bar{w}_{i,j}(t) = \lim_{h \to 0} \frac{p_{i,j}(t, t+h) - \delta(i,j)}{h}, \tag{2.1}$$

where $\delta(\cdot, \cdot)$ is the Kronecker delta. $\bar{w}_{i,j}(t)$ is called the infinitesimal generator of the chain or the jump rate from state e_i to e_j. In addition, the following assumption is imposed.

$$\sup_{t \geq 0} \max_{i,j} \bar{w}_{i,j}(t) < \infty. \tag{2.2}$$

Then, we have the semimartingale representation of the Markov chain, X_t, (see [5])

$$X_t = X_0 + \int_0^t G^\top(s) X_s ds + \phi(t), \tag{2.3}$$

where $\phi(t) = [\phi_1(t), \cdots, \phi_N(t)]^\top$ is an N-dimensional square integrable martingale with a right continuous trajectory, and the generator matrix $G(t) := [\bar{w}_{i,j}(t)]_{i,j=1\cdots,N}$.

The Markov process X_t defined above is assumed to be unobservable and is used to represent the economic regime. The hidden economic regime determines the rates of return on risky assets. We assume that the financial market consists of three instruments defined as follows: Under $\mathbb{P}$, let $W_t^{(1)}$ and $W_t^{(2)}$ be independent $\mathbb{G}-adapted$ Brownian motions.

- Risk-free bank account: The dynamics of its price process is defined to be

$$dB_t = rB_t dt, \quad B_0 = 1. \tag{2.4}$$

- Stock security: Its price dynamics is defined to be

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t^{(1)}, \quad S_0 = s^\circ, \tag{2.5}$$

  where

$$\mu_t = \langle \mu, X_t \rangle, \quad \sigma_t \equiv \sigma,$$

  and $\mu = [\mu^1, \cdots, \mu^N]^\top$ is a constant vector of the rates of return in different economic regimes. Here, we use $\langle \cdot, \cdot \rangle$ to denote the inner product of two vectors.

- Defaultable security: Before default, the defaultable security has a similar price dynamics as the stock security, but it becomes worthless at the moment of default. Let τ denote the default time and $H_t := 1_{\tau \leq t}$ denote the default process. The default time is defined by rescaling an independent exponential random variable, χ (see section 5.1 in [6] for detail).

$$\tau = \inf \left\{ t \in R^+ : \int_0^t h_u du \geq \chi \right\}. \tag{2.6}$$

The hazard process, h_t, is determined by the economic regime as follows:

$$h_t := \langle h, X_t \rangle ,$$

where $h = [h^1, \cdots, h^N]^\top$ is a vector of hazard rates at different economic states. The semimartingale representation of the default process, which can be found in section 6.5 of [6], is

$$H_t = \int_0^t \bar{H}_{u^-} h_u du + \xi_t = \int_0^{t\wedge\tau} h_u du + \xi_t \tag{2.7}$$

where ξ_t is a $\mathbb{G}$-martingale under $\mathbb{P}$, and $\bar{H}_u := 1 - H_u$. For any stochastic process, X_t, we use $X_{t^-} = \lim_{s\to t^-} X_s$ to denote its left limit.

Based on the default process, the dynamics of the defaultable security is defined to be

$$dP_t = P_{t^-}(a(t, X_t)dt + \upsilon_t dW_t^{(2)} - dH_t), \quad P_0 = P^\circ, \tag{2.8}$$

where

$$\upsilon_t \equiv \upsilon,$$

and we assume that

$$\int_0^T a^2(t, e_i)dt < \infty, \quad \forall\, T > 0 \text{ and } i \in \{1, \ldots, N\}. \tag{2.9}$$

For convenience, when dealing with the relative portfolio, we define the pre-defult log-price process as $Y_t = (\log S_t, \log P_t)^\top$. Its dynamics is given by

$$dY_t = \vartheta_t dt + \Sigma_t dW_t,$$

where

$$W_t := \begin{pmatrix} W_t^{(1)} \\ W_t^{(2)} \end{pmatrix}, \quad \Sigma_t := \begin{pmatrix} \sigma_t & 0 \\ 0 & \upsilon_t \end{pmatrix}, \text{ and } \vartheta_t := \begin{pmatrix} \mu_t - \frac{\sigma_t^2}{2} \\ a(t, X_t) - \frac{\upsilon_t^2}{2} \end{pmatrix}.$$

As mentioned earlier, the investor's filtration is defined to be

$$\mathcal{G}_t^I := \sigma(S_u, P_u, H_u; u \leq t). \tag{2.10}$$

2.1.2 The Utility Maximization Problem

The goal of managing the portfolio is to maximize the investor's expected utility at the fixed terminal time, T, by allocating her wealth in the three different types of instruments defined in the previous section during the period $[0, T]$.

Let the portfolio process, $\nu = \{(\nu_t^B, \nu_t^S, \nu_t^P)\}_{t \geq 0}$, denote the investor's holding positions on the three instruments at time t. Here, a negative value in ν represents a short position. The value process, V_t^ν, is defined to be

$$V_t^\nu = \nu_t^B B_t + \nu_t^S S_t + \nu_t^P \mathbf{1}_{\tau > t} P_t. \tag{2.11}$$

We assume that the investor does not invest additional capital into or consume from the portfolio. Thus, we get the self-financing dynamics.

$$dV_t = \nu_t^B dB_t + \nu_t^S dS_t + \nu_t^P \mathbf{1}_{\tau > t} dP_t. \tag{2.12}$$

Remark: The self-financing condition above is different from the conventional one introduced in the first chapter due to the extra $\mathbf{1}_{\tau > t}$. The portfolio process is often assumed to be left continuous when there are jumps in the securities' prices. Otherwise, we can long (or short) the security at its jump time to achieve an unrealistic gain. In our case, $\nu_t^P \mathbf{1}_{\tau > t}$ is assumed to be 0 at the default time τ, which is equivalent to ignoring the loss from default. As a result, the value process, V_t, will be continuous ($\mathbb{P} - a.s.$), even if the default event has happened. This is often considered as a violation. However, in our case, the pre- and post-default problems will be solved separately, and (2.12) does correctly describe the pre- and post-default dynamics. The only violation point is at the default time, and we do not allow an unrealistic gain by setting the holding position to be 0 instead of a large negative number. In this case, such construction will still give a valid trading strategy. We emphasize this issue here because it will be addressed in later research and ignored in this thesis. However, this thesis will provide a useful partial solution to the whole problem.

The relative portfolio is defined to be

$$\pi_t^B := \frac{\nu_t^B B_t}{V_{t^-}^{\nu}}, \quad \pi_t^S := \frac{\nu_t^S S_t}{V_{t^-}^{\nu}}, \quad \pi_t^P = \frac{\nu_t^P P_t}{V_{t^-}^{\nu}} \mathbf{1}_{\tau > t}. \tag{2.13}$$

Using the condition $\pi^B + \pi^P + \pi^S = 1$, and the dynamics of price processes defined in section 2.1.1, we can rewrite the dynamics of the value process as

$$\frac{dV_t^\pi}{V_{t^-}^\pi} = rdt + \pi_t^S (\mu_t - r)\, dt + \pi_t^S \sigma_t dW_t^{(1)} + \pi_t^P (a(t, X_t) - r)\, dt + \pi_t^P \upsilon_t dW_t^{(2)}, \quad V_0^\pi = v, \tag{2.14}$$

where $v \in (0, \infty)$ is the initial wealth.

Our goal is to choose a $\mathbb{G}^I - adapted$ strategy $\pi = (\pi^S, \pi^P)^\top$ that maximize the expected terminal utility of the risk-averse investor, which is defined to be

$$J(v, \pi, T) := \frac{1}{\gamma} \mathbb{E}^{\mathbb{P}} \left[(V_T^\pi)^\gamma \right], \tag{2.15}$$

where $\gamma \in (0, 1)$ is a given fixed value that reflects the diminishing return to the risk-averse investor. With the notation $V_t^\gamma := (V_t^\pi)^\gamma$, the dynamics of the utility process is given by

$$V_t^\gamma = v^\gamma \exp\left(\gamma \int_0^t \pi_s^\top \Sigma_s dW_s - \gamma \int_0^t \eta_s ds - \frac{\gamma^2}{2} \int_0^t \pi_s^\top \Sigma_s \Sigma_s^\top \pi_s ds \right), \tag{2.16}$$

where

$$\eta_t = -r + \pi_t^S (r - \langle \mu, X_t \rangle) + \pi_t^P (r - a(t, X_t)) + \frac{1-\gamma}{2} \pi_t^\top \Sigma_t^\top \Sigma_t \pi_t. \tag{2.17}$$

2.2 The New Model

As shown in the previous section, the volatilities, σ_t and υ_t, are assumed to be constant. However, in reality, the volatility is not necessarily a constant through all economic regimes. For example, the stock prices are usually more volatile during recessions [7]. The constant volatility assumption is necessary for the original model to comply with the economic regime's unobservable assumption. If the volatility is a one-to-one function of X_t, the unobservable assumption will be violated since X_t will be observed through the quadratic variation of the log price process, which converges ($\mathbb{P} - a.s.$) to the integrated volatility [8]. To solve this problem, it is natural to model the volatility process by an observable process, O_t, that is stochastically determined by X_t.

Notice that not only the states of X_t need to remain unobservable, but also its jump times. Investors generally do not know when the economy changes states. So, there should be a random time lag between a jump time of X_t and the response time of O_t. Further, the observable process, O_t, may be viewed as an investor's exogenous view about the current state of the economy, developed through policies, macroeconomic variables, or other information available. In each case, there is a time lag between a change in X_t and the corresponding change in O_t. This time lag is known as the recognition lag in Economics.

With the motivations above, we develop a new model in the following subsection.

2.2.1 Economic Regime

We first define the hidden economic regime X_t and the observable process O_t.

Definition 2.2.1 *Let $\{(O_t, X_t)\}_{0 \le t}$ be a two dimensional continuous-time Markov chain with finite states, where X_t is the hidden process, and O_t is observable. The two dimensional Markov chain has $M \times N$ different pairs of states. We assume that the two dimensions, X and O, do not change together in one transition. For each dimension,*

- *X_t has N different states, $\{e_1, e_2, \cdots, e_N\}$, where $e_i = (0, \cdots, 1, \cdots, 0)^\top \in \mathbb{R}^N$. Let $\bar{w}_{i,j}(t)$ denotes the jump rate from $(O = u_a, X = e_i)$ to $(O = u_a, X = e_j)$, for $i \neq j$, at time t, and it is the same for all different a's.*

- *O_t has M different states, $\{u_1, u_2, \cdots, u_M\}$, where $u_i = (0, \cdots, 1, \cdots, 0)^\top \in \mathbb{R}^M$. Let $w_{a,b}^{(k)}(t)$ denotes the jump rate from $(O = u_a, X = e_k)$ to $(O = u_b, X = e_k)$, for $a \neq b$, at time t.*

For the initial distribution, let O_0 have a degenerate marginal distribution at its observed value, and independent to X_0. X_0 has initial probabilities $p^{\circ,i} > 0$, $\forall i = 1, \cdots, N$.

The assumption that the two dimensions, X and O, do not change together in one jump is realistic, since investors have lagged responses to economic changes.

Further, if the initial distribution assumption is not satisfied, we can make our analysis conditional on all the information available at time 0, which will simplify the problem into our assumed case. Namely, $p^{\circ,i}$ can be viewed as the conditional probability given all information available at time 0.

We should notice that the dimensions of the two dimensional Markov chain is superfluous from a mathematical perspective. We can represent the process (O_t, X_t) as a univariate Markov chain, Z_t with $M \times N$ states. Let

$$Z_t = [Z_{1,1}(t), Z_{1,2}(t), \cdots, Z_{a,i}(t), \cdots, Z_{M,N}(t)]^\top \tag{2.18}$$

where

$$Z_{a,i}(t) = \begin{cases} 1 & (O_t, X_t) = (u_a, e_i) \\ \\ 0 & o.w. \end{cases}.$$

Then, similar to the previous section, we have the following representation.

$$Z_t = Z_0 + \int_0^t A^\top(s) Z_s ds + \varphi(t),$$

where $\varphi(t) = [\varphi_{1,1}(t), \varphi_{1,2}(t), \cdots, \varphi_{a,i}(t), \cdots, \varphi_{M,N}(t)]^\top$ is a $M \times N$ dimensional martingale. $A(t) := [\alpha_{a,b}^{(i,j)}(t)]$ is the generator matrix with

$$\alpha_{a,b}^{(i,j)}(t) = \begin{cases} 0 & a \neq b,\ i \neq j \\ \\ w_{a,b}^{(i)}(t) & a \neq b,\ i = j \\ \\ \bar{w}_{i,j}(t) & a = b,\ i \neq j \\ \\ w_{a,a}^{(i)}(t) + \bar{w}_{i,i}(t) & a = b,\ i = j \end{cases} \tag{2.19}$$

on its $[a(N-1)+i]^{th}$ row and $[b(N-1)+j]^{th}$ column, where $w_{a,a}^{(i)} = -\sum_{c \neq a} w_{a,c}^{(i)}$ and $\bar{w}_{i,i} = -\sum_{k \neq i} \bar{w}_{i,k}$.

We will also impose the following mild assumption for later defining the change of measure:

$$\sup_{t \geq 0} \max_{a,b,i,j} \alpha_{a,b}^{(i,j)}(t) < \infty. \tag{2.20}$$

For each dimension of Z_t, we have, for all $a = 1, \cdots, M$ and $i = 1, \cdots, N$,

$$Z_{a,i}(t) = Z_{a,i}(0) + \int_0^t \alpha_{O_s,a}^{(X_s,i)}(s) ds + \varphi_{a,i}(t), \tag{2.21}$$

where we abuse the notation in the superscript and subscript by leting X_t and O_t represent the corresponding index, i and j, of their states e_i and u_j, respectively.

2.2.2 The Marginal Distribution of X_t

To verify that the new model is a generalized version of the original one, the following property must be satisfied.

Proposition 2.2.1 *Under the construction in definition 3.2.1, $\{X_t\}_{t\geq 0}$ is a Markov chain marginally with jump rates $\bar{w}_{i,j}(t)$.*

Proof For $s > t > u \geq 0$,

$$\begin{aligned}
&\mathbb{P}(X_s = e_j | X_t = e_i, X_u = e_k) \\
=&\sum_{a=1}^{M} \mathbb{P}(X_s = e_j | X_t = e_i, O_t = u_a, X_u = e_k)\mathbb{P}(O_t = u_a | X_t = e_i, X_u = e_k) \\
=&\sum_{a=1}^{M} \mathbb{P}(X_s = e_j | X_t = e_i, O_t = u_a)\mathbb{P}(O_t = u_a | X_t = e_i, X_u = e_k) \qquad (2.22)
\end{aligned}$$

Because of the balanced design that the dynamics of X_t are the same for all different states in O_t, we have

$$\begin{aligned}
\mathbb{P}(X_s = e_j | X_t = e_i, O_t = u_1) = \cdots &= \mathbb{P}(X_s = e_j | X_t = e_i, O_t = u_M) \\
&= \mathbb{P}(X_s = e_j | X_t = e_i).
\end{aligned}$$

Plug into (2.22), we get

$$\begin{aligned}
\mathbb{P}(X_s = e_j | X_t = e_i, X_u = e_k) &= \mathbb{P}(X_s = e_j | X_t = e_i) \sum_{a=1}^{M} \mathbb{P}(O_t = u_a | X_t = e_i, X_u = e_k) \\
&= \mathbb{P}(X_s = e_j | X_t = e_i). \qquad (2.23)
\end{aligned}$$

Thus, X_t is a markov chain marginally.

Similarly, the jump rates are given by

$$
\begin{aligned}
&\lim_{h\to 0}\frac{\mathbb{P}(X_{t+h}=e_j|X_t=e_i)-\delta(i,j)}{h}\\
=&\lim_{h\to 0}\frac{\sum_{a=1}^{M}\mathbb{P}(X_{t+h}=e_j|X_t=e_i,O_t=u_a)\mathbb{P}(O_t=u_a|X_t=e_i)-\delta(i,j)}{h}\\
=&\lim_{h\to 0}\frac{\sum_{a=1}^{M}\left[\mathbb{P}(X_{t+h}=e_j|X_t=e_i,O_t=u_a)-\delta(i,j)\right]\mathbb{P}(O_t=u_a|X_t=e_i)}{h}\\
=&\sum_{a=1}^{M}\mathbb{P}(O_t=u_a|X_t=e_i)\lim_{h\to 0}\frac{\left[\mathbb{P}(X_{t+h}=e_j|X_t=e_i)-\delta(i,j)\right]}{h}\\
=&\sum_{a=1}^{M}\mathbb{P}(O_t=u_a|X_t=e_i)\bar{w}_{i,j}(t)=\bar{w}_{i,j}(t).
\end{aligned}
$$

Therefore, we finish the proof. ■

The Proposition 2.2.1 suggests that the hidden process will still have the representation (2.3). Moreover, the assumption (2.2) can be deduced from (2.20). For each dimension of $X_t=[X_1(t),\cdots,X_N(t)]^\top$, we have

$$X_i(t)=X_i(0)+\int_0^t \bar{w}_{X_s,i}ds+\phi_i(t),\quad \forall i=1,\cdots,N, \tag{2.24}$$

where we notice that $\phi_i(t)=\sum_{a=1}^{M}\varphi_{a,i}(t)$.

2.2.3 The Dynamics of O_t

Let $O_t=[O_1(t),O_2(t),\cdots,O_M(t)]^\top$ with each dimension serving as an indicator of the corresponding state. i.e.

$$O_a(t)=\mathbf{1}_{O_t=u_a},\quad \forall\, a=1,\cdots,M.$$

Then, we have the representation

$$
\begin{aligned}
O_a(t) = \sum_{i=1}^{N} Z_{a,i}(t) &= \sum_{i=1}^{N} Z_{a,i}(0) + \sum_{i=1}^{N} \int_0^t \alpha_{O_s,a}^{(X_s,i)}(s)ds + \sum_{i=1}^{N} \varphi_{a,i}(t) \\
&= O_a(0) + \int_0^t \sum_{i=1}^{N} \alpha_{O_s,a}^{(X_s,i)}(s)ds + \varphi_a(t) \\
&= O_a(0) + \int_0^t w_{O_s,a}^{(X_s)}(s)ds + \varphi_a(t) \\
&= O_a(0) + \int_0^t [(G^{(X_s)})^\top O_s]_a ds + \varphi_a(t), \qquad (2.25)
\end{aligned}
$$

where $\varphi_a(t) = \sum_{i=1}^{N} \varphi_{a,i}(t)$ is a martingale, and $G^{(i)}(t) = [w_{a,b}^{(i)}]_{1\le a,b\le M}$ is a $M \times M$ matrix. So,

$$
O_t = O_0 + \int_0^t (G^{(X_s)})^\top O_s ds + [\varphi_1(t), \cdots, \varphi_M(t)]^\top. \qquad (2.26)
$$

We find that it is more convenient to represent the observable process O_t using a counting process, K_t, which counts the number of visits to each state. Let $K_t = [K_1(t), \cdots, K_M(t)]^\top$, where $K_a(t)$ is the number of visits to state u_a over the time $[0, t]$ counting the initial state. Then,

$$
\begin{aligned}
K_a(t) &= O_a(0) + \int_0^t (1 - O_a(s^-))dO_a(s) \\
&= O_a(0) + \int_0^t (1 - O_a(s^-))w_{O_s,a}^{(X_s)}ds + \int_0^t (1 - O_a(s^-))d\varphi_a(s) \\
&=: O_a(0) + \int_0^t r_a(s)ds + M_a(t), \qquad (2.27)
\end{aligned}
$$

where $M_a(t)$ is a margingale. K_t is adapted to $\mathcal{G}_t^O$, the filtration generated by O_t.

At time t, O_t has the state $u_{\hat{a}}$ where $\hat{a} = \arg\min_a \inf\{s : K_a(t) - K_a(s^-) = 1\}$, with the convention $K_a(0^-) = 0$ and $\inf \emptyset = \infty$ for all a. So, O_t is also adapted to $\mathcal{G}_t^K$, the filtration generated by K_t. Then, we can replace O_t with K_t in our later analysis since

$$
\mathcal{G}_t^O = \mathcal{G}_t^K \qquad (2.28)
$$

2.2.4 The New Market Model and Optimization Problem

The observable process, O_t, is introduced to enable time-varying volatilities. Now, instead of assuming σ_t and υ_t to be fixed constants, we let

$$\sigma_t := \langle \sigma, O_t \rangle \quad \upsilon_t := \langle \upsilon, O_t \rangle , \tag{2.29}$$

where $\sigma := [\sigma^{(1)}, \cdots, \sigma^{(M)}]^\top$ and $\upsilon := [\upsilon^{(1)}, \cdots, \upsilon^{(M)}]^\top$ are constant vectors consisting of different volatilities corresponding to the observable states. Our change will only affect the dynamics in the original model through the volatility process. So, the new volatilities will not change the representation of the dynamics introduced in section 2.1 since the change will not violate the assumption of Itô's formula.

Since we have an additional observation, O_t, over X_t, the investor's filtration now becomes,

$$\mathcal{G}_t^I := \sigma(S_u, P_u, H_u, O_u; u \leq t) = \sigma(S_u, P_u, H_u, K_u; u \leq t). \tag{2.30}$$

We require the control process π_t, to be adapted to the newly defined investor's filtration, and the goal is still to maximize the objective function (2.15). The hidden process X_t is adapted to the underlying filtration $\mathbb{G}$, but not the investor's filtration $\mathbb{G}^I$. So this is an optimal control problem with partial observation. In next chapter, we will reduce the problem to a complete observation problem.

Now, it's easy to see that if $\sigma^{(1)} = \cdots = \sigma^{(M)}$ and $\upsilon^{(1)} = \cdots = \upsilon^{(M)}$, the market model will be the same as the original model. If we further assume that $w_{a,b}^{(1)} = \cdots = w_{a,b}^{(N)}$, for all $a, b = 1, \cdots, M$, by (2.26), the observable process will be independent of X_t. This is just the case in the original model. So, under the special case, our solution should be the same as in the original paper. This is confirmed by our solution in next chapter.

3. The Problem Reduction

In the first section, we find the dynamics of the filtered probabilities of the economic regimes. Then, we present an equivalent complete observation control problem in the second section.

3.1 Filtered Probabilities

In this part, we will derive the filtered probabilities,

$$p_t^i = \mathbb{P}(X_t = e_i | \mathbb{G}^I) \quad \forall\, i = 1, \cdots N, \tag{3.1}$$

using a new probability measure $\hat{\mathbb{P}}$. Under the new measure, the hidden process X_t is independent of the investor's filtration.

To define the change of measure, we first introduce some notations. Let $[\cdot]$ and $[\cdot,\cdot]$ denote the quadratic variation and quadratic covariation, respectively. We denote the stochastic exponential of L_t as $\mathcal{E}_t(L)$. We first solve some stochastic exponentials for later use (see [1] for the first two solutions):

- For a continuous Itô process Y_t, let $L_t = \int_0^t \theta_s^\top dY_s$, where θ_s is $\mathbb{G}$-predictable. Then,

$$\mathcal{E}_t(L) = \exp\left(\int_0^t \theta_u^\top dY_u - \frac{1}{2}\int_0^t \theta_u^\top \theta_u d[Y]_u \right). \tag{3.2}$$

- For a martingale ξ_s as defined in (2.7), let $L_t = \int_0^t \iota_s d\xi_s$, where ι_s is $\mathbb{G}$-predictable, with $\iota > -1$. Then,

$$\mathcal{E}_t(L) = \exp\left(\int_0^t \log(1+\iota_s) dH_s - \int_0^{t\wedge\tau} \iota_s h_s ds \right). \tag{3.3}$$

- For a martingale $M_a(t)$ as defined in(2.27), let $L_t = \int_0^t \iota_s dM_a(s)$, where ι_s is $\mathbb{G}$-predictable, with $\iota > -1$. Then,

$$\mathcal{E}_t(L) = \exp\left(\int_0^t \log(1+\iota_s)dK_s - \int_0^t \iota_s r_a(s)ds\right) \tag{3.4}$$

Only the last stochastic exponential is not given in the original paper [1], so we will check it here. Since $M_a(t)$ is a finite variation process with finite jumps on finite interval, and so does $L_t = \int_0^t \iota_s(dK_a(s) - r_a(s)ds)$. By Lemma 4.4.1 in [6],

$$\begin{aligned}
\mathcal{E}_t(L) &= e^{L_t^c} \prod_{0<u\le t}(1+\Delta L_u) \\
&= \exp\left(-\int_0^t \iota_s r_a(s)ds\right)\cdot \exp\left(\sum_{0<u\le t}\log(1+\Delta L_u)\right) \\
&= \exp\left(-\int_0^t \iota_s r_a(s)ds + \sum_{0<u\le t}\log(1+\iota_u\Delta K_u)\right) \\
&= \exp\left(-\int_0^t \iota_s r_a(s)ds + \int_0^t \log(1+\iota_s)dK_s\right)
\end{aligned}$$

where, in the last step, we use the fact that K_s has jump size 1.

Then, we are ready to introduce the new measure $\hat{\mathbb{P}}$ on $(\Omega, \mathbb{G})$. It is defined using a density process as follows:

$$\begin{aligned}
\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{G}_t} &:= \mathcal{E}_t\left(\int_0^\cdot -\vartheta_s^\top \Sigma_s^{-1}dW_s\right)\mathcal{E}_t\left(\int_0^\cdot \frac{1-h_{s-}}{h_{s-}}d\xi_s\right)\prod_{a=1}^M \mathcal{E}_t\left(\int_0^\cdot \frac{1-r_a(s^-)}{r_a(s^-)}dM_a(s)\right) \\
&= \mathcal{E}_t\left(M^{(w)}\right)\mathcal{E}_t\left(M^{(\xi)}\right)\prod_{a=1}^M \mathcal{E}_t\left(M^{(a)}\right) =: \rho_t^{(w)}\rho_t^{(\xi)}\prod_{a=1}^M \rho_t^{(a)} = \rho_t
\end{aligned} \tag{3.5}$$

More specifically, using the solutions of stochastic exponentials above, we have,

- $$\rho_t^{(w)} = \mathcal{E}_t(M^{(w)}) = \exp\left(-\int_0^t \vartheta_s^\top \Sigma_s^{-1}dW_s - \frac{1}{2}\int_0^t \vartheta_s^\top(\Sigma_s\Sigma_s)^{-1}\vartheta_s ds\right);$$

- $$\rho_t^{(\xi)} = \mathcal{E}_t(M^{(\xi)}) = \exp\left(-\int_0^t \log(h_{s-})\,dH_s - \int_0^{t\wedge\tau}(1-h_{s-})ds\right);$$

- $\forall\, a = 1, \cdots, M$

$$\rho_t^{(a)} = \mathcal{E}_t(M^{(a)}) = \exp\left(-\int_0^t \log\left(r_a(s^-)\right) dK_a(s) - \int_0^{t\wedge\tau} (1 - r_a(s^-))ds \right).$$

Then, notice that $M^{(w)}$ is a continuous martingale. $M^{(\xi)}$ and $M^{(a)},\ a = 1, \cdots, M$, are purely discontinuous martingales. So,

$$[M^{(w)}, M^{(\xi)}]_t = [M^{(w)}, M^{(a)}]_t = 0, \quad \forall\, a = 1, \cdots, M. \tag{3.6}$$

Also, each of $M^{(\xi)}$ and $M^{(a)},\ a = 1, \cdots, M$, can be written as a sum of an absolutely continuous process (dt part) and a pure jump process. $M^{(a)}$ can only have finite jumps over $[0, T]$, which coincide with the jumps of the finite activity Markov chain O_t. Also, τ is constructed using an independent exponential random variable χ, which has 0 probability measure on any finite subset of $[0, T]$. So, there is no simultaneous jump of $M^{(\xi)}$ and $M^{(a)}$ ($\mathbb{P} - a.s.$). Also, by construction, for all $a \neq b$, $M^{(a)}$ and $M^{(b)}$ have no simultaneous jump. We get, for all $a, b = 1, \cdots, M,$ and $a \neq b$,

$$[M^{(a)}, M^{(\xi)}]_t = [M^{(a)}, M^{(b)}]_t = 0. \tag{3.7}$$

Next, we use the following formula for some martingales A and B:

$$\mathcal{E}_t\left(A\right)\mathcal{E}_t\left(B\right) = \mathcal{E}_t\left(A + B + [A, B]\right).$$

Since all the quadratic covariations are 0 as shown above, we get

$$\rho_t = \mathcal{E}_t\left(M^{(w)} + M^{(\xi)} + \sum_{a=1}^{M} M^{(a)} \right) =: \mathcal{E}_t\left(M\right). \tag{3.8}$$

So, we have the dynamics

$$\rho_t = 1 + \int_0^t \rho_{s-}(dM^{(w)}(s) + dM^{(\xi)}(s) + \sum_{a=1}^{M} dM^{(a)}(s)). \tag{3.9}$$

Next, we show that $\hat{\mathbb{P}}$ is a well-defined probability measure by checking $E^{\mathbb{P}}(\rho_T) = 1$ using the generalized Novikov's condition [9]:

$$\mathbb{E}^{\mathbb{P}}\left[e^{\frac{1}{2}\langle M^c, M^c\rangle_T + \left\langle M^d, M^d\right\rangle_T}\right] < \infty, \tag{3.10}$$

where M^c and M^d are the continuous and purely discontinuous parts of the martingale M. We use $\langle\cdot,\cdot\rangle_T$ denote the compensator of the quadratic variation at time T.

For the continuous part, we have that

$$\langle M^{(w)}, M^{(w)}\rangle_T = \int_0^T \vartheta_s^\top (\Sigma_s \Sigma_s)^{-1} \vartheta_s ds \tag{3.11}$$

is bounded, using the assumption (2.9).

The purely discontinuous part is

$$\begin{aligned}\langle M^d, M^d\rangle_T &= \langle M^{(\xi)} + \sum_{a=1}^{M} M^{(a)}, M^{(\xi)} + \sum_{a=1}^{M} M^{(a)}\rangle_T \\ &= \langle M^{(\xi)}, M^{(\xi)}\rangle_T + \sum_{a=1}^{M} \langle M^{(a)}, M^{(a)}\rangle_T.\end{aligned} \tag{3.12}$$

Then, we consider each term separately. Note that,

$$\langle M^{(\xi)}, M^{(\xi)}\rangle_T = \int_0^T \left[\frac{(1-h_s)^2}{h_s}\right] \bar{H}_s ds \tag{3.13}$$

is bounded due to the boundness of h_t and $\bar{H}_t$. Furthermore,

$$\langle M^{(a)}, M^{(a)}\rangle_T = \int_0^T \left[\frac{(1-r_a(s^-))^2}{r_a(s^-)}\right] ds, \text{ for all } a = 1, \cdots, M, \tag{3.14}$$

is also bounded since $r_a(t)$ is bounded because of (2.20) and (2.27). We have shown that all the compensators are bounded, which verifies the condition (3.10). So, $\hat{\mathbb{P}}$ is a valid probability measure. Next, to show that X_t is independent of $\mathbb{G}^I$ under $\hat{\mathbb{P}}$, we use the following proposition.

Proposition 3.1.1 *Under the probability measure* $\hat{\mathbb{P}}$ *defined in* (3.5),

$$\hat{W}_t = W_t + \int_0^t \Sigma_s^{-1} \vartheta_s ds \tag{3.15}$$

is a Brownian motion, and

$$\hat{\xi}_t = \xi_t - \int_0^{t\wedge\tau}(1-h_s)ds = H_t - \int_0^t \bar{H}_{s^-}ds \tag{3.16}$$

and

$$\hat{M}_a(t) = M_a(t) - \int_0^t (1-r_a(s^-))ds = K_a(t) - K_a(0) - t \tag{3.17}$$

are $\mathbb{G}^I$ *martingales.*

Proof The proof is an analog to the proof in section 5.3 of [6]. (3.15) and (3.16) are obvious form the proof in the book, using the fact that all the quadratic covariations are 0 as shown earlier in (3.6) and (3.7). (3.17) is verified as follows:

$$\begin{aligned}
d(\rho_t\hat{M}_a(t)) &= \hat{M}_a(t^-)d\rho_t + \rho_{t-}d\hat{M}_a(t) + d[\rho,\hat{M}_a(\cdot)]_t \\
&= \hat{M}_a(t^-)d\rho_t + \rho_{t-}dM_a(t) - \rho_{t-}(1-r_a(t^-))dt + \rho_{t-}\frac{1-r_a(t^-)}{r_a(t^-)}dK_a(t) \\
&= \hat{M}_a(t^-)d\rho_t + \rho_{t-}(1+\frac{1-r_a(t^-)}{r_a(t^-)})dM_a(t)
\end{aligned}$$

$\rho_t\hat{M}_a(t)$ is a $\mathbb{G}$-martingale under $\mathbb{P}$, so $\hat{M}_a(t)$ is a $\mathbb{G}$-martingale under $\hat{\mathbb{P}}$. Since $\hat{M}_a(t)$ is also adapted to $\mathbb{G}^I$, we finish the proof. ■

Moreover, under $\hat{\mathbb{P}}$,

$$dY_t = \Sigma_t d\hat{W}_t = \Sigma(t,O_t)d\hat{W}_t; \tag{3.18}$$

$$dH_t = \bar{H}_{t^-}dt + d\hat{\xi}_t; \tag{3.19}$$

$$dK_a(t) = dt + d\hat{M}_a(t),\ \forall\ a=1,\cdots,M. \tag{3.20}$$

So, K_t (therefore O_t), H_t, and Y_t are independent of X_t under $\hat{\mathbb{P}}$.

Next, we can easily find the inverse density process,

$$\frac{d\mathbb{P}}{d\hat{\mathbb{P}}}|_{\mathcal{G}_t} = U_t^{(w)}U_t^{(\xi)}\prod_{a=1}^{M}U_t^{(a)} = U_t,$$

where

$$U_t^{(w)} = \exp\left(\int_0^t \vartheta_s^\top \Sigma_s^{-1} dW_s + \frac{1}{2}\int_0^t \vartheta_s^\top (\Sigma_s \Sigma_s)^{-1} \vartheta_s ds\right) = \mathcal{E}_t\left(\int_0^\cdot \vartheta_s^\top \Sigma_s^{-1} d\hat{W}_s\right),$$

$$U_t^{(\xi)} = \exp\left(\int_0^t \log\left(h_{s-}\right) dH_s + \int_0^{t\wedge\tau} (1-h_{s-}) ds\right) = \mathcal{E}_t\left(\int_0^\cdot (h_{s-}-1) d\hat{\xi}_s\right),$$

and

$$U_t^{(a)} = \exp\left(\int_0^t \log\left(r_a(s^-)\right) dK_a(s) + \int_0^t (1-r_a(s^-)) ds\right) = \mathcal{E}_t\left(\int_0^\cdot (r_a(s^-)-1) d\hat{M}_a(s)\right).$$

Next, we are ready to provide the solution to the filtering problem.

Proposition 3.1.2 *The normalized filtered probabilities $p_t := (p_t^1, \dots, p_t^N)^\top$ have the dynamics:*

$$\begin{aligned} dp_t^i = &\sum_{\ell=1}^N \bar{w}_{\ell,i}(t) p_t^\ell dt + p_t^i (\vartheta(t,e_i)^\top - \hat{\vartheta}(t,p_t)^\top)(\Sigma_t \Sigma_t^\top)^{-1}(dY_t - \hat{\vartheta}(t,p_t)dt) \\ &+ p_{t-}^i \frac{h^i - \hat{h}(p_{t-})}{\hat{h}(p_{t-})}\left(dH_t - \hat{h}(p_{t-})\bar{H}_{t-}dt\right) + p_{t-}^i \sum_{a=1}^M \frac{r_a(t^-,e_i) - \hat{r}_a(t^-,p_{t-})}{\hat{r}_a(t^-,p_{t-})}(dK_a(t) - \hat{r}_a(t^-,p_{t-})dt) \end{aligned} \tag{3.21}$$

with the initial probabilities, $p_0^i = p^{\circ,i}$.

Here, for any stochastic process $r_t = r(t, X_t)$ or $h_t = h(X_t)$, as a function of X_t, unless explicitly specified, we introduce the notation

$$\hat{r}(t,p_t) = \sum_{i=1}^N r(t,e_i) p_t^i \quad \text{or} \quad \hat{h}(p_t) = \sum_{i=1}^N h(e_i) p_t^i.$$

Remark: If $w_{a,b}^{(1)} = \cdots = w_{a,b}^{(N)}$, for all $a, b = 1, \cdots, M$, $r_a(t)$ will no longer be a function of X_t. In this case, $r_a(t,e_i) - \hat{r}_a(t^-,p_{t-}) = 0$, we get the same result as the proposition 3.1 in the original model [1].

Proof We start with the unnormalized probability process

$$q_t^i = E^{\hat{\mathbb{P}}}[U_t X_i(t)|\mathcal{G}_t^I].$$

As shown in (2.24),

$$\begin{aligned} X_i(t) &= X_i(0) + \int_0^t \bar{w}_{X_s,i} ds + \phi_i(t) \\ &= X_i(0) + \int_0^t \sum_{l=1}^N \bar{w}_{l,i} X_l(s) ds + \phi_i(t) \end{aligned}$$

Still, since all quadratic covariation terms are 0, we have the dynamics of U_t,

$$dU_t = U_{t-} \left(\vartheta_t^\top \Sigma_t^{-1} d\hat{W}_t + (h_{t-} - 1) d\hat{\xi}_t + \sum_{a=1}^M (r_a(t^-) - 1) d\hat{M}_a(t) \right). \qquad (3.22)$$

Since Under $\hat{\mathbb{P}}$, Y_t, H_t, and K_t (also O_t) are independent to X_t (therefore also to $X_i(t)$), we have

$$[\hat{W}, X_i(\cdot)]_t = [\hat{\xi}, X_i(\cdot)]_t = [\hat{M}_a(\cdot), X_i(\cdot)]_t = 0 \quad \forall a = 1, \cdots, M.$$

So,

$$\begin{aligned} U_t X_i(t) =& X_i(0) + \int_0^t X_i(s^-) dU_s + \int_0^t U_{s-} dX_i(s) \\ =& X_i(0) + \int_0^t X_i(s^-) U_{s-} \left(\vartheta_s^\top \Sigma_s^{-1} d\hat{W}_s + (h_{s-} - 1) d\hat{\xi}_s + \sum_{a=1}^M (r_a(s^-) - 1) d\hat{M}_a(s) \right) \\ &+ \int_0^t U_{s-} \left(\sum_{l=1}^N \bar{w}_{l,i} X_l(s) ds + d\phi_i(s) \right). \end{aligned}$$

Take the conditional expectations w.r.t. $\mathcal{G}_t^I$. Then, by Lemma 3.2 in chapter 7 of [10], we can change the order of the integral and conditional expectation. So, we get

$$\begin{aligned} E^{\hat{\mathbb{P}}}[U_t X_i(t)|\mathcal{G}_t^I] =& p_0^i + \int_0^t E^{\hat{\mathbb{P}}}[X_i(s) U_s \vartheta_s^\top \Sigma_s^{-1} |\mathcal{G}_t^I] d\hat{W}_s + \int_0^t E^{\hat{\mathbb{P}}}[X_i(s^-) U_{s^-} (h_{s-} - 1)|\mathcal{G}_t^I] d\hat{\xi}_s \\ &+ \sum_{a=1}^M \int_0^t E^{\hat{\mathbb{P}}}[X_i(s^-) U_{s-} (r_a(s^-) - 1)|\mathcal{G}_t^I] d\hat{M}_a(s) + \int_0^t E^{\hat{\mathbb{P}}}[U_s \sum_{l=1}^N \bar{w}_{l,i} X_l(s)|\mathcal{G}_t^I] ds \\ &+ E^{\hat{\mathbb{P}}}[\int_0^t U_{s-} d\phi_i(s)|\mathcal{G}_t^I]. \end{aligned}$$

i.e.

$$
\begin{aligned}
q_t^i =& p_0^i + \int_0^t q_s^i \vartheta(s, e_i)^\top \Sigma_s^{-1} d\hat{W}_s + \int_0^t q_{s^-}^i (h^i - 1) d\hat{\xi}_s \\
&+ \int_0^t \sum_{a=1}^M q_{s^-}^i (r_a(s^-, e_i) - 1) d\hat{M}_a(s) + \int_0^t \sum_{l=1}^N \bar{w}_{l,i} q_s^l ds
\end{aligned} \tag{3.23}
$$

Next, find the normalized filtered probabilities. Let

$$
S_t = \sum_{i=1}^N q_t^i. \tag{3.24}
$$

For conditional expectations, since $U_t = \frac{d\mathbb{P}}{d\hat{\mathbb{P}}}|_{\mathcal{G}_t}$, we have the formula

$$
p_t^i = E^{\mathbb{P}}[X_i(t)|\mathcal{G}_t^I] = \frac{E^{\hat{\mathbb{P}}}[U_t X_i(t)|\mathcal{G}_t^I]}{E^{\hat{\mathbb{P}}}[U_t|\mathcal{G}_t^I]} = \frac{q_t^i}{S_t} \tag{3.25}
$$

So,

$$
dp_t^i = q_{t-}^i df(S_t) + f(S_{t-}) dq_t^i + d\left[\frac{1}{S}, q^i\right]_t \tag{3.26}
$$

Then we drive each term in (3.26). First, we have

$$
dS_t = \sum_{i=1}^N q_t^i \vartheta(t, e_i)^\top \Sigma_t^{-1} d\hat{W}_t + \sum_{i=1}^N q_{s^-}^i (h^i - 1) d\hat{\xi}_t + \sum_{i=1}^N \sum_{a=1}^M q_{t^-}^i (r_a(t^-, e_i) - 1) d\hat{M}_a(t). \tag{3.27}
$$

Next, let $f(x) = 1/x$. Use Itô's formula,

$$
f(S_t) = \frac{1}{p_0^i} + \int_0^t f'(S_{s-}) dS_s + \frac{1}{2}\int_0^t f''(S_{s-}) d[S^c]_s + \sum_{0<s\le t} (f(S_s) - f(S_{s-}) - f'(S_{s-})\Delta S_s). \tag{3.28}
$$

For the continuous part,

$$
\begin{aligned}
f''(S_{t-}) d[S^c]_t &= \frac{2}{S_{t-}^3} \left(\sum_{j=1}^N \sum_{k=1}^N q_t^j \vartheta(t, e_j)^\top \Sigma_t^{-1} \Sigma_t^{-1} \vartheta(t, e_k) q_t^k \right) dt \\
&= \frac{2}{S_{t-}} \left(\sum_{j=1}^N \sum_{k=1}^N p_t^j \vartheta(t, e_j)^\top \Sigma_t^{-1} \Sigma_t^{-1} \vartheta(t, e_k) p_t^k \right) dt.
\end{aligned}
$$

For the jump part, since there is no simultaneous jump in H_t and K_t,

$$
\frac{\Delta S_t}{S_{t-}} = (\hat{h}(p_{t-}) - 1)\Delta H_t + \sum_{a=1}^M (\hat{r}_a(t^-, p_{t-}) - 1)\Delta K_a(t).
$$

Then,

$$\frac{\Delta S_t}{S_t} = \frac{(\hat{h}(p_{t-}) - 1)}{\hat{h}(p_{t-})}\Delta H_t + \sum_{a=1}^{M} \frac{(\hat{r}_a(t^-, p_{t-}) - 1)}{\hat{r}_a(t^-, p_{t-})}\Delta K_a(t).$$

We also have that

$$f(S_t) - f(S_{t-}) = -\left(\frac{(\hat{h}(p_{t-}) - 1)}{\hat{h}(p_{t-})}\Delta H_t + \sum_{a=1}^{M} \frac{(\hat{r}_a(t^-, p_{t-}) - 1)}{\hat{r}_a(t^-, p_{t-})}\Delta K_a(t)\right)\frac{1}{S_{t-}},$$

and

$$f'(S_{t-})\Delta S_t = -\frac{(\hat{h}(p_{t-}) - 1)}{S_{t-}}\Delta H_t - \sum_{a=1}^{M} \frac{(\hat{r}_a(t^-, p_{t-}) - 1)}{S_{t-}}\Delta K_a(t).$$

Plug into (3.28), we get

$$\begin{aligned}
df(S_t) =& -\frac{1}{S_{t-}^2}\left(\sum_{i=1}^{N} q_t^i \vartheta(t, e_i)^\top \Sigma_t^{-1} d\hat{W}_t + \sum_{i=1}^{N} q_{s-}^i (h^i - 1) d\hat{\xi}_t + \sum_{i=1}^{N}\sum_{a=1}^{M} q_{t-}^i (r_a(t^-, e_i) - 1) d\hat{M}_a(t)\right) \\
&+ \frac{1}{S_{t-}}\left(\sum_{j=1}^{N}\sum_{k=1}^{N} p_t^j \vartheta(t, e_j)^\top \Sigma_t^{-1}\Sigma_t^{-1}\vartheta(t, e_k) p_t^k\right) dt \\
&- \frac{1}{S_{t-}}\left(\frac{(\hat{h}(p_{t-}) - 1)}{\hat{h}(p_{t-})}\Delta H_t + \sum_{a=1}^{M} \frac{(\hat{r}_a(t^-, p_{t-}) - 1)}{\hat{r}_a(t^-, p_{t-})}\Delta K_a(t)\right) \\
&+ \frac{(\hat{h}(p_{t-}) - 1)}{S_{t-}}\Delta H_t + \sum_{a=1}^{M} \frac{(\hat{r}_a(t^-, p_{t-}) - 1)}{S_{t-}}\Delta K_a(t) \\
=& -\frac{1}{S_{t-}}\left(\sum_{i=1}^{N} p_t^i \vartheta(t, e_i)^\top \Sigma_t^{-1} d\hat{W}_t\right) + \frac{1}{S_{t-}}\left(\sum_{j=1}^{N}\sum_{k=1}^{N} p_t^j \vartheta(t, e_j)^\top \Sigma_t^{-1}\Sigma_t^{-1}\vartheta(t, e_k) p_t^k\right) dt \\
&- \frac{1}{S_{t-}}\frac{(\hat{h}(p_{t-}) - 1)}{\hat{h}(p_{t-})}\left(\Delta H_t - \hat{h}(p_{t-})\bar{H}_{t-} dt\right) \\
&- \sum_{a=1}^{M} \frac{1}{S_{t-}}\frac{(\hat{r}_a(t^-, p_{t-}) - 1)}{r_a(t^-, p_{t-})}\left(\Delta K_a(t) - \hat{r}_a(t^-, p_{t-}) dt\right).
\end{aligned}$$

Lastly, We also need the following quadratic covariation.

$$d\left[\frac{1}{S}, q^i\right]_t = -\left(\sum_{i=1}^{N} p_t^i \vartheta(t, e_i)^\top \Sigma_t^{-1}\right)(\Sigma_t^\top)^{-1}\vartheta(s, e_i) p_s^i dt + \Delta\frac{1}{S_t}\Delta q_t^i, \tag{3.29}$$

with

$$
\begin{aligned}
\Delta\frac{1}{S_t}\Delta q_t^i =& -\left(\frac{(\hat{h}(p_{t-})-1)}{\hat{h}(p_{t-})}\Delta H_t + \sum_{a=1}^{M}\frac{(\hat{r}_a(t^-,p_{t-})-1)}{\hat{r}_a(t^-,p_{t-})}\Delta K_a(t)\right) \\
&\times\left(p_{t-}^i(h^i-1)\Delta H_t + \sum_{a=1}^{M}p_{t-}^i(r_a(t^-,e_i)-1)\Delta K_a(t)\right) \\
=& -\frac{(\hat{h}(p_{t-})-1)}{\hat{h}(p_{t-})}p_{t-}^i(h^i-1)\Delta H_t - \sum_{a=1}^{M}\frac{(\hat{r}_a(t^-,p_{t-})-1)}{\hat{r}_a(t^-,p_{t-})}p_{t-}^i(r_a(t^-,e_i)-1)\Delta K_a(t).
\end{aligned}
\tag{3.30}
$$

Using the expressions driven above, we get

$$
\begin{aligned}
q_{t-}^i df(S_t) =& -p_t^i\left(\hat{\vartheta}(t,p_t)^\top\Sigma_t^{-1}d\hat{W}_t\right) + p_t^i\left(\hat{\vartheta}(t,p_t)^\top\Sigma_t^{-1}\Sigma_t^{-1}\hat{\vartheta}(t,p_t)\right)dt \\
&- p_{t-}^i\frac{(\hat{h}(p_{t-})-1)}{\hat{h}(p_{t-})}\left(\Delta H_t - \hat{h}(p_{t-})\bar{H}_{t-}dt\right) \\
&- p_{t-}^i\sum_{a=1}^{M}\frac{(\hat{r}_a(t^-,p_{t-})-1)}{r_a(t^-,p_{t-})}\left(\Delta K_a(t) - \hat{r}_a(t^-,p_{t-})dt\right); \\
f(S_{t-})dq_t^i =& p_t^i\vartheta(t,e_i)^\top\Sigma_t^{-1}d\hat{W}_t + p_{t-}^i(h^i-1)d\hat{\xi}_t \\
&+ \sum_{a=1}^{M}p_{t-}^i(r_a(t^-,e_i)-1)d\hat{M}_a(t) + \sum_{l=1}^{N}\bar{w}_{l,i}q_t^l dt
\end{aligned}
$$

Substituting into (3.26) gives the desired result. The initial condition is direct from (3.23). ■

3.2 An Equivalent Complete Observation Control Problem

In this section, We will show that the control problem with partial observation is equivalent to a complete observation control problem under the new probability measure $\hat{\mathbb{P}}$.

Let us start by noticing that the objective function can be written as

$$
\begin{aligned}
\frac{1}{\gamma}\mathbb{E}^{\mathbb{P}}\left[V_T^\gamma\right] &= \frac{v^\gamma}{\gamma}\mathbb{E}^{\hat{\mathbb{P}}}\left[e^{-\gamma\int_0^T\eta_s ds+\gamma\int_0^T\pi_s^\top\Sigma_s dW_s-\frac{\gamma^2}{2}\int_0^T\pi_s^\top\Sigma_s\Sigma_s^\top\pi_s ds}U_T\right] \\
&= \frac{v^\gamma}{\gamma}\mathbb{E}^{\hat{\mathbb{P}}}\left[L_T\right],
\end{aligned}
\tag{3.31}
$$

where

$$L_t := \mathcal{E}_t\left(\int_0^{\cdot} Q(s, X_s)^\top \Sigma_s d\hat{W}_s\right) \exp\left(\int_0^t -\gamma\eta_s ds\right) U_t^{(\xi)} \prod_{a=1}^{M} U_t^{(a)} \tag{3.32}$$

and

$$\begin{aligned} Q(s, e_i) &:= (\Sigma_s \Sigma_s^\top)^{-1} \vartheta(s, e_i) + \gamma\pi_s \\ &= \left(\frac{1}{\sigma_t^2}\left(\mu^i - \frac{\sigma_t^2}{2}\right) + \gamma\pi_s^S, \frac{1}{\upsilon_t^2}\left(a(t, e_i) - \frac{\upsilon_t^2}{2}\right) + \gamma\pi_s^P\right)^\top. \end{aligned} \tag{3.33}$$

Recall that η_s is defined in (2.17). The dynamic of process L_t above still depends on X_t. Next, we define $\hat{L}_t$ whose dynamics are fully determined by the processes that are observed by the investor. Let

$$\begin{aligned} \hat{L}_t := &\mathcal{E}_t\left(\int_0^{\cdot} \hat{Q}(s, p_s)^\top \Sigma_s d\hat{W}_s\right) \exp\left(\int_0^t -\gamma\hat{\eta}(s, p_s) ds\right) \\ &\mathcal{E}_t\left(\int_0^{\cdot} (\hat{h}(p_{t^-}) - 1) d\hat{\xi}_s\right) \prod_{a=1}^{M} \mathcal{E}_t\left(\int_0^{\cdot} (\hat{r}_a(s^-, p_{t^-}) - 1) d\hat{M}_a(s)\right) \end{aligned} \tag{3.34}$$

where

$$\hat{\eta}(t, p_t) = \sum_{i=1}^{N} \eta(t, e_i) p_t^i = -r + \pi_t^S(r - \hat{\mu}(p_t)) + \pi_t^P\left(r - \hat{a}(t, p_t)\right) + \frac{1-\gamma}{2}\left(\sigma_t^2(\pi_t^S)^2 + \upsilon_t^2(\pi_t^P)^2\right)$$

$$\hat{Q}(t, p_t) = \sum_{i=1}^{N} Q(t, e_i) p_t^i = \left(\frac{1}{\sigma_t^2}\left(\hat{\mu}(p_t) - \frac{\sigma_t^2}{2}\right) + \gamma\pi_t^S, \frac{1}{\upsilon_t^2}\left(\hat{a}(t, p_t) - \frac{\upsilon_t^2}{2}\right) + \gamma\pi_t^P\right)^\top.$$

We can further rewrite the objective function as an expection of the $\mathbb{G}^I - \textit{adapted}$ process $\hat{L}_t$.

Proposition 3.2.1

$$J(v, \pi, T) = \frac{v^\gamma}{\gamma}\mathbb{E}^{\hat{\mathbb{P}}}\left[\hat{L}_T\right]. \tag{3.35}$$

Proof Similar to the proof of the filtered probabilities, let

$$q_t^i = E^{\hat{\mathbb{P}}}[L_t X_a(t) | \mathcal{G}_t^I].$$

We can show that

$$q_t^i = \hat{L}_t p_t^i \quad (\hat{\mathbb{P}} - a.s.) \tag{3.36}$$

Then,

$$\begin{aligned}
J(v,\pi,T) &= \frac{v^\gamma}{\gamma}\mathbb{E}^{\hat{\mathbb{P}}}\left[L_T\right] = \frac{v^\gamma}{\gamma}\mathbb{E}^{\hat{\mathbb{P}}}\left[\mathbb{E}^{\hat{\mathbb{P}}}\left[L_T \middle| \mathcal{G}_T^I\right]\right] \\
&= \frac{v^\gamma}{\gamma}\sum_{i=1}^{N}\mathbb{E}^{\hat{\mathbb{P}}}\left[\mathbb{E}^{\hat{\mathbb{P}}}\left[L_T X_i(t) \middle| \mathcal{G}_T^I\right]\right] \\
&= \frac{v^\gamma}{\gamma}\sum_{i=1}^{N}\mathbb{E}^{\hat{\mathbb{P}}}\left[q_T^i\right] = \frac{v^\gamma}{\gamma}\sum_{i=1}^{N}\mathbb{E}^{\hat{\mathbb{P}}}\left[\hat{L}_T p_T^i\right] \\
&= \frac{v^\gamma}{\gamma}\mathbb{E}^{\hat{\mathbb{P}}}\left[\hat{L}_T\right].
\end{aligned}$$

■

Since the dynamics of $\hat{L}_t$ is fully determined by the observable processes. By (3.35), we reduced our partially observed control problem to a problem with complete observation.

4. Conclusion

In this thesis, we provide a model under which the hidden economic regime determines the rates of return on a stock security and a defaultable security. In our model, the price volatilities are also stochastically determined by the hidden economic regime. The new model is a generalization of the original model introduced by Capponi et al. [1], where they assume constant volatilities. The observable process we introduced captures the lagged and noisy nature of the investor's response to the change in the economic regime.

Then, we study the optimal portfolio problem. We solve the filtering problem by converting the observable process into a counting process. Next, using the filtered probabilities, we reduce the optimal control problem with partial knowledge to one with complete observation. In future research, we will further solve the proposed problem, conduct necessary numerical analysis, and test the model empirically.

REFERENCES

[1] Agostino Capponi, José E Figueroa-López, and Andrea Pascucci. Dynamic credit investment in partially observed markets. *Finance and Stochastics*, 19(4):891–939, 2015.

[2] Tomas Björk. *Arbitrage theory in continuous time.* Oxford university press, 2009.

[3] Bernt Oksendal. *Stochastic differential equations: an introduction with applications.* Springer Science & Business Media, 2013.

[4] Robert Ševilevič Liptser and Albert Nikolaevič Širjaev. *Statistics of random processes: I. General theory*, volume 1. Springer Science & Business Media, 2001.

[5] Robert J Elliott, Lakhdar Aggoun, and John B Moore. *Hidden Markov models: estimation and control*, volume 29. Springer Science & Business Media, 2008.

[6] Tomasz R Bielecki and Marek Rutkowski. *Credit risk: modeling, valuation and hedging.* Springer Science & Business Media, 2013.

[7] G William Schwert. Business cycles, financial crises, and stock volatility. In *Carnegie-Rochester Conference series on public policy*, volume 31, pages 83–125. Elsevier, 1989.

[8] Henry P McKean Jr. a free boundary problem for the heat equation arising from a problem of mathermatical economics. *Industrial Management Review*, 6:32–39, 1965.

[9] Philip Protter, Kazuhiro Shimbo, et al. No arbitrage and general semimartingales. In *Markov processes and related topics: a Festschrift for Thomas G. Kurtz*, pages 267–283. Institute of Mathematical Statistics, 2008.

[10] Eugene Wong and Bruce Hajek. Stochastic processes. In *Stochastic Processes in Engineering Systems*, pages 37–73. Springer, 1985.