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Strongly Correlated Systems Under High Magnetic Field: A Mixed Landau Levels Description for Fractional Quantum Hall Effect

Sumanta Bandyopadhyay
Washington University in St. Louis

WASHINGTON UNIVERSITY IN ST.LOUIS
Department of Physics

Dissertation Examination Committee:
Alexander Seidel, Chair
Jeroen van den Brink
Renato Feres
Zohar Nussinov
Michael Ogilvie
Gerardo Ortiz

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by
Sumanta Bandyopadhyay

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The last five years of my life has been a wonderful voyage through the realm of the exciting phenomena in emergent physical systems. In this long journey, I have realized now, I was always at home, Washington University in St Louis. Calm and comforting campus life of our university, especially the openminded, welcoming and highly interactive culture of physics dept., has helped me getting matured in multiple ways while making my journey as smooth and enjoyable as possible. I will be always thankful to Washington University for these five best years of my life. The graduate program of physics dept was so skillfully designed that, I have never imagined of attending such a vast set of excellent lectures in different fields of physics. Professor Carl Bender's courses on asymptotic gave me an entirely new perspective to approach any research level problem. Professor Tanmay Bachaspathy's course on topological defects gave me a comprehensive vision of the different applications of similar ideas of physics in entirely different energy scales. The course on phase transition in statistical mechanics, taught by professor Gerardo Ortiz gave me a solid understanding of the field. I am very thankful to the physics department, for letting me take a course on conformal field theory with professor Alexander Seidel. This course has helped me to develop a better understanding of two-dimensional physics, the central theme of my research. These five years of my introduction to research in physics would never have been the same without the guidance of my thesis advisor

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Sumanta Bandyopadhyay

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Dedicated to those people, who find it useful.

ABSTRACT OF THE DISSERTATION

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by

Sumanta Bandyopadhyay

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Alexander Seidel, Chair

Strong correlation among electrons under high magnetic field gives rise to an entirely new arena of emergent physics, namely fractional quantum Hall effect. Such systems have entirely different elementary degrees of freedom and generally, demand non-perturbative approaches to develop a better understanding. In the literature, there are several analytical methodologies and numerical toolkits available to study such a system. Clustering of zeros, parent Hamiltonian, off-diagonal order parameter, parton construction, matrix product states are to be named among a few of those popular methodologies in the existing literature. Most of these methods work well in the lowest Landau level or holomorphic wavefunction framework. It is, however, imperative to develop such methodology to study systems with Landau levels mixing to study more exotic as well as experimentally relevant states. In this work, we have developed particular methodologies, which denounce the traditional importance of the analytic properties of first quantized model wavefunction thereby extend the existing parent Hamiltonian, topological order-parameter, matrix product states descriptions to mixed Landau level systems. Such extension produces a deeper, compact and holistic understanding of universal physics of

exotic phases in strongly correlated systems from the microscopic viewpoint, as well as produces interesting new results.

Our second quantized/ non-analytic approach allows us to construct the "entangled Pauli principle", a guidebook to extract universal/topological properties such as braiding statistics, fractional charge quantization, topological degeneracy of the ground states starting from a relatively simple many-body wavefunction, "root pattern" of fractional quantum Hall state. Such an entangled Pauli principle can be derived from a microscopic parent Hamiltonian setting, thereby provide us a potential tool to probe the non-universal physics in quantum Hall fluids as well. Essentially, entangled Pauli principle is the "DNA" of fractional quantum Hall states. Using this guiding principle, we have shown ground states with non-abelian excitations, such as Majorana fermion or Fibonacci fermion can be stabilized for two-particle interaction. Fibonacci fermion supports universal quantum gates, thereby a potential candidate for the topologically protected universal quantum computer. Entangled Pauli principle, along with a recently developed topological order parameter for composite fermions, gives rise to Parent Hamiltonian description for composite fermions as well.

Chapter 1

Introduction

[1] In 1879, Edwin Hall, a graduate student at Johns Hopkins University, first observed "Hall effect" phenomena in a thin gold leaf [6]. From the modern point of view, this phenomenon gives a rather mundane observation, once an out of the plane magnetic field B is applied to a constant current flow along the x-direction, a voltage is induced along the y- direction in the gold leaf. One, however, must note that such a discovery precedes the discovery of the electron. Until that time, electrical measurements provided only the carrier density and mobility product, and the separation of these two important physical quantities had to rely on other difficult measurements. The discovery of the Hall effect enabled a direct measure of the carrier density. The polarity of this transverse Hall voltage proved that it is in fact electrons (negative charge career) that are physically moving in an electric current. This earliest variant of Hall effect can be easily explained using classical

[1] In this chapter, we will review the concepts behind integer quantum Hall effect, starting from the classical Hall effect. Integer quantum physics can be described by non-interacting electrons under high magnetic field. Understanding of integer quantum Hall, however, is crucial for the understanding of interacting electron system under high magnetic field. Parts of this chapter has been motivated from the materials given in the book named "Composite Fermions" by J.K. Jain [4], and the "Lectures on Quantum Hall Effect" by David Tong [5].

linear response theory or Drude model. One can, in fact, calculate the resistivity matrix in two dimensions.

$$\rho = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{xy} & \rho_{yy} \end{pmatrix}; \quad \rho_{yy} = \rho_{xx} = \frac{m}{ne^2\tau}; \quad \rho_H = \rho_{xy} = \frac{B}{ne} \tag{1.1}$$

Longitudinal resistivity depends on the scattering time τ, the effective mass of the electron m, charge density n and the square of effective carrier charge e. ρ_{xx} goes to zero as scattering processes become less important ($\tau \to \infty$), while off-diagonal resistivity (Hall resistivity) ρ_H goes linearly with the magnetic strength B. One should, however, realize that once the scattering process gets less important, Drude model starts to break down. Once the magnetic field is very high, electrons get pinned down to small cyclotron orbits, thus no longer get governed by thermal physics or scattering. More than hundred years after Edwin Hall's discovery, in 1980, von Klitzing measured Hall conductivity under magnetic field 18 T in a sample prepared by Dorda and Pepper. One can easily notice that both longitudinal and Hall resistivity (voltage is shown in the graph) no longer follow the classical explanation based on the Drude model. Hall resistivity stays on a plateau for a range of magnetic field, before jumping to the next plateau. On the plateau, the Hall resistivity has the following form,

$$\rho_H = \frac{2\pi\hbar}{ne^2}; \quad n \in \mathbb{N} \tag{1.2}$$

This phenomenon, however, has been already been explained by Ando [7] in terms of Landau levels. He has suggested that the electrons in impurity bands, arising from short range scatterers, do not contribute to the Hall current; whereas the electrons in the Landau level give rise to the same Hall current as that obtained when all the electrons are in the level and can move freely. These Landau levels, as shown be Lev Landau, have the

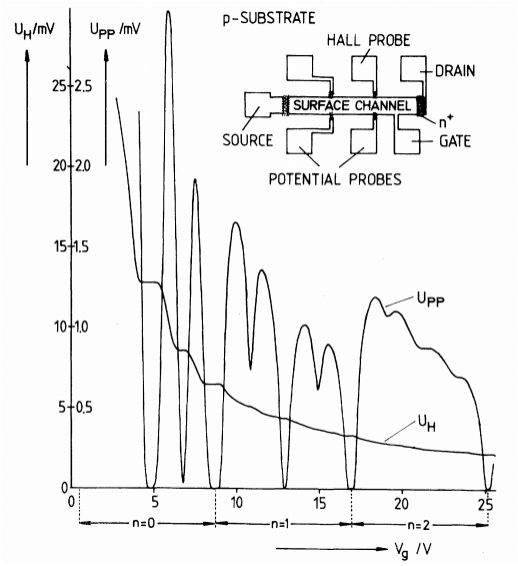


Figure 1.1: [Graph is taken from [1]] Recordings of the Hall voltage U_H and the voltage drop between the potential probes, U_{PP}, as a function of the gate voltage V at T = 1.5 K. The constant magnetic field (B) is 18 T and the source-drain current, I, is 1 μA. The inset shows a top view of the device with a length of L =400 μm, a width of W =50 μm, and a distance between the potential probes of L_{PP}= 130 μm.

purely quantum origin, thus Eq. (1.2), describes Quantum Hall effect. As ρ_H in eq. (1.2), is characterized by integer N, this phenomenon is named Integer quantum Hall (IQH) effect. In the rest of the introduction, we will talk about IQH systems. Understanding of such systems helps us also to create the foundation of strongly correlated systems under high magnetic field.

1.1 Electron under high magnetic field: Landau level

The presence of $\hbar$ in eq. (1.2) suggests the quantum nature of electron prevails the physics under a high magnetic field. It is imperative to study an electron moving under high

magnetic field in an entirely quantum mechanical framework. Such a system for a low-mobility (non-relativistic) electron can be described by following Hamiltonian,

$$H = \frac{1}{2m}\frac{\mathfrak{ß}^2}{2m} = \frac{1}{2m}(\mathbf{p} + e\mathbf{A}/c)^2; \quad \nabla \times \mathbf{A} = B\hat{z} \text{ (Out of the plane magnetic field)} \tag{1.3}$$

Where mechanical momentum $\mathfrak{ß}$ depends on the canonical momentum $\mathbf{p}$ as well as vector potential $\mathbf{A}$. Any physical property of this Hamiltonian won't depend on the choice of gauge. It is, however, crucial to notice that the particular geometry of the system naturally suggests some particular choice of gauge. We will solve the Hamiltonian with three different gauge choices to elaborate on this point. In order to avoid notational inconvenience, we will set the following quantities to unity.

$$l = \sqrt{\frac{c}{eB}} = 1, \quad \hbar = 1, \quad m = 1 \tag{1.4}$$

1.1.1 Landau gauge $\mathbf{A} = -B(y, 0, 0)$

Under this particular gauge choice, Hamiltonian has no x dependence, thus momentum along x-direction, $p_x = k_x$ is a good quantum number. Under this gauge Hamiltonian becomes,

$$H = \frac{1}{2}(p_y^2 + (y - k_x)^2) \Rightarrow E_n = (n - 1/2) \quad n \in \mathbb{N} \tag{1.5}$$

Notice that energy E_n does not depend on k_x, thus any specific n accommodates a large number of degeneracy for all allowed values of k_x. Each n denotes different Landau level. For each such Landau level, the associated wavefunctions are,

$$\eta_{n,k_x} = \frac{e^{ik_x x}}{\sqrt[4]{\pi(2^n n!)^2}} e^{-\frac{1}{2}(y-k_x)^2} H_n(y - k_x); \quad H_n(.) \text{ are Hermite polynomials} \tag{1.6}$$

This wavefunction is delocalized along x direction and localized along y around $y = k_x$. From the wavefunction's structure, it is clear that this gauge naturally allows us to impose periodic boundary condition $k_x = \frac{2\pi}{L_x}$. Thus, Landau gauge is a natural choice for cylindrical geometry with circumference L_x. One must appreciate at this point, continuous spectrum of Fermi-sea becomes equally spaced highly degenerate Landau levels (LL). This is a non-perturbative consequence of the magnetic field. This particular feature is gauge-independent observation.

1.1.2 Symmetric gauge $\mathbf{A} = \frac{B}{2}(-y, x, 0)$

Under this particular gauge, eq. (1.2) becomes

$$H = \frac{1}{2}\left((-i\partial_x - y/2)^2 + (-i\partial_y + x/2)^2\right) \tag{1.7}$$

In order to exploit this particular symmetry, we will go to complex co-ordinates

$$z = x + iy = re^{i\theta}, \quad \bar{z} = x - iy = re^{i\theta} \tag{1.8}$$

In this gauge, we have,

$$H = (a^\dagger a + 1/2); \quad L = -i\partial_\theta = a^\dagger a - b^\dagger b; \quad [H, L] = 0 \tag{1.9}$$

Last commutation relation between H and angular momentum L gives rise to the macroscopic degeneracy for this Hamiltonian. Thus it serves the purpose of the linear momentum k_x in the Landau gauge or cylinder geometry. a and b are defined in the following manner,

$$a = \frac{1}{\sqrt{2}}(z/2 + 2\partial_{\bar{z}}); \quad b = \frac{1}{\sqrt{2}}(\bar{z}/2 + 2\partial_z) \tag{1.10}$$

Thus our Hamiltonian has been transformed into a familiar form of harmonic oscillator, with additional degeneracy in angular momentum basis. Hence, the eigenfunctions for this Hamiltonian can be written as,

$$|n,m\rangle = \frac{(b^\dagger)^{n+m}(a^\dagger)^n}{\sqrt{(m+n)!n!}}\,|0,0\rangle\,; \quad E_n = (n+1/2); \quad a\,|0,0\rangle = b\,|0,0\rangle = 0 \tag{1.11}$$

Thus LLshas a macroscopic degeneracy in terms of angular momentum quantum number m. From the above equation it is clear that m can be integer $\geq -n$ for n^{th} LL. One can write LLs in co-ordinate basis as,

$$\eta_{n,m}(z,\bar{z}) = \langle z,\bar{z}|\,|n,m\rangle = \frac{(-1)^n}{\sqrt{2\pi}}\sqrt{\frac{n!}{2^m(m+n)!}}z^m L_n^m(z\bar{z}/2)e^{-z\bar{z}/4}; \tag{1.12}$$

$L_n^m(.)$ are associated Laguerre polynomials. These wave functions are peaked around a circular region with radius $r = z\bar{z} = \sqrt{2m}$. Hence, in disk geometry with a given radius, r number of states allowed in one LL is given by $m = r^2/2$. This again suggests macroscopic degeneracy of Landau levels.

1.1.3 Singular gauge: Spherical geometry

In this thesis, we will discuss spherical geometry, a finite manifold in a limited way. It is, however, very much illuminating to study quantum Hall physics in this particular geometry. In other geometries, each LL has infinite degeneracy in the absence of confining potential at the edge. Due to finite size and absence of an edge in spherical geometry, filled LLs are unambiguously defined. In this geometry, however, in order to introduce out-of-the-plane magnetic field, we must introduce a monopole charge of $2Q$, which invokes

singularity in the vector potential.

$$\mathbf{B} = \frac{2Q}{4\pi R^2}\hat{r}; \qquad A_{\pm} = -\frac{Q}{R}(\cot\theta \pm \csc\theta)\hat{\phi} \tag{1.13}$$

Where the magnetic field is perpendicular to the surface of the sphere of radius R. For all future reference, we will assume $R = |Q|$, unless otherwise specified. Vector potential A_+ is defined on the upper half of the sphere, with a singularity at the north pole ($\theta = 0$) and A_- is defined in the lower half of the sphere. From the symmetry of the geometry, it is apparent that angular momentum is a good quantum number. One can indeed construct the following angular momentum algebra,

$$L_z = -i\partial_\phi; \quad L_\pm = e^{\pm\phi}[\pm\partial_\theta + i\cot\theta\partial_\phi + Q\csc\theta] \tag{1.14}$$

In the language of angular momentum, Hamiltonian in eq. (1.2), will take the following form,

$$H = \frac{1}{2|Q|}(L^2 - Q^2) \tag{1.15}$$

L^2 is the total angular momentum operator. Thus eigenvectors of this Hamiltonian will simultaneously diagonalize L^2, L_z and Hamiltonian itself. The solution of the above system can be written in terms of monopole harmonics Y_{Qlm}.

$$L^2 Y_{Qlm} = l(l+1)Y_{Qlm}; \quad L_z Y_{Qlm} = (m-Q)Y_{Qlm}; \quad H Y_{Qlm} = (l(l+1) - Q^2)/2|Q| \tag{1.16}$$

Note that, we have chosen the monopole charge to be $2Q$, hence Q can take both integer and half-integer values. Moreover, due to the semi-positive nature of Hamiltonian (eq. (1.2)) last equation ensures, $l(l+1) \geq Q^2$. Thus, for n^{th} Landau level, we get maximum

of l to be $|Q| + n$. In the large Q limit[2] , we get back the familiar form of Landau level energy, $E_n = (n + 1/2)$.

1.2 Integer quantum Hall plateau: Landau level quantization

In 1981, Laughlin has argued [8] IQH plateau, is related to Landau level quantization. In order to follow his argument, let us introduce a test vector potential $\mathbf{A}_t$ in cylindrical geometry with circumference L. We will set $\mathbf{A}_t$ to zero at the end. Our Hamiltonian will look like,

$$H(\phi_t) = \frac{1}{2}(p + A + A_t)^2; \quad A_t = \frac{\alpha\phi_0}{L}x \tag{1.17}$$

Where ϕ_0 is magnetic flux quantum, which we will set to unity. Now, $\Psi(x, y)$ is a solution of the original Hamiltonian $H(\phi_t = 0)$, $\Psi^t(x, y)$ is a solution of $H(\phi_t)$.

$$\Psi^t(x, y) = e^{i2\pi\alpha x/L}\Psi(x, y); \quad \psi(x, y) = \frac{e^{ik_x x}}{\sqrt[4]{\pi(2^n n!)^2}} e^{-\frac{1}{2}(y-k_x)} H_n(y - k_x) \tag{1.18}$$

Due to the periodic boundary condition $x = x + L$, we have

$$k_x = \frac{2\pi}{L}(n - \alpha), \quad n \in \mathbb{N} \tag{1.19}$$

As we change α adiabatically from zero to unity, n changes by 1. As we have already discussed, these states are localized around $y = k_x$, changing k_x by unity moves the states along the y direction. Keeping this phenomenon in mind, let us consider n filled LLs. If we increase $alpha$ from zero to 1, it will move each single particle state to their right, thus forcing n states to move from one edge to another. This will result in a current $I = nV_H$

[2] As $R = |Q|$ large Q implies large radius, which can be thought as planar geometry.

for a given potential difference V_H, between two edges. Thus Hall resistance will be quantized at the plateau,

$$R = 1/n \tag{1.20}$$

The stability of the plateau is explained by the introduction of impurities. Such a discussion is outside the scope of this thesis. At this end, we should realize the following fact. We have so far talked about single particle under high magnetic field. However, the last equation makes sense only if there were filled Landau levels. It is thus imperative to study N-particle wavefunction of electrons with Pauli exclusion statistics under high magnetic field.

1.3 Integer Quantum Hall States: Filled Landau levels

In the last section, we have talked about the importance of filled LLs in order to explain the integer quantum Hall effect. In order to understand the notion of filled LLs, let us first construct N electron system under high magnetic field. At this point, we will assume electrons are strongly bound to their cyclotron orbits and well separated such that they do not interact among themselves. One such electron can be described by,

$$\langle x, y|\, c^{\dagger}_{n,m}\, |0\rangle = \eta_{n,m}(x, y) \tag{1.21}$$

Where n and m are good quantum numbers, which can be associated with physical quantities. For the sake of completeness, we will choose disk geometry with symmetric gauge for out of the plane magnetic field. In this particular choice of basis, n is Landau level index and m is angular momentum of the particle. The exact form of $\eta_{n,m}$ is given by eq.

(1.12). Hence N non-interactive electrons system can be described by,

$$\Psi = \langle \{\mathbf{x}, \mathbf{y}\} | \, |\Psi\rangle = \langle \{\mathbf{x}, \mathbf{y}\} | \prod_{i=1}^{N} c^{\dagger}_{n_i,m_i} \, |0\rangle = \mathcal{A}(\prod_{i=1}^{N} \eta_{n_i,m_i}(x_i, y_i)) \tag{1.22}$$

$\{\mathbf{x}, \mathbf{y}\}$ is the set of the coordinates of all N particles. Where total antisymmetry operation $\mathcal{A}$ is applied to take care of the Pauli exclusion principle. Such anti-symmetry can be expressed as the Slater determinant.

$$\Psi = \begin{vmatrix} \eta_{n_1,m_1}(x_1, y_1) & \cdots & \cdots & \eta_{n_1,m_1}(x_N, y_N) \\ \vdots & \ddots & \ddots & \vdots \\ \eta_{n_N,m_N}(x_1, y_1) & \cdots & \cdots & \eta_{n_N,m_N}(x_N, y_N) \end{vmatrix} \tag{1.23}$$

In order to describe n filled LLs for N electrons, we must make sure to occupy all of the angular momenta up to m in each LLs. Thus in disk geometry, we can describe n filled LL states by,

$$\Psi_1 = \begin{vmatrix} \eta_{0,0}(z_1, \bar{z}_1) & \cdots & \cdots & \eta_{0,0}(z_N, \bar{z}_N) \\ \vdots & \ddots & \ddots & \vdots \\ \eta_{n,m}(z_1, \bar{z}_1) & \cdots & \cdots & \eta_{n,m}(z_N, \bar{z}_N) \end{vmatrix} = \begin{vmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ \bar{z}_1^n z_1^{m+n} & \cdots & \cdots & \bar{z}_N^n z_N^{m+n} \end{vmatrix} e^{-\sum_{i=1}^{N} z_i \bar{z}_i / 4} \tag{1.24}$$

The last step of simplification is the result of a straightforward application of the multi-linearity of the determinants. We will, however, further discuss the importance of this innocuous and simple representation n-filled Landau level. For all future purpose, we will avoid mentioning the trivial exponential part, unless otherwise needed. This wave-function, as discussed in the earlier section, successfully describes IQH systems. At this

end, we will wrap the discussion of IQH system with some special properties of the lowest Landau level filled wavefunction.

$$\Psi_1 = \begin{vmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ z_1^m & \cdots & \cdots & z_N^m \end{vmatrix} = \prod_{i>j}(z_i - z_j) \tag{1.25}$$

The last step of the above expression is based on the following fact that, once we get rid of all the anti-holomorphic parts, the above Slater determinant became Vandermonde determinant. We will exploit this particular feature in multiple scenarios in the later part of this thesis.

1.3.1 Strong interactions: Beyond integer quantum Hall systems

So far, we have considered only low mobility electrons. Under a high magnetic field, such electrons get pinned down to small cyclotron orbit. It is reasonable, for such instances, ignore electron interaction. In a clean sample with high mobility electrons, such picture breaks down. Electrons tend to accommodate themselves into larger cyclotron orbit and start interacting with each other. Such interactions try to move electron further apart and it becomes less energetically favorable to occupy all the angular momentum states in a given Landau level.

It turns out, such interactions under high magnetic field give rise to highly non-trivial phenomena, like fractional charge excitation, anyonic statistics, etc. Explaining such phenomena demands nonperturbative approaches. In the next section, we will discuss a few existing analytical toolkits to study some relatively simple fractional plateaus. In the process of reviewing several different procedures, we will discuss deeper connection among

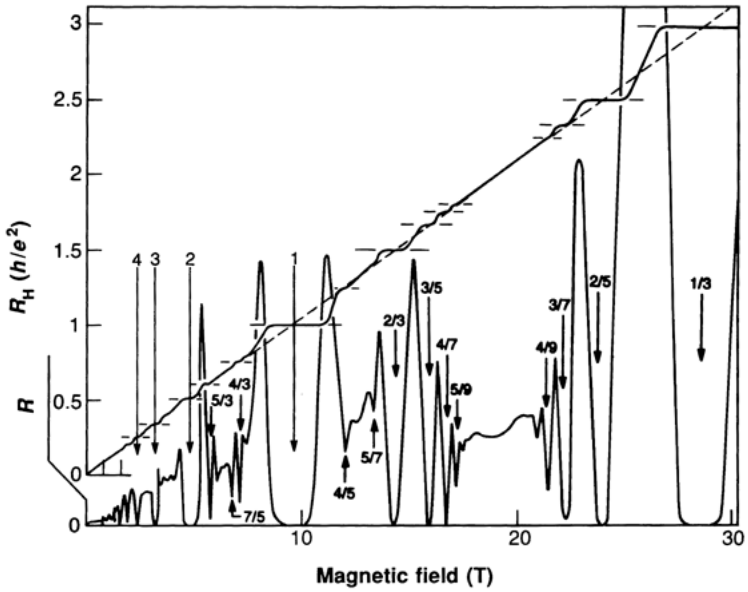


Figure 1.2: [The graph is taken from [2]] In a clean sample with high mobility electrons, integer plateau starts to disappear. However, new plateaus start to emerge at fractional filling fractions. This was first observed by Tsui and Stormer using samples prepared by Gossard [3].

those methods. In the latter chapters, we will use such understanding to develop such methods for more exotic plateaus.

Chapter 2

A Simple Fractional Quantum Hall state: Laughlin's State

[3] In 1982 Tsui, Stormer and Gossard [3] first observed the existence of plateau at 1/3 filling fraction. Following the same argument Laughlin [8] gave for integer quantum Hall effect, one should anticipate a plateau at 1/3 must be related to an elementary charge excitation 1/3. As we know that no single electron cannot have fractional charge, at this point, one must consider the possibility of the emergence of 1/3 quasiparticle from the electron-electron interaction. Now, the barest interaction, two electrons can have, is Coulomb interaction. Coulomb interaction, being long range, is exponentially hard to

[3] In this chapter, we will discuss an important class of strongly correlated systems, under a high magnetic field. This class of systems is named after Laughlin's state [9]. Bob Laughlin has given a holomorphic parton construction of quantum Hall states with fractional charge excitations. His construction successfully describes the first fractional quantum Hall states with 1/3 excitations. In this chapter, we will review Laughlin's states, with several different methodologies, existing in the literature. All of these methods have been well established for Laughlin state. In this chapter, we will show that all of these methods are connected, even in a microscopic sense. Most of the results in this section have been established previously [10, 11]. Our method, however, is a little different (but equivalent) from them and can be generalized to more complicated cases, as discussed in the latter chapters. The new concepts, developed in these chapters are results from a collaboration with L. Chen and K. Yang from National magnetic Lab in Tallahassee [12] and an ongoing project with M. Schossler and A. Seidel.

calculate the exact ground state for such a system. One would ideally think of tackling such a problem by slowly cranking up the interacting energy, starting from zero. In this case, however, such a method would fail measurably. Due to the high magnetic field, all of the electrons are pinned into cyclotron orbit, such that their kinetic energy is frozen. Hence, we have only potential energy in the system. We cannot start from very small interaction energy, as there is nothing else to compare to. When interaction is switched off, there is an infinite degeneracy of ground states degeneracy. This degeneracy gets lifted as soon as the interaction is switched on. It is just impossible to guess the physics from the perturbation of free particle theory. The bad news is, the problem cannot be solved exactly either, for Coulomb potential. This stalemate was broken by Bob Laughlin, who came up with an ansatz [9] for this particular problem. For the symmetric gauge, out of the plane magnetic field, his proposed wavefunction has the following form,

$$\psi = \prod_{i>j}(z_i - z_j)^m e^{-\sum_i z_i \bar{z}_i/4}; \quad \Psi_1 = \prod_{i>j}(z_i - z_j); \quad m = 3 \text{ for } 1/3 \text{ filling fraction.} \tag{2.1}$$

Ψ_1 is the integer quantum Hall state in the lowest Landau level. We have dropped the Gaussian factor for the notational convenience. Hence, one can think of Laughlin's ansatz as a product of m parton states. Each parton state describes integer quantum Hall states for N number of fermions with $1/m$ charge. These partons are not, however, physical degrees of freedom of the system. A physical electron with unit charge quanta can be constructed as a product of m such partons. This wavefunction is numerically tested to have a large overlap with the ground state of the Coulomb interaction between electrons under high magnetic field. Later in this section, we will establish this state as a uniform density state for $1/m$ filled LLs. Before investigating Laughlin's state further, we will further elaborate some of the universal properties of this state. Let us start by comparing ψ with IQH state Ψ_1. For a given number of particles, ψ accommodates m times larger

exponent of particle co-ordinates z_is. In the earlier section, we have seen, a Landau level wavefunction associated with z_i^q mostly lives on a circle of radius $\sqrt{2q}$. Having higher exponent of z_i is ψ compared to Ψ_1, immediately implies (assuming uniform density) electrons are more separated in $1/m$ plateaus. This can already be expected from the fact that electrons try to minimize the interaction energy by increasing distance among themselves. Hence, we expect, moving electrons further apart by multiplying symmetric polynomials of z_is should be energetically allowed in an infinite disk. In a finite disk, however, such operation will move electrons towards the edge and electrons will start experiencing the confining potential near the edge. Thus we can generate excitations in the Laughlin state by multiplying it with the symmetric polynomials of z_is. At this end, let us consider the following operation,

$$U(z)\psi = \prod_{i=1}^{N}(z - z_i) \prod_{i>j>0}^{N} (z_i - z_j)^m \Rightarrow U(z_0)^m\psi = \prod_{i>j\geq 0}^{N} (z_i - z_j)^m; \tag{2.2}$$

$U(z)$ multiplies ψ by different orders of symmetric polynomials in z_is and thus increase the angular momentum of the state. We can immediately see, m such operations of $U(z_0)$ together on ψ introduces another co-ordinate z_0 in the system. z_0 experiences the same status of other particle co-ordinates z_is without a particle associated with it. Now, a similar scenario can be created by using a particle annihilation operator $\mathcal{U}(z)$, which will introduce a hole in the system. $\mathcal{U}(z)$ can be written as,

$$\mathcal{U}(z) = \sum_n a_n z^n \frac{e^{-z\bar{z}/4}}{\sqrt{2\pi 2^n n!}} \tag{2.3}$$

Where a_n, a second quantized operator, destroys a particle with angular momentum n in lowest LL disk geometry. As we identify $U(z)^m$ creates a hole in ψ, we will identify the operation of $U(z)$ on ψ as the creation of a quasihole to the system. Densest Laughlin's

state, as well as its quasihole excitations, govern the universal physics of the system. Without a guiding principle, however, it is a non-trivial problem to all possible quasihole excitations for a given confining potential at the edge. We will address this question later in this section. At this end, we must highlight that $\mathcal{U}^{\dagger}(z)$ is increase particle number N by 1, while $U(z)$ increases total angular momentum M. Hence, the operation $\mathcal{O}(z) = \mathcal{U}^{\dagger}(z)U(z)^{m}$ breaks symmetry generated by $N/2 + M/mN$, while keeping $N/2 - M/mN$ symmetry unbroken. Thus $\mathcal{O}(z)$ classify ψ in a different topological class for a particular value of m. This topological order parameter $\mathcal{O}(z)$ [13], actually governs the construction of ψ in a recursive way.

$$\psi_{N\text{-particle}} = \int d^2z \mathcal{O}(z)\psi_{N-1\text{-particle}} \tag{2.4}$$

Due to the first and second quantized mixed notation used in the definition of $\mathcal{O}(z)$, it is a daunting task to evaluate such quantity. We will address more on this order parameter calculation, later in this section. At this point, however, we must realize such topological order parameter sets this state different from a free particle picture. In order to elaborate on this particular feature, let us take $m = 3$ Laughlin's state for two particles only.

$$(z_2 - z_1)^3 = \begin{vmatrix} 1 & 1 \\ z_1^3 & z_2^3 \end{vmatrix} + 3 \begin{vmatrix} z_1 & z_2 \\ z_1^2 & z_2^2 \end{vmatrix} = \langle \mathbf{r_1}, \mathbf{r_2}| \left(c_{0,0}^{\dagger}c_{0,3}^{\dagger} + 3c_{0,1}^{\dagger}c_{0,2}^{\dagger}\right)|0\rangle \tag{2.5}$$

As discussed in earlier section, $c_{n,m}^{\dagger}$ creates a particle (in disk geometry) in n^{th} LL with angular momentum m. Unlike IQH states, already for two particles, we have two Slater determinants, with fixed (for a particular geometry and gauge choice) coefficients in front of them. This is an entangled state, and the number of Slater determinants increases exponentially with the number of particles. Before discussing other properties of this entangled state further, we must understand how to generate all of this Slater determinants

starting from a single configuration. We will call this special configuration, the dominance pattern of the system. In the next part of this section, we will justify the name dominance pattern and how to construct full Laughlin's state from the dominance pattern.

2.1 Dominance pattern and construction of Laughlin's state

[4] Laughlin's state as Eq. (2.5) suggested, can be written as an expansion in the basis of N particle slater determinant $|S\rangle$.

$$\psi = \sum_S C_S |S\rangle \tag{2.6}$$

For a compact notation, we will represent each such Slater determinant by a binary string. As a concrete example, let us take the first determinant in eq. (2.5),, constructed from of 0^{th} and 3^{rd} exponent of z_1 and z_2. A string representation of this state will be 1001, i.e, 0^{th} and 3^{rd} position will be occupied in the binary string. $(z_1 - z_2)^3$ in this language will look like,

$$(z_1 - z_2)^3 := (1001) + 3(0110) \tag{2.7}$$

Such a notation, actually gives a faithful representation of the two-dimensional quantum Hall system in 1-d string, by getting rid of any dynamical degrees of freedom. We will refer to this sting as guiding center coordinates. Depending on the different choice of gauge and geometry, the guiding center co-ordinates will have a different interpretation. In symmetric gauge, such description has a one-to-one correspondence with angular momentum, while they can be interpreted as linear momentum in Landau gauge, cylinder geometry. Unless otherwise stated, we choose symmetric gauge as the preferred gauge

[4] [10, 14–17]

for our system. Thus we will use angular momentum basis and guiding center basis interchangeably. Thus, Laughlin's wavefunction ψ can be uniquely expanded in the basis of this binary strings $(\{n_i\})$.

$$|\psi\rangle := \sum_{\{n_i\}} C_{\{n_i\}} |\{n_i\}\rangle \tag{2.8}$$

From the properties of anti-symmetric polynomial, one should readily understand, the total angular momentum of each of these binary strings is constant, however, they do have different second moments. We call binary strings with the highest second-moment dominance pattern or root partition. For Laughlin state, there is unique such state where all the evenly spaced angular momentum orbital is occupied. Later, we will see, once we take a very thin cylinder limit [18–22], Laughlin state reduces to this root partition, an unentangled direct product state.

For a set of given binary strings, with fixed angular momentum (first moment), root partition is defined as a binary string with the highest second moment. For Laughlin's $1/m$ state, root partition is defined uniquely by a string where particles are evenly spaced, or more concretely a string with following pattern, $...1000...(m-1)$zeros$...0001000...(m-1)$zeros$...0001....$ This state has the highest angular momentum, thus cannot be expanded further. For example, Laughlin's $1/3$ state have ...1001001001... root partition. This implies ...1010000101... cannot exist in the Slater determinant expansion of $1/3$ state, even if this particular state has same angular momentum as root partition.

For the time being, we will concentrate on the problem at hand, how to construct the full basis for Laughlin's state in terms of binary strings. At this point, we will introduce inward squeezing, a two-body operator in guiding-center basis.

$$SQ_{in}(x) = \sum_{i<j} c^{\dagger}_{0,i-x} c^{\dagger}_{0,j+x} c_{0,i} c_{0,j} \quad \forall x > 0 \tag{2.9}$$

This operator decreases the second moment while acting on any state while keeping the total angular momentum constant. Thus, we can formally define the root partition with this operator,

$$\begin{aligned} &\langle \psi | \, |\{n\}\rangle \neq 0 \text{ and } \langle \psi | \, |\{n_i\}\rangle \neq 0, \ \langle \{n\}| \, SQ_{in}(x) \, |\{n_i\}\rangle = 0 \forall \{x, i\} \\ &\Rightarrow |\{n\}\rangle = \text{root partition.} \end{aligned} \tag{2.10}$$

In the latter part of this section, we will show that the entire basis set for Laughlin's state and it's quasihole excitations can be generated starting from root partition. Furthermore, all of the Laughlin's state and its quasihole excitations can be uniquely identified in terms of the root partition. In the latter part of this thesis, we will argue that all of the topological information for exotic fractional Quantum Hall systems (FQH), are actually embedded in the root partition itself. We will refer root partition as the "DNA" of many exotic FQH states.

2.2 Parent Hamiltonian: A guiding principle to construct dominance pattern

We have already talked about the importance of electron-electron interaction in FQH system. Bare interactions, two electrons can have is given by Coulomb potential. We, however, talk about a more general scenario, where two electrons can have any translation and rotation invariant interaction $V(|\mathbf{r}_i - \mathbf{r}_j|)$, where $\mathbf{r}_i$ is the co-ordinate of the i^{th} electron. Any two body interaction $V(\mathbf{r}_i - \mathbf{r}_j)$ can be expanded as [23]

$$V(\mathbf{r}_i - \mathbf{r}_j) = \sum_{m=0}^{+\infty} V_m \, L_m(-\nabla^2)\delta^{(2)}(\mathbf{r}_i - \mathbf{r}_j), \tag{2.11}$$

where $L_m(x)$ are Laguerre polynomials. The expansion coefficients V_m can be determined from the specific form of the interaction,

$$V_m = \int \frac{d^2k}{(2\pi)^2} \tilde{V}(\mathbf{k}) \, L_m(k^2) \, e^{-k^2}, \tag{2.12}$$

where $\tilde{V}(\mathbf{k})$ is the Fourier transform of the potential. As $m = 0$ onsite potential is already taken care of by Pauli exclusion principle for spin-less electrons [5]. For the lowest order approximation, we will assume our effective potential as,

$$V_1 = \sum_{i>j} \nabla_i^2 \delta^{(2)}(\mathbf{r}_i - \mathbf{r}_j) \tag{2.13}$$

[5] Due to the high magnetic field, it is logical to assume spin of the electron is aligned towards magnetic field. We can safely assume the spin degree of electrons is frozen.

It is shown in [24,25], a similar Hamiltonian indeed captures all the universal physics for Laughlin 1/3 state. We will study this Hamiltonian in a complex basis.

$$z = x + iy = re^{i\theta}; \quad \bar{z} = x - iy = re^{-i\theta} \tag{2.14}$$

In this basis, V_1 will have the following form,

$$V_1 = \sum_{i>j} \partial_{z_i}\partial_{\bar{z}_i}\delta(z_i - z_j)\delta(\bar{z}_i - \bar{z}_j) \tag{2.15}$$

In order to qualify this as a good approximation of electron-electron interaction, this should be repulsive or positive-semidefinite in nature. Let us first verify such property for any pair of particles with coordinates $(z_1, \bar{z}_1)$ for the first particle and $(z_2, \bar{z}_2)$ for the second particle.

$$\begin{aligned}
& \langle\phi|\, V_1 \,|\phi\rangle \\
= & \int d^2z_1 d^2z_2 (\partial_{\bar{z}_1}\partial_{z_1}\delta(z_1 - z_2)\delta(\bar{z_1} - \bar{z_2}))\bar{\phi}(z_1, z_2, \bar{z_1}, \bar{z_2})\phi(z_1, z_2, \bar{z_1}, \bar{z_2}) \\
= & \int d^2z_1 d^2z_2 (\partial_{\bar{z_1}}\partial_{z_1}\delta(z_1 - z_2)\delta(\bar{z_1} - \bar{z_2}))\bar{\phi}(z_1, z_2, \bar{z_1}, \bar{z_2})\phi(z_1, z_2, \bar{z_1}, \bar{z_2}) \\
= & \int d^2z_1 d^2z_2 \delta(z_1 - z_2)\delta(\bar{z_1} - \bar{z_2})(\partial_{\bar{z_2}}\bar{\phi}(z_1, z_2, \bar{z_1}, \bar{z_2})\partial_{z_2}\phi(z_1, z_2, \bar{z_1}, \bar{z_2}) + \\
& \bar{\phi}(z_1, z_2, \bar{z_1}, \bar{z_2})\partial_{\bar{z_2}}\partial_{z_2}\phi(z_1, z_2, \bar{z_1}, \bar{z_2}) + \partial_{\bar{z_2}}\partial_{z_2}\partial_{\bar{z_2}}\bar{\phi}(z_1, z_2, \bar{z_1}, \bar{z_2})\phi(z_1, z_2, \bar{z_1}, \bar{z_2})) \\
= & \int d^2z_1 d^2z_2 \delta(z_1 - z_2)\delta(\bar{z_1} - \bar{z_2})(\partial_{\bar{z_2}}\bar{\phi}(z_1, z_2, \bar{z_1}, \bar{z_2})\partial_{z_2}\phi(z_1, z_2, \bar{z_1}, \bar{z_2})) \geq 0
\end{aligned} \tag{2.16}$$

The last line, we get by applying the Fermionic property of ϕ, namely,

$$\delta(z_1 - z_2)\phi(z_1, z_2, \bar{z}_1, \bar{z}_2) = 0 \tag{2.17}$$

Which is just a fancy way of saying, no two particles are allowed in the same position. Where ϕ is any well-behaved wavefunction, i.e, ϕ and ϕ' either goes to zero at infinity or periodic. Notice such interaction for two particle does not depend on their center-of-mass co-ordinate. For two particle state, V_1 can be simplified in terms of their relative co-ordinate ($z = z_1 - z_2$).

$$V_1 = \partial_z \partial_{\bar{z}} \delta(z)\delta(\bar{z}) \tag{2.18}$$

Thus we get

$$\langle \phi | \, V_1 \, | \phi \rangle = \int d^2 z \delta(z)\delta(\bar{z} - \bar{z})(\partial_{\bar{z}} \bar{\phi}(z, \bar{z}) \partial_z \phi(z, \bar{z})) \tag{2.19}$$

In order to have zero energy or ground state, $\partial_z \phi(z, \bar{z}$ must have at least first order of zeros either in z or $\bar{z}$. This will ensure two particle ground state must have the following form,

$$\phi_{G.S} = (z_1 - z_2)^2 f(z_1, \bar{z}_1, z_2, \bar{z}_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) g(z_1, \bar{z}_1, z_2, \bar{z}_2) + (\bar{z}_1 - \bar{z}_2)^2 h(z_1, \bar{z}_1, z_2, \bar{z}_2) \tag{2.20}$$

Where f, g and h are arbitrary functions. Once we introduce the anti-symmetry due to fermionic statistics, we can further simplify the ground state as,

$$\phi_{G.S} = (z_1 - z_2)^2 f(z_1, \bar{z}_1, z_2, \bar{z}_2) + (\bar{z}_1 - \bar{z}_2)^2 g(z_1, \bar{z}_1, z_2, \bar{z}_2) \tag{2.21}$$

Where f and g are anti-symmetric functions. At this point, if we project V_1 to lowest LL. $\bar{z}$ dependence will go away. Thus for any two particle state lowest LL projected Hamiltonian will have the ground state of the following form,

$$V_1 \to \mathcal{P}_{LLL} V_1 \mathcal{P}_{LLL} \implies \phi_{G.S} = (z_1 - z_2)^2 f(z_1, z_2) = (z_1 - z_2)^3 f_s(z_1, z_2) \tag{2.22}$$

In the last line, we have replaced antisymmetric function f by $(z_1 - z_2) f_s$, where f_s is a non-singular symmetric function. This definition is true for any two pair of particles. Hence, N particle ground state of a projected V_1 will have the following form,

$$\phi_{G.S} = \prod_{i>j}^{N} (z_i - z_j)^3 f_s(\{z_1, z_2, \ldots, z_N\}) \tag{2.23}$$

One can immediately see, for $f_s = 1$, we recover Laughlin's wavefunction for $1/3$ state. However, V_1 not only stabilizes Laughlin's state, but we can also construct any number of ground state by choosing different symmetric functions f_s. In plain sight, it seems like a bug. On the contrary, this is, in fact, a very important artifact of this kind of Hamiltonians. In order to fully understand this feature, we must realize, any higher order polynomial in $f(z)$, would actually increase the angular momentum of this state and create quasihole excitations to the system. One can indeed use Schur's Lemma of the symmetric polynomial to show eq. (2.23) describes entire spectrum of quasihole excitations of Laughlin's $1/3$ state [26] for a fixed angular momentum. For future reference, we will describe this property as "counting in zero mode paradigm". Thus, V_1 does not only stabilizes the Laughlin's $1/3$ state, in the ground state it does accommodate a complete set of the quasihole excitations on the densest state. Thus V_1 does successfully capture the entire physics of zero mode paradigm for $1/3$ state. We will refer to such Hamiltonian as parent Hamiltonian of the FQH state.

Parent Hamiltonians for fractional quantum Hall states are positive semidefinite, frustration-free Hamiltonian, which stabilizes the relevant fractional quantum Hall states as the densest ground state and describes the correct "zero mode paradigm" for such state. Zero mode paradigm dictates the number of all possible quasihole excitations on the densest ground state for a given total angular momentum.

Before we reconstruct the zero mode paradigm for Laughlin's state starting from V_1 let us realize, we have so far defined the interaction V_1 for the symmetric gauge in the disk geometry. We, however, claim V_1 describes the universal properties of the Laughlin's state, which are topological in nature. Hence, we must convince ourselves that, such universal properties does not depend on a particular representation of V_1 for a given gauge choice. In order to settle this issue, it is instructive to express V_1 in guiding center basis, such that, we get rid of all the dynamical momentum. As V_1 does not depend on the center-of-mass coordinates, center-of-mass angular momentum $2J$ is a good quantum number for such interaction. Furthermore, due to the semi-positive nature of V_1, we can write,

$$V_1 := \sum_J T_J^\dagger T_J \tag{2.24}$$

In this language, we have reduced the two-dimensional problem to a one-dimensional chain. The cost of such reduction, however, is the following. The short-range interaction

V_1 is replaced by long-range interaction T_J. One can show,

$$T_J = \sum_{x \geq 0} \eta_J(x) c_{0,J-x} c_{0,J+x} \tag{2.25}$$

The exact form of η depends on the choice of geometry, but for the sake of generality, we will consider a generic form of η for time being. Any wavefunction $|\psi\rangle$ must be annihilated by T_J for all J. We will prove the root partition associated with the densest ground state $|\psi\rangle_0$ is ...100100100.. in the following lemmas,

Lemma 1: There is no 11 or 101 in root partitions of any zero mode (ground state) $|\psi\rangle$.

We will use the method of contradiction and the property that any root pattern is, by definition, non-expandable. Now let us assume that a root partition $|\{n_{root}\}\rangle$ contains the string 101 in which 0 has angular momentum j. Then $|\{n_{root}\}\rangle$ can be written as $|\{n_{root}\}\rangle = c^\dagger_{0,J+1} c^\dagger_{0,J-1} |\{n'\}\rangle$, such that,

$$\langle\{n'\}| c^\dagger_{0,J-x} c^\dagger_{0,J+x} |\psi\rangle = 0 \forall |x| > 1 \tag{2.26}$$

Otherwise $|\{n_{root}\}\rangle$ will be expandable. Thus keeping only $x = \pm 1$ terms we get,

$$\langle\{n'\}| T_J |\psi\rangle = \eta_J(1) \langle\{n_{root}\}| |\psi\rangle \neq 0 \tag{2.27}$$

$|\psi\rangle$ being the ground state, the above equation cannot be correct. Thus $|\psi\rangle$ cannot have 101 root partition. Similar thing can also be proved for 11 root partition.

Lemma 2: 1001 root partition is allowed in the root partitions of a zero mode $|\psi\rangle$.

For a ground state $|\psi\rangle$ with the root partition 1001, can be written as

$$|\psi\rangle = (\alpha c^{\dagger}_{0,J-3/2} c^{\dagger}_{0,J+3/2} + \beta c^{\dagger}_{0,J-1/2} c^{\dagger}_{0,J+1/2}) \, |\{n'\}\rangle + \text{ other orthogonal terms.} \quad (2.28)$$

Now, $|\psi\rangle$ being the ground state, we have,

$$\langle\{n'\}| \, T_J \, |\psi\rangle = 0 \implies \alpha\eta_J(3/2) + \beta\eta_J(1/2) = 0 \quad (2.29)$$

The above equation accommodates unique non-trivial solution for $alpha$. Thus 1001 root partition is allowed in the ground state. We, however, already shown $(z_1 - z_2)^3$ is a ground state of V_1, which indeed has 1001 root partition (see eq. (2.7)). This concludes the proof of the existence of 1001 root partition in the ground state. Hence, the densest ground state has a unique root partition ...100100100100... Using a similar argument as Lemma 2, one can show, 1000...0001 is also allowed in the ground state root partition. At this point let us ponder on the implications of the above results. For simple electrons, we have the Pauli exclusion principle, which states that no two electrons can occupy the same guiding center coordinate. For the root partition, we have super exclusion rule, no two electrons can occupy three consecutive sites. Such a rule is called generalized Pauli principle [14, 18, 19, 21, 27–32].

> Generalized Pauli principle (GPP) uniquely characterizes the root partitions for Laughlin states. For $1/m$ Laughlin's state, GPP does not allow more than one electron in consecutive m sites.

The densest root pattern, ...100100100..., ensures only 1/3 of the Laughlin state is occupied. Furthermore, if we assume charge neutrality for the densest state by a uniform

background charge. Each unoccupied site has $1/3$ charge. With this understanding let us put our parent Hamiltonian V_1 in some confining potential. In disk geometry, angular momentum m is related with a wavefunction picked at $\sqrt{2m}$ radius. Hence, total angular momentum can serve as a physical confining potential.

$$H = V_1 + \Delta L; \quad (\Delta L = 0 \text{ for densest state}) \tag{2.30}$$

$\Delta L = 0$ has unique root partition ...1001001001, associated with Laughlin's $1/3$ state. $\Delta L = 1$ can also be achieved uniquely in terms of root partition ...10010010001. Where we have introduces one '0' at the right end. One such '0' has an extra $1/3$ charge, thus introduces a fractional charge excitation to the densest ground state. This root partition is associated with Laughlin's $1/3$ state, multiplied by symmetric N-particle polynomial of order 1. Such polynomial is also uniquely given by,

$$f_1 = \sum_{i=1}^{N} z_i \tag{2.31}$$

For $\Delta L = 2$, we have two ways to construct root partition, ...100100100001 and ...10010001001. There is, however, exactly two independent polynomials of order 2. Namely,

$$f_2 = \sum_{i=1}^{N} z_i^2 \quad \& \quad g_2 = \sum_{i \neq j}^{N} z_i z_j \tag{2.32}$$

Counting all possible quasihole configurations for a particular ΔL, one can see, quasihole configurations have a one-to-one correspondence with particular sets of zeros inserted to the 1-d chain of the densest root partition. we can bosonize these '0' fields in a 1-d chain, and expect a chiral boson to describe the entire quasihole/ edge physics for the Laughlin's state. As, we already have argued these '0' fields have $1/m$ ($m = 3$ for current case) charge, with $1/m$ charge, we expect the chiral boson field to carry $1/m$

charge as well. Later in this section, we will construct a low energy theory at the edge of the system, which will exactly produce the same $1/m$ charged bosonic description. This section, we will, however, conclude our discussion by emphasizing on the fact, that we have a parent Hamiltonian V_1 for Laughlin's state, which stabilizes the $1/3$ state in disk geometry (or in geometry with equivalent topology) as the unique ground state. At the same time, produces a unique description of quasiholes or low energy excitations in a gauge independent way. We can write such parent Hamiltonian, as a positive semi-definite, two particle long range, frustration-free 1-d lattice Hamiltonian, of the following form.

$$V_1 = \sum_J T_J^\dagger T_J; \quad T_J = \sum_{x \geq 0} \eta(x)_J c_{0,J-x} c_{J+x} \tag{2.33}$$

The form of η will vary depending on geometry. In a particular choice of geometry (infinite thick cylinder), η assumes very simple form, $\eta_J(x) = x$. In this particular geometry, let us propose the following parent Hamiltonian, V_M, for Laughlin's $1/M$ state.

In the next section, we will see that above Hamiltonian indeed qualify as the parent Hamiltonian for the Laughlin state. At this point, however, we will reconstruct the Read's order parameter (see eq. 2.4) and comments to on the equivalence of such order parameter description with the parent Hamiltonian picture.

2.3 Read's order parameter approach to study Laughlin's state

Laughlins seminal wave function and subsequent hierarchical constructions [24,33] opened the door for a theoretical understanding of the FQH effect. Due to the exotic nature of

these states, several nonperturbative approaches has been developed to study such states. Read's order parameters [13] in terms of recursion formula in the particle number basis, one of the most promising methods so far exist. This method, although uniquely classifies Laughlin's state topologically, does not get much popularity due to cumbersome use of a mixed first quantized as well as second quantized language. In this section, we will streamline the construction of the recursion formula entirely in second quantized, guiding center basis. In the due process, we will see establish that such recursion formula gives an equivalent description to parent Hamiltonian formalism. Such an exercise was previously done in [11, 34]. We, however, intend to start our construction from a little different point of view, related to the next chapter of this thesis. Laughlin's $1/M+1$ state for N-particle can be described in the following way,

$$\psi_{1/M+1} = \prod_{i>j}(z_i - z_j)^M = \prod_{i>j}(z_i - z_j)^M \Psi_1 = J_N \Psi_1^{\{N\}} \tag{2.34}$$

Where $\Psi_1^{\{N\}}$ describes the non-interactive electrons in the IQH plateau at filling fraction 1. We call J_N the composite fermion vortex attachment operator. This nomenclature will be justified in the next chapter. The heart of this paper will be a second-quantized formula, recursive in particle number N, for the composite fermion vortex attachment operator

$$\begin{aligned} \hat{J}_N : &\psi(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N) \\ &\to N \prod_{1 \le i < j \le N} (z_i - z_j)^M \psi(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N), \end{aligned} \tag{2.35}$$

where M is an even number that we will usually leave implicit, the z_i are the particle's complex coordinates, and we leave room for a (N-dependent) normalization factor N that we will not be interested in. For pedagogical reasons, we will begin our discussion by focusing on the lowest LL ($n = 1$) in this section. A second-quantized recursion relation for

the Laughlin state was given earlier in Ref. [34]. The main difference between the latter and the developments in this section will be that here we establish the recursion *directly* for the Jastrow vortex-attachment operator $\hat{J}_N$ itself. This will descend to the earlier recursion for the Laughlin state. However, the extension to the operator $\hat{J}_N$ will prove essential to the generalization of the recursion formulas to unprojected Jain states (the case $n > 1$).

In this section, we are working entirely lowest LL basis. In order to avoid cumbersome notation, we will drop LL index from particle creation/annihilation operators in this section. Considering, for now, $n = 1$, recall that the N-particle Laughlin state, may be written as

$$|\psi_N\rangle = \hat{J}_N |\Omega_N\rangle , \tag{2.36}$$

where $|\Omega_N\rangle = c_0^\dagger c_1^\dagger c_2^\dagger \cdots c_{N-1}^\dagger |0\rangle$ is an integer quantum Hall state for fermions, and the Bose-Einstein condensate $|\Omega_N\rangle = (c_0^\dagger)^N |0\rangle$ for bosons, we will see that a recursion of $\hat{J}_N$ will descend to recursion of the Laughlin state. (Here, $|0\rangle$ denotes the vacuum state.) Analogous statements will be true for $n > 1$ (Jain states).

The object $\hat{J}_N$ in Eq. (2.35) can be interchangeably viewed as an operator and as a symmetric polynomial in N variables. As such, it can be written as $J_N(z_1, \ldots, z_N)$ or $J_N(p_1, \ldots, p_N)$ (we will stick to the latter), where the $p_k = \sum_{i=1}^N z_i^k$ are power-sum symmetric polynomials. As a by-product, we will clarify the relation between J_N and such power-sum symmetric polynomials, again via recursion. At the operator level, we may then also write

$$\hat{J}_N = J_N(\hat{p}_1, ..., \hat{p}_N), \tag{2.37}$$

where the $\hat{p}_k$ are operator representations of the p_k that facilitate the multiplication of first-quantized wave functions with the symmetric polynomial p_k. Such representations have been discussed at some length in Refs. [10, 11, 34]. They depend slightly on the geometry

(and LL basis), where, with the conventions of the "thick cylinder", one simply has

$$\hat{p}_k = \sum_m c^\dagger_{m+k} c_m \qquad (k \geq 0). \tag{2.38}$$

In this section, we will adopt these thick cylinder conventions for simplicity. Other geometries differ from the above only by normalization conventions that can be implemented via the replacements

$$c_m \to \mathcal{N}_m c_m \,, \quad c^\dagger_m \to \mathcal{N}_m^{-1} c^\dagger_m \,, \tag{2.39}$$

This can be facilitated via the similarity transformation

$$D^{-1}(\ )D, \quad D = \exp(\sum_m \ln(\mathcal{N}_m) c^\dagger_m c_m).$$

We give the normalization constants $\mathcal{N}_m$ for various relevant geometries in Table 2.1. The electron creation/annihilation operators $c^\dagger_m$, c_m refer to lowest LL orbitals with angular momentum m about the quantization axis. The results in this section will be stated in a manner that is valid for both bosonic as well as fermionic commutation relations, except where explicitly stated otherwise. The sum in Eq. (2.38) is generally unrestricted, but we will use the convention $c^\dagger_m = c_m = 0$ for $m < 0$ for the cylinder and disk geometry (thus rendering the cylinder "half-infinite"), and analogous appropriate restrictions for the sphere. It should be emphasized that for many of our purposes, the "first-quantized" interpretation of the operators Eq. (2.38) as power-sum symmetric polynomials do not matter, but indeed the definition (2.38) and the resulting algebraic properties are all that we need. For example, it is trivial to verify that the operators (2.38) all commute ($k \geq 0$!). However, whenever definiteness is required, the term "symmetric polynomial" means a polynomial in the complex coordinates z_i only in disk geometry. On the cylinder, it

means a polynomial in the quantities $\xi_i = \exp(\kappa z_i)$, where κ is the inverse radius of the cylinder. Analogous statements can be made for the spherical geometry, which we will not use explicitly in this work, but refer the reader to Ref. [10] for further details in this context. The reader who wishes to focus on the disk should always have the substitutions (2.39) in mind, which do not affect any of the following algebra.

According to a well-known theorem in algebra, any symmetric polynomial $\mathcal{P}(z_1 \dots z_N)$ in N variables can be uniquely expressed through a polynomial in $p_1, \dots, p_N$ ($\mathcal{P} = \mathbf{P}(p_1, \dots, p_N)$). This includes the p_k for $k > N$. Note, however, that the operators Eq. (2.38) are defined for *any* particle number N, and the aforementioned polynomial relations between the $p_{k\in\{1\dots N\}}$ and the $p_{k>N}$ carry over to the $\hat{p}_k$ only within subspaces of particle number $\leq N$. Similarly, it is convenient to define $\hat{p}_0 = \sum_m c_m^\dagger c_m \equiv \hat{N}$, which, for fixed particle number N, can be viewed as representing a constant (degree zero) polynomial.

Alternatively, any symmetric polynomials in N variables $\mathcal{P}(z_1 \dots z_N)$ can be generated from elementary symmetric polynomials $e_k = \sum_{1\leq i_1<\dots<i_k\leq N} z_{i_1} \cdot \dots \cdot z_{i_k}$, $1 \leq k \leq N$, i.e., $\mathcal{P} = P(e_1, \dots, e_N)$, with P a polynomial. Again, we may ask what second quantized operator facilitates multiplication with e_k. These are [11, 34]

$$\begin{aligned} \hat{e}_k &= \frac{1}{k!} \sum_{l_1,\dots,l_k} c^\dagger_{l_1+1} c^\dagger_{l_2+1} \cdots c^\dagger_{l_k+1} c_{l_k} \cdots c_{l_2} c_{l_1}, \\ &\text{with} \quad \hat{e}_0 := \mathbb{1}, \end{aligned} \tag{2.40}$$

given here again for the simple thick cylinder conventions, with disk conventions as detailed in Eq. (2.39) and Table 2.1. It is worth noting that unlike the p_k, the e_k vanish automatically for $k > N$. This is respected by the operators $\hat{e}_k$, which automatically vanish on any state with particle number $N < k$. The $\hat{e}_k$ and the $\hat{p}_k$ are related by the Newton-Girard

Table 2.1: Normalization constants $\mathcal{N}_m$ for various geometries. κ is the inverse radius of the cylinder $\kappa = 1/R_y$. R is the radius of the sphere and N_Φ is the number of flux quanta threading the sphere.

	disk	cylinder	sphere
$\mathcal{N}_m$	$\frac{1}{\sqrt{2^m m!}}$	$\exp(-\frac{1}{2}\kappa^2 m^2)$	$\frac{1}{(2R)^{m+1}}\sqrt{\binom{N_\Phi}{m}}$

formulas,

$$\hat{e}_k = \frac{1}{k}\sum_{d=1}^{k}(-1)^{d-1}\hat{p}_d\hat{e}_{k-d}. \tag{2.41}$$

These can be directly derived [11] from the operator definitions (2.38) and (2.40), without any reference to the "polynomial interpretation" of these operators. Clearly, Eq. (2.41) is invariant under the similarity transformation leading to Eq. (2.39), thereby seen to be geometry independent even if we did not know about its meaning in terms of polynomials. Eq. (2.41) may first be used for $k \leq N$ to express all $\hat{e}_{k\leq N}$ through $\hat{p}_{k\leq N}$. Subsequently, letting $\hat{e}_{k>N} \equiv 0$, it can be used to explicitly obtain the identities for the $\hat{p}_{k>N}$ in terms of the $\hat{p}_{k\leq N}$ mentioned above, valid within the subspace of particle number $\leq N$. Independent of N, it is also obvious from these relations that the $\hat{e}_k$ commute with one another (as the $\hat{p}_k$ do), and also commute with all of the $\hat{p}_k$ (for the same reason).

We will now derive a second-quantized recursive formula for $\hat{J}_N$, which turns out to be straightforward to generalize to higher Landau levels. At the polynomial level, we will also clarify the relation between the Laughlin-Jastrow factor Eq. (2.35) and power-sum symmetric polynomials. More precisely, we will give a recursive operator definition of $\hat{J}_N$ *both* through electron creation/annihilation operators as well as in terms of polynomial expressions in the p_k.

We begin by stating a technical lemma.

Lemma 0. Let $\mathbf{P}(p_0, p_1, \ldots, p_N)$ be a polynomial in $N+1$ variables. The operator $\mathbf{P}(\hat{p}_0, \hat{p}_1, \ldots, \hat{p}_N)$ obtained by substituting the operators $\hat{p}_k$, Eq. (2.38), for p_k satisfies

$$\begin{aligned} & c_k^\dagger \mathbf{P}(\hat{p}_0, \hat{p}_1, \ldots, \hat{p}_N) \\ = & \sum_{l_0, l_1, \ldots, l_N} \frac{(-1)^{l_0+l_1+\cdots l_N}}{l_0! l_1! \cdots l_N!} \left(\partial_{p_0}^{l_0} \cdots \partial_{p_N}^{l_N} \mathbf{P} \right) (\hat{p}_0, \hat{p}_1, \ldots, \hat{p}_N) \\ & \times c^\dagger_{k+l_1+2l_2+\cdots+Nl_N}. \end{aligned} \tag{2.42}$$

Note: We will often be interested only in the action of operators such as $\mathbf{P}$ within the subspace of fixed particle number N. In this context it may not be warranted to have explicit dependence on $\hat{p}_0$, which is then just a constant, and representing the constant part of $\mathbf{P}$ through $\hat{p}_0$ may be considered redundant/unnecessary. It is, however, easy to specialize the lemma to the case of no dependence on $\hat{p}_0$.

Proof of Lemma 0: We start by noting

$$[c_k^\dagger, \hat{p}_r] = -c^\dagger_{r+k}, \tag{2.43}$$

trivially obtained from (2.38), for both fermions and bosons. We first prove Eq. (2.42) for the case of powers of the form $\mathbf{P} = \hat{p}_r^d$, by induction in d, then prove the case of general polynomials by induction in N. For this proof, we will not distinguish between the variables p_r and the operators $\hat{p}_r$ for notational convenience. Considering now $\mathbf{P} = p_l^d$, we see that Eq. 2.42 is trivially satisfied for $d = 0$. Assuming Eq. (2.42) is satisfied for p_r^{d-1},

we have

$$
\begin{aligned}
& c_k^\dagger p_r^d \\
&= [c_k^\dagger, p_r] p_r^{d-1} + p_r (c_k^\dagger p_r^{d-1}) \\
&= -c_{k+r}^\dagger p_r^{d-1} + p_r \sum_l \frac{(-1)^l}{l!} \left(\partial_{p_r}^l p_r^{d-1} \right) c_{k+rl}^\dagger \\
&= \sum_l \frac{(-1)^l}{l!} \left(l \partial_{p_r}^{l-1} p_r^{d-1} + p_r \partial_{p_r}^l p_r^{d-1} \right) c_{k+rl}^\dagger \\
&= \sum_l \frac{(-1)^l}{l!} \left(\partial_{p_r}^l p_r^d \right) c_{k+rl}^\dagger,
\end{aligned}
\tag{2.44}
$$

where we used induction in the third and fourth line, and $\partial_x^l x^d = l \partial_x^{l-1} x^{d-1} + x \partial_x^l x^{d-1}$ in the last. Having proven Eq. 2.42 for simple powers of the p_r, we now prove it for general polynomials by simple induction in N. By linearity, it is sufficient to consider monomials. Assume hence that Eq. 2.42 is true for $\mathbf{P} = p_{N-1}^{m_{N-1}} \cdots p_0^{m_0}$. We have

$$
\begin{aligned}
& c_k^\dagger p_N^{m_N} p_{N-1}^{m_{N-1}} \cdots p_0^{m_0} \\
=& \sum_{l_N} \frac{(-1)^{l_N}}{l_N!} (\partial_{p_N}^{l_N} p_N^{m_N}) c_{k+Nl_N}^\dagger p_{N-1}^{m_{N-1}} \cdots p_0^{m_0} \\
=& \sum_{l_N, l_{N-1}, \ldots, l_0} \frac{(-1)^{l_0 + l_1 + \cdots l_N}}{l_0! l_1! \cdots l_N!} \\
& \times \left(\partial_{p_0}^{l_0} \cdots \partial_{p_N}^{l_N} p_N^{m_N} p_{N-1}^{m_{N-1}} \cdots p_0^{m_0} \right) c_{k+l_1+2l_2+\cdots+Nl_N}^\dagger.
\end{aligned}
\tag{2.45}
$$

This concludes our induction proof □.

We now define some useful operators:

$$
\begin{aligned}
& \hat{S}_\ell = (-1)^\ell \sum_{n_1+n_2+\cdots+n_M=\ell} \hat{e}_{n_1} \hat{e}_{n_2} \cdots \hat{e}_{n_M} \quad \text{for} \quad \ell \geq 0, \\
& \hat{S}_\ell = 0 \quad \text{for} \quad \ell < 0.
\end{aligned}
\tag{2.46}
$$

Note that, again, the $\hat{S}_\ell$ also depend on M, the "flux attachment" parameter defined in Eq. (2.35), which we usually leave implicit. In the next section, we will connect these S_ℓ operators to the physcial operator in a rational conformal field theory. Such a connection will establish a one-to-one correspondence between this order parameter recursion relation and the matrix product treatment of Laughlin's state. It is important at this point to note that, $\hat{S}_\ell$ obey a M-deformed Newton-Gerard formula.

$$\hat{S}_\ell \;=\; (-1)^\ell \sum_{n_1+n_2+\cdots+n_M=\ell} \hat{e}_{n_1}\hat{e}_{n_2}\cdots\hat{e}_{n_M} = \frac{(-1)^\ell}{l} \sum_{n_1+n_2+\cdots+n_M=\ell} (n_1+n_2+\cdots+n_M)\hat{e}_{n_1}\hat{e}_{n_2}\cdots\hat{e}_{n_M}$$

This can be further simplified as,

$$\hat{S}_\ell \;=\; \frac{M(-1)^\ell}{l} \sum_{n_1+n_2+\cdots+n_M=\ell} n_1\hat{e}_{n_1}\hat{e}_{n_2}\cdots\hat{e}_{n_M} \tag{2.47}$$

Using the Newton-Gerard relation for e_{n_1}, we can write,

$$\hat{S}_\ell = \frac{M(-1)^\ell}{\ell} \sum_{d=1}^{\ell} \sum_{n+n_2+\cdots+n_M=\ell-d} \hat{p}_d(-1)^{d-1}\hat{e}_n\hat{e}_{n_2}\cdots\hat{e}_{n_M} = \frac{M}{\ell}\sum_{d=1}^{\ell}(-1)^{d-1}\hat{p}_d\hat{S}_{\ell-d} \tag{2.48}$$

With the help of these $\hat{s}_\ell$, we now define the following operator recursion:

$$\begin{aligned} \hat{J}_0 &= \mathbb{1}, \\ \hat{J}_N &= \frac{1}{N}\sum_{r\geq 0}\sum_{m\geq 0} c^\dagger_{m+r}\hat{S}_{M(N-1)-r}\hat{J}_{N-1}c_m, \end{aligned} \tag{2.49}$$

From this definition, it is not immediately obvious that the operator $\hat{J}_N$ is of the form Eq. (2.37), i.e., is a polynomial in the $\hat{p}_{k\leq N}$. Our first goal will be to prove precisely that. This then has two important consequences: 1. Any operator that commutes with all the $\hat{p}_k$

also commutes with $\hat{J}_N$ and moreover, 2. the operator $\hat{J}_N$ acts on N-body wave functions via multiplication with a certain symmetric polynomial, since all the $\hat{p}_k$ have this property. We will then establish that this polynomial is, up to a normalization, the Laughlin-Jastrow flux-attachment factor, Eq. (2.35).

To see this, we assume $\hat{J}_{N-1} = J_{N-1}(\hat{p}_1, \ldots, \hat{p}_{N-1})$, J_{N-1} a polynomial. This induction assumption is obviously true for $\hat{J}_0$. We may then use Eq. (2.42) to get the following:

$$\begin{aligned}\hat{J}_N =& \frac{1}{N}\sum_{r,m}\sum_{l_1,\ldots,l_{N-1}} \frac{(-1)^{l_1+\cdots l_{N-1}}}{l_1!\cdots l_{N-1}!} \\ & \times \left(\partial_{p_1}^{l_1}\cdots\partial_{p_{N-1}}^{l_{N-1}} S_{M(N-1)-r} J_{N-1}\right)\Big|_{p_1\to\hat{p}_1,\ldots} \\ & \times c^\dagger_{m+r+l_1+2l_2+\cdots+(N-1)l_{N-1}} c_m \\ =& \frac{1}{N}\sum_{r}\sum_{l_1,\ldots,l_{N-1}} \frac{(-1)^{l_1+\cdots l_{N-1}}}{l_1!\cdots l_{N-1}!} \\ & \times \left(\partial_{p_1}^{l_1}\cdots\partial_{p_{N-1}}^{l_{N-1}} S_{M(N-1)-r} J_{N-1}\right)\Big|_{p_1\to\hat{p}_1,\ldots} \\ & \times \hat{p}_{r+l_1+2l_2+\cdots+(N-1)l_{N-1}}.\end{aligned} \tag{2.50}$$

In writing the above, S_ℓ is a polynomial such that $\hat{S}_\ell = S_\ell(\hat{p}_1, \ldots, \hat{p}_{N-1})$ *when acting on states of* $N-1$ *particles or less.* We can always achieve this, as explained earlier, by expressing the $\hat{e}_{k\le N-1}$ through the $\hat{p}_{k\le N-1}$ in Eq. (2.46), and letting the $\hat{e}_{k\ge N}$ equal to zero. (Note that if $\hat{J}_N$ acts on N-particle states, then $\hat{J}_{N-1}$ in Eq. (2.49) acts on $N-1$ particle states.) We may similarly express all the terminal $\hat{p}$-operators in the last line of Eq. (2.50) through the $\hat{p}_{k\le N}$. With these replacements, the difference between Eq. (2.49) and Eq. (2.50) strictly speaking vanishes only on states with particle number $\le N$. However, since we will exclusively be interested in the action of $\hat{J}_N$ on states with N particles, this difference can be ignored in the following. Anticipating that the last two equations really define

the composite fermion operator (2.35), we see that Eq. (2.50), viewed as an equation for symmetric polynomials (i.e., omitting hats) gives a recursive definition of the (even M) Laughlin-Jastrow factor in terms of power-sum symmetric polynomials. In this polynomial sense, Eq. (2.50) must of course be correct independent of the number of LLs kept, *unlike* the operator definitions given in this section, which so far stand only for the lowest LL. Working backwards from Eq. (2.50), we will be able to generalize the operator recursion (2.49) to higher Landau levels.

Before we do this, we give applications of Eq. (2.49) within the lowest LL, and in doing so, establish correspondence with Eq. (2.35). Consider now fermions and the N-particle state

$$|\psi_N\rangle = \hat{J}_N c_0^\dagger c_1^\dagger \cdots c_{N-1}^\dagger |0\rangle . \tag{2.51}$$

We will use Eq. (2.35) to re-establish a recursive relation for this state, from which, via Ref. [34] it is then known that Eq. (2.51) defines the densest zero mode of a pseudopotential Hamiltonian (for $M = 2$, the V_1 Haldane pseudo-potential), thus identifying it uniquely as the $1/(M+1)$-Laughlin-state.

From the definition of $\hat{J}_N$ in Eq. 2.49, we can prove the following identity

$$c_r \hat{J}_N = \sum_m \hat{S}_{M(N-1)-r+m} \hat{J}_{N-1} c_m . \tag{2.52}$$

The proof of Eq. 2.52 is given in Appendix A.1. Using Eq. 2.52, we obtain

$$\begin{aligned} c_r |\psi_N\rangle = & \sum_m \hat{S}_{M(N-1)-r+m} \hat{J}_{N-1} (-1)^m \\ & \times c_0^\dagger \cdots c_{m-1}^\dagger c_{m+1}^\dagger \cdots c_{N-1}^\dagger |0\rangle . \end{aligned} \tag{2.53}$$

We observe that $c_0^\dagger \cdots c_{m-1}^\dagger c_{m+1}^\dagger \cdots c_{N-1}^\dagger |0\rangle$ is just

$$\hat{e}_{N-1-m} c_0^\dagger c_1^\dagger \cdots c_{N-2}^\dagger |0\rangle \tag{2.54}$$

using the definition of $\hat{e}_k$ in Eq. 2.40. Thus we have

$$c_r |\psi_N\rangle = \sum_m \hat{S}_{M(N-1)-r+m} (-1)^m \hat{e}_{N-1-m} |\psi_{N-1}\rangle \tag{2.55}$$

in which we have used that $\hat{J}_{N-1}$, being a polynomial in the $\hat{p}_k$, commutes with $\hat{e}_{N-1-m}$. The latter can be written more suggestively after defining

$$\begin{aligned} \hat{S}_\ell^\sharp &= (-1)^\ell \sum_{n_1+n_2+\cdots+n_{M+1}=\ell} \hat{e}_{n_1} \hat{e}_{n_2} \cdots \hat{e}_{n_{M+1}} \quad \text{for} \quad \ell \geq 0, \\ \hat{S}_\ell^\sharp &= 0 \quad \text{for} \quad \ell < 0, \end{aligned} \tag{2.56}$$

i.e., $\hat{S}_\ell^\sharp$ is defined just as $\hat{S}_\ell$ but with the odd number $M+1$ replacing the even number M. With this we can rewrite Eq. (2.55) as

$$c_r |\psi_N\rangle = (-1)^{N-1} \hat{S}^\sharp_{(M+1)(N-1)-r} |\psi_{N-1}\rangle , \tag{2.57}$$

which, up to a constant $(-1)^{N-1}$ amounting to a convention, is the same as that obtained in Ref. [34] for the Laughlin state with filling fraction $1/(M+1)$. This formula and its generalizations will be crucial in much of the following. It should be read as follows: The operator c_r creates a (charge 1) hole of well-defined angular momentum. Due to bulk-edge correspondence, such a hole can always be interpreted as an edge excitation of the $N-1$ particle incompressible state, though one possibly living deeply in the bulk of the system. As we've explained elsewhere, [11, 34] the operator $\hat{S}_\ell^\sharp$ and the $\hat{e}_k$ it is composed of should be thought of as generators of such edge excitations when acting

on the incompressible state. To make these notions more precise, one may consider a pseudo-potential Hamiltonian of the form [24]

$$H = V_1 + V_3 + \ldots + V_{M-1}, \tag{2.58}$$

where the positive operator V_k is (proportional to) the kth Haldane pseudo-potential. It is well-known that the $1/(M+1)$-Laughlin state is the densest zero energy mode (zero mode) of this Hamiltonian, and one may *define* quasi-hole/edge excitations as the set of all other zero modes of the same Hamiltonian. It is easy to see [34] that the left-hand side of Eq. (2.57) is a zero mode if $|\psi_N\rangle$ is, and the $\hat{e}_k$ can be shown [11] to generate a complete set of zero modes of the same particle number when acting on the incompressible $1/(M+1)$-Laughlin state. Eq. (2.57) is the precise way to express the charge-1 quasi-hole $c_r|\psi_N\rangle$ in this manner.

At this point, a recursion for the Laughlin state can be obtained following the logic of Ref. [34]. Applying the operator $c_r^\dagger$ to Eq. (2.57) and summing over r produces a factor of the particle number N on the left hand side. Dividing by this factor gives

$$|\psi_N\rangle = \frac{1}{N}\sum_r (-1)^{N-1}\, c_r^\dagger \hat{S}^\sharp_{(M+1)(N-1)-r}\, |\psi_{N-1}\rangle \ . \tag{2.59}$$

This recursion, with $|\psi_1\rangle = c_0^\dagger|0\rangle$, has been shown in Ref. [34] to give the densest (lowest angular momentum) zero mode of the Hamiltonian (2.58), thus uniquely identifying the $|\psi_N\rangle$, Eq. (2.51), as the $1/(M+1)$-Laughlin state (defined up to an overall constant). As we have shown above, the effect of the operator $\hat{J}_N$ on *any* N-particle state is the multiplication of the state's wave function with a fixed symmetric polynomial $J_N(p_1, \ldots, p_N)$. We may find this polynomial by looking at Eq. (2.51), which we now know to be the Laughlin

state. From this equation, we thus have

$$N\prod_{i<j}(z_i - z_j)^{M+1} = J_N(p_1,\ldots,p_N)\prod_{i<j}(z_i - z_j)\,, \tag{2.60}$$

where the left hand side is the $1/(M+1)$-Laughlin state, on the right hand side we used that $c_0^\dagger c_1^\dagger \cdots c_{N-1}^\dagger |0\rangle$ in Eq. (2.51) is just a Vandermonde determinant, and we dropped Gaussian factors on both sides. This determines the polynomial $J_N(p_1,\ldots,p_N)$ to be the Laughlin-Jastrow factor in Eq. (2.35). The same derivation is possible for bosons with very few changes.

We will end this section by pointing out such order parameter recursion relation, we have discussed so far, can be used to construct parent Hamiltonian description. In order to establish such claim, we should show that the ground state of our proposed parent Hamiltonian V_M is spanned by Laughlin's $1/M$ state as well as all quasihole excitations of it. Now,

$$V_M = \sum_{m\le M} T^{m\dagger}T^m_J; \quad T^m_J = \sum_{x\ge 0} x^m c_{0,J-x}c_{J+x} \tag{2.61}$$

One can easily prove,

$$[T^m_J, \hat{p}_d] = T^m_{J-d/2} \implies \text{If } T^m_J|\psi\rangle = 0 \implies T^m_J(\hat{p}_d|\psi\rangle) = 0. \tag{2.62}$$

Thus our proposed Hamiltonian is indeed had the infinite number of ground state, given that at least one ground state exists. Furthermore, any generic fermionic bilinear $\mathcal{T}_J$ satisfies the following relation,

$$\mathcal{T}_J = \frac{1}{2}\sum_{m,k}[\mathcal{T}_J, c^\dagger_{m,k}]c_{m,k}\,. \tag{2.63}$$

Now let us assume, a N particle state $|\psi_N\rangle$ is a ground state of our Hamiltonian, the above equation suggests the following relation [34],

$$|\psi_{N+1}\rangle = \frac{1}{N+1}\sum_m c_m^\dagger c_m |\psi_{N+1}\rangle \implies T_J^m |\psi_{N+1}\rangle = \frac{2}{N+1}|\psi_{N+1}\rangle\,, \text{ if } T_J^m |\psi_{N+1}\rangle \quad (2.64)$$

Thus, if we establish Laughlin's $1/M$ state as two particle ground state of the V_m parent Hamiltonian, we prove that the N particle Laughlin's state and all of the quasihole excitations live in the ground state of our proposed Hamiltonian. Now a two-particle Laughlin's state can be written as,

$$|\psi_2\rangle_{1/M} = \frac{1}{2}\sum_{r\geq 0}(-1)^{M-r}\binom{M}{r} c_r^\dagger c_{M-r}^\dagger |0\rangle \quad (2.65)$$

One can show [34],

$$T_J^m = \delta_{M,2J}(-1)^{M+2J}\sum_x x^m(-1)^x\binom{M}{x}|0\rangle = 0 \;\; \forall m < M \quad (2.66)$$

The last relation establishes $1/M$ Laughlin state is indeed a ground state of the parent Hamiltonian V_M. Now, in order to establish this state as the densest ground state, one must construct the GPP [10] for V_M interaction. At this end, one indeed sees V_M serves as the parent Hamiltonian of the $1/M$ Laughlin's state. While proving this result we establish that order parameter recursion formula, indeed, gives rise to an equivalent description to the parent Hamiltonian approach.

2.4 Conformal field theory/ matrix product state description of Laughlin's states

In this last section of the chapter, we want to elucidate the connection of the different microscopic, many-body techniques, to the effective field theory for Laughlin's state. We will start our discussion by pointing out the following relations,

$$\psi_{1/M} = \prod_{i>j}(z_i - z_j)^M e^{-\sum_i z_i \bar{z}_i}/4; \quad \langle \phi(z)\phi(z')\rangle_{plane} = -log(z - z') \tag{2.67}$$

The first equation, states the good old Laughlin's state, while the second equation gives the two-point correlation functions for free field chiral boson ϕ. Hence, one can write the Laughlin's state [35] as,

$$\psi_{1/M} = \left\langle \exp[i\sqrt{M}(\sum_{a=1}^{N}\phi(z_a) - \rho\int d^2z\phi(z)] \right\rangle \tag{2.68}$$

$\phi(z)$ is a chiral bosonic field with compactification radius $2\pi\sqrt{M}$. The second term in the exponential comes from, charge neutrality condition, which is imposed by a constant background charge ρ. It can be shown [36], this back-ground charge gives rise to the trivial Gaussian factor in the wavefunction. For the rest of the discussion, we will assume this term is present implicitly. We are more interested, however, to the analysis of the first term in the exponential. One can write Laughlin state as N-point correlation function,

$$\psi_{1/M} = \langle V(z_1)V(z_2)...V(z_N)\rangle_N; \quad V(z) =: \exp[i\sqrt{M}\phi(z)] : \tag{2.69}$$

, Where $V(z)$ are physically relevant electron operators. : . : defines the normal ordered or path ordered product, depending on the choice of coordinates. $\langle\rangle_N$ defines the charge

neutral N point correlation. One can mode expand, $V(z)$ as,

$$V(z) = \sum_{n\in\mathbb{Z}} V_{-n-h} z^n \tag{2.70}$$

$h = M/2$ is the conformal charge of $V(z)$ [37,38]. Thus,

$$\psi_{1/M} := \sum_{\{\lambda_i\}} \langle V_{-\lambda_1-h} V_{-\lambda_2-h} \dots V_{-\lambda_N-h}\rangle_N z_1^{\lambda_1} z_2^{\lambda_2} \dots z_N^{\lambda_N} |0\rangle \tag{2.71}$$

Rearranging the above equation in Slater determinant basis, one can write $\psi_{1/M}$ is entirely second quantized, guiding center language.

$$\psi_{1/M} = \sum_{\{\lambda_i\}} \langle V_{-\lambda_1-h} V_{-\lambda_2-h} \dots V_{-\lambda_N-h}\rangle_N c_{\lambda_1}^\dagger c_{\lambda_2}^\dagger \dots c_{\lambda_N}^\dagger \tag{2.72}$$

The above equation gives an exact matrix product state definition, for Laughlin's state. In order to see that, explicity, one must realize these V_λ operators are defined in terms of chiral bosonic operator, $\phi(z)$. One can mode expand $\phi(z)$ [37],

$$\phi(z) = \phi_0 - i a_0 \log(z) + i \sum_{n\neq 0} \frac{1}{n} a_n z^{-n}; \quad [a_n, a_{-m}] = n\delta_{n,m}; \quad [a_n, V_{-\lambda-h}] = \sqrt{M} V_{-\lambda+n-h} \tag{2.73}$$

The first two terms of this expansion takes care of charge neutrality. As we have already assume the charge neutrality, we will ignore first two terms of the expansion, for the discussion in this section. Hence $V_{-\lambda-h}$ can be expressed a matrix in an infinite auxiliary basis $|\{n_i\}\rangle$. Where $|\{n_i\}\rangle$ can be defined as,

$$|\{n_i\}\rangle \sim \prod_{j>0, j\in\{n_i\}} a_{-j} |0\rangle \tag{2.74}$$

$|0\rangle$ is defined as a vacuum, which gets annihilate by all the a_js for $j > 0$. Using the definition of $V(z)$, let us expand it in terms of the auxiliary field, a_ns.

$$V(z) = e^{-\sqrt{M}\sum_n \frac{a_n}{n} z^{-n}} = \sum_\lambda V_{-\lambda-h} z^\lambda \tag{2.75}$$

Taking derivative on both sides, we get

$$V_{-\lambda-h} = -\frac{M}{-\lambda} \sum_{n\geq 1} \frac{a_n}{\sqrt{M}} v_{-\lambda-h-n} \tag{2.76}$$

The Last equation is the same M-deformed Newton-Gerard algebra, constructed for quasi-hole operator $\hat{p}_d$ and physically relevant elementary symmetric polynomial operator $\hat{S}_\ell$ (see eq. (2.48)). Such a similarity is not actually a coincidence. One can indeed show, matrix product state has a one-to-one correspondence to the order parameter recursion relation [6].

At this point, we want to conclude this chapter with the following message.

[6] M Schossler, S Bandyopadhyay, A Seidel, manuscript under preparation

Electron interaction under a high magnetic field gives rise to exotic emergent phases. In order to understand such phases, one often needs non-perturbative approaches. In this chapter, we have reviewed different methodologies, existing in the literature. Each of these methodologies is motivated from a different perspective of the physical systems, under study. For a broad class of such exotic phases, namely, Laughlin states, however, we have shown that such methods are not quite independent. One such method rather can be constructed starting from another.

Chapter 3

More Exotic Fractional Quantum Hall states: Composite Fermions

[7] laughlin's construction of ansatz wavefunction does a wonderful job in describing fractional Hall plateaus at $1/M$ filling. Existing methods, such as parent Hamiltonian description, order parameter recursion, conformal field theory/ matrix product construction, also give us a comprehensive understanding of the non-perturbative physics behind such exotic phases. This would be a perfect ending of the story, only if there were no plateaus at other filling fractions. There are, however, many more states, experimentally observed, but does not follow Laughlin's state paradigm. Figure. 1.2, indeed shows

[7] In this chapter, we will develop the extension of the existing methods to composite fermions. Existing methods, such as parent Hamiltonian, topological order parameter, bosonization from conformal field theory, are well established in analytical wavefunction framework of Laughlin's state. In this chapter, we denounce the importance of analytic properties of the lowest Landau level wavefunction in each methodology and extend the idea to higher Landau levels. Such an extension naturally explains many other, more exotic, fractional quantum Hall states. Concepts and results in this section are reproduced from two separate collaborations: One with L. Chen and K. Yang from National Magnetic Lab, Tallahassee [12, 39]. Another collaboration with L.Chen, Z. Nussinov, G. Ortiz (manuscript under preparation).

such plateaus at filling fraction $2/5, 3/7\ 4/9$. Following the logic behind Laughlin's construction, one can, however, construct a large set of state [40], namely, Jain's composite fermions. These states are numerically verified to have a large overlap with the wavefunctions at the plateau with a filling fraction of the form $n/(nM+1)$. Such states can explain most of the plateau, observed in experiments. Understanding the physics behind such states is imperative to develop a comprehensive idea about FQH states. It is, however, worth reviewing the variational construction of these states, before investigating them further. Jain's composite fermion N-particle wavefunction for $n/(nM+1)$ can be constructed as,

$$\psi_{n/(nM+1)} = J_N \Psi_1 \tag{3.1}$$

Where, Ψ_1 is the n-LL filled N-particle IQH wavefunction, given by Eq. (1.24). J_N is the composite fermion vertex attachment operator, given by Eq. (2.35). Under the symmetric gauge in the planar geometry, J_N is given by the product of M Jastrow factors for N particle. For $n = 1$, the lowest LL, one can restore the Laughlin's construction of the wavefunction. In this brief introduction to composite fermion, however, a reader can realize one difficulty, we must face while studying such state. For any $n > 1$, LLs lose their analytic properties (see Eq. (1.12)). Higher LLs depend on z as well as $\bar{z}$. Connecting such state to an N-point correlation function for chiral CFT is an non-trivial task. One can construct a holomorphic variant of composite fermion by projecting it down to lowest LL state. Such a construction numerically have good overlap with the actual wavefunction. This lowest LL projection, however, has no known close tractable polynomial description. A lattice of Read's topological order parameters [41] for a composite fermion state can be determined in the field theoretical sense. Such an order parameter actually helps us to construct an effective field theory [42] for composite fermions but a connection of relevant field theory to microscopic description remains unclear. Due to the nonanalytic nature,

until recently [12], no parent Hamiltonian description was not there in the literature. Only for 2/5 state, such a parent Hamiltonian (Trugman-Kivelson Hamiltonian) was known from numerical calculations.

The above discussion should convince readers that, most if not of all the well established non-perturbative approaches, which worked in harmony for Laughlin's state, measurably fails for composite fermions. In this chapter, we will remedy this, by denouncing the importance of the analytic structure of the wavefunction and reformulating the existing methodology in second-quantized guiding center coordinates. We will start our discussion by rigorously establishing [39] Trugman- Kivelson the parent Hamiltonian for 2/5 state.

3.1 Genralized Trugman-Kivelson Hamiltonian: A Second quantized description in disk geometry for 2 LLs

In this section and a few after that, we will be concerned with the two-body Trugman-Kivelson interaction [25]

$$H = P_n \, \nabla_1^2 \delta \left(x_1 - x_2 \right) \delta \left(y_1 - y_2 \right) P_n \, , \tag{3.2}$$

projected onto the first n Landau levels via an orthogonal projection operator P_n, focusing on the case where $n = 2$. For $n = 1$, it is well known that this interaction agrees, up to a factor, with the V_1 Haldane pseudopotential. [24] The case $n = 2$ was identified by Rezayi and MacDonald [43] as a parent Hamiltonian for the Jain-2/5 state, where at the same time, the kinetic energy is quenched not only within individual Landau levels, but the splitting between the lowest and first excited Landau level is set to zero. Here we will

mainly be concerned with the properties of this ($n = 2$) Hamiltonian. Results for the case $n = 3$ have appeared recently. [44] The extension of the methods developed below to general n is left to a forthcoming paper [8].

As a starting point, we establish a second quantized form of the Hamiltonian in various geometries, beginning with the disk geometry. For positive, angular momentum conserving two-particle operators, the second quantized many-body Hamiltonian is generally [10] of the form

$$H = \sum_{k=1}^{M} \sum_{J} \mathcal{T}_J^{(k)\dagger} \mathcal{T}_J^{(k)}, \tag{3.3}$$

where $\mathcal{T}_J^{(k)} = \sum_x f_{i,j}^k(J,x) c_{i,J-x} c_{j,J+x}$ destroys a pair of particles with well defined angular momentum $2J$, $c_{i,m}$ is an electron destruction operator for a state in the ith Landau level (LL) with angular momentum m, and $f_{i,j}^k(J,x)$ is a form factor defining the operator $\mathcal{T}_J^{(k)}$. In Eq. (3.3), The sum over R is over integer and half-odd integer values, and x in the definition of $\mathcal{T}_J^{(k)}$ is *either* over integer *or* half-odd integer, depending on J (i.e., $2x \equiv 2J$ mod 2). In the most general case, the number M of families of $\mathcal{T}$-operators can be infinite.

We now work out the connection between Eqs. (3.2) and (3.3) specializing to $n = 2$ Landau levels (carrying Landau level indices 0 and 1, respectively). To this end, we recall the wave functions for a single particle in the disk with angular momentum $L_z = m$ in the lowest and first excited LLs under symmetric gauge,

$$\eta_{0,m}(z) = \frac{z^m e^{-|z|^2/4l_B^2}}{\sqrt{2\pi 2^m l_B^{2m+2} m!}} \tag{3.4}$$

and

$$\eta_{1,m}(z) = \frac{\left(\bar{z} z^{m+1} - 2l_B^2 (m+1) z^m\right) e^{-|z|^2/4l_B^2}}{\sqrt{2\pi 2^{m+2} l_B^{2m+6} (m+1)!}}, \tag{3.5}$$

[8] M. T. Ahari, S. Bandyopadhyay, Z. Nussinov, A. Seidel, G. Ortiz, Manuscript under preparation

respectively, where $z = x + iy$ is the complex coordinate on the disk, and l_B is magnetic length $\sqrt{\hbar/eB}$. As an immediate consequence, we have the following analytic structure for general *two-particle* wave functions projected onto the first two LLs,

$$\begin{aligned}\psi(z_1, z_2) =& \Big(C_{00}(z_1, z_2) + \bar{z}_1 C_{10}(z_1, z_2) + \bar{z}_2 C_{01}(z_1, z_2) \\ &+ \bar{z}_1 \bar{z}_2 C_{11}(z_1, z_2) \Big) e^{-\frac{|z_1|^2}{4l_B^2} - \frac{|z_2|^2}{4l_B^2}},\end{aligned} \tag{3.6}$$

where $C_{00}(z_1, z_2), C_{10}(z_1, z_2), C_{01}(z_1, z_2)$ and $C_{11}(z_1, z_2)$ are holomorphic functions of z_1 and z_2. For two-particle states, it is generally advantageous to phrase expressions in terms of a center-of-mass coordinate $z_c = (z_1 + z_2)/2$ and a relative coordinate $z_r = z_1 - z_2$, and their complex conjugates $\bar{z}_c$, $\bar{z}_r$. Furthermore, in this paper, we will be exclusively considering fermions. Then, Eq. (3.6) can be recast as

$$\begin{aligned}\psi(z_c, z_r) =& \Big(d_{00}(z_c, z_r) + \bar{z}_c d_{10}(z_c, z_r) + \bar{z}_r d_{01}(z_c, z_r) \\ &+ (\bar{z}_c^2 - \bar{z}_r^2/4) d_{11}(z_c, z_r) \Big) e^{-\frac{|z_c|^2}{2l_B^2} - \frac{|z_r|^2}{8l_B^2}},\end{aligned} \tag{3.7}$$

where $d_{00}(z_c, z_r), d_{10}(z_c, z_r), d_{01}(z_c, z_r)$ and $d_{11}(z_c, z_r)$ are holomorphic functions of z_r and z_c with well-defined parity in z_r. Specifically, antisymmetry dictates that $d_{00}(z_c, z_r), d_{10}(z_c, z_r), d_{11}(z_c, z_r)$ are odd in z_r whereas $d_{01}(z_c, z_r)$ is even in z_r. It will be beneficial to work with an orthogonal basis of two-particle states that preserve as far as possible a factorization into center-of-mass and relative parts. Note that unlike the lowest LL, higher Landau levels are *not* invariant subspaces of the relative or center-of-mass angular momentum operators individually, hence unlike in the lowest LL, there are no good quantum numbers associated with these observables. This is related to the presence of the last term in Eq. (3.7). We thus

write:

$$\begin{aligned}\psi(z_c, z_r) = \sum_{J,\ell} \Big\{ & a_{J,\ell}\, \eta^r_{0,\ell}(z_r)\eta^c_{0,2J-\ell}(z_c) + \\ & b_{J,\ell}\, \eta^r_{0,\ell}(z_r)\eta^c_{1,2J-\ell}(z_c) + \\ & c_{J,\ell}\, \eta^r_{1,\ell}(z_r)\eta^c_{0,2R-\ell}(z_c) + \\ & d_{J,\ell}\, \left(\eta^r_{0,\ell}(z_r)\eta^c_{2,2J-\ell}(z_c) - \eta^r_{2,\ell-2}(z_r)\eta^c_{0,2J+2-\ell}(z_c)\right)/\sqrt{2} \Big\},\end{aligned} \tag{3.8}$$

where functions $\eta^r_{k,m}(z_r)$ and $\eta^c_{k,m}(z_c)$ are obtained from $\eta_{k,m}(z)$ via $l_B \to \sqrt{2} l_B$ and $l_B \to l_B/\sqrt{2}$, respectively, ℓ is restricted to odd integers, the $k = 0, 1$ Landau level wave functions were given above, and those for $k = 2$ are also needed:

$$\begin{aligned}&\eta_{2,m}(z) = e^{-|z|^2/4l_B^2} \\ &\times \frac{z^m(\bar{z}^2 z^2 - 4l_B^2(m+2)\bar{z}z + 4l_B^4(m+2)(m+1))}{\sqrt{2\pi 2^{m+5} l_B^{2m+10}(m+2)!}}.\end{aligned} \tag{3.9}$$

It is easy to see that Eq. (3.8) reproduces the analytic structure of Eq. (3.7). Moreover, for sufficiently rapidly decaying $\psi(z_c, z_r)$, which we will always assume, any such $\psi(z_c, z_r)$ can be expanded in the form Eq. (3.8), which follows from completeness properties of the η-functions.

One may see that the Hamiltonian Eq. (3.2) is positive (semi-definite) for general n, which will be made explicit for $n = 2$ below. Therefore, as in the more familiar case $n = 1$, any zero modes are exact ground states. One may further see easily that the familiar analyticity requirements for zero modes for $n = 1$ generalize as follows. For the two-particle state (3.8) *not* to be annihilated by H (i.e., to have any non-zero matrix elements within the image of P_n), its polynomial expansion (not including the Gaussian term) must have terms that are at most linear in z_r, $\bar{z}_r$. With this in mind, working at fixed angular

momentum $L_z = 2R$ at the moment, we see that all non-zero eigenstates of H must be contained in the six-dimensional subspace spanned by the following states,

$$\eta^r_{1,-1}(z_r)\eta^c_{0,2J+1}(z_c)\,, \tag{3.10a}$$

$$\eta^r_{0,1}(z_r)\,\eta^c_{0,2J-1}(z_c)\,, \tag{3.10b}$$

$$\eta^r_{0,1}(z_r)\,\eta^c_{1,2J-1}(z_c)\,, \tag{3.10c}$$

$$\frac{\left(\eta^r_{0,1}(z_r)\,\eta^c_{2,2J-1}(z_c) - \eta^r_{2,-1}(z_r)\,\eta^c_{0,2J+1}(z_c)\right)}{\sqrt{2}}, \tag{3.10d}$$

$$\eta^r_{1,1}(z_r)\,\eta^c_{0,2J-1}(z_c)\,, \tag{3.10e}$$

$$\frac{\left(\eta^r_{0,3}(z_r)\,\eta^c_{2,2J-3}(z_c) - \eta^r_{2,1}(z_r)\,\eta^c_{0,2R-1}(z_c)\right)}{\sqrt{2}}\,, \tag{3.10f}$$

while its orthogonal complement (for given J) is spanned by states already annihilated by H. It follows from this that the Hamiltonian may be written in the form

$$H = \sum_J \sum_{i,j=1}^{6} m_{i,j} {T_J^{(i)}}^\dagger T_J^{(j)} \tag{3.11}$$

where the operators ${T_J^{(i)}}^\dagger$, $i = 1 \ldots 6$, create the states in Eq. (3.10). Specifically, in second quantized form, these operators read:

$$T_J^{(1)} = \frac{1}{2^{J+1/2}} \sum_{x=-J}^{J+1} \sqrt{\binom{2J+1}{J+x}}\, c_{1,J-x} c_{0,J+x}, \tag{3.12a}$$

$$T_J^{(2)} = -\frac{1}{2^J} \sum_{x=-J}^{J} x \sqrt{\frac{1}{J}\binom{2J}{J+x}}\, c_{0,J-x} c_{0,J+x}, \tag{3.12b}$$

$$T_J^{(3)} = \frac{1}{2^{J+1/2}} \sum_{x=-J}^{J+1} (1-2x) \sqrt{\frac{1}{2J+1}\binom{2J+1}{J+x}} \times c_{1,J-x} c_{0,J+x}, \tag{3.12c}$$

$$T_J^{(4)} = -\frac{1}{2^{J+1/2}} \sum_{x=-J-1}^{J+1} x \sqrt{\frac{1}{2J+2}\binom{2J+2}{J+1+x}} \times c_{1,J-x} c_{1,J+x}, \tag{3.12d}$$

$$T_J^{(5)} = \frac{1}{2^J} \sum_{x=-J}^{J+1} \left(2x^2 - 2x - J\right) \sqrt{\frac{1}{2J\,(2J+1)}\binom{2J+1}{J+x}} \times c_{1,J-x} c_{0,R+x}, \tag{3.12e}$$

$$T_J^{(6)} = -\frac{1}{2^J\sqrt{3}} \sum_{x=-J-1}^{J+1} (2x^3 - (3J+2)x) \times \sqrt{\frac{1}{2J(2J+1)(2J+2)}\binom{2J+2}{J+1+x}} \times c_{1,J-x} c_{1,J+x}. \tag{3.12f}$$

As before, x is summed over (half)integers when J is (half)integer. Possible values for $J \pm x$ are non-negative for Landau level index $i = 0$, and are greater than or equal to -1 for $i = 1$, to accommodate for the $L_z = -1$ angular momentum state in the first excited Landau level. One may check that these operators satisfy $\langle 0| \, T_J^{(n)} {T_{J'}^{(m)}}^\dagger \, |0\rangle = \delta_{n,m}\delta_{J,J'}$, as expected from the orthonormality of first quantized wave functions used in this analysis. The matrix elements m_{ij} in Eq. (3.11) turn out to be independent of J, and can be read of the following expression:

$$\begin{aligned} H = & \frac{1}{4\pi}\sum_J {T_J^{(1)}}^\dagger T_J^{(1)} + \frac{3}{8\pi}\sum_J {T_J^{(4)}}^\dagger T_J^{(4)} \\ & + \frac{1}{4\pi}\sum_J ({T_J^{(1)}}^\dagger T_J^{(4)} + \text{h.c.}) + \frac{1}{4\pi}\sum_J {T_J^{(3)}}^\dagger T_J^{(3)} \\ & + \frac{1}{4\pi}\sum_J {T_J^{(2)}}^\dagger T_J^{(2)} + \frac{1}{2\pi}\sum_J {T_J^{(5)}}^\dagger T_J^{(5)} \\ & + \frac{3}{8\pi}\sum_J {T_J^{(6)}}^\dagger T_J^{(6)} - \frac{\sqrt{2}}{4\pi}\sum_J ({T_J^{(2)}}^\dagger T_J^{(5)} + \text{h.c.}) \\ & - \frac{\sqrt{6}}{8\pi}\sum_J ({T_J^{(2)}}^\dagger T_J^{(6)} + \text{h.c.}) + \frac{\sqrt{3}}{4\pi}\sum_J ({T_J^{(5)}}^\dagger T_J^{(6)} + \text{h.c.}). \end{aligned} \tag{3.13}$$

It further turns out that only four of the six eigenvalues of the m-matrix are non-zero, having values $\frac{5\pm\sqrt{17}}{16\pi}$, $\frac{1}{4\pi}$, and $\frac{9}{8\pi}$, respectively. Eigenstates corresponding to these non-zero eigenvalues are: $\frac{\sqrt{2}}{2\sqrt{17\mp\sqrt{17}}}((-1 \pm \sqrt{17}){T_J^{(1)}}^\dagger + 4{T_J^{(4)}}^\dagger) \, |0\rangle$, ${T_J^{(3)}}^\dagger \, |0\rangle$ and $(-\sqrt{2}{T_J^{(2)}}^\dagger + 2{T_J^{(5)}}^\dagger + \sqrt{3}{T_J^{(6)}}^\dagger) \, |0\rangle \, /3$. If we denote the latter by $\mathcal{T}_J^{(1)\dagger} \, |0\rangle$, $\mathcal{T}_J^{(4)\dagger} \, |0\rangle$, $\mathcal{T}_J^{(3)\dagger} \, |0\rangle$ and $\mathcal{T}_J^{(2)\dagger} \, |0\rangle$, then the Hamiltonian can be written in diagonal form:

$$\begin{aligned} H = & \frac{5+\sqrt{17}}{16\pi}\sum_J \mathcal{T}_J^{(1)\dagger}\mathcal{T}_J^{(1)} + \frac{5-\sqrt{17}}{16\pi}\sum_J \mathcal{T}_J^{(4)\dagger}\mathcal{T}_J^{(4)} \\ & + \frac{1}{4\pi}\sum_J \mathcal{T}_J^{(3)\dagger}\mathcal{T}_J^{(3)} + \frac{9}{8\pi}\sum_J \mathcal{T}_J^{(2)\dagger}\mathcal{T}_J^{(2)}. \end{aligned} \tag{3.14}$$

After rescaling of the $\mathcal{T}_J$-operators, this is of the form (3.3) with $M = 4$. The Hamiltonian (3.14) is manifestly the sum of positive (which we will always take to mean semi-definite) terms. A direct consequence of this is that any zero mode of the Hamiltonian (3.14), must be a simultaneous zero energy eigenstate of each positive term $\mathcal{T}_J^{(k)\dagger}\mathcal{T}_J^{(k)}$, and, to this end, must be annihilated by each individual operator $\mathcal{T}_J^{(k)}$. Any zero modes $|\psi_0\rangle$ thus obeys the zero mode condition

$$\mathcal{T}_J^{(i)} |\psi_0\rangle = 0 \tag{3.15}$$

for $i = 1, 2, 3, 4$ and for any integer or half integer J. Equivalently, zero modes are annihilated by $T_J^{(1)}$, $T_J^{(4)}$, $T_J^{(3)}$ and $\mathcal{T}_J^{(2)}$, leading to a slightly more convenient reformulation of the zero mode condition:

$$T_J^{(1)} |\psi_0\rangle = 0, \tag{3.16a}$$

$$T_J^{(3)} |\psi_0\rangle = 0, \tag{3.16b}$$

$$T_J^{(4)} |\psi_0\rangle = 0, \tag{3.16c}$$

$$\mathcal{T}_J^{(2)} |\psi_0\rangle = 0. \tag{3.16d}$$

This generalizes the familiar statement for $n = 1$ Landau level, where the V_1 Haldane pseudopotential is a two-body projection operator onto states of relative angular momentum 1. Presently, for $n = 2$, and for given pair angular momentum $2J$, the spectral decomposition of the Trugman-Kivelson interaction involves four two-particle projection operators, each associated to a one dimensional eigenspace spanned by $T_J^{(i)\dagger} |0\rangle$, $i = 1...4$.

Note that it is no longer possible to ascribe definite relative angular momentum quantum numbers to these states. Note also that the four coefficients in Eq. (3.14) may be replaced with any positive numbers without affecting the zero mode structure of the theory.

3.2 Derivation of general properties of root partitions in disk geometry for $2/5$ state: Entangled Pauli Principle

With the second quantized form of the parent Hamiltonian, we are now in a position to analyze properties of what we will call general dominance patterns of zero modes of this Hamiltonian. To this end, we will utilize a recently developed method [10] to extract dominance patterns of zero modes directly from the parent Hamiltonian, without any need for studying presupposed wave functions. This has the advantage that since rules for root patterns are arrived at directly as properties of the Hamiltonian, these rules immediately provide rigorous constraints on the zero mode counting for the respective Hamiltonian. In particular, upper bounds for the number of zero modes are immediately available (which we will subsequently show to be saturated), and in particular claims about the unprojected Jain state as the unique densest zero modes of its parent Hamiltonian are immediately established (and in some geometries, refined). Such claims have appeared earlier in the literature, [43, 45] but, by our reading, have so far been based on numerics, and were thus limited to finite particle number. The present treatment will be free of such limitations.

We begin by clarifying what we mean by a dominance pattern. The notion of a dominance pattern has mainly appeared in the literature in the context of single component states, where dominance patterns are essentially simple product states associated with more complicated quantum Hall trial wave functions. The present situation involves

Landau level mixing and is more akin to that in multi-component states, which is more complicated and was described in Refs. [46–49].

We first remind the reader of what has been termed a "non-expandable" basis state [10] in the expansion of a zero mode,

$$|\psi_0\rangle = \sum_{\{n\}} C_{\{n\}} |\{n\}\rangle \,. \tag{3.17}$$

Here, each $|\{n\}\rangle$ is a basis state created by a product of single particle creation operators $c^\dagger_{i,m}$. We will call a basis state $|\{n\}\rangle$ in Eq. (3.17) non-expandable if it enters the expansion with non-zero coefficient $C_{\{n\}}$ and it cannot be obtained from any other such basis state $|\{n'\}\rangle$, also having $C_{\{n'\}} \neq 0$, through "inward-squeezing" processes [50]. That is,

$$|\{n\}\rangle \neq c^\dagger_{l_1,j} c^\dagger_{l_2,i} c_{l_3,i-x} c_{l_4,j+x} \ldots |\{n'\}\rangle \,, \tag{3.18}$$

where a single inward squeezing process is a center-of-mass conserving inward pair hopping satisfying $i - x < i \leq j < j + x$, the $l_1 ... l_4$ are *arbitrary* Landau level indices (thus generalizing the standard notion of inward squeezing for single Landau level one-component states), and the dots represent a multiplicative string of any finite number of such inward squeezing terms.

The existence of non-expandable states in any occupancy number spectral decomposition of the form (3.17) follows from the finiteness of the number of states available at given angular momentum. (We may, of course, limit the discussion to zero modes of well-defined angular momentum without loss of generality). It turns out, as we will show below for the present case, that such non-expandable states are subject to certain quite restrictive rules. We will first describe the more familiar situation for a single component, lowest

LL states. In this context, the rules governing non-expandable product states have been referred to as generalized Pauli principles(GPPs). [50–53] Product states satisfying these rules are generally known as dominance patterns or root patterns. Every zero mode contains at least one non-expandable root pattern in its orbital occupancy number spectral decomposition (3.17). Typically, a clever basis of zero modes may be chosen in a manner that there is precisely one such non-expandable root pattern per zero mode. It then follows from the above that every $|\{n\}\rangle$ appearing in the zero mode's decomposition (3.17) may be obtained from its unique root pattern through inward squeezing processes. This then establishes a one-to-one correspondence between root patterns and zero modes. It is worth pointing out that while this correspondence has been discussed for a large class of single component quantum Hall states, [50–54] this was usually done by analysis of special analytic clustering conditions attributed to first-quantized zero mode wave functions. The very notion of clustering conditions may be less clear in the presence of Landau level mixing. Related to this, while for single component states root patterns always represent simple, non-entangled product states, we find it useful to relax this notion in the multi-component or multi-Landau-level situation of interest here. Indeed, the analysis of multi-component states [46,47] suggests the following generalization: We will distinguish between dominance patterns and "root states". Dominance patterns are certain strings of symbols subject to rules we will work out below (lemmas 1-6). To each dominance pattern, we can associate a root state, which will be a fairly simple linear combination of product states $|\{n\}\rangle$, but one possibly featuring some local entanglement. It will then follow from the rules below that the non-expandable Slater-determinants $|\{n\}\rangle$ appearing in any zero mode must appear as linear combinations of root states. Again, a clever basis of zero modes can be chosen, where each zero mode is associated with exactly one dominance pattern or one root state. This does, however, no longer imply that the zero mode features just a single non-expandable Slater determinant in its expansion (3.17).

We note again that "entangled root states" as described above have appeared earlier in the context of multicomponent quantum Hall states. [46, 47] In this context, other approaches to defining dominance patterns have been brought forth as well. [49] The approach taken here is such that, while no reference to a "thin torus" like geometry is made, our definition of a root state will necessarily agree with that based on the thin torus limit. The thin torus approach has been explored for the multi-component states discussed in Refs. [46, 47] using first quantized analytic wave functions. In the following, however, we argue that a more efficient and general approach to studying the structure of root states is to forgo first quantized wave functions and work with a second quantized form of the zero mode condition as in Eq. (3.15). We find this particularly true in problems where degrees of freedom beyond pure guiding centers are present, e.g. spin and/or Landau level degrees of freedom. To this end, we generalize the method introduced in Ref. [10] for single Landau level, single component states to states living in multiple Landau levels.

In the following, we will write second quantized wave functions in terms of a string of numbers, e.g., $1_i01_j010...$, where 1_i stands for an occupied orbital in the i^{th} LL, 1 represents a particle in any of the two LLs (and possibly different LLs for different occurrences of 1) and 0 stands for an unoccupied orbital. Here, orbitals are arranged in the order of ascending angular momenta stating with -1. Before proceeding to our main results, we will state and prove a few lemmas. For definiteness, we find it useful to refer to any non-expandable Slater determinant $|\{n\}\rangle$ appearing in a zero mode as a "root pattern". The root state of the zero mode is then the state obtained by keeping only root patterns in Eq. (3.17). A basis for all possible root states can then be labeled by certain dominance patterns (formal strings of symbols), as we will see below.

Lemma 1: There is no 1_101_1 in root patterns of any zero mode $|\psi_0\rangle$.

Proof. We will use the method of contradiction and the property that any root pattern is, by definition, non-expandable. Now let us assume that a root pattern $|\{n_{\text{root}}\}\rangle$ contains the string 101 in which 0 has angular momentum j. Then $|\{n_{\text{root}}\}\rangle$ can be written as $|\{n_{\text{root}}\}\rangle = c^\dagger_{1,j+1}c^\dagger_{1,j-1}|\{n'\}\rangle$. For $|x| > 1$, $c^\dagger_{1,j+x}c^\dagger_{1,j-x}|\{n'\}\rangle$ must have zero coefficient in the spectral decomposition of $|\psi_0\rangle$, i.e., $\langle\{n'\}| c_{1,j-x}c_{1,j+x}|\psi_0\rangle = 0$ for $|x| > 1$, otherwise $|\{n_{\text{root}}\}\rangle$ would be expandable. Thus, keeping only the $x = \pm 1$ terms, $\langle\{n'\}| Q^{(4)}_j |\psi_0\rangle = -2^{1/2-j}\sqrt{\binom{2j+2}{j+2}/(2j+2)}\langle\{n_{\text{root}}\}|\psi_0\rangle$, which is non-zero. This, however, contradicts the zero mode condition Eq. (3.16c). Thus, $1_1 0 1_1$ must be excluded from any root pattern. ■

Using precisely the same logic, and the respectively appropriate zero mode condition, we may further obtain the following 2 lemmas:

Lemma 2: There is no $1_1 1_1$ in root patterns of the zero mode.

Lemma 3: A root pattern cannot feature any simultaneous occupancy of both lowest and first excited Landau level orbitals of given angular momentum $j \geq 0$.

We then have the following stronger version of *Lemma 2*:

Lemma 4: There is no 11 in root patterns of any zero mode $|\psi_0\rangle$.

Proof. According to *Lemma 2*, there is no $1_1 1_1$ in any root pattern, so possible configurations of 11 are $1_1 1_0$, $1_0 1_1$ and $1_0 1_0$. Thus we consider $|\psi_0\rangle = (\gamma_{0,0}c^\dagger_{0,j}c^\dagger_{0,j+1} + \gamma_{0,1}c^\dagger_{0,j}c^\dagger_{1,j+1} + \gamma_{1,0}c^\dagger_{0,j+1}c^\dagger_{1,j})|\{n'\}\rangle +$ orthogonal terms where the first three terms are root patterns. As in the above, Eq.(3.16a) and Eq.(3.16b) then lead to $\sqrt{j+1}\gamma_{0,1} + \sqrt{j+2}\gamma_{1,0} = 0$ and $-\sqrt{j+1}\gamma_{0,1} + \sqrt{j+2}\gamma_{1,0} = 0$, respectively. Thus both $\gamma_{0,1}$ and $\gamma_{1,0}$ are zero. We then use Eq.(3.16d) to find that $\gamma_{0,0}$ is also zero. ■

The following Lemma states that 101 is allowed in root patterns, but requires local entanglement between the 1-sites of the resulting root state:

Lemma 5: If x0x appears in root patterns of a zero mode $|\psi_0\rangle$, then the proportions of coefficients of root patterns having $1_1 01_0$, $1_0 01_0$, and $1_0 01_1$ with all other occupancies the same are $2 : \sqrt{j+2} : -\sqrt{j}$, where j is the angular momentum of the "0" in 101.

Proof. We can write $|\psi_0\rangle = (\alpha_{0,0} c^\dagger_{0,j-1} c^\dagger_{0,j+1} + \alpha_{0,1} c^\dagger_{0,j-1} c^\dagger_{1,j+1} + \alpha_{1,0} c^\dagger_{1,j-1} c^\dagger_{0,j+1} + \beta_{0,1} c^\dagger_{0,j} c^\dagger_{1,j}) |\{n'\}\rangle +$ orthogonal terms. In the latter expression, the first three terms define three x0x root patterns related as in the statement of the lemma, whereas the fourth term is inward squeezed from these root patterns. Note that $1_1 01_0$ must be absent in root patterns because of *Lemma 1.* Using Eqs.(3.16a), (3.16b) and (3.16d) in a manner analogous to the proofs of the preceding lemmas, we find that $\alpha_{1,0} = -\alpha_{0,1}\sqrt{j}/\sqrt{j+2}$, $\beta_{0,1} = -2\alpha_{0,1}\sqrt{j}/\sqrt{j+2}$ and $\alpha_{0,1} = \alpha_{0,0}\sqrt{j+2}/2$. ■

Note that in the special case $j = 0$, $1_0 01$ is impossible, and the Lemma implies that 101 cannot occur at the very beginning of a root pattern.

The next Lemma involves three particles at a time. Such rules are known from single component states only in the case of 3-body Hamiltonians, but can arise here because of root state entanglement:

Lemma 6: There is no 10101 in root patterns of a zero mode $|\psi_0\rangle$.

Proof. From the first four Lemmas, the only allowed 10101 in root patterns are $1_1 01_0 01_1$, $1_1 01_0 01_0$, $1_0 01_1 01_0$, $1_0 01_0 01_1$ and $1_0 01_0 01_0$. If we assume that the angular momentum of the first orbital in the above patterns is j, then from *Lemma 5.*, the proportions of the coefficients of $1_1 01_0 01_0$, $1_1 01_0 01_1$ and $1_1 01_1 01_0$ are $2 : \sqrt{j+4} : -\sqrt{j+2}$. As $1_1 01_1 01_0$ is excluded from root patterns by virtue of *Lemma 1*, therefore $1_1 01_0 01_0$, $1_1 01_0 01_1$ are also

excluded. Using the same trick, remaining three possible configurations are excluded form root patterns as well. ■

The last Lemma will be proven later:

Lemma 7: There are no constraints on the occurrence of 1001 is in root patterns, that is, 1_i001_j, and likewise for more than two zeros between occupied orbitals.

Lemma 7 is listed here for completeness, as together with the remaining lemmas, it gives a complete set of rules for the construction of root states in one-to-one correspondence with the zero modes of the Hamiltonian. That all the root states allowed by these rules do indeed correspond to a zero mode follows only from explicit construction of such zero modes, and will be discussed below. We will refer to these rules as entangled Pauli principle.

In the presence of multiple Landau levels, root partition is no longer uniquely defined. Some particular linear combination of all of the root partition could well be able to uniquely characterize the zero mode properties. We will refer to that particular "entangled" linear combination of root partition as root pattern. Root patterns, always obey a set of superselection rules, determined by the parent Hamiltonian. We will call this set of rules entangled Pauli principle (EPP) [55]. For 2/5 composite fermion, such rule is composed of Lemma 1-6.

The constraints imposed by Lemmas 1-6, on the other hand, can then be used to rigorously imply that the set of zero modes thus constructed is complete. It may be instructive,

though, to see why the logic used to derive Lemmas 1-6 does not give additional constraints in the situation relevant to Lemma 7. To briefly show this, we may write $|\psi_0\rangle = (ac^\dagger_{0,j}c^\dagger_{0,j+3} + bc^\dagger_{0,j}c^\dagger_{1,j+3} + dc^\dagger_{1,j}c^\dagger_{0,j+3} + ec^\dagger_{1,j}c^\dagger_{1,j+3} + fc^\dagger_{0,j+1}c^\dagger_{0,j+2} + gc^\dagger_{0,j+1}c^\dagger_{1,j+2} + hc^\dagger_{1,j+1}c^\dagger_{0,j+2} + ic^\dagger_{1,j+1}c^\dagger_{1,j+2})\,|\{n'\}\rangle +$ orthogonal terms as in the proofs of Lemmas 4 and 5. *Lemma 7* is then related to the fact that there are eight unknown coefficients and four zero mode conditions (3.16).

We may now make precise the notion of a dominance pattern. Any root pattern satisfying Lemmas 1-4 and 6 defines a formal string of symbols "0" and "$1_{i=0,1}$" as discussed above. The first character in such a string cannot be 1_0, and the Lemmas translate into the requirements that any 1_i in such a string may have no nearest and at most one next nearest neighbor other than 0, and 1_101_1 is further disallowed. If in all possible such strings, we send any occurrence of 1_i01_j, to 101, we will call the resulting set of strings the dominance patterns consistent with Lemmas 1-6. Examples are shown in Table 3.1. Alternatively, we can characterize the set of all possible dominance patterns as all possible concatenations of the strings 0, 1_i00 and 10100, with the leading character not being 1_0. We will refer to these concatenation rules as the GPP for dominance patterns, though this may be a slight abuse of terminology, as dominance patterns are not generally in one-to-one correspondence with product states. However, we may identify dominance patterns with certain states in the Fock space, consisting of the unique (up to an overall factor) a linear combination of all root patterns associated to it that also satisfies *Lemma 5*. Lemmas 1-6 can then be summarized as saying that any root state of a zero mode must be a linear combination of states obtained from dominance patterns via this identification. Since the identification yields states of well-defined particle number N and angular momentum L, we can obviously assign quantum numbers N and L to any dominance pattern.

Using these notions, we are able to arrive at the following important theorem(s) about the zero mode counting of the Hamiltonian (3.2), where in the following, we will always imply the case $n = 2$ and disk geometry:

Theorem 1 At given particle number N and given angular momentum L, the number of linearly independent zero modes of the Hamiltonian (3.2) is no greater than the number of dominance patterns satisfying the GPP.

Proof. Assume that the number of linearly independent zero modes is greater than the number of dominance patterns satisfying the GPP. Then it is possible to make a non-trivial linear combination $|\psi_0\rangle$ of such zero modes that are orthogonal to all states identified with these dominance patterns. Hence $P|\psi_0\rangle = 0$, where P is the orthogonal projection onto the subspace spanned by all states associated with dominance patterns. On the other hand, since $|\psi_0\rangle$ is a zero mode, the definition of a root state and the lemmas imply $|\psi_0\rangle = |\mathsf{root}\rangle + |\mathsf{rest}\rangle$ where $|\mathsf{root}\rangle$ is non-zero, $P|\mathsf{root}\rangle = |\mathsf{root}\rangle$, and $\langle\mathsf{root}|\mathsf{rest}\rangle = 0$. This contradicts $\langle\mathsf{root}|P|\psi_0\rangle = 0$. ■

As a result, we immediately have the following

Corollary 1.1 For given particle number N, there exist no zero modes of the Hamiltonian (3.2) at angular momentum $L < L_e(N) := 5/4N^2 - 2N$ for N even, and at angular momentum $L < L_o(N) := 5/4(N-1)^2 + 1/2(N-3)$ for N odd. If a zero mode exists at $L = L_o(N)$, it is unique, whereas for N even, a zero mode at $L = L_e(N)$ can be at most doubly degenerate.

Proof. The densest possible dominance patterns consistent with the GPP are, respectively, $1_1$0010100101...00101 for N odd, and $1_1$0010100101...00101001$_{i=0,1}$ for N even (see also Fig. 3.2), where "densest" means in particular that no consistent dominance patterns exist at

smaller angular momenta than the ones corresponding to these patterns, which can be seen to be $L_e(N)$ for even N and $L_o(N)$ for odd N. Hence the statement is a special case of Theorem 1. ■

For any zero mode, let l_{max} be the highest angular momentum among the single particle orbitals that are at least partially occupied in that zero mode, i.e., that have $\langle \sum_i c^\dagger_{i,l} c_{i,l} \rangle \neq 0$. Then we finally have

Corollary 1.2 Any zero mode of the Hamiltonian (3.2) has $l_{\mathsf{max}} \geq 5(N-1)/2 - 1$ for N odd, and $l_{\mathsf{max}} \geq 5N/2 - 3$ for N even. Any zero modes satisfying these bounds have angular momentum $L_o(N)$ or $L_e(N)$, respectively, and in particular the statements about degeneracy from Corollary 1.1 apply.

Proof. Any $|\{n\}\rangle$ appearing in a zero mode either appears in its root state or can be obtained via inward squeezing from some other Slater determinants appearing in the root state. Hence the l_{max} of the zero mode is the same as that of its root state, which in turn is the highest occupied orbital among dominance patterns contributing to the root state. For given N, the dominance patterns of smallest l_{max} are those referenced in the proof of Corollary 1.1, and these have the l_{max} values given in the statement of Corollary 1.2, which hence follows. ■

If we define the filling factor ν of a zero mode as N/l_{max}, then Corollary 1.2 implies that the densest (highest) filling factor for which zero modes exist is bounded from above by $2/5$ in the thermodynamic limit. This bound is, of course, saturated, as the corresponding wave function is known. [40, 43] So far, the statements derived here constitute upper bounds on the number of zero modes of the Hamiltonian (3.2). In the following, we will be concerned with the question whether these bounds are saturated, and how the resulting zero mode counting is related to the mode counting in the effective edge theory.

Table 3.1: Some dominance patterns consistent with Lemmas 1-6 for $N = 9$ particles. The leading position corresponds to single particle angular momentum $L_z = -1$ and can only be 0 (empty) or 1_1 (first excited Landau level). a) Unique dominance pattern at smallest angular momentum $L = 83$. b)-e) All consistent patterns with $\Delta L = 1$ relative to the ground state. f)-n) All consistent patterns with $\Delta L = 2$. o) A consistent pattern with higher $\Delta L = 19$. As is shown in the text, the number of consistent patterns at given ΔL equals the dimension of the zero mode subspace of the $n = 2$ Hamiltonian Eq. (3.2).

a)	$1_1001010010010100101$
b)	$1_1001010010100101001_1001_1$
c)	$1_1001010010100101001_0001_0$
d)	$1_1001010010100101001_0001_1$
e)	$1_1001010010100101001_1001_0$
f)	$1_100101001010010100101$
g)	$1_1001010010100101001_10001_1$
h)	$1_1001010010100101001_10001_0$
i)	$1_1001010010100101001_00001_1$
j)	$1_1001010010100101001_00001_0$
k)	$1_100101001010011_0010100 1_1$
l)	$1_100101001010011_0010100 1_0$
m)	$1_100101001010010_0010100 1_1$
n)	$1_100101001010010_0010100 1_0$
o)	$1_1001_10010100001_0010100011_1001_1$

3.3 Entangled Pauli principle on the sphere for $2/5$ state

In this section, we wish to make contact with previous studies that seem to have focused on the sphere. [43, 45] One question that has been addressed by earlier works is the uniqueness of the ground state whenever the number of flux quanta is chosen to be $2s = 5/2N - \mathcal{S}$ where $\mathcal{S} = 4$ is the topological shift of the Jain=2/5 state. This requires the particle number N to be even. We have shown above that for even N there generally is no unique ground state in the disk geometry. However, the statement is nonetheless correct on the sphere. While earlier confirmations of this uniqueness seem to have rested

at least in part on numerics for finite particle number, the methods established above suggest several routes to establish this fact analytically. Indeed, the statement becomes immediate once lemmas 1-6 have been translated to the sphere. For this we will also have to briefly discuss the second quantized form of the $n = 2$ Hamiltonian on the sphere, which we also believe to be of benefit for future reference.

We first remind the reader that a sphere threaded by $2s$ flux quanta has a Landau level structure where the ith Landau level has $2(s+i)+1$ orbitals. [24] Moreover, the ith Landau level transforms under rotations according to the spin $s_n = s + i$ representation of SU(2). Working with eigenstates of the z-component of angular momentum, basis states within a given Landau level thus vary from $L_z = -s - i$ to $s + i$. Specializing to $n = 2$, this means that not only the smallest possible L_z is unique to the first excited Landau level (as is $L_z = -1$ in the disk geometry), but so is the largest L_z. The situation is depicted in Fig.3.1. We see that boundary conditions on the left end are then exactly the same as on the right. When the filling factor is given by $2s = 5/2N - 4$, the application of Lemmas 1-6 then leads to a unique dominance pattern. By Theorem 1, this, in turn, yields the uniqueness, as a zero mode, of the corresponding Jain-2/5 state on the sphere. Likewise, there cannot be any zero modes for $2s < 5/2N - 4$, due to the impossibility to construct permissible dominance patterns under such conditions.

To establish the above, we now turn to the second quantized presentation of $n = 2$ Hamiltonian on the sphere. We will work with the stereographic projection of the sphere introduced in this context in Ref. [56]:

$$z = \tan\frac{\theta}{2}e^{-i\phi}, \tag{3.19}$$

where θ and ϕ are the usual polar and azimuthal angles on the sphere, respectively. With this, the rotationally invariant volume element on the sphere becomes $\sin\theta\, d\theta d\phi = \sqrt{g(z)}dzd\bar{z}$ with $g(z) = (1+z\bar{z})^{-4}$. The rotationally invariant analog of Eq. (3.2) is then

$$H = P_n \frac{\partial_{z_1}\partial_{\bar{z}_1}\delta(z_1 - z_2)\delta(\bar{z}_1 - \bar{z}_2)}{\sqrt{g(z_1)g(z_2)}} P_n \,. \tag{3.20}$$

Moreover, using the gauge $A = -\frac{2s}{e}\cot\theta\hat{e}_\phi$, the relevant lowest and first excited Landau level single particle states have wave functions

$$\begin{aligned} \eta_{0,m}(z) &= \mathcal{N}_{0,m}\, z^{s-m} G_0(z,\bar{z}), \\ \eta_{1,m}(z) &= \mathcal{N}_{1,m}\, [(1+s+m)z\bar{z} - (1+s-m)] z^{s-m} G_1(z,\bar{z}) \end{aligned} \tag{3.21}$$

where the normalization factors are

$$\begin{aligned} \mathcal{N}_{0,m} &= \sqrt{(2s+1)!/[(s+m)!(s-m)!]}, \\ \mathcal{N}_{1,m} &= \sqrt{(2s+3)!/[2(1+s)(1+s+m)!(1+s-m)!]} \end{aligned}$$

and furthermore $G_n(z,\bar{z}) = \bar{z}^{s/2}/[z^{s/2}(1+z\bar{z})^{s+n}]$.

In studying the effect of Eq. (3.20) on two-particle states of well-defined total angular momentum L, one easily observes that H annihilates all states with $L < 2s-1$. This is so because all such states are proportional to at least a third power of $(z_1 - z_2)$ (see Appendix B for detailed calculations). (With the rotational invariance, it is sufficient to observe that all states with total $L_z < 2s-1$ have this property when either z_1 or z_2 are sent to the North pole at $z = 0$.) It further turns out that for two fermions in the lowest two Landau levels, there are two representation with $L = 2s+1$, one representation with $L = 2s$, and three representations with $L = 2s-1$, as one easily finds by focusing on highest weight

states with $L = L_z$. The corresponding six highest weight states are, respectively,

$$|1\rangle = c^\dagger_{0,s} c^\dagger_{1,s+1} |0\rangle \,,$$

$$|2\rangle = c^\dagger_{0,s} c^\dagger_{0,s-1} |0\rangle \,,$$

$$|3\rangle = (\sqrt{\tfrac{s}{1+2s}} c^\dagger_{0,s} c^\dagger_{1,s} - \sqrt{\tfrac{1+s}{1+2s}} c^\dagger_{0,s-1} c^\dagger_{1,s+1}) |0\rangle,$$

$$|4\rangle = c^\dagger_{1,s+1} c^\dagger_{1,s} |0\rangle \,, \tag{3.22}$$

$$|5\rangle = (\sqrt{\tfrac{2s-1}{2(1+4s)}} c^\dagger_{0,s} c^\dagger_{1,s-1} - \sqrt{\tfrac{(4s^2-1)}{2s(1+4s)}} c^\dagger_{0,s-1} c^\dagger_{1,s}$$

$$+ \sqrt{\tfrac{(1+2s)(1+s)}{2s(1+4s)}} c^\dagger_{0,s-2} c^\dagger_{1,s+1}) \, |0\rangle \,.$$

$$|6\rangle = (\sqrt{\tfrac{1+s}{1+4s}} c^\dagger_{1,s+1} c^\dagger_{1,s-2} - \sqrt{\tfrac{3s}{1+4s}} c^\dagger_{1,s} c^\dagger_{1,s-1}) \, |0\rangle \,,$$

There is an obvious correspondence between the above six states and the six states identified in Eqs. (3.12) for the disk geometry. Hence we expect that there are still two zero modes contained in the subspace spanned by these six states, as happened in the disk geometry. Taking into account the lower L_z descendants of these states, this will then lead to four non-zero energy two-particle states for given $L_z = 2R$, except for extremal values of L_z. Working first at the highest level, one finds that there are two zero modes among the $L = 2s - 1$ states $|1\rangle$, $|5\rangle$, and $|6\rangle$, and non-zero energy eigenstates correspond to the linear combinations

$$
\begin{aligned}
|\tilde{1}\rangle \;&= \frac{\sqrt{2}}{(17s^2+6s+1)^{1/4}\sqrt{s+1}} \\
&\Bigg(\frac{\sqrt{(s+1)\sqrt{17s^2+6s+1}-(s^2+4s+1)}}{2}\,|1\rangle + \\
&\quad \frac{s\sqrt{(2s+1)(2s+3)}}{\sqrt{(s+1)\sqrt{17s^2+6s+1}-(s^2+4s+1)}}\,|4\rangle \Bigg),
\end{aligned}
\tag{3.23a}
$$

$$
\begin{aligned}
|\tilde{2}\rangle \;&= -\frac{\sqrt{s(2s+1)(4s+1)}}{(s+1)\sqrt{6(6s-1)}}\,|2\rangle \\
&+ \frac{\sqrt{(2s+1)(2s-1)(2s+3)}}{(s+1)\sqrt{3(6s-1)}}\,|5\rangle \\
&+ \frac{\sqrt{s}(2s+3)}{(s+1)\sqrt{2(6s-1)}}\,|6\rangle\,,
\end{aligned}
\tag{3.23b}
$$

$$
\begin{aligned}
|\tilde{4}\rangle \;&= \frac{\sqrt{2}}{(17s^2+6s+1)^{1/4}\sqrt{s+1}} \\
&\Bigg(-\frac{\sqrt{(s+1)\sqrt{17s^2+6s+1}+(s^2+4s+1)}}{2}\,|1\rangle + \\
&\quad \frac{s\sqrt{(2s+1)(2s+3)}}{\sqrt{(s+1)\sqrt{17s^2+6s+1}+(s^2+4s+1)}}\,|4\rangle \Bigg),
\end{aligned}
\tag{3.23c}
$$

and $|\tilde{3}\rangle = |3\rangle$, with $L = 2s-1$, $2s+1$, $2s+1$, and $2s$, respectively. This implies the following form of the $n = 2$ Hamiltonian on the sphere,

$$
\begin{aligned}
H = \frac{1}{4\pi} \sum_{J\in\{-s-1,-s-\frac{1}{2},\ldots,s+1\}} \\
\Bigg(\frac{6(2s+1)(6s-1)}{(16s^2-1)} \mathcal{T}_J^{(2)\dagger}\mathcal{T}_J^{(2)} + \frac{2(2s+3)}{4s+1}\mathcal{T}_J^{(3)\dagger}\mathcal{T}_J^{(3)} \\
+ \frac{2(2s+3)(-\sqrt{17s^2+6s+1}+5s+2)}{(4s+1)(4s+3)}\mathcal{T}_J^{(4)\dagger}\mathcal{T}_J^{(4)} \\
+ \frac{2(2s+3)(\sqrt{17s^2+6s+1}+5s+2)}{(4s+1)(4s+3)}\mathcal{T}_J^{(1)\dagger}\mathcal{T}_J^{(1)} \Bigg),
\end{aligned}
\tag{3.24}
$$

where we have also made explicit the eigenvalues corresponding to the eigenstates in Eq. (3.23), and introduced two-particle projection operators $\mathcal{T}_J^{(i)\dagger}\mathcal{T}_J^{(i)}$ onto two-particle states $\mathcal{T}_J^{(i)\dagger}|0\rangle$ that, at the appropriate highest weight value of L_z, correspond to the states $|\tilde{j}\rangle$, $j = 1\ldots4$. To be more explicit, we first define similar operators $T_J^{(i)\dagger}$ that correspond

in the same manner to the two particle states $|j\rangle$, $j = 1 \ldots 6$, Eq. (3.22):

$$
\begin{aligned}
T_J^{(1)} &= \sum_x \langle s, J+x; s+1, J-x | 2s+1, 2J \rangle \, c_{1,J-x} c_{0,J+x} \\
T_J^{(2)} &= \frac{1}{\sqrt{2}} \sum_x \langle s, J+x; s, J-x | 2s-1, 2J \rangle \, c_{0,J-x} c_{0,J+x} \\
T_J^{(3)} &= \sum_x \langle s, J+x; s+1, J-x | 2s, 2J \rangle \, c_{1,J-x} c_{0,J+x} \\
T_J^{(4)} &= \frac{1}{\sqrt{2}} \sum_x \langle s+1, J+x; s+1, J-x | 2s+1, 2J \rangle \\
&\qquad c_{1,J-x} c_{1,J+x} \\
T_J^{(5)} &= \sum_x \langle s, J+x; s+1, J-x | 2s-1, 2J \rangle \\
&\qquad c_{1,J-x} c_{0,J+x} \\
T_J^{(6)} &= \frac{1}{\sqrt{2}} \sum_x \langle s+1, J+x; s+1, J-x | 2s-1, 2J \rangle \\
&\qquad c_{1,J-x} c_{1,J+x}
\end{aligned}
\tag{3.25}
$$

Here, $\langle j1, m1; j2, m2 | j, m \rangle$ is a Clebsch-Gordan coefficient. From Eq. (3.25), we then form operators $\mathcal{T}_J^{(i)}$ in a manner exactly as shown in Eq. (3.23). We observe that the zero mode condition can still be cast in the form of Eq. (3.16). It is further worth noting that in the limit $s \to \infty$, Eq. (3.24) recovers the form of Eq. (3.73) for the infinite disk geometry.

We are now in a perfect position to transcribe Lemmas 1-6 to the situation on the sphere. Upon reviewing the logic underlying the proofs of these lemmas, one finds that these hold generically for Hamiltonians of the form Eqs. (3.14), (3.24), provided that certain coefficients at distances $|x| \leq 1$ are non-zero in the T-operators, in this case, Eq. (3.25), as well as certain determinants involving these coefficients, which describe the linear relations used in the proofs of the lemmas. For the sphere, the relevant Clebsch-Gordan coefficients at $j_1 - j_2 - j \leq 3$ can be obtained from a standard sum [57, 58] that never has

more than four terms, which especially for small $|x| \leq 1$ are similar and can be combined into manageable closed forms. One thus verifies that the coefficients of Eq. (3.25) satisfy all the above-mentioned non-vanishing conditions for Lemmas 1-6 to hold. As a result, the only detail about these Lemmas that must be modified are the precise ratios in Lemma 5. Here we state this modified version:

Lemma 5 (sphere) If 101 appears in root patterns of a zero mode $|\psi_0\rangle$, then the proportions of coefficients of root patterns having !0!, !01, and 10! with all other occupancies the same are $2\sqrt{2s+3}$:$\sqrt{(s-j+2)(s+j)}$:$-\sqrt{(s+j+2)(s-j)}$, where j is the angular momentum of the "0" in 101.

Again we note that one recovers the proportions stated earlier for the disk geometry upon taking the limit $s, j \to \infty$ with $s - j$ finite.

Of course, the new Lemma 5 does not change the zero mode counting on the sphere in terms of dominance patterns, for which the only relevant modification is the boundary condition discussed initially and in Fig. 3.1. As explained, the above in particular confirms that the Jain-2/5 state satisfying $2s = 5/2N - 4$ is the unique zero mode at this particular filling factor, with no zero modes existing at larger filling factor.

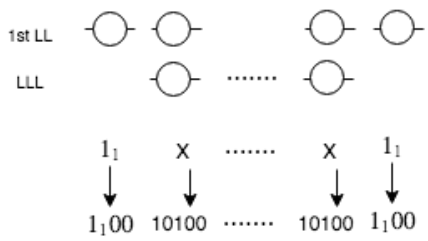


Figure 3.1: Same as Fig. 3.2, but for the sphere, where the first excited LL has one more orbital at both maximum and minimum L_z, for both electrons and composite fermions. Shown (bottom line) is the resulting unique dominance pattern for a sphere satisfying $2s = \frac{5}{2}N - 4$, where $2s$ is the number of flux quanta penetrating the sphere.

3.4 Absence of parent Hamiltonian for projected $2/5$ state

In the above, we have established a description in terms of dominance patterns for the zero modes of the parent Hamiltonian of the unprojected Jain-$2/5$ state. In doing so, we have further developed techniques to extract rules governing such patterns directly from a Hamiltonian principle. We found that, like in other examples [46, 47] where additional degrees of freedom beyond guiding centers are present, dominance patterns are not necessarily product states, but are subject to rules requiring simple entanglement under various circumstances. These rules may be thought of as further generalizations of conventional GPPs describing product states. The rules we found are nonetheless sufficiently simple to serve in zero mode counting, and we have in fact proven that this procedure correctly gives the dimension of the zero mode space at given angular momentum and particle number. We have established this for both the disk and spherical geometries, and demonstrated that zero mode counting at fixed angular momentum and particle number

– but with no restriction on quantum numbers describing relative occupancy of CF Landau levels or associated "winding numbers" in the effective edge theory – is in agreement with the mode counting of the conformal field theory describing the edge physics.

The general approach followed in this paper emphasizes the study of FQH parent Hamiltonians using second quantized methods in a context in which traditionally first quantized language has been given preference. Indeed, only recently the second quantized presentation of FQH Hamiltonians has become a subject of interest in its own right. [10, 11, 34, 59] For one thing, it can be argued that this approach more readily gives access to spectral properties at finite energies. [60] For another, the second quantized approach seems to be effective also in unraveling the zero mode structure of special Hamiltonians, as the present example demonstrates. We emphasize again that few examples seem to have been studied systematically in this regard where the wave function is not described by holomorphic polynomials, i.e., is not contained within the lowest Landau level. The advantage of our approach is that it directly ties the zero mode structure to a GPP for dominance patterns. Such close ties between GPPs and Hamiltonians satisfying a zero mode paradigm may, in fact, explain why parent Hamiltonians have not been found in certain settings. For example, in the case of Jain states that are projected onto the lowest Landau level, the methods presented here strongly suggest that a parent Hamiltonian satisfying the zero mode paradigm would also lead to a GPP consistent with the effective edge theory. That is, to a set of rules governing the fusion of certain local building blocks on a one-dimensional lattice that leads to a densest possible state at the correct filling factor, and yields the correct zero mode counting at larger angular momenta. We conjecture that such a GPP is not possible for the Jain-2/5 state if the particles subject to the GPP have only the angular momentum (or guiding center) degrees of freedom of a single Landau level, with no additional degrees of freedom present(such as spin, Landau level indices,

etc...). More generally, we conjecture that this is true for any state with an edge theory rich enough to comprise at least two branches of non-interacting chiral bosons: It appears that a "plain vanilla", single component GPP cannot be combinatorially rich enough to account for such edge theories. On the other hand, how such GPPs are possible when additional degrees of freedom are present was seen here for the case of additional Landau level degrees of freedom. Similar, but distinct GPPs are implicit in Ref. [46] for, e.g., the (two-component) Halperin (332)-state, which has filling factor 2/5 but a different topological shift than the Jain-2/5 state. We leave the proof of this conjecture as a challenge for future work.

It may be worth noting that, despite our emphasis on edge physics, there is no sharp distinction between edge and (quasi-hole type) bulk excitations from the point of view of dominance patterns. This is of course expected in any microscopic theory and is a consequence of the holographic principle. General bulk excitations in Abelian FQH states can be organized into a 'lattice of excitations', [41] which is two-dimensional in the present case and accommodates both charged and neutral excitations. It is quite clear, e.g., that defects of the form $\ldots 101001_0 00101\ldots$, $\ldots 101001_1 00101\ldots$, represent excitations of the same charge 1/5, but differ by a neutral excitation. They would then have the same statistics. [41] The results of the present paper also lay the basis to study such properties of bulk excitations, in particular pertaining to their statistics, in terms of dominance patterns using the coherent state method of earlier works. [61–63]

We point out that our results also rigorously imply certain properties of the lowest LL *projected* Jain-2/5 state, and, more generally, CF states of the form (3.26). On the sphere, e.g., all Slater determinants contributing to the projected Jain-2/5 state must be obtainable via inward squeezing from the dominance pattern $1_1 000100101\ldots\ 101001_1$. This pattern, of course, does by itself not appear in the projected Jain-2/5 state, as the first and last

occupied orbital belongs to the first excited LL. The projected Jain-2/5 state was studied from this point of view before in Ref. [64], where a different dominance pattern was identified that becomes "non-expandable" in our terminology after projection. The general pattern $\ldots 1_0 01_0 001_1 01_0 001_0 01_0 \ldots$ has also appeared in a thin torus study of the lowest LL projected Coulomb interaction [30].

While we have focused on the case of the Jain-2/5 parent Hamiltonian for definiteness, the validity of our approach is certainly not limited to this case or those presented earlier along similar lines. [10, 11, 34] In particular, generalization to more than two-body Hamiltonians is certainly possible. An obvious direction for future exploration is the case of larger n in Eq. (3.2), especially $n = 3$, which leads to physics at $\nu = 1/2$. [44] Filling factors of the form 1/2+integer have traditionally been fruitful ground for a great wealth of proposals of Abelian, non-Abelian, and gapless states [36, 45, 65–68], and are recently again actively investigated from a particle-hole symmetric point of view, [69] the latter having inspired interesting new wave-functions. [70] Even beyond the realm of FQH physics, attractive features of frustration-free lattice Hamiltonians that are not necessarily finite ranged but feature a "center-of-mass-conservation" symmetry has long been advertised. [18, 71] We are hopeful that the methods developed here will make major contributions to the general study of such Hamiltonians, the general n case of Eq. (3.2) being a particular example.

3.5 Explicit construction of zero mode counting and edge theory for $2/5$ composite fermion, starting from parent Hamiltonian description

3.5.1 Zero mode counting

As argued in the previous sections, the zero mode condition derived from a good quantum Hall parent Hamiltonian will not only characterize the incompressible quantum fluid sufficiently uniquely but also encode the proper edge theory of the system. The rules derived in the preceding section thus far only suggest a certain zero mode structure, but, with the exception of (the yet unproven) Lemma 7, only constrain this structure without guaranteeing the existence of any zero modes. It is, however, worth noting that all of this was derived from the second quantized operators $T_J^{(i)}$ alone, and, if we took Lemma 7 for granted, the entire zero mode structure in terms of dominance patterns would follow correctly from this analysis. To prove Lemma 7 and thus establish the complete zero mode structure of Eq. (3.2) with $n = 2$, we briefly make contact with the first quantized presentation of zero modes, though at least in part we will see below that an operator-based approach could also be envisioned.

The analysis of Sec. 3.1 implies that a sufficient (and necessary) property of any zero mode is that the associated analytic many-body wave function contains the factor $(z_i - z_j)^2$ for all i, j (see in particular discussion following Eq. (3.9)). This is, in fact, a quite special property of the cases $n = 1$ and $n = 2$ of Eq. (3.2). More generally, zero modes of Eq. (3.2) may be linear combinations of terms containing the factors $(z_i - z_j)^2$, $(z_i - z_j)(\bar{z}_i - \bar{z}_j)$, and $(\bar{z}_i - \bar{z}_j)^2$, which, by symmetry, must be true for all i, j. That is, a zero

mode vanishes at least to second order in the separation of any pair of coordinates. For $n \leq 2$, however, the third term is prohibited by Landau-level projection, and the second then always necessitates another factor of $z_i - z_j$ by anti-symmetry, such that the first term still covers all possible cases for having a second-order zero. This renders the $n = 2$ of Eq. (3.2) rather special. While the presence of the first excited Landau level allows terms in $\bar{z}_i$ to be present in the wave function, the zero mode condition can thus be stated only in terms of the holomorphic variables z_i. Indeed, it is only for $n \leq 2$ that the ground state of Eq. (3.2) is in the Jain sequence of states. [44]

Thanks to the work done in the preceding section, for now, it will do to note that divisibility of the wave function by $\psi_{1/2} = \prod_{i<j}(z_i - z_j)^2$, the bosonic $\nu = 1/2$ Laughlin-Jastrow factor, is a sufficient criterion for a wave function to be a zero mode. In our present approach, the necessity of this criterion (for $n = 2$), i.e., the completeness of the resulting zero mode space, will be inferred from Theorem 1. This route will set the stage for the larger n Hamiltonians as well. As an added benefit, this will establish the one-to-one correspondence between dominance patterns satisfying the rules given above and zero modes of the Hamiltonian.

We thus consider zero mode wave functions of the form $\psi_{1/2} p(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N)$, where p is an arbitrary polynomial of the requisite anti-symmetry and at most first order in the $\bar{z}_i$ (so as for $\psi_{1/2}\, p$ to be contained within the first two Landau levels), and we drop the obligatory Gaussian factor for simplicity. It is clear that a suitable basis for these polynomials is given by $S_{\{n\}}(z_1, \bar{z}_1, \ldots)$, where $S_{\{n\}}$ is a Slater determinant of single particle states in the lowest and first excited Landau level, with occupancies determined by a set

of occupancy numbers $\{n\}$. [9] Hence we wish to study zero modes of the form

$$\psi_{1/2}(z_1, \dots) S_{\{n\}}(z_1, \bar{z}_1, \dots) \,. \tag{3.26}$$

We note that zero modes of this form are naturally viewed as composite fermion (CF) states, where any fermion forms a composite object with two flux quanta. In particular, if the CF-occupancy configuration $\{n\}$ is chosen to represent two equally filled Landau levels, one recovers the Jain-2/5 state, and one easily verifies that this state saturates the bounds of the Corollaries of the last section. Therefore, the Jain-2/5 state is the densest zero mode of Eq. (3.2) for $n = 2$, unique up to the twofold degeneracy mentioned in Corollary 1.1 (see below).

We emphasize that while notationally similar to the *electron* occupancy numbers $\{n\}$ labeling basis states in Eq. (3.17), the labels $\{n\}$ represent *composite fermion* occupancy numbers and must be well distinguished from the labels $\{n\}$. To analyze the dominance patterns underlying the zero modes (3.26), we make use of well-known rules [72] for products of polynomials with known root patterns, generalized to the case where non-holomorphic variables (or more than a single Landau level) are present. Every CF-Slater determinant configuration $S_{\{n\}}(z_1, \bar{z}_1, \dots)$ is naturally its own root state, as it is the only Slater determinant appearing in its wave function. The associated CF-occupancy pattern $\{n\}$ may now be thought of as a string made up of characters X, 0 and $1_{i=0,1}$. The last three characters have the analogous meaning as in our notation for root patterns of full zero mode wave functions (but refer to CFs), and X now means a double occupancy of the associated angular momentum state in both Landau levels. As before, the first character can only be 1_1 or 0, see Fig.3.2. Moreover, as is well known, [73] the bosonic Laughlin

[9] If there were any doubts as to the completeness of these Slater determinants for present purposes, this would follow below from the fact that all possible dominance patterns are obtained in this way.

factor $\psi_{1/2}$ has a root state given by the pattern $1_0 01_0 0 ! 1_0 01_0 0 \ldots$. Dominance patterns may generally be associated to partitions $l_N + l_{N-1} + \ldots + l_1 = L$, where $l_i \geq l_{i+1}$ is the angular momentum of the ith particle in the pattern, and L is the total angular momentum of the pattern. When two wave functions whose root states have dominance patterns with partitions $\{l_i\}$ and $\{l'_i\}$, respectively, are multiplied, the resulting wave function has a root state whose dominance pattern has the partition $\{l_i + l'_i\}$. It is easy to see that these rules when applied to the present situation, imply that the multiplication of $\psi_{1/2}$ by the Slater determinant $S_{\{n\}}$ leads to a wave function with a dominance pattern obtained from the pattern associated to $\{n\}$ as follows. The character 1_0 is replaced with $1_0 00$, ($1_0 \rightarrow 1_0 00$, rule 1). An X in the CF-pattern corresponds to the case where $l_i = l_{i+1}$ in the associated partition, signifying two particles with identical angular momenta but different Landau level indices. The resulting ambiguity in ordering these two particles leads to the situation described as 101 in the dominance pattern of the resulting zero mode, i.e., we have the rule $X \rightarrow$10100 (rule 2). That the underlying configurations $1_0 01_0$, $1_0 01_1$, and $1_1 01_0$ indeed occur with the ratios claimed by Lemma 5 could be verified directly from Eq. (3.26), but this is not necessary, since Eq. (3.26) is definitely a zero mode, and then the proof of Lemma 5 applies. A "1_1" in the CF-pattern associated to $S_{\{n\}}$ leads to at least two root patterns in the root state of Eq. (3.26), one obtained from the replacement $1_1 \rightarrow 1_1 00$ (rule 3.a), and one from $1_1 \rightarrow 1_0 00$ (rule 3.b). However, it is clear that if we ignore rule 3.b for the moment, rules 1-3.a establish a one-to-one correspondence (see Fig.3.2) between CF-occupation number patterns $\{n\}$ of N particles occupying orbitals with angular momentum up to l_{max} and permissible dominance patterns of N particles occupying orbitals with angular momentum up to $l_{\mathsf{max}} + 2(N-1)$ (where the addition of $2(N-1)$ can be thought of as being due to flux attachment.) Let us now denote a dominance pattern satisfying the GPP of the preceding section by p and the associated root state by $|p\rangle$. Let us choose an ordering of these patterns such that the number of 1_1s in the pattern increases

monotonously for patterns associated with the same partition $\{l_i\}$. Furthermore, we may order patterns associated to different partitions according to increasing $S(\{l_i\}) := \sum_i l_i^2$. (Note that these requirements do not specify the order uniquely; however, any ordering in compliance will do.) Finally, let us order the CF-occupancy patterns $\{n\}$ in the same way, by means of the one-to-one correspondence. We then see that the matrix

$$C_{p,\{n\}} = \langle p|\psi_{1/2}S_{\{n\}}\rangle \tag{3.27}$$

is upper triangular[10] with non-zero diagonal and thus invertible. Therefore, new linear combinations of the $|\psi_{1/2}S_{\{n\}}\rangle$ can be found such that the new overlap matrix with the $\langle p|$'s is the identity.[11] This implies that for each dominance pattern p satisfying the GPP, there is a superposition of zero modes of the form (3.26) that is dominated precisely by the associated root state $|p\rangle$, with no other of the states $|p'\rangle$ present in its spectral decomposition (3.17). This establishes both the completeness of zero modes of the form Eq. (3.26) (by Theorem 1), and, moreover, the following stronger version of Theorem 1:

Theorem 2 At given particle number N and given angular momentum L, the number of linearly independent zero modes of the Hamiltonian (3.2) is exactly equal to the number of dominance patterns satisfying the GPP.

3.5.2 Edge mode counting

We will now discuss that the counting of zero modes at a given angular momentum and particle number that follows from the construction of dominance patterns above agrees

[10] For, let p_n be the pattern that is associated to n. Then by design, any p' different from p_n but having the same partition $\{l_i\}$ must come before p_n in order for the overlap (3.27) to be non-zero. Likewise, any such p' corresponding to a different partition $\{l_i'\}$ would be obtainable from the dominant pattern p_n via inward squeezing, and thus have smaller $S(\{l_i'\})$.

[11] I.e., linear combinations with coefficients given by the columns of the inverse of the matrix Eq. (3.27).

with the counting of edge states in the effective edge theory. We will argue that there is a weaker and a stronger version of this statement. The weaker version, often found in the literature, is concerned with the number of zero modes/edge modes $\mathcal{N}(\Delta L)$, where ΔL is the angular momentum relative to the ground state at fixed particle number. In the thermodynamic limit of large particle number N, this quantity is not expected to depend (much) on N. We will see that the counting problem defined by $\mathcal{N}(\Delta L)$ can be conveniently addressed in terms of CF-patterns. However, the quantity $\mathcal{N}(\Delta L)$ is not sensitive to all aspects of the K-matrix describing the edge theory. Indeed, the K-matrix of any Jain state is congruent to a matrix of the form $K' = W^T K W = mJ_n + \mathbb{1}$, [74] where J_n is an $n \times n$ matrix of ones, and W is an $SL(n, \mathbb{Z})$ matrix. K' has precisely one eigenvalue different from 1, which is non-degenerate with eigenvector t describing charged excitations. The quantity $\mathcal{N}(\Delta L)$ is only sensitive to neutral excitations orthogonal to t, which always lie in the eigenvalue 1 eigenspace of K'. In particular, $\mathcal{N}(\Delta L)$ does not distinguish between Jain states that have the same number of edge branches. (For example, $\mathcal{N}(\Delta L)$ does not distinguish different Laughlin states; see, e.g., the discussion in Ref. [75].) In contrast, we may consider the number of zero modes $\mathcal{N}(N, L)$ at given particle number and given *total* angular momentum, which, among other things, also keeps track in absolute terms of how angular momentum changes with particle number. We will show that this quantity, when evaluated for the present microscopic Hamiltonian, captures all aspects of the K-matrix of the edge theory.

To make things concrete, we consider the edge theory of the Jain-2/5 states in the form [76]

$$H = \frac{1}{4\pi} \int dx\, V_{ij} : \partial_x \phi_i \partial_x \phi_j : - \frac{\mu_i}{2\pi} \int dx\, \partial_x \phi_i \,, \tag{3.28}$$

where $i, j = 1, 2$ describe two bosonic edge modes through phase fields $\phi_i(x)$ and associated densities $\rho_i = \frac{1}{2\pi} \partial_x \phi_i$, satisfying the Kac-Moody algebra $[\rho_i(x), \rho_j(x')] = (K^{-1})_{ij} \frac{i}{2\pi} \partial_{x'} \delta(x -$

x'). The colons imply normal ordering with respect to finite momentum modes defined below. K_{ij} is a characteristic matrix that together with the charge vector t_i defines the edge theory. The Jain- or hierarchy-2/5 edge can be described by $K = \left(\begin{smallmatrix} 3 & 2 \\ 2 & 3 \end{smallmatrix}\right)$ [74] and $t = (1, 1)$, where t is defined such that $\rho_e = \sum_i t_i \rho_i$ represents the physical electron charge. In the following, we will pay special attention to the zero momentum modes of the densities ρ_i, which we will write as $N_i/(2\pi R)$, where R is the radius of the quantum Hall fluid. Physical operators must respect the integer character of the N_i. [76] We note in passing that close formal relations [36] between the edge theory conformal blocks and CF wave functions have been explored in detail in Ref. [77].

Eq. (3.28) describes an edge with general interaction matrix V_{ij} between densities and with general chemical potentials μ_i coupling to the integer charges N_i. The latter control both the total particle number as well as the radial spatial separation between the two edge branches, which, in the limit of large separation, define two individual edges between a 2/5-phase and a 1/3-(Laughlin-)phase and between a 1/3-phase and vacuum, respectively. On general grounds, [78] a close relation is expected between the spectrum of the edge Hamiltonian and the angular momentum operator of the fluid, if the interactions are so tuned that the edge theory is conformally invariant. This requires all edge modes to travel with the same velocity v. Is is easy to see that this can be achieved by letting $V_{ij} = vK_{ij}$, leading to the equation of motion $\partial_t \rho_i + v\partial_x \rho_i = 0$. With this, we then look at the mode expansion of Eq. (3.28):

$$\begin{aligned} H &= \frac{v}{2R}(3N_0^2 + 3N_1^2 + 4N_0N_1) - \mu_0 N_0 - \mu_1 N_1 + \frac{v}{R} P, \\ P &= \sum_{j=0,1} \sum_{n>0} n\, b_{j,n}^\dagger b_{j,n}\,. \end{aligned} \tag{3.29}$$

Here, the $b^\dagger_{j,n}$ ($b_{j,n}$) are appropriate linear combinations of the positive (negative) Fourier components of the $\rho_i(x)$ satisfying $[b_{j,n}, b^\dagger_{j',n'}] = \delta_{j,j'}\delta_{n,n'}$, $n = 1, 2, \ldots$.

For the purpose of comparing the dimensions of zero mode spaces and edge mode spaces for various sectors, it is useful to identify the quantum numbers N_0, N_1 of the edge theory with the CF-numbers in the lowest and first excited LL, respectively, in zero modes of the form (3.26). We first appeal to the one-to-one correspondence between CF-occupancy patterns of fixed N_i and excitations of the edge theory, likewise for fixed N_i. This is a standard result in bosonization, [79] applied here to the case of two chiral branches. Let us denote the CF-state with "densest" (minimum angular momentum) CF-occupancy pattern for given N_i by $|N_0, N_1\rangle_{\mathsf{CF}}$. Then the one-to-one correspondence between CF-states and edge states at fixed N_i applies to all CF-states whose angular momentum relative to $|N_0, N_1\rangle_{\mathsf{CF}}$ is smaller than a cutoff given by particle number: $\Delta L \lesssim N_i$ (c.f., e.g., Ref. [11]). That is, the number of such CF zero modes of given N_i and ΔL relative to $|N_0, N_1\rangle_{\mathsf{CF}}$ is equal to the number of edge states described by Eq. (3.29) of fixed N_i and "edge momentum" $P = \Delta L$.

We note, however, that counting at fixed N_i is an artificial constraint from the point of view of the microscopic theory, as these quantum numbers do not correspond to any local (or even Hermitian) conserved quantities in the microscopic theory. Moreover, counting subject to this constraint contains no information about the K-matrix (except for its dimension). To make a statement that is both more physical and stronger, we now claim that for proper choice of chemical potentials μ_i and up to a scale factor v/R we will let equal to 1, for any given particle number $N = N_0 + N_1$, the degeneracies of the eigenvalues of the angular momentum operator of the macroscopic theory, projected onto the zero mode subspace of Eq. (3.2), are exactly the same as the degeneracies of the energy eigenvalues of the edge Hamiltonian Eq. (3.29). That is, the number $\mathcal{N}(N, L)$ introduced above for the microscopic Hamiltonian is identical to the degeneracy of the energy $E = L$

of Eq. (3.29) for given $N = N_0 + N_1$. Loosely speaking, the edge Hamiltonian Eq. (3.29) *is* the zero-mode-projected angular momentum operator of the microscopic theory.

It is sufficient to show that edge states with $P = 0$ and given $N = N_0 + N_1$ have an energy equal to the angular momentum of the CF "vacua" $|N_0, N_1\rangle_{\mathsf{CF}}$ defined above. For then, it follows that all states identified within each N_0, N_1 sector via bosonization must also have identical eigenvalues for, respectively, energy (in Eq. (3.29)) and angular momentum (in the microscopic theory). The choice of μ_i for which this is true is totally determined by the requirement that $N_0 = N_1 = 1$ leads to angular momentum $L = 1$ in the microscopic theory, whereas $N_0 = 0$, $N_1 = 1$ leads to $L = -1$, giving $\mu_0 = 3/2$, $\mu_1 = 5/2$ in Eq. (3.29) ($v/R = 1$, $P = 0$). It thus suffices to show that the minimum angular momentum states $|N_0, N_1\rangle_{\mathsf{CF}}$ have L equal to

$$L_{\mathsf{min}} = \frac{3}{2}(N_0 + N_1)(N_0 + N_1 - 1) - N_1(N_0 + 1)\,. \tag{3.30}$$

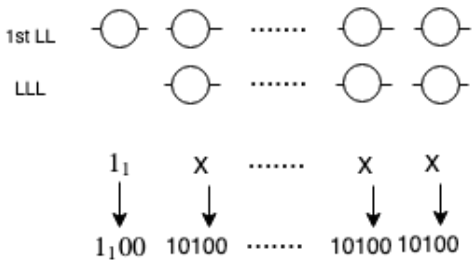
That this is indeed the case can easily be established considering first the densest pattern for given $N_0 + N_1$ (e.g. a) in Table 3.1) and then proceeding by induction to general values of $N_1 - N_0$ (examples are b) and c) in Table 3.1 for $N_0 = 3$, $N_1 = 6$ and $N_0 = 5$, $N_1 = 4$, respectively). Alternatively, the statement also can be followed from Eq. (3.26).

The above establishes that the counting of microscopic zero modes at given particle number N and angular momentum L is exactly the same as that of energy eigenmodes in an appropriately scaled edge Hamiltonian describing the 2/5-edge. While the counting can be done in terms of CF-patterns, as expected in any system that can be understood in terms of non-interacting CFs, we have shown that counting can be done equally well in terms of dominance patterns. In this regard, it is worth noting that CF occupancy patterns as defined above manifestly encode only *changes* in angular momentum at *fixed* particle

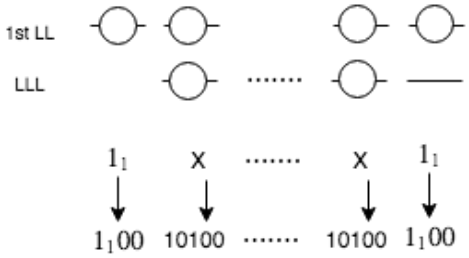
number. Obtaining the absolute angular momentum of a CF-state described by a given CF occupancy pattern requires additional information about the number of flux quanta each composite fermion carries. In contrast, the total angular momentum of the associated (root) state is manifest in dominance patterns. The set of rules governing the composition of valid dominance patterns can thus be interpreted as a set of minimal rules to construct the quantity $\mathcal{N}(N, L)$ from certain local building blocks (see discussion above Theorem 1 and caption of Table 3.1). The fact that this then reproduces edge mode counting is the property that one expects a good GPP to have. We thus find that the present Hamiltonian does not only fully fall into the "zero mode paradigm" expected of special quantum Hall parent Hamiltonians, but is also linked to a GPP which facilitates the pertinent counting. It should be clear that our arguments leading from FQH Hamiltonians admitting zero modes to GPPs governing dominance patterns have a very general character. If such a Hamiltonian satisfies the zero mode paradigm, the implied GPP must then reproduce edge mode counting from local rules as demonstrated above. We will argue below that this general connection between the existence of zero modes and GPPs imposes useful constraints on settings in which "good" (zero mode paradigm) parent Hamiltonians may be constructed. We caution, however, that there are modified versions of this paradigm, as, e.g., realized in the parent Hamiltonian of the anti-Pfaffian state. [80, 81] Here, the equivalent of zero mode counting would describe an edge with a $\nu = 1$ integer quantum Hall state, as opposed to vacuum.

We note that the quantity $\mathcal{N}(N, L)$ is in principle robust to sufficiently weak rotationally invariant perturbations. Here, "weak" means sufficiently small compared to the gap separating low-energy modes from the rest of the spectrum at given L. Under such

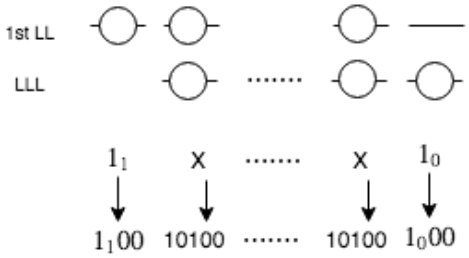
conditions, $\mathcal{N}(N, L)$ may thus even survive some degree of edge reconstruction. However, it is clear that this quantity is directly meaningful only in exceptionally clean systems. The more robust features of edge mode counting can be probed experimentally in momentum-resolved tunneling. [29, 82–86]



(a)



(b)



(c)

Figure 3.2: Composite fermion occupancy patterns and resulting dominance patterns. Three different cases are shown. Level diagrams show composite fermion occupancies, followed by a more symbolic composite fermion occupancy pattern and the associated dominance pattern as explained in text. a) corresponds to the densest (minimum angular momentum) zero mode for odd particle number, followed by the two configurations corresponding to the doubly degenerate densest zero modes for even particle number (b) and c)). Note that only the dominance patterns manifestly encode the total angular momentum of the state. More general dominance patterns consistent with Lemmas 1-6, and thus in one-to-one correspondence with zero modes (see text), are shown in Table 3.1.

3.6 Concluding remarks on $2/5$ state parent Hamiltonian

So far in this paper, we have further developed a method to extract GPPs governing zero modes of a FQH parent Hamiltonian directly from its second quantized form. In particular, we have demonstrated that such principles apply to states involving higher Landau levels and provided an in-depth analysis of the zero mode structure of the Jain-$2/5$ state parent Hamiltonian and its realization through certain dominance patterns. As in earlier works focusing on single Landau level physics, we have identified single-particle operators that generate zero modes. Our approach does, somewhat uncharacteristically, emphasize the second quantized presentation of parent Hamiltonians, which we developed in detail for the Jain-$2/5$ state for the disk and sphere geometries. The cylinder geometry can be treated similarly, with implications for the torus. This represents one route to a presentation of the physics that manifestly exposes the dynamics of the guiding centers and retains dynamical momenta only to the extent that they have not been eliminated by Landau level projection. These aspects seem to be much in keeping with a line of thought recently put forth by Haldane. [87] A powerful strategy in exploring correlated electron physics is to stabilize special wave functions associated with certain fixed points in the phase diagram via local Hamiltonians. For the phases described by Jain states, lowest Landau level projected versions of Jain states, or manifestly projected hierarchy states, are sometimes thought to be the proper fixed point wave functions, since they are compatible with the strong field limit. We have presented arguments here why a local parent Hamiltonian for these states may not be possible, at least not if we want it to fall within the usual zero mode paradigm. It is then reassuring that the existing parent Hamiltonian for the unprojected Jain-$2/5$ state does fall into this paradigm, as we argued in great detail. The Hamiltonian studied here is the $n = 2$ special case of a family of Trugman-Kivelson

interactions projected onto n Landau levels. We expect that the methodology developed here will be of great value to shed light on the case of a larger n.

3.7 Zero mode generators for $2/5$ state

While results from the preceding section establish the full zero mode structure of the Jain-2/5 state parent Hamiltonian, we mention here an alternative approach more in line with our general philosophy of working with the operator algebras of the second quantized problem. Such an approach has been carried out earlier by some of us [10, 11, 34] for the Laughlin states and their parent Hamiltonians. One attractive feature of this approach is its resulting in a "microscopic bosonization dictionary", where operators present in the effective edge theory are identified with second-quantized microscopic operators that interact with the microscopic Hamiltonian in exactly the way expected from the effective theory. Another motivation to consider this route is the fact that in the single Landau level example of Refs. [10, 11, 34], Read's order parameter of the Laughlin state [13] appeared naturally (in a fully second quantized form). Clearly, an analogous construction for the Jain-2/5 state would be of great interest. Here we will report some preliminary results regarding this approach, leaving details for future work.

We begin by identifying four sets of single particle "zero mode generators":

$$P_d^{(1)} = \sum_{r=-1}^{+\infty} \sqrt{\frac{(r+d)!}{(r+1)!}} c_{0,r+d}^\dagger c_{1,r} \quad d \geq 1, \tag{3.31}$$

$$\begin{aligned} P_d^{(2)} = & \sum_{r=0}^{+\infty} \sqrt{\frac{(r+d)!}{r!}} c_{0,r+d}^\dagger c_{0,r} \\ & + \sum_{r=-1}^{+\infty} \sqrt{\frac{(r+d+1)!}{(r+1)!}} c_{1,r+d}^\dagger c_{1,r} \quad d \geq 0, \end{aligned} \tag{3.32}$$

$$
\begin{aligned}
P_d^{(4)} = &\sum_{r=0}^{+\infty} \Big(\sqrt{\frac{(r+d+1)!}{r!}} c^\dagger_{1,r+d} c_{0,r} \\
&+ (r+d+1) \sqrt{\frac{(r+d)!}{r!}} c^\dagger_{0,r+d} c_{0,r} \Big) \\
&- \sum_{r=-1}^{+\infty} \Big((r+1) \sqrt{\frac{(r+d+1)!}{(r+1)!}} c^\dagger_{1,r+d} c_{1,r} \\
&+ (r+1)(r+d+1) \sqrt{\frac{(r+d)!}{(r+1)!}} c^\dagger_{0,r+d} c_{1,r} \Big) \quad d \geq -1.
\end{aligned}
\tag{3.33}
$$

These generalize the single set of zero mode generators identified for the $n = 1$ (Laughlin-state) case earlier. [10,11,34] Their algebraic properties can be summarized as follows. By themselves, the $P_d^{(i)}$ form a graded Lie-algebra, where the grading is furnished by the label d. Explicitly, this means that $[P_d^{(i)}, P_{d'}^{(j)}]$ is a linear combination of $P_{d+d'}^{(k)}$, $k = 1 \ldots 4$. This graded Lie-algebra can be extended by the $T_R^{(i)}$, or, alternatively, the operators appearing on the left-hand side of Eq. (3.16) defining the zero mode condition, where the grading is now provided by the label $-2R$. While commutators between different $T_R^{(i)}$ of course vanish, commutators of the form $[T_R^{(i)}, P_d^{(j)}]$ give linear combinations of $T_{R-d/2}^{(k)}$, $k = 1 \ldots 4$. This last property justifies the term "zero mode generators". It assures that, when any $P_d^{(i)}$ acts on a zero mode $|\psi\rangle$ (and does not give zero), it generates another zero mode, because all commutators $[T_R^{(i)}, P_d^{(j)}]$ vanish inside the zero mode subspace. [10] Note also that $P_d^{(i)}$ increases the angular momentum of the zero mode by d. It thus clear that the $P_d^{(i)}$ have properties that are similar to those of the mode operators $b_{i,d}^\dagger$ ($i = 0, 1$)in the effective edge theory. This leads to the obvious question of why we found more than two sets of $P_d^{(i)}$ operators. Although we must carefully distinguish between electron and CF occupancy numbers, it is clear that the operator $P_d^{(1)}$ gradually depopulates the first excited Landau level. This will also reduce the number of CFs in the first excited Landau level. Note that the operator is nilpotent (for fixed particle number): A sufficiently large power of $P_d^{(1)}$ will certainly annihilate the state. We may thus interpret $P_d^{(1)}$ as an operator

that creates edge excitations of the kind generated by the operators $b^{\dagger}_{i,d}$ in the effective edge theory, but at the same time lowers the quantum number $N_1 - N_0$. To identify zero mode operators that, like the operators $b^{\dagger}_{i,d}$ create independent branches of edge excitations that do *not* affect $N_1 - N_0$, we must find two commuting linear combinations of the $P^{(i)}_d$ that are *not* nilpotent. These criteria are satisfied by $dP^{(1)}_d + P^{(2)}_d$ and $P^{(3)}_d$. The other two linear combinations of the $P^{(i)}_d$ operators will correspond to operators in the edge theory that do change the quantum number $N_1 - N_0$ (or else are not independent of the former). We have indeed shown that $P^{(4)}_0$ can be used to connect one of the two degenerate lowest angular momenta zero modes at even particle number(see Sec. 3.5) to the other. These considerations make it feasible that by acting with combinations of products of the operators $P^{(i)}_d$ on a lowest angular momentum zero mode, we can generate all zero modes at fixed particle number. Moreover, in Ref. [34] we have succeeded in constructing a microscopic operator that, when acting on the smallest angular momentum zero modes in the $n = 1$ (Laughlin) case, leads to the corresponding zero mode with the total particle number increased by 1. This can be interpreted as a microscopic realization of the operator of the edge theory that raises the quantity $N_0 + N_1$. It is here where the connection with the order parameter of the Laughlin state can be made. Establishing such a connection is, however, uncontrollably hard in the canonical LL basis. In the next section, we will establish a pseudo-fermion basis, which explicitly captures underlying symmetries of composite fermion, thus turns out to be a natural basis for the problem.

3.8 A pseudo-fermion description: A natural basis for composite fermion

In the last few sections, we have given a microscopic parent Hamiltonian description for $2/5^{th}$ state. Our method not only predict the densest ground state as the $2/5^{th}$ composite fermion. In the ground state projected basis, this commutes with a large set of quasihole algebra. Such commutations rigorously capture all of the universal properties of the $2/5$ composite fermion. The above construction, however, is not easy to extrapolate to arbitrary Landau levels, due to involved mathematical calculations, associated with it. In this section, we will show, such an apparent mathematical complexity is just an artifact of choosing a "wrong basis" for the problem. In last section, we have identify the "zero mode paradigm" in terms of two commuting operators $p_k = kP_k^{(1)} + P_k^{(2)}$ and $P_k^{(3)}$, which create independent branches of edge excitations that do *not* change number of particles in each Landau levels. The operator p_d, also appear for the lowest Landau level, while construction of order parameter recursion relation for Laughlin's state. Before proceeding further towards construction of a "natural" basis for composite fermion, let us review the order parameter recursion relation for Laughlin's state.

The central ingredient was the recursion for the Jastrow (CF flux attachment) operator $\hat{J}_N$, Eq. (2.49). The key to the generalization of this recursion to higher-LL CF states is the fact that this recursion is the operator manifestation of a polynomial recursion, which we have formally expressed as (2.50). This last equation must remain valid since in any number of LLs the (M-dependent) Jastrow factor is always represented by the same symmetric polynomial in the holomorphic coordinates. As we emphasized earlier, the second-quantized operators associated with the multiplication with such polynomials somewhat depend on the geometry in question, at least when the standard orbital basis for that geometry

is used. At the same time, they depend on the number of Landau levels kept. The goal is now to work out the second quantized operator equations of the last section for the case of multiple LLs, especially the recursion Eq. (2.49). Our strategy will be to work backwards from Eq. (2.50), which is essentially a statement about polynomials and which therefore holds independent of the number of LLs. The glue between these two equations was the general Eq. (2.42), which flows from the elementary Eq. (2.43). We thus begin by re-establishing relations concerning the operators associated with power-sum and elementary symmetric polynomials. We will consider $n = 2$ first, from which the general structure will become obvious. For lowest LL case, we used thick cylinder conventions for pedagogical reasons. In the presence of multiple Landau levels, the advantage of this geometry is less immediate, and hence we will start by working in disk geometry. The following treatment will specialize in a re-derivation of most of the results for Laughlin's state in disk geometry when all the higher LL creation/annihilation operators are set equal to zero.

We start by giving the equation for the operator $\hat{p}_k$, which again describes the multiplication with the polynomial $\sum_{i=1}^{N} z_i^k$. As before, these are single particle operators, and can be straightforwardly worked out in second quantization from their first quantized definition. Using Eq. (3.31),

$$\begin{aligned}\hat{p}_k = &\sum_{r=0}^{+\infty} \sqrt{\frac{(r+k)!}{r!}} c^\dagger_{0,r+k} c_{0,r} + \sum_{r=-1}^{+\infty} k \sqrt{\frac{(r+k)!}{(r+1)!}} c^\dagger_{0,r+k} c_{1,r} \\ &+ \sum_{r=-1}^{+\infty} \sqrt{\frac{(r+k+1)!}{(r+1)!}} c^\dagger_{1,r+k} c_{1,r}.\end{aligned} \tag{3.34}$$

Here, the operator $c_{m,r}$ now refers to the orbital with angular momentum r in the mth LL, with $r \geq -m$. An inconvenience is the fact that the commutator $[c^\dagger_{m,r}, \hat{p}_k]$ is not diagonal in m, i.e., in general produces terms referring to Landau levels other than m. This precludes straightforward generalization of Eq. (2.42), which rests on the simple form of Eq. (2.43). However, one can rewrite the Eq. (3.34) as

$$\begin{aligned}\hat{p}_k = &\sum_{r=0}^{+\infty} \sqrt{\frac{(r+k)!}{r!}} c^\dagger_{0,r+k}(c_{0,r} - \sqrt{r+1}c_{1,r}) \\ &+ \sum_{r=-1}^{+\infty} \sqrt{\frac{(r+k+1)!}{(r+1)!}}(c^\dagger_{1,r+k} + \sqrt{r+k+1}c^\dagger_{0,r+k})c_{1,r}.\end{aligned} \tag{3.35}$$

It turns out that the operators made explicit in this factorization have favorable commutation relations. We introduce

$$\tilde{c}^*_{a,r} = \sum_b A(r)_{ab} c^\dagger_{b,r}; \quad \tilde{c}_{a,r} = \sum_b A(r)^{-1}_{ba} c_{b,r}, \tag{3.36}$$

where

$$A(r) = \begin{pmatrix} \sqrt{r!} & 0 \\ (1+r)\sqrt{r!} & \sqrt{(1+r)!} \end{pmatrix}, \tag{3.37}$$

and note that $\tilde{c}^*_{i,r} \neq \tilde{c}^\dagger_{i,r}$, but we still have anti-commutation relations

$$\begin{aligned}\{\tilde{c}_{i,r}, \tilde{c}^*_{j,r'}\} &= \delta_{i,j}\delta_{r,r'} \\ \{\tilde{c}_{i,r}, \tilde{c}_{j,r'}\} &= \{\tilde{c}^*_{i,r}, \tilde{c}^*_{j,r'}\} = 0.\end{aligned} \tag{3.38}$$

The restriction $r \geq -i$ of the $c_{i,r}$, $c^\dagger_{i,r}$-operators carries over to the $\tilde{c}_{i,r}$, $\tilde{c}^*_{i,r}$-operators, As usual, we will use the convention $\tilde{c}_{i,r} = \tilde{c}^*_{i,r} = 0$ whenever r lies outside this range. The significance of the operators $\tilde{c}^*_{i,r}$ is that they create the non-orthogonal, non-normalized

single particle states $z^{i+r}\bar{z}^i$ (Gaussians omitted). This gives

$$\hat{p}_k = \sum_{a=0,1} \sum_{r=-a}^{+\infty} \tilde{c}^*_{a,r+k} \tilde{c}_{a,r} \tag{3.39}$$

such that

$$[\tilde{c}^*_{a,r}, \hat{p}_k] = -\tilde{c}^*_{a,r+k}, \tag{3.40}$$

which is analogous to Eq. (2.43), with the "LL level like" basis label a a pure spectator. We still have $\hat{p}_0 = \hat{N}$. Observe that if we specialize to a single LL, the transformation (3.36) facilitates just the similarity transformation discussed in the preceding section. The only difference is that here we do not view this as an "active" transformation between different geometries, but rather as a "passive" change of basis, involving a non-orthonormal basis (though still orthogonal for $n = 1$). In this basis, not only the zero modes operators, but all of the two-body annihilation operators (Eq. (3.16)) also transforms into simple looking expression. later, we will discuss the explicit form of those operators in the pseudo fermion, $\tilde{c}$ basis. At this time, we want to point out to the entangled Pauli principle 2/5 state on this basis.

Entangled Pauli principle (EPP) for $2/5^{th}$ state in pseudo-fermion language can be simplified in terms of two rules.

a) No double occupancy or nearest-neighbor (NN) occupancy in guiding-center coordinates is allowed.

b) Next -NN, occupancy is allowed only if they form spin singlet state.

In the pseudofermion language, $\tilde{c}_{1,m}$ denotes $\uparrow$ and $\tilde{c}_{0,m}$ denotes for a spin $1/2$ algebra. Thus root pattern $2/5$ state looks like,

$$...\boxed{\uparrow 0 \downarrow}00\boxed{\uparrow 0 \downarrow}00\boxed{\uparrow 0 \downarrow}... \tag{3.41}$$

Where $\boxed{\uparrow 0 \downarrow}$ denotes the singlet state. We will discuss this emergent $SU(2)$ symmtetry for 2-LLs in a larger context of $SU(N)$ symmetry for n-LL case.

At this point, it is necessary to make contact with the order parameter recursion relation for the composite fermions, in a microscopic level. While constructing such order parameter in the first quantized method is indeed a daunting task. We will, however, use guiding-center coordinates in the new pseudofermion basis. This will give us such order parameter calculation as a simple algebraic extension of the Laughlin's case. The implication of constructing such an order parameter is huge. Using a similar exercise done for Laughlin's state (see Eq. (2.61)-Eq. (2.66)), one can immediately construct parent Hamiltonian for all of the composite fermions.

3.9 Order parameter recursion formulas for multiple Landau level composite fermion states

3.9.1 Operator recursion

With this new expression for the $\hat{p}_k$, it is straightforward to adapt the operators for the elementary symmetric polynomials:

$$\begin{aligned}\hat{e}_k = \frac{1}{k!} \sum_{a_1,\ldots,a_k=0,1} \sum_{l_1,\ldots,l_k} & \tilde{c}^*_{a_1,l_1+1}\tilde{c}^*_{a_2,l_2+1}\cdots\tilde{c}^*_{a_k,l_k+1} \\ & \times \tilde{c}_{a_k,l_k}\cdots\tilde{c}_{a_2,l_2}\tilde{c}_{a_1,l_1} \\ \text{for} \quad & k>0, \\ \hat{e}_0 = \mathbb{1}, \qquad & \hat{e}_k = 0 \quad \text{for} \quad k<0.\end{aligned} \tag{3.42}$$

Indeed, the $\hat{e}_k$ and $\hat{p}_k$ still satisfy the Newton-Girard formula Eq. (2.41). Given that the $\hat{p}_k$ represent power-sum symmetric polynomials, this again uniquely identifies the $\hat{e}_k$ in the above equation as representing elementary symmetric polynomials. Owing to Eqs. (3.39) and (3.40), the proof that Newton-Girard equations are satisfied is a straightforward generalization of that given in Ref. [11] for the LLL. Details are given in Appendix A.2.

In a similar vein, one then easily generalizes Eq. (2.42) to the present situation, using the same procedure as in Sec. 3.8:

$$\begin{aligned}
&\tilde{c}^*_{a,k}\mathbf{P}(\hat{p}_0,\hat{p}_1,\ldots,\hat{p}_N)\\
&=\sum_{l_0,l_1,\ldots,l_N}\frac{(-1)^{l_0+l_1+\cdots l_N}}{l_0!l_1!\cdots l_N!}\left(\partial^{l_0}_{p_0}\cdots\partial^{l_N}_{p_N}\mathbf{P}\right)(\hat{p}_0,\hat{p}_1,\ldots,\hat{p}_N)\\
&\times\tilde{c}^*_{a,k+l_1+2l_2+\cdots+Nl_N}.
\end{aligned}\tag{3.43}$$

With this it is a simple task to carry out the program described at the beginning of this section: We take the last line of Eq. (2.50) as the recursive definition of the $\hat{J}_N$ operator, with $\hat{J}_0 = \mathbb{1}$. From this we easily obtain, using the generalized Eq. (2.42), a generalized version of the operator recursion (2.49) :

$$\begin{aligned}
\hat{J}_0 =&\mathbb{1},\\
\hat{J}_N =&\frac{1}{N}\sum_a\sum_{r\geq 0}\sum_{m\geq -a}\tilde{c}^*_{a,m+r}\hat{S}_{M(N-1)-r}\hat{J}_{N-1}\tilde{c}_{a,m},
\end{aligned}\tag{3.44}$$

Lastly, just as in the preceding section, and as explained in Appendix A.1, we obtain from this the generalization of Eq. (2.52):

$$\tilde{c}_{a,r}\hat{J}_N=\sum_{m\geq -a}\hat{S}_{M(N-1)-r+m}\hat{J}_{N-1}\tilde{c}_{a,m}.\tag{3.45}$$

With all the key ingredients in hand, let us now construct the densest composite fermion states occupying two Landau levels, also known as Λ-levels (ΛLs) in this context. [88] These are just the Jain states at filling factor $2/(2M+1)$. We define

$$
\begin{aligned}
|\psi_{2N}\rangle \sim & \hat{J}_{2N} c^\dagger_{1,-1} c^\dagger_{0,0} c^\dagger_{1,0} \ldots c^\dagger_{0,N-2} c^\dagger_{1,N-2} c^\dagger_{0,N-1} |0\rangle \,, \\
|\psi_{2N+1}\rangle \sim & \hat{J}_{2N+1} c^\dagger_{1,-1} c^\dagger_{0,0} c^\dagger_{1,0} \ldots c^\dagger_{0,N-2} c^\dagger_{1,N-2} \\
& \times c^\dagger_{0,N-1} c^\dagger_{1,N-1} |0\rangle
\end{aligned}
\tag{3.46}
$$

for particle number $2N$ and $2N+1$, respectively. It is easy to see that, up to normalization factors, these are exactly equal to

$$
\begin{aligned}
|\psi_{2N}\rangle = & \hat{J}_{2N} \tilde{c}^*_{1,-1} \tilde{c}^*_{0,0} \tilde{c}^*_{1,0} \ldots \tilde{c}^*_{0,N-2} \tilde{c}^*_{1,N-2} \tilde{c}^*_{0,N-1} |0\rangle \,, \\
|\psi_{2N+1}\rangle = & \hat{J}_{2N+1} \tilde{c}^*_{1,-1} \tilde{c}^*_{0,0} \tilde{c}^*_{1,0} \ldots \tilde{c}^*_{0,N-2} \tilde{c}^*_{1,N-2} \\
& \times \tilde{c}^*_{0,N-1} \tilde{c}^*_{1,N-1} |0\rangle \,,
\end{aligned}
\tag{3.47}
$$

which we use to fix the normalization. We note that for $M = 2$ this defines precisely the Jain-2/5 state, for which again a local pseudo-potential Hamiltonian can be given, such that the states (3.47) are densest zero modes. [39, 43] It can be shown that the set of all (N-particle) zero modes of this Hamiltonian is precisely the *range* of the operator $\hat{J}_N$, that is, the set generated from states obtained when $\hat{J}_N$ acts on general N-particle Slater determinants, [39] as opposed to only the *densest* (lowest angular momentum) Slater determinants used in the definitions (3.47). For the cases, $M > 2$ and/or $n > 2$, there exist, to our knowledge, no local parent Hamiltonians with similar properties in the literature, and we leave there discussion as an interesting problem for the future. For these cases, we will simply *define* the N-particle zero mode space as the range of the operator $\hat{J}_N$.

3.9.2 Zero mode generators

Before we further apply the results of this section, we need to introduce a larger set of operators that we will think of as "zero mode generators". Also, we use this opportunity to generalize the setting of the preceding subsection from 2 to a general number of n LLs. This is straightforward in principle. Essentially, all it takes is to generalize Eq. (3.36) by means of an appropriate $n \times n$ matrix $A(r)$. The explicit form of $A(r)$ is given in Appendix A.3.

In the following, we will be interested in the generalization of the recursive formulas for the ($n = 1$) Laughlin state to the n-ΛL composite fermion states, in particular, (3.47) for $n = 2$. In addition to the operator recursion (3.44), this requires an understanding of zero mode generators, i.e., operators like the $\hat{e}_k$ and $\hat{p}_k$ that generate more (possibly, all) zero modes when acting on the "incompressible" (densest, or smallest angular momentum) zero mode. To this end, in the n LL system, one can construct n^2 different operators which will satisfy a modified Newton-Girard formula, namely,

$$\hat{p}_k^{a,b} = \sum_{r=-b}^{+\infty} \tilde{c}_{a,r+k}^* \tilde{c}_{b,r} \tag{3.48}$$

such that

$$\hat{e}_k^{a,b} = \frac{1}{k} \hat{p}_1^{a,b} \hat{e}_{k-1}^{a,b} + \frac{\delta_{a,b}}{k} \sum_{d=2}^{k} (-1)^{d-1} \hat{p}_d^{a,b} \hat{e}_{k-d}^{a,b}, \tag{3.49}$$

where $\hat{e}_k^{a,b}$ can be written explicitly,

$$\begin{aligned} \hat{e}_k^{a,b} = & \frac{1}{k!} \sum_{l_1,\ldots,l_k=-b}^{+\infty} \tilde{c}_{a,l_1+1}^* \tilde{c}_{a,l_2+1}^* \cdots \tilde{c}_{a,l_k+1}^* \\ & \times \tilde{c}_{b,l_k} \cdots \tilde{c}_{b,l_2} \tilde{c}_{b,l_1}. \end{aligned} \tag{3.50}$$

The proof of these (modified) Newton-Girard formulae is given in Appendix A.2. It is through the introduction of these new operators that our formalism offers a true advantage over a first quantized language of polynomials. Unlike the $\hat{p}_k$, $\hat{e}_k$, Eqs. (3.48), (3.50) have no particularly natural presentation in polynomial language (see below), but still, have the favorable algebraic properties discussed here.

The significance of these operators is the following. First, we identify the operators $\hat{p}_d^{a,a}$ as the operators that send first quantized expressions of the form $\bar{z}^a z^{a+\ell}$ to $\bar{z}^a z^{a+\ell+d}$. It is then clear that the operator

$$\hat{p}_d = \sum_{a,b} \delta_{a,b} \hat{p}_d^{a,b}. \tag{3.51}$$

multiplies any single particle wave function by z^d, and, in the general many-particle context, can be identified as the operator associated with the power-sum polynomial p_d as before. Similarly, the operators $\hat{e}_k$ associated with elementary symmetric polynomials are obtained as the trace over the $\hat{e}_k^{a,b}$, as, by Eq. (3.50), the $\hat{e}_k$ and $\hat{p}_k$ then satisfy Newton-Girard relations. In order to further motivate the physical meaning of $\hat{p}_d^{a,b}$, let us look into their commutation relations,

$$[\hat{p}_k^{a,b}, \hat{p}_{k'}^{b',a'}] = \delta_{b,b'} \hat{p}_{k+k'}^{a,a'} - \delta_{a,a'} \hat{p}_{k+k'}^{b',b}. \tag{3.52}$$

This immediately implies

$$[\hat{p}_k^{a,b}, \hat{p}_{k'}] = 0. \tag{3.53}$$

$\hat{p}_k^{a,b}$s construct a $SU(n)$ layered algebra, with $\hat{p}_k$ being the center of the group.

For the composite fermion operator $\hat{J}_N$, on the other hand, we will always use the recursion Eq. (2.50) as the defining property. Therefore, as before, the $\hat{J}_N$ are always expressible

through the $\hat{p}_k$. The last equation then gives

$$[\hat{J}_N, \hat{p}_k^{a,b}] = [\hat{J}_N, \hat{e}_k^{a,b}] = 0, \tag{3.54}$$

where, for the $\hat{e}_k^{a,b}$, we have used the fact that by the relations (3.50), we can express all of the latter through the $\hat{p}_k^{a,b}$. As explained/defined above, the space of all zero modes is precisely the range of the operator $\hat{J}_N$. Eqs. (3.54) then say that the zero mode space is *invariant* under the action of the $\hat{p}_k^{a,b}$ or $\hat{e}_k^{a,b}$. That is, when any of these operators act on a zero mode, a new zero mode results. It is for this reason that we think of these operators as zero mode generators. It is further true that we can generate any N-particle zero modes by repeatedly acting with these generators on certain incompressible (lowest angular momentum) zero modes ψ_N, such as the Laughlin state or a Jain state. In this sense we can think of both the $\hat{p}_k^{a,b}$ as well as the $\hat{e}_k^{a,b}$ (separately) as a complete set of zero mode generators.

We close this section by remarking that with the generalized $A(r)$-matrix of Appendix A.3, Eqs. (3.44) and (3.45) generalize without change to $n > 2$ LLs.

3.9.3 Recursion formulas for general composite fermion states

Let us consider the second quantized composite fermion wave function at the filling fraction $\nu = \frac{n}{Mn+1}$ for $N = n(L_{\max} + (n-1)/2) + q$ particles with $1 \leq q \leq n$,

$$|\psi_N\rangle = \hat{J}_N |\Psi_N\rangle , \tag{3.55}$$

where the wave function $|\Psi_N\rangle$ corresponds to the state in which orbitals $\tilde{c}^*_{r,j}$ with indices $r = 0, 1...q-1$ are filled up to angular momentum $j = L_{\max}$, and in which orbitals $\tilde{c}^*_{r,j}$ with

indices $r = q, q+1, ..., n-1$ are filled up to angular momentum $j = L_{\max} - 1$. Explicitly,

$$\begin{aligned}
|\Psi_N\rangle = & \tilde{c}^*_{n-1,-(n-1)} \tilde{c}^*_{n-1,-(n-2)} \tilde{c}^*_{n-1,-(n-3)} \cdots \tilde{c}^*_{n-1,L_{\max}-1} \\
& \times \tilde{c}^*_{n-2,-(n-2)} \tilde{c}^*_{n-2,-(n-3)} \cdots \tilde{c}^*_{n-2,L_{\max-1}} \\
& \times \cdots \\
& \times \tilde{c}^*_{q,-q} \tilde{c}^*_{q,-q+1} \cdots \tilde{c}^*_{q,L_{\max}-1} \\
& \times \tilde{c}^*_{q-1,-(q-1)} \cdots \tilde{c}^*_{q-1,L_{\max}} \\
& \times \cdots \\
& \times \tilde{c}^*_{0,0} \cdots \tilde{c}^*_{0,L_{\max}} |0\rangle \, .
\end{aligned} \tag{3.56}$$

By abuse of terminology, we will now refer to the index r in $\tilde{c}^*_{r,j}$ as a Λ-level index, and to the orbitals created by $\tilde{c}^*_{r,j}$ with fixed r as a Λ-level. Let us introduce a state $|\Psi_N^{m,k}\rangle$, where we have created a hole in the k-th ΛL at angular momentum m with $k = 0, 1...n-1$. With Eq. (3.45), we have

$$\tilde{c}_{k,r} |\psi_N\rangle = \sum_{m \geq -k} \hat{S}_{M(N-1)-r+m} \hat{J}_{N-1} \tilde{c}_{k,m} |\Psi_N\rangle = \sum_{m \geq -k} \hat{S}_{M(N-1)-r+m} \hat{J}_{N-1} |\Psi_N^{m,k}\rangle \, . \tag{3.57}$$

Now we need to relate $|\Psi_N^{m,k}\rangle$ to some zero mode generator acting on $|\Psi_{N-1}\rangle$, where the only difference between $|\Psi_{N-1}\rangle$ and $|\Psi_N\rangle$ is that the orbital at $L_{\max}$ in $q-1$-th ΛL in $|\Psi_{N-1}\rangle$ is vacant. What's required is that the zero mode generator moves the particle from the orbital corresponding to $\tilde{c}^*_{k,m}$ to that corresponding to $\tilde{c}^*_{q-1,L_{\max}}$ in $|\Psi_{N-1}\rangle$. For the sake of conciseness, we will simply say moving the particle from $\tilde{c}^*_{k,m}$ to $\tilde{c}^*_{q-1,L_{\max}}$ and similarly for other processes involving moves of particles.

As seen in Fig. 3.3, we need to consider three cases, (i) $k > q-1$, (ii) $k < q-1$ and (iii) $k = q-1$. In case (i), $k > q-1$, the first step is to act with $\hat{p}_1^{q-1,k}$ on $|\Psi_{N-1}\rangle$ so that one

particle is moved from $\tilde{c}^*_{k,L_{\max}-1}$ to $\tilde{c}^*_{q-1,L_{\max}}$. The second step is to further act $\hat{e}^{k,k}_{L_{\max}-m-1}$ on the resultant state to move all the particles in k-th ΛL beginning with $\tilde{c}^*_{k,m}$ and ending with $\tilde{c}^*_{k,L_{\max}-2}$ to the right such that their angular momenta all increase by 1. This is reflected by the following identity,

$$\hat{e}^{k,k}_{L_{\max}-m-1}\hat{p}_1^{q-1,k}\left|\Psi_{N-1}\right\rangle = (-1)^{f(m)}\left|\Psi_N^{m,k}\right\rangle, \tag{3.58}$$

where $f(m) = (n-q+1)(2L_{\max}+q+n-2)/2 - L_{\max} - m$.

This leads to

$$\tilde{c}_{k,r}\left|\psi_N\right\rangle = \sum_{m\geq -k}(-1)^{f(m)}\hat{S}_{M(N-1)+m-r}\hat{e}^{k,k}_{L_{\max}-m-1}\hat{p}_1^{q-1,k}\left|\psi_{N-1}\right\rangle, \tag{3.59}$$

where we have used the commutation relations (3.53) and (3.54).

Case (iii), $k = q-1$, is very similar, only that no action with a $\hat{p}$-type operator is necessary. We obtain

$$\tilde{c}_{q-1,r}\left|\psi_N\right\rangle = \sum_{m\geq -(q-1)}(-1)^{f(m)}\hat{S}_{M(N-1)+m-r}\hat{e}^{q-1,q-1}_{L_{\max}-m}\left|\psi_{N-1}\right\rangle. \tag{3.60}$$

In case (ii), $k < q-1$, the first step is to act $\hat{p}_0^{q-1,k}$ on $\left|\Psi_{N-1}\right\rangle$ so that one particle is moved from $\tilde{c}^*_{k,L_{\max}}$ to $\tilde{c}^*_{q-1,L_{\max}}$. Then we act $\hat{e}^{k,k}_{L_{\max}-m}$ on the resultant state to move all the particles in the k-th ΛL beginning with $\tilde{c}^*_{k,m}$ and ending with $\tilde{c}^*_{k,L_{\max}-1}$ to the right such that their angular momenta all increase by 1. The overall phase picked up in the process differs by -1 from the formula given for the other two cases. Thus we have

$$\tilde{c}_{k,r}\left|\psi_N\right\rangle = \sum_{m\geq -k}(-1)^{f(m)-1}\hat{S}_{M(N-1)+m-r}\hat{e}^{k,k}_{L_{\max}-m}\hat{p}_0^{q-1,k}\left|\psi_{N-1}\right\rangle. \tag{3.61}$$

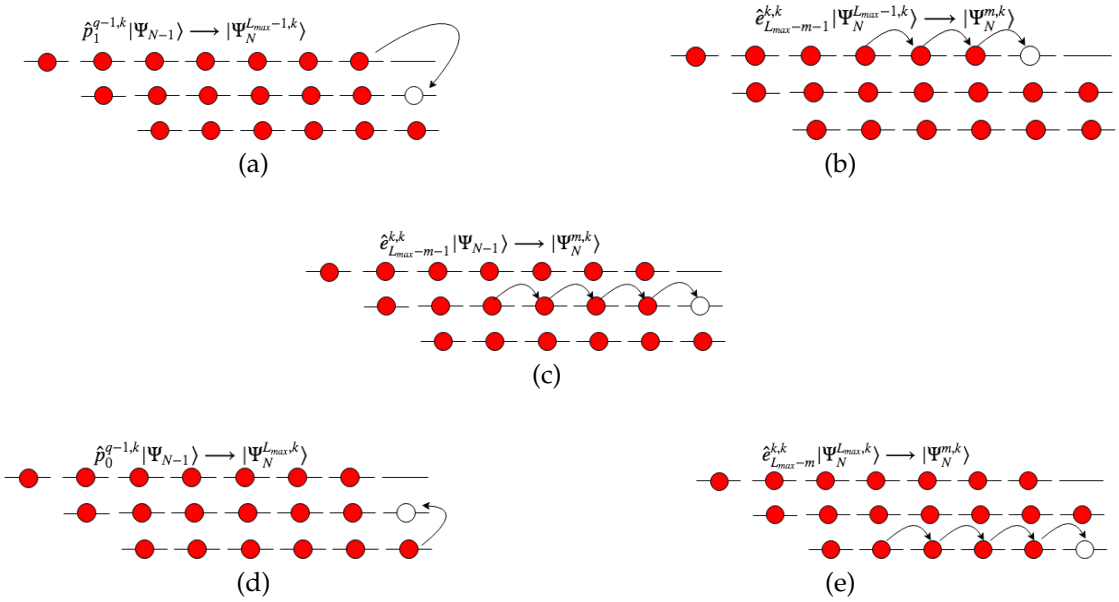


Figure 3.3: Connecting bare fermion Slater determinants $|\Psi_{N-1}\rangle$ (integer quantum Hall) and $|\Psi_N^{m,k}\rangle$ (one hole) via zero mode generators. Shown are visualizations of the processes used in Eqs.(3.59)-(3.61). All three relevant cases (see main text) are illustrated for $n = 3$ Landau levels.

It is now clear that we can repeat the logic that led to the recursion (2.59) for the higher ΛL composite fermion states: To this end, we simply state

$$|\psi_N\rangle = \frac{1}{N}\sum_{k,r} c^\dagger_{k,r} c_{k,r}\,|\psi_N\rangle = \frac{1}{N}\sum_{k,r} \tilde{c}^*_{k,r}\tilde{c}_{k,r}\,|\psi_N\rangle\ . \tag{3.62}$$

In here, we simply replace $\tilde{c}_{k,r}\,|\psi_N\rangle$ with Eqs. (3.59)-(3.61). This gives the desired recursion of $|\psi_N\rangle$ in terms of $|\psi_{N-1}\rangle$. In the following section, we apply these results to the special case $n = 2$ again.

3.10 Recursion formulas for $n = 2$ Λ-level composite fermion states

In the last section, we have constructed recursive formulas for any second quantized composite fermion wave functions. In this section, we will further simplify these formulas for composite fermion states involving two ΛLs. In Sec. 3.11, we will prove, for the special case $M = 2$ describing the Jain-2/5 state, that this state is indeed the densest zero mode of its parent Hamiltonian of the general form (2.58). Our proof differs from a previous one [39] in that it makes no use whatsoever of the polynomial structure of the state's first quantized wave function, but rests entirely on the operator algebra developed here. There, we will also comment further on the connection between the quasihole operators given in Ref. [39] and those in this paper. In a similar vein, we will show how to extract the filling factor of the CF-states using the present, "polynomial free" apparatus. In this section, we'll find it convenient to denote the particle number as $2N$ and $2N+1$, respectively, for the even and odd case, as in Eq. (3.47) above.

We now use Eq. (3.60), specializing to $n = 2$, $k = 1$, $q = 2$ and $L_{\max} = N - 1$ for $2N + 1$ particles. This gives

$$\tilde{c}_{1,r} \left|\psi_{2N+1}\right\rangle = \sum_{m=-1}^{N-1} (-1)^{1-m} \hat{S}_{2MN+m-r} \hat{e}^{1,1}_{N-1-m} \left|\psi_{2N}\right\rangle . \tag{3.63}$$

The above can be put into a concise form,

$$c_{1,r} \left|\psi_{2N+1}\right\rangle = (-1)^N \sqrt{\frac{(r+1)!}{N!}} \hat{S}^{\sharp 1,1}_{(2M+1)N-1-r} \left|\psi_{2N}\right\rangle , \tag{3.64}$$

where

$$\hat{S}_\ell^{\sharp a,b} = \sum_m (-1)^m \hat{S}_{\ell-m} \hat{e}_m^{a,b} \,, \tag{3.65}$$

a definition that we may adopt for any n. In the last equation, we have also replaced $\tilde{c}_{1,r}$ with the operator $c_{1,r}$, referring to the original (orthonormal) basis.

Similarly using Eq. (3.61), we obtain

$$\tilde{c}_{0,r} \left|\psi_{2N+1}\right\rangle = (-1)^{N-1} \hat{S}^{\sharp 0,0}_{(2M+1)N-1-r} \hat{p}_0^{1,0} \left|\psi_{2N}\right\rangle . \tag{3.66}$$

This leads to

$$c_{0,r} \left|\psi_{2N+1}\right\rangle = (-1)^N \sqrt{\frac{r!}{N!}} \left((r+1) \hat{S}^{\sharp 1,1}_{(2M+1)N-1-r} - \hat{S}^{\sharp 0,0}_{(2M+1)N-1-r} \hat{p}_0^{1,0} \right) \left|\psi_{2N}\right\rangle , \tag{3.67}$$

Note that

$$c_{1,r} \left|\psi_{2N+1}\right\rangle = c_{0,r} \left|\psi_{2N+1}\right\rangle = 0 \quad \text{for} \quad r > (2M+1)N - 1 \tag{3.68}$$

as, by definition, $S_\ell^{\sharp a,b}$ vanishes for $\ell < 0$. This establishes that the highest occupied orbital in $\left|\psi_{2N+1}\right\rangle$ has angular momentum $\ell_{\mathsf{max}} \leq (2M+1)N - 1$. Moreover, since $S_0^{\sharp a,b} = \mathbb{1}$, Eq. (3.64) for $r = (2M+1)N - 1$ gives that the orbital created by $c^\dagger_{1,(2M+1)N-1}$ is certainly occupied in the state $\left|\psi_{2N+1}\right\rangle$, as long as $\left|\psi_{2N}\right\rangle$ is not zero. In particular, the state $\left|\psi_{2N+1}\right\rangle$ does not vanish as long as $\left|\psi_{2N}\right\rangle$ doesn't. Assuming this for the moment, we find $\ell_{\mathsf{max}} = (2M+1)N - 1$. Defining the filling factor as the particle number $2N+1$ divided by ℓ_{max}, we see that the filling factor approaches $2/(2M+1)$ in the thermodynamic limit, as expected. Similar arguments carry over to larger n.

In the same way of obtaining Eqs. 3.64 and 3.67, we obtain

$$c_{1,r}\,|\psi_{2N}\rangle = \sqrt{\frac{(r+1)!}{(N-1)!}}\hat{S}^{\sharp 1,1}_{(2M+1)N-M-2-r}\hat{p}_1^{0,1}\,|\psi_{2N-1}\rangle \tag{3.69}$$

and

$$c_{0,r}\,|\psi_{2N}\rangle = \sqrt{\frac{r!}{(N-1)!}}\left((r+1)\hat{S}^{\sharp 1,1}_{(2M+1)N-M-2-r}\hat{p}_1^{0,1} - \hat{S}^{\sharp 0,0}_{(2M+1)N-M-1-r}\right)|\psi_{2N-1}\rangle\,. \tag{3.70}$$

Again, we can immediately see that $c_{1,r}\,|\psi_{2N}\rangle$ vanishes for $r > (2M+1)N - M - 2$ and $c_{0,r}\,|\psi_{2N}\rangle$ vanishes for $r > (2M+1)N - M - 1$. On the other hand, $c_{0,(2M+1)N-M-1}\,|\psi_{2N}\rangle$ is proportional to $|\psi_{2N-1}\rangle$. In particular, $|\psi_{2N}\rangle$ is nonzero if $|\psi_{2N-1}\rangle$ is. Together with the observation below Eq. (3.68), this establishes inductively that the states $|\psi_{2N}\rangle$, $|\psi_{2N+1}\rangle$ do not vanish (even if we did not know the meaning of the operator $\hat{J}_N$ in first quantization), and that $\ell_{\mathsf{max}} = (2M+1)N - 1$ for $|\psi_{2N+1}\rangle$ and $\ell_{\mathsf{max}} = (2M+1)N - M - 1$ for $|\psi_{2N}\rangle$.

Now we use Eqs. 3.64, 3.67 and the identity Eq. (3.62) to get a recursive formula

$$\begin{aligned}
|\psi_{2N+1}\rangle = {} & \frac{(-1)^N}{(2N+1)\sqrt{N!}}\sum_{r=-1}^{(2M+1)N-1}\sqrt{(r+1)!}\,c^\dagger_{1,r}\hat{S}^{\sharp 1,1}_{(2M+1)N-1-r}\,|\psi_{2N}\rangle \\
& + \frac{(-1)^N}{(2N+1)\sqrt{N!}}\sum_{r=0}^{(2M+1)N-1}(r+1)\sqrt{r!}\,c^\dagger_{0,r}\hat{S}^{\sharp 1,1}_{(2M+1)N-1-r}\,|\psi_{2N}\rangle \\
& - \frac{(-1)^N}{(2N+1)\sqrt{N!}}\sum_{r=0}^{(2M+1)N-1}\sqrt{r!}\,c^\dagger_{0,r}\hat{S}^{\sharp 0,0}_{(2M+1)N-1-r}\hat{p}_0^{1,0}\,|\psi_{2N}\rangle\,.
\end{aligned} \tag{3.71}$$

Likewise, we can also obtain $|\psi_{2N}\rangle$ from $|\psi_{2N-1}\rangle$,

$$
\begin{aligned}
|\psi_{2N}\rangle = & \frac{1}{2N\sqrt{(N-1)!}} \sum_{r=-1}^{(2M+1)N-M-2} \sqrt{(r+1)!}\, c^{\dagger}_{1,r} \hat{S}^{\sharp 1,1}_{(2M+1)N-M-2-r} \hat{p}_1^{0,1} |\psi_{2N-1}\rangle \\
& + \frac{1}{2N\sqrt{(N-1)!}} \sum_{r=0}^{(2M+1)N-M-2} (r+1)\sqrt{r!}\, c^{\dagger}_{0,r} \hat{S}^{\sharp 1,1}_{(2M+1)N-M-2-r} \hat{p}_1^{0,1} |\psi_{2N-1}\rangle \\
& - \frac{1}{2N\sqrt{(N-1)!}} \sum_{r=0}^{(2M+1)N-M-1} \sqrt{r!}\, c^{\dagger}_{0,r} \hat{S}^{\sharp 0,0}_{(2M+1)N-M-1-r} |\psi_{2N-1}\rangle \,.
\end{aligned}
\tag{3.72}
$$

The above recursions, together with the expressions of local charge-1 holes through zero-mode generators acting on an incompressible state, as well as their $n > 2$ generalizations of the preceding section, are the central results of this paper.

3.11 Proof of the zero mode properties from guiding-center coordinate viewpoint

The construction of parent Hamiltonians for FQH states has traditionally emphasized analytic clustering properties of special wave functions. Obstructions for successfully doing this, so far, for most composite fermion states have been discussed by some of us. [39] In short, we argued that a successful parent Hamiltonian satisfying the zero mode paradigm discussed in the introduction is possible in principle only for *unprojected* CF states, such as discussed in this paper. (There may, of course, be parent Hamiltonians outside this paradigm. [89]) On the other hand, Landau level mixing makes it harder to harvest nice analytic clustering properties for the construction of a parent Hamiltonian. A notable exception is the case $n = M = 2$, leading to the Jain 2/5-state. An extensive discussion of its parent Hamiltonian was given in Ref. [39]. There, some of the framework established in this paper has been anticipated, as well as the fact that the zero mode properties

of the 2/5-parent Hamiltonian can be understood as a purely algebraic consequence of the second-quantized operators that can be used to define it (Eq. (3.73) below) and their interplay with the zero mode generators extensively discussed here. Indeed, this approach allows one to establish properties of parent Hamiltonians while "forgetting" the analytic properties of the associated first-quantized many-body wave functions. While this is somewhat counter to traditional construction principles in FQH physics, we argue this to be fruitful in the context of CF states with $n \geq 2$, where parent Hamiltonians are somewhat scarce. This approach also resonates with the manifestly guiding-center-projected language recently advocated by Haldane. [87] While in Ref. [39] we did not elaborate on how to establish the zero mode properties of the 2/5-Hamiltonian in such a purely algebraic manner, here we are in a perfect position to do so. We begin by presenting the Hamiltonian in more general form as the sum of four two-particle projection operators at each pair-angular-momentum $2J$,

$$\begin{aligned} H = E^{(1)} \sum_J \mathcal{T}_J^{(1)\dagger} \mathcal{T}_J^{(1)} + E^{(2)} \sum_J \mathcal{T}_J^{(2)\dagger} \mathcal{T}_J^{(2)} \\ + E^{(3)} \sum_J \mathcal{T}_J^{(3)\dagger} \mathcal{T}_J^{(3)} + E^{(4)} \sum_J \mathcal{T}_J^{(4)\dagger} \mathcal{T}_J^{(4)}. \end{aligned} \tag{3.73}$$

Here,

$$\mathcal{T}_J^{(\lambda)} = \sum_{x,m_1,m_2} \eta^{(\lambda)}_{J,x,m_1,m_2} c_{m_1,J-x} c_{m_2,J+x} \tag{3.74}$$

is a fermion bilinear that destroys a pair of particles of angular momentum $2J$. The details of the form factors $\eta^{(\lambda)}_{J,x,m_1,m_2}$ are of no importance in the following, but will be given in Appendix C.1. The $E^{(\lambda)}$ are positive constants that are arbitrary in principle, but may be chosen so as to give the Hamiltonian a simple "Trugman-Kivelson" form [25] in first

quantization, see again Appendix C.1 for this choice. Note that the sum over J goes over integers and half-odd-integers, and x-sums in the $\mathcal{T}$-operators are restricted so that $J \pm x$ are integers.

From the positivity of each of the four terms in the Hamiltonian (3.73), it follows that the zero mode property is equivalent to the following

$$\mathcal{T}_J^{(\lambda)} |\psi_{\text{zm}}\rangle = 0, \quad \text{for } \lambda = 1, 2, 3, 4. \tag{3.75}$$

The zero mode property of the Jain-2/5 state as given by Eq. (3.55) (for $M = n = 2$), with the *recursively defined* composite fermion operator $\hat{J}_N$, Eq. (3.44), then rests on the following properties :

1. The operators identified in Sec. 3.9.2 are zero mode generators precisely in the strict sense defined at the end of Sec. 3.9.1: Namely, they leave invariant the zero mode space defined in terms of the Hamiltonian through Eq. (3.16). We show this in Appendix A.2.

2. The operators $\mathcal{T}_J^{(\lambda)}$ satisfy

$$\mathcal{T}_J^{(\lambda)} = \frac{1}{2} \sum_{m,k} [\mathcal{T}_J^{(\lambda)}, c_{m,k}^\dagger] c_{m,k} \,. \tag{3.76}$$

This is a generic property of the fermion bilinears, and does not depend on the form factors $\eta_{J,x,m_1,m_2}^{(\lambda)}$.

3. The 2-particle CF state $|\psi_{N=2}\rangle$ is a zero mode, allowing an "induction beginning".

We begin by demonstrating property 3. Since $|\psi_{N=0}\rangle = |0\rangle$, we get $|\psi_{N=1}\rangle = c_{1,-1}^\dagger |0\rangle$ and $|\psi_{N=2}\rangle = (\sqrt{2} c_{1,-1}^\dagger c_{0,2}^\dagger + 2 c_{0,1}^\dagger c_{1,0}^\dagger - \sqrt{2} c_{0,0}^\dagger c_{1,1}^\dagger - 4 c_{0,0}^\dagger c_{0,1}^\dagger) |0\rangle$ using Eqs. 3.71 and 3.72.

It is trivial to see that $|\psi_{N=0}\rangle$, and $|\psi_{N=1}\rangle$ are zero modes, and indeed $|\psi_{N=2}\rangle$ can also straightforwardly shown to satisfy the zero mode conditions Eq. (3.75), using the explicit formulas for the $\mathcal{T}_J^{(\lambda)}$ given in Appendix A.2. (Note that this only requires the relatively simple special cases with $J = 1/2$.) Now assuming $|\psi_{2N}\rangle$ $(N \geq 1)$ is a zero mode, we immediately find

$$\mathcal{T}_J^{(\lambda)} c_{1,k} |\psi_{2N+1}\rangle = 0 \tag{3.77a}$$

and

$$\mathcal{T}_J^{(\lambda)} c_{0,k} |\psi_{2N+1}\rangle = 0, \tag{3.77b}$$

since on the right hand sides of Eqs. (3.64) and (3.67), all operators are zero mode generators, acting on the zero mode $|\psi_{2N}\rangle$, thus giving another zero mode.

Acting with $\mathcal{T}_J^{(\lambda)}$, $\lambda = 1, 2, 3, 4$ on the identity (3.62) with particle number being $2N+1$ instead of N, and then using Eq. (3.77), we obtain

$$\begin{aligned} \mathcal{T}_J^{(\lambda)} |\psi_{2N+1}\rangle &= \frac{1}{2N+1} \sum_k [\mathcal{T}_J^{(\lambda)}, c_{0,k}^\dagger] c_{0,k} |\psi_{2N+1}\rangle \\ &+ \frac{1}{2N+1} \sum_k [\mathcal{T}_J^{(\lambda)}, c_{1,k}^\dagger] c_{1,k} |\psi_{2N+1}\rangle \\ &= \frac{2}{2N+1} \mathcal{T}_J^{(\lambda)} |\psi_{2N+1}\rangle , \end{aligned} \tag{3.78}$$

where in the last line, we have used Eq. (3.76). This implies that $|\psi_{2N+1}\rangle$ satisfies the zero mode condition Eq. (3.75). The induction step from odd particle number $2N+1$ to even particle number $2N+2$ proceeds analogously, with the help of Eqs. (3.69), (3.70), thus concluding the induction proof for the zero mode property of $n = M = 2$ ($\nu = 2/5$) Jain-state. Using the methods of Ref. [39], which we later characterized as making use of an "entangled Pauli principle"(EPP), [55] we can also establish that these are the densest possible (highest filling factor or smallest angular momentum) zero modes (see

Ref. [55] for details). Aside from the EPP, the only ingredients needed are knowledge of the total angular momentum of the CF state as defined in Eq. (3.55), and/or its highest occupied orbital, all of which is either manifest or follows from the discussion in Sec. 3.10. In particular, as we have shown here, none of this requires knowledge of the analytic structure of the first quantized Jain-2/5 state wave function.

One may envision that the results of this section readily generalize to other CF state, for which, to the best of our knowledge, so far no (zero-mode-paradigm) parent Hamiltonians have been discussed in the literature, with the exception of the case $n = 1$. This requires identification of the proper set of operators $\mathcal{T}^{(\lambda)}$ that generalize the algebraic features discussed here and in Appendix A.2 to larger n and M, which will require a larger set of such operators. We will comment on this interesting problem in the latter section.

3.12 Microscopic Bosonization for composite fermions

In this brief section, we make contact with an observation made in Ref. [39] (and earlier for Laughlin states in Ref. [11]). This is the fact that the zero mode generators $\hat{p}_k^{m,m}$ (no summation implied), Eq. (3.48), formally look like bosonic modes generating excitations in the m-th branch of a free chiral fermion edge theory. Indeed, a zero mode at small angular momentum k relative to the incompressible ground state must be interpreted as a low energy edge excitation. This can be made concrete by considering a confining potential proportional to total angular momentum, which may be added to the parent Hamiltonian – in those cases where one is known – without changing the eigenstates of the system. Our result can then be considered a microscopic form of bosonization –

the identification of generators of eigenstates for the *microscopic* Hamiltonian with corresponding counterparts in the effective edge theory. To make this case, we must argue that the $\hat{p}_k^{m,m}$ in some sense generate a *complete set* of low energy modes. In this case, we can unambiguously deduce the effective edge theory from exact properties of the microscopic parent Hamiltonian. We note that the latter is quite non-trivial even for the Laughlin-state parent Hamiltonians using conventional polynomial methods. [26]

In Ref. [39] we conjectured that the operators formed by products of the $\hat{p}_k^{m,m}$ do indeed generate a complete set of zero modes (not just at small angular momentum) for the Jain-2/5 parent Hamiltonian when acting on the Jain-2/5 state $|\psi_N\rangle$. With the results of this work, this becomes an easy corollary. To this end, we first note that a complete set of zero modes is given by

$$\hat{J}_N |\Phi\rangle , \tag{3.79}$$

where $|\Phi\rangle$ is any N-particle state within the first n LLs. Specifically for the Jain-2/5 state ($n = M = 2$), we established the densest zero mode in the preceding Section, which is of the form (3.79). The general statement for *all possible* zero modes can either be established in first quantization or, using EPP-based methods and knowledge of the densest zero-mode, in second quantization. See Ref. [39] for details. Here we want to show that all zero modes, of given total particle number N, are obtained by acting on the densest zero mode, $|\psi_N\rangle = \hat{J}_N |\Psi_N\rangle$, Eq. (3.55), with sums of products of the operators $\hat{p}_k^{a,b}$. (For $n = 1$, pertinent considerations were carried out earlier, [11] using somewhat different methods.) We first focus on such zero modes where the $|\Phi\rangle$ in Eq. (3.79) has the same particle number *in each* Λ*-level* as the integer quantum Hall state $|\Psi_N\rangle$. For this, we may restrict ourselves to the operators $\hat{p}_k^{m,m}$. Since we have established that these operators commute with $\hat{J}_N$, the statement is thus simply that each fermion state $|\Phi\rangle$, with given particle number in each of n ΛLs equal to that in the state $|\Psi_N\rangle$ can be expressed as $|\Psi_N\rangle$ acted upon by

sums of products of the $\hat{p}_k^{m,m}$. For $a = b = m$, these operators now act on ΛLs *exactly* as the ones that appear in the bosonization dictionary. The fact that these operators, within each branch (ΛL) m, generate the full fermionic subspace of the same particle number when acting on the "vacuum" present in $|\Psi_N\rangle$ is a well-known theorem in bosonization. Here, we need a version of this theorem at finite particle number, which is also readily available. [11, 90] Similarly, it is easy to see that the operators $\hat{p}_k^{a,b}$, $k \geq 0$, which likewise commute with $\hat{J}_N$, can be used to generate an arbitrary imbalance in particle number between the occupied ΛLs in $|\Psi_N\rangle$, without introducing any holes into any of these ΛLs. By the same reasoning, when acting on these states with all possible combinations of the $\hat{p}_k^{m,m}$, we generate the full Fock space of n ΛLs at fixed particle number. Note that the relative ease with which we can establish this property here crucially depends on having control of the relationship between the operator $\hat{J}_N$ and the operators $\hat{p}_k^{a,b}$, in particular, their trivial commutators.

The above considerations may serve as an alternative proof [39] for the fact that the Jain-2/5 parent Hamiltonian falls into the "zero mode paradigm": Counting of zero modes at given angular momentum Δk relative to the "incompressible state" (densest zero mode) reproduces exactly the mode counting in an associated conformal edge theory.

3.13 Composite fermion state order parameters

The question of off-diagonal long-range order has been an influential subject in the theory of the Hall effect, leading, in particular, to a description in terms of effective Ginzburg-Landau type actions [13, 41, 91, 92]. Beyond this theoretical use, non-local order parameters could in principle be useful in practical numerical calculations, serving as diagnostics for the myriad possible phases in the fractional quantum Hall regime. Unfortunately, a

number of reasons seem to have prohibited widespread use of this approach. For one, there is the problem of efficient evaluation of non-local objects such as

$$\mathcal{O}(z) := (\Psi(z)^\dagger)^p \prod_i (z - z_i)^q \,, \tag{3.80}$$

where the z_i are the complex electron coordinates, and $\Psi(z)^\dagger$ is a local electron creation operator. This order parameter is expected to characterize the order of all composite fermion states with "single particle condensates" at filling fraction $\nu = p/q$. [13,41] The non-locality of this object and the mixed first-second quantized definition make numerical evaluation challenging, though, making use of special properties of spherical geometry, related order parameters have been evaluated for 8-particle systems. [92] We are not aware of any attempt to numerically evaluate Eq. (3.80) on the cylinder, which is arguably the preferred geometry for DMRG. What is more important, the order parameter (3.80) is by itself still a rather crude diagnostic. Already for composite fermion states in n ΛLs, a multiplet of n independent order parameters is expected to exist, which can be given precise meanings in suitable variational wave functions, [41] and which are the basis for field theoretic and/or Ginzburg-Landau level descriptions . [41,42] Except for Eq. (3.80), which is always a member of the "lattice" [41] of order parameters, we are, however, not aware of a general definition of these order parameters as operators acting on the microscopic Fock space.

The results of the preceding sections allow us to address these obstacles in the following way. We will be able to express order parameters such as Eq. (3.80) in a full second quantized form that is directly applicable to planar, spherical, and cylinder geometries, respectively. What's more, for $n > 1$ composite fermion states we will do the same for an n-tuplet of generators of the order parameter lattice, all of whose members will create

charge 1 and are thus more elementary than Eq. (3.80), which creates charge $p > 1$ for $n > 1$.

A close connection between quantum Hall-type order parameters and the developments of this paper could be surmised on the basis that Read wrote the Laughlin state as

$$\left(\int dz (\mathcal{O}(z) \right)^N |\mathsf{vac}\rangle$$

, which leads to the Laughlin state recursion Eq. (2.59), albeit in a mixed first/second quantized guise. We will immediately discuss the general case $n \geq 1$. We start with an argument similar to one made by Read [13] for the Laughlin state and, originally, leading up to the special order parameter (3.80). We will, however, start by working in the orbital basis. Consider the correlation function of the orbital density $\rho_r = \sum_k c^\dagger_{k,r} c_{k,r}$,

$$\langle \psi_{N+1} | \rho_r \rho_{r'} | \psi_{N+1} \rangle \longrightarrow \langle \rho_r \rangle \langle \rho_{r'} \rangle \sim \nu^2 \,, \tag{3.81}$$

where, on the right hand side, we take the limit of large $|r - r'|$ and expect that correlations decay exponentially, causing the un-connected correlator to approach a non-zero constant equal to the square of the filling factor ν. As argued by Read, electron destruction operators such as $c_{k,r}$ acting on $|\psi_{N+1}\rangle$ generally should give a state that can be thought of as q quasi-hole operators, fused at the same location, acting on the incompressible state $|\psi_N\rangle$. Here we use the fact that in the presence of a special Hamiltonian as discussed above, this notion becomes entirely sharply defined in a microscopic sense. Indeed, since $|\psi_N\rangle$ is a zero mode of the Hamiltonian, then so is $c_{k,r} |\psi_{N+1}\rangle$, as all the $c_{k,r}$ commute with all fermion bilinears $\mathcal{T}^{(\lambda)}_R$. $c_{k,r} |\psi_{N+1}\rangle$ is thus always *uniquely* expressible in any basis of N-particle zero modes. Moreover, one may prefer to think of N-particle zero modes as being generated by appropriate operators acting on the N-particle incompressible state.

While in some abstract sense, such operators may always exist, here we have already unambiguously defined them via concrete expressions involving only microscopic electron creation and annihilation operators, Eqs. (3.59)-(3.61). More concisely, we have shown that

$$\tilde{c}_{k,r}\,|\psi_{N+1}\rangle = S^{\sharp k,k}_{MN-r-\delta+L_{\max}(N+1,n)} R_{N,n,k}\,|\psi_N\rangle\,, \tag{3.82}$$

where $R_{N,n,k}$ is a local operator (in the orbital basis) that may be inferred from Eqs. (3.59)-(3.61), along with $\delta \in \{0,1\}$. Writing $\rho_r = \sum_k \tilde{c}^*_{k,r}\tilde{c}_{k,r}$ as in Eq. (3.62), Eq. (3.81) takes on the form

$$\langle\psi_N|\mathcal{O}^\dagger_r\mathcal{O}_{r'}|\psi_N\rangle \longrightarrow \nu^2\,, \tag{3.83}$$

where

$$\mathcal{O}_r = \sum_k \tilde{c}^*_{k,r} S^{\sharp k,k}_{MN-r-\delta+L_{\max}(N+1,n)} R_{N,n,k}\,. \tag{3.84}$$

The object $\mathcal{O}_r$ therefore exhibits off-diagonal long-range order (ODLRO). Eq. (3.84) is closely related to Eq. (3.80) only for $p = 1$. It is different for $p > 1$, as it adds only one particle overall whereas Eq. (3.80) adds p particles. More importantly, Eq. (3.80) is just a single point in an "order parameter lattice" that has n generators. [41] In contrast, it stands to reason that in Eq. (3.84) each term for given k contributes to the ODLRO. In fact, this is of a kind with an SU(n) symmetry discussed in Ref. [41] on the basis of variational wave functions, and which moreover can be seen to be a property of the zero mode spaces associated to *all* composite fermion states, given appropriate parent Hamiltonians [12]. (This is quite a robust property of $n > 1$ special Hamiltonians, and generalizes even to more complicated "parton" states. [55]) It is thus natural to define

$$\mathcal{O}_{k,r} = \tilde{c}^*_{k,r} S^{\sharp k,k}_{MN-r-\delta+L_{\max}(N+1,n)} R_{N,n,k}\,. \tag{3.85}$$

[12] S. Bandyopadhyay, G. Ortiz, Z. Nussinov and A. Seidel; manuscript under preparation.

and identify this family of n operators for $k = 0 \ldots n-1$ as the generators of the order parameter lattice, which exhibit ODLRO in the orbital degree of freedom r.

Several remarks are in order. For one, the operators $R_{N,n,k}$ are a consequence of choosing a particular edge configuration for the incompressible state $|\psi_N\rangle$, which is not uniquely determined in general by the requirement of attaining minimum angular momentum within the zero mode space. These operators must be kept if Eq. (3.83) is to be *exact* for the given composite fermion state $|\psi_N\rangle$ as defined above. However, the ODLRO is expected to be a property of all states in the same phase and is not expected to rely on the choices leading to the $R_{N,n,k}$-operators. (Note that in Eq. (3.60), $R_{N,n,k}$ is proportional to the identity anyway, and is proportional to the single body operators $\hat{p}_0^{q-1,k}$, $\hat{p}_1^{q-1,k}$ in the other cases, respectively.) In the same vein, the parameter $\delta \in \{0,1\}$ is irrelevant to the ODLRO. We may thus settle for the slightly more streamlined variant

$$\mathcal{O}'_{k,r} = \tilde{c}^*_{k,r} S^{\sharp k,k}_{MN-r+L_{\mathsf{max}}(N+1,n)} \,. \tag{3.86}$$

Note that although we have arrived at a reasonably compact definition for these operators using an n-Landau level framework, all of these operators remain meaningful, non-trivial, and independent when projected onto the lowest Landau level. To see this, observe that the $S^\sharp$ operator in Eq. (3.86) creates a (charge 1) quasi-hole in the k-th composite fermion Λ-Level, at orbital location r. One expects such states for different k to remain linearly independent even after lowest-LL projection. For a $n > 1$ composite fermion state there are n-distinct ways of creating a charge 1 hole at given (orbital or real space) location. These n distinct ways are encoded in the $S^\sharp$-operators, whose relation to electron creation/annihilation operators is explicitly given here, and which remain distinct objects whether or not we choose to lowest-Landau-level-project. In the spirit of Ref. [41], to create an order parameter, these n distinct types of holes can then be filled by the action of

any electron creation operator, in particular, one in the lowest Landau level. Note that in particular the creation operator $\tilde{c}^*_{k,r}$ of Eq. (3.86) always has a non-zero component in the lowest Landau level. The relevance of the order parameters (3.86) is thus by no means limited to the mixed-Landau-level setting used here to derive them. After lowest-Landau-level-projection, the k-labels refer to Λ-levels in the original, purely emergent sense of the term. [4]

It should be emphasized that the two processes in (3.86) are very different, where the $S^\sharp$-operator creates a hole via flux insertion into one of the Λ-levels, but without changing overall particle number, a highly non-local operation. In contrast, this hole is then filled by a local electron creation operator. In Eq. (3.86), both the hole and the subsequently inserted particle are localized in orbital space. If desired, it is easy to construct corresponding order parameters with both electron and hole localized in real space (but the latter still facilitated by a non-local operator). If a local electron destruction operator $\hat{\psi}_j(z)$ is obtained via

$$\hat{\psi}_j(z) = \sum_{k,r} \mathcal{F}_{k,r,j}(z)\tilde{c}_{k,r}\,, \tag{3.87}$$

where $\mathcal{F}_{k,r,j}(z)$ depends in straightforward ways on the matrix $A(r)_{ab}$ defined in Eq. (3.36) and the Landau level basis wave functions, the desired order parameter is given by

$$\mathcal{O}'_j(z) = \hat{\psi}^\dagger_j(z)\sum_{k,r}\mathcal{F}_{k,r,j}(z)S^{\sharp k,k}_{MN-r+L_{\max}(N+1,n)}\,. \tag{3.88}$$

Eq. (3.88) is obtained following strictly the same logic leading up to Eq. (3.86). However, since for $j > 0$, $\hat{\psi}^\dagger_j(z)$ now does create a state orthogonal to the lowest Landau level, projection to the lowest Landau level now necessitates replacing $\hat{\psi}^\dagger_j(z)$ with $\hat{\psi}^\dagger_0(z)$. In view of the discussion above, this should not affect the ODLRO of these operators.

3.14 Concluding remarks on construction of composite fermion order parameter recursion relation

We have developed a comprehensive formalism to discuss composite fermions in Hilbert space. The heart of this formalism is a presentation of the Laughlin-Jastrow flux attachment operator in terms of second quantized electron creation and annihilation operators. This allows us, in particular, to define certain operations that *add* fermions to an incompressible composite fermion ground state, as well as general operations that remove them, while staying in the composite fermion sector of the Hilbert space. As a result, we can define Jain composite fermions states recursively in the orbital basis, generalizing similar recursions for Laughlin states. This operator-based approach has several advantages. The properties of parent Hamiltonians, where they exist, can be rigorously established. This, in particular, establishes edge theories microscopically on much more than variational grounds. n-component order parameters for the Jain composite fermion phases can be microscopically defined, i.e., their relation to microscopic electron creation and annihilation operators is fully specified, and their meaning thus extended from a variational subspace to the full Hilbert space.

We expect that this work will spur further developments in particular along several interesting directions: One is the construction of new special parent Hamiltonians for mixed Landau-level wave functions. This includes all of the Jain states, but also other, more exotic quantum Hall states including parton states. [45, 55, 93, 94] Indeed, the present work and the treatment [55] by some of us of the non-Abelian Jain-221 state can both be regarded as different natural extensions of earlier work on the Jain-2/5 state. [34] It, therefore, seems likely that further extensions of the formalism developed here to non-Abelian

states are possible. This formalism, in connection with the idea of "entangled Pauli principles" (EPP) that naturally extends the notion of "generalized Pauli principles" [50, 51] or thin torus patterns [18, 19, 21, 46, 47, 54, 95–99], represent a powerful new framework to construct and study FQH parent Hamiltonians from the point of view of infinite-range frustration free one-dimensional lattice models, as opposed to analytic wave functions. This may further turn out to be beneficial when studying spectral properties of such models at non-zero energy, [60] or making a connection between EPPs and braiding statistics. [63, 100, 101] Another exciting prospect is the further development of non-local order parameters as a numerical diagnostic and theoretical tool. We leave these as interesting problems for future work.

3.15 Is there a parent Hamiltonian for composite fermions

Fractional quantum Hall effect (FQHE) was discovered in 1982. Since then, these phenomena have enjoyed the maximum attention in various fields of physics. FQHE has not only introduced the idea of topological phases, but it has also served as a potential starting point for the description of strongly correlated many-body physics. Latter application has started with the Laughlin's prediction of FQHE wavefunction, which very accurately predicts the lowest energy states for the $1/3$ filling fraction. Laughlin's construction predicts, in general, $1/m$ filling fractions, for odd m. Haldane has constructed the pseudo-potential description for Laughlin state, thus given a many-body starting point for physics starting from $1/m$ FQHE states. There are, however, many states which lack a many-body Hamiltonian description as the starting point but successfully predicts various filling fractions. Composite fermions are the poster child for such states. For $2/5^{th}$ state, it was predicted that Trugman-Kivelson Hamiltonian (H_{TK}) gives the densest zero energy mode

(zero mode). A newly developed technique, entangled Pauli principle, (EPP) indeed establishes H_{TK} as the parent Hamiltonian for the unprojected Jain's $2/5^{th}$ state. EPP gives the constraints on the arrangements of particles in the root states or dominant pattern for the allowed ground states. It has been shown previously, those constraints are sufficient to describe clustering properties of the entire ground state, without emphasizing any analytic properties of the zeros of the ground states. Hence, EPP turns out to be a natural description for higher Landau level (LL) physics. Using this EPP, it has been established filling fraction $1/2$, not $3/7$, is the densest zero mode for H_{TK}, when projected to three Landau levels. Hence, H_{TK} failed to serve as the parent Hamiltonian in higher Landau levels.

In the rest of this chapter, we present a parent, frustration-free, quantum Hall Hamiltonian, for Jain's composite fermions main sequence (filling fractions, $1/3$, $2/5$, $3/7$...) as the densest zero modes for different Landau level projection. For example, if one projects our Hamiltonian to three Landau levels, one should get filling fraction $3/7$, instead of $1/2$. This ground state should be uniquely determined by the $3/7$ CF wavefunction. Furthermore, all the zero mode excitations of this state should agree with the quasihole excitations for $3/7$ CF state. In order to establish, that our proposed Hamiltonian is indeed the desired parent Hamiltonian for the CF states, we use the organizing EPP. However, in order to maintain the simplicity of our calculation, we have decided to express our EPP in a non-canonical basis of Landau levels. This basis was first introduced in constructing the Read's string order parameter for CFs [12].

Our proposed Hamiltonian is second-quantized, positive semi-definite and frustration free. We will also give similar Hamiltonian in cylinder geometry, in second quantized form. In the conclusion part, we will discuss, the apparent extension of our Hamiltonian to any CF filling fraction. In the next section, we will summarize the composite fermions,

we have discussed so far. In this way, we will highlight some important concepts, which are essential for the construction of parent Hamiltonian.

3.16 Composite fermion state in second quantization.

The unprojected Jain state at filling factor $n/(Mn+1)$, M an even number, can be defined in disk geometry as

$$\Psi_{n,M}(N) = \prod_{1\leq i<j\leq N} (z_i - z_j)^M \Phi_n(N), \tag{3.89}$$

where $\Phi_n(N)$ denotes in integer quantum Hall state of N particles in n LLs, and the $z_i = x_i + iy_i$ are the particles complex coordinates. $\Phi_n(N)$ is by definition a state of minimum angular momentum for given n and N, where ambiguities at the edge may arise for certain N that we will resolve in a manner to be made precise below.

While Eq. (3.89) has a "clustering property", where the wave function has a $(M+1)$th order zero when two particles converge to the same point. However, only for $n = 2$ does Eq. (3.89) represent the densest (lowest angular momentum) wave function(s) having this property, which is why for $n = M = 2$, there is a by now well-documented parent Hamiltonian [39, 91]. For any $n > 2$, Eq. (3.89) does not optimally use the dependence on the complex conjugates $\bar{z}_i$ of the particle coordinate to be the densest state with such a clustering property, this honor going, in general, to interesting non-Abelian wave functions [93, 94]. It is thus clear that the clustering property does, but itself, insufficiently characterize free composite fermion states. It is thus highly non-trivial to enforce "free composite fermion" behavior via solvable local Hamiltonians. Here we solve this problem by utilizing a characterization of this behavior that eschews first-quantized polynomial description. A complete, alternative characterization has been given by some of

us [12] in terms of an algebra of second quantized operators that can be understood as "zero mode generators". We begin by summarizing the nuts and bolts of this formalism.

3.17 Operator description of composite fermions states.

In first quantization, an orbital $\phi_{n,\ell}$ in the nth Landau level, $n = 0, 1 \ldots$, with angular momentum ℓ is a superposition of monomials of the form $m_{p,\ell} = \bar{z}^p z^{\ell+p}$ with $p \leq n$. (We omit obligatory Gaussian factors). Higher LL many-body wave functions such as Eq. (3.89) may be expanded in $m_{r,\ell}$, adorned with additional particle indices. A significant advantage of the first quantized presentation is the fact that this expansion is essentially *geometry independent*, assuming that we limit ourselves to zero genus geometries (disk, cylinder, sphere). This is so since there is a one-to-one correspondence between the wave functions in these geometries, once $\bar{z}$, z (for the disk) are replaced with suitable functions of co-ordinates respecting the boundary conditions of the other geometries. In other words, variation wave functions such as Eq. (3.89) are described by the same polynomials in the genus 0 geometries. To obtain a manifestly geometry independent language, and to the extent that the successful construction of a parent Hamiltonian is a direct consequence of the underlying polynomial structure, however complicated, it proves advantageous to make the monomials $m_{p,\ell}$ the essential degrees of freedom of the second quantized formalism also. We thus introduce operators $\tilde{c}_{p,\ell}$, $\tilde{c}^*_{p,\ell}$ satisfying canonical anti-commutation relations

$$\{\tilde{c}_{p,\ell}, \tilde{c}_{p',\ell'}\} = \{\tilde{c}^*_{p,\ell}, \tilde{c}^*_{p',\ell'}\} = 0, \quad \{\tilde{c}_{p,\ell}, \tilde{c}^*_{p',\ell'}\} = \delta_{p,p'}\delta_{\ell,\ell'}, \tag{3.90}$$

where the $\tilde{c}^*_{p,\ell}$ create an electron in the orbital $m_{p,\ell}$. These orbitals are *not* normalized or orthogonal (for fixed ℓ), and hence $\tilde{c}^*_{p,\ell}$ and $\tilde{c}_{p,\ell}$ are not Hermitian conjugates, but this will present no obstacle in the following. If desired, at the end we may always return to the

creation/annihilation operators $c_{n,\ell}$, $c^\dagger_{n,\ell}$ of the orbitals $\phi_{n,\ell}$ via

$$c_{n,\ell} = \sum_p A(\ell)_{n,p} \tilde{c}_{p,\ell} \,, \quad c^\dagger_{n,\ell} = \sum_p A(\ell)^{-1}_{n,p} \tilde{c}^*_{p,\ell} \,. \tag{3.91}$$

The matrix $A(\ell)$ is the only geometry-dependent aspect of this formalism.

The considerable advantage of the second quantized formalism, especially for multiple Landau levels, lies in the fact that gives us control over an algebra of "zero mode generators" that we arguably do not have in first quantization. It is also much more conducive to recursive schemes in particle number which will now heavily pursue. To this end we introduce the following operators, which we will think of zero mode generators in a sense to be made precise:

$$\hat{p}^{a,b}_k = \sum_r \tilde{c}^*_{a,r+k} \tilde{c}_{b,r} \tag{3.92}$$

The operators in Eq. (3.92) generate an algebra (via taking sums and/or products) that we will denote by $\mathcal{Z}$. The significance of this algebra is manifold [12]. It allows for a definition of composite fermion states recursive in particle number, quite distinct from the recently fashionable matrix product presentation of fractional quantum Hall states [35,37,38], but in essence a generalization of Read's expression of the Laughlin state through an order parameter [13]. Indeed, it allows for a microscopic definition of a complete set of order parameters for composite fermion states. In the present context, it will turn out that algebra $\mathcal{Z}$ generates all possible zero energy modes (zero modes) when acting on the incompressible ground state. In that sense they are related to a first quantized formalism discussed by Stone [26] for the Laughlin state, possible there because $\sum_{a=0}^{n-1} p^{a,a}_k$ (which, for $n = 1$ Landau level, is really all Eq. (3.92) boils down to) has a simple first quantized interpretation: It multiplies many-body wave functions with power-sum symmetric polynomials

$p_z = \sum z_i^k$. Here, however, we need the full set $p_k^{a,b}$, which does *not*, in general, have a straightforward first quantized-interpretation.

Consider now Eq. (3.89). To resolve the "edge ambiguity" mentioned above, we will define the Slater determinant by successively filling the state $m_{p,\ell}$ with lowest available ℓ that has lowest not-yet-occupied p. We seek to establish a parent Hamiltonian such that Eq. (3.89), which we now also suitably write $|\Psi_{n,M,N}\rangle$, is a densest zero mode of this Hamiltonian. Since general zero modes will describe edge excitations and, deeper in the bulk, quasi hole excitations, one has the intuition [13] that $\tilde{c}_{r,\ell}\,|\Psi_{n,M,N}\rangle$, is a zero mode of the Hamiltonian, namely, one describing a cluster of quasi-holes of total charge 1 inserted into the N-particle incompressible CF-state for given M and n. Anticipating that this is so, then, with the properties of the $\hat{p}_k^{a,b}$ as advertised, we must be able to interpret this as a higher angular momentum zero mode generated on top of the incompressible state $|\Psi_{n,M,N-1}\rangle$, or,

$$\tilde{c}_{r,\ell}\,|\Psi_{n,M,N}\rangle = \hat{Z}_{n,M,N,r,\ell}\,|\Psi_{n,M,N-1}\rangle\ , \tag{3.93}$$

where $\hat{Z}_{n,M,N,r}$ is a suitable element of the algebra $\mathcal{Z}$. Indeed, the relation between $\hat{Z}_{n,M,N,r}$ and the generators Eq. (3.92) was made explicit in [12], but will not be needed in the following.

3.18 Parent Hamiltonian for Composite Fermions.

With the above construction of order parameter recursion realtion, we are now ready to present the following Hamiltonian as the parent Hamiltonian for composite fermion with

filling fraction, $n/(nM+1)$

$$H_{M,n} = \sum_{\substack{r<M,J \\ 0\le a<n \\ 0\le b<n}} T^{\dagger r}_{a,b,J} T^{r}_{a,b,J}, \quad T^{r}_{a,b,J} = \sum_{x} x^r \tilde{c}_{J+x} \tilde{c}_{J-x} \tag{3.94}$$

where the ${T^r_{a,b,J}}^\dagger T^r_{a,b,J}$ are suitable generalizations of pseudopotentials, whose relation to Haldane pseudo-potentials for $a = b = 0$ we discussed in [10]. While it may not be obvious in the $\tilde{c}$-basis, we discuss in Appendix - B.2 that these pseudopotentials are in fact local. Note that for fermions, $T^r_{a,b,J}$ vanishes for even r and $a = b$, giving us $\frac{1}{2}Mn^2$ different pseudo-potentials at each pair angular momentum $2J$. In writing the above, we tacitly use the convention $c_{n,\ell} \equiv 0$ for $n + \ell < 0$. A key observation is that the operators $T^r_{a,b,J}$ and $p^{a,b}_k$ satisfy the following commutation relation:

$$[T^r_{a,b,J}, p^{a',b'}_k] = T^r_{a,b',J+k/2}\delta_{b,a'} - T^r_{b',b,J-k/2}\delta_{a,a'} \tag{3.95}$$

This justifies the notion that the $p^{a,b}_k$ are "zero mode generators": The condition for $|\psi\rangle$ to be a zero mode of the positive Hamiltonian (3.94) reads $T^r_{a,b,J}|\psi\rangle = 0$ for all r, J, a, b. The commutator (3.95) thus clearly vanishes within the zero mode subspace. Therefore, any $p^{a,b}_k$ acting on $|\psi\rangle$ immediately generates another zero mode, with angular momentum increased by k. In the following, we first wish to establish (i) that the Jain state $|\Psi_{n,M,N}\rangle$ is a zero mode of Eq. (3.94), and (ii) we want to establish all the zero modes of Eq. (3.94).

We will achieve these goals via radical departure from established paradigms, i.e., not paying attention whatsoever to "analytic clustering properties". We will do so by utilizing the properties of second-quantized operator algebras established thus far and in the following. It is worth noting that while the $p_k = \sum_a p^{a,a}_k$ have a simple interpretation in first-quantized polynomial language [12], this holds less so for the $p^{a,b}_k$. For (i), we give a

simple induction proof in N which extends that of [34]. We give the induction step first, assuming that $|\Psi_{n,M,N-1}\rangle$ is known to be a zero mode. One easily verifies

$$T^r_{a,b,J} = \frac{1}{2}\sum_{m,k}[T^r_{a,b,J}, c^\dagger_{m,k}]c_{m,k}\,. \tag{3.96}$$

together with the fact that $Z_{n,M,r,\ell}$ is a zero mode generator gives

$$T^r_{a,b,J}\,|\Psi_{n,M,N}\rangle = \frac{N}{2}T^r_{a,b,J}\,|\Psi_{n,M,N}\rangle \implies T^r_{a,b,J}\,|\Psi_{n,M,N}\rangle = 0 \;\forall\; N > 2. \tag{3.97}$$

So far, the only special property of the T^r, $0 \leq r < M$ we have used is that $Z_{n,M,r,\ell}$ is a zero mode generator as defined above. All that's left to do is to establish an induction beginning for $N = 2$ (see Appendix- B.3 for $N = 2$ particle proof).

3.19 Entangled Pauli Principle for composite fermion parent Hamiltonian

In the last section, we have shown our proposed parent Hamiltionian, is positive semi-definite, frustration-free Hamiltonian, which stabilizes composite fermions for filling fraction $n/(nM+1)$ in n Landau levels. Moreover, this Hamiltonian is local and commmutes with all of the quasihole operators in the zero mode subspace. To establish composite fermion state as the "unique" [13] ground state with maximum possible filling fraction, the only thing left to prove that the Eq. (3.94) is indeed a good parent Hamiltonian. In order to establish that, let us start with the idea of entangled Pauli principle to prove the following results

[13] in disk geometry or in other topological equivalent geometry.

a) The root pattern of the densest ground state is unique up to boundary condition.

b) Filling fraction of the densest ground state is exactly same with the composite fermion's filling fraction.

In the rest of this section, we will prove above statements for $M = 2$. Our proof can be generalized to any any integer M, in a straight-forward manner. To begin with the proof, we will make contact with the binary string representation of of N-body wavefunction for n Landau levels [39, 55]. FOllowing the notation, developed in earlier sections, an 11 string in n Landau levels, can be represented as,

$$11 = \sum_{a,a} \alpha_{a,b} 1_a 1_a = \sum_{a,b<n} \alpha_{ab} \tilde{c}^*_{a,J-1/2} \tilde{c}^*_{b,J+1/2} \left|\{l\}\right\rangle \tag{3.98}$$

Where $\left|\{l\}\right\rangle$ is the occupation basis in the guiding center/ angular momentum co-ordinates. a, a are Landau level indices and $2J$ is the total angular momentum for two particles. Now, If 11 is allowed in the root pattern of a ground state ψ_0. We will have,

$$\left\langle\{n\}\right| \tilde{c}_{n_1,J-k} \tilde{c}_{n_2,J-k} \left|\psi_0\right\rangle = 0 \ \forall \ k > 1/2; \quad T^{0/1}_{a,b,J} \left|\psi_0\right\rangle = 0 \ \forall a, b < n \tag{3.99}$$

Where the first condition ensures the non-expandablity of the root state, the latter condition implies $\left|\psi_0\right\rangle$ is ground state. a and b are Landau level indices, each can take n different values. This is easy to see there are n^2 free parameters, namely α_{ab} to satisfy n^2[14] constraints given In terms of $T^{0/1}_{a,b,J}$ for given total angular momentum $2J$. Thus we get all the α_{ab}s set to zero, or in other words, 11 is not allowed in the root state of the ground state. Similarly, one can show any denser state than 11, i.e, two particle in same angular momentum is not allowed in the root state as well. Now, let us look into 101 state in the

[14]Total number of constraints are $n^2 M/2$. We have taken $M = 2$ for concreteness.

root. This state can be defined in the similar fashion,

$$\begin{aligned} 101 &= \sum_{a,b} \beta_{ab} 1_a 01_b + \sum_{a>b} \gamma_{ab} 02_{a,b} 0 \\ &= \sum_{a,b} \beta_{ab} \tilde{c}^*_{a,J-1} \tilde{c}^*_{b,J+1} \left|\{l\}\right\rangle \\ &+ \sum_{a>b} \gamma_{ab} \tilde{c}^*_{a,J} \tilde{c}^*_{b,J} \left|\{l\}\right\rangle \end{aligned} \tag{3.100}$$

s.t,

$$\left\langle\{l\}\right| \tilde{c}_{a,J-k} \tilde{c}_{b,J-k} \left|\psi_0\right\rangle = 0 \ \ \forall \ \ k > 1; \quad T^{0/1}_{a,b,J} \left|\psi_0\right\rangle = 0 \ \ \forall a, b < n \tag{3.101}$$

Where, γ_{ab} is there due to inward squeezing. Now, applying the constraints arising from $T^{0/1}$, we get,

$$\beta_{ab} = -\beta_{ba} \tag{3.102}$$

Hence, we get total $n(n-1)/2$ free parameters $\beta_{[a,b]}$. Where $[a,b]$ stands for ordered pair of a, b. That is said, 101 root pattern can occur in the n Landau level projected ground state in $\binom{n}{2}$ ways. Now let us look into three particle root pattern 10101. Which can be defined in the similar pattern,

$$\begin{aligned} 10101 \ \ &= \ \ \sum_{a,b,d<n} \beta_{abd} \tilde{c}^*_{a,J-2} \tilde{c}^*_{b,J} \tilde{c}^*_{d,J+2} \left|\{n\}\right\rangle \\ &+ \ \ \text{Inward squeezed terms} \end{aligned} \tag{3.103}$$

Using the two particle constraints derived from eq. 3.102, one can write 10101 pattern $\binom{n}{3}$ different ways in terms $\beta_{[a,b,d]}$. Where $[a,b,d]$ stands for ordered set of a, b and d. Inductively one can prove, a 1010101.. chain of m particles $m < n$ can present in n Landau level ground state of in $\binom{n}{m}$ ways. for $m > n$, $\binom{n}{m}$=0 implies there is no 101010.. root pattern for more than n particles, living in n Landau levels. When $m = n$, we will have a

unique root pattern up to a overall phase,

$$\beta_{a_1 a_2 \ldots a_n} = \epsilon_{a_1 a_2 \ldots a_n} \tag{3.104}$$

$\epsilon_{a_1 a_2 \ldots a_n}$ is Levi-Civita matrix. Thus 1010101.. pattern of n particle will form a $SU(n)$ singlet in n Landau level root pattern. Hence, we can summerize the entangled Pauli principle for our parent Hamiltonian ($M = 2$ case),

Entangled Pauli principle (EPP):
a) No double occupancy or nearest-neighbor (NN) occupancy in guiding-center coordinates is allowed.
b) Next - NN, occupancy is allowed for n particles only if they form $SU(N)$ singlet.
Thus densest root pattern will look like,

$$\ldots \boxed{101010101..01}\, 00\, \boxed{101010101..01}\, 00\, \boxed{101010101..01} \ldots \tag{3.105}$$

Where $\boxed{101010101..01}$ represents a $SU(n)$ singlet. Notice the densest root pattern, has filling fraction $n/(2n+1)$, exactly matches the n Landau level composite fermion filling fraction for $M = 2$ flux attachment.

At this point, one must realize the for any arbitrary M, the only change in the above calculation will be number of constraint equations will be $Mn^2/2$. hence the entangled Pauli principle will have following form,

Entangled Pauli principle:

a) No double occupancy in M consecutive sites in guiding center coordinate is allowed.

b) M - NN, occupancy is allowed for n particles only if they form $SU(n)$ singlet.

Where, M - NN occupancy refers to two occupied states ("1") must be separated by $M-1$ unoccupied states "0".

Notice the densest root pattern, allowed by above entangled Pauli principle, is unique. It has filling fraction $n/(nM+1)$, exactly matches the n Landau level composite fermion filling fraction. Thus we establish *Eq.*(3.94) as a valid parent Hamiltonian for composite fermions.

Chapter 4

Even More Exotic Fractional Quantum Hall states: Non-abelian stattistics

[15]

4.1 Introduction

The fractional quantum Hall (FQH) regime exhibits an astonishing wealth of interacting topological phases. A rich theoretical framework describing such phases has historically nucleated around a construction principle for holomorphic lowest Landau level (LL) wavefunctions [9] and fruitful generalizations to the non-holomorphic, multi-LL situation, with optional subsequent lowest-LL projection [40]. This variational principle has proven invaluable in driving the development of field-theoretic descriptions of both the

[15] In this chapter, we have further developed the idea of entangled Pauli principle for non-abelian phases. The content of this chapter is reproduced from a collaboration with my Ph.D. advisor Alexander Seidel, L. Chen from National magnetic lab, Z. Nussinov, G. Ortiz and M. T. Ahari from G. Ortiz's group in Indiana university. [55].

bulk and the edge physics and their intimate relation [36, 102]. One may take the point of view that a complete many-body theory of any correlated phase of matter requires, in addition to the aforementioned ingredients, a microscopic Hamiltonian granting analytic access to its low energy sector, reproducing key aspects of the field-theoretic description of such a phase. Such "parent Hamiltonians" do exist for many [24, 25, 103–105] FQH-liquids but lack for even more. Notably, to our knowledge, they are absent for most Jain states, which are regarded as fundamental both theoretically and experimentally.

In this Letter, we argue that the lack of microscopic Hamiltonians stabilizing representative variational wavefunctions for FQH-phases stems from complexities associated with non-holomorphic variational states. These include unprojected Jain states [40] and more general "parton" constructions [45, 106]. In these cases, lowest-LL projection leads to sufficiently intractable wavefunctions to preclude the construction of parent Hamiltonians. Moreover, the unprojected, multi-LL variational states still lack many "analytic clustering" properties that were instrumental in the construction of parent Hamiltonians for many lowest-LL states [24, 25, 103]. For these reasons, even in those cases where parent Hamiltonians have been proposed for multi-LL states, rigorous analytic results are usually lacking. This is particularly true for zero-mode counting, from which the case for incompressibility at special filling factors is usually made. We will develop principles to study the zero-mode properties of frustration-free multiple-LL parent Hamiltonians on the same footing as for similar single-LL Hamiltonians. Our second-quantized framework de-emphasizes analytic clustering properties [10], which are arguably less useful in the multi-LL situation, as we will demonstrate. This lack of emphasis on analytic properties, in favor of a "guiding-center based" description, was recently advocated for various reasons [11, 34, 60, 87, 107, 108]. Our approach connects with the topical investigation of frustration free lattice Hamiltonians and their matrix-product ground states (MPS), with

the important additional feature that it extends to non-local lattice Hamiltonians and, in principle, MPS of infinite bond dimension [35, 37, 109].

The heart of our framework consists in further elaboration on the concept of a "generalized Pauli principle" (GPP), various guises of which play an important role in discussing the structure of single-LL wavefunctions [14, 18–21, 46, 53, 72, 110, 111]. Our extension not only provides a foundation based on Hamiltonian principles but also generalizes to multiple LLs. The latter will naturally lead to what we coin "entangled Pauli principles" (EPPs), which, in addition to the now familiar rules for GPPs, permit MPS-like entanglement at "root level" encoding the quantum fluid's "DNA". We argue this generalization to be key in yielding microscopic Hamiltonian descriptions to possibly all FQH-phases. We demonstrate our approach in detail for the parent Hamiltonian of the Jain-221-state [94]. By rigorously establishing the zero-mode structure of this Hamiltonian, we make direct contact both with bulk topological and edge conformal properties. As a byproduct, this affords a case where simple two-body interactions stabilize a non-Abelian FQH-state, in contrast to better-known higher-body, single-LL cases [112, 113].

4.2 Parent Hamiltonian.

Consider the n-LL projected "Trugman-Kivelson" interaction for fermions,

$$H_{\mathsf{TK}} = \sum_{i<j} P_n \, \partial_{z_i}\partial_{\bar{z}_i}\delta(z_i - z_j)\delta(\bar{z}_i - \bar{z}_j) \, P_n \,, \tag{4.1}$$

where $z_i = x_i + iy_i$ is the coordinate of the ith particle, and $\bar{z}_i$ its complex conjugate. For general projection P_n onto the subspace spanned by the lowest n LLs, this interaction is positive (semi-)definite. If the n-LLs are energetically quenched [43], as is in multi-layer

graphene [94, 114, 115], the ground states of the resulting Hamiltonian can be characterized as zero-energy modes (zero-modes). For any n, the wavefunctions of such zero-modes will have at least second order zeros as pairs of particles coalesce into the same point. For both n=1 and n=2, this is equivalent to the polynomial wavefunction being divisible by the Laughlin-Jastrow factor $\prod_{i<j}(z_i - z_j)^2$. This was realized early on for n=1 [24, 25] and leads to the stabilization of the 1/3-Laughlin-state and its quasi-hole excitations. The n=2 case was extensively discussed recently [39]. For $n \geq 3$, zero-modes can only be characterized as polynomials belonging to the ideal generated by $(z_i - z_j)^2$ and $(\bar{z}_i - \bar{z}_j)^2$ for some fixed $i \neq j$, in addition to being anti-symmetric. This makes the characterization of all possible zero-modes considerably more challenging. For the case n=3, we will establish that the space of all zero-modes is linearly generated by all wavefunctions of the form

$$\psi = \prod_{i<j}(z_i - z_j)\, D_1\, D_2\,, \tag{4.2}$$

where D_1 and D_2 are the polynomial (in $\{z_i, \bar{z}_i\}$) parts of two Slater determinants each comprised of lowest and first excited LL states, and we omit obligatory Gaussian factors. It is easy to see that states of the form (4.2) are zero-modes of the n=3-Hamiltonian. The "Jain-221" state, where $D_1 = D_2$ is the Slater-determinant of smallest possible angular momentum in the first two LLs for given particle number N, was conjectured to be the densest zero-mode [94]. We will show that the set of all possible wavefunctions of the form (4.2) is overcomplete and establish rules for the selection of a complete set of zero-modes as an EPP on dominance patterns.

4.3 Entangled Pauli Principle.

Our starting point is a second-quantized form of Eq. (4.1) for n=3, in disk geometry, which we present in the general [10] form

$$\hat{H}_{\mathsf{TK}} = \sum_{J} \sum_{\lambda=1}^{8} E_\lambda \mathcal{T}_J^{(\lambda)\dagger} \mathcal{T}_J^{(\lambda)}. \tag{4.3}$$

The $\mathcal{T}_J^{(\lambda)}$ annihilate a pair of particles of angular momentum $2J$, with $J = 0, \frac{1}{2}, 1, \ldots$, $\mathcal{T}_J^{(\lambda)} = \sum_{x,m_1,m_2} \eta^{\lambda}_{J,x,m_1,m_2} c_{m_1,J-x} c_{m_2,J+x}$ and Eq. (4.3) may be viewed as a weighted (by E_λ) sum over eight two-particle projection operators at each J. Note that x is (half-odd)-integer if J is (half-odd)-integer, and $c_{m,j}$ destroys a fermion in the mth LL, m=0,1,2, at angular momentum ("site") $j \geq -m$. The η-symbols and the positive E_λ can be efficiently derived for general n [16], and are given for n=3 in Appendix - C.1. Consider the Slater-determinant decomposition of any N-particle zero-mode

$$|\psi\rangle = \sum C_{m_1,j_1;\ldots;m_N,j_N} c^\dagger_{m_1,j_1} \ldots c^\dagger_{m_N,j_N} |0\rangle \equiv \sum C_S |S\rangle \ . \tag{4.4}$$

General arguments [10,39] imply that there are "non-expandable" Slater-determinants $|S\rangle$ in such an expansion that are pivotal in the analysis of any zero-mode of Eq. (4.3): These are those states $|S\rangle$ in Eq. (4.4) with non-zero C_S that cannot be obtained from a $|S'\rangle$ with non-zero $C_{S'}$ through an inward-squeezing [14] process: $|S\rangle \neq c^\dagger_{m_1,j_1} c^\dagger_{m_2,j_2} c_{m'_2,j_2+x} c_{m'_1,j_1-x} |S'\rangle$, where $j_1 < j_2$, $x > 0$. We define the state obtained from the zero-mode (4.4) by keeping only the non-expandable part as the "root state" $|\psi_{\mathsf{root}}\rangle$ of $|\psi\rangle$. The root state is closely related to the thin torus limit [18–21, 116], and is generally subject to simple rules usually

[16] M. T. Ahari, S. Bandyopadhyay, Alexander Seidel, Zohar Nussinov and Gerardo Ortiz, manuscript under preparation

known as GPPs in the single-LL context. We will show that the zero-mode condition leads to a generalization thereof in the present case, which we call EPP.

We begin by demonstrating that a state $|S\rangle$ in $|\psi_{\text{root}}\rangle$ may not have a double occupancy at any given j. Otherwise, $|\psi_{\text{root}}\rangle = \sum_{m_1,m_2} \alpha_{m_1 m_2} c^\dagger_{m_1,j} c^\dagger_{m_2,j} |\tilde{S}\rangle + |\text{rest}\rangle$, with $|\text{rest}\rangle$ being orthogonal to each of the leading terms, and $|\tilde{S}\rangle$ an $N-2$ particle Slater-determinant with no j-mode occupied. The zero-mode condition amounts to [10, 39] $\mathcal{T}_J^{(\lambda)} |\psi\rangle = 0$ for all J, λ. Then, in

$$0 = \langle\psi|{\mathcal{T}^{(\lambda)}_{J=j}}^\dagger|\tilde{S}\rangle = \sum_{x,m_1,m_2} (\eta^\lambda_{j,x,m_1,m_2})^* \langle\psi|c^\dagger_{m_2,j+x} c^\dagger_{m_1,j-x}|\tilde{S}\rangle \tag{4.5}$$

, the $x \neq 0$ terms must already give zero, otherwise the x=0 terms would by definition not appear in $|\psi_{\text{root}}\rangle$. One thus obtains the eight conditions

$$\sum_{m_1,m_2} \eta^\lambda_{j,0,m_1,m_2} \alpha_{m_1,m_2} = 0 \quad (\lambda = 1 \ldots 8)\,. \tag{4.6}$$

Since there are only three independent numbers $\alpha_{m_1,m_2} = -\alpha_{m_2,m_1}$, and the x=0 η-symbols are sufficiently (see Appendix - C.1) linearly independent, one finds that all α_{m_1,m_2} vanish. One can similarly rule out triple occupancies in $|\psi_{\text{root}}\rangle$. Likewise, one may evaluate possibilities for nearest-neighbor occupancies in $|\psi_{\text{root}}\rangle$. Applying the same method to the similar expression (J half-odd integer) $|\psi_{\text{root}}\rangle = \sum_{m_1,m_2} \beta_{m_1 m_2} c^\dagger_{m_1,J-\frac{1}{2}} c^\dagger_{m_2,J+\frac{1}{2}} |\tilde{S}\rangle + |\text{rest}\rangle$, there are eight constraints on the nine constants $\beta_{m_1 m_2}$,

$$\sum_{m_1,m_2} \eta^\lambda_{J,1/2,m_1,m_2} \beta_{m_1,m_2} = 0 \quad (\lambda = 1 \ldots 8)\,. \tag{4.7}$$

There is a unique solution to these equations which thus determines any nearest-neighbor pair in $|\psi_{\text{root}}\rangle$ to be in a certain entangled state. In evaluating constraints at root level for

pairs further separated, we must also take into account inward squeezed configurations of the pair. Writing

$$|\psi\rangle = \sum_{m_1,m_2} \gamma_{m_1 m_2} c^\dagger_{m_1,J-1} c^\dagger_{m_2,J+1} |\tilde{S}\rangle + \alpha_{m_1 m_2} c^\dagger_{m_1,J} c^\dagger_{m_2,J} |\tilde{S}\rangle + |\mathsf{rest}\rangle \tag{4.8}$$

, where the first term is non-expandable, we obtain eight conditions in the twelve constants γ_{m_1,m_2}, $\alpha_{m_1,m_2} = -\alpha_{m_2,m_1}$. After eliminating the latter, these result in five conditions on the γ_{m_1,m_2}:

$$\sum_{m_1,m_2} \Omega^\mu_{J,m_1,m_2} \gamma_{m_1,m_2} = 0 \quad (\mu = 1 \ldots 5)\,, \tag{4.9}$$

with Ω a function of the η's at x=0,1/2. The constraints derived so far require any two particles in a root state to be entangled when in configurations $\ldots 11 \ldots$ or $\ldots 101 \ldots$, where 0 denotes an empty site, 1 denotes a single occupancy (in any LL), and consecutive entries denote states with consecutive j. We now ask what these constraints imply for clusters of more than two particles.

4.4 Emergent SU(2)-symmetry.

Let us apply to $|\psi_{\mathsf{root}}\rangle$ a non-unitary (but invertible) single-particle transformation $\hat{V}$ such that $c^\dagger_{m,j} = \hat{V}^{-1} d^\dagger_{m-1,j} \hat{V} = v_{m,s_z} d^\dagger_{s_z,j}$, where $s_z = 0, \pm 1$ is interpreted as the SU(2)-label of a spin-1 particle, as detailed in Appendix - C.2. In the new basis, Eq. (4.7) requires any nearest-neighbor 11-pair in $\hat{V}|\psi_{\mathsf{root}}\rangle$ to form a singlet. Clearly, then, it cannot be entangled with any other particle. This is consistent with Eqs. (4.7), (4.9) only if any such pair is separated by at least two zeros from any other particle in $|\psi_{\mathsf{root}}\rangle$. Moreover, Eq. (4.9) takes on a form implying that any 101-configuration is orthogonal to the spin-2-sector. The satisfiability of this condition for N-particles separated by individual empty sites is tantamount to the problem of finding ground states of an open AKLT-chain [117].

Table 4.1: Survey of all dominance patterns with angular momentum ΔL=3 above the ground state for odd particle number. The total number including "spin degeneracy" allowed by AKLT-entanglement or due to isolated occupied sites is 33, in agreement with Table 4.2. The corresponding densest state ($\Delta L = 0$) has the pattern 100110011 . . . 110011, where the boundary condition at the left end is explained in Appendix - C.2

Patterns	Degeneracy
$100...110011001_{s_z}0001_{s_z}$	3×3
$100...1100110001_{\sigma_L}01_{\sigma_R}$	4
$100...11001_{\sigma_L}0101_{\sigma_R}001_{s_z}$	4×3
$100...11001_{\sigma_L}01_{\sigma_R}0011$	4
$100...1_{\sigma_L}0101010101_{\sigma_R}$	4

To label such a structure, we use the notation $\ldots 1_{\sigma_L}0101\ldots0101_{\sigma_R}\ldots$ where $\sigma_{L,R} = \pm$ denote the boundary spin-1/2 degrees of freedom of an AKLT ground state. Aside from the aforementioned entangled 11- and 101-blocks, a root state may have singly occupied sites surrounded by at least two empty sites on either side. Such sites may be in any of the three LLs, or in any "spin state" after the $\hat{V}$-map. We denote such configurations by $\ldots 001_{s_z}00\ldots$. All of these observations imply that a complete set of (rotated) root states is afforded by product states of entangled units of the 11- and $1_{\sigma_L}0\ldots01_{\sigma_R}$(AKLT)-type, and of 1_{s_z}-units, all separated by at least two empty sites. We refer to the resulting patterns as "dominance patterns" compatible with an EPP.

The SU(2)-structure discussed here is not limited to the root level but emerges in the full zero-mode sector of the Hamiltonian [118]. Indeed, we identified global SU(2)-generators S_ν, $\nu = x, y, z$ that leave the zero-mode sub-space invariant (see Appendix -C.3). Consequently, zero-modes can be organized into irreps of this SU(2)-symmetry, as suggested by the root structure and associated dominance patterns.

4.5 Braiding statistics.

Recently, multi-LL wavefunctions have been discussed on the torus [119]. If the dominance patterns established here are understood as "thin torus (TT) patterns", there exists a well-defined "coherent state" method to associate braiding statistics to the excitations of the underlying state [54,63,100,101]. In this regard, we first observe that if we discard the subscripts $\sigma_{R,L}$ and s_z in the dominance patterns satisfying the EPP, the resulting reduced patterns of 1s and 0s satisfy the GPP associated with TT/dominance patterns of the ν=1/2 Moore-Read (MR) Pfaffian state: There are no more than two 1s in any four adjacent sites. In particular, the densest such patterns, $\ldots 11001100 \ldots$, and $\ldots 10101010 \ldots$, signify the six-fold torus degeneracy of the MR-state in the usual way [21]. We assume that the EPP remains meaningful on the torus and governs TT-limits of zero-modes of Eq. (4.1) and that the usual assumptions about adiabatic continuity [18] into the TT-limit hold. Then, in the presence of periodic boundary conditions, the discussion of ground state degeneracy carries over from the MR-case, and the torus degeneracy of the n=3 Hamiltonian will be six. However, any charge-1/4 quasi-hole excitation, represented by the familiar domain walls between 1010 and 1100-patterns, will carry an additional spin-1/2 described by a σ-label. So long as we fix the state of this spin (say, $\uparrow$) for all quasi-holes, the coherent state method will make the same predictions for the statistics as in the MR case [63,101]. That is, one finds that each quasi-hole carries a Majorana-fermion, and braiding two such quasi-holes is described by an operator $\theta_{ij} = \exp(i\theta_m - (-1)^m \frac{\pi}{4}\gamma_i\gamma_j)$, where γ_k is the Majorana operator of the kth quasi-hole, and θ_m is a phase only determined up to one of eight possible values by the coherent state method, as reported earlier for the ν=1 bosonic MR-state [63,101]. Elsewhere we will show that, for the fermions, the method yields $\theta_m = \frac{m\pi}{4}$, $m = 0 \ldots 7$. This is consistent with $\theta = \frac{\pi}{4}$ [120] for the ν=1/2 MR-state, but it seems

possible that the 221-state discussed here realizes a different allowed phase which, presumably, can be determined from the CFT proposed in [45, 68, 121]. The SU(2)-symmetry discussed above can, however, be used to argue that this phase does not depend on the spin-state of the quasi-holes, and the full braid operator is given simply by $\theta_{ij} X_{ij}$, where X_{ij} exchanges the spin of the ith and jth quasi-holes.

4.6 Zero-mode counting and edge physics.

General principles [10, 39] (see Appendix - C.4) imply that at any angular momentum L, the number of possible dominance patterns sets an upper-bound on the number of linearly independent zero-modes. This bound was derived as a a necessary condition on root states (the EPP). As such it applies to a large class of Hamiltonians of the form Eq. (4.1) and can be generalized to Hamiltonians with a different number of terms, internal degrees of freedom, or multi-body interactions. That there are, however, indeed as many zero-modes as admitted by the EPP depends strongly on the details of the Hamiltonian. To establish this for the n=3 Hamiltonian (4.1), we must show that to each dominance pattern allowed by the EPP, there is a zero-mode with the corresponding root state. We show in Appendix - C.4 that indeed, for every dominance pattern one can construct one such zero-mode from the states (4.2). This then necessarily yields a complete set of zero-modes. It is easy to show that the (odd N) Jain-221 state has $|\psi_{\text{root}}\rangle$ corresponding to the densest possible (minimum angular momentum) pattern consistent with the EPP: 10011001100110011 . . . (the leading orbital may not be entangled as shown in Appendix - C). This establishes that the Jain-221 state is the densest possible zero-mode since there are no allowed dominance patterns at the higher filling factor, or smaller L at given N. Note that the topological shift on the sphere, which further distinguishes candidate ν=1/2 states and in principle relates to Hall viscosity [78, 122], is likewise efficiently encoded in

Table 4.2: Number of modes for a given number of "quanta" relative to the ground state. Quanta refers to angular momentum in the case of microscopic zero-modes, and energy in the effective edge theory (4.10). The counting agrees for at least up to four quanta, and for ΔL=3, is shown in detail in Table 4.1 in terms of patterns. The chemical potential in (4.10) is chosen to give equality between total ground state angular momentum and total edge energy for any $\Delta L \ll N$.

ΔL or ΔE	0	1	2	3	4
N odd	1	4	14	33	77
N even	3	7	22	50	115

this pattern. The existence of the densest filling factor (here: 1/2) permitting zero-modes usually hints at incompressibility. This is particularly so if the edge theory encoded in the zero-mode counting is a unitary rational conformal field theory (CFT). Using patterns, we have full control over zero-mode counting. Let $\mathcal{N}(\Delta L)$ be the number of zero-modes of Eq. (4.1) at angular momentum ΔL relative to the ground state, where $\Delta L \ll N$. One may ask [123, 124] if $\mathcal{N}(\Delta L)$ agrees with the number of states having ΔL energy quanta in some CFT. In the presence of suitable chemical potential terms, one may find [39] complete agreement, for $\Delta L \ll N$, between the degeneracies of some CFT Hamiltonian and of the total angular momentum operator $\hat{L}$ within the zero-mode sector of a special Hamiltonian, for any fixed particle number N (N being identified with a suitable conserved quantity of the CFT). For $\Delta L \leq$4, we verified such agreement between the mode counting determined by our EPP and the mode counting in a 1+1d edge theory of the form [45,121]

$$H = \sum_{i=0,1} H_{b,i}(\Phi_i) + H_f(\gamma) - \frac{5}{2} N_0 \,. \tag{4.10}$$

Here, Φ_i are free chiral bosons of compactification radii $\frac{1}{2}$ and 1, respectively, γ is a Majorana field in the anti-periodic sector, all modes are co-propagating, N_i is the winding number of Φ_i, and the parity of the number of occupied Majorana modes must be opposite to $N_0 + N_1$. Except for the chemical potential term, Eq. (4.10) is the $U(1) \times SU(2)_2$-edge-CFT

first ascribed to the Jain-221 state in Refs. [45, 68, 121], notably different from other non-Abelian candidate states at half-filling, such as the Pfaffian [36] or anti-Pfaffian [80, 81]. Table 4.2 describes the above mode-counting agreement when N_0 is identified with the particle number N. Detailed counting for the number of zero-modes at ΔL=3 in terms of patterns is shown in Table 4.1.

4.7 Conclusion.

Our framework enables controlled access to numerous quasi-exactly solvable quantum-many-body Hamiltonians with LL mixing. We argued that the ability to deal with LL mixing is *essential* to establish microscopic models for a more comprehensive set of phases in the FQH-regime. To give an important and concrete example, a substantial number of results were obtained with a special focus on the n=3 LL projected Trugman-Kivelson Hamiltonian: i) Generalized Pauli principles of lowest-LL model wavefunctions become "entangled" in the presence of LL degrees of freedom. ii) This establishes a link between a large class of FQH-states, in particular "parton-like" states, and MPS of *finite* bond dimension. The latter is in turn linked to 1D symmetry protected topological phases, in our example, the Haldane phase [125, 126]. iii) EPPs can be used for efficient and, as we show, *rigorous* zero-mode counting. In particular, they establish densest zero-modes, which typically remains the only direct analytic evidence for the incompressible character of certain model FQH-states, here, the Jain-221 state. iv) Through direct zero-mode counting, we confirmed a "zero-mode paradigm" for Eq. (4.1), i.e., the edge theory of Eq. (4.1) (n=3) is a $U(1) \times SU(2)_2$-CFT. v) We identified an emergent SU(2)-symmetry under which the zero-mode spaces of Eq. (4.1) and many of its generalizations remain invariant. vi) We

demonstrated how microscopically derived EPP-dominance patterns encode bulk topological properties, notably braiding statistics, which are of Ising/Majorana-type for the Jain-221 state.

The above establishes the emergence of non-Abelian topological phases based on a solvable *two*-body interaction, which has potentially interesting implications for trilayer graphene. Our findings straightforwardly generalize to bosons, where Eq. (4.1) becomes a pure *contact* interaction. It was demonstrated [127], at least for n=1, that such contact interactions in an optical lattice with engineered band-structure lead to exactly the same zero-modes found in the continuum. Our results thus imply that a controlled route to non-Abelian phases, using only realistic two-body contact interactions, is feasible. Interestingly, many of these findings generalize to n=4, where a new parton state emerges [17] supporting Fibonacci-type anyons that facilitate universal fault-tolerant quantum computation [120].

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Appendix A

Order parameter recursion relation

A.1 Proof of Eq. (2.52)

Now we prove Eq. 2.52 by induction. It is trivial to see that it is satisfied for $N = 0, 1$. Now assume

$$c_r \hat{J}_{N-1} = \sum_m \hat{S}_{M(N-2)-r+m} \hat{J}_{N-2} c_m \tag{A.1}$$

is true. The induction hinges on the following two identities,

$$c_r \hat{S}_\ell = \sum_{k=0}^{M} (-1)^k \binom{M}{k} \hat{S}_{\ell-k} c_{r-k}, \tag{A.2}$$

$$\hat{S}_\ell c_r^\dagger = \sum_{k=0}^{M} (-1)^k \binom{M}{k} c_{r+k}^\dagger \hat{S}_{\ell-k}, \tag{A.3}$$

which one easily obtains from the definition of the $\hat{S}_\ell$ operators, Eq. (2.46) with the aid of the following two commutators,

$$[c_r, \hat{e}_n] = \hat{e}_{n-1} c_{r-1}, \tag{A.4}$$

$$[\hat{e}_n, c_r^\dagger] = c_{r+1}^\dagger \hat{e}_{n-1}. \tag{A.5}$$

Then, using the definition in Eq. 2.49 and the identity Eq. A.2 we have

$$\begin{aligned} c_r \hat{J}_N = & \frac{1}{N} \sum_m \hat{S}_{M(N-1)-r+m} \hat{J}_{N-1} c_m \\ & - \frac{1}{N} \sum_{m,r'} \sum_{k=0}^{M} (-1)^k \binom{M}{k} c_{r'+m}^\dagger \hat{S}_{M(N-1)-r'-k} \\ & \times c_{r-k} \hat{J}_{N-1} c_m. \end{aligned} \tag{A.6}$$

Henceforth, the indices of sums r, r', m, m' go from 0 to $+\infty$ unless otherwise noted. We can separate the above sum in k from 0 to M into two partial sums(one is from 0 to $M-1$ and another is $k = M$) and then use Eq. A.1 to get

$$\begin{aligned} c_r \hat{J}_N = & \frac{1}{N} \sum_m \hat{S}_{M(N-1)-r+m} \hat{J}_{N-1} c_m \\ & - \frac{1}{N} \sum_{m',m,r'} \sum_{k=0}^{M-1} (-1)^k \binom{M}{k} c_{r'+m}^\dagger \hat{S}_{M(N-1)-r'-k} \\ & \times \hat{S}_{M(N-2)-r+k+m'} \hat{J}_{N-2} c_{m'} c_m \\ & - \frac{1}{N} \sum_{m',m,r'} c_{r'+m}^\dagger \hat{S}_{M(N-1)-r+m'} \hat{S}_{M(N-2)-r'} \\ & \times \hat{J}_{N-2} c_{m'} c_m. \end{aligned} \tag{A.7}$$

In the third term of the above, we have exchange the order of two commuting $\hat{S}$ operators. We can further move $\hat{S}_{M(N-1)-r+m'}$ to the left of $c_{r'+m}^\dagger$ using the identity Eq. A.3. After

doing this, we have

$$
\begin{aligned}
c_r \hat{J}_N = & \frac{1}{N} \sum_m \hat{S}_{M(N-1)-r+m} \hat{J}_{N-1} c_m \\
& - \frac{1}{N} \sum_{m',m,r'} \sum_{k=0}^{M-1} (-1)^k \binom{M}{k} c^\dagger_{r'+m} \hat{S}_{M(N-1)-r'-k} \\
& \times \hat{S}_{M(N-2)-r+k+m'} \hat{J}_{N-2} c_{m'} c_m \\
& + \frac{1}{N} \sum_{m'} \hat{S}_{M(N-1)-r+m'} \\
& \times \Big(\sum_{m,r'} c^\dagger_{r'+m} \hat{S}_{M(N-2)-r'} \hat{J}_{N-2} c_m \Big) c_{m'} \\
& + \frac{1}{N} \sum_{m',m,r'} \sum_{k=1}^{M} (-1)^k \binom{M}{k} c^\dagger_{r'+m+k} \hat{S}_{M(N-2)-r'} \\
& \times \hat{S}_{M(N-1)-r+m'-k} \hat{J}_{N-2} c_{m'} c_m .
\end{aligned}
\tag{A.8}
$$

The third term in the above is just

$$
\frac{N-1}{N} \sum_{m'} \hat{S}_{M(N-1)-r+m'} \hat{J}_{N-1} c_{m'} \tag{A.9}
$$

using Eq. 2.49. Combined with the first term, it gives the desired result. The second term cancels with the fourth term after we make change of variables $k = M - k', r' = r'' - k = r'' - M + k'$ in the fourth term and use the fact that $\hat{S}_\ell \equiv 0$ for $l > (N-2)M$ when acting on states with particle number $N-2$. This concludes our induction proof of Eq. 2.52.

Furthermore, generalizing the above proof of Eq. 2.52 to the case of n Landau levels by using notations in Eq. 3.36 with $A(r)$ given in Appendix A.3 and using the following generalization of Eqs. A.2 and A.3,

$$
\tilde{c}_{a,r} \hat{S}_\ell = \sum_{k=0}^{M} (-1)^k \binom{M}{k} \sqrt{\frac{(a+r)!}{(a+r-k)!}} \hat{S}_{\ell-k} \tilde{c}_{a,r-k}, \tag{A.10}
$$

$$\hat{S}_\ell \tilde{c}^*_{a,r} = \sum_{k=0}^{M} (-1)^k \binom{M}{k} \sqrt{\frac{(a+r+k)!}{(a+r)!}} \tilde{c}^*_{a,r+k} \hat{S}_{\ell-k}, \tag{A.11}$$

we easily arrive at Eq. 3.45 using the same method.

A.2 Zero Mode Generators

In Ref. [39], we have obtained in second-quantized form the parent Hamiltonian for the unprojected Jain 2/5 state,

$$\begin{aligned} H = & E^{(1)} \sum_R \mathcal{T}_R^{(1)\dagger} \mathcal{T}_R^{(1)} + E^{(2)} \sum_R \mathcal{T}_R^{(2)\dagger} \mathcal{T}_R^{(2)} \\ & + E^{(3)} \sum_R \mathcal{T}_R^{(3)\dagger} \mathcal{T}_R^{(3)} + E^{(4)} \sum_R \mathcal{T}_R^{(4)\dagger} \mathcal{T}_R^{(4)}, \end{aligned} \tag{A.12}$$

where $E^{(1)} = \frac{5+\sqrt{17}}{16\pi}$, $E^{(2)} = \frac{9}{8\pi}$, $E^{(3)} = \frac{1}{4\pi}$, $E^{(4)} = \frac{5-\sqrt{17}}{16\pi}$.

The bilinear $\mathcal{T}$-operators are given by $\mathcal{T}_R^{(\lambda)} = \sum_{x,m_1,m_2} \eta^{(\lambda)}_{R,x,m_1,m_2} c_{m_1,R-x} c_{m_2,R+x}$ with

$$\begin{aligned} \eta^{(1)}_{R,x,m_1,m_2} =& \frac{\sqrt{2}}{2\sqrt{17-\sqrt{17}}} \left(\frac{(-1+\sqrt{17})}{2^{R+1/2}} \sqrt{\binom{2R+1}{R+x}} \delta_{m_1,1}\delta_{m_2,0} - \frac{4x}{2^{R+1/2}} \sqrt{\frac{1}{2R+2}\binom{2R+2}{R+1+x}} \delta_{m_1,1}\delta_{m_2,1} \right), \\ \eta^{(2)}_{R,x,m_1,m_2} =& \frac{1}{2^R 3} \Big(\sqrt{2}\, x \sqrt{\frac{1}{R}\binom{2R}{R+x}} \delta_{m_1,0}\delta_{m_2,0} + 2(2x^2-2x-R) \sqrt{\frac{1}{2R(2R+1)}\binom{2R+1}{R+x}} \delta_{m_1,1}\delta_{m_2,0} \\ & - (2x^3-(3R+2)x) \sqrt{\frac{1}{2R(2R+1)(2R+2)}\binom{2R+2}{R+1+x}} \delta_{m_1,1}\delta_{m_2,1} \Big), \\ \eta^{(3)}_{R,x,m_1,m_2} =& \frac{1-2x}{2^{R+1/2}} \sqrt{\frac{1}{2R+1}\binom{2R+1}{R+x}} \delta_{m_1,1}\delta_{m_2,0}, \\ \eta^{(4)}_{R,x,m_1,m_2} =& \frac{\sqrt{2}}{2\sqrt{17+\sqrt{17}}} \left(\frac{(-1-\sqrt{17})}{2^{R+1/2}} \sqrt{\binom{2R+1}{R+x}} \delta_{m_1,1}\delta_{m_2,0} - \frac{4x}{2^{R+1/2}} \sqrt{\frac{1}{2R+2}\binom{2R+2}{R+1+x}} \delta_{m_1,1}\delta_{m_2,1} \right). \end{aligned} \tag{A.13}$$

We have found four classes of one-body zero mode generators in Ref. [39], which leave invariant the zero mode space of the above Hamiltonian,

$$\begin{aligned}
\hat{P}_d^{(1)} &= \sum_{r=-1}^{+\infty} \sqrt{\frac{(r+d)!}{(r+1)!}} c_{0,r+d}^{\dagger} c_{1,r}, \\
\hat{P}_d^{(2)} &= \sum_{r=0}^{+\infty} \sqrt{\frac{(r+d)!}{r!}} c_{0,r+d}^{\dagger} c_{0,r} + \sum_{r=-1}^{+\infty} \sqrt{\frac{(r+d+1)!}{(r+1)!}} c_{1,r+d}^{\dagger} c_{1,r}, \\
\hat{P}_d^{(3)} &= \sum_{r=-1}^{+\infty} \Big((r+d+1) \sqrt{\frac{(r+d)!}{(r+1)!}} c_{0,r+d}^{\dagger} c_{1,r} + \sqrt{\frac{(r+d+1)!}{(r+1)!}} c_{1,r+d}^{\dagger} c_{1,r} \Big), \\
\hat{P}_d^{(4)} &= \sum_{r=0}^{+\infty} \Big(\sqrt{\frac{(r+d+1)!}{r!}} c_{1,r+d}^{\dagger} c_{0,r} + (r+d+1) \sqrt{\frac{(r+d)!}{r!}} c_{0,r+d}^{\dagger} c_{0,r} \Big) \\
&\quad - \sum_{r=-1}^{+\infty} \Big((r+1) \sqrt{\frac{(r+d+1)!}{(r+1)!}} c_{1,r+d}^{\dagger} c_{1,r} + (r+1)(r+d+1) \sqrt{\frac{(r+d)!}{(r+1)!}} c_{0,r+d}^{\dagger} c_{1,r} \Big).
\end{aligned} \tag{A.14}$$

The fact that they are indeed zero mode generators results from the non-trivial commutation relations $[\mathcal{T}_R^{(\lambda)}, \hat{p}_d^{(i)}] = \sum_{\lambda'=1}^{4} \alpha_{\lambda,\lambda',i,R,d} \mathcal{T}_{R-\frac{d}{2}}^{(\lambda')}$ for $\lambda, i = 1,2,3,4$, where $\alpha_{\lambda,\lambda',i,R,d}$ is a coefficient depending on $\lambda, \lambda', i, R, d$.

Simple calculations show that $\hat{p}_d^{a,b}$s and $\hat{p}_d$ in the main article are essentially equivalent to the above zero mode generators. In deed, we have

$$\begin{aligned}
&\hat{p}_d^{0,0} = \hat{P}_d^{(2)} + d\hat{P}_d^{(1)} - \hat{P}_d^{(3)}, \ \hat{p}_d^{0,1} = \hat{P}_d^{(1)}, \ \hat{p}_d^{1,0} = \hat{P}_d^{(4)}, \\
&\hat{p}_d^{1,1} = \hat{P}_d^{(3)}, \ \hat{p}_d = \hat{p}_d^{0,0} + \hat{p}_d^{1,1} = \hat{P}_d^{(2)} + d\hat{P}_d^{(1)}.
\end{aligned} \tag{A.15}$$

As shown in Eq. 3.52, $\hat{p}_d^{a,b}$s form a closed graded Lie algebra, $[\hat{p}_k^{a,b}, \hat{p}_{k'}^{b',a'}] = \delta_{b,b'}\hat{p}_{k+k'}^{a,a'} - \delta_{a,a'}\hat{p}_{k+k'}^{b',b}$. Now If we define $Q_R^{(1)}$ and $Q_R^{(4)}$ as linear combinations of $\mathcal{T}_R^{(1)}$ and $\mathcal{T}_R^{(4)}$:

$$\begin{aligned} Q_R^{(1)} &= \sqrt{\frac{1}{34}\left(17-\sqrt{17}\right)}\mathcal{T}_R^{(1)} - \sqrt{\frac{1}{34}\left(17+\sqrt{17}\right)}\mathcal{T}_R^{(4)}, \\ Q_R^{(4)} &= \sqrt{\frac{1}{34}\left(17+\sqrt{17}\right)}\mathcal{T}_R^{(1)} + \sqrt{\frac{1}{34}\left(17-\sqrt{17}\right)}\mathcal{T}_R^{(4)}, \end{aligned} \tag{A.16}$$

the zero mode condition Eq. 3.16 becomes

$$\begin{aligned} \mathcal{T}_R^{(\lambda)}\left|\psi_{\mathrm{zm}}\right\rangle &= 0, \quad \text{for } \lambda = 2,3, \\ Q_R^{(\lambda')}\left|\psi_{\mathrm{zm}}\right\rangle &= 0, \quad \text{for } \lambda' = 1,4. \end{aligned} \tag{A.17}$$

It is easy to verify that $\hat{p}_d^{a,b}$ are indeed zero mode generators by virtue of following commutators:

$$[Q_R^{(1)}, \hat{p}_d^{0,0}] = 2^{1-\frac{d}{2}}\sqrt{\frac{(2R+1)!}{(2R-d+1)!}}\,Q_{R-\frac{d}{2}}^{(1)}, \tag{A.18a}$$

$$\begin{aligned} [\mathcal{T}_R^{(2)}, \hat{p}_d^{0,0}] =& 2^{(1-d)/2}\sqrt{\frac{(2R-1)!}{(2R-d+1)!}}\Big(\frac{2d(d-1)}{3}Q_{R-\frac{d}{2}}^{(1)} \\ &+ \sqrt{2(2R-d)(2R-d+1)}\mathcal{T}_{R-\frac{d}{2}}^{(2)} \\ &+ d(d-1)Q_{R-\frac{d}{2}}^{(4)}\Big). \end{aligned} \tag{A.18b}$$

$$[\mathcal{T}_R^{(3)}, \hat{p}_d^{0,0}] = 2^{1-\frac{d}{2}}\sqrt{\frac{(2R)!}{(2R-d)!}}\,\mathcal{T}_{R-\frac{d}{2}}^{(3)}, \tag{A.18c}$$

$$[Q_R^{(4)}, \hat{p}_d^{0,0}] = 2^{1-\frac{d}{2}}\sqrt{\frac{(2R+1)!}{(2R-d+1)!}}\,Q_{R-\frac{d}{2}}^{(4)}, \tag{A.18d}$$

$$[Q_R^{(1)}, \hat{p}_d^{0,1}] = 0, \tag{A.18e}$$

$$[\mathcal{T}_R^{(2)}, \hat{p}_d^{0,1}] = -\frac{2^{(3-d)/2}}{3}\sqrt{\frac{(2R-1)!}{(2R-d+1)!}}\Big((d-1)Q^{(1)}_{R-\frac{d}{2}} + \sqrt{2R-d+1}\mathcal{T}^{(3)}_{R-\frac{d}{2}} + 2(d-1)Q^{(4)}_{R-\frac{d}{2}}\Big), \tag{A.18f}$$

$$[\mathcal{T}_R^{(3)}, \hat{p}_d^{0,1}] = 2^{1-\frac{d}{2}}\sqrt{\frac{(2R)!}{(2R-d+1)!}}\, Q^{(4)}_{R-\frac{d}{2}}, \tag{A.18g}$$

$$[Q_R^{(4)}, \hat{p}_d^{0,1}] = 0, \tag{A.18h}$$

$$[Q_R^{(1)}, \hat{p}_d^{1,0}] = 2^{-\frac{d}{2}}\sqrt{\frac{(2R+1)!}{(2R-d+1)!}}\Big((d+1)Q^{(1)}_{R-\frac{d}{2}} + (2R+1)\sqrt{2R-d+1}\mathcal{T}^{(3)}_{R-\frac{d}{2}}\Big), \tag{A.18i}$$

$$\begin{aligned}[\mathcal{T}_R^{(2)}, \hat{p}_d^{1,0}] =& \frac{2^{(1-d)/2}}{3}\sqrt{\frac{(2R-1)!}{(2R-d+1)!}} \\ &\Big((1+d)R(1+2R)Q^{(1)}_{R-\frac{d}{2}} \\ &+ 3\sqrt{2(2R-d)(2R-d+1)}\mathcal{T}^{(2)}_{R-\frac{d}{2}} \\ &- R(1+2d-2R)\sqrt{2R-d+1}\mathcal{T}^{(3)}_{R-\frac{d}{2}} \\ &- 2(d+1)R(-2+d-4R)Q^{(4)}_{R-\frac{d}{2}}\Big),\end{aligned} \tag{A.18j}$$

$$\begin{aligned}[\mathcal{T}_R^{(3)}, \hat{p}_d^{1,0}] =& 2^{-1-\frac{d}{2}}\sqrt{\frac{(2R)!}{(2R-d+1)!}} \\ &\Big(-2(1+d)(2R+1)Q^{(1)}_{R-\frac{d}{2}} \\ &- 3\sqrt{2(2R-d)(2R-d+1)}\mathcal{T}^{(2)}_{R-\frac{d}{2}} \\ &+ 2(1+d)\sqrt{2R-d+1}\mathcal{T}^{(3)}_{R-\frac{d}{2}} \\ &+ (d^2-d-4-4R^2-4dR-10R)Q^{(4)}_{R-\frac{d}{2}}\Big),\end{aligned} \tag{A.18k}$$

$$[Q_R^{(4)}, \hat{p}_d^{1,0}] = 2^{-\frac{d}{2}} \sqrt{\frac{(2R+1)!}{(2R-d+1)!}} \Big(-(d+1) Q^{(1)}_{R-\frac{d}{2}} + \sqrt{2R-d+1}\, \mathcal{T}^{(3)}_{R-\frac{d}{2}} \Big), \tag{A.18l}$$

$$[Q_R^{(1)}, \hat{p}_d^{1,1}] = 2^{-\frac{d}{2}} \sqrt{\frac{(2R+1)!}{(2R-d+1)!}} \Big(Q^{(1)}_{R-\frac{d}{2}} - Q^{(4)}_{R-\frac{d}{2}} \Big), \tag{A.18m}$$

$$[\mathcal{T}_R^{(2)}, \hat{p}_d^{1,1}] = \frac{2^{(3-d)/2}}{3} \sqrt{\frac{(2R-1)!}{(2R-d+1)!}} \Big(dR Q^{(1)}_{R-\frac{d}{2}} - R\sqrt{2R-d+1}\, \mathcal{T}^{(3)}_{R-\frac{d}{2}} + 2dR Q^{(4)}_{R-\frac{d}{2}} \Big), \tag{A.18n}$$

$$[\mathcal{T}_R^{(3)}, \hat{p}_d^{1,1}] = 2^{-\frac{d}{2}} \sqrt{\frac{(2R)!}{(2R-d+1)!}} \Big(dQ^{(1)}_{R-\frac{d}{2}} + \sqrt{2R-d+1}\, \mathcal{T}^{(3)}_{R-\frac{d}{2}} + (1+2d+2R) Q^{(4)}_{R-\frac{d}{2}} \Big), \tag{A.18o}$$

$$[Q_R^{(4)}, \hat{p}_d^{1,1}] = 2^{1-\frac{d}{2}} \sqrt{\frac{(2R+1)!}{(2R-d+1)!}}\, Q^{(4)}_{R-\frac{d}{2}}, \tag{A.18p}$$

Now we will prove that $\hat{e}_k$ defined in Eq. 3.42 satisfies the Newton-Girard formula Eq. A.20, therefore is a k-body zero mode generator as it can be expressed in terms of $\hat{p}_d$ with $d = 1, ...k$. As a result, $\hat{S}_\ell$ is also a zero mode generator by its definition. To prove the Newton-Girard formula, we can write down $\hat{e}_k$ in terms of $\hat{e}_{k-1}$,

$$\hat{e}_k = \frac{1}{k} \sum_{n,l} \sqrt{l+n+1}\, \tilde{c}^*_{n,l+1} \hat{e}_{k-1} \tilde{c}_{n,l}. \tag{A.19}$$

Using the commutator $[\hat{e}_k, \tilde{c}_{n,l}] = -\hat{e}_{k-1}\sqrt{l+n}\,\tilde{c}_{n,l-1}$ to move $\hat{e}$ operator all the way to the right of $\tilde{c}$ operators, one can arrive at the Newton-Girard formula

$$\hat{e}_k = \frac{1}{k}\sum_{d=1}^{k}(-1)^{d-1}\hat{p}_d\hat{e}_{k-d}. \tag{A.20}$$

In the same way, one can use $[\hat{e}^{a,b}_k, \tilde{c}_{b,l}] = -\delta_{a,b}\sqrt{l+b}\,\hat{e}^{a,b}_{k-1}\tilde{c}_{b,l-1}$ to obtain a modified Newton-Girard formula

$$\hat{e}^{a,b}_k = \frac{1}{k}\hat{p}^{a,b}_1\hat{e}^{a,b}_{k-1} + \frac{\delta_{a,b}}{k}\sum_{d=2}^{k}(-1)^{d-1}\hat{p}^{a,b}_d\hat{e}^{a,b}_{k-d}. \tag{A.21}$$

Consequently, $\hat{e}^{a,b}_k$ are also k-body zero mode generators since they can be expressed in terms of either $\hat{p}^{a,b}_1$ or $\hat{p}^{a,a}_d$ with $d = 1, ...k$.

With Eq. A.18 and the above (modified) Newton-Girard formulae, we immediately see that $\hat{S}$ and $\hat{e}^{a,b}$ are zero mode generators.

A.3 $A(r)$ matrix for n LLs

Now we generalize the transformation matrix $A(r)$ for 2 LLs to the case of n LLs. Its entries are

$$A(r)_{i,j} = \frac{1}{(i-j)!}\sqrt{\frac{(i+r)!i!}{(j+r)!j!}}, \tag{A.22}$$

as obtained straightforwardly by expanding disk Landau level wave functions in powers of z and $\bar{z}$. Thus, it is a lower triangular matrix with all the diagonal entries being 1. Its inverse is easily found out to be a lower triangular matrix with all the diagonal entries being 1 as well,

$$A^{-1}(r)_{i,j} = \frac{(-1)^{i+j}}{(i-j)!}\sqrt{\frac{(i+r)!i!}{(j+r)!j!}}. \tag{A.23}$$

Appendix B

Parent Hamiltonian Construction

B.1 Parent Hamiltonian Construction on sphere and zero modes for two Landau levels

B.1.1 Basis Transformation

Consider spinor basis, $u = \cos\frac{\theta}{2}e^{i\phi/2}$, $v = \sin\frac{\theta}{2}e^{-i\phi/2}$. Let us define another co-ordinate $z = 2R\frac{v}{u} = 2R\tan\frac{\theta}{2}e^{-i\phi}$. It is straight forward to construct the inverse function.

$$\theta = 2\arctan\frac{\sqrt{z\bar{z}}}{2R}; \qquad \phi = \frac{i}{2}(\ln z - \ln\bar{z}) \tag{B.1}$$

Hence,

$$\theta_z = \frac{\frac{1}{2R}\sqrt{\frac{\bar{z}}{z}}}{1+\frac{z\bar{z}}{4R^2}}; \quad \theta_{\bar{z}} = \frac{\frac{1}{2R}\sqrt{\frac{z}{\bar{z}}}}{1+\frac{z\bar{z}}{4R^2}}; \quad \phi_z = \frac{i}{2z}; \quad \phi_{\bar{z}} = -\frac{i}{\bar{z}} \tag{B.2}$$

Now, let us consider the metric on the surface of a sphere of radius R. $g_{\theta\theta} = R^2$, $g_{\phi\phi} = R^2 \sin^2\theta$. Where we can write $\sin^2\theta$ in terms of $z, \bar{z}$.

$$\sin^2\theta = \frac{2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{(\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2})^2} = \left(\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)^2 = \frac{1}{4R^2}\frac{4z\bar{z}}{(1+z\bar{z}/4R^2)^2} \tag{B.3}$$

Now, we can calculate the matrix elements for the metric in $z, \bar{z}$.

$$\begin{aligned}
g_{zz} &= g_{\theta\theta}\theta_z^2 + g_{\phi\phi}\phi_z^2 = \frac{\bar{z}}{4z(1+z\bar{z}/4R^2)^2} - \frac{\bar{z}}{4z(1+z\bar{z}/4R^2)^2} = 0; && \text{(B.4)}\\
\Rightarrow g_{\bar{z}\bar{z}} &= 0 && \text{(B.5)}\\
\implies g_{z\bar{z}} &= g_{\bar{z}z} = g_{\theta\theta}\theta_z\theta_{\bar{z}} + g_{\phi\phi}\phi_z\phi_{\bar{z}} && \text{(B.6)}\\
g_{z\bar{z}} &= \frac{1}{4(1+z\bar{z}/4R^2)^2} + \frac{1}{4(1+z\bar{z}/4R^2)^2} = \frac{1}{2(1+z\bar{z}/4R^2)^2} && \text{(B.7)}\\
\sqrt{|g|} &= \frac{1}{2(1+z\bar{z}/4R^2)^2} && \text{(B.8)}
\end{aligned}$$

Now in the lowest Landau level, $\psi_0(u,v) = u^{s+m}v^{s-m} = \frac{v^{2s}(2R)^{s+m}}{z^{s+m}}$. But,

$$v\bar{v} = \sin^2\frac{\theta}{2} = 1 - \frac{1}{1+tan^2\frac{\theta}{2}} = \frac{\frac{z\bar{z}}{4R^2}}{\left(1+\frac{z\bar{z}}{4R^2}\right)^2} \tag{B.9}$$

Let us define the inner product,

$$\int dzd\bar{z}\frac{1}{2(1+z\bar{z}/4R^2)^2}\frac{(v\bar{v})^{2s}(4R^2)^{s+m}}{(z\bar{z})^{s+m}} = \frac{1}{2}\int dzd\bar{z}\frac{(z\bar{z}/4R^2)^{s-m}}{(1+z\bar{z}/4R^2)^{2(s+1)}} \tag{B.10}$$

Let us take $2R = 1$ from now on. We can define $\psi_0(z)$ on new basis as, $\frac{z^{s-m}}{(1+z\bar{z})^{s+1}}$. Where, $2s$ is the magnetic flux quanta and m is L_z value.

B.1.2 Angular Momentum Operator

$$L = -(e_\phi \partial_\theta - e_\theta \frac{1}{\sin\theta}\partial_\phi) + (eRe_r \times A) + se_r \tag{B.11}$$

By the choice $A = -\frac{S}{eR}\cot\theta e_\phi$. Where e_r, e_θ, e_ϕ are unit vectors along the r, θ and ϕ directions. Hence,

$$L_Z = -(e^z_\phi \partial_\theta - e^z_\theta \frac{1}{\sin\theta}\partial_\phi) + (e^z_\theta) + se^z_r = -i\partial_\phi = -iz_\phi \partial_z - i\bar{z}_\phi \partial_{\bar{z}} = \bar{z}\partial_{\bar{z}} - z\partial_z \tag{B.12}$$

However, when we act with this L_Z on our new wavefunction, we get the eigenvalue as $m - s$. Hence, let us redefine $L_Z = \bar{z}\partial_{\bar{z}} - z\partial_z + s$. For two particle, we can write,

$$L_Z = \bar{z}_1\partial_{\bar{z}_1} - z_1\partial_{z_1} + \bar{z}_2\partial_{\bar{z}_2} - z_2\partial_{z_2} + 2s \tag{B.13}$$

B.1.3 Laplacian in $z, \bar{z}$ co-ordinate

General Lapalacian operator, $\nabla^2 f = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial)jf)$. But

$$g_{ij} = \frac{1}{2(1+z\bar{z})^2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow g^{ij} = 2(1+z\bar{z})^2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \sqrt{|g|}g^{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{B.14}$$

Hence, Laplacian in $z, \bar{z}$ co-ordinate will become, $\frac{2}{\sqrt{|g|}}\partial_z\partial_{\bar{z}}$.

B.1.4 V_1 Potential in $z, \bar{z}$ co-ordinate

$$\int dz d\bar{z} \delta(z)\delta(\bar{z}) = 1 \Rightarrow \int dz d\bar{z} \nabla^2 \delta(z)\delta(\bar{z})\psi^*(z,\bar{z})\phi(z,\bar{z}) \tag{B.15}$$

$$= 2\int dz d\bar{z} \frac{1}{\sqrt{|g|}} \partial_z \partial_{\bar{z}} \delta(z)\delta(\bar{z})\psi^*(z,\bar{z})\phi(z,\bar{z}) \tag{B.16}$$

Hence, we can write V_1 potential as,

$$\langle\psi|V_1|\phi\rangle = \int d^2z_1 d^2z_2 \partial_{z_1}\partial_{\bar{z}_1}\delta(z_1 - z_2)\delta(\bar{z}_1 - \bar{z}_2)\frac{1}{\sqrt{|g|}}\psi^*(z_1,z_2)\phi(z_1,z_2) \tag{B.17}$$

$$= \int d^2z_1 d^2z_2 \delta(z_1 - z_2)\delta(\bar{z}_1 - \bar{z}_2)\partial_{z_1}\partial_{\bar{z}_1}\frac{1}{\sqrt{|g|}}\psi^*(z_1,z_2)\phi(z_1,z_2) \tag{B.18}$$

$$\int d^2z_1 d^2z_2 \delta(z_r)\delta(\bar{z}_r)\partial_{z_r}\partial_{\bar{z}_r}\frac{1}{\sqrt{|g|}}\psi^*(z_r,z_c)\phi(z_r,z_c) \tag{B.19}$$

Where, $z_r = z_1 - z_2$, $z_c = \frac{z_1+z_2}{2}$. Also, $[V_1, L_Z] = 0$.

B.1.5 Two particle wavefunctions in lowest and first Landau levels

before going to two particle picture let us review the single particle wave functions. In LLL,

$$\psi_0 = u^{s+m}v^{s-m} \Rightarrow \frac{z^{s-m}}{(1+z\bar{z})^{s+1}} = \frac{z^{s-m}(1+z\bar{z})}{(1+z\bar{z})^{s+2}}$$

In the first Landau level,

$$\psi_1 = \frac{z^{s-m}((1+s+m)z\bar{z} - (1+s-m))}{(1+z\bar{z})^{s+2}}$$

Each of these wavefunctions have $L_Z = m$. Hence, two particle wavefunctions can be written for lowest two Landau levels in the following way,

$$\psi^{(1,2)} = \frac{A(z_1,z_2) + B(z_1,z_2)\bar{z}_1 + C(z_1,z_2)\bar{z}_2 + D(z_1,z_2)\bar{z}_1\bar{z}_2}{(1+z_1\bar{z}_1)^{s+2}(1+z_1\bar{z}_1)^{s+2}} \tag{B.20}$$

$$= \frac{A(z_c,z_r) + B(z_c,z_r)\bar{z}_c + C(z_c,z_r)\bar{z}_r + D(z_c,z_r)\bar{z}_c\bar{z}_r}{(1+z_1\bar{z}_1)^{s+2}(1+z_1\bar{z}_1)^{s+2}} \tag{B.21}$$

Where, A, B, C, D are holomorphic functions. Just following the similar argument from the disk geometry, we can Taylor expand each of them in z_r and look for only those terms which have non-zero contribution to V_1 potential.

$$\psi^{(1,2)} = \frac{\begin{array}{c} A_0(z_c)(z_1-z_2) + B_0(z_c)(z_1-z_2)(\bar{z}_1+\bar{z}_2) + C_0(z_c)(\bar{z}_1-\bar{z}_2) + \\ C_1(z_c)(z_1-z_2)^2(\bar{z}_1-\bar{z}_2) + D_0(z_c)(z_1-z_2)\bar{z}_1\bar{z}_2 + D_1(z_c)(z_1-z_2)^3\bar{z}_1\bar{z}_2 \end{array}}{(1+z_1\bar{z}_1)^{s+2}(1+z_1\bar{z}_1)^{s+2}} \tag{B.22}$$

Unlike plane, we can not separate the denominator (eqv to Gaussian part in the plane) in z_r and z_c basis. Hence, for sphere we are again going back to the particle basis. Now, in order to have $L_z = 2s - m$, we should choose the following,

$$A_0(z_c) = a_0(z_1+z_2)^{m-1}; \quad B_0(z_c) = b_0(z_1+z_2)^m; \quad C_0(z_c) = c_0(z_1+z_2)^{m+1}; \tag{B.23}$$

$$C_0(z_c) = c_1(z_1+z_2)^{m-1}; \quad D_0(z_c) = d_0(z_1+z_2)^{m+1}; \quad D_1(z_c) = d_1(z_1+z_2)^{m-1}; \tag{B.24}$$

$$\Rightarrow \psi^{(1,2)} = (z_1-z_2)^{m-1}\frac{az_1 + bz_1^2\bar{z}_1 + cz_1^2\bar{z}_2 + dz_1\bar{z}_1z_2 + ez_1^3\bar{z}_1z_2 + fz_1^2\bar{z}_1z_2\bar{z}_2 - (z_1 \leftrightarrow z_2)}{(1+z_1\bar{z}_1)^{s+2}(1+z_1\bar{z}_1)^{s+2}} \tag{B.25}$$

Where, $a = a_0$, $b = b_0 + c_0 + c_1$, $c = b_0 - c_0 - c_1$, $d = 2(c_0 - c_1)$, $e = d_0 + d_1$, $f = d_0 - 3d_1$ $\psi^{(1,2)}$ be further simplified as,

$$\sum_{r=0}^{m-1} \binom{m-1}{r} \frac{\begin{array}{c} az_1^{r+1}z_2^{m-r-1} + b(z_1\bar{z}_1)z_1^{r+1}z_2^{m-r-1} + cz_1^{r+2}(z_2\bar{z}_2)z_2^{m-r-2} + d(z_1\bar{z}_1)z_1^r z_2^{m-r} \\ + e(z_1\bar{z}_1)z_1^{r+2}(z_2\bar{z}_2)z_2^{m-r-2} + f(z_1\bar{z}_1)z_1^{r+1}(z_2\bar{z}_2)z_2^{m-r-1} - (z_1 \leftrightarrow z_2) \end{array}}{(1+z_1\bar{z}_1)^{s+2}(1+z_1\bar{z}_1)^{s+2}} \tag{B.26}$$

Now, we will expand this in terms of orthonormal basis. Let us define, $\eta_0(z,r) = \frac{z^r}{(1+z\bar{z})^{s+2}}$, $\eta_1(z,r) = \frac{((2+2s-r)z\bar{z}-(1+r))z^r}{(1+z\bar{z})^{s+2}}$

$$\int dzd\bar{z}\eta_0(z,r)\eta_0(\bar{z},r') = \delta_{r,r'}\frac{\pi}{\binom{2s+3}{r+1}(r+1)} \tag{B.27}$$

$$= \frac{1}{(\mathcal{N}_0^r)^2}; \int dzd\bar{z}\eta_1(z,r)\eta_1(\bar{z},r') \tag{B.28}$$

$$= \delta_{r,r'}\frac{\pi(2s+2-r)}{\binom{2s+2}{r+2}(r+2)} = \frac{1}{(\mathcal{N}_1^r)^2} \tag{B.29}$$

We also have, $\int dzd\bar{z}\eta_0(z,r)\eta_1(\bar{z},r') = 0$. Now Let us redefine,

$$\mathcal{N}_0^r\eta_0(z,r) = \frac{z^r}{(1+z\bar{z})^{s+2}};\ \ \mathcal{N}_1^r\eta_1(z,r) = \frac{((2+2s-r)z\bar{z}-(1+r))z^r}{(1+z\bar{z})^{s+2}} \tag{B.30}$$

$$\Rightarrow \frac{(z\bar{z})z^r}{(1+z\bar{z})^{s+2}} = \frac{\mathcal{N}_1^r\eta_1(z,r) + (1+r)\mathcal{N}_0^r\eta_0(z,r)}{2+2s-r} \tag{B.31}$$

Notice, neither η_0 nor η_1 are physical particles in lowest or first Landau levels. When $s \to \infty$, η_0 represents the LLL while η_1 represents first Landau level. But this limit is physically meaningful when $R \to \infty$.This limit will restore the planar geometry. Nonethe-less, $\eta_0(r)$ and $\eta_1(r)$ are eigenvectors of L_Z with eigenvalues r. Hence, V_1 potential will not mix ηs for different r. However, it is perfectly okay to mix between same r. Now, we can rewrite $\psi^{(1,2)}$, in terms of ηs.

B.2 Proof of the locality in real space for the composite fermion parent Hamiltonian

We will argue about the locality of our parent Hamiltonian by making a connection to the positive semi-definite V_M potentials, given by,

$$V_M = \sum_{m \leq M,\ i>j} a_m \nabla_i^{2m} \delta^{(2)}(\mathbf{r}_i - \mathbf{r}_j) \tag{B.32}$$

∇_i^2 is the laplacian in the coordinate of the i^{th} particle. This is a highly localized interaction Hamiltonian. We, however, studied these Hamiltonians quite extensively, in certain projected scenario of Laughlin's state. It can be shown [10], when we project these Hamiltonians in lowest Landau levels (with some $\mathcal{P}_{LLL}$ operators), we get the following second quantized operators,

$$\mathcal{P}_{LLL} V_M \mathcal{P}_{LLL} := \sum_{m \leq M} \sum_J T_J^{m\dagger} T_J^m; \qquad T_J^m = \sum_x x^m c_{0,J+x} c_{0,J-x} \tag{B.33}$$

These annihilation operators are exactly $T^m_{a,b,J}$ operators in in Eq. (3.94) for a and b set to zero. These operators, however, have been derived from local interaction terms, thus $T^m_{a,b,J}$ operators are indeed coming from local interaction terms for $a = 0$ and $b = 0$. This concludes first part of the proof. Now, let us consider another set of operators, $p_k^{a,b}$

$$p_0^{a,b} = \sum_r \tilde{c}^* a, r \tilde{c}_r \tag{B.34}$$

These are single body operators, describe "hopping" in virtual degrees of freedom (Λ-levels). All of those Λ-levels with fixed angular momentum is localized in space. Hence,

$p_0^{a,b}$ won't be able to change any non-local interaction to local interaction. However,

$$[p^{a,0}, T^m_{a,0,J}] = T^m_{0,0,J} \tag{B.35}$$

This concludes $T^m_{a,0,J}$ for all odd m can be generated from local interaction. Applying the same logic once more, and using the fact that $T^m_{a,0,J}$s are generated from local interaction, one can argue all $T^m_{a,b,J}$s for odd m can be generated from local interactions.

The above argument, however, fails for even powers of m. It is due to the fact that $T^m_{0,0,J}$s are zero for even m. In order to complete the argument, we must look into the projected form of Eq. (B.32) in two Landau levels. It has been calculated explicitly for V_1 case while constructing parent Hamiltonian for $2/5$ composite fermion. Zero mode of that Hamiltonian can be written in terms of four annihilation operators, given by Eq. (3.16).

In this part of the proof, we should take those four equations and write them in the pseudo-fermion basis. We will get following set of four equations for any ground state $|\psi\rangle$.

$$T^1_{0,0,J}|\psi\rangle = T^1_{1,0,J}|\psi\rangle = T^1_{1,1,J}|\psi\rangle = T^0_{1,0,J}|\psi\rangle = 0 \tag{B.36}$$

$T^m_{a,b,J}$ are given by eq. (3.94) for arbitrary a, b and m. Now all of the above four equations are derived from a local interaction and we have already argued each $T^1_{a,b,J}$ can be generated from local interaction. Thus we conclude $T^0_{1,0,J}$ can be generated from local interaction. Using the same technique on other V_Ms (second Landau level projected), one can argue $T^m_{1,0,J}$ can be constructed from a local interaction term. Now, using the commutation relation

$$[p^{a,0}, T^m_{a,0,J}] = T^m_{0,0,J} \tag{B.37}$$

and reconstructing the steps from $m =$ odd case, one can indeed show that all $T^m_{a,b,J}$s for odd m can be generated from local interactions.

B.3 Composite fermions are the zero modes of the parent Hamiltonian given by Eq. (3.94)

In this section we will complete the proof that composite fermions are indeed the zero modes of the parent Hamiltonian given by Eq. (3.94). As argued in the main text, it is sufficient to prove the result for two-particle case. $N > 2$ particle case can be proven by induction. Jain's composite fermion for two particles and can be written down as,

$$\psi = (z_1 - z_2)^M(\bar{z}_1^n\bar{z}_2^{n-1} - \bar{z}_1^n\bar{z}_2^{n-1}) = \frac{1}{2}\sum_r \binom{M}{r}(z_1^r z_2^{M-r} + z_2^r z_1^{M-r})(\bar{z}_1^n\bar{z}_2^{n-1} - \bar{z}_1^n\bar{z}_2^{n-1})$$
$$\Rightarrow |\psi\rangle = \frac{1}{2}\sum_r \binom{M}{r}(\tilde{c}^*_{n,r-n}\tilde{c}^*_{n-1,M+1-r-n} + \tilde{c}^*_{n,M-r-n}\tilde{c}^*_{n-1,r+1-n})|0\rangle \tag{B.38}$$

For the fermionic states, M is always even integer in order to preserve the total antisymmetry. Now let us act $T_{r,a,b,J}$ on the above state, where,

$$T_{r,a,b,J} = \sum_x x^r \tilde{c}_{a,J+x}\tilde{c}_{b,J-x}, \quad r < M \tag{B.39}$$

$$T_{r,n,n-1,J}|\psi\rangle = 2\delta_{M+1,2(J+n)}(-1)^{3(J+n)}\sum_x \left(x^r(-1)^x\binom{2(J+n)-1}{J+n+x}\right)|0\rangle \tag{B.40}$$

Now replacing $J + n + x$ by x' gives,

$$T_{r,n,n-1,J}|\psi\rangle = 2\delta_{2M+1,2(J+n)}(-1)^{2M+1}\sum_{x'}(x' - J - n)^r(-1)^{x'}\binom{2M}{x'}|0\rangle \tag{B.41}$$

The above equation is identically zero [34] for all $r < M$. Thus we prove composite fermions are indeed the ground state of the Hamiltonian given by Eq. (3.94).

Appendix C

Entangled Pauli Principle for non-abelian states

C.1 Second quantization in disk geometry for n Landau level projected two-body Hamiltonian

We will present a general method to project H_{TK} onto the lowest N_L Landau levels, specializing to the $N_L = 3$ case.

Since we want to project a two-body Hamiltonian, we construct an appropriate two-fermion basis

$$\Phi_{JI} \;=\; G^{n_1,n_2}_{(-1)^{m+1}}\Phi^{m}_{0[J+(n_1+n_2)/2]}, \tag{C.1}$$

where, Φ_{0J}^{m} is a lowest Landau level state of two particles with relative angular momentum m and total angular momentum $2J$,

$$\Phi_{0J}^{m} = \frac{2^{-J}}{\sqrt{(2J-m)!\,m!}}\,(b_1^\dagger + b_2^\dagger)^{2J-m}(b_1^\dagger - b_2^\dagger)^m\Phi_0, \tag{C.2}$$

elevated to higher Landau levels by the operator $G_{\pm}^{n_1,n_2}$ with $0 \leq n_i \leq N_L - 1$ and,

$$G_{\pm}^{n_1,n_2} \;=\; \frac{1}{\sqrt{n_1!n_2!\,2(1+\delta_{n_1,n_2})}}(a_1^{\dagger n_1}a_2^{\dagger n_2} \pm a_1^{\dagger n_2}a_2^{\dagger n_1}). \tag{C.3}$$

I encodes a multi-index consisting of the quantum numbers n_1, n_2, and m as per Table C.1, and Φ_0 is the two-particle vacuum of the ladder operators $a_{1,2}$, $b_{1,2}$ associated to dynamical momenta and guiding centers, respectively, which can be defined in symmetric gauge as

$$a_i = \frac{1}{\sqrt{2}}(\frac{z_i}{2\ell} + 2\ell\partial_{\bar{z}_i})\;,\; a_i^\dagger = \frac{1}{\sqrt{2}}(\frac{\bar{z}_i}{2\ell} - 2\ell\partial_{z_i}), \tag{C.4}$$

$$b_i = \frac{1}{\sqrt{2}}(\frac{\bar{z}_i}{2\ell} + 2\ell\partial_{z_i})\;,\; b_i^\dagger = \frac{1}{\sqrt{2}}(\frac{z_i}{2\ell} - 2\ell\partial_{\bar{z}_i}), \tag{C.5}$$

$i = 1, 2$. The latter satisfy the canonical bosonic algebra

$$[a_i, a_j^\dagger] = \delta_{ij} = [b_i, b_j^\dagger]\;,\; [a_i, b_j] = [a_i^\dagger, b_j^\dagger] = 0. \tag{C.6}$$

Note that Eq. (C.1) is even (odd) in n_1, n_2 for m odd (even), thereby always producing a state that's odd under the exchange of particle coordinates.

We are interested in establishing the Fock-space representation of H_{TK} projected onto the subspace of the three lowest Landau levels, $0 \leq n_i \leq 2$, generated by the basis Φ_{JI}. Note

Table C.1: Triplets (n_1, n_2, m) for any given state Φ_I with $I = 1, 2, ..., 18$.

I	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
n_1	0	0	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2
n_2	0	1	0	0	1	1	0	0	0	0	1	1	1	1	1	2	2	2
m	1	0	2	1	1	3	1	3	0	2	1	3	0	2	4	1	3	5

that the latter are orthogonal by construction. It further turns out that H_{TK} annihilates all states with $m > n_1 + n_2 + 1$. For any fixed J, its nonzero eigenvalues and eigenstates can therefore be obtained by diagonalizing H_{TK} within the subspace defined by the 18 I-indices listed in Table C.1. Moreover, the relevant matrix elements can be shown to be *independent* of J. We will thus omit the J-index from now on when no confusion may arise. The 18-dimensional subspace defined in Table C.1 contains all positive eigenvalue eigenstates for two particles at given J. Straightforward but tedious diagonalization yields that there are only eight such states (with all orthogonal states, even within this subspace, having zero energy), as listed in Table C.2. We formally write these eigenstates as

$$\Psi_\lambda = \sum_I \alpha_I^\lambda \Phi_I, \tag{C.7}$$

with λ an index associated to the eight positive eigenvalues E_λ and coefficients α_I^λ made explicit in Table C.2. Passing to a second quantized language is now easy. We write the two-particle states (C.1) as $|\Phi_{JI}\rangle = T_{JI}^\dagger |0\rangle$, with $|0\rangle$ the vacuum of the Fock space. The two-particle creation operators $T_{JI}^\dagger$ can be written as [18]

$$T_{JI}^\dagger = \frac{1}{\sqrt{2(1+\delta_{n_1,n_2})}} \sum_k \eta_{k+\frac{n_2-n_1}{2}}(J + \frac{n_1+n_2}{2}, m_I) \; c_{n_1,J-k}^\dagger c_{n_2,J+k}^\dagger, \tag{C.8}$$

[18] M. T. Ahari, S. Bandyopadhyay, Alexander Seidel, Zohar Nussinov and Gerardo Ortiz, manuscript under preparation

where $-J - n_2 \leq k \leq J + n_1$ for the infinite plane[19]. In Eq. (C.8), $c^\dagger_{n,x}$ creates an electron in LL n with angular momentum x, and the form factor $\eta_k(J,m)$ is the identical to the one already appearing in the lowest Landau level case [10],

$$\begin{aligned}\eta_p(J,m) &= 2^{-J+1/2}\sqrt{\frac{(J-p)!\,(J+p)!}{(2J-m)!\,m!}}(-1)^{m+J-p} \\ &\times \sum_{r=0}^{J-p}(-1)^r \binom{2J-m}{r}\binom{m}{J-p-r}.\end{aligned} \tag{C.9}$$

One can write the states of Eq. (C.7) as

$$|\lambda\rangle_J = \sum_I \alpha_I^\lambda\, T^\dagger_{JI}\,|0\rangle = \mathcal{T}_J^{(\lambda)\dagger}\,|0\rangle\,, \tag{C.10}$$

where

$$\begin{aligned}\mathcal{T}_J^{(\lambda)} &= \sum_{I,k}\frac{\alpha_I^\lambda}{\sqrt{2(1+\delta_{n_1,n_2})}}\eta_{k+\frac{n_2-n_1}{2}}(J+\frac{n_2+n_1}{2},m_I) \\ &\quad c_{n_2,J+k}c_{n_1,J-k} \\ &:= \sum_{k,n_1,n_2}\eta^\lambda_{J,k,n_1,n_2}\,c_{n_2,J+k}c_{n_1,J-k}\end{aligned} \tag{C.11}$$

and we have made contact with the η^λ-symbols defined in the main text, letting

$$\begin{aligned}\eta^\lambda_{J+\frac{n_2+n_1}{2},k,n_1,n_2} = &\sum_I \frac{\alpha_I^\lambda}{\sqrt{2(1+\delta_{n_1,n_2})}} \\ &\eta_{k+\frac{n_2-n_1}{2}}(J+\frac{n_2+n_1}{2},m_I).\end{aligned} \tag{C.12}$$

[19] For a finite size disk with L available states, the last inequality must be replaced with $-\min(J, L-1-J) \leq k \leq \min(J, L-1-J)$.

Now the two-particle Hamiltonian can now be written manifestly in terms of its spectral decomposition,

$$H_{\mathsf{TK}} = \sum_{J} \sum_{\lambda=1}^{8} E_{\lambda\ J} |\lambda\rangle\langle\lambda|_J, \tag{C.13}$$

and generalized, as usual, to a many-body Hamiltonian by dropping the projection $|0\rangle\langle 0|$ onto the vacuum that, upon use of Eq. (C.10), would otherwise follow the action of $\mathcal{T}_J^{(\lambda)}$:

$$\hat{H}_{\mathsf{TK}} = \sum_{J} \sum_{\lambda=1}^{8} E_\lambda\, \mathcal{T}_J^{(\lambda)\dagger} \mathcal{T}_J^{(\lambda)}. \tag{C.14}$$

Table C.2: Eigenvalues and Eigenvectors of the projected H_{TK}. Overall normalization factors in the column to the right are omitted. They are straightforward but tedious to calculate, and are not needed throughout the paper.

$4\pi E_\lambda$	Eigenvectors
$\frac{87}{8}$	$\Psi_1 = 2\sqrt{\frac{2}{15}}\Phi_1 - \frac{4}{\sqrt{15}}\Phi_3 - \frac{2}{\sqrt{5}}\Phi_6 + \frac{2}{\sqrt{5}}\Phi_8 + 2\sqrt{\frac{2}{5}}\Phi_{15} + \Phi_{18}$
$\frac{3}{4}(6+\sqrt{17})$	$\Psi_2 = -\frac{\sqrt{6}(363+89\sqrt{17})}{3(469+113\sqrt{17})}\Phi_2 - \sqrt{\frac{2}{3}}\Phi_5 + \frac{\sqrt{6}\left(7\sqrt{17}+23\right)}{33\sqrt{17}+141}\Phi_7 + \frac{31\sqrt{17}+129}{\sqrt{6}\left(11\sqrt{17}+47\right)}\Phi_{14} + \Phi_{17}$
$\frac{15}{4}$	$\Psi_3 = -\frac{2}{\sqrt{3}}\Phi_4 + 2\sqrt{\frac{2}{3}}\Phi_{10} + \Phi_{12}$
$\frac{1}{8}(9+\sqrt{33})$	$\Psi_4 = \Phi_9 + \frac{1+\sqrt{33}}{4\sqrt{2}}\Phi_{11}$
$\frac{3}{4}(6-\sqrt{17})$	$\Psi_5 = \frac{\sqrt{6}\left(89\sqrt{17}-363\right)}{3\left(469-113\sqrt{17}\right)}\Phi_2 - \sqrt{\frac{2}{3}}\Phi_5 + \frac{\sqrt{6}\left(7\sqrt{17}-23\right)}{33\sqrt{17}-141}\Phi_7 + \frac{31\sqrt{17}-129}{\sqrt{6}\left(11\sqrt{17}-47\right)}\Phi_{14} + \Phi_{17}$
$\frac{1}{16}(9+\sqrt{57})$	$\Psi_6 = \Phi_{13} + \frac{5+\sqrt{57}}{4\sqrt{2}}\Phi_{16}$
$\frac{1}{8}(9-\sqrt{33})$	$\Psi_7 = \Phi_9 + \frac{1-\sqrt{33}}{4\sqrt{2}}\Phi_{11}$
$\frac{1}{16}(9-\sqrt{57})$	$\Psi_8 = \Phi_{13} + \frac{5-\sqrt{57}}{4\sqrt{2}}\Phi_{16}$

C.2 Construction of EPP from microscopic Hamiltonian

In this section, we provide some additional details for the derivation of the EPP from the second-quantized zero mode condition associated with the microscopic Hamiltonian derived in the preceding section. We reproduce this zero mode condition here as

$$\mathcal{T}_J^{(\lambda)} |\psi\rangle = 0 \quad \forall J, \lambda. \tag{C.15}$$

Note that equivalent reformulations of these conditions can be given in terms of arbitrary new (linearly independent) linear combinations of the $\mathcal{T}_J^{(\lambda)}$. From Table C.2, is easy to see that the T_{JI} with $I = 9, 11, 13, 16$ must all individually annihilate any zero mode. Moreover, from $\mathcal{T}_J^{(2)}$ and $\mathcal{T}_J^{(5)}$, we may make new linear combinations

$$\begin{aligned} \tilde{\mathcal{T}}_J^{(2)} &= T_{J,2} - 2T_{J,7} - \frac{1}{2}T_{J,14}, \\ \tilde{\mathcal{T}}_J^{(5)} &= T_{J,5} + T_{J,7} - T_{J,14} - \sqrt{\frac{3}{2}}T_{J,17}, \end{aligned} \tag{C.16}$$

so that we may rephrase the zero mode condition for a ket $|\psi\rangle$ equivalently by saying that $|\psi\rangle$ is annihilated by each of the eight operators in the set

$$Z_J = \{\mathcal{T}_J^{(1)}, \tilde{\mathcal{T}}_J^{(2)}, \mathcal{T}_J^{(3)}, \tilde{\mathcal{T}}_J^{(5)}, T_{J,9}, T_{J,11}, T_{J,13}, T_{J,16}\}$$

, for all J. This considerably simplifies the resulting equations.

We first turn to Eq. (5) of the main text, which we rephrase here for the operators in the set Z_J:

$$\sum_{n_1,n_2} \eta^{(\tilde{\lambda})}_{J,0,n_1,n_2} \alpha_{n_1,n_2} = 0 \ \ (\lambda = 1 \ldots 8)\,. \tag{C.17}$$

where $\tilde{\lambda}$ now indexes the members of the set Z_J, and $\eta^{(\tilde{\lambda})}$ is the associated form factor. The goal is to show that these have only trivial solutions. Since there are only three independent variables $\alpha_{n_1,n_2} = -\alpha_{n_2,n_1}$, it is sufficient to focus on three members of Z_J. The $T_{J,9}$-equation in (C.17) then readily implies $\alpha_{02} = 0$ (cf. Table C.1), and the $T_{J,13}$-equation implies $\alpha_{12} = 0$. Finally, consider the $\mathcal{T}_J^{(2)}$-equation. Since $\eta_0(J,m) = 0$ in Eq. (C.8) for m odd, the only contributions to this equation can come from $T_{J,2}$ and $T_{J,14}$ (Tables C.1 and C.2). However, that of $T_{J,14}$ also vanishes, since $\alpha_{12} = 0$ is already known. This gives $\alpha_{01} = 0$.

We may likewise put Eq. (6) of the main text into a form that references the form factors associated to the operator set Z_J:

$$\sum_{n_1,n_2} \eta^{(\tilde{\lambda})}_{J,1/2,n_1,n_2} \beta_{n_1,n_2} = 0 \quad (\lambda = 1 \dots 8)\,. \tag{C.18}$$

The resulting eight linear equations have the following solution, unique up to a scale factor:

$$\begin{aligned} \beta_{22} = \beta_{12} = \beta_{21} = \beta_{10} = 0, \beta_{20} = 1, \beta_{11} = -\sqrt{2}, \\ \beta_{01} = \frac{\sqrt{8}}{\sqrt{J+1}}, \beta_{02} = \frac{\sqrt{J+3}}{\sqrt{J+1}}, \beta_{00} = \frac{\sqrt{2(J+2)}}{\sqrt{J+1}}. \end{aligned} \tag{C.19}$$

At root level, as explained in the main text, this uniquely fixes any nearest neighbor occupied orbitals to be in a certain entangled state. Upon the local change of basis detailed in the next section, we can understand this state as a "singlet" formed by two spin-1 degrees of freedom. In the dominance patterns that we use to encode root states, this two-orbital entangled state is simply represented as $\dots 11 \dots$.

Last, we also consider the situation of occupied next-nearest neighbor orbitals in some more detail. As in the main text, consider a zero mode of the form

$$|\psi\rangle = \sum_{n_1,n_2} \gamma_{n_1 n_2} c^\dagger_{n_1,J-1} c^\dagger_{n_2,J+1} |\tilde{S}\rangle + \alpha_{n_1 n_2} c^\dagger_{n_1,J} c^\dagger_{n_2,J} |\tilde{S}\rangle + |\mathsf{rest}\rangle\,, \tag{C.20}$$

where $|\tilde{S}\rangle$ is an $N-2$ particle Slater-determinant that has all orbitals with angular momenta J, $J\pm 1$ vacant, $|\mathsf{rest}\rangle$ is orthogonal to the first two terms, and the first term is non-expandable. The condition

$$0 = \langle\psi|{\mathcal{T}^{(\tilde{\lambda})}_J}^\dagger|\tilde{S}\rangle = \sum_{x,n_1,n_2} (\eta^{(\tilde{\lambda})}_{J,x,n_1,n_2})^* \langle\psi| c^\dagger_{n_2,j+x} c^\dagger_{n_1,j-x} |\tilde{S}\rangle \tag{C.21}$$

then leads to the conditions

$$\sum_{n_1,n_2} \left(\eta^{(\tilde{\lambda})}_{J,1,m_1,m_2}\gamma_{n_1,n_2} + \eta^{(\tilde{\lambda})}_{J,0,m_1,m_2}\alpha_{n_1,n_2}\right) = 0\,, \tag{C.22}$$

where again only the $x=0$ and $x=1$ terms can contribute, as the presence of any other terms would imply that the γ_{n_1,n_2}-terms could be obtained via inward squeezing, contrary to assumption. From these eight equations, the three variables $\alpha_{n_1,n_2} = -\alpha_{n_2,n_1}$ may be eliminated, leaving five equations for the coefficients γ_{n_1,n_2} that constrain the entanglement of second-nearest neighbor occupied orbitals at root level:

$$\begin{aligned}
&\gamma_{22} = 0, \quad \gamma_{21} + \sqrt{\frac{2+J}{4+J}}\gamma_{12} = 0,\\
&\gamma_{00} - \frac{2}{\sqrt{3+J}}\gamma_{01} + \sqrt{\frac{1+J}{3+J}}\gamma_{11} + \sqrt{\frac{18}{(3+J)(4+J)}}\gamma_{02} - \sqrt{\frac{8(1+J)}{(3+J)(4+J)}}\gamma_{12} = 0\,,\\
&\sqrt{3+J}\gamma_{10} + \sqrt{1+J}\gamma_{01} - \sqrt{\frac{2}{(4+J)}}\gamma_{12} - \sqrt{\frac{8(1+J)}{(4+J)}}\gamma_{02} = 0,\\
&\gamma_{20} + \sqrt{\frac{2+J}{3+J}}\left(\gamma_{11} - \frac{2}{\sqrt{4+J}}\gamma_{12} - \sqrt{\frac{1+J}{4+J}}\gamma_{02}\right) = 0\,.
\end{aligned} \tag{C.23}$$

The intuitive meaning of these equations will again become clearer in the following section. There are four solutions to these five equations in nine variables, which we formally label as $1_\uparrow 01_\uparrow$, $1_\uparrow 01_\downarrow$, $1_\downarrow 01_\uparrow$, and $1_\downarrow 01_\downarrow$. A dominance pattern containing one of these strings $\ldots 001_{\sigma_1} 01_{\sigma_2} 00 \ldots$ corresponds to a root state where the two orbitals indicated by the 1's in the pattern are in a pure entangled state corresponding to one of the four solutions. Conversely, in any root state of a zero mode, the state of any two next-nearest-neighbor occupied orbitals must always be in the four-dimensional subspace defined by these four solutions. As long as no member of the pair has any other nearest or next-nearest neighbor orbitals occupied, there are no further constraints affecting the pair. However, if one member had a nearest neighbor occupied, as in the string 1011, equations (C.23) constraint the first pair, while equations (C.19) constrain the second. There are no solutions to the combined set of equations, thus there are no dominance patterns of the 1011 kind. Similarly, the string 111 can be ruled out, and a 11 configuration must thus always be separated by 00 on either side from all the other orbitals, in any legitimate dominance pattern. The only remaining case of interest is that of consecutive strings of next nearest neighbors. In such strings, Eqs. (C.23) must be applied to each next-nearest-neighbor pair. We will see in the next section that the resulting equations, applied to any string of consecutively occupied next-nearest-neighbor orbitals separated by terminal 00 units from all other orbitals, still result in four solutions. We will show this below by showing that solutions have an MPS-structure that's of a kind with ground states in the AKLT model. The resulting dominance patterns are thus again naturally labeled by strings $\ldots 001_{\sigma_1} 0101 \ldots 10101_{\sigma_2} 00 \ldots$, where only the terminal 1s carry a spin-1/2 index labeling a boundary degree of freedom. In all, we have shown that states appearing at root level for any zero mode can be decomposed into mutually non-entangled units of the following kinds: 1. Nearest neighbor pairs 11 governed by Eqs. (C.19), 2. next-nearest

neighbor strings $1_{\sigma_1}01\ldots101_{\sigma_2}$, and 3. isolated occupied sites 1_{s_z}, where s_z may be interpreted either as a label for the three Landau levels or, alternatively, a spin-1 label to be discussed in the following section. All these units must be separated by at least two unoccupied sites from one another. Special consideration must be given to the orbitals with negative angular momenta $j = -1$ and $j = -2$. Carrying out the above analysis with the special constraint in mind that there is only one such orbital for $j = -2$ and two such orbitals for $j = -1$, one obtains the boundary condition that at root level, apart from being unoccupied, the $j = -2$ orbital may only occupy isolated $1_{s_z=\mathsf{max}}$ unit. Again, the latter must again be separated by at least two zeros from all other units. Similarly, the $j = -1$ orbital may only be in a 1_{s_z} state, with s_z assuming the top two values, or may be the left end of a $1_{\sigma_L}0101\ldots$ pattern with σ_L fixed to $\uparrow$. This completes the set of rules that all dominance patterns and their associated root states are subject to.

We emphasize that thus far, the above rules represent necessary conditions on root states. Below we establish that to each permissible dominance pattern, there is precisely one zero mode that has the associated root state. Since zero modes form a linear space, the root state of a generic zero mode may, of course, as well be a superposition of root states associated with the dominance patterns characterized above.

C.3 Emergent SU(2)-Symmetry

We now discuss an emergent SU(2)-symmetry within the zero mode sector that also sheds the entangled Pauli principle discussed in the proceeding section in a simpler light. To this end, we temporarily limit the discussion to the Fock space $\mathcal{F}_+$ associated to orbitals of angular momentum index $j \geq 0$ (and, as before, LL index $0 \leq n \leq 2$). We consider the following single particle operators acting within this space, which we define in first

quantization through their action on the *polynomial part* of the wave function via

$$\begin{aligned} S_z &= \sum_i (\bar{z}_i \partial_{\bar{z}_i} - 1), \\ S_- &= \sum_i \frac{1}{z_i} \partial_{\bar{z}_i}, \quad S_+ = \sum_i z_i \bar{z}_i (2 - \bar{z}_i \partial_{\bar{z}_i}). \end{aligned} \tag{C.24}$$

If the action on full wave functions, including Gaussian factors, is desired, a simple shift $\partial_{\bar{z}_i} \to \partial_{\bar{z}_i} + \frac{1}{4} z_i$ may be performed. In this section, we will omit Gaussian factors for simplicity.

One checks without difficulty that the operators (C.24) satisfy the su(2)-algebra $[S_+, S_-] = 2S_z$, $[S_z, S_\pm] = \pm S_\pm$, albeit without having the properties under Hermitian conjugation that are usually taken for granted in physics. This is irrelevant to the representation theory of this algebra, and in any case the representation within $\mathcal{F}_+$ can be unitarized by using the following single particle basis:

$$z^j, \quad \sqrt{2} z^{j+1} \bar{z}, \quad z^{j+2} \bar{z}^2 . \tag{C.25}$$

In this basis, it is manifest that each angular momentum $j \geq 0$ is associated to a triplet of LL orbitals that transforms under the spin-1 representation of the operators (C.24). The usual Landau level basis is obtained by applying to the above, written as a column vector, the matrix

$$V = \frac{1}{\sqrt{2\pi 2^j j!}} \begin{pmatrix} 1 & 0 & 0 \\ -\sqrt{j+1} & \frac{1}{2\sqrt{j+1}} & 0 \\ \frac{\sqrt{(j+1)(j+2)}}{\sqrt{2}} & -\frac{\sqrt{j+2}}{\sqrt{2}\sqrt{(j+1)}} & \frac{1}{4\sqrt{2}\sqrt{(j+1)(j+2)}} \end{pmatrix}, \tag{C.26}$$

whose matrix elements v_{m,s_z} are referenced in the main text to define operators $d^\dagger_{s_z,j}$. The latter just create the single particle states (C.25). From Eq. (C.25) it is also clear that the space $\mathcal{F}_+$ is invariant under the action of the generators (C.24). If we define $\mathcal{F}_+^0$ as the subspace of zero modes

that are contained in $\mathcal{F}_+$, we want to show next that $\mathcal{F}_+^0$ is *also* invariant under the action of the generators. These operators thus generate an *emergent* (since the Hamiltonian is not invariant) symmetry within the zero mode subspace $\mathcal{F}_+^0$.

It is sufficient to analyze this question for two-body wave functions. Take S_- and act on a two body wave function in $\mathcal{F}_+^0$, which we express as a polynomial $\psi(Z, \bar{Z}, z, \bar{z})$ in the center-of-mass and relative coordinates $Z = \frac{1}{2}(z_1 + z_2)$, $z = z_1 - z_2$ and their complex conjugates. Being a zero mode, ψ has a third order zero in z, $\bar{z}$ for any Z, $\bar{Z}$. Moreover, since S_- certainly preserves analyticity for $|z| < 2|Z|$, $|\bar{z}| < 2|\bar{Z}|$, and contains only single derivatives, $S_-\psi$ must still have at least a second order zero in z, $\bar{z}$ for any $Z, \bar{Z} \neq 0$. As S_- also preserves oddness under $z \to -z$, $\bar{z} \to -\bar{z}$, $S_-\psi$ must in fact *still* have a third order zero in z, $\bar{z}$ for any $Z, \bar{Z} \neq 0$. On the other hand, since $\psi \in \mathcal{F}_+$, $S_-\psi$ is still in $\mathcal{F}_+$, and is still analytic everywhere (in fact polynomial). If in its expansion

$$S_-\psi = \sum_{m,n\geq 0} z^n \bar{z}^m g_{mn}(Z, \bar{Z}) \tag{C.27}$$

there is any non-zero term with $n + m < 3$, then $g_{m,n}(Z, \bar{Z})$ is a polynomial of non-zero degree and must be finite at some $Z, \bar{Z} \neq 0$. At such $Z, \bar{Z}$, $S_-\psi$ would then *not* have a third order zero in z, $\bar{z}$, contradicting the foregoing. Therefore, all g_{mn} with $m + n < 3$ vanish, and $S_-\psi$ is in $\mathcal{F}_+^0$. The cases S_z and S_+ can be treated similarly (and without paying special attention to $Z, \bar{Z} = 0$). $\mathcal{F}_+^0$ is thus invariant under the generators (C.24).

We emphasize that the notion of an emergent SU(2) symmetry is not an artifact of the restriction to $\mathcal{F}_+$. Note that any zero mode of well-defined total angular momentum (thus finite spatial extent) will, up to exponentially small terms, lie in $\mathcal{F}_+^0$ after a sufficiently large spatial translation T. Action with the modified generators $\tilde{S}_i = T^\dagger S_i T$ will preserve the zero mode property, up to terms that can be made exponentially small. Note that the $\tilde{S}_i$ are still *local* operators (though no longer angular momentum preserving). Related to that, the construction of the generators (C.24) naturally extends to the cylinder geometry. There, the singularity at $z_i = 0$ (for the disk geometry) is automatically pushed to infinity.

The single particle orbitals (C.25) may be extended to $j \geq -2$, with the additional constraint that orbitals with negative exponents are to be discarded. The resulting set of orbitals is then a non-orthogonal basis of the three lowest landau levels (associated to the $d^\dagger_{s_z,j}$ operators of the main text, where $s_z \geq \max(-1, -1-j)$). It is natural to analyze the conditions (C.19) and (C.23) in this basis. It is straightforward to show that Eq. (3.102) precisely expresses that any 11 factor of a root state must be a singlet under the su(2) algebra (C.24). Moreover, Eq. (C.23) mandates that any neighboring particles in a $\ldots 101 \ldots$ factors must have total spin 0 or spin 1 (i.e., after introduction of an inner product for which the orbitals (C.25) are orthonormal, any 101 in a root state must be orthogonal to spin 2). This is precisely the zero mode condition of the famous AKLT-model [117]. The claims about the MPS-structure and number of solutions to the constraints (C.23) made above and in the main text are immediate consequences of this observation.

C.4 Construction of Ground states and quasiholes from parton structures

We emphasize that the results of the preceding two sections only impose necessary conditions on the existence of zero modes of the Hamiltonian (1) of the main text: A priori, the existence of a pattern composed of the units and according to the rules established in the foregoing does not guarantee the existence of a zero mode whose root state is described by this pattern. Together with a construction principle for such zero modes, however, the EPP governing root states has far reaching consequences. In particular, if for every allowed dominance pattern a zero mode can be constructed whose root state precisely corresponds to this pattern, it follows that the wave functions so constructed are a *complete* set of zero modes. This has been established by some of us earlier [10] and generalizes effortlessly to the present, multi-Landau-level context [39]. We will apply this reasoning now to the case at hand. Consider thus wave functions of the form (2) of the

main text, or

$$\psi = \prod_{i<j} (z_i - z_j)\, D_1\, D_2\,, \tag{C.28}$$

where D_1 and D_2 can be taken to be Slater-determinants consisting only of the *first two* types of orbitals in Eq. (C.25). In this section, we again find it advantageous to work with the single particle basis (C.25), and omit all Gaussian factors. If now we take $D_1 = D_2$ equal to the "densest" Slater-determinant $\mathcal{D}$, where for some N, all orbitals admissible orbitals z^j ($j \geq 0$) and $\bar{z} z^{j+1}$ ($j \geq -1$) with $j \leq (N-1)/2$ are occupied then the resulting zero mode is dominated by the root state with the densest dominance pattern, i.e. $1_2 001100110011\ldots$; here the subscript 2 indicates that the leading particle resides in the second excited Landau level (as it must, having $j = -2$). The pattern is "densest" in the sense that for a given N, no pattern of smaller total angular momentum is possible, nor any pattern whose largest occupied orbital has smaller (single particle) angular momentum. It immediately follows that the zero mode with $D_1 = D_2 = \mathcal{D}$ *is* the densest zero mode as conjectured earlier in [94], for any odd N. For, any zero mode of the same N but smaller total angular momentum or smaller highest occupied orbital would necessarily have a root states with the same properties, and this root state could then not satisfy the EPP.

This reasoning can be extended to show that the zero modes (C.28) form a(n) (over-)complete set of zero modes. In algebraic terms, this proves the quite non-trivial theorem that the set of all polynomials in $z_i, \bar{z}_i$, with the requisite anti-symmetry, at most second order in any $\bar{z}_i$, *and* having at least third order zeros as $z_i \to z_j$, $\bar{z}_i \to \bar{z}_j$ is already linearly generated by the states of the form (C.28), i.e., Jastrow-factor times a product of two Slater determinants in z_i, $\bar{z}_i$, *each* at most linear in any $\bar{z}_i$. Clearly, this statement has useful generalizations to other parton states involving higher Landau levels and similarly constructed parent Hamiltonians, which we will leave for future work.

The detailed argument proceeds as follows. Below we construct for every dominance pattern d allowed by the EPP a state ψ_d of the form (C.28) such that the root state of ψ_d is precisely $|d\rangle$, i.e., the root state associated with the pattern d. The construction is such that $\langle d'|\psi_d\rangle$ may be non-zero for

some $d' \neq d$, however, the matrix $\langle d'|\psi_d\rangle$ will have a triangular structure with non-zero diagonal, and thus be invertible. This ensures the completeness of the ψ_d. For one, it trivially implies the linear independence of the ψ_d. What's more, to any zero mode $|\phi\rangle$ we may then construct a linear combination $|\tilde{\phi}\rangle$ of the ψ_d such that $\langle d|\tilde{\phi}\rangle = \langle d|\phi\rangle$ for all root states $|d\rangle$ allowed by the EPP. This means that $|\tilde{\phi}\rangle - |\phi\rangle$ is a zero mode that's orthogonal to all permissible root states. This is only possible if $|\phi\rangle = |\tilde{\phi}\rangle$. Thus $|\phi\rangle$ is already a linear combination of the ψ_d.

We proceed with the construction of ψ_d. We introduce the short hand notation $(\ell)_i{}^n = z_i^{\ell+n}\bar{z}_i^n$ for the monomials (C.25), not including the normalization, which is inessential for present purposes. We will use the notation

$$\{(\ell_1)_1{}^{n_1}\dots(\ell_N)_N{}^{n_N}\} \tag{C.29}$$

for anti-symmetrized products of these monomials, where we will always insist that $\ell_i \leq \ell_{i+1}$. The D_1, D_2 in (C.28) are of this form, with the additional constraint that $n_i \leq 1$. There is a simple rule describing "dominance" for a product of two Slater-determinants of this form, first stated for the lowest LL case [72] ($n_i = 0$), but easily generalized to $n_i \geq 0$ [39]. This is that in the expansion of the product of $\{(\ell_1)_1{}^{n_1}\dots(\ell_N)_N{}^{n_N}\}$ and $\{(\ell'_1)_1{}^{n'_1}\dots(\ell'_N)_N{}^{n'_N}\}$ into Slater determinants, there is a non-expandable Slater determinant of the form $\{(\ell_1+\ell'_1)_1{}^{n_1+n'_1}\dots(\ell_N+\ell'_N)_N{}^{n_N+n'_N}\}$. The key novel feature for multiple LLs is that while the rule $\ell_i \leq \ell_{i+1}$, $\ell'_i \leq \ell'_{i+1}$ fixes the angular momenta $\ell_i + \ell'_i$ of "dominant" (non-expandable) Slater determinants in the product, in the case of multiple degenerate ℓ_i, the order of the associated n_i is arbitrary. The dominance-rule can be applied to any such ordering, leading to all the different non-expandable Slater determinants in the product, all of which have the same angular momentum quantum numbers or occupied lattice positions, but differ in the LL-related indices $n_i + n'_i$. This phenomenon precisely leads to the root state entanglement we know to be required, in general, of zero modes! The rule can be straightforwardly generalized to products of three Slater determinants. Note that one may write the Jastrow-factor in (C.28) as $\mathcal{J} = \{(0_1)_0(1_2)_0\dots((N-1)_N)_0\}$, making this rule straightforwardly applicable to Eq. (C.28). Table C.3 shows how any of the three building blocks of the EPP can be mapped onto units in D_1 and D_2 such that the root state of $\mathcal{J}D_1D_2$ will contain this building block at the right

position. It is worth considering the $1_{\sigma_L}01\ldots1_{\sigma_R}$ block. The product rule described above when applied to $\mathcal{J}D_1D_2$ as given in the table readily implies that the resulting orbital pattern at root level, without regard to LL-indices, is $101\ldots1$. One may now argue that the rule of Table C.3 results in the AKLT-type MPS structure described in the proceeding sections in two slightly different ways. One may check directly that the permissible permutations of the n, n'-indices described above reproduce the advertized MPS structure. Alternatively, it is sufficient to point out that, all other parts of D_1 and D_2 staying the same, the rule of Table C.3 results in four linearly independent zero modes with the $101\ldots1$ orbital pattern at root level. By the necessary criteria of the proceeding two sections, the entanglement structure at root level *must* then be consistent with the four AKLT-MPS states (or linearly independent linear combinations thereof).

We have thus constructed a set of zero modes $\{\psi_d\}$ of the form (C.28), where for any dominance pattern d conforming to the EPP, ψ_d is dominated by the root state $|d\rangle$ associated to d. To establish the completness property of these zero modes, as explained above, we need only consider the matrix $\langle d'|\psi_d\rangle$. We follow the argument of [39]. Diagonal elements are non-zero by construction. Moreover, for $\langle d'|\psi_d\rangle$ to be non-zero for some $d' \neq d$, d' must be obtainable from d by the inward-squeezing processes defined in the main text. Such processes always strictly decrease the value of the "moment"

$$M = \sum_j \sum_n j^2 c^\dagger_{n,j} c_{n,j}\,, \tag{C.30}$$

of which all $|d\rangle$ are eigenstates. Thus, if we order the $|d\rangle$ according to increasing M, the matrix $\langle d'|\psi_d\rangle$ is upper triangular, hence invertible. This completes the proof of the one-to-one correspondence between zero modes and dominance patterns.

Table C.3: Rules for distributing the building blocks of the EPP over corresponding units in D_1 and D_2, Eq. (C.28). The leading particle in the EPP block is assumed to be the ith particle and occupying the orbital with angular momentum j. A "free" 1_{s_z}-block leads to two singly occupied orbital in both D_1 and D_2 (see text for notation). A 11-block leads to double occupied orbitals in both D_1 and D_2. In a 101-block of k-particles, third row, there is no real freedom in choosing most of the $n_{1\ldots k},\ n'_{1\ldots k} \in \{0,1\}$, as most orbitals will be doubly occupied, for both D_1 and D_2. However, for both k even and odd, among D_1 and D_2 there will be exactly one singly occupied orbital at the left, and exactly one singly occupied orbital at the right. I.e., among the n_1, n'_1, exactly one is free, say n_1, and may be identified with σ_L via $n_1 = \sigma_L + \frac{1}{2}$. The analogous statement holds for n_k, n'_k and σ_R. Observing that adjacent EPP-blocks in the same pattern are padded from one another by double zeros, it is easy to see the corresponding units in the Slater-determinants do not overlap in orbital space, both for D_1 and for D_2, respectively.

EPP building block	D_1	D_2
$\ldots 1_{s_z} \ldots$	$\{\ldots (\lfloor \frac{j-i+1}{2} \rfloor)_i^{\lfloor \frac{s_z+1}{2} \rfloor} \ldots\}$	$\{\ldots (\lceil \frac{j-i+1}{2} \rceil)_i^{\lceil \frac{s_z+1}{2} \rceil} \ldots\}$
$\ldots 11 \ldots$	$\{\ldots (\lfloor \frac{j-i+1}{2} \rfloor)_i^0 (\lfloor \frac{j-i+1}{2} \rfloor)_{i+1}^1 \cdots\}$	$\{\ldots (\lceil \frac{j-i+1}{2} \rceil)_i^0 (\lceil \frac{j-i+1}{2} \rceil)_{i+1}^1 \cdots\}$
$\ldots 1_{\sigma_L} 01 \ldots 01_{\sigma_R} \ldots$	$\{\ldots (\lfloor \frac{j-i+1}{2} \rfloor)_i^{n_1} (\lfloor \frac{j-i+2}{2} \rfloor)_{i+1}^{n_2} \cdots (\lfloor \frac{j-i+k}{2} \rfloor)_{i+k}^{n_k} \cdots\}$	$\{\ldots (\lceil \frac{j-i+1}{2} \rceil)_i^{n'_1} (\lceil \frac{j-i+2}{2} \rceil)_{i+1}^{n'_2} \cdots (\lceil \frac{j-i+k}{2} \rceil)_{i+1}^{n'_k} \cdots\}$

C.5 Braiding Statistics from EPP

One can demonstrate [46, 47] that the dominance patterns as defined here agree with the thin cylinder limiting form of analytic trial wave functions, and are of course likewise expected to agree with thin torus limits, as demonstrated in many cases (e.g. [18,19]). Moreover, it is generally found that the thin torus limit of zero modes of parent Hamiltonians (such as Eq. (1) of the main text) is adiabatically connected to zero modes of a "thick" (therefore, two-dimensional) torus. This adiabatic continuity can be exploited via the "coherent state method" to extract braiding statistics from microscopic rules governing dominance patterns [63], here the EPP. We will present some key steps of this method as applied to the present case, and leave details, regarding statistics and general torus wave functions, for future publications.

As argued in the main text, the topological information ingrained in the EPP for the Jain-221 state is highly analogous to similar data for the $\nu = 1/2$ Moore-Read state. Hence, the task is essentially to generalize earlier discussions [63, 101] for bosons at $\nu = 1$ to fermions at $\nu = 1/2$. The heart of the method is a "topological table" as given by Table. C.4. This table illustrates how features of dominance patterns associated with states of few quasiholes (here: two) determine a coherent state Ansatz that is used to extract phases associated with two basic types of operations. Translations (T) describe transitions between different "types" or topological sectors under orbital (magnetic) translations. The rightmost column (F) describes exchange processes between quasiholes along topologically non-trivial paths. In the table, patterns are shown without the spin-1/2 degrees of freedom, which we choose identical for all domain-walls (represented as $|$ for additional clarity) associated to quasi-holes. F operations translate the first domain ($|_1$) wall to the second one ($|_2$), while the latter will be translated to the position of first domain wall around one of the "holes" of the torus.

Having identified and labeled topological sectors for two quasi-holes as in the table, we may now be interested in the braid matrix for the adiabatic exchange of two quasi-holes (these must be

Table C.4: Topological Table. η is the particle number parity. Column T shows the phase and the new sector, respectively, one gets once T is applied on the given sector to the left. Column to is analogous for F. For illustrative purposes, we note that $T|2\rangle = -(-1)^{\eta}|3\rangle$ where $|2\rangle$ denotes a coherent state in sector 2.

Sector	Domain walls	T	F
1	$1010\|_1 01100110\|_2 01010$	1,2	$1, 5+2\eta$
2	$01010\|_1 01100110\|_2 0101$	$(-1)^{1+\eta}, 3$	$1, 6+2\eta$
3	$101010\|_1 01100110\|_2 010$	1,4	$(-1)^{1+\eta}, 7-2\eta$
4	$0101010\|_1 01100110\|_2 01$	$(-1)^{1+\eta}, 1$	$(-1)^{1+\eta}, 8-2\eta$
5	$110\|_1 010101010\|_2 01100$	1,6	$(-1)^{1+\eta}, 2$
6	$0110\|_1 010101010\|_2 0110$	1,7	1,3
7	$00110\|_1 010101010\|_2 011$	$(-1)^{1+\eta}, 8$	1,4
8	$100110\|_1 010101010\|_2 01$	$(-1)^{1+\eta}, 5$	1,1

thought of as localized in space via the coherent stat Ansatz, see [63] for details). Locality imposes stringent constraints on what matrix element may in principle be non-zero. Generally, only those matrix elements can be non-zero whose associated patterns in Table C.4 differ only *in between* the domain walls, but not to the left or right of the domain walls [63]. Moreover, taking into account translational symmetry this dictates the following general structure of the braid matrix:

$$\Gamma = \begin{pmatrix} a & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & b & 0 & 0 & 0 & 0 \\ b' & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & b' & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c \end{pmatrix} \tag{C.31}$$

One may piece together the information of Table C.4 with the above structure and additional requirements from S-duality [98] on the torus, following the method of [63]. This fixes all entries

down to a set of eight possible solutions, all related by abelian phases and complex conjugation. In particular, one finds $b = \pm ia$, $b' = -b^*$, which is essentially responsible for a description in terms of Majorana fermions as mentioned in the main text. The operation of braiding on patterns with more than two quasi-holes is generated by applying the rules given for two quasi-holes to any pair of neighboring domain-walls in the associated patterns. Details will be given in a future publication.

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