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Index Theory for Toeplitz Operators on Algebraic Spaces

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Contents

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Preface

This dissertation is about the abstract Toeplitz operators obtained by compressing the multishifts of the usual Hilbert spaces of analytic functions onto co-invariant subspaces generated by polynomial functions. These operators were introduced by Arveson in regard to his multivariate dilation theory for spherical contractions [6, 7, 8, 10, 11]. The main technical issue here is essential normality, addressed in Arveson's conjecture. If this conjecture holds true then the fundamental tuple of Toeplitz operators associated to a polynomial ideal I can be thought as *noncommutative coordinate functions* on the variety defined by I intersected with the boundary of the unit ball. This interpretation suggests operator-theoretic techniques to study certain algebraic spaces. More specifically, we are interested in Douglas' index problem. These topics are discussed in Chapter 1.

In the special case of monomial ideals we give a new proof for Arveson's essential normality conjecture, also answer Douglas' index problem. Our main construction is a certain resolution (in the sense of homological algebra) of Hilbert modules. These are discussed in Chapter 2.

Thinking of the fundamental tuple of Toeplitz operators as noncommutative coordinate functions, we start applying them to study the isolated singularities of algebraic hypersurfaces. The main extra operator-theoretic ingredient here is a unitary operator, the holonomy of a certain Gauss-Manin connection induced by the monodromy of the singularity. We want to understand how this unitary operator interacts with the Toeplitz operators. This study could lead to an analytic way for detecting exotic smooth structures on odd-dimensional spheres. These are discussed in Chapter 3.

A list of the fundamental notations and conventions used throughout the dissertation is provided on pages vii-x; specially the fundamental concept of the essential normality of a Hilbert module is defined there.

Notations and conventions

- $\mathbb{N}$ is the set of nonnegative integers.
- $a_n \approx b_n$ (respectively, $a_n \ll b_n$), for number sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$, means that the ratio a_n/b_n converges to a nonzero (respectively, possibly zero) finite complex number.
- m is an arbitrary positive integer fixed throughout the whole dissertation.
- $\mathbb{B}_m$ and $\overline{\mathbb{B}}_m$ are respectively the open and closed unit balls in $\mathbb{C}^m$.
- A always denotes the algebra $\mathbb{C}[z_1, \ldots, z_m]$ of polynomials in m variables.
- Let $I \subseteq A$ be an ideal.
  - $V(I) \subseteq \mathbb{C}^m$ is the zero variety defined by I.
  - $X_I := V(I) \cap \partial\overline{\mathbb{B}}_m$.
- $C(X)$ denotes the C*-algebra of complex-valued functions on topological space X.
- The term *algebraic spaces* generally refers to those ringed spaces whose structure sheaf might contain nonzero nilpotent germs. Examples are Grothendick's schemes [64] and Grauert's nonreduced analytic spaces [58, Chapter 1][72, Section 43].
- We try to denote Hilbert spaces by uppercase calligraphic letters like $\mathcal{A}$, $\mathcal{H}$, $\mathcal{I}$, $\mathcal{M}$, $\mathcal{N}$. However, we follow the usual tradition of denoting the Lebesgue, Sobolev, Hardy

and Bergman spaces by L, W, H and L_a, respectively. (See pages ix-x.) All abstract Hilbert spaces are assumed to be separable.

- The term *operator*, unless otherwise stated, refers to linear continuous maps between Hilbert spaces.

- We try to denote C*-algebras by uppercase fraktur letters like $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{K}$, $\mathfrak{Q}$, $\mathfrak{T}$.

  $\mathfrak{B}(\mathcal{H})$ is the C*-algebra of operators on the Hilbert space $\mathcal{H}$.

  $\mathfrak{K}(\mathcal{H})$ is the C*-algebra of compact operators on the Hilbert space $\mathcal{H}$.

  $\mathfrak{Q}(\mathcal{H}) := \mathfrak{B}(\mathcal{H})/\mathfrak{K}(\mathcal{H})$ is the Calkin algebra of the Hilbert space $\mathcal{H}$.

- There is a one-to-one correspondence between commuting m-tuples of operators $T := (T_1, \ldots, T_m)$ acting on a Hilbert space $\mathcal{H}$ and Hilbert A-module structures (in the sense of Arveson [11]) on $\mathcal{H}$. This correspondence is given by representing each polynomial $p(z_1, \ldots, z_m) \in A$ by the operator $p(T_1, \ldots, T_m)$. Conversely, T is identified with the m-tuple $(M_{z_1}, \ldots, M_{z_m}) \in \mathfrak{B}(\mathcal{H})^m$ of multiplication operators by the coordinate functions, and is called the fundamental tuple of Toeplitz operators on the Hilbert A-module $\mathcal{H}$. The C*-algebra generated by $\{1, T_1, \ldots, T_m\} \cup \mathfrak{K}(\mathcal{H})$ is denoted by $\mathfrak{T}(\mathcal{H})$ and is called the Toeplitz C*-algebra of the Hilbert A-module $\mathcal{H}$. Based on this equivalence, the properties of T are attributed to $\mathcal{H}$ and vice versa. For example, $\mathcal{H}$ is called essentially normal if $[T_i, T_j^*] \in \mathfrak{K}(\mathcal{H})$ for all $i, j = 1, \ldots, m$.

- $\mathcal{H} \otimes \mathbb{C}^r$, r positive integer, is the r-fold inflation of Hilbert space $\mathcal{H}$, implemented by the tensor product or direct sum $\bigoplus_r \mathcal{H}$ of Hilbert spaces. An operator $T \in \mathfrak{B}(\mathcal{H})$ naturally induces the inflation $T \otimes 1 \in \mathfrak{B}(\mathcal{H} \otimes \mathbb{C}^r)$.

- Let $\Omega \subseteq \mathbb{C}^m$ be a smoothly bounded domain.

  - $W^s_{\text{hol}}(\Omega)$, $s \in \mathbb{R}$, is the Bergman-Sobolev space consisting of all holomorphic functions in the L^2 Sobolev space $W^s(\Omega)$. (See [53, 80, 19].) These are also known as the holomorphic Sobolev spaces.

– $W^s_{\text{hol}}(\partial\overline{\Omega})$, $s \in \mathbb{R}$, is the Hardy-Sobolev space consisting of all functions in the L^2 Sobolev space $W^s(\partial\overline{\Omega})$ whose Poisson extension to Ω is not only harmonic but also holomorphic. Alternatively, it is the closure in $W^s(\partial\overline{\Omega})$ of the boundary values of holomorphic functions on Ω which are continuous up to the boundary [25, 53]. Note that $W^s_{\text{hol}}(\partial\overline{\Omega})$ is isometrically isomorphic to the Bergman-Sobolev space $W^{s+\frac{1}{2}}_{\text{hol}}(\Omega)$ through the Poisson extension and trace map [53].

– $H^2(\partial\overline{\Omega}) := W^0_{\text{hol}}(\partial\overline{\Omega})$ is the Hardy space [75, 93, 98, 32].

– $L^2_{a,s}(\Omega)$, $s > -1$, is the weighted Bergman space consisting of holomorphic functions f on Ω such that $\int_\Omega |f(z)|^2 \rho(z)^s dV(z) < \infty$, where $\rho(z)$ is a positively signed smooth defining function for Ω (equivalently, the distance function $\operatorname{dist}(z, \partial\Omega)$), and $dV(z)$ is the Lebesgue measure on Ω normalized such that $\int_\Omega \rho(z)^s dV(z) = 1$. (See [19, 53, 104].) Note that $L^2_{a,s}(\Omega) = W^{-\frac{s}{2}}_{\text{hol}}(\Omega)$ as sets with equivalent norms [81, 53, 19].

– $L^2_a(\Omega) := L^2_{a,0}(\Omega)$ is the (unweighted) Bergman space [75, 98, 97].

- H^2_m is the Drury-Arveson space of analytic functions on $\mathbb{B}_m$, the one with the reproducing kernel $(1 - \langle z, w\rangle)^{-1}$. (See [6][2, Chapter 41].) It has the standard orthonormal basis $\left\{(n!/|n|!)^{-1/2} z^n : n \in \mathbb{N}^m\right\}$. It is also known as the m-shift or symmetric Fock space.

- $\mathcal{H}^{(s)}_m$, $s \in \mathbb{R}$, is the Besov-Sobolev space of analytic functions on $\mathbb{B}_m$, the one with the reproducing kernel

$$K_s(z, w) := \begin{cases} (1 - \langle z, w\rangle)^{-s-m-1}, & s > -m-1, \\ (-s-m)^{-1} F\left(1, 1; 1-s-m; \langle z, w\rangle\right), & s \le -m-1, \end{cases}$$

where $F(a, b; c; \zeta) := \sum_{q\in\mathbb{N}} \frac{(a)_q (b)_q}{(c)_q q!} \zeta^q$ is the hypergeometric function, and $(x)_y := \frac{\Gamma(x+y)}{\Gamma(x)}$ is the Pochhammer symbol. (See [20, 102, 53, 3, 104]; our parameter $s + m + 1$ is q in [20], $\alpha + m + 1$ in [102, 53], and 2σ in [3]; [104] only studies the $s = -m - 1$ case.)

$\mathcal{H}_m^{(s)}$ has the standard orthonormal basis $\left\{\omega_s(n)^{-1/2}z^n : n \in \mathbb{N}^m\right\}$ where

$$\omega_s(n) := \begin{cases} \frac{n!(s+m)!}{(|n|+s+m)!}, & s > -m-1, \\ \frac{n!(-s-m)_{|n|+1}}{(|n|!)^2}, & s \leq -m-1. \end{cases}$$

Note that $\omega_s(n) \approx \frac{n!}{|n|!(|n|+1)^{s+m}}$ for each $s \in \mathbb{R}$. (We will not need the reproducing kernel, but this equivalent norm is enough for our purposes.) We have the identifications:

$$\mathcal{H}_m^{(s)} = \begin{cases} \text{the Bergman-Sobolev space } W_{\text{hol}}^{-\frac{s}{2}}(\mathbb{B}_m) \text{ (as sets with equivalent norms)}, & s \in \mathbb{R}, \\ \text{the Hardy-Sobolev space } W_{\text{hol}}^{-\frac{s+1}{2}}(\partial\overline{\mathbb{B}}_m) \text{ (as sets with equivalent norms)}, & s \in \mathbb{R}, \\ \text{the Drury-Arveson space } H_m^2 \text{ (as sets with equal norms)}, & s = -m, \\ \text{the Hardy space } H^2(\partial\overline{\mathbb{B}}_m) \text{ (as sets with equal norms)}, & s = -1 \\ \text{the weighted Bergman space } L_{a,s}^2(\mathbb{B}_m) \text{ (as sets with equal norms)}, & s > -1. \end{cases}$$

- $\sigma_e(T)$, T operator, is the essential spectrum of T. When T is an m-tuple of commuting operators acting on a common Hilbert space, $\sigma_e(T)$ is the essential Taylor spectrum [2, Chapter 42][84]. For Hilbert A-module $\mathcal{H}$, $\sigma_e(\mathcal{H})$ is the essential Taylor spectrum associated to the fundamental tuple of Toeplitz operators of $\mathcal{H}$.

- As the basic setting of this dissertation, in Section 1.1, we associate to each homogeneous ideal $I \subseteq A$ the following objects:

  - $\overline{I}$, $\mathcal{Q}_I$, $I^\perp$ (Hilbert A-modules)
  - $\mathfrak{T}_I := \mathfrak{T}(I^\perp)$ (C*-algebra)
  - $\tau_I \in K_1(X_I)$ (odd K-homology class; only defined if $I^\perp$ is essentially normal.)

Chapter 1

Introduction

This chapter introduces and motivates the main analytic objects we work on throughout the dissertation. More specifically, in Section 1.1 to each polynomial ideal we associate abstract Toeplitz operators, Hilbert modules, C*-algebras and (conjectural) odd K-homology classes, and state Arveson's essential normality as well as Douglas' index conjectures about them. Sections 1.2 and 1.3, respectively, try to reveal Arveson's and Douglas' path to their conjectures. Some variants of Arveson's conjecture are discussed in Section 1.4. Section 1.5 gives a summary of results in this dissertation.

1.1 The basic setting, Arveson's conjecture, Douglas' index problem

The commutative algebra of polynomial ideals $I \subseteq A := \mathbb{C}[z_1, \ldots, z_m]$ is reflected in the geometry of their corresponding affine subvarieties $V \subseteq \mathbb{C}^m$. More specifically, there is a complete algebro-geometric duality between radical ideals and Zariski-closed subspaces [64, I.1.4], which extends to a complete duality between general ideals and closed subschemes [64, II.5.10, II.2.6, II.4.10]. The main idea of these dualities is to realize the elements of the quotient A/I as *functions* living on (some enlargement of) the zero variety V. Multivariate operator theory adds a third analytic facet to this duality through the C*-algebra of abstract Toeplitz operators, which we now describe [10, 11].

The goal is to realize the quotient A/I with operator-theoretic means. We work with the Drury-Arveson space H^2_m, but the constructions below make sense for any of the analytic Hilbert spaces on page ix. The closure $\overline{I}$ of I inside H^2_m is invariant under the action of multiplication operators $M_p \in \mathfrak{B}(H^2_m)$, $p \in A$. This makes $\overline{I}$ a Hilbert A-submodule of H^2_m. The quotient Hilbert space $\mathcal{Q}_I := H^2_m/\overline{I}$ has a natural Hilbert module structure given by $p \cdot (f + \overline{I}) = pf + \overline{I}$, $p \in A$, $f \in H^2_m$. Transporting this action to the orthogonal complement

$$H^2_m \ominus \overline{I} = I^\perp \cong \mathcal{Q}_I$$

makes $I^\perp$ a Hilbert A-module. Alternatively, this module structure is given by the compression of multiplication operators:

$$T_p := P_{I^\perp} M_p|_{I^\perp} \in \mathfrak{B}(I^\perp), \quad p \in A, \tag{1.1}$$

where $P_{I^\perp}$ is the orthogonal projection onto $I^\perp$. These compressed shifts T_p are called Toeplitz operators associated to I. The Toeplitz C*-algebra

$$\mathfrak{T}_I := \mathfrak{T}(I^\perp)$$

generated by $\{1\} \cup \{T_p : p \in A\} \cup \mathfrak{K}(I^\perp)$ is the analytic facet we talked about. Arveson [8, 10], based on his work in multivariate dilation theory, conjectured:

Conjecture 1 (Arveson). *$I^\perp$ is essentially normal, in other words $\mathfrak{T}_I/\mathfrak{K}$ is abelian.*

Suppose momentarily that this conjecture holds true. Also assume that I is homogeneous. Then the maximal ideal space of $\mathfrak{T}_I/\mathfrak{K}$ is homeomorphic via the mapping $\varphi \mapsto (\varphi(T_{z_1}), \dots, \varphi(T_{z_m}))$ to the essential Taylor spectrum of $(T_{z_1}, \dots, T_{z_m})$, which is itself identified [37, Corollary 3.10][61, Theorem 5.1] as $X_I := V(I) \cap \partial\overline{\mathbb{B}}_m$. (Note that this early identification indicates how some geometric information is retrieved from the analytic facet. More is on the way.) The Gelfand-Naimark duality then gives the following short exact sequence of C*-algebras:

$$0 \to \mathfrak{K}(I^{\perp}) \hookrightarrow \mathfrak{T}_I \to C(X_I) \to 0. \tag{1.2}$$

Let

$$\tau_I := [\mathfrak{T}_I]$$

be the equivalence class represented by this exact sequence in the odd K-homology group $K_1(X_I)$ of Brown-Douglas-Fillmore [29, 30]. Douglas [45] (also see [14, Section 25]) asked for an explicit computation of this element in other topological or geometric realizations of K-homology:

Problem 2 (Douglas)**.** *Suppose I is homogeneous and $I^{\perp}$ is essentially normal. Identify $\tau_I \in K_1(X_I)$.*

More specifically, in the same paper he conjectured that:

Conjecture 3 (Douglas)**.** *Let I be the vanishing ideal of a variety $V \subseteq \mathbb{C}^m$ which intersects $\partial\overline{\mathbb{B}}_m$ transversally. Then $I^{\perp}$ is essentially normal, and its induced extension class is identified with the fundamental class of X_I, namely the extension class induced by the Spin^c Dirac operator associated to the natural Cauchy-Riemann structure of X_I.*

By analogy with the Atiyah-Singer index theorem one expects that this conjecture would lead to new connections between geometry and operator theory.

Let us review some results about Conjecture 1, Problem 2 and Conjecture 3. (See also [2, Chapter 41].) Conjecture 1 has been proved for the following cases:

- I is monomial [10, 46];
- I is principal[1] [61, 50, 54, 55];

[1]If I is nonhomogeneous the conjecture has only been verified for the Besov-Sobolev spaces $\mathcal{H}_m^{(s)}$ in the range $s \in (-1, \infty) \cup ((-3, \infty) \cap [-m, \infty))$.

- I is homogeneous and $m \leq 3$ [61];

- I is homogeneous and $\dim V(I) \leq 1$ [61];

- I has a stable generating set $\{p_1, \ldots, p_k\}$ of homogeneous polynomials in the sense that there exists $C > 0$ such that every $q \in I$ can be written as $q = \sum_{1 \leq j \leq k} r_j p_j$ with $r_j \in A$ and $\|r_j p_j\|_{H^2_m} \leq C\|q\|_{H^2_m}$ [91];

- I is the vanishing ideal of a homogeneous variety smooth away from the origin [53]. Also see [49, 51, 101].

In regard to Problem 2 and Conjecture 3 we mention two results. Guo and Wang [61] computed τ_I for $m \leq 2$, and proved that it is nontrivial for $m \leq 3$. Douglas, Tang and Yu [49] verified Conjecture 3 for complete intersection varieties with only isolated singularities:

Theorem 4 (Douglas-Tang-Yu). *Let I be a not necessarily homogeneous ideal of A which is generated by polynomials $p_1, \ldots, p_c$ satisfying: (1) $c \leq m - 2$; (2) $V(I)$ intersects $\partial\overline{\mathbb{B}}_m$ transversally; and (3) the Jacobian matrix $(\partial p_i / \partial z_j)$ is of maximal rank on X_I. Then $I^\perp \subseteq L^2_a(\mathbb{B}_m)$ is isomorphic as Hilbert A-module to $L^2_{a,c}(V(I) \cap \mathbb{B}_m)$[2], both are essentially normal, and their induced extension class is identified with the fundamental class of X_I.*

Remark 5. We have followed Arveson to denote his compressed multiplications (1.1) by T_p [10]. S_p is also used in the literature [61, 53]. ■

Remark 6. Here are two reasons why we refer to the compressed multiplications T_p as Toeplitz operators. First recall that a classical Toeplitz operator is of the form

$$T_f = P_{\mathcal{M}} M_f|_{\mathcal{M}}$$

acting on the Hardy or Bergman space $\mathcal{M} \subseteq L^2(\mathbb{B}_m)$, where $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$, and M_f is the multiplication by function f living in some symbol class

[2]Note that $V(I) \cap \mathbb{B}_m$ might have finitely many isolated singularities. Accordingly, the Bergman functions are defined as L^2 functions which are holomorphic on $V(I) \cap \mathbb{B}_m \setminus \{\text{singularities}\}$.

say $C(\overline{\mathbb{B}}_m)$.[3] Boutet de Monvel introduced the so-called generalized Toeplitz operators of the form

$$T_Q = PQ : W^s_{\text{hol}}(\partial\overline{\Omega}) \to W^{s-n}_{\text{hol}}(\partial\overline{\Omega})$$

acting between Hardy-Sobolev spaces, where $s \in \mathbb{R}$, Ω is a smoothly bounded strongly pseudoconvex domain inside some reduced complex-analytic space with no singularity on $\partial\overline{\Omega}$, symbol Q is an arbitrary pseudodifferential operator of order n on $\partial\overline{\Omega}$, and $P : W^{s-n}(\partial\overline{\Omega}) \to W^{s-n}_{\text{hol}}(\partial\overline{\Omega})$ is the (Szegö) orthogonal projection [25, 26].

(1) Engliš and Eischmeier [53], in the special case that I is the vanishing ideals of a homogeneous variety smooth away from the origin, linked Arveson's compressed multiplications acting on $I^\perp \subseteq H^2_m$ to Boutet de Monvel's generalized Toeplitz operators of order zero acting on Hardy space $H^2(X_I) = W^0_{\text{hol}}(X_I)$.

(2) From the K-homology point of view there is a common generalization of all operators T_p, T_f, T_Q (Q of zero order) above as well as self-adjoint elliptic pseudodifferential operators, the so-called abstract Toeplitz operators [66, Definition II.7.7]. (See also [43, Page 23][14, Sections 20 and 21][15].) Here is this notion. Let $\rho : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ be a $*$-representation of a C*-algebra $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$, and let $P \in \mathfrak{B}(\mathcal{H})$ be a projection. Assume that: *P essentially intertwines ρ in the sense that $[P, \rho(a)]$ is compact for any $a \in \mathfrak{A}$.* Then the abstract Toeplitz operator $T_a \in \mathfrak{B}(P\mathcal{H})$ with symbol $a \in \mathfrak{A}$ is defined[4] by $T_a = P\rho(a)|_{P\mathcal{H}}$. ■

Remark 7. When the ideal $I \subseteq A$ is homogeneous, the C*-algebra generated by $\{1\} \cup \{T_p : p \in A\}$ is irreducible[5], hence contains $\mathfrak{K}(I^\perp)$ if $I^\perp$ is essentially normal [61, Page 923][44, Theorem 5.39]. ■

Finally, we gather several useful facts about essential normality which will be used

[3] Some general references: [9, 24, 42, 44, 57, 85, 97, 98, 103]

[4] Note that then the mapping $\mathfrak{A} \to \mathfrak{Q}(P\mathcal{H})$, $a \mapsto T_a + \mathfrak{K}$, is in fact a $*$-homomorphism, hence induces an element of the odd K-homology group $K^1(\mathfrak{A})$. The notion of abstract Toeplitz operators is so general that the extension classes they induce represent the whole $K^1(\mathfrak{A})$ when $\mathfrak{A}$ is commutative [66, Proposition II.7.10].

[5] Namely it has no proper reducing closed subspace.

freely in the future.

Proposition 8 (Arveson-Douglas)**.** *(a) Let $I \subseteq A$ be a homogeneous ideal, and let $P \in \mathfrak{B}(H^2_m)$ and $Q := 1 - P$ be the orthogonal projections onto $\overline{I}$ and $I^\perp$, respectively. Then $\overline{I}$ is essentially normal if and only if $I^\perp$ is essentially normal if and only if all $[M_{z_\alpha}, P]$, $\alpha = 1, \ldots, m$, are compact if and only if all $PM_{z_\alpha}Q$ are compact if and only if all $[M_{z_\alpha}, Q]$ are compact if and only if all $QM^*_{z_\alpha}P$ are compact.*

(b) Let $\mathcal{M}$ and $\mathcal{N}$ be isomorphic Hilbert A-modules. Then $\mathcal{M}$ is essentially normal if and only of $\mathcal{N}$ is; if so then they represent the same odd K-homology class.

(c) Let $\mathcal{M}$ be an essentially normal Hilbert A-module, and let $\mathcal{N} \subseteq \mathcal{M}$ be a submodule. Then $\mathcal{N}$ is essentially normal if and only if the quotient module $\mathcal{M}/\mathcal{N}$ is.

(d) Let $\Psi : \mathcal{A}_1 \to \mathcal{A}_2$ be a closed-range Hilbert A-module map between essentially normal Hilbert modules. Then the kernel and range of Ψ are essentially normal.

Proof. (a) Our reference is [10, Theorem 4.3]. Recall that an operator T is compact if and only if T^* is so if and only if TT^* is so. Let the module action of $p \in A$ on H^2_m, $\overline{I}$ and $I^\perp$ be denoted by operators M_p, R_p and T_p, respectively. For brevity set $M_\alpha := M_{z_\alpha}$, $R_\alpha := R_{z_\alpha}$ and $T_\alpha := T_{z_\alpha}$. The last four statements are easily seen to be equivalent. Here are the reasons. Since $\overline{I}$ is invariant under M_α we have $PM_\alpha P = M_\alpha P$. Then

$$[M_\alpha, P] = M_\alpha P - PM_\alpha = PM_\alpha P - PM_\alpha = -PM_\alpha Q.$$

The equality $P + Q = 1$ gives $[M_\alpha, P] = -[M_\alpha, Q]$. Also note that $(PM_\alpha Q)^* = QM^*_\alpha P$.

For the rest we use the fact that H^2_m is essentially normal [6], namely that all $[M_\alpha, M^*_\beta]$ are compact. With a little abuse of language, one says that, as mappings from H^2_m to $\overline{I}$, $R_\alpha P$ and $R^*_\beta P$ equal $PM_\alpha P = M_\alpha P$ and $PM^*_\beta P$, respectively. Then

$$\begin{aligned}[R_\alpha, R^*_\beta]P &= M_\alpha PM^*_\beta P - PM^*_\beta M_\alpha P \sim M_\alpha PM^*_\beta P - PM_\alpha M^*_\beta P = [M_\alpha, P]M^*_\beta P \\ &= -PM_\alpha QM^*_\beta P = -(PM_\alpha Q)(QM^*_\beta P) = -(PM_\alpha Q)(PM_\beta Q)^* = -[M_\alpha, P][M_\beta, P]^*,\end{aligned}$$

where $\sim$ denotes equality modulo compacts. This identity shows that all $[R_\alpha, R_\beta^*]$ are compact if and only if all $[M_\alpha, P]$ are so. The rest of the proof is dual. As mappings from H_m^2 to $I^\perp$, $T_\alpha Q$ and $T_\beta^* Q$ equal $QM_\alpha Q$ and $QM_\beta^* Q = M_\beta^* Q$, respectively, and we have the identity

$$[T_\alpha, T_\beta^*]Q \sim [M_\beta, Q]^*[M_\alpha, Q]$$

which proves that all $[T_\alpha, T_\beta^*]$ are compact if and only if all $[M_\alpha, Q]$ are so.

(b, c, d) Refer respectively to [49, Proposition 4.4], [46, Theorem 2.1], [45, Theorem 2.2]. ■

1.2 Arveson's motivation

In operator theory, like many other areas of mathematics, the classification problem namely finding models for the operators of an appropriately chosen class as well as developing a complete set of easily computable unitary invariants to distinguish among those models, is of utmost importance [63, Section 45][64, Section I.8]. Here are some results in this direction:

- The spectral theorem together with the associated spectral multiplicity theory classifies normal multioperators up to unitary equivalence.[6] The complete classifier here is the cardinal-valued multiplicity function.

- Weyl, von Neumann and Berg showed that two normal operators are essentially unitarily equivalent exactly when they have the same essential spectrum [36, 39.8]. (Moreover, any compact subspace of the complex plane is the essential spectrum of some normal operator.) The odd K-homology functor of Brown-Douglas-Fillmore classifies essentially normal multioperators up to essential unitary equivalence. (See Section 1.3 below.)

[6]Some references: [5, Section II.2][39, II.3.6][35, Section IX.10][63][28][52, Page 919][88, VII.6][62, VII.22].

- For each positive integer n there is a certain set of less than 4^{n^2} numerical invariants which completely classifies $n \times n$ complex matrices up to unitary equivalence [87, Theorem 2].

- The so-called model theory of Nagy-Foiaş classifies completely nonunitary single contractions up to unitary equivalence. The complete classifier here is the operator-valued characteristic function [95, Chapter 6].

Inspired by the work of Nagy-Foiaş above, Arveson [6] developed a model theory for multivariate spherical contractions. Here are the details. Assume an m-tuple $T = (T_1, \ldots, T_m)$ of commuting operators acting on a common Hilbert space $\mathcal{H}$. Consider the completely positive map $P_T : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ given by $P_T(X) = \sum_{1 \leq j \leq m} T_j X T_j^*$. T is called a spherical contraction (also row contraction or m-contraction in the literature) if $P_T(1) \leq 1$. For simplicity we are going to state Arveson's model theorem only for pure finite-rank spherical contractions. Purity means $P_T^n(1) \to 0$ in the strong operator topology as $n \to \infty$. The rank of T is defined as the rank of the defect operator $\Delta_T := \sqrt{1 - P_T(1)}$. Arveson showed [6, Section 8][2, Chapter 41]:

Theorem 9 (Arveson)**.** *Let H_m^2 be the Hilbert space of analytic functions on $\mathbb{B}_m$ obtained by completing the polynomial vector space $\mathbb{C}[z_1, \ldots, z_m]$ with respect to the inner product $\langle z^\alpha, z^\beta \rangle = \delta_{\alpha,\beta} \frac{\alpha!}{|\alpha|!}$. Then any pure m-contraction T of finite rank r is unitarily equivalent to the r-fold inflation $M_z \otimes 1 \in \mathfrak{B}(H_m^2 \otimes \mathbb{C}^r)$ of the canonical multi-shift $M_z := (M_{z_1}, \ldots, M_{z_m}) \in \mathfrak{B}(H_m^2)^m$ compressed to the orthogonal complement of a $(M_z \otimes 1)$-invariant subspace $\mathcal{M} \subseteq H_m^2 \otimes \mathbb{C}^r$. In notations $T \cong P_{\mathcal{M}^\perp} M_z \otimes 1|_{\mathcal{M}^\perp}$. $\mathcal{M}$ is determined uniquely up to unitary equivalence by the unitary equivalence class of T.*

Arveson's next goal was to develop unitary invariants to distinguish among these models. First he [7] constructed a real-valued *curvature* invariant for (pure) finite rank spherical contractions. This invariant is computed in two different ways in [59, Theorem 5.2] and [56, Theorem 4.5], but is far from being a complete classifier.[7]

[7]The formula in the second work equates the curvature of T with $(-1)^m$ times the Fredholm index of

Later Arveson [8] associated an abstract Dirac operator D_T to any commuting m-tuple of operators T acting on a common Hilbert space $\mathcal{H}$. D_T acts on $\mathcal{H} \otimes \bigwedge \mathbb{C}^m$, where $\bigwedge \mathbb{C}^m$ is the exterior algebra made into a Hilbert space by declaring the 2^m wedge products

$$1,\ e_1,\ \ldots,\ e_m,\ e_1 \wedge e_2,\ \ldots,\ e_1 \wedge e_m,\ \ldots,\ e_1 \wedge \cdots \wedge e_m$$

to be orthonormal for the standard (or any other) orthonormal basis $\{e_1, \ldots, e_m\}$ of $\mathbb{C}^m$. D_T is defined to be $d + d^*$ where $d = \sum_{1 \le j \le m} T_j \otimes C_j$ and the creation operator C_j acts[8] as $e_j \wedge -$. For example, when $m = 1$, D_T acts as matrix

$$\begin{bmatrix} 0 & T^* \\ T & 0 \end{bmatrix}$$

on $\mathcal{H} \oplus \mathcal{H}$. Two commuting m-tuples are unitarily equivalent exactly when their corresponding Dirac operators are *isomorphic* in the sense that there is a unitary operator which intertwines not only Dirac operators but also the canonical Clifford structures used to define them [8, Theorem A]. In other words, D_T is a complete classifier for commuting multioperators, and fills a position analogous to the Nagy-Foiaş characteristic function of a single contraction (see also [21, 22] for more natural generalizations of the characteristic function.) However it is not a *computable* classifier, especially, its spectrum is complicated [8, Page 60]. To extract a computable invariant from D_T, Arveson studied its Fredholmness. Like any other operator, D_T is Fredholm exactly when $D_T D_T^*$ is so.

$T - \lambda$ for any $\lambda \in \mathbb{B}_m \setminus \sigma_e(T)$. They also showed that $\mathbb{B}_m \cap \sigma_e(T)$ is contained in a complex hypersurface.

[8] Note that d is exactly the differential map of the Koszul complex used to define the Taylor spectrum of T. This is responsible for the fact that T is Fredholm (in the sense that $\ker(d)/\mathrm{im}(d)$ is finite dimensional) if and only if D_T is so.

We have:

$$\begin{aligned}
D_T D_T^* &= dd^* + d^*d + d^2 + d^{*2} \\
&= \sum T_j T_k^* \otimes C_j C_k^* + \sum T_k^* T_j \otimes C_k^* C_j, \quad d^2 = d^{*2} = 0 \text{ by CAR} \\
&= \sum T_j T_k^* \otimes (\delta_{jk} 1 - C_k^* C_j) + \sum T_k^* T_j \otimes C_k^* C_j, \quad \text{by CAR} \\
&= \left(\sum T_j T_j^*\right) \otimes 1 - \sum [T_j, T_k^*] \otimes C_k^* C_j,
\end{aligned}$$

where CAR stands for the canonical anticommutation relations

$$C_j C_k + C_k C_j = 0, \quad C_j C_k^* + C_k^* C_j = \delta_{jk} 1, \quad j, k = 1, \dots, m.$$

Here comes the main observation [10, Proposition 1.1]: *Let T be a finite rank spherical contraction. Then $\sum T_j T_j^* = P_T(1) = 1 - \Delta_T^2$ equals the identity minus a finite rank operator, hence Fredholm. Therefore, the computation above shows that a sufficient condition for the Fredholmness of D_T is the essential normality of T.* He already knew that the canonical shift M_z of H_m^2 was essentially normal [6], and verified that they remained essentially normal after passing to the quotient by monomial ideals. All these (and maybe more) lead him to his essential normality conjecture in Section 1.1.

Remark 10. Grothendieck used homological algebra, based on the notions of ringed spaces, sheaves of $\mathcal{O}_X$-modules, resolutions and derived functors, as his language and machinery to develop the foundations of algebraic geometry. Even before him many important results in algebra and geometry were homological in essence, for example Hilbert syzygy theorem, the coherence theorems of Oka, Cousin problems, Cartan theorems A and B, and Serre's GAGA paper. Eventually, Grothendieck obtained important group-valued invariants for algebraic spaces through sheaf cohomology. Arveson [7, Page 174] and Douglas [48, Page 1], among many others, had in mind to bring into multivariate operator theory the power of homological algebra. Especially, Arveson's model theory suggests that H^2_m is the analytic analogue of the free rank one (algebraic) A-module.[9] On the other hand, Grothendieck was also pioneer applying homotopical algebra techniques to algebraic geometry. This leads to K-theoretic invariants for spaces, which will be discussed in Section 1.3 in the context of operator theory. ■

1.3 Douglas' motivation

Recall the classification result of Weyl-von Neumann-Berg in Section 1.2. Brown, Douglas and Fillmore [29, 30] solved the more natural problem of classifying essentially normal multioperators up to essential unitary equivalence[10]. For simplicity we are going to introduce their classifier for single essentially normal operators, but the definitions naturally generalize to essentially normal multioperators.

Let $T \in \mathfrak{B}(\mathcal{H})$ be an essentially normal operator. Thinking in terms of C*-algebras, T induces the short exact sequence

$$0 \to \mathfrak{K}(\mathcal{H}) \hookrightarrow \mathfrak{T} \to C(X) \to 0, \tag{1.3}$$

where $\mathfrak{T}$ is the C*-algebra generated by $\{1, T\} \cup \mathfrak{K}(\mathcal{H})$, and X is the maximal ideal space of $\mathfrak{T}/\mathfrak{K}$. One says that $\mathfrak{T}$ is an extension of $\mathfrak{K}$ by $C(X)$ (also an extension of $C(X)$ by $\mathfrak{K}$ in the literature!). Note that X is naturally identified with the essential spectrum of T, especially, $X \subseteq \mathbb{C}$ and we have access to the coordinate function z. Thinking in terms of representations, the data in (1.3) is exactly equivalent to the pointed $*$-monomorphism $C(X) \to \mathfrak{Q}(\mathcal{H})$ which sends the coordinate function $z|_X$ to the class $T + \mathfrak{K}$ of T in the Calkin algebra. Essentially because the automorphisms of $\mathfrak{K}$ are spatially implemented [39, Page 253], if $T' \in \mathfrak{B}(\mathcal{H}')$ is another essentially normal operator, T and T' are essentially unitarily equivalent if and only if T' has the same essential spectrum X, and

[9] While there is only one algebraic free module of rank one (namely A itself), there are many inequivalent Hilbert modules which can replace H^2_m, the so-called graded completions of A [11, Definition 2.2].

[10] Some other references: [23, 39, 43, 66].

the induced short exact sequences of T and T' are equivalent in the sense that there exists a $*$-isomorphism $\mathfrak{T} \to \mathfrak{T}'$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{K}(\mathcal{H}) & \hookrightarrow & \mathfrak{T} & \longrightarrow & C(X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathfrak{K}(\mathcal{H}') & \hookrightarrow & \mathfrak{T}' & \longrightarrow & C(X) & \longrightarrow & 0, \end{array}$$

commutes. There is a corresponding notion of equivalence for $*$-monomorphisms. Therefore, by its very definition, the set

$$K_1(X) := \{[\mathfrak{T}]\}$$

of all these equivalence classes of short exact sequences (or $*$-monomorphisms; or even $*$-homomorphisms [66, 2.6.3]) is a complete classifier for essentially normal operators with essential spectrum X.

Forgetting about operators, this latter definition makes sense for any topological space X; just replace $\mathfrak{T}$ by any C*-algebra of operators on a Hilbert space which fits into the short exact sequence (1.3). Brown-Douglas-Fillmore made K_1 a functor from the category of compact metrizable spaces into the category of abelian groups, and then used the methodology of algebraic topology (especially, pairing with the topological K-theory functor K^1, the axioms of generalized Steenrod homology theory, etc.) to compute it in some cases including spheres and planar subspaces. There is a non-canonically split short exact sequence, called the universal coefficient theorem, which computes $K_1(X)$ in terms of the topological K-theory groups $K^0(X)$ and $K^1(X)$ [66, VII.6.1][89]. (See also [69].) To sum up our presentation of $K_1(X)$ so far: *for $X \subseteq \mathbb{C}^m$ compact, $K_1(X)$ is the universal complete classifier of essentially normal m-multioperators with essential Taylor spectrum X up to essential unitary equivalence.*

Besides the operator-theoretic interpretation above[11], the elements of $K_1(X)$ has

[11] In terms of Kasparov's bivariant theory $K_1(X) = KK^1(C(X), \mathbb{C})$ [71, 23].

found other geometric and topological realizations such as:

1. self-adjoint elliptic pseudodifferential operators (for X closed smooth manifold) [14, Sections 6 and 24][15]; (See also [12, 70].)

2. the so called topological K-cycles; each such cycle is a triple (M, φ, E) consisting of odd-dimensional closed Spin^c manifold M, continuous map $\varphi : M \to X$, and complex vector bundle E over M [14, 18].

The link between two latter realizations is the standard construction of the twisted Spin^c Dirac operator in differential geometry [14, Section 17][97, 12.8]. We sketch the link between operator-theoretic K-homology and the first realization above[12]. Let $D : C^\infty(X; E) \to C^\infty(X; E)$ be an elliptic pseudodifferential operator of positive order acting on the smooth sections of a complex vector bundle E over closed manifold X. Fix some smooth positive density on X and a smooth Hermitian (inner product) structure on E such that D is symmetric, namely formally self-adjoint. Most important examples are the Laplace ($dd^* + d^*d$, $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$), de Rham ($d + d^*$), Dolbeault ($\overline{\partial} + \overline{\partial}^*$), and Dirac type operators [13][97, 10.1]. D, as an unbounded operator on the Lebesgue space $L^2(X; E)$, has an orthonormal basis of eigenfunctions[13], hence essentially self-adjoint[14], and we denote its unique self-adjoint extension again by D. Let $L^2_+(X; E) \subseteq L^2(X; E)$ be the spectral subspace corresponding to $[0, \infty)$, namely the range of the orthogonal projection $P := \chi_{[0,\infty)}(D)$. Let $T_f \in \mathfrak{B}(L^2_+(X; E))$, $f \in C(X)$, be the compression of the multiplication operator $M_f \in \mathfrak{B}(L^2(X; E))$. Since M_f and P are pseudodifferential operators of order zero [94][96, 12.1.3], the commutator $[P, M_f]$ is pseudodifferential of order ≤ -1, hence compact. Therefore the mapping $C(X) \to \mathfrak{Q}(L^2_+(X; E))$, $f \mapsto T_f + \mathfrak{K}$ is $*$-homomorphic, hence we get an element of $K_1(X)$ that will be denoted by $[D]$.

Here is the definition of the fundamental class in Conjecture 3. Let $Y \subseteq \mathbb{C}^m$ be a smooth closed oriented real hypersurface. The Cauchy-Riemann structure on Y induces

[12]Some references: [14, Sections 6 and 20][15][66, II.8.c, X.6]
[13]Some references: [97, 7.10][41, 23.35.2][92, 8.3][86, XI.14][79, III.5.8][66, X.4.6].
[14]Some references: [40, 2.2.10][62, 9.25][31][66, X.2.6].

a canonical Spin^c structure on it [97, 10.8], hence a Spin^c Dirac operator D_Y, hence an element $[D_Y] \in K_1(Y)$ by the construction in the previous paragraph. This element is called the fundamental class of Y.

We can now motivate Conjecture 3. Baum and Douglas [14] defended the viewpoint that an index theorem, namely a formula for the index of a naturally occurring Fredholm operator in terms of the underlying topological information, should be understood as an isomorphism between different realizations of K-homology. They (together with M. E. Taylor) [14, 15] put into this framework the index theorems of Grothendieck-Riemann-Roch (possibly singular projective algebraic varieties over $\mathbb{C}$), Atiyah-Singer (elliptic pseudodifferential operators on closed manifolds), Connes (transversally elliptic differential operators on foliated manifolds) and Boutet de Monvel (classical Toeplitz operators on strongly pseudoconvex domains)[15]. Specially, [15, Proposition 4.5] generalizes Boutet de Monvel's index theorem to certain classes of smoothly bounded weakly pseudoconvex domains inside complex manifolds (no singularity is allowed). This result identifies the extension class represented by the C*-algebra of continuous-symbol Toeplitz operators with the fundamental class (induced by the Spin^c Dirac operator). Conjecture 3 is the analogous statement for possibly singular algebraic varieties. Maybe this is why Douglas [61, Page 910] suggested that one needs a generalization of the calculus of pseudodifferential operators to the context of algebraic spaces in order to resolve Conjecture 1.

Remark 11. In retrospect, one observes that while the classification of tuples of normal operators up to unitary equivalence via spectral theory relies on measure theory as its

[15]Here is a concrete formula from Boutet de Monvel's work [25, 60, 99]. For any smoothly bounded strongly pseudoconvex domain $\Omega \subseteq \mathbb{C}^m$ and any smooth matrix-valued function $F \in C^\infty\left(\overline{\Omega}; M_{n\times n}(\mathbb{C})\right)$, $n \geq 1$, if $F|_{\partial\overline{\Omega}}$ is invertible, then the Toeplitz operator $T_F \in \mathfrak{B}\left(L^2_a(\Omega)^n\right)$ with symbol F is Fredholm, and its index is given by

$$-\frac{(m-1)!}{(2\pi i)^m(2m-1)!}\int_{\partial\overline{\Omega}} \text{trace}\left(F^{-1}\left(dF\right)^{2m-1}\right).$$

When $\Omega \subseteq \mathbb{C}$ is the unit disk this statement reduces to the classical Toeplitz index theorem usually attributed to Gohberg-Krein [9, 4.4.3][44, 7.26].

fundamental tool, the corresponding perturbation problem of the classification of tuples of essentially normal operators up to essential unitary equivalence via Brown-Douglas-Fillmore theory uses algebraic topology as its fundamental tool [66, Page 15][2, Page 967]. From the representation theory standpoint, these two theories classify $*$-representations of $C(X)$ in $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{Q}(\mathcal{H})$, respectively [35, IX.1.14]. ■

1.4 Some variations of Arveson's conjecture

This section gathers several variations of Conjecture 1 that will be needed in the future. Arveson's original statement of his conjecture was [8, Problem 2][10, Conjecture A]:

Conjecture 12 (Arveson)**.** *Let $\mathcal{M}$ be a homogeneous Hilbert A-submodule of $H^2_m \otimes \mathbb{C}^r$, $r > 0$. Then $\mathcal{M}^\perp$ is essentially normal.*

Note that the Hilbert module structure on $\mathcal{M}^\perp$ is by the compressions $T_p := P_{\mathcal{M}^\perp} M_p \otimes 1|_{\mathcal{M}^\perp}$, $p \in A$. Homogeneity (or gradedness) means that $\mathcal{M}$ contains all homogeneous components of its elements[16]; then automatically $\mathcal{M} = \overline{\mathcal{M} \cap (A \otimes \mathbb{C}^r)}$, hence $\mathcal{M}$ is generated by finitely many homogeneous (vector-valued) polynomials according to Hilbert basis theorem. Arveson insisted on his conjecture even for nonhomogeneous submodules generated by finitely many polynomials [10, Conjecture B]. However an example of Gleason, Richter and Sundberg [56, Page 72] shows that the conjecture can not be extended to general submodules.[17] "A question seemingly beyond current techniques is whether a submodule of $L^2_a(\mathbb{B}_m)$ is essentially normal if and only if it is finitely generated" [50, Page 3179].

Although we do not refer to it but in the literature when people talk about the *Arveson-Douglas Conjecture* they mean the following [2, Page 1165]:

[16] The monomial elements $z^n \otimes \xi \in H^2_m \otimes \mathbb{C}^r$ are declared to be homogeneous of degree $|n|$.

[17] They in fact found a pure 2-contraction of rank 1 which is not Fredholm, hence not essentially normal.

Conjecture 13 (Arveson-Douglas). *Let $\mathcal{M}$ be a homogeneous Hilbert A-submodule of $H^2_m \otimes \mathbb{C}^r$, $r > 0$. Then $\mathcal{M}^\perp$ is p-essentially normal for all $p > \dim \mathcal{M}$.*

Here, p-essential normality means that all commutators $[T_{z_i}, T^*_{z_j}]$, $i, j = 1, \dots, m$, are Schatten p-summable, namely $|[T_{z_i}, T^*_{z_j}]|^p$ are trace class. $\dim \mathcal{M}$ is the complex dimension of the variety that $\mathcal{M}$ lives above namely $V(\mathrm{Ann}(\mathcal{M})) \subseteq \mathbb{C}^m$. Algebraically, $\dim \mathcal{M}$ equals one plus the degree of the Hilbert polynomial of $H^2_m \otimes \mathbb{C}^r/\mathcal{M}$ [64, I.7.5].

Douglas [45] even suggests Conjecture 13 for $L^2_a(\Omega)$, $\Omega \subseteq \mathbb{C}^m$ smoothly bounded strongly pseudoconvex domain, instead of H^2_m. The convexity assumption can not be dropped: even the Bergman space over the bidisk is not essentially normal. A complete characterization of all essentially normal homogeneous submodules of the Bergman space on the unit polydisk is given in [100].

Remark 14. Arveson [11] showed that to prove Conjecture 12 it suffices to verify it for homogeneous submodules generated by linear vector-valued polynomials. Shalit [91] showed that to prove Conjecture 12 it suffices to verify Conjecture 1 for homogeneous ideals generated by quadratic polynomials.

1.5 A summary of the results in this dissertation

There are two sets of new results in this dissertation, arranged in Chapters 2 and 3:

- *Chapter 2*. For an arbitrary monomial ideal $I \subseteq \mathbb{C}[z_1, \dots, z_m]$, we resolve $\overline{I} \subseteq H^2_m$ through essentially normal Hilbert modules and Hilbert module maps between them (Theorem 15):

  $$0 \to \overline{I} \hookrightarrow \mathcal{A}_0 \overset{\Psi_0}{\to} \mathcal{A}_1 \overset{\Psi_1}{\to} \cdots \overset{\Psi_{k-1}}{\to} \mathcal{A}_k \to 0.$$

  Together with Proposition 8.(d) it gives a new proof for Arveson's essential normality conjecture. (Compare [10, 46].) Each $\mathcal{A}_q$, $q = 0, \dots, k$, has a tractable geometry as the Hilbert space of square-integrable analytic sections of a Hermitian vector bundle on a disjoint union of subsets of $\mathbb{B}_m$. As an application of this resolution we derive an index formula for τ_I (Theorem 16):

$$\tau_I := \left[\mathfrak{T}(I^\perp)\right] = \sum_{q=1}^{k}(-1)^{q-1}[\mathfrak{T}(\mathcal{A}_q)] \quad \text{in} \quad K_1\left(\sigma_e^1 \cup \cdots \cup \sigma_e^k\right).$$

This answers Douglas' index problem in the special case of monomial ideals. Some ideas to extend these results are discussed in Section 2.8.

- *Chapter 3*. To analytically study the monodromy of an isolated singularity at the origin on an algebraic hypersurface $V(f) \subseteq \mathbb{C}^m$, we consider the perturbed 1-parameter family of principal ideals $I(t) := \langle f - \epsilon e^{it}\rangle$, $t \in \mathbb{R}$, $\epsilon > 0$ small enough. The family $I(t)^\perp \subseteq H_m^2$ of associated Hilbert modules, as a subbundle of the trivial bundle $H_m^2 \times \mathbb{R}$, is naturally equipped with a metric connection. Of special interest is the holonomy of this connection, a (conjecturally) unitary operator $U \in \mathfrak{B}(I(0)^\perp)$, and the way it interacts with the Toeplitz algebra $\mathfrak{T}_{I(0)}$. Our study is at a preliminary stage. In this chapter we propose a program to study the conjectural holonomy operator by formulating a series of reasonable conjectures (Conjectures 31, 32, and 34). We are able to test these conjectures for our toy model $f := z_1^k$, $k \in \mathbb{N}$. One of our guidelines is the classical work of Milnor in singularity theory [83]. We hope to eventually get hands on the Brieskorn polynomials $f := \sum_{1 \leq l \leq m} z_l^{b_l}$, $b_l \geq 2$; this study could eventually lead to an analytic way for detecting exotic smooth structures on odd-dimensional spheres (Section 3.1). Some potential directions for future works are discussed in Section 3.5.

Chapter 2

A Toeplitz index theorem for monomial ideals

In this chapter, we [47] give an answer to Douglas' index problem for the special case of monomial ideals. One reason why we care about monomial ideals is that a comprehensive understanding of the phenomena appearing in this generically nonradical case may lead to new results beyond the recently established ones about radical ideals [53, 49, 51]. (Notice that the ideal assumed in Theorem 4 is necessarily radical [49, Page 325].)

One concept we extensively put into action in this chapter is that of jets. Also we prefer to work with the Bergman space $L^2_a(\mathbb{B}_m)$ instead of the m-shift space H^2_m, although our results hold for the latter. The reason is that we need weights to make our differentiation maps between Bergman spaces bounded.[1] In the future we wish to understand in a more abstract framework the rigid structures present in this chapter.

Here is an outline of this chapter. Section 2.1 motivates and states the main results: a resolution of the closure of a monomial ideal by essentially normal Hilbert modules, and its resulting K-homology index formula. Section 2.2 introduces the main building blocks of our resolution: the so-called boxes, and their associated Hilbert modules. The

[1]Beatrous (among others) taught us that the Bergman weights and Sobolev differentiability indices do compensate for each other [19]. One realization of this idea is the identification $\mathcal{H}^{(s)}_m = W^{-\frac{s}{2}}_{\text{hol}}(\mathbb{B}_m)$, $s \in \mathbb{R}$, between the Besov-Sobolev and Bergman-Sobolev spaces, already mentioned on page x.

resolution is constructed in Section 2.3, the proof that it is in fact a resolution comes in Section 2.4. Section 2.5 proves the index formula. Sections 2.6 and 2.7 are devoted to examples. Section 2.8 gives some directions for future works.

2.1 The main results

We motivate our main results by considering the simplest nonradical ideal $I := \langle z_1^2 \rangle \subseteq \mathbb{C}[z_1, z_2]$. Here the quotient $\mathbb{C}[z_1, z_2]$-module $\mathcal{Q}_I = L_a^2(\mathbb{B}_2)/\overline{I}$ can be identified with the direct sum $\mathcal{A}_1 := L_{a,1}^2(\mathbb{B}_1) \oplus L_{a,2}^2(\mathbb{B}_1)$ in the following way. If $\Psi_0 : L_a^2(\mathbb{B}_2) \to \mathcal{A}_1$ is defined by

$$f \mapsto \left(f|_{z_1=0}, \frac{\partial f}{\partial z_1}\Big|_{z_1=0} \right),$$

then one can easily find a Hilbert $\mathbb{C}[z_1, z_2]$-module structure on $\mathcal{A}_1$ which makes

$$0 \to \overline{I} \hookrightarrow L_a^2(\mathbb{B}_2) \overset{\Psi_0}{\to} \mathcal{A}_1 \to 0 \tag{2.1}$$

into an exact sequence of Hilbert modules; the Hilbert module structure on $\mathcal{A}_1$ is:

$$z_1 \cdot (X(z_2), Y(z_2)) = (0, X(z_2)), \quad z_2 \cdot (X(z_2), Y(z_2)) = (z_2 X(z_2), z_2 Y(z_2)).$$

Computation with the standard orthonormal basis shows that $L_a^2(\mathbb{B}_2)$ and $\mathcal{A}_1$ are essentially normal, hence (2.1) is a resolution of $\overline{I}$ by essentially normal Hilbert modules and bounded module homomorphisms between them. Just the existence of such a resolution, by Proposition 8, implies the essential normality of $\overline{I}$ and $\mathcal{Q}_I$. Furthermore, $\mathcal{Q}_I$ and $\mathcal{A}_1$ are isomorphic as Hilbert modules, so their Toeplitz extension classes $[\mathfrak{T}(\mathcal{Q}_I)]$ and $[\mathfrak{T}(\mathcal{A}_1)]$ are identified.

From the geometrical point of view, the resolution (2.1) organizes jets $z^i \notin I$ living on the variety $V(I)$ in different co-syzygy levels in order to co-present $\overline{I}$. More generally, we can prove that:

Theorem 15. *Let $I \subseteq A = \mathbb{C}[z_1, \ldots, z_m]$ be a monomial ideal. Let $\overline{I}$ be the closure of I in $L_a^2(\mathbb{B}_m)$. Then there exist a positive integer k, essentially normal Hilbert A-modules $\mathcal{A}_0 := L_a^2(\mathbb{B}_m)$, $\mathcal{A}_1, \ldots, \mathcal{A}_k$, and Hilbert A-module morphisms $\Psi_q : \mathcal{A}_q \to \mathcal{A}_{q+1}$, $q = 0, \ldots, k-1$, such that*

$$0 \to \overline{I} \hookrightarrow \mathcal{A}_0 \overset{\Psi_0}{\to} \mathcal{A}_1 \overset{\Psi_1}{\to} \cdots \overset{\Psi_{k-1}}{\to} \mathcal{A}_k \to 0 \tag{2.2}$$

is exact.

We will explain later that each $\mathcal{A}_q$ is a direct sum of weighted Bergman spaces over lower dimensional balls, hence can be geometrized as the Hilbert space of square-integrable analytic sections of a Hermitian vector bundle on a disjoint union of subsets of $\mathbb{B}_m$. Regardless of the fine structure of the modules and maps in resolution (2.2), just the existence of such an exact sequence implies the essential normality of $\overline{I}$ and $\mathcal{Q}_I$ via repeated applications of Proposition 8. This is a new proof for Arveson's conjecture in the special case of monomial ideals. (Compare [10, 46].) With some extra work we deduce the following theorem which answers Douglas' index problem for monomial ideals:

Theorem 16. *Assume the notations of Theorem 15. For each q, let $\mathfrak{T}(\mathcal{A}_q)$ and σ_e^q be respectively the Toeplitz C^*-algebra and the essential Taylor spectrum associated to the Hilbert A-module $\mathcal{A}_q$. Then the identification*

$$\tau_I = \sum_{q=1}^{k} (-1)^{q-1} [\mathfrak{T}(\mathcal{A}_q)] \tag{2.3}$$

holds in $K_1\left(\sigma_e^1 \cup \cdots \cup \sigma_e^k\right)$. (Recall that $\tau_I := \left[\mathfrak{T}(I^\perp)\right]$ is the Toeplitz class associated to $I^\perp$.)

Our developments in this chapter is another attempt to apply homological algebra to multivariate operator theory. (Recall Remark 10. See also [48][2, Chapters 38 and 39].)

2.2 Boxes and their associated Hilbert modules

This section introduces and studies the main building blocks in the construction of the resolution (2.2).

Some notations

Recall that the weighted Bergman space $L^2_{a,s}(\mathbb{B}_m)$, $s > -1$, has the standard orthonormal basis:

$$\left\{ z^{\mathfrak{n}} := \frac{z_1^{n^1} \dots z_m^{n^m}}{\sqrt{\omega_s(\mathfrak{n})}} \ : \ \mathfrak{n} = (n^1, \dots, n^m) \in \mathbb{N}^m \right\}, \tag{2.4}$$

where

$$\omega_s(\mathfrak{n}) := \frac{\mathfrak{n}!(s+m)!}{(|\mathfrak{n}| + s + m)!}.$$

For each positive integer q, let $S_q(m)$ denote the set of all q-shuffles of the set $\{1, \dots, m\}$, namely

$$S_q(m) := \left\{ \mathfrak{j} := (j^1, \dots, j^q) \in \mathbb{Z}^q : 1 \le j^1 < j^2 < \dots < j^q \le m \right\}.$$

Whenever necessary we identify shuffles in $S_q(m)$ with subsets of $\{1, \dots, m\}$ of size q. This enables us to talk about the union, intersection, etc. of shuffles of $\{1, \dots, m\}$ with themselves and with other subsets of $\{1, \dots, m\}$.

Boxes and their associated Hilbert modules

To each $\mathfrak{j} = (j^1, \dots, j^q) \in S_q(m)$ and $\mathfrak{b} = (b^1, \dots, b^q) \in \mathbb{N}^q$, we associate the *box*

$$\mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}} := \left\{ (n^1, \dots, n^m) \in \mathbb{N}^m : n^{j^i} \le b^i \text{ for } i = 1, \dots, q \right\}.$$

To each box $\mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}}$, we associate the Hilbert space $\mathcal{H}_{\mathfrak{j}}^{\mathfrak{b}} \subseteq L^2_a(\mathbb{B}_m)$ consisting of functions $X = \sum_{\mathfrak{n} \in \mathbb{N}^m} X_{\mathfrak{n}} z^{\mathfrak{n}}$ satisfying $X_{\mathfrak{n}} = 0$ for $\mathfrak{n} \notin \mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}}$. In other words, $\mathcal{H}_{\mathfrak{j}}^{\mathfrak{b}}$ is the orthogonal complement

$$\mathcal{H}_{\mathfrak{j}}^{\mathfrak{b}} = L^2_a(\mathbb{B}_m) \ominus \left\langle z_{j^1}^{b^1+1}, \dots, z_{j^q}^{b^q+1} \right\rangle.$$

An element $X \in \mathcal{H}_{\mathfrak{j}}^{\mathfrak{b}}$ has the Taylor expansion

$$X = \sum_{n^{j^1} \le b^1, \ldots, n^{j^q} \le b^q} X_{n^1 \cdots n^m} z^{\mathfrak{n}}.$$

The general construction of Section 1.1 makes $\mathcal{H}_{\mathfrak{j}}^{\mathfrak{b}}$ a Hilbert A-module. More explicitly, its fundamental tuple of Toeplitz operators are given by

$$T_{z_i}^{\mathfrak{j},\mathfrak{b}}(z^{\mathfrak{n}}) := \begin{cases} z_i z^{\mathfrak{n}}, & \text{if } (n^1, \ldots, n^{i-1}, n^i + 1, n^{i+1}, \ldots, n^m) \in \mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}}, \\ 0, & \text{otherwise.} \end{cases}, \quad i = 1, \ldots, m.$$

Some properties of the Hilbert modules associated to boxes

Lemma 17. *Each $\mathcal{H}_{\mathfrak{j}}^{\mathfrak{b}}$ is essentially normal.*

Proof. Let P be the orthogonal projection from $L_a^2(\mathbb{B}_m)$ onto $\mathcal{H}_{\mathfrak{j}}^{\mathfrak{b}}$, and $M_{z_i} \in \mathfrak{B}(L_a^2(\mathbb{B}_m))$, $i = 1, \ldots, m$, the multiplication by z_i. According to Proposition 8.(a), it suffices to check that each $[M_{z_i}, P]$ is compact. For each $\mathfrak{n} \in \mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}}$, we have

$$PM_{z_i}(z^{\mathfrak{n}}) = \begin{cases} \sqrt{\frac{\omega_0(n^1 \cdots n^i + 1 \cdots n^m)}{\omega_0(n^1 \cdots n^m)}} z^{n^1 \cdots n^i + 1 \cdots n^m}, & \text{if } (n^1 \cdots n^i + 1 \cdots n^m) \in \mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}}, \\ 0, & \text{otherwise,} \end{cases}$$

$$M_{z_i}P(z^{\mathfrak{n}}) = \begin{cases} \sqrt{\frac{\omega_0(n^1 \cdots n^i + 1 \cdots n^m)}{\omega_0(n^1 \cdots n^m)}} z^{n^1 \cdots n^i + 1 \cdots n^m}, & \text{if } (n^1 \cdots n^i \cdots n^m) \in \mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the coefficients $\sqrt{\cdots}$ appear because of the normalization assumption in (2.4). Therefore

$$[M_{z_i}, P](z^{\mathfrak{n}}) = \begin{cases} \sqrt{\frac{\omega_0(n^1 \cdots b^l + 1 \cdots n^m)}{\omega_0(n^1 \cdots b^l \cdots n^m)}} z^{n^1 \cdots n^i + 1 \cdots n^m}, & \text{if } (n^1 \cdots n^i \cdots n^m) \in \mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}} \text{ and} \\ & \exists l \text{ such that } i = j^l, n^i = b^l, \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$\frac{\omega_0(n^1\cdots b^l+1\cdots n^m)}{\omega_0(n^1\cdots b^l\cdots n^m)}\to 0 \quad \text{as} \quad \|(n^1,\ldots,b^l,\ldots,n^m)\|\to\infty,$$

we can conclude that $[M_{z_i},P]$ is compact. ∎

Lemma 18. *The intersections of boxes are again boxes.*

Proof. It suffices to consider only two boxes $\mathbf{B}_{\mathfrak{j}_i}^{\mathfrak{b}_i}$, $i=1,2$, with $\mathfrak{j}_i=(j_i^1,\ldots,j_i^{q_i})$ and $\mathfrak{b}_i=(b_i^1,\ldots,b_i^{q_i})$. Let $\mathfrak{j}:=(j^1,\ldots,j^q)\in S_q(m)$ be the union of $\mathfrak{j}_1$ and $\mathfrak{j}_2$. Define $\mathfrak{b}:=(b^1,\ldots,b^q)\in\mathbb{N}^q$ by

$$b^l:=\begin{cases}\min(b_1^{s_1},b_2^{s_2}), & j^l=j_1^{s_1}=j_2^{s_2}\in\mathfrak{j}_1\cap\mathfrak{j}_2,\\ b_1^{s_1}, & j^l=j_1^{s_1}\in\mathfrak{j}_1\setminus\mathfrak{j}_2,\\ b_2^{s_2}, & j^l=j_2^{s_2}\in\mathfrak{j}_2\setminus\mathfrak{j}_1.\end{cases}$$

It is easy to check that $\mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}}=\mathbf{B}_{\mathfrak{j}_1}^{\mathfrak{b}_1}\cap\mathbf{B}_{\mathfrak{j}_2}^{\mathfrak{b}_2}$. ∎

The geometry of the Hilbert modules associated to boxes

Consider the Hilbert module $\mathcal{H}_{\mathfrak{j}}^{\mathfrak{b}}$ associated to the box $\mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}}$. Set

$$\mathbb{B}_{\mathfrak{j}}:=\{(z_1,\ldots,z_m)\in\mathbb{B}_m:\ z_{j^1}=\cdots=z_{j^q}=0\}.$$

Observe that $\mathbb{B}_{\mathfrak{j}}$ is the unit ball inside the space

$$\{z\in\mathbb{C}^m: z_{j^1}=\cdots=z_{j^q}=0\}\cong\mathbb{C}^{m-q}.$$

Consider the Hilbert space

$$\widetilde{\mathcal{H}}_{\mathfrak{j}}^{\mathfrak{b}}:=\bigoplus_{\substack{\mathfrak{i}=(i^1,\ldots,i^q)\in\mathbb{N}^q\\ i^1\le b^1,\ldots,i^q\le b^q}} L^2_{a,q+|\mathfrak{i}|}(\mathbb{B}_{\mathfrak{j}}),$$

and the map $R^{\mathfrak{b}}_{\mathrm{j}} : \mathcal{H}^{\mathfrak{b}}_{\mathrm{j}} \to \widetilde{\mathcal{H}}^{\mathfrak{b}}_{\mathrm{j}}$ given by sending $X \in \mathcal{H}^{\mathfrak{b}}_{\mathrm{j}}$ to $Y = \sum Y^{\mathrm{i}}$, $Y^{\mathrm{i}} \in L^2_{a,q+|\mathrm{i}|}(\mathbb{B}_{\mathrm{j}})$ defined by

$$Y^{\mathrm{i}} = \left(\frac{m!}{i!(m+|i|)!}\right)^{\frac{1}{2}} \frac{\partial^{|i|} X}{\partial z^{i^1}_{j^1} \cdots \partial z^{i^q}_{j^q}}\bigg|_{\mathbb{B}_{\mathrm{j}}}.$$

A straightforward computation with the orthonormal basis shows that: $R^{\mathfrak{b}}_{\mathrm{j}}$ *is an isomorphism of Hilbert spaces.*

Now consider the trivial vector bundle $E^{\mathfrak{b}}_{\mathrm{j}} := \mathbb{C}^{(b^1+1)\cdots(b^q+1)} \times \mathbb{B}_{\mathrm{j}}$ over $\mathbb{B}_{\mathrm{j}}$, together with its standard constant frame

$$\{e_{\mathrm{i}} : \mathrm{i} = (i^1, \ldots, i^q) \in \mathbb{N}^q, i^1 \leq b^1, \ldots, i^q \leq b^q\}.$$

Put the following Hermitian structure on $E^{\mathfrak{b}}_{\mathrm{j}}$:

$$\langle e_{\mathrm{i}}, e_{\mathrm{i}'}\rangle(z) = \delta_{\mathrm{i},\mathrm{i}'} \left(1 - |z|^2\right)^{q+|\mathrm{i}|}, \quad z \in \mathbb{B}_{\mathrm{j}}.$$

The Hilbert space $\widetilde{\mathcal{H}}^{\mathfrak{b}}_{\mathrm{j}}$ can be identified with the Bergman space of the L^2-holomorphic sections of $E^{\mathfrak{b}}_{\mathrm{j}}$. Consider the Toeplitz algebra $\mathfrak{T}(E^{\mathfrak{b}}_{\mathrm{j}})$ generated by matrix-valued Toeplitz operators on the Bergman space of L^2-holomorphic sections. Under the isomorphism $R^{\mathfrak{b}}_{\mathrm{j}}$, one can easily identify the Toeplitz algebra generated by $\left\{T^{\mathrm{j},\mathfrak{b}}_{z_i} : i = 1, \ldots, m\right\}$ with $\mathfrak{T}(E^{\mathfrak{b}}_{\mathrm{j}})$.

2.3 The construction of the resolution

This section constructs the resolution in Theorem 16. Let the ideal $I \subseteq A$ be generated by distinct monomials

$$z^{\alpha_i}, \quad \alpha_i := (\alpha^1_i, \ldots, \alpha^m_i) \in \mathbb{N}^m, \quad i = 1, \ldots, l.$$

Let the complementary space $\mathsf{C}(I) \subseteq \mathbb{N}^m$ be the set of the exponents of monomials that do not belong to I. Note that the set of monomials belonging to I is a basis of I as a complex vector space [65, Theorem 1.1.2]. Also note that a monomial u belongs to I if and only if there is a monomial v such that $u = vz^{\alpha_i}$ for some $i = 1, \ldots, l$. (See [65, Proposition 1.1.5].) Contrapositively, $z_1^{n^1} \cdots z_m^{n^m} \in \mathsf{C}(I)$ if and only for each $i = 1, \ldots, l$ there is $s_i \in \{1, \ldots, m\}$ such that $n^{s_i} < \alpha_i^{s_i}$. Consider the finite collection

$$S(\alpha_1, \ldots, \alpha_l) := \{1, \ldots, m\}^l$$

of l-tuples $\mathfrak{s} = (s_1, \ldots, s_l)$ such that $1 \leq s_i \leq m$ for all i. For each $\mathfrak{s}$, let $\mathfrak{j}_\mathfrak{s}$ be the shuffle associated to the set $\{s_1, \ldots, s_l\}$. For each $j \in \mathfrak{j}_\mathfrak{s}$, let b_j be the minimum of all $\alpha_i^{s_i} - 1$, $i = 1, \ldots, l$, such that $s_i = j$. Set $\mathfrak{b}_\mathfrak{s} := (b_j)_{j \in \mathfrak{j}_\mathfrak{s}}$. The following symbolic logic computation shows that: $\mathsf{C}(I)$ *is the union of boxes* $\mathbf{B}_{\mathfrak{j}_\mathfrak{s}}^{\mathfrak{b}_\mathfrak{s}}$, $\mathfrak{s} \in S(\alpha_1, \ldots, \alpha_l)$.

$$\begin{aligned} z_1^{n^1} \cdots z_m^{n^m} \in \mathsf{C}(I) &\leftrightarrow \left(n^1 < \alpha_1^1 \vee \cdots \vee n^m < \alpha_1^m\right) \wedge \cdots \wedge \left(n^1 < \alpha_l^1 \vee \cdots \vee n^m < \alpha_l^m\right) \\ &\leftrightarrow \bigvee_{(s_1, \ldots, s_l) \in \{1, \ldots, m\}^l} \left(n^{s_1} < \alpha_1^{s_1} \wedge \cdots \wedge n^{s_l} < \alpha_l^{s_l}\right). \end{aligned}$$

The construction of modules $\mathcal{A}_q$

From now on and throughout Sections 2.3 and 2.4, fix a finite collection of boxes

$$\mathbf{B}_{\mathfrak{j}_i}^{\mathfrak{b}_i}, \quad i = 1, \ldots, k, \tag{2.5}$$

such that their union equals $\mathsf{C}(I)$. For each $I \subseteq \{1, \ldots, k\}$ (note that we are using the symbol I for two purposes), let

$$\mathbf{B}_{\mathfrak{j}_I}^{\mathfrak{b}_I} := \bigcap_{i \in I} \mathbf{B}_{\mathfrak{j}_i}^{\mathfrak{b}_i}$$

denote the intersection box (Lemma 18). Each box $\mathbf{B}_{\mathfrak{j}_I}^{\mathfrak{b}_I}$ has a corresponding Hilbert module $\mathcal{H}_{\mathfrak{j}_I}^{\mathfrak{b}_I}$ as introduced in Section 2.2. For each $q = 1, \ldots, k$, set:

$$\mathcal{A}_q := \bigoplus_{I\in S_q(k)} \mathcal{H}_{\mathfrak{j}_I}^{\mathfrak{b}_I}.$$

For convenience, we use $\mathcal{A}_0$ to denote the Bergman space $L_a^2(\mathbb{B}_m)$. Note that each Hilbert space $\mathcal{A}_q$ is equipped with a Hilbert A-module structure from the corresponding A-module structure on each component $\mathcal{H}_{\mathfrak{j}_I}^{\mathfrak{b}_I}$. It follows from Lemma 17 that each $\mathcal{A}_q$ is essentially normal.

The construction of maps Ψ_q

To explain our construction, we start with a few examples with a small number k of boxes (2.5).

When $k = 1$, there is only one box $\mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}}$. We have two Hilbert modules $\mathcal{A}_0 = L_a^2(\mathbb{B}_m)$ and $\mathcal{A}_1 = \mathcal{H}_{\mathfrak{j}}^{\mathfrak{b}}$. The map $\Psi_0 : \mathcal{A}_0 \to \mathcal{A}_1$ is defined by sending $X \in \mathcal{A}_0$ to $Y \in \mathcal{A}_1$ given by

$$Y_{\mathfrak{n}} = \begin{cases} X_{\mathfrak{n}}, & \mathfrak{n} \in \mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}}, \\ 0, & \text{otherwise.} \end{cases}$$

When $k = 2$, there are two boxes $\mathbf{B}_{\mathfrak{j}_1}^{\mathfrak{b}_1}$ and $\mathbf{B}_{\mathfrak{j}_2}^{\mathfrak{b}_2}$. Let $\mathbf{B}_{\mathfrak{j}_{12}}^{\mathfrak{b}_{12}}$ denote their intersection. We have three Hilbert modules $\mathcal{A}_0 = L_a^2(\mathbb{B}_m)$, $\mathcal{A}_1 = \mathcal{H}_{\mathfrak{j}_1}^{\mathfrak{b}_1} \oplus \mathcal{H}_{\mathfrak{j}_2}^{\mathfrak{b}_2}$, and $\mathcal{A}_2 = \mathcal{H}_{\mathfrak{j}_{12}}^{\mathfrak{b}_{12}}$. The map $\Psi_0 : \mathcal{A}_0 \to \mathcal{A}_1$ is defined by sending $X \in \mathcal{A}_0$ to $(Y^1, Y^2) \in \mathcal{A}_1$, given by

$$Y_{\mathfrak{n}}^1 := \begin{cases} X_{\mathfrak{n}}, & \mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_1}^{\mathfrak{b}_1}, \\ 0, & \text{otherwise,} \end{cases}, \qquad Y_{\mathfrak{n}}^2 := \begin{cases} X_{\mathfrak{n}}, & \mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_2}^{\mathfrak{b}_2}, \\ 0, & \text{otherwise.} \end{cases}$$

The map $\Psi_1 : \mathcal{A}_1 \to \mathcal{A}_2$ is defined by sending $(X^1, X^2) \in \mathcal{A}_1$ to $Y \in \mathcal{A}_2$ given by

$$Y_{\mathfrak{n}} = \begin{cases} X_{\mathfrak{n}}^2 - X_{\mathfrak{n}}^1, & \mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_{12}}^{\mathfrak{b}_{12}}, \\ 0, & \text{otherwise.} \end{cases}$$

For arbitrary k, in order to define $\Psi_q : \mathcal{A}_q \to \mathcal{A}_{q+1}$, $q = 0, \ldots, k-1$, first consider the following maps $f^i_{q+1} : S_{q+1}(k) \to S_q(k)$, $i = 1, \ldots, q+1$. As usual, elements of $S_{q+1}(k)$ are identified with subsets $I_{q+1} \subseteq \{1, \ldots, k\}$ of size $q+1$. $f^i_{q+1}(I_{q+1})$ is the subset of $\{1, \ldots, k\}$ obtained by dropping the i-th smallest element in I_{q+1}. The map Ψ_q is defined by sending $X = \sum_{I_q \in S_q(k)} X^{I_q} \in \mathcal{A}_q$, $X^{I_q} \in \mathcal{H}^{\mathfrak{b}_{I_q}}_{\mathfrak{j}_{I_q}}$, to $Y = \sum_{I_{q+1} \in S_{q+1}(k)} Y^{I_{q+1}} \in \mathcal{A}_{q+1}$, $Y^{I_{q+1}} \in \mathcal{H}^{\mathfrak{b}_{I_{q+1}}}_{\mathfrak{j}_{I_{q+1}}}$, given by

$$\left(Y^{I_{q+1}}\right)_{\mathfrak{n}} = \begin{cases} \sum_{i=1}^{q+1} (-1)^{i-1} \left(X^{f^i_{q+1}(I_{q+1})}\right)_{\mathfrak{n}}, & \mathfrak{n} \in \mathbf{B}^{\mathfrak{b}_{I_{q+1}}}_{\mathfrak{j}_{I_{q+1}}}, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 19. Similar to the explanation in Section 2.2, each Hilbert module $\mathcal{A}_q$, $q = 1, \ldots, k$, can be identified with the Bergman space of the L^2-holomorphic sections of a Hermitian vector bundle on a disjoint union of subsets of $\mathbb{B}_m$. Under this identification, the module morphisms Ψ_q, $q = 0, \ldots, k-1$, can be realized as restriction maps of jets of holomorphic sections to the subsets. Although this geometric picture is not used heavily in what follows but we believe that such a geometric picture will play a crucial role in the future study about more general ideals. ■

2.4 The proof of Theorem 15

In this section we step-by-step check that the construction of Section 2.3 is a resolution asserted in Theorem 15.

Proposition 20. *Each Ψ_q is bounded.*

Proof. We write $X \in \mathcal{A}_q$ as a sum

$$X = \sum_{I_q \in S_q(k)} X^{I_q}, \qquad X^{I_q} \in \mathcal{H}^{\mathfrak{b}_{I_q}}_{\mathfrak{j}_{I_q}}.$$

By definition, $\Psi_q(X) = \sum_{I'_{q+1}} Y^{I'_{q+1}}$, where $Y^{I'_{q+1}} \in \mathcal{H}_{\mathfrak{j}_{I'_{q+1}}}^{\mathfrak{b}_{I'_{q+1}}}$ is given by

$$Y_{\mathfrak{n}}^{I'_{q+1}} = \begin{cases} \sum_{i=1}^{q+1} (-1)^{i-1} X_{\mathfrak{n}}^{f_{q+1}^i(I'_{q+1})}, & \mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_{I'_{q+1}}}^{\mathfrak{b}_{I'_{q+1}}}, \\ 0, & \text{otherwise.} \end{cases}$$

The norm of $\Psi_q(X)$ is computed as

$$\begin{aligned} \|\Psi_q(X)\|^2 = \sum_{I'_{q+1}} \left\| Y^{I'_{q+1}} \right\|^2 &= \sum_{I'_{q+1}} \sum_{\mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_{I'_{q+1}}}^{\mathfrak{b}_{I'_{q+1}}}} \left| Y_{\mathfrak{n}}^{I'_{q+1}} \right|^2 \\ &= \sum_{I'_{q+1}} \sum_{\mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_{I'_{q+1}}}^{\mathfrak{b}_{I'_{q+1}}}} \left| \sum_{i=1}^{q+1} (-1)^{i-1} X_{\mathfrak{n}}^{f_{q+1}^i(I'_{q+1})} \right|^2 \\ &\le \sum_{I'_{q+1}} \sum_{\mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_{I'_{q+1}}}^{\mathfrak{b}_{I'_{q+1}}}} (q+1) \left| X_{\mathfrak{n}}^{f_{q+1}^i(I'_{q+1})} \right|^2, \quad \text{by the Cauchy-Schwartz inequality} \\ &\le \sum_{I'_{q+1}} \sum_{\mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_{I_q}}^{\mathfrak{b}_{I_q}}} (q+1) \left| X_{\mathfrak{n}}^{I_q} \right|^2, \quad \text{as } \mathbf{B}_{\mathfrak{j}_{I'_{q+1}}}^{\mathfrak{b}_{I'_{q+1}}} \subseteq \mathbf{B}_{\mathfrak{j}_{I_q}}^{\mathfrak{b}_{I_q}} \\ &\le (k-q)(q+1) \sum_{I \in S_q(k)} \sum_{\mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_{I_q}}^{\mathfrak{b}_{I_q}}} \left| X_{\mathfrak{n}}^{I_q} \right|^2, \\ &\quad \text{as every } I_q \text{ is contained in at most } (k-q) \text{ number of } I'_{q+1} \\ &= (k-q)(q+1)\|X\|^2. \end{aligned}$$

■

Proposition 21. *Each Ψ_q is a module homomorphism.*

Proof. For each $I \in S_q(k)$ and $X^I \in \mathcal{H}_{\mathfrak{j}_I}^{\mathfrak{b}_I}$, $\Psi_q(X^I)$ has the form

$$\sum_{1 \le s \le k,\ s \notin I} (-1)^{\text{sign}(I,s)} Y^{I \cup \{s\}},$$

where $Y^{I\cup\{s\}} \in \mathcal{H}_{\mathfrak{j}_{I\cup\{s\}}}^{\mathfrak{b}_{I\cup\{s\}}}$, s is the α-th smallest number in $I \cup \{s\}$, sign(I, s)=$\alpha - 1$, and the function $Y^{I\cup\{s\}}$ is given by

$$Y_{\mathfrak{n}}^{I\cup\{s\}} = \begin{cases} X_{\mathfrak{n}}^{I}, & \mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_{I\cup\{s\}}}^{\mathfrak{b}_{I\cup\{s\}}}, \\ 0, & \text{otherwise.} \end{cases}$$

Fix $p \in \{1, \ldots, m\}$. The z_p action on $\mathcal{H}_{\mathfrak{j}_I}^{\mathfrak{b}_I}$ is implemented by

$$T_{z_p}^{\mathfrak{j}_I,\mathfrak{b}_I}\left(X^I\right)_{n_1\cdots n_p+1\cdots n_m} = \begin{cases} \sqrt{\frac{\omega_0(n_1\cdots n_p+1\cdots n_m)}{\omega_0(n_1\cdots n_m)}}X^I_{n_1\cdots n_p\cdots n_m}, & p \notin \mathfrak{j}_I, \\ \sqrt{\frac{\omega_0(n_1\cdots n_p+1\cdots n_m)}{\omega_0(n_1\cdots n_m)}}X^I_{n_1\cdots n_p\cdots n_m}, & p = j^s \in \mathfrak{j}_I,\ n_p + 1 \le b^s, \\ 0, & \text{otherwise.} \end{cases}$$

From this one observes that the operator $T_{z_p}^{\mathfrak{j}_I,\mathfrak{b}_I}$ preserves the component $\mathcal{H}_{\mathfrak{j}_I}^{\mathfrak{b}_I}$. Similarly, the z_p action on $\mathcal{H}_{\mathfrak{j}_{I\cup\{s\}}}^{\mathfrak{b}_{I\cup\{s\}}}$ is realized by

$$\begin{aligned} &T_{z_p}^{\mathfrak{j}_{I\cup\{s\}},\mathfrak{b}_{I\cup\{s\}}}\left(Y^{I\cup\{s\}}\right)_{n_1\cdots n_p+1\cdots n_m} \\ &= \begin{cases} \sqrt{\frac{\omega_0(n_1\cdots n_p+1\cdots n_m)}{\omega_0(n_1\cdots n_m)}}Y^{I\cup\{s\}}_{n_1\cdots n_p\cdots n_m}, & p \notin \mathfrak{j}_I,\quad p \neq s, \\ \sqrt{\frac{\omega_0(n_1\cdots n_p+1\cdots n_m)}{\omega_0(n_1\cdots n_m)}}Y^{I\cup\{s\}}_{n_1\cdots n_p\cdots n_m}, & p = j^t \in \mathfrak{j}_{I\cup\{s\}},\ n_p + 1 \le b^t, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Using the definition of $\Psi_q(X^I)$, one can directly check that on each component $\mathcal{H}_{\mathfrak{j}_{I\cup\{s\}}}^{\mathfrak{b}_{I\cup\{s\}}}$, we have

$$\left(\Psi_q\left(T_{z_p}^{\mathfrak{j}_I,\mathfrak{b}_I}\left(X^I\right)\right)\right)^{I\cup\{s\}} = T_{z_p}^{\mathfrak{j}_{I\cup\{s\}},\mathfrak{b}_{I\cup\{s\}}}\left(\Psi_q\left(X^I\right)^{I\cup\{s\}}\right),$$

which shows that Ψ_q is compatible with the A-module structure. ■

Lemma 22. $\overline{I} = \ker(\Psi_0)$.

Proof. If $f \in I$, then f has no nonzero component in any of the boxes $\mathbf{B}_{\mathfrak{j}_{\mathfrak{s}}}^{\mathfrak{b}_{\mathfrak{s}}}$, $\mathfrak{s} \in S(\alpha_1, \ldots, \alpha_l)$, hence $f \in \ker(\Psi_0)$. This shows that $\overline{I} \subseteq \ker(\Psi_0)$. Conversely, suppose $f \in \ker(\Psi_0)$. Consider the Taylor expansion $f = \sum_{\mathfrak{n}\in\mathbb{N}^m} f_{\mathfrak{n}} z^{\mathfrak{n}}$. As $\Psi_0(f) = 0$, by the definition of Ψ_0, for any $i = 1, \ldots, k$, and any $\mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_i}^{\mathfrak{b}_i}$, $f_{\mathfrak{n}} = 0$. For any positive integer M, let f_M be the truncation of the Taylor expansion of f by requiring $n^1, \ldots, n^m < M$, namely

$$f_M := \sum_{\substack{\mathfrak{n} \in \mathbb{N}^m \\ n^1 < M, \ldots, n^m < M}} f_{\mathfrak{n}} z^{\mathfrak{n}}.$$

It is not hard to see that f_M is a polynomial, and has no component in the boxes $\mathbf{B}_{\mathfrak{j}_1}^{\mathfrak{b}_1}, \ldots, \mathbf{B}_{\mathfrak{j}_k}^{\mathfrak{b}_k}$. By the construction of the boxes $\mathbf{B}_{\mathfrak{j}_1}^{\mathfrak{b}_1}, \ldots, \mathbf{B}_{\mathfrak{j}_k}^{\mathfrak{b}_k}$, f_M belongs to the ideal I. As $M \to \infty$, f_M converges to f in $L_a^2(\mathbb{B}_m)$. Therefore $f \in \overline{I}$. ∎

Proposition 23. $\mathrm{Im}(\Psi_{q-1}) \subseteq \ker(\Psi_q)$, $q = 1, \ldots, k$.

Proof. For each $I \in S_{q-1}(k)$ and $X^I \in \mathcal{H}_{\mathfrak{j}_I}^{\mathfrak{b}_I}$, the image of X^I under Ψ_{q-1} is of the form

$$\sum_{1 \leq s \leq k,\ s \notin I} (-1)^{\mathrm{sign}(I,s)} Y^{I \cup \{s\}},$$

where $Y^{I\cup\{s\}} \in \mathcal{H}_{\mathfrak{j}_{I\cup\{s\}}}^{\mathfrak{b}_{I\cup\{s\}}}$, s is the α-th smallest number in $I \cup \{s\}$, $\mathrm{sign}(I, s)$=$\alpha - 1$, and the function $Y^{I\cup\{s\}}$ has the form

$$Y_{\mathfrak{n}}^{I\cup\{s\}} = \begin{cases} X_{\mathfrak{n}}^I, & \mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_{I\cup\{s\}}}^{\mathfrak{b}_{I\cup\{s\}}}, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the image of $Y^{I\cup\{s\}}$ under Ψ_q equals

$$\sum_{1 \leq t \leq k,\ t \notin I\cup\{s\}} (-1)^{\mathrm{sign}(I\cup\{s\},t)} Z^{I\cup\{s,t\}},$$

where $Z^{I\cup\{s,t\}} \in \mathcal{H}_{\mathfrak{j}_{I\cup\{s,t\}}}^{\mathfrak{b}_{I\cup\{s,t\}}}$, t is the β-th smallest number in $I\cup\{s,t\}$, $\mathrm{sign}(I\cup\{s\}, t)$=$\beta-1$, and the function $Z^{I\cup\{s,t\}}$ is given by

$$Z_{\mathfrak{n}}^{I\cup\{s,t\}} = \begin{cases} Y_{\mathfrak{n}}^{I\cup\{s\}}, & \mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_{I\cup\{s,t\}}}^{\mathfrak{b}_{I\cup\{s,t\}}}, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned}\Psi_q\left(\Psi_{q-1}\left(X^I\right)\right) &= \sum_{1\le s\le k,\ s\notin I} (-1)^{\mathrm{sign}(I,s)}\Psi_q\left(Y^{I\cup\{s\}}\right)\\ &= \sum_{1\le s\le k,\ s\notin I} (-1)^{\mathrm{sign}(I,s)}\left(\sum_{1\le t\le k,\ t\notin I\cup\{s\}} (-1)^{\mathrm{sign}(I\cup\{s\},t)} Z^{I\cup\{s,t\}}\right)\\ &= \sum_{1\le s\ne t\le k,\ s,t\notin I} (-1)^{\mathrm{sign}(I,s)+\mathrm{sign}(I\cup\{s\},t)} Z^{I\cup\{s,t\}}\\ &= \sum_{1\le s<t\le k,\ s,t\notin I} \left((-1)^{\mathrm{sign}(I,s)+\mathrm{sign}(I\cup\{s\},t)} + (-1)^{\mathrm{sign}(I,t)+\mathrm{sign}(I\cup\{t\},s)}\right) Z^{I\cup\{s,t\}}.\end{aligned}$$

When $s<t$, it is not hard to check that:

$$\mathrm{sign}(I,s) = \mathrm{sign}(I\cup\{t\},s), \qquad \mathrm{sign}(I\cup\{s\},t) = \mathrm{sign}(I,t)+1.$$

Therefore, $\Psi_q(\Psi_{q-1}(X^I)) = 0$. ■

Proposition 24. $\mathrm{Im}(\Psi_0) \supseteq \ker(\Psi_1)$.

Proof. Consider $X := (X^1,\dots,X^p) \in \ker(\Psi_1)$. Define the function $\xi \in \mathcal{A}_0$ by

$$\xi_{\mathfrak{n}} := \begin{cases} X^s_{\mathfrak{n}}, & \text{there is } s \text{ such that } \mathfrak{n} \in \mathbf{B}^{\mathfrak{b}_s}_{\mathfrak{j}_s},\\ 0, & \text{otherwise.}\end{cases}$$

Note that ξ is well-defined because if there are two s and t such that $\mathfrak{n}$ belongs to both $\mathbf{B}^{\mathfrak{b}_s}_{\mathfrak{j}_s}$ and $\mathbf{B}^{\mathfrak{b}_t}_{\mathfrak{j}_t}$, then the $\mathcal{H}^{\mathfrak{b}_{st}}_{\mathfrak{j}_{st}}$ component of $\Psi_1(\xi)$ equals $X^s_{\mathfrak{n}} - X^t_{\mathfrak{n}} = 0$ by the assumption $X \in \ker(\Psi_1)$. Also note that $\|\xi\|^2 = \|X^1\|^2 + \cdots + \|X^p\|^2$, hence $\xi \in \mathcal{A}_0$. Clearly, $\Psi_0(\xi) = X$. ■

Proposition 25. $\mathrm{Im}(\Psi_{q-1}) \supseteq \ker(\Psi_q)$, $q = 1,\dots,k$.

Proof. We prove the proposition by induction on k. For $k=1$, we consider the map $\Psi_0 : \mathcal{A}_0 \to \mathcal{A}_1$. Computing with the orthonormal basis, it is not hard to observe that

$\mathcal{A}_1$ can be identified with a closed subspace of $\mathcal{A}_0 = L_a^2(\mathbb{B}_m)$, and the map Ψ_0 is the corresponding orthogonal projection map. Therefore Ψ_0 is surjective. Suppose that

$$\mathrm{Im}(\Psi_{q-1}) \supseteq \ker(\Psi_q), \quad q = 1, \ldots, k, \quad 1 \le k < p.$$

We prove the statement for $k = p$. The case $q = 1$ is proved in Proposition 24, so we are left with the cases $2 \le q \le p$. We consider the following two collections of $p-1$ boxes:

1. the first $p-1$ boxes
$$\left\{ \mathbf{B}_{\mathfrak{j}_1}^{\mathfrak{b}_1}, \ldots, \mathbf{B}_{\mathfrak{j}_{p-1}}^{\mathfrak{b}_{p-1}} \right\}.$$
Applying the construction in Section 2.3 to these boxes, we get the Hilbert modules $\mathcal{A}_s^1$ together with the Hilbert module maps $\Psi_s^1 : \mathcal{A}_s^1 \to \mathcal{A}_{s+1}^1$, $s = 1, \ldots, p-2$. Set $\mathcal{A}_p^1 := \{0\}$ and $\Psi_{p-1}^1 = 0$;

2. the intersection of the first $p-1$ boxes with the last one $\mathbf{B}_{\mathfrak{j}_p}^{\mathfrak{b}}$
$$\left\{ \mathbf{B}_{\mathfrak{j}_{1p}}^{\mathfrak{b}_{1p}}, \ldots, \mathbf{B}_{\mathfrak{j}_{p-1p}}^{\mathfrak{b}_{p-1p}} \right\}.$$
Applying the construction in Section 2.3 to these boxes, we get the Hilbert modules $\mathcal{A}_s^2$ together with the Hilbert module maps $\Psi_s^2 : \mathcal{A}_s^2 \to \mathcal{A}_{s+1}^2$, $s = 1, \ldots, p-2$. Set $\mathcal{A}_p^2 := \{0\}$ and $\Psi_{p-1}^2 = 0$.

By the induction assumption we know that

$$\mathrm{Im}\left(\Psi_{q-1}^1\right) \supseteq \ker\left(\Psi_q^1\right), \quad \mathrm{Im}\left(\Psi_{q-1}^2\right) \supseteq \ker\left(\Psi_q^2\right), \quad q = 1, \ldots, p-1.$$

Define a map $\Phi_t : \mathcal{A}_t^1 \to \mathcal{A}_t^2$ by

$$\Phi_t(X^I) = Y^{I \cup \{p\}}, \quad I \in S_t(p-1),$$

where $Y^{I\cup\{p\}}$ denotes the component corresponding to the intersection of the boxes $\mathbf{B}_{\mathfrak{j}_{i_1 p}}^{\mathfrak{b}_{i_1 p}}, \ldots, \mathbf{B}_{\mathfrak{j}_{i_t p}}^{\mathfrak{b}_{i_t p}}$ with

$$Y_{\mathfrak{n}}^{I\cup\{p\}} := \begin{cases} (-1)^t X_{\mathfrak{n}}^I, & \mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_{I\cup\{p\}}}^{\mathfrak{b}_{I\cup\{p\}}}, \\ 0, & \text{otherwise.} \end{cases}$$

Similar to Proposition 21, Φ_t is an A-module map. We leave the detail to the reader.

With the construction above, we can easily check the following identities.

1. $\mathcal{A}_q = \mathcal{A}_q^1 \oplus \mathcal{A}_{q-1}^2$ for $q = 2, \ldots, p$;
2. $\Psi_q = \begin{pmatrix} \Psi_q^1 & 0 \\ \Phi_q & \Psi_{q-1}^2 \end{pmatrix}$ for $q = 2, \ldots, p-1$.

These identifications are used to prove $\mathrm{Im}(\Psi_{q-1}) \supseteq \ker(\Psi_q)$. We split the proof into three cases.

1. $q = 2$.

   Suppose $(X_1, X_2) \in \mathcal{A}_2^1 \oplus \mathcal{A}_1^2 = \mathcal{A}_2$ is in the kernel of Ψ_2. By the identification above for Ψ_q, we have

   $$\Psi_2^1(X_1) = 0, \qquad \Phi_2(X_1) + \Psi_1^2(X_2) = 0.$$

   By the induction assumption, $\ker(\Psi_2^1) \subseteq \mathrm{Im}(\Psi_1^1)$. So there exists $Y_1 \in \mathcal{A}_1^1$ such that $\Psi_1^1(Y_1) = X_1$. By Proposition 23 for the morphism $\Psi_\bullet$, we have

   $$\begin{aligned}(0,0) &= \Psi_2\big(\Psi_1(Y_1, 0)\big) = \Psi_2\big(\Psi_1^1(Y_1), \Phi_1(Y_1)\big) \\ &= \Big(\Psi_2^1\big(\Psi_1^1(Y_1)\big), \Phi_2\big(\Psi_1^1(Y_1)\big) + \Psi_1^2\big(\Phi_1(Y_1)\big)\Big), \quad \Psi_1^1(Y_1) = X_1,\ \Psi_2^1\big(\Psi_1^1(Y_1)\big) = 0 \\ &= \Big(0, \Phi_2(X_1) + \Psi_1^2\big(\Phi_1(Y_1)\big)\Big).\end{aligned}$$

   Therefore, $\Phi_2(X_1) + \Psi_1^2\big(\Phi_1(Y_1)\big) = 0$. Consider $X_2' := X_2 - \Phi_1(Y_1)$. We have

   $$\Psi_1^2(X_2') = \Psi_1^2(X_2 - \Phi_1(Y_1)) = \Psi_1^2(X_2) - \Psi_1^2(\Phi_1(Y_1)) = \Psi_1^2(X_2) + \Phi_2(X_1) = 0,$$

since

$$\Psi_2(X_1, X_2) = (\Psi_2^1(X_1), \Phi_2(X_1) + \Psi_1^2(X_2)) = 0.$$

Using the property that $\Psi_1^2(X_2') = 0$, we construct an element $Y_2 \in \mathcal{H}_{\mathfrak{j}_p}^{\mathfrak{b}_p}$ by setting

$$(Y_2)_{\mathfrak{n}} := \begin{cases} (X_2'^{ip})_{\mathfrak{n}}, & \mathfrak{n} \in \mathbf{B}_{\mathfrak{j}_{ip}}^{\mathfrak{b}_{ip}} \text{ for some } \ i = 1, ..., p-1, \\ 0, & \text{otherwise.} \end{cases}$$

As $\Psi_1^2(X_2') = 0$, the definition above of Y_2 is independent of the choices of i. It is not hard to check the norm of Y_2 is bounded. (Arguments are similar to the proof of Proposition 24.) Therefore, $Y_2 \in \mathcal{H}_{\mathfrak{j}_p}^{\mathfrak{b}_p} \subseteq L_a^2(\mathbb{B}_m)$ and $\Psi_0^2(Y_2) = X_2'$.

In summary, we have constructed an element $(Y_1, Y_2) \in \mathcal{A}_1 = \mathcal{A}_1^1 \oplus \mathcal{H}_{\mathfrak{j}_p}^{\mathfrak{b}_p}$ which satisfies

$$\Psi_1(Y_1, Y_2) = \big(\Psi_1^1(Y_1), \Phi_1(Y_1) + \Psi_0^2(Y_2)\big) = (X_1, \Phi_1(Y_1) + X_2') = (X_1, X_2).$$

Therefore $(X_1, X_2) \in \mathrm{Im}(\Psi_1)$.

2. $q = 2, \dots, p-1$.

   Suppose $(X_1, X_2) \in \mathcal{A}_q^1 \oplus \mathcal{A}_{q-1}^2 = \mathcal{A}_q$ is in the kernel of Ψ_q. By the identification above for Ψ_q, we have

   $$\Psi_q^1(X_1) = 0, \qquad \Phi_q(X_1) + \Psi_{q-1}^2(X_2) = 0.$$

   Since $\mathrm{Im}(\Psi_{q-1}^1) \supseteq \ker(\Psi_q^1)$ there is $Y_1 \in \mathcal{A}_{q-1}^1$ such that $X_1 = \Psi_{q-1}^1(Y_1)$. Since $\Psi_q(\Psi_{q-1}(Y_1, 0)) = 0$ we have $\Phi_q(X_1) + \Psi_{q-1}^2(\Phi_{q-1}(Y_1)) = 0$. Therefore

   $$\Psi_{q-1}^2\left(X_2 - \Phi_{q-1}(Y_1)\right) = 0.$$

   Since $\mathrm{Im}\left(\Psi_{q-2}^2\right) \supseteq \ker\left(\Psi_{q-1}^2\right)$ there exists $Y_2 \in \mathcal{A}_{q-2}^2$ such that

   $$\Psi_{q-2}^2(Y_2) = X_2 - \Phi_{q-1}(Y_1).$$

Therefore we have found $(Y_1, Y_2) \in \mathcal{A}_q$ satisfying

$$\Psi_{q-1}(Y_1, Y_2) = \big(\Psi^1_{q-1}(Y_1), \Phi_{q-1}(Y_1) + \Psi_{q-2}(Y_2)\big) = (X_1, X_2).$$

3. $q = p$.

   Notice that $\mathcal{A}_p$ is the same as $\mathcal{A}^2_{p-1}$. Since $\Psi^2_{p-2} : \mathcal{A}^2_{p-2} \to \mathcal{A}^2_{p-1}$ is surjective, it follows that

$$\Psi_{p-1} : \mathcal{A}_{p-1} = \mathcal{A}^1_{p-1} \oplus \mathcal{A}^2_{p-2} \to \mathcal{A}_p = \mathcal{A}^2_{p-1}$$

   is also surjective.

All cases are exhausted. ■

2.5 The proof of Theorem 16

To deduce the index formula in Theorem 16 from the resolution in Theorem 15, we need the following proposition and its corollary.

Proposition 26. *Consider the following exact sequence of essentially normal Hilbert A-modules and Hilbert module maps between them:*

$$0 \to \mathcal{M}_1 \overset{W_1}{\to} \mathcal{M}_2 \overset{W_2}{\to} \mathcal{M}_3 \to 0.$$

Suppose that the essential spectra of $\mathcal{M}_i$, $i = 1, 2, 3$, is contained in the closed unit ball $\overline{\mathbb{B}}_m$, and let

$$\alpha_i : C(\overline{\mathbb{B}}_m) \to \mathfrak{Q}(\mathcal{M}_i)$$

be the $*$-representation of $C(\overline{\mathbb{B}}_m)$ on the Calkin algebra induced by the essential normality of $\mathcal{M}_i$. There are coisometries $U : \mathcal{M}_2 \to \mathcal{M}_1$ and $V : \mathcal{M}_2 \to \mathcal{M}_3$ such that

$$UV^* = 0 = VU^*, \quad U^*U + V^*V = 1,$$

and they commute with A-module structures up to compact operators in the sense that

$$[U]\alpha_2[U]^* = \alpha_1, \qquad [V]\alpha_2[V]^* = \alpha_3,$$

where $\alpha_i(p) = [T_p^i] \in \mathfrak{Q}(\mathcal{M}_i)$, $p \in A$, is the equivalence class of the multiplication operator $T_p^i \in \mathfrak{B}(\mathcal{M}_i)$.

Proof. As W_2 is surjective, $W_2W_2^*$ is positive definite. Consider the polar decomposition $W_2 = A_3V$ with positive definite $A_3 = \sqrt{W_2W_2^*}$ and coisometry V. Since W_2 is a module homomorphism, for each $p \in A$ we have

$$A_3VT_p^2 = W_2T_p^2 = T_p^3W_2 = T_p^3A_3V,$$

where T_p^2 and T_p^3 are the multiplication operators on $\mathcal{M}_2$ and $\mathcal{M}_3$ associated to p. Since $\mathcal{M}_2$ and $\mathcal{M}_3$ are essentially normal, T_p^2 and T_p^3 are normal in the respective Calkin algebras. By the Fuglede-Putnam theorem

$$A_3V(T_p^2)^* = W_2\left(T_p^2\right)^* = \left(T_p^3\right)^* W_2 = \left(T_p^3\right)^* A_3V,$$

all equations modulo compact operators. Taking adjoints:

$$T_p^2V^*A_3 = V^*A_3T_p^3.$$

Multiplying on the left by A_3V:

$$A_3VT_p^2V^*A_3 = A_3VV^*A_3T_p^3 = A_3^2T_p^3.$$

Since $A_3VT_p^2 = T_p^3A_3V$, we conclude from the equation above that

$$A_3VT_p^2V^*A_3 = T_p^3A_3VV^*A_3 = T_p^3A_3^2 = A_3^2T_p^3.$$

Since A_3 is positive definite it is safe to conclude

$$T_p^3 A_3 = A_3 T_p^3.$$

This commutativity plus the equation $A_3 V T_p^2 = T_p^3 A_3 V$ gives

$$V T_p^2 = T_p^3 V.$$

Since $VV^* = 1$, we have

$$V T_p^2 V^* = T_p^3,$$

which is exactly

$$V \alpha_2 V^* = \alpha_3.$$

The derivation of $U\alpha_2 U^* = \alpha_1$ is similar. Here are the details. Since W_1 is injective, $W_1^* W_1$ is positive definite. Consider the polar decomposition $W_1 = W A_1$ with $A_1 = \sqrt{W_1^* W_1}$ and $W : \mathcal{M}_1 \to \mathcal{M}_2$ is an isometry. A similar argument as above for W_2 shows that modulo compact operators, for any $p \in A$,

$$A_1 T_p^1 = T_p^1 A_1,$$

and

$$W^* T_p^2 W = T_p^1.$$

Setting $U := W^*$ we have $U T_p^2 U^* = T_p^1$ and $UU^* = 1$, which shows

$$U \alpha_2 U^* = \alpha_1.$$

Since $W_2 W_1 = A_3 V U^* A_1 = 0$, by the invertibility of A_1 and A_3, we get $VU^* = 0$. Therefore, $U^* U$ and $V^* V$ are commuting orthogonal projections on $\mathcal{M}_2$. To prove that their sum is the identity operator, it is sufficient to prove that the kernel of their sum is trivial. Suppose $\xi \in \mathcal{M}_2$ such that $U^*U\xi + V^*V\xi = 0$. Then $U^*U\xi = V^*V\xi = 0$, hence $U\xi = V\xi = 0$. Then $W_2\xi = A_3V\xi = 0$, and $W_1^*\xi = A_1U\xi = 0$. Therefore $\xi \in \ker(W_2)$, and by exactness, there is $\eta \in W_1$ such that $W_1\eta = \xi$. As $W_1^*\xi = 0$, $W_1^*W_1\eta = 0$, hence $\xi = W_1\eta = 0$. ■

Still assume the notations in Proposition 26. Let σ_e^i be the essential spectrum of the Hilbert module $\mathcal{M}_i$. The morphisms α_1 and α_2 factor into $*$-monomorphisms $C(\sigma_e^1) \to \mathfrak{Q}(\mathcal{M}_1)$ and $C(\sigma_e^2) \to \mathfrak{Q}(\mathcal{M}_2)$, respectively. By Proposition 26, $\alpha_1 = [U]\alpha_2[U]^*$. The composition of $[U]\alpha_2[U]^*$ with α_1^{-1} is a $*$-homomorphism $C(\sigma_e^2) \to C(\sigma_e^1)$, hence we get a natural map $\sigma_e^1 \to \sigma_e^2$. Similar arguments give a natural map $\sigma_e^3 \to \sigma_e^2$. By the functoriality of K_1, α_1 and α_3 induce classes $[\alpha_1]$ and $[\alpha_3]$ in $K_1(\sigma_e^2)$. Putting all equations

$$UU^* = 1 = VV^*, \quad UV^* = 0 = VU^*, \quad U^*U + V^*V = 1,$$

$$[U]\alpha_2[U]^* = \alpha_1, \quad [V]\alpha_2[V]^* = \alpha_3,$$

together we get:

Corollary 27. *Assume the notations in Proposition 26. In $K_1(\sigma_e^2)$ we have the formula*

$$[\alpha_2] = [\alpha_1] + [\alpha_3],$$

where $[\alpha_1]$ and $[\alpha_3]$ are identified as classes in $K_1(\sigma_e^2)$ by the coisometries U and V.

The proof of Theorem 16

The rest of this section is devoted to the proof of Theorem 2.3. To do this we are going to decompose the long exact sequence in Theorem 2.4 into short exact sequences and apply Corollary 27. The details follow.

Consider the resolution of $\overline{I}$ is Theorem 15. For each $q = 1, \ldots, k$, we introduce the following closed subspace of $\mathcal{A}_q$:

$$\mathcal{A}_q^- := \mathrm{Im}(\Psi_{q-1}) = \ker(\Psi_q).$$

As Ψ_{k-1} is surjective $\mathcal{A}_k^- = \mathcal{A}_k$. Since $\Psi_q : \mathcal{A}_q \to \mathcal{A}_{q+1}$ is a morphism of A-modules, the kernel $\mathcal{A}_q^- = \ker(\Psi_q)$ is naturally an A-module. Furthermore, we have the following exact sequence of Hilbert A-modules:

$$0 \to \mathcal{A}_q^- \to \mathcal{A}_q \to \mathcal{A}_{q+1}^- \to 0, \quad q = 1, \dots, k-1, \tag{2.6}$$

where the first map is the inclusion, and the second map is Ψ_q.

Lemma 28. *Each Hilbert A-module $\mathcal{A}_q^-$ is essentially normal.*

Proof. When $q = k-1$, as Ψ_{k-1} is surjective, we have the short exact sequence

$$0 \to \mathcal{A}_{k-1}^- \to \mathcal{A}_{k-1} \to \mathcal{A}_k \to 0.$$

Since both $\mathcal{A}_{k-1}$ and $\mathcal{A}_k$ are essentially normal A-modules (Theorem 15), by Proposition 8, $\mathcal{A}_{k-1}^-$ is essentially normal. Repeating this argument for the exact sequence

$$0 \to \mathcal{A}_{k-2}^- \to \mathcal{A}_{k-1} \to \mathcal{A}_{k-1}^- \to 0,$$

we conclude that $\mathcal{A}_{k-2}^-$ is also essentially normal. We are done by induction. ∎

Let σ_e^q, $q = 1, \dots, k$, be the essential spectrum of $\mathcal{A}_q$, and let α_q (resp. α_q^-) be the associated $*$-monomorphism $C(\sigma_e^q) \to \mathfrak{Q}(\mathcal{A}_q)$ (resp. $C(\sigma_e^{q-}) \to \mathfrak{Q}(\mathcal{A}_q^-)$) induced by essential normality. Applying Corollary 27 to the short exact sequence (2.6) gives

$$[\alpha_q] = [\alpha_q^-] + [\alpha_{q+1}^-] \quad \text{in} \quad K_1(\sigma_e^q).$$

When $q = k-1$, $\mathcal{A}_k^- = \mathcal{A}_k$, and we have

$$[\alpha_{k-1}] = [\alpha_{k-1}^-] + [\alpha_k] \quad \text{in} \quad K_1\left(\sigma_e^{k-1}\right).$$

Similarly, for $q = k-2$, we get

$$[\alpha_{k-2}] = [\alpha_{k-2}^-] + [\alpha_{k-1}^-] \quad \text{in} \quad K_1\left(\sigma_e^{k-2}\right).$$

Combining the previous two equations we conclude that

$$[\alpha_{k-1}] + [\alpha_{k-2}^-] = [\alpha_k] + [\alpha_{k-2}] \quad \text{in} \quad K_1\left(\sigma_e^{k-1} \cup \sigma_e^{k-2}\right),$$

by pushing forward the respective equations in $K_1\left(\sigma_e^{k-1}\right)$ and $K_1\left(\sigma_e^{k-2}\right)$ into the ones in $K_1\left(\sigma_e^{k-1} \cup \sigma_e^{k-2}\right)$ via the natural inclusion maps $\sigma_e^{k-1}, \sigma_e^{k-2} \hookrightarrow \sigma_e^{k-1} \cup \sigma_e^{k-2}$. By induction we get

$$[\alpha_1^-] = [\alpha_1] - [\alpha_2] + \ldots + (-1)^{k-1}[\alpha_k] \quad \text{in} \quad K_1(\sigma_e^1 \cup \cdots \cup \sigma_e^k). \tag{2.7}$$

On the other hand, the the exact sequence

$$0 \to \overline{I} \to L_a^2(\mathbb{B}_m) \to \mathcal{A}_1^- \to 0,$$

establishes a natural Hilbert A-module isomorphism between the quotient

$$\mathcal{Q}_I := \frac{L_a^2(\mathbb{B}_m)}{\overline{I}} \cong I^\perp$$

and $\mathcal{A}_1^-$, hence by Proposition 8 we get $\tau_I := [I^\perp] = [\alpha_1^-]$. Together with (2.7) this gives the index formula in Theorem 2.5.

2.6 Examples

This section gives examples for the resolution constructed in Section 2.3.

Example 29. Consider the ideal $I := \langle z_1^2 z_2^2 \rangle \subseteq \mathbb{C}[z_1, z_2]$. The exponents of monomials in $I = \langle z_1^2 z_2^2 \rangle$ comprise the region

$$\left\{(n^1, n^2) \in \mathbb{N}^2 : n^1, n^2 \geq 2\right\}.$$

Here, there is only one $\alpha = (2, 2)$. We have two boxes:

$$\mathbf{B}_{\mathrm{j}_1}^{\mathfrak{b}_1} := \left\{(n^1, n^2) : n^1 \leq 1\right\}, \quad \mathbf{B}_{\mathrm{j}_2}^{\mathfrak{b}_2} := \left\{(n^1, n^2) : n^2 \leq 1\right\}.$$

The intersection $\mathbf{B}_{\mathrm{j}_{12}}^{\mathfrak{b}_{12}} := \mathbf{B}_{\mathrm{j}_1}^{\mathfrak{b}_1} \cap \mathbf{B}_{\mathrm{j}_2}^{\mathfrak{b}_2}$ equals $\{(n^1, n^2) : n^1, n^2 \leq 1\}$.

The Hilbert module $\mathcal{A}_1$ is the direct sum of two modules $\mathcal{A}_1^1$ and $\mathcal{A}_1^2$, where $\mathcal{A}_1^1 \subseteq L_a^2(\mathbb{B}_2)$ is the submodule spanned by $\{z_2^n, z_1 z_2^n : n \in \mathbb{N}\}$, and $\mathcal{A}_1^2 \subseteq L_a^2(\mathbb{B}_2)$ is the submodule spanned by $\{z_1^n, z_1^n z_2 : n \in \mathbb{N}\}$. The Hilbert module $\mathcal{A}_2$ is the subspace of $L_a^2(\mathbb{B}_2)$ spanned by $\{1, z_1, z_2, z_1 z_2\}$. It is easy to see that $\mathcal{A}_2 = \mathcal{A}_1^1 \cap \mathcal{A}_1^2$. ■

Example 30. Consider the ideal $I := \langle z_1^p z_2^q, z_1^r z_2^s \rangle \subseteq \mathbb{C}[z_1, z_2]$, $p, q, r, s \in \mathbb{N}$, $r < p$ and $q < s$. The complementary space $\mathsf{C}(I) \subseteq \mathbb{N}^2$ is the blue region in the Figure 2.1.

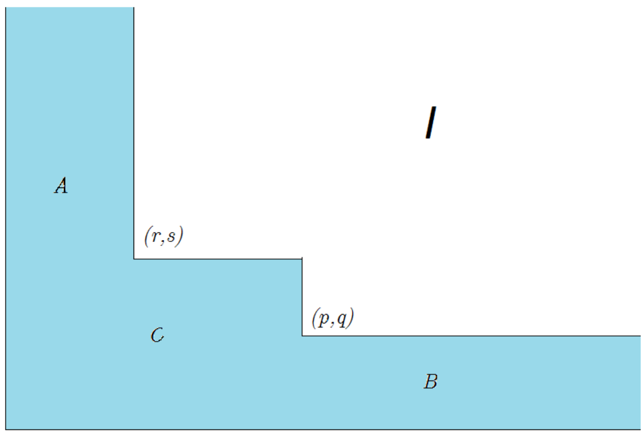

Figure 2.1: Staircase diagram corresponding to $I = \langle z_1^p z_2^q, z_1^r z_2^s \rangle$.

Here, $\alpha_1 = (p,q)$ and $\alpha_2 = (r,s)$, and $S(\alpha_1,\alpha_2)$ consists of four pairs $(1,1)$, $(1,2)$, $(2,1)$ and $(2,2)$. The boxes associated to these arrays are:

1. For $\mathfrak{s} = (1,1)$, the box $\mathbf{B}_{\mathrm{j}_{11}}^{\mathfrak{b}_{11}}$ is $\{(n^1,n^2) : n^1 < r\}$;

2. For $\mathfrak{s} = (1,2)$, the box $\mathbf{B}_{\mathrm{j}_{12}}^{\mathfrak{b}_{12}}$ is $\{(n^1,n^2) : n^1 < p, n^2 < s\}$;

3. For $\mathfrak{s} = (2,1)$, the box $\mathbf{B}_{\mathrm{j}_{21}}^{\mathfrak{b}_{21}}$ is $\{(n^1,n^2) : n^1 < r, n^2 < q\}$;

4. For $\mathfrak{s} = (2,2)$, the box $\mathbf{B}_{\mathrm{j}_{22}}^{\mathfrak{b}_{22}}$ is $\{(n^1,n^2) : n^2 < q\}$.

In Figure (2.1), the boxes $\mathbf{B}_{\mathrm{j}_{11}}^{\mathfrak{b}_{11}}$, $\mathbf{B}_{\mathrm{j}_{12}}^{\mathfrak{b}_{12}}$ and $\mathbf{B}_{\mathrm{j}_{22}}^{\mathfrak{b}_{22}}$ are respectively marked as region A, C, and B. Since $\mathbf{B}_{\mathrm{j}_{21}}^{\mathfrak{b}_{21}}$ is contained in $\mathbf{B}_{\mathrm{j}_{12}}^{\mathfrak{b}_{12}}$, we do not need to include $\mathbf{B}_{\mathrm{j}_{21}}^{\mathfrak{b}_{21}}$ in our construction. However, we still get a resolution of $\overline{I}$.

The Hilbert space $\mathcal{A}_1$ is the direct sum of three spaces $\mathcal{A}_1^{11}$, $\mathcal{A}_1^{12}$ and $\mathcal{A}_1^{22}$, where $\mathcal{A}_1^{11} \subseteq L_a^2(\mathbb{B}_2)$ is the subspace spanned by $\left\{z_2^n, z_1 z_2^n, \ldots, z_1^{r-1} z_2^n : n \in \mathbb{N}\right\}$, $\mathcal{A}_1^{12} \subseteq L_a^2(\mathbb{B}_2)$ is the finite dimensional subspace spanned by

$$\begin{array}{llll} 1, & z_1, & \ldots & z_1^{p-1}, \\ z_2, & z_1 z_2, & \ldots & z_1^{p-1} z_2, \\ \vdots & \vdots & \ddots & \vdots \\ z_2^{s-1}, & z_1 z_2^{s-1}, & \ldots & z_1^{p-1} z_2^{s-1}, \end{array}$$

and $\mathcal{A}_1^{22} \subseteq L_a^2(\mathbb{B}_2)$ is the subspace spanned by $\left\{z_1^n, z_1^n z_2, \ldots, z_1^n z_2^{q-1} : n \in \mathbb{N}\right\}$. The Hilbert space $\mathcal{A}_2$ is the direct sum of three spaces $\mathcal{A}_1^{11} \cap \mathcal{A}_1^{12}$, $\mathcal{A}_1^{11} \cap \mathcal{A}_1^{22}$ and $\mathcal{A}_1^{12} \cap \mathcal{A}_1^{22}$. The Hilbert space $\mathcal{A}_3 \subseteq L_a^2(\mathbb{B}_2)$ is the subspace spanned by

$$\begin{array}{llll} 1, & z_1, & \ldots & z_1^{r-1}, \\ z_2, & z_1 z_2, & \ldots & z_1^{r-1} z_2, \\ \vdots & \vdots & \ddots & \vdots \\ z_2^{q-1}, & z_1 z_2^{q-1}, & \ldots & z_1^{r-1} z_2^{q-1}. \end{array}$$

■

2.7 A nonmonomial ideal

This section discusses a nonmonomial ideal which can be reduced to monomials after a biholomorphic change of variables.

Consider the ideal $I := \langle z_1^2, z_3 - z_2^2 \rangle \subseteq \mathbb{C}[z_1, z_2, z_3]$. The biholomorphic mapping

$$T : \mathbb{C}^3 \to \mathbb{C}^3, \quad (z_1, z_2, z_3) \mapsto (\zeta_1, \zeta_2, \zeta_3) = \left(z_1, z_2, z_3 - z_2^2\right)$$

changes I to $I' := \langle \zeta_1^2, \zeta_3 \rangle \subseteq \mathbb{C}[\zeta_1, \zeta_2, \zeta_3]$. The unit ball $\mathbb{B}_3$ is mapped to the domain

$$\Omega = \left\{(\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3 : |\zeta_1|^2 + |\zeta_2|^2 + |\zeta_3 + \zeta_2^2|^2 < 1\right\}.$$

Also, the Hilbert spaces $L_a^2(\mathbb{B}_3)$ and $\overline{I}$ are mapped isomorphically to $L_a^2(\Omega)$ and the closure $\overline{I'} \subseteq L_a^2(\Omega)$, respectively. These identifications are also valid as Hilbert modules over the polynomial rings with three variables.

Since I' is monomial, we can apply the construction of Section 2.3 with $\mathbb{B}_3$ replaced by Ω. In the following we check that this gives a resolution of $\overline{I'}$. Here we have only one box

$$\mathbf{B}_{\mathfrak{j}}^{\mathfrak{b}} = \{(n_1, n_2, 0) \in \mathbb{N}^3 : n_1 \leq 1\}, \quad \mathfrak{j} = (1, 3), \quad \mathfrak{b} = (1, 0).$$

Consider the subdomain

$$\Omega_{\mathfrak{j}} = \{(\zeta_1, \zeta_2, \zeta_3) \in \Omega : \zeta_1 = \zeta_3 = 0\},$$

which can be identified with the planar domain

$$\{z_2 \in \mathbb{C} : \ |\zeta_2|^2 + |\zeta_2|^4 < 1\}.$$

Consider the weighted Bergman space $L_{a,s}^2(\Omega_{\mathfrak{j}})$, $s > -1$, of analytic functions on $\Omega_{\mathfrak{j}}$ which are square integrable with respect to the measure $(1 - |\zeta_2|^2 - |\zeta_2|^4)^s \, \mathrm{d}V_{\Omega_{\mathfrak{j}}}$, where $\mathrm{d}V_{\Omega_{\mathfrak{j}}}$ is the normalized Lebesgue measure on Ω_j. Our resolution is

$$0 \to \overline{I'} \to L^2_a(\Omega) \xrightarrow{\Psi'} \mathcal{A}' \to 0, \tag{2.8}$$

where

$$\mathcal{A}' = L^2_{a,2}(\Omega_j) \oplus L^2_{a,3}(\Omega_j),$$

and Ψ' is given by

$$\Psi'(f) := \left(f|_{\zeta_1=\zeta_3=0}, \frac{\partial f}{\partial \zeta_1}\bigg|_{\zeta_1=\zeta_3=0} \right).$$

The module structure on $\mathcal{A}'$ is given by

$$\zeta_1 \cdot (X, Y) = (0, X), \quad \zeta_2 \cdot (X, Y) = (\zeta_2 X, \zeta_2 Y), \quad \zeta_3 \cdot (X, Y) = (0, 0).$$

for $(X, Y) \in \mathcal{A}'$. The monomials $\{\zeta_2^i : i \in \mathbb{N}\}$ form an orthogonal basis for both $L^2_{a,2}(\Omega_j)$ and $L^2_{a,3}(\Omega_j)$, and a straightforward computation with them shows that $\mathcal{A}'$ is essentially normal. Arguments similar to the ones in Section 2.5 show that (2.8) is an exact sequence of Hilbert modules with bounded module maps.

Under the inverse mapping T^{-1}, the resolution (2.8) gives the following resolution for $\overline{I}$:

$$0 \to \overline{I} \to L^2_a(\mathbb{B}_3) \to \mathcal{A} \to 0,$$

where $\mathcal{A}$ is the analogue of $\mathcal{A}'$ with Ω_j replaced by $T^{-1}(\Omega_j) = \{(0, z_2, z_2^2) \in \mathbb{B}_3\}$. Finally, we can conclude that $\overline{I}$ and its associated quotient $\mathcal{Q}_I$ are both essentially normal, with the following index formula for the Toeplitz extension:

$$[\mathfrak{T}(\mathcal{Q}_I)] = \left[\mathfrak{T}\left(L^2_{a,2}(\Omega_j) \oplus L^2_{a,3}(\Omega_j)\right)\right].$$

2.8 Some potential future directions

Here are some directions for future works:

1. We aim to prove the analogue of Theorem 15 for the egg domains of the form

$$\left\{\sum a_j |z_j|^{p_j} < 1\right\}, \quad a_j, p_j > 0,$$

   instead of the unit ball $\{\sum |z_j|^2 < 1\}$. Explicit formulas for the orthonormal basis of the Bergman spaces on such domains [38] will be useful. It is also interesting to generalize this theorem to Reinhardt domains of the form

$$\{\psi\left(|z_1|, \dots, |z_m|\right) < 1\},$$

   where $\psi : [0, \infty)^m \to [0, \infty)$ is a smooth function, monotonically increasing in each argument. Now proving the essential normality of $\mathcal{A}_q$ needs ideas from harmonic analysis in the same spirit as [51].

2. Recall from Section 1.2 that Arveson originally formulated his essential normality conjecture for homogeneous submodules $\mathcal{M} \subseteq H^2_m \otimes \mathbb{C}^r$ (Conjecture 12) instead of the multiplicity-free version $I^\perp \subseteq H^2_m$ (Conjecture 1). It is interesting to find the analogue of the resolution (2.2) in this generality, and understand its geometry. Now $\mathcal{M}^\perp$ can be geometrized as a Hilbert space of the holomorphic sections of a vector bundle or more generally a sheaf over the algebraic variety $V(\mathrm{Ann}(\mathcal{M})) \subseteq \mathbb{C}^m$.

3. For a monomial ideal I, the intersection $V(I) \cap \partial\overline{\mathbb{B}}_m$ is singular in general. Several notions of fundamental class has been defined for singular algebraic varieties [16, 17, 90]. It is interesting to relate the right hand side of the equation (2.3) to these characteristic classes.

Chapter 3

A Gauss-Manin connection in noncommutative geometry and its holonomy

In this chapter we initiate a project of using the Toeplitz algebras $\mathfrak{T}_I$ of Section 1.1 to study hypersurface singularities. More specifically, to analytically study the monodromy of an isolated singularity at the origin on an algebraic hypersurface $V(f) \subseteq \mathbb{C}^m$, $f \in A = \mathbb{C}[z_1, \ldots, z_m]$, we consider the perturbed 1-parameter family of principal ideals $I(t) := \langle f - \epsilon e^{it} \rangle \subseteq A$, $t \in \mathbb{R}$, $\epsilon > 0$ small enough. The family $I(t)^\perp \subseteq H^2_m$ of associated Hilbert A-modules, as a subbundle of the trivial bundle $H^2_m \times \mathbb{R}$, comes equipped with a natural metric connection. The holonomy of this connection, a (conjecturally) unitary operator $U \in \mathfrak{B}(I(0)^\perp)$, is the main object of study in this chapter. Of special concern is the interaction of U with the Toeplitz algebra $\mathfrak{T}_{I(0)}$. We are currently at the stage of setting the foundations for this study mostly through formulating reasonable conjectures (Conjectures 31, 32 and 34). In Section 3.1 the motivation of our study is presented. Some singularity theory backgrounds in differential topology are gathered in Section 3.2. A proposal about the holonomy operator U is presented with conjectures in Section 3.3. In Section 3.4 we examine the proposal of Section 3.3 on the toy model $f := z_1^k \in \mathbb{C}[z_1, z_2]$,

$k \geq 2$, and verify all the conjectures in this special case. Some potential directions for future works are discussed in Section 3.5.

3.1 Motivation

It is famous that there are exactly 28 oriented smooth structures, up to orientation preserving diffeomorphisms, on the topological 7-dimensional sphere $\mathbb{S}^7$ [73]. Putting the standard one aside the rest are called exotic spheres. Of all the numerous constructions of exotic spheres in the literature we are interested in the following algebraic one discovered by Brieskorn [27, 68]. He showed that as j varies on $1, 2, \ldots, 28$, if $\epsilon > 0$ is chosen small enough such that the zero set $V(f_j)$ of the polynomial

$$f_j := z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6j-1} \in \mathbb{C}[z_1, \ldots, z_5] \tag{3.1}$$

intersects transversally in $\mathbb{C}^5$ with the sphere $\mathbb{S}^9_\epsilon = \{\|z\| = \epsilon\}$ of radius ϵ centered at the origin, then the intersection $K_j := V(f_j) \cap \mathbb{S}^9_\epsilon$ is homeomorphic to $\mathbb{S}^7$, but with its naturally induced orientation and smooth structure, represents all 28 oriented smooth classes mentioned above. One way to distinguish among these structures is to use the so-called Milnor monodromy map associated to the isolated singularity of $V(f_j) \subseteq \mathbb{C}^5$ at the origin. More specifically, the Milnor map gives rise to a numerical invariant, called the Milnor number, which equals $12j - 4$, hence completely classifies all the oriented smooth structures on $\mathbb{S}^7$ realized by Brieskorn varieties.

It is interesting to find operator-theoretic invariants capable of detecting exotic spheres [45, Page 381]. (See [33, 34] for an operator-theoretic study of smooth structures on Spin manifolds.) Theorem 4 (Section 1.1) applied to the principal ideal $I_j := \langle f_j \rangle \subseteq \mathbb{C}[z_1, \ldots, z_5]$ says that the Toeplitz class τ_{I_j} is the same as the fundamental class of K_j. However K_j supports only one Spin^c structure because of topological reasons (vanishing of the first and second cohomologies [79, Page 392]). Therefore Theorem 4, at least in the natural setting that comes into mind, can not classify smooth structures. In Section 3.3 we suggest a noncommutative analogue of the Milnor monodromy map which we hope could eventually lead to an invariant that detects exotic spheres.

3.2 A review of the Milnor fibration in singularity theory

Let $f \in A$ be a complex polynomial in m variables such that the origin is an isolated singular point of the hypersurface $V(f) \subseteq \mathbb{C}^m$. An interesting example to have in mind is the Brieskorn polynomials (3.1). Let $\mathbb{B}_\epsilon \subseteq \mathbb{C}^m$ be the open ball of radius ϵ around the origin, and set $\mathbb{S}_\epsilon := \partial\overline{\mathbb{B}}_\epsilon$. To study the topology of $K := V(f) \cap \mathbb{S}_\epsilon$, Milnor brought the perturbed family $V(f - c) \cap \mathbb{B}_\epsilon$ of spaces into the scene, where the complex parameter c moves on a small circle around the origin [83]. Here we summarize some of his and other mathematicians' results. It is helpful to have Figure 3.1 in mind.

1. There exists $\epsilon_0 > 0$ such that for any $0 < \epsilon < \epsilon_0$, $V(f)$ intersects $\mathbb{S}_\epsilon$ transversally, hence K is a smooth manifold. From now on fix such a sufficiently small ϵ.

2. Topologically, $V(f) \cap \mathbb{B}_\epsilon$ is a cone over K.

3. The mapping

$$\varphi : \mathbb{S}_\epsilon \setminus K \to \mathbb{S}^1, \quad z \mapsto f(z)/|f(z)|$$

   is a smooth fiber bundle called the Milnor fibration. Consider the fibers $F_t := \varphi^{-1}(e^{it})$, $t \in [0, 2\pi]$. The homotopy lifting property of fibrations induces the Milnor monodromy map $h_t : F_0 \to F_t$, clearly a homeomorphism. It induces the homomorphism

$$(h_t)_* : H_{m-1}(F_0; \mathbb{C}) \to H_{m-1}(F_t; \mathbb{C})$$

   at the middle homology level. Set $h := h_{2\pi}$.

4. The closure of each fiber F_t inside $\mathbb{S}_\epsilon$ is a smooth $(2m-2)$-dimensional manifold with boundary, with the interior F_t and boundary K. Intuitively, the fibers F_t embrace K the same way as the pages of an open book embrace the spine.

5. Each fiber F_t is diffeomorphic to $V(f-c) \cap \mathbb{B}_\epsilon$, where c is a small enough complex number.

6. Each fiber F_t is homotopic to a bouquet of $(m-1)$-dimensional spheres. The number of these sphere, namely the middle Betti number of the fibers, is strictly positive. It is denoted by μ and is called the Milnor number.

7. Here are two other topological and algebraic characterizations of μ: (1) μ is the multiplicity of 0 as an isolated zero of the system of equations $\frac{\partial f}{\partial z_1} = \cdots = \frac{\partial f}{\partial z_m} = 0$, namely the topological degree of the map $\mathbb{S}_\epsilon \to \mathbb{S}^{2m-1}$ sending z to the normalization of the Jacobian $df = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_m}\right)$. (2) μ is the complex vector space dimension of the quotient of the polynomial algebra $\mathbb{C}[z_1, \ldots, z_m]$ by the ideal $\left\langle \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_m} \right\rangle$. [4, Chapter 5]

8. F_t is not contractible, and K is not an unknotted sphere in $\mathbb{S}_\epsilon$.

9. K is homeomorphic to a sphere (namely $\mathbb{S}^{2m-3}$) exactly when $\det(1-h_*) = \pm 1$, where h_* is the linear map induced by h at the middle homology level.

One can say more for Brieskorn polynomials $f = \sum_{1 \leq l \leq m} z_l^{b_l}$, $b_l \geq 2$:

1. The Milnor number μ equals $\prod_{1 \leq l \leq m} (b_l - 1)$.

2. Each fiber F_t is homotopic to the join of the finite cyclic groups corresponding to the b_l-th roots of unity, $l = 1, \ldots, m$.

3. The eigenvalues (counting multiplicity) of the middle homology induced Milnor map h_* are the products $\omega_1 \omega_2 \cdots \omega_m$ where each ω_l ranges over all b_l-th roots of unity that are not equal to 1.

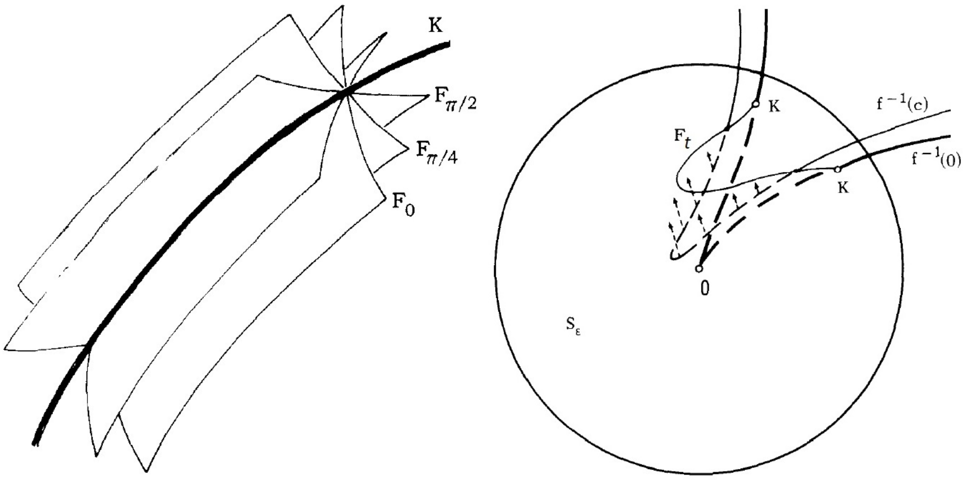

Figure 3.1: Milnor fibration [83].

3.3 The holonomy operator U

Suppose a polynomial $f \in A = \mathbb{C}[z_1, \ldots, z_m]$ which vanishes at the origin, and has the origin as an isolated critical point. In geometric terms the origin is an isolated singularity of the hypersurface $V(f) \subseteq \mathbb{C}^m$. Consider the family of principal ideals

$$I(t) := \langle f - \epsilon e^{it} \rangle \subseteq A, \quad t \in \mathbb{R},$$

where ϵ is a fixed sufficiently small positive real number. We think of t as the time variable. Let $P_t \in \mathfrak{B}(H^2_m)$ be the orthogonal projection onto $I(t)^\perp$. Let

$$p : \mathcal{I}^\perp \to \mathbb{R}, \quad \mathcal{I}^\perp := \biguplus \left\{ I(t)^\perp \subseteq H^2_m : t \in \mathbb{R} \right\} \subseteq \mathbb{R} \times H^2_m, \quad p(I(t)) = \{t\},$$

and

$$P : \mathbb{R} \to \mathfrak{B}\left(H^2_m\right), \quad P := (P_t),$$

be respectively the assembly of Hilbert spaces $I(t)^\perp$ and projections P_t into a rough[1] Hilbert bundle and a rough map between Banach spaces. Topologize $\mathcal{I}^\perp \subseteq \mathbb{R} \times H^2_m$ with the subspace topology.

[1]Namely we are putting continuity or smoothness considerations momentarily aside.

For the rest of this section we assume without proof that:

Conjecture 31. *p is a smooth Hilbert bundle.*[2]

At the moment we can only verify this conjecture for our toy model of Section 3.4 (see Theorem 39). Note that since the base space of p is contractible, even the weaker assumption that p is a topological vector bundle implies that it is trivial [67, IV.2.5][77, Corollary 1], hence automatically smooth, and this smooth structure is unique up to smooth vector bundle isomorphisms [67, IV.3.5]. The set of all (smooth) sections of p is denoted by $C^\infty(\mathbb{R};\mathcal{I}^\perp)$.

Unfortunately P is not smooth in general. (See Section 3.4.3 for a discussion.) Thinking of P as a rough connection between nearby fibers $I(t)^\perp$, imitating the standard construction of the Levi-Civita connection for subbundles of Hilbert bundles [74, Example 1.5.14][97, Volume II, Page 540] gives us a rough covariant derivative:

$$D\xi(t) = P_t\left(\frac{d\xi}{dt}\right), \quad \xi \in C^\infty(\mathbb{R};\mathcal{I}^\perp). \tag{3.2}$$

Note that D is called a covariant derivative because it satisfies the Leibniz rule:

$$D(g\xi)(t) = g'(t)\xi(t) + g(t)D(\xi)(t), \quad \forall g \in C^\infty(\mathbb{R};\mathbb{C}), \quad \forall \xi \in C^\infty(\mathbb{R};\mathcal{I}^\perp).$$

The D-flat sections of p are those $\xi \in C^\infty(\mathbb{R};\mathcal{I}^\perp)$ which satisfy the evolution equation

$$D\xi(t) = 0, \quad \forall t \in \mathbb{R}. \tag{3.3}$$

For the rest of this section we assume without proof that:

Conjecture 32. *The parallel transport equation (3.3) has a unique solution on $t \in \mathbb{R}$ for each initial value $\xi(0) \in I(0)^\perp$.*

---

[2] C^2-smoothness is enough for our purposes. Standard references for infinite-dimensional differential geometry are [74, 76, 78]. See also [2, Chapter 7].

At the moment we can only verify this conjecture for our toy model of Section 3.4 (see Theorem 38). The resulting holonomy map

$$U_t : I(0)^{\perp} \to I(t)^{\perp}, \quad t \in \mathbb{R},$$

is the one sending the initial value $\xi(0)$ of flat section ξ to its time-t value $\xi(t)$.

Proposition 33. *Each U_t is unitary.*

Proof. Linearity is immediate from the uniqueness assumption of Conjecture 32. Assume a flat section ξ. Since ξ and $d\xi/dt$ are orthogonal, we have

$$0 = 2\left\langle \xi(t), \frac{d\xi}{dt} \right\rangle = \frac{d}{dt}\left\|\xi(t)\right\|^2 = \frac{d}{dt}\left\|U_t\xi(0)\right\|^2,$$

therefore

$$\left\|U_t\xi(0)\right\| = \left\|U_0\xi(0)\right\| = \left\|\xi(0)\right\|.$$

This shows that U_t is an isometry. For each $\tau \in \mathbb{R}$, the inverse of $U_\tau : \xi(0) \mapsto \xi(\tau)$ is given by the parallel translation $\eta(0) \mapsto \eta(\tau)$ along the flat section $\eta(t) := \xi(\tau - t)$. Note that we are again using the uniqueness assumption of Conjecture 32. ■

We are specially interested in $U := U_{2\pi} \in \mathfrak{B}(I(0)^{\perp})$. This is our noncommutative analogue of the Milnor monodromy map $h : F_0 \to F_0$ of Section 3.2. We expect:

Conjecture 34. *U acts by conjugation on the Toeplitz algebra $\mathfrak{T}_{I(0)}$ in the sense that $U\mathfrak{T}_{I(0)}U^* \subseteq \mathfrak{T}_{I(0)}$.*

At the moment we can only verify this conjecture for our toy model of Section 3.4 (see Theorem 44).

Remark 35. Assuming Conjecture 34, we get an induced map $K\left(\mathfrak{T}_{I(0)}\right) \to K\left(\mathfrak{T}_{I(0)}\right)$ at the K-homology level. ■

Remark 36. Since $\mathcal{I}^\perp$ is trivial a rough section of p is an expression of the form

$$\xi := \sum_{n\in\mathbb{N}^m} x_n z^n, \quad z^n := \frac{z_1^{n_1}\cdots z_m^{n_m}}{\sqrt{\omega(n)}}, \quad \omega(n) := \|z_1^{n_1}\cdots z_m^{n_m}\|^2_{H^2_m} = \frac{n!}{|n|!},$$

where each $x_n : \mathbb{R} \to \mathbb{C}$ is a function of t, and $\xi(t) := \sum x_n(t) z^n$ is formally orthogonal to whole $I(t)$ for each t. Smooth sections are those rough sections such that for each $t \in \mathbb{R}$, each series $\xi^{(l)}(t) := \sum_{n\in\mathbb{N}^m} \frac{d^l x_n}{dt^l} z^n$, $l \in \mathbb{N}$, of term-by-term time derivatives lives in H^2_m, and $\left\|\xi^{(l)}(t+h) - \xi^{(l)}(t) - h\xi^{(l+1)}(t)\right\|_{H^2_m} \to 0$ as $h \to 0$. ■

3.4 A toy model

We use the notations of Section 3.3, more specifically, polynomial f, small positive number ϵ, Hilbert bundle $\mathcal{I}^\perp$ (more precisely, $p : \mathcal{I}^\perp \to \mathbb{R}$) and holonomy operator U. Fix integer $k \geq 2$. For the toy model $f := z_1^k \in \mathbb{C}[z_1, z_2]$, we find explicit formulas for U and the fundamental Toeplitz operators associated to $I(0)^\perp$, and study their interaction.

3.4.1 The holonomy operator U

We first find an explicit smooth orthonormal frame for our Hilbert bundle

$$\mathcal{I}^\perp = \biguplus \left\{ \langle z_1^k - \epsilon e^{it} \rangle^\perp \subseteq H^2_2 : t \in \mathbb{R} \right\}.$$

We start with a computational lemma.

Lemma 37. *Let E be a complex number with $|E| < 1$. Set*

$$F := E^{\frac{1}{k}},$$

$$\zeta_j := e^{i\frac{2\pi}{k}j}, \quad j = 0, \ldots, k-1,$$

$$a_j := 1 - \zeta_j F, \quad j = 0, \ldots, k-1.$$

(1) We have:

$$\sum_{q\in\mathbb{N}} \binom{n+r+kq}{n} E^q = k^{-1} F^{-r} \sum_{j=0}^{k-1} \zeta_j^{-r} a_j^{-n-1},$$

$$\sum_{q\in\mathbb{N}} \binom{n+r+kq}{n} E^q q = k^{-2} F^{-r} \left(-r \sum_{j=0}^{k-1} \zeta_j^{-r} a_j^{-n-1} + F(n+1) \sum_{j=0}^{k-1} \zeta_j^{-r+1} a_j^{-n-2} \right),$$

$$\sum_{q\in\mathbb{N}} \binom{n+r+kq}{n} E^q q^2 = k^{-3} F^{-r} \times$$

$$r^2 \sum_{j=0}^{k-1} \zeta_j^{-r} a_j^{-n-1} + F(1-2r)(n+1) \sum_{j=0}^{k-1} \zeta_j^{-r+1} a_j^{-n-2} + F^2(n+1)(n+2) \sum_{j=0}^{k-1} \zeta_j^{-r+2} a_j^{-n-3}.$$

(2) For any positive integer l we have the asymptotic formula:

$$\sum_{q\in\mathbb{N}} \binom{n+r+kq}{n} E^q q^l \approx n^l (1-F)^{-n}, \qquad n \to \infty.$$

Proof. (1) Note that the sequence of numbers

$$\psi_q := k^{-1} \sum_{j=0}^{k-1} \zeta_j^{q-r}, \quad q \in \mathbb{N},$$

equals 1 when q has remainder r modulo k, and zero otherwise. Therefore

$$\begin{aligned} \sum_{q\in\mathbb{N}} \binom{n+r+kq}{n} E^{r+kq} &= \sum_{q\in\mathbb{N}} \binom{n+q}{n} Z_q E^q \\ &= k^{-1} \sum_{q\in\mathbb{N}} \sum_{j=0}^{k-1} \binom{n+q}{n} \zeta_j^{q-r} E^q \\ &= k^{-1} \sum_{j=0}^{k-1} \zeta_j^{-r} \left(1 - \zeta_j E\right)^{-n-1}, \end{aligned}$$

where we have used the negative binomial formula

$$\sum_{q\in\mathbb{N}} \binom{n+q}{n} G^q = (1-G)^{-n-1}$$

in the last line. This gives the first formula. The other two are followed by differentiation with respect to E.

(2) By induction find a general formula for the left hand side, and then note that when $n \to \infty$ the dominant summand in each $\sum_{0\leq j\leq k-1} \zeta_j^{-r+m} a_j^{-n-m-1}$, $m \in \mathbb{N}$, is the one with smallest $|a_j|$, so the one with $j = 0$. ■

Having this lemma at hand we can prove:

Theorem 38. *Set*

$$F := \epsilon^{\frac{2}{k}},$$

$$\zeta_j := e^{i\frac{2\pi}{k}j}, \quad j = 0, \dots, k-1,$$

$$a_j := 1 - \zeta_j F, \quad j = 0, \dots, k-1,$$

$$J := \left\{(r,n) \in \mathbb{N}^2 : 0 \leq r \leq k-1\right\}.$$

(1) A smooth orthogonal frame for the Hilbert bundle $\mathcal{I}^\perp$ is given by

$$\alpha := \left\{ \alpha_{r,n}(t) := \sum_{q\in\mathbb{N}} \omega_{r+kq,n}^{-1} \epsilon^q e^{-iqt} z_1^{r+kq} z_2^n \ : \ (r,n) \in J,\ t \in \mathbb{R} \right\}, \tag{3.4}$$

where

$$\omega_{m,n} = \|z_1^m z_2^n\|_{H_2^2}^2 = \binom{m+n}{m}^{-1}.$$

(2) A smooth orthonormal frame for the Hilbert bundle $\mathcal{I}^\perp$ is given by

$$\beta := \left\{ \beta_{r,n}(t) := \frac{\alpha_{r,n}(t)}{\|\alpha_{r,n}(t)\|} \ : \ (r,n) \in J,\ t \in \mathbb{R} \right\}, \tag{3.5}$$

where

$$\|\alpha_{r,n}(t)\|^2 = \sum_{q\in\mathbb{N}} \omega_{r+kq,n}^{-1}\epsilon^{2q} = F^{-r}k^{-1}\sum_{j=0}^{k-1}\zeta_j^{-r}a_j^{-n-1}. \tag{3.6}$$

(3) The holonomy operator $U_t : I(0)^\perp \to I(t)^\perp$ acts diagonally by

$$U_t\left(\beta_{r,n}(0)\right) = e^{if_{r,n}t}\beta_{r,n}(t), \tag{3.7}$$

where frequencies $f_{r,n}$ are given by

$$f_{r,n} = \frac{\sum_{q\in\mathbb{N}} \omega_{r+kq,n}^{-1} q\epsilon^{2q}}{\sum_{q\in\mathbb{N}} \omega_{r+kq,n}^{-1}\epsilon^{2q}} = \frac{-r\sum\limits_{j=0}^{k-1}\zeta_j^{-r}a_j^{-n-1} + F(n+1)\sum\limits_{j=0}^{k-1}\zeta_j^{-r+1}a_j^{-n-2}}{k\sum\limits_{j=0}^{k-1}\zeta_j^{-r}a_j^{-n-1}}. \tag{3.8}$$

(4) When $n\to\infty$, $f_{r,n}$ varies asymptotically like:

$$f_{r,n} \approx \frac{F}{k(1-F)}n + \frac{F}{k(1-F)} - \frac{r}{k}. \tag{3.9}$$

(5) A smooth orthonormal parallel frame for the Hilbert bundle $\mathcal{I}^\perp$ is given by

$$\gamma := \left\{\gamma_{r,n}(t) := e^{if_{r,n}t}\beta_{r,n}(t) \ : \ (r,n)\in J,\ t\in\mathbb{R}\right\}. \tag{3.10}$$

Proof. (1) We first check the smoothness. For comparison purposes observe that any one-variable power series of the form

$$\sum_{q\in\mathbb{N}} R(q)\zeta^q, \quad R\in\mathbb{C}[\zeta] \text{ a polynomial in single variable } \zeta, \tag{3.11}$$

has radius of convergence equal to one, hence absolutely and uniformly convergent on any compact subset of the open unit disk of the complex ζ-plane. The formal power series of term-by-term time derivative of each $\alpha_{r,n}$ of order $l\in\mathbb{N}$, as well as its H_2^2-norm are given by:

$$\frac{d^l\alpha_{r,n}}{dt^l} := \sum_{q\in\mathbb{N}} \binom{kq+r+n}{n} (-iq)^l e^{-iqt}\epsilon^q z_1^{r+kq} z_2^n,$$

$$\left\|\frac{d^l\alpha_{r,n}}{dt^l}\right\|_{H_2^2}^2 := \sum_{q\in\mathbb{N}} \binom{kq+r+n}{n} q^{2l}\epsilon^{2q}.$$

Comparison with (3.11) shows that for any $\epsilon < 1$ and any $t \in \mathbb{R}$, each $\frac{d^l\alpha_{r,n}}{dt^l}(t)$ is an analytic function on $\mathbb{B}_2$ with finite H_2^2-norm, hence lives in H_2^2. That $\alpha_{r,n}$ lives in $I(t)^\perp$ is immediate from our derivation of $\alpha_{r,n}$ in the next paragraph, but here is a direct verification. For each $(M,N) \in \mathbb{N}^2$, $\alpha_{r,n}(t)$ and $z_1^M z_2^N \left(z_1^k - \epsilon e^{it}\right)$ has no monomial in common (hence orthogonal) except when $N = n$ and r equals the remiander of M in division by k. For this exceptional case, assuming $M = kQ + r$, $Q \in \mathbb{N}$, we have

$$\left\langle \alpha_{r,n}, z_1^M z_2^N \left(z_1^k - \epsilon e^{it}\right)\right\rangle = \epsilon^{Q+1}e^{-i(Q+1)t} - \epsilon^Q e^{-iQt}\epsilon e^{-it} = 0.$$

By Taylor's theorem, we have

$$\begin{aligned}\left\|\alpha_{r,n}(t+h) - \alpha_{r,n}(t) - h\frac{d\alpha_{r,n}}{dt}(t)\right\|^2 &= \sum_{q\in\mathbb{N}} \binom{kq+r+n}{n} \epsilon^{2q} \left|e^{-iq(t+h)} - e^{-iqt} + hiqe^{-iqt}\right|^2 \\ &\le \sum_{q\in\mathbb{N}} \binom{kq+r+n}{n} \epsilon^{2q} \left(\frac{h^2}{2!}q^2\right)^2,\end{aligned}$$

which shows that $\alpha_{r,n} : \mathbb{R} \to H_2^2$ is first-order differentiable. The same line of arguments proves the smoothness.

Next we show that sections of $\mathcal{I}^\perp$ are linear combinations of $\alpha_{r,n}$. A section of $\mathcal{I}^\perp$ has the form

$$\xi(t) = \sum_{m,n\ge 0} x_{m,n}(t) z_1^m z_2^n, \tag{3.12}$$

and satisfies the orthogonality equations:

$$0 = \left\langle \xi(t), z_1^m z_2^n \left(z_1^k - \epsilon e^{it}\right)\right\rangle = x_{m+k,n}\omega_{m+k,n} - x_{m,n}\omega_{m,n}\epsilon e^{-it}, \quad \forall m,n \ge 0,$$

or equivalently

$$x_{m+k,n}\omega_{m+k,n} = x_{m,n}\omega_{m,n}\epsilon e^{-it}, \quad \forall m,n \geq 0. \tag{3.13}$$

Assuming

$$X_{m,n} := x_{m,n}\omega_{m,n},$$

this latter recursive equation becomes

$$X_{m+k,n} = X_{m,n}\epsilon e^{-it},$$

hence

$$X_{r+kq,n} = X_{r,n}\epsilon^q e^{-iqt}, \quad r = 0,1,\ldots,k-1, \quad q,n = 0,1,2,\ldots. \tag{3.14}$$

This shows that

$$\{X_{r,n} : (r,n) \in J\}$$

are basic Taylor coefficients of ξ in the sense that they linearly determine all the other coefficients, and there are no nontrivial linear equations among them. Note that $\alpha_{r,n}$ *is the section with* $X_{r,n} = 1$*, and all other basic coefficients vanish.* Working backwards, this shows that (3.4) is a basis for $I(t)^\perp$. Any two $\alpha_{r,n}$ and $\alpha_{r',n'}$, $(r,n) \neq (r',n')$, are orthogonal because they have no monomials in common, and we know that monomials constitute an orthogonal basis for H^2_m.

(2) Lemma 37 gives (3.6). Since $\|\alpha_{r,n}(t)\|$ does not depend on t, the rest follows immediately from part (1).

(3) A flat section of $\mathcal{I}^\perp$ has the form $\eta(t) = \sum y_{m,n} z_1^m z_2^n$ such that $\dot\eta = \sum \dot y_{m,n} z_1^m z_2^n$ lives in $I(t)^{\perp\perp}$ for each t. In other words the inner product $\langle \dot\eta, \xi\rangle_{H^2_2}$ is zero for every section ξ as in (3.12). Equivalently, in terms of Taylor coefficients, we have

$$\sum_{m,n} \dot y_{m,n} \overline{X}_{m,n} = 0,$$

for all $X_{m,n}$ satisfying (3.14). Rewriting this in terms of basic Taylor coefficients we get

$$\sum_{\substack{0\le r<k\\ q,n\ge 0}} \dot{y}_{r+kq,n}\overline{X}_{r,n}\epsilon^q e^{iqt} = 0.$$

Since this is true for any choice of basic coefficients $X_{r,n}$, $(r,n)\in J$, we should have

$$\sum_{q\in\mathbb{N}} \dot{y}_{r+kq,n}\epsilon^q e^{iqt} = 0, \quad (r,n)\in J. \tag{3.15}$$

Since η is a section its Taylor coefficients satisfy

$$y_{r+kq,n} = y_{r,n}\frac{\omega_{r,n}}{\omega_{r+kq,n}}\epsilon^q e^{-iqt}, \quad (r,n)\in J, \quad q\in\mathbb{N}.$$

(Recall (3.13).) Plugging this into (3.15) yields

$$\sum_{q\in\mathbb{N}} \left(\dot{y}_{r,n} - iqy_{r,n}\right)\frac{\omega_{r,n}}{\omega_{r+kq,n}}\epsilon^{2q} = 0, \quad (r,n)\in J.$$

Therefore we have the explicit evolution laws

$$\dot{y}_{r,n} = y_{r,n} i f_{r,n}, \quad (r,n)\in J, \tag{3.16}$$

where

$$f_{r,n} = \frac{\sum_{q\in\mathbb{N}}\binom{n+r+kq}{n}q\epsilon^{2q}}{\sum_{q\in\mathbb{N}}\binom{n+r+kq}{n}\epsilon^{2q}}. \tag{3.17}$$

Evolution equations (3.16) are solved as

$$y_{r,n}(t) = y_{r,n}(0)e^{if_{r,n}t}, \quad (r,n)\in J,$$

hence (3.7). Lemma 37 computes $f_{r,n}$.

(4) When $n\to\infty$ the dominant summands in the numerator and denominator of

$f_{r,n}$ in (3.8) are those with the smallest $|a_j|$, so those with $j = 0$. Therefore $f_{r,n}$ varies asymptotically like

$$f_{r,n} \approx \frac{-ra_0^{-n-1} + F(n+1)a_0^{-n-2}}{ka_0^{-n-1}} = -\frac{r}{k} + \frac{F}{k(1-F)}(n+1).$$

(5) By (3.7), we have

$$U_t(\gamma_{r,n}(0)) = \gamma_{r,n}(t),$$

hence γ is a parallel frame. Smoothness is the result of the smoothness of β and the asymptotic formula (3.9) for $f_{r,n}$. ■

Theorem 39. *The Conjecture 31 holds true for the toy model.*

Proof. Assume $\mathcal{I}^\perp \subseteq \mathbb{R} \times H_2^2$ with the subspace topology as a rough Hilbert bundle over $\mathbb{R}$. Recall that $J := \{(r,n) \in \mathbb{N}^2 : 0 \leq r \leq k-1\}$ is the index set of the orthonormal frame β in Theorem 38. Since each $\beta(t)$, $t \in \mathbb{R}$, is an orthonormal basis for the fiber $I(t)^\perp$, the mapping

$$\Phi : \mathbb{R} \times l^2(J) \to \mathcal{I}^\perp, \quad \big(t, (a_{r,n})_{(r,n)\in J}\big) \mapsto \left(t, \sum_{(r,n)\in J} a_{r,n}\beta_{r,n}(t)\right),$$

trivializes $\mathcal{I}^\perp$ as a topological vector bundle, namely Φ is a homeomorphism and the triangle

$$\begin{array}{ccc} \mathbb{R} \times l^2(J) & \xrightarrow{\Phi} & \mathcal{I}^\perp \\ {\scriptstyle \mathrm{pr}_{\mathbb{R}}}\downarrow & \swarrow{\scriptstyle p} & \\ \mathbb{R} & & \end{array}$$

commutes. Since this trivialization is given by a single chart, it also gives $\mathcal{I}^\perp$ the structure of a smooth vector bundle. ■

Remark 40. (1) Lemma 37 also gives the following formula for $\alpha_{r,n}$:

$$\alpha_{r,n}(t) = F^{-\frac{r}{2}} k^{-1} e^{-\frac{irt}{k}} z_2^n \sum_{j=0}^{k-1} \zeta_j^{-r} \left(1 - \zeta_j F^{\frac{1}{2}} e^{-\frac{it}{k}} z_1\right)^{-n-1}.$$

(2) If one thinks of the unitary operator $U_t : I(0)^\perp \to I(t)^\perp$ as an integral operator

$$U_t\eta(z) = \int_{w\in\mathbb{B}_2} K(z,w)\eta(w)dw,$$

then, since U_t acts diagonally on the orthonormal basis $\beta_{r,n}$ with the corresponding eigenvalues $e^{itf_{r,n}}$, the kernel is given by

$$K(z,w) = \sum_{(r,n)\in J} e^{itf_{r,n}}\beta_{r,n}(z)\overline{\beta}_{r,n}(w).$$

Plugging from Theorem 38 and the previous part we get:

$$K(z,w) = \sum_{(r,n)\in J} e^{itf_{r,n}}\left(z_2\overline{w}_2\right)^n \frac{\sum\limits_{j,l=0}^{k-1} \zeta_{j-l}^{-r}\left(\left(1-\zeta_j F^{\frac{1}{2}}e^{-\frac{it}{k}}z_1\right)\left(1-\zeta_{-l}F^{\frac{1}{2}}e^{\frac{it}{k}}\overline{w}_1\right)\right)^{-n-1}}{\sum\limits_{j=0}^{k-1}\zeta_j^{-r}\left(1-\zeta_j F\right)^{-n-1}}.$$

We will not need this expression in this dissertation. ■

3.4.2 The interaction of U with the Toeplitz algebra

Consider the Toeplitz algebra $\mathfrak{T}_{I(0)}$ associated to the ideal $I(0) = \langle z_1^k - \epsilon\rangle \subseteq \mathbb{C}[z_1,z_2]$. It is the C*-algebra generated by $\{1, T_{z_1}, T_{z_2}\}\cup\mathfrak{K}(I(0)^\perp)$ where T_{z_j}, $j=1,2$, is multiplication by coordinate function z_j compressed to $I(0)^\perp$. For brevity we set $T_j := T_{z_j}$, $j=1,2$.

Proposition 41. *Assume the notations of Theorem 38. T_1, T_2 and their adjoints are weighted shifts given by:*

$$T_1\beta_{r,n} = F^{\frac{1}{2}}\frac{\left(\sum_{j=0}^{k-1}\zeta_j^{-r}a_j^{-n-1}\right)^{\frac{1}{2}}}{\left(\sum_{j=0}^{k-1}\zeta_j^{-r-1}a_j^{-n-1}\right)^{\frac{1}{2}}}\beta_{r+1,n},$$

$$T_2\beta_{r,n} = \frac{\left(\sum_{j=0}^{k-1}\zeta_j^{-r}a_j^{-n-1}\right)^{\frac{1}{2}}}{\left(\sum_{j=0}^{k-1}\zeta_j^{-r}a_j^{-n-2}\right)^{\frac{1}{2}}}\beta_{r,n+1},$$

$$T_1^*\beta_{r,n} = F^{\frac{1}{2}} \frac{\left(\sum_{j=0}^{k-1} \zeta_j^{-r+1} a_j^{-n-1}\right)^{\frac{1}{2}}}{\left(\sum_{j=0}^{k-1} \zeta_j^{-r} a_j^{-n-1}\right)^{\frac{1}{2}}} \beta_{r-1,n},$$

$$T_2^*\beta_{r,n} = \frac{\left(\sum_{j=0}^{k-1} \zeta_j^{-r} a_j^{-n}\right)^{\frac{1}{2}}}{\left(\sum_{j=0}^{k-1} \zeta_j^{-r} a_j^{-n-1}\right)^{\frac{1}{2}}} \beta_{r,n-1}.$$

Proof. Each $\beta_{r,n}$ is a sum of monomials $z_1^{r+kq} z_2^n$, $q \geq 0$. Since distinct monomials are orthogonal to each other in H_2^2, $z_1\beta_{r,n}$ is orthogonal to all elements $\beta_{r',n'}$ of our orthonormal basis except for $\beta_{r+1,n}$. Therefore $T_1\beta_{r,n}$ is just the orthogonal projection of $z_1\beta_{r,n}$ onto $\beta_{r+1,n}$, namely

$$T_1\beta_{r,n} = \langle z_1\beta_{r,n}, \beta_{r+1,n}\rangle \beta_{r+1,n}.$$

To compute the weight $\langle z_1\beta_{r,n}, \beta_{r+1,n}\rangle$, substitute $\beta_{r,n}$ from Theorem 38 and apply Lemma 37:

$$\begin{aligned}\langle z_1\beta_{r,n}, \beta_{r+1,n}\rangle &= \frac{\sum_{q\in\mathbb{N}} \omega_{r+kq,n}^{-1} \epsilon^{2q}}{F^{-r-\frac{1}{2}} k^{-1} \left(\sum_{j=0}^{k-1} \zeta_j^{-r} a_j^{-n-1}\right)^{\frac{1}{2}} \left(\sum_{j=0}^{k-1} \zeta_j^{-r-1} a_j^{-n-1}\right)^{\frac{1}{2}}} \\ &= F^{\frac{1}{2}} \frac{\left(\sum_{j=0}^{k-1} \zeta_j^{-r} a_j^{-n-1}\right)^{\frac{1}{2}}}{\left(\sum_{j=0}^{k-1} \zeta_j^{-r-1} a_j^{-n-1}\right)^{\frac{1}{2}}}.\end{aligned}$$

The rest is straightforward. ∎

Proposition 42. *Assume the notations of Theorem 38. We have:*

$$U^*T_1U\beta_{r,n} = e^{i2\pi(f_{r,n}-f_{r+1,n})} T_1\beta_{r,n},$$

$$U^*T_1^*U\beta_{r,n} = e^{i2\pi(f_{r,n}-f_{r-1,n})} T_1^*\beta_{r,n},$$

$$U^* T_2 U \beta_{r,n} = e^{i2\pi(f_{r,n} - f_{r,n+1})} T_2 \beta_{r,n},$$

$$U^* T_2^* U \beta_{r,n} = e^{i2\pi(f_{r,n} - f_{r,n-1})} T_2^* \beta_{r,n}.$$

Proof. Immediate from Proposition 41. ■

We need to understand the asymptotic behavior of the factors appearing in Proposition 42 when n grows large. Recalling the asymptotic formula (3.9) for $f_{r,n}$ one expects:

Lemma 43. *Assume the notations of Theorem 38. Then*

$$f_{r,n} - f_{r-1,n} \to -\frac{1}{k}, \qquad f_{r,n} - f_{r,n-1} \to \frac{F}{k(1-F)},$$

as $n \to \infty$.

Proof. By (3.8), $k\,(f_{r,n} - f_{r,n-1})$ equals

$$\frac{-r\sum_{j=0}^{k-1} \zeta_j^{-r} a_j^{-n-1} + F(n+1)\sum_{j=0}^{k-1} \zeta_j^{-r+1} a_j^{-n-2}}{\sum_{j=0}^{k-1} \zeta_j^{-r} a_j^{-n-1}} - \frac{-r\sum_{l=0}^{k-1} \zeta_l^{-r} a_l^{-n} + Fn\sum_{l=0}^{k-1} \zeta_l^{-r+1} a_l^{-n-1}}{\sum_{l=0}^{k-1} \zeta_l^{-r} a_l^{-n}}$$

$$= \frac{Fn\sum_{j,l=0}^{k-1} \zeta_j^{-r+1} a_j^{-n-2} \zeta_l^{-r} a_l^{-n} - Fn\sum_{j,l=0}^{k-1} \zeta_j^{-r} a_j^{-n-1} \zeta_l^{-r+1} a_l^{-n-1} + F\sum_{j,l=0}^{k-1} \zeta_j^{-r+1} a_j^{-n-2} \zeta_l^{-r} a_l^{-n}}{\sum_{j,l=0}^{k-1} \zeta_j^{-r} a_j^{-n-1} \zeta_l^{-r} a_l^{-n}}.$$

We need to find the dominant terms in the numerator and denominator of the latter fraction when n grows large. The dominant summand in the denominator is the one with smallest $|a_j||a_l|$, which is the one with $j = l = 0$, namely

$$\zeta_0^{-r+1} a_0^{-n-1} \zeta_0^{-r} a_0^{-n} = (1-F)^{-2n-1}.$$

We have three summations in the numerator with dominant terms

$$Fn(1-F)^{-2n-2}, \quad n(1-F)^{-2n-2} \quad \text{and} \quad F(1-F)^{-2n-2},$$

respectively. The first two cancel each other, and all the remaining summands in the first two summations are dominated by the dominant term of the denominator $(1-F)^{-2n-1}$. Therefore the dominant term of the numerator is $F(1-F)^{-2n-2}$. Therefore

$$\lim_{n\to\infty} k\left(f_{r,n} - f_{r,n-1}\right) = \lim_{n\to\infty} \frac{F(1-F)^{-2n-2}}{(1-F)^{-2n-1}} = \frac{F}{1-F}.$$

Using (3.8), $k\left(f_{r,n} - f_{r-1,n}\right)$ equals

$$\frac{-r\sum \zeta_j^{-r} a_j^{-n-1} + F(n+1)\sum \zeta_j^{-r+1} a_j^{-n-2}}{\sum \zeta_j^{-r} a_j^{-n-1}} \\ - \frac{-(r-1)\sum \zeta_l^{-r+1} a_l^{-n-1} + F(n+1)\sum \zeta_l^{-r+2} a_l^{-n-1}}{\sum \zeta_l^{-r+1} a_l^{-n-1}}$$

$$= \frac{-\sum \zeta_j^{-r}\zeta_l^{-r+1}(a_j a_l)^{-n-1} + F(n+1)\left(\sum \zeta_j^{-r+1} a_j^{-n-2}\zeta_l^{-r+1} a_l^{-n} - \zeta_j^{-r}\zeta_l^{-r+2}(a_j a_l)^{-n-1}\right)}{\sum \zeta_j^{-r} a_j^{-n-1}\zeta_l^{-r+1} a_l^{-n-1}}.$$

When n grows large the dominant terms in the numerator and denominator of the latter fraction are

$$-(1-F)^{-2n-2} + F(n+1)\times\left(\text{exponentially smaller than } (1-F)^{-2n-2}\right) \quad \text{and} \quad (1-F)^{-2n-2},$$

respectively. Therefore $k\left(f_{r,n} - f_{r,n-1}\right)$ tends -1. ∎

Proposition 42 and Lemma 43 gives:

Theorem 44. *As before $F := \epsilon^{\frac{2}{k}}$. The unitary operator U acts by conjugation on the*

Toeplitz algebra $\mathfrak{T}_{I(0)}$ in the sense that $U^*\mathfrak{T}_{I(0)}U \subseteq \mathfrak{T}_{I(0)}$. In more details

$$U^*T_1U - e^{i\frac{2\pi}{k}}T_1, \quad U^*T_1^*U - e^{-i\frac{2\pi}{k}}T_1^*, \quad U^*T_2U - e^{-i\frac{2\pi F}{k(1-F)}}T_2, \quad U^*T_2^*U - e^{i\frac{2\pi F}{k(1-F)}}T_2^*,$$

are all compact.

3.4.3 The smoothness of P

Recall the projection assembly map $P : \mathbb{R} \to \mathfrak{B}(H_2^2)$ acting between Banach spaces. We now prove what we mentioned before:

Proposition 45. *P is not smooth.*

Proof. According to Theorem 38.(4), each

$$\delta_{r,n} := e^{if_{r,n}t}\alpha_{r,n}, \quad (r,n) \in J,$$

is a flat section of $\mathcal{I}^\perp$, namely satisfies the equations

$$P_t\delta_{r,n}(t) = \delta_{r,n}(t), \quad P_t\dot{\delta}_{r,n}(t) = 0.$$

Suppose by contradiction that P is smooth. Differentiating the first equation and plugging from the second gives

$$\dot{P}_t\delta_{r,n}(t) = \dot{\delta}_{r,n}(t).$$

However the ratio

$$\frac{\left\|\dot{\delta}_{r,n}(t)\right\|}{\|\delta_{r,n}(t)\|} = \frac{\|if_{r,n}\alpha_{r,n}(t) + \dot{\alpha}_{r,n}(t)\|}{\|\alpha_{r,n}(t)\|} = \left(\frac{\sum_{q\in\mathbb{N}}\omega_{r+kq,n}^{-1}\epsilon^{2q}(f_{r,n}-q)^2}{\sum_{q\in\mathbb{N}}\omega_{r+kq,n}^{-1}\epsilon^{2q}}\right)^{\frac{1}{2}}$$
$$= \left(f_{r,n}^2 - 2f_{r,n}\frac{\sum_{q\in\mathbb{N}}\omega_{r+kq,n}^{-1}\epsilon^{2q}q}{\sum_{q\in\mathbb{N}}\omega_{r+kq,n}^{-1}\epsilon^{2q}} + \frac{\sum_{q\in\mathbb{N}}\omega_{r+kq,n}^{-1}\epsilon^{2q}q^2}{\sum_{q\in\mathbb{N}}\omega_{r+kq,n}^{-1}\epsilon^{2q}}\right)^{\frac{1}{2}} \quad (3.18)$$

asymptotically behaves like $n^{\frac{1}{2}}$ as $n \to \infty$, hence $\dot{P}_t$ would be unbounded. Here are more details. By Lemma 37 and the asymptotic formula for $f_{r,n}$ in (3.9), the three consecutive terms a, b and c in the last expression $(a-b+c)^{1/2}$ in (3.18), asymptotically behave like $a_2n^2 + a_1n$, $b_2n^2 + b_1n$ and $c_2n^2 + c_1n$, where a_j, b_j and c_j are nonzero constants (with respect to n) satisfying $a_2 - b_2 + c_2 = 0$ and $a_1 - b_1 + c_1 \neq 0$. (Here by saying that a behaves asymptotically like $a_2n^2 + a_1n$ we mean that $a \approx n^2$, $a - a_2n^2 \approx n$ and $a - a_2n^2 - a_1n \ll 1$. Likewise for b and c.) This shows that $(a-b+c)^{1/2}$ asymptotically behaves like $n^{1/2}$. This contradiction shows that P is not even first differentiable. ■

We can fix this problem by using weights to compensate for differentiation [19]. More precisely, viewing the Drury-Arveson space $H_2^2 = \mathcal{H}_2^{(-2)}$ as a member of the Besov-Sobolev scale $\mathcal{H}_2^{(s)}$, $s \in \mathbb{R}$, of Hilbert spaces, we have:

Theorem 46. *(1) The modification $\widetilde{P} : \mathbb{R} \to \mathfrak{B}\left(\mathcal{H}_2^{(-2)}, \mathcal{H}_2^{(4)}\right)$ of P where $\widetilde{P}_t$ is the composition of P_t with the inclusion $\mathcal{H}_2^{(-2)} \hookrightarrow \mathcal{H}_2^{(4)}$ is first differentiable.*

(2) Suppose positive integer l and positive real σ. Then the modification $\widetilde{P} : \mathbb{R} \to \mathfrak{B}\left(\mathcal{H}_2^{(-2)}, \mathcal{H}_2^{(2l+1+\sigma)}\right)$ of P where $\widetilde{P}_t$ is the composition of P_t with the inclusion $\mathcal{H}_2^{(-2)} \hookrightarrow \mathcal{H}_2^{(2l+1+\sigma)}$ is l-th differentiable.

Proof. (1) We have a corresponding version of Theorem 38 for $\mathcal{H}_2^{(4)}$ instead of $H_2^2 = \mathcal{H}_2^{(-2)}$, where $\omega_{r+kq,n}$ is replaced by

$$\widetilde{\omega}_{r+kq,n} := \left\| z_1^{r+kq} z_2^n \right\|_{\mathcal{H}_2^{(4)}} = \frac{(r+kq)!n!6!}{(r+kq+n+6)!} = S(n)\omega_{r+kq,n+6},$$

and

$$S(n) := \frac{6!n!}{(n+6)!} \approx n^{-6}.$$

Let

$$\left\{ e_{m,n} := \omega_{m,n}^{-\frac{1}{2}} z_1^m z_2^n \;:\; (m,n) \in \mathbb{N}^2 \right\} \quad \text{and} \quad \left\{ \widetilde{e}_{m,n} := \widetilde{\omega}_{m,n}^{-\frac{1}{2}} z_1^m z_2^n \;:\; (m,n) \in \mathbb{N}^2 \right\}$$

be the time-independent standard orthonormal bases of $\mathcal{H}_2^{(-2)}$ and $\mathcal{H}_2^{(4)}$ respectively. We first compute the matrix coefficients of $\widetilde{P}_t$ with respect to these bases. Note that for each $(m,n) \in \mathbb{N}^2$, $e_{m,n}$ is orthogonal to all members of the orthonormal frame $\widetilde{\beta}$ except for $\widetilde{\beta}_{r,n}$, where

$$m = kQ + r, \quad Q, r \in \mathbb{N}, \quad 0 \leq r < k,$$

is the unique division of m by k. Therefore

$$\begin{aligned}
\widetilde{P}_t\left(e_{m,n}\right) &= \langle e_{m,n}, \beta_{r,n}\rangle \, \beta_{r,n} = \omega_{m,n}^{-\frac{1}{2}} \left\|\alpha_{r,n}\right\|^{-2} \langle z_1^m z_2^n, \alpha_{r,n}\rangle \, \alpha_{r,n} \\
&= \omega_{m,n}^{-\frac{1}{2}} \left\|\alpha_{r,n}\right\|^{-2} \epsilon^{Q} e^{iQt} \sum_{q\in\mathbb{N}} \omega_{r+kq,n}^{-1} \widetilde{\omega}_{r+kq,n}^{\frac{1}{2}} \epsilon^{q} e^{-iqt} \widetilde{e}_{r+kq,n} \\
&= \frac{\sum_q \omega_{m,n}^{-\frac{1}{2}} \omega_{r+kq,n}^{-1} \widetilde{\omega}_{r+kq,n}^{\frac{1}{2}} \epsilon^{Q+q} e^{i(Q-q)t} \widetilde{e}_{r+kq,n}}{\sum_q \omega_{r+kq,n}^{-1} \epsilon^{2q}} \\
&= \sqrt{S(n)} \frac{\sum_q \omega_{m,n}^{-\frac{1}{2}} \omega_{r+kq,n}^{-1} \omega_{r+kq,n+6}^{\frac{1}{2}} \epsilon^{Q+q} e^{i(Q-q)t} \widetilde{e}_{r+kq,n}}{\sum_q \omega_{r+kq,n}^{-1} \epsilon^{2q}}.
\end{aligned}$$

Therefore the formal matrix $\dot{\widetilde{P}}_t$ of entry-by-entry differentiation of P_t equals

$$\dot{\widetilde{P}}_t\left(e_{m,n}\right) := \sqrt{S(n)} \frac{\sum_q \omega_{m,n}^{-\frac{1}{2}} \omega_{r+kq,n}^{-1} \omega_{r+kq,n+6}^{\frac{1}{2}} \epsilon^{Q+q} i(Q-q) e^{i(Q-q)t} \widetilde{e}_{r+kq,n}}{\sum_q \omega_{r+kq,n}^{-1} \epsilon^{2q}}.$$

From this expression the Hilbert-Schmidt norm of $\dot{\widetilde{P}}_t$ can be computed as:

$$
\begin{aligned}
\left\|\dot{\widetilde{P}}_t\right\|_{\mathrm{HS}}^2 &:= \sum_{n,r} S(n) \frac{\sum_{Q,q} \omega_{r+kQ,n}^{-1}\omega_{r+kq,n}^{-2}\omega_{r+kq,n+6}\epsilon^{2Q+2q}(Q-q)^2}{\left(\sum_q \omega_{r+kq,n}^{-1}\epsilon^{2q}\right)^2} \\
&\leq \sum_{n,r} S(n) \frac{\sum_{Q,q} \omega_{r+kQ,n}^{-1}\omega_{r+kq,n}^{-1}\epsilon^{2Q+2q}\left(Q^2+q^2\right)}{\left(\sum_q \omega_{r+kq,n}^{-1}\epsilon^{2q}\right)^2} = \sum_{n,r} 2S(n) \frac{\sum_q \omega_{r+kq,n}^{-1}\epsilon^{2q}q^2}{\sum_q \omega_{r+kq,n}^{-1}\epsilon^{2q}} \\
&\approx \sum_n S(n)n^2 \approx \sum_n n^{-4} < \infty. \qquad \text{By Lemma 37}
\end{aligned}
$$

This especially shows that $\dot{\widetilde{P}}_t$ is bounded (in operator norm) [9, 2.8.4][97, Volume I, A.6]. With the same line of arguments along with Taylor's theorem, for any $h \in \mathbb{R}$ we have:

$$
\begin{aligned}
\left\|\widetilde{P}_{t+h} - \widetilde{P}_t - h\dot{\widetilde{P}}_t\right\|_{\mathrm{HS}}^2 &\leq \sum_{n,r} S(n) \frac{\sum_{Q,q} \omega_{r+kQ,n}^{-1}\omega_{r+kq,n}^{-2}\omega_{r+kq,n+6}\epsilon^{2Q+2q}(Q-q)^4\left(\frac{h^2}{2!}\right)^2}{\left(\sum_q \omega_{r+kq,n}^{-1}\epsilon^{2q}\right)^2} \\
&\ll \sum_{n,r} S(n)h^4 \frac{\sum_q \omega_{r+kq,n}^{-1}\epsilon^{2q}q^4}{\sum_q \omega_{r+kq,n}^{-1}\epsilon^{2q}} \ll \sum_n S(n)h^4n^4 = h^4\sum_n n^{-2}.
\end{aligned}
$$

This finishes the proof that $\widetilde{P}$ is first differentiable.

(2) Imitating the proof in (1), set

$$
\widetilde{\omega}_{r+kq,n} := \left\|z_1^{r+kq}z_2^n\right\|_{\mathcal{H}_2^{(2l+1+\sigma)}} = \frac{(r+kq)!n!(2l+3+\sigma)!}{(r+kq+n+2l+3+\sigma)!} = S(n)\omega_{r+kq,n+2l+3+\sigma},
$$

where

$$
S(n) := \frac{(2l+3+\sigma)!n!}{(n+2l+3+\sigma)!} \approx n^{-2l-3-\sigma}.
$$

For any $j = 1, \ldots, l$, and any $h \in \mathbb{R}$, we get estimates:

$$
\left\|\frac{d^j\widetilde{P}_t}{dt^j}\right\|_{\mathrm{HS}}^2 \ll \sum_n S(n)n^{2j} \approx \sum_n n^{-3-\sigma-2(l-j)} < \infty,
$$

$$\left\| \frac{d^{j-1}\widetilde{P}_{t+h}}{dt^{j-1}} - \frac{d^{j-1}\widetilde{P}_t}{dt^{j-1}} - h\frac{d^j\widetilde{P}_t}{dt^j} \right\|_{\text{HS}}^2 \ll \sum_n S(n) h^{2j+2} n^{2j+2} \approx h^{2j+2} \sum_n n^{-1-\sigma-2(l-j)},$$

which implies that $\widetilde{P}$ is l-th differentiable. ■

Remark 47. Recall the identification $\mathcal{H}_m^{(s)} = W_{\text{hol}}^{-\frac{s}{2}}(\mathbb{B}_m)$, $s \in \mathbb{R}$, between Besov-Sobolev and Bergman-Sobolev spaces (Page x). Theorem 46.(2) says that by taking l-th derivative of P we lose differentiability by order no worse than $l+2$. We do not know whether this estimate of differentiability loss is optimal. ■

Remark 48. Suppose a section $\xi \in C^\infty(\mathbb{R}, \mathcal{I}^\perp)$. Proposition 46.(2) shows that $\frac{d^l P_t}{dt^l}(\xi(t))$ lives in $\mathcal{H}_2^{(2l+1+\sigma)}$. Similar arguments shows that for each $s \leq -2$, if $\xi(t) \in \mathcal{H}_2^{(s)}$ then $\frac{d^l P_t}{dt^l}(\xi(t))$ in fact lives in $\mathcal{H}_2^{(s+2l+3+\sigma)}$. Here is a corollary. If S denotes the set of all sections ξ such that for each t, $\xi(t)$ and all its time derivatives live in $\bigcap_{s\in\mathbb{R}} \mathcal{H}_2^{(s)}$, then the connection D (3.2) maps S to itself. ■

3.5 Some potential future directions

Here are some directions for future works:

1. Study Conjectures 31, 32 and 34 for general ideals. In particular we plan to extend our study of the toy model $f := z_1^k$ of Section 3.4 to the Brieskorn polynomials $f := \sum_{1\leq l\leq m} z_l^{b_l}$, $b_l \geq 2$.

2. It is interesting to study the asymptotic behavior of the unitary operator U when $\epsilon \to 0$. More specifically, note that in Theorem 44 there appears the *phase factor* $\exp \frac{2\pi i F}{k(1-F)}$ where $F = \epsilon^{\frac{2}{k}}$. When $\epsilon \to 0$, this factor varies like

$$\exp\left(\frac{2\pi i}{k}\epsilon^{\frac{2}{k}}\right).$$

For another toy model $f := z_1 z_2 \in \mathbb{C}[z_1, z_2]$, our computations (not included in this dissertation) shows that the phase factor equals

$$\exp\left(2\pi i\frac{1-\sqrt{1-4\epsilon^2}}{\sqrt{1-4\epsilon^2}}\right) = \exp\left(4\pi i\epsilon^2 + O\left(\epsilon^4\right)\right).$$

It is desirable to understand these phase factors in the general case.

3. It is interesting to extend the study in this chapter about isolated singularities on hypersurfaces to complete intersection analytic sets. See [82].

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