Spring 5-15-2019

Limits and Singularities of Normal Functions.

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Limits and Singularities of Normal Functions

by

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A dissertation presented to

The Graduate School

of Washington University in

partial fulfillment of the

requirements for the degree

of Doctor of Philosophy

May 2019

St. Louis, Missouri

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Acknowledgments

First and foremost I would like to thank my advisor Professor Matt Kerr. Throughout five years we had constant meetings and discussions. It was really fortunate that I could receive much significant advice and share his vision of algebraic cycles. He also gave me the opportunities to join conferences.

I also would like to thank my past advisor Professor Tomohide Terasoma in Japan, who firstly suggested me to observe the interesting example of $K3$ surfaces.

Professor James Lewis and Xi Chen also gave me advice about this research at a workshop in Toronto. They help me to make a future research plan by generalizing the result in this thesis. I would appreciate them.

When I came to St.Louis, Professor Neithalath Mohan Kumar supported me to take appropriate courses and prepare for starting this research project. He also often answered for my questions about algebraic geometry. I am grateful for his advice.

Finally I would like to thank my friends and the staff of the Department of Mathematics, Washington University.

Tokio Sasaki

Wasington University, May 2019

Abstract of the Dissertation

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Tokio Sasaki

Doctor of Philosophy in Mathematics,

Washington University in St. Louis, 2019.

Professor Matt Kerr, Chair

On a projective complex variety X, constructing indecomposable higher Chow cycles is an interesting question toward the Hodge conjecture, motives, and other arithmetic applications. A standard method to determine whether a given higher cycle is indecomposable or not is to consider it as a general fiber of a degenerate family of higher cycles, and observe the asymptotic behaviors of the associated higher normal functions. In this thesis, we introduce some known examples of indecomposable cycles and a new method to detect the linearly independence of $\mathbb{R}$-regulator indecomposable K_1-cycles which is based on the singularities and limits of admissible normal functions with real coefficients. We also construct a collection of higher Chow cycles on certain surfaces in $\mathbb{P}^3$ of degree $d \geq 4$ which degenerate to an arrangement of d planes in general position. By applying our method, we show that these higher Chow cycles are enough to show the surjectivity of the real regulator map when $d = 4$. Hence our construction gives a new explicit proof of the Hodge-$\mathcal{D}$-Conjecture for a certain type of $K3$ surfaces. As an application, we also construct new examples of non trivial elements in the Griffiths groups on a certain Calabi-Yau threefold, which is a general fiber of a Tyurin degeneration arising from two reflexive polytopes. Since these Calabi-Yau manifolds and (higher or usual algebraic) cycles are totally derived from the combinatorial geometry of these polytopes, we expect that their dual polytopes encodes the mirror objects via mirror symmetry.

1. Introduction

Let X be a smooth quasi-projective variety over $\mathbb{C}$. An algebraic cycle on X is defined as a locally finite formal linear sum of subvarieties on X. The quotient of the groups of algebraic cycles by an adequate equivalence relation such as rational, algebraic, numerical, and homological equivalence defines an invariant of X and finding non trivial elements of such a quotient is interest in Algebraic Geometry. When we consider the rational equivalence, which identifies two algebraic cycles given as the zero and pole of a rational function over a one dimensional higher subvariety, the quotient is called the Chow group $CH^p(X)$ (of codimension p cycles). The celebrated Hodge Conjecture states the surjectivity of the cycle class map from the Chow group $CH^p(X)$ with the rational coefficients to the Hodge cycle class $\mathrm{Hdg}(X)$.

For a closed subvariety Z in X and its complement U, their Chow groups defines the localization exact sequence

$$CH^p(Z) \to CH^p(X) \to CH^p(U) \to 0.$$

To extend this sequence to a long exact sequence, we need an extended notion of the Chow group, which is called the higher Chow groups $CH^p(X, n)$. Roughly speaking, an element of higher Chow groups (which is often called a higher cycle) is a algebraic cycle on $X \times \square^n$ of codimension p with $\square^n := (\mathbb{P}^1 \setminus \{1\})^n$ on an appropriate position, with the information of "boundaries" coming from the pullback-pushforward images of $0, \infty \in \square^1$ (See Chapter 2 for the precise definition). The higher Chow groups $CH^p(X, n)$ are isomorphic to the motivic cohomology $H^{2p-n}_{\mathcal{M}}(X, \mathbb{Q}(p))$ for a smooth variety and have a natural bigraded product structure which is compatible with the cup products on the cohomology of each realization, and contains the image of $CH^{p-1}(X)$ for $CH^p(X, 1)$ via this product structure. A higher cycle in this image is called decomposable, and we may consider it as an apparent higher one and essentially arising from a usual algebraic cycle. Hence to show a given higher cycle is indecomposable is a deeper question than its non-triviality.

The aim of this paper is to introduce some known constructions of indecomposable cycles and show a new method to detect the linear independence of indecomposable K_1-cycles on the nearby fiber by using the asymptotic behaviors of real regulators. We also find a systematic construction of a collection of higher Chow cycles on a certain type of families $\{X_t\}_{t\in\mathbb{P}^1}$ of degree d surfaces in $\mathbb{P}^3$. This construction gives a new explicit proof of the Hodge-$\mathcal{D}$-Conjecture when $d = 4$. The Hodge-$\mathcal{D}$-Conjecture is a generalization of the Hodge Conjecture: We can generalize the cycle map from the Chow group to the higher Chow groups $CH^p(X, n)$, and by extending the coefficient to $\mathbb{R}$, we obtain the real regulator map $r^{p,n}_{\mathcal{D},\mathbb{R}} : CH^p(X, n) \otimes \mathbb{R} \to H^{2p-n}_{\mathcal{D}}(X, \mathbb{R}(p))$. Hodge-$\mathcal{D}$-Conjecture states the surjectivity of $r^{p,n}_{\mathcal{D},\mathbb{R}}$. Unfortunately this conjecture is false for general projective varieties, but it is still open (and expected to hold) for X defined over $\overline{\mathbb{Q}}$ ([1]). The most significant result is due to X. Chen and J. Lewis and introduced in Chapter 3. They proved that the Hodge-$\mathcal{D}$-Conjecture holds for (analytically) general polarized $K3$ surfaces in the moduli space by observing the deformation of the higher cycles along the degeneration of the general $K3$ to a special one with Picard number 20 (which is called Bryan-Leung $K3$ surface).

While the existence of sufficiently many higher cycles is abstractly contained in Chen and Lewis's work, our construction is completely explicit and concrete. The precise construction is given in Section 5.1, but roughly the type of surfaces we consider has the form

$$X_t \colon L_1 L_2 \cdots L_d + t M_1 M_2 \cdots M_d = 0 \subset \mathbb{P}^3$$

with general $t \in \mathbb{P}^1$ and linear forms L_i, M_l in general position. Then each intersection $L_i \cap M_l$ defines a line on X_t, which is constant even when we move t. By choosing intersecting three lines of this type with the boundaries at $L_i \cap L_j \cap M_l$, we can construct a higher Chow cycle $\gamma_{ijk,l} \in CH^2(X_t, 1)$ so that its support is just a union of three lines. By changing the roles of the linear forms L and M, we also can construct another type of higher Chow cycle $\delta_{i,lmn}$, and moreover each line $L_i \cap M_l$ as an algebraic cycle also defines an element λ_{il} of $CH^2(X_t, 1)$ in the naive way. Theorem 7 in Chapter 5 states that these higher cycles $\{\gamma_{ijk,l}\}, \{\delta_{i,lmn}\}, \{\lambda_{il}\}$ are enough to prove the Hodge-$\mathcal{D}$-Conjecture for $d = 4$ and general choices of t, L_i, M_s.

Our method is based on the theory of limits and singularities of admissible normal functions. After a resolution of singularities and change of the coordinates, we may consider $\{X_t\}_{t\in\Delta^*}$ as a semistable degeneration to the simple normal crossing divisor X_0 with smooth fibers over the punctual unit disc $\Delta^* = \Delta \setminus \{0\}$. The Abel-Jacobi values of $\gamma_{ijk,l}$ and $\delta_{i,lmn}$ as families of higher Chow cycles define holomorphic sections of the intermediate Jacobian bundle over Δ^*, which are examples of admissible normal functions. Roughly speaking, the *limit* of the admissible normal function associated to a family of higher Chow cycles describes the limiting behavior of the Abel-Jacobi value as t approaches to 0. However, generally the degeneration of the family of higher Chow cycles may not be a higher Chow cycle, since it may have some obstructions coming from the singularities. Such an obstruction can be described as another invariant, which is called the *singularity* of the admissible normal function.

Recently, the limiting behaviors of complex valued admissible (or usual) normal functions has been studied ([2] [3] [4]) and it is not difficult to show that the singularity invariants factor through the projection to the real regulator. In fact, for general d we show that each $\gamma_{ijk,l}$ has non-trivial singularities and moreover $\{\gamma_{ijk,l}\} \cup \{\lambda_{il}\}$ span the codomain $\mathrm{Hdg}(\mathrm{Coker}N)$ of this invariant, where N denotes the log monodromy action around $t = 0$ (Theorem 5 in Chapter 5) and this implies that $\{\gamma_{ijk,l}\}$ span a 19 dimensional subspace of $H^{1,1}_{\mathbb{R}}$.

On the other hand, the limit invariants are typically killed by the projection to the real regulator. As our new method, in Section 5.3 and 5.4 we dig further into the asymptotic behavior of the real regulator to recover these limits. We show not only the non-triviality of the limit of $\delta_{i,lmn}$ when $d = 4$, but also the limit of its real regulator value is linearly independent from that of $\{\frac{1}{\log(t)}\gamma_{ijk,l}\}$. These results give explicit proof of the Hodge-$\mathcal{D}$-Conjecture for this type of $K3$ surface. We also remark that our construction itself yields a collection of higher cycles on the surface X_t of general degree $d \geq 4$. While the Hodge-$\mathcal{D}$-Conjecture is known to be false for very general surfaces in $\mathbb{P}^3$ of degree ≥ 5, it is still an interesting problem to determine subfamilies on which the conjecture holds. Though we are not able to prove Hodge-$\mathcal{D}$-Conjecture yet for X_t of general degree, at least each of $\gamma_{ijk,l}$ is still $\mathbb{R}$-regulator indecomposable (cf. Section 2.3 for the definition). It suggests that the higher cycles $\{\gamma_{ijk,l}\}, \{\delta_{i,lmn}\}, \{\lambda_{il}\}$ may indicate an explicit proof.

As another application, we also introduce a new construction of threefolds with non-trivial Griffiths groups from $\gamma_{ijk,l}$. The Griffiths group $\mathrm{Griff}^p(X)$ of a projective variety X is defined by the quotient $CH^p_{hom}(X)/CH^p_{alg}(X)$ by the subgroup $CH^p_{alg}(X)$ of cycles which are algebraically equivalent to zero. Cycles in $\mathrm{Griff}^2(X)$ and their normal functions provide the B-model for Morrison and Walcher's work on the open mirror symmetry ([5]). Meanwhile, C. Doran, A. Harder and A. Thompson introduced a non-toric mirror scenario involving Tyurin degenerations, in which the Calabi-Yau threefolds degenerate to a union of quasi-Fano threefolds intersecting along a $K3$ surface ([6]). Key to studying open mirror symmetry in the latter setting would be to construct K_1-cycles on the $K3$ surface which are limits of K_0-cycles on the nearby Calabi-Yau threefold (this is called "going-up" in the theory of the K-theory elevator which is introduced in [4]). For this construction, one will need totally concrete K_1-cycles on the $K3$ surface, and this point is an advantage of our explicit proof of the existence of $\mathbb{R}$-regulator indecomposable cycles. In Section 6.1, starting from a general degree d surface X_{t_0} defined as above, we construct a semistable degeneration family $\mathcal{Y}$ of threefolds, which is an example of Tyurin degeneration when $d = 4$. Its singular fiber Y_0 consists of the union of the product $X_{t_0} \times \mathbb{P}^1$ and two blown up copies of $\mathbb{P}^3$, meeting along two copies of X_{t_0} (The picture before taking the blow up is drawn in Figure 2).

Applying the theory of the K-theory elevator, we can shift the higher Chow cycle $\gamma_{ijk,l}$ in the intersection $X_{t_0} \times \mathbb{P}^1$ of Y_0 to an algebraic cycle in one of the blown up $\mathbb{P}^3$, which is a fiber of a family of algebraic cycles $\mathcal{C}_{ijk,l}$ on $\mathcal{Y}$. $\mathbb{R}$-regulator indecomposability of $\gamma_{ijk,l}$ implies the non-triviality of the general fiber of $\mathcal{C}_{ijk,l}$ in the Griffiths groups. Therefore this yields a new example exhibiting the connection between the algebraically non-trivial cycles and $\mathbb{R}$-regulator indecomposable cycles.

This thesis is organized as follows. In Chapter 2, we briefly recall the definitions of the higher Chow cycles, indecomposable cycles, real regulator map, and the statement of Hodge-$\mathcal{D}$-conjecture. We also introduce the KLM formula, which is an essential tool in computing the Abel-Jacobi maps. In Chapter 3, we introduce the construction of indecomposable cycles by A. Collino on Jacobians of hyperellpitic curves, and the proof of Hodge-$\mathcal{D}$-Conjecture by X. Chen and J. Lewis. Their approaches to show the indecomposablity is based on finding a specialization of the constructed higher cycles and consider the limiting behavior of associated admissible normal functions or real regulator values. More generally, these limit and singularity invariants of admissible normal functions are explained in Chapter 4. Chapter 5 is the body of this paper. We define the family of surfaces $\mathcal{X} = \{X_t\}$ and construct the specific higher Chow cycles $\gamma_{ijk,l}$, $\delta_{i,lmn}$, and λ_{il} on this family. Then we show the non-triviality of these invariants for the above higher Chow cycles respectively, and prove the Hodge-$\mathcal{D}$-Conjecture for our case in Section 5.4. Finally, we construct a threefold with non-trivial Griffiths groups starting from X_{t_0} as another application in Chapter 6.

2. Higher Chow Groups and Indecomposable cycles

2.1 Higher Chow Groups

Throughout this paper, we fix the base field to be $\mathbb{C}$ and an algebraic variety means an integral separated scheme of finite type over $\mathbb{C}$. Firstly, we recall the definition of the higher Chow groups. See [7] for the original construction with algebraic simplexes and [8] for the cubical version, which we use here. The algebraic n-cube is defined by

$$\square^n := (\mathbb{P}^1 \setminus \{1\})^n.$$

For each i $(0 \leq i \leq n)$, there is the ith-face map $\rho_i^\epsilon \colon \square^{n-1} \hookrightarrow \square^n$ with $\epsilon = 0, \infty$ defined by the embedding $(z_1, z_2, \ldots, z_{n-1}) \mapsto (z_1, z_2, \ldots, z_{i-1}, \epsilon, z_i, \ldots, z_n)$. The facet $\partial_i^\epsilon \square^n$ is defined by the image of ρ_i^ϵ and more generally the face $\partial_I^{\underline{\epsilon}} \square^n$ for each $I \subset \{0, \ldots, n\}$ and $\underline{\epsilon} = \{\epsilon(i)\}_{i \in I}$ is defined by $\bigcap_{i \in I} \partial_i^{\epsilon(i)} \square^n$. We also denote $\partial \square^n := \bigcap_{i \in I} \bigcup_{\epsilon = 0, \infty} \partial_i^\epsilon \square^n$.

Let X be a quasi-projective variety. For $p, n \in \mathbb{Z}_{\geq 0}$, $\mathcal{C}^p(X, n)$ is defined as the free abelian group generated by subvarieties of $X \times \square^n$ of codimension p which intersects each $X \times \partial_I^{\underline{\epsilon}} \square^n$ properly. It contains the subgroup $\mathcal{D}^p(X, n)$ which is generated by the pullbacks of cycles via face projections $X \times \square^n \twoheadrightarrow X \times \square^{n-|I|}$, and we denote the quotient

$\mathcal{C}^p(X,n)/\mathcal{D}^p(X,n)$ by $\mathcal{Z}^p(X,n)$. Then $\mathcal{Z}^p(X,\bullet)$ becomes a chain complex with the well-defined boundary map

$$\partial \coloneqq \sum_i (-1)^i((\rho_i^0)^* - (\rho_i^\infty)^*) \colon \mathcal{Z}^p(X,n) \to \mathcal{Z}^p(X,n-1).$$

An element of $\mathcal{Z}^p(X,n)$ is called a *precycle* on X. The *higher Chow groups* are defined by taking the homology of this complex:

$$CH^p(X,n) \coloneqq H_n(\mathcal{Z}^p(X,\bullet)).$$

Note that $CH^p(X) = CH^p(X,0)$ by the definition.

When X is smooth, there is another expression via the Gersten-Milnor resolution for $CH^p(X,1)$:

$$CH^p(X,1) \cong H_1\Big(\bigoplus_{cd_x Z=p-2} K_2^M(\mathbb{C}(Z)) \to \bigoplus_{cd_x Z=p-1} K_1^M(\mathbb{C}(Z)) \to \bigoplus_{cd_x Z=p} K_0^M(\mathbb{C}(Z))\Big).$$

Here, $K_p^M(k)$ is the p-th Milnor K-theory of a field k. Since $K_0^M(\mathbb{C}(Z)) \cong \mathbb{Z}$ and $K_1^M(\mathbb{C}(Z)) = \mathbb{C}(Z)^*$, each element of $CH^p(X,1)$ can be represented by a formal sum $\sum(f_i, Z_i)$ with a codimension $(p-1)$ subvariety Z_i and a rational function f_i over Z_i such that $\sum_i \operatorname{div}(f_i) = 0$. Taking the quotient by the image of the Tame symbols, we obtain $CH^p(X,1)$. More specifically, the graph of $f_i|_{Z_i \setminus f_i^{-1}(1)}$ as a subvariety of $X \times (\mathbb{P}^1 \setminus \{1\})$ defines an element of $\mathcal{Z}^p(X,1)$.

Notation 1 *We consider only non-torsion higher cycles in this paper. For this reason,* ***we use the notation*** $CH^p(X,n)$ ***for the rational coefficient higher Chow groups*** $CH^p(X,n) \otimes \mathbb{Q}$ ***from now on****.*

2.2 Hodge-$\mathcal{D}$-Conjecture

For a subring $\mathbb{A} \subset \mathbb{R}$, the Deligne complex is defined by a complex of sheaves on X

$$\mathbb{A}_{\mathcal{D}}(p) \colon \mathbb{A}(p) \to \mathcal{O}_X \to \Omega_X^1 \to \ldots \to \Omega_X^{p-1}.$$

Here $\mathbb{A}(p) \coloneqq \mathbb{A}(2\pi\sqrt{i})^p$. Then the *Deligne cohomology* is defined by the hypercohomology

$$H^i_{\mathcal{D}}(X, \mathbb{A}(p)) \coloneqq \mathbb{H}^i(\mathbb{A}_{\mathcal{D}}(p))$$

and Bloch defined a cycle class map

$$cl^{p,n}_{\mathcal{D}} \colon CH^p(X, n) \to H^{2p-n}_{\mathcal{D}}(X, \mathbb{Q}(p)).$$

In the case of $n = 0$, $cl^p_{\mathcal{D}} \coloneqq cl^{p,0}_{\mathcal{D}}$ can be considered as the unified map of the usual cycle class map cl^p to the Hodge class $\mathrm{Hdg}^p(X)$ and the Abel-Jacobi map AJ^p to the intermediate Jacobian $J^p(X)$. More precisely, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & CH^p_{\mathrm{hom}}(X) & \longrightarrow & CH^p(X) & \longrightarrow & CH^p(X)/CH^p_{\mathrm{hom}}(X,n) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle AJ^p} & & \downarrow{\scriptstyle cl^p_{\mathcal{D}}} & & \downarrow{\scriptstyle cl^p} & & \\ 0 & \longrightarrow & J^p(X) & \longrightarrow & H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p)) & \longrightarrow & \mathrm{Hdg}^p(X) & \longrightarrow & 0 \end{array}$$

with exact rows.

For a quasi-projective variety U, we can define the higher cycle class map to the generalized Hodge class

$$\begin{aligned} cl^{p,n} \colon CH^p(U, n) \to \mathrm{Hdg}^{p,n}(U) &\coloneqq \mathrm{Hdg}(H^{2p-n}(U, \mathbb{Q})(p)) \\ &\coloneqq \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^{2p-n}(U, \mathbb{Q})(p)) \end{aligned}$$

and the higher Abel-Jacobi map from $CH^p_{\mathrm{hom}}(U,n)(:= \mathrm{Ker}(cl^{p,n}))$ to the generalized intermediate Jacobian

$$AJ^{p,n}\colon CH^p_{\mathrm{hom}}(U,n) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H^{2p-n-1}(U,\mathbb{Q})(p)).$$

By replacing the Deligne cohomology to the absolute Hodge cohomology ([9], Section 2), we also can define the cycle class map $cl^{p,n}_{\mathcal{H}}$ and obtain the generalization of the above commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & CH^p_{\mathrm{hom}}(U,n) & \longrightarrow & CH^p(U,n) & \longrightarrow & CH^p(U,n)/CH^p_{\mathrm{hom}}(U,n) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle AJ^{p,n}} & & \downarrow{\scriptstyle cl^{p,n}_{\mathcal{H}}} & & \downarrow{\scriptstyle cl^{p,n}} & & \\ 0 & \longrightarrow & J^{p,n}(U) & \longrightarrow & H^{2p}_{\mathcal{H}}(U,\mathbb{Q}(p)) & \longrightarrow & \mathrm{Hdg}^{p,n}(U) & \longrightarrow & 0. \end{array}$$

The composition $CH^p(U,n) \to CH^p(U,n)/CH^p_{\mathrm{hom}}(U,n) \xrightarrow{cl^{p,n}} \mathrm{Hdg}^{p,n}(U)$ is also often denoted by just $cl^{p,n}$. For a smooth projective variety X, however, each cohomology class has the pure Hodge structure and hence $H^{2p-n}(X,\mathbb{Z})(p)$ has no weight 0 graded pieces up to torsion. Thus $\mathrm{Hdg}^{p,n}(X) = \{0\}$ and the diagram turns into

$$\begin{array}{ccc} CH^p_{\mathrm{hom}}(X,n) & = & CH^p(X,n) \\ \downarrow{\scriptstyle AJ^{p,n}} & & \downarrow{\scriptstyle cl^{p,n}_{\mathcal{D}}} \\ J^{p,n}(X) & = & H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p)). \end{array}$$

The vanishing of the generalized Hodge class clearly shows that we cannot state the Hodge conjecture for the higher case as the surjectivity of $cl^{p,n}$. Instead, we consider the composition of the natural surjection $H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p)) \to H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p))$ after $cl^{p,n}_{\mathcal{D}}$, which is called the *real regulator map*

$$r^{p,n}_{\mathcal{D}}\colon CH^p(X,n) \to H^{2p-n}_{\mathcal{D}}(X,\mathbb{R}(p)).$$

Then a version of Beilinson's Hodge-$\mathcal{D}$-Conjecture is

Conjecture 1 (Hodge-$\mathcal{D}$-Conjecture) *For a smooth variety X over $\bar{\mathbb{Q}}$,*

$$r^{p,n}_{\mathcal{D},\mathbb{R}} := r^{p,n}_{\mathcal{D}} \otimes \mathbb{R} \colon CH^p(X,n) \otimes \mathbb{R} \to H^{2p-n}_{\mathcal{D}}(X, \mathbb{R}(p))$$

is surjective.

Note that $H^{2p-1}_{\mathcal{D}}(X, \mathbb{R}(p)) \cong H^{p-1,p-1}_{\mathbb{R}}(X)(p-1) := H^{p-1,p-1}(X, \mathbb{R}) \otimes \mathbb{R}(p-1)$ ([10], Section 3).

Remark 1 *The same statement for quasi-projective varieties over $\mathbb{C}$ is known to be false. See [1].*

2.3 Indecomposable Cycles

The higher Chow groups have a product structure

$$CH^p(X,n) \otimes CH^q(X,m) \to CH^{p+q}(X, n+m)$$

which is compatible with the cup products in the Deligne cohomology and the real regulator map. Since it is known that $CH^1(X,1) \cong H^1_{\mathcal{D}}(X, \mathbb{Q}(1)) \cong \mathbb{C}^*$, especially we obtain a map

$$\mathbb{C}^* \otimes CH^{p-1} \to CH^p(X,1). \tag{2.1}$$

The image $CH^p_{\text{dec}}(X,1)$ of the above map (2.1) is called the *subgroup of the decomposable cycles* and the *group of indecomposable cycles* is defined by the quotient

$$CH^p_{\text{ind}}(X,1) := CH^p(X,1)/CH^p_{\text{dec}}(X,1).$$

If especially the real regulator image $r^{p,1}_{\mathcal{D},\mathbb{R}}(\gamma)$ of an element $\gamma \in CH^p(X,1)$ is not in the image $\mathrm{Im}(H^1_{\mathcal{D}}(X,\mathbb{R}(1)) \otimes H^{2p-2}_{\mathcal{D}}(X,\mathbb{R}(p-1)) \xrightarrow{\mu} H^{2p-1}_{\mathcal{D}}(X,\mathbb{R}(p))) \cong \mathbb{R} \otimes \mathrm{Hdg}^{p-1}(X)$, we say that γ is $\mathbb{R}$-*regulator indecomposable.* Clearly $\mathbb{R}$-regulator indecomposable cycles are indecomposable.

The Deligne cohomology $H^{2p-n}_{\mathcal{D}}(X,\mathbb{A}(p))$ can be also defined as the $(-r)$th cohomology of the Deligne cohomology complex

$$\mathcal{M}^\bullet := Cone\{\mathcal{C}^{2p+\bullet}_X(X,\mathbb{A}(p)) \oplus F^p\mathcal{D}^{2p+\bullet}_X(X) \xrightarrow{\epsilon - l} \mathcal{D}^{2p+\bullet}_X(X)\}[-1]$$

with the sheaves of topological chains and distributions on X. Here, ϵ maps to the associated current and l is the natural embedding. On the other hand, the complex of precycles $\mathcal{Z}^p(X,\bullet)$ has a subcomplex $\mathcal{Z}^p_{\mathbb{R}}(X,\bullet)$ of cycles meeting real faces properly such that the inclusion is a (rational) quasi-isomorphism. The *KLM-formula* [11] is a map of complexes $\mathcal{Z}^p(X,-\bullet) \to \mathcal{M}^\bullet$ defined by

$$Z \to (2\pi i)^{p-n}((2\pi i)^n T_Z, \Omega_Z, R_Z),$$

and indicating $AJ^{p,n}$. Here, each of T_Z, Ω_Z, R_Z is essentially defined by the pushforward-pull back image of the following current on $\square^r := (\mathbb{P}^1 \setminus \{1\})^r$ respectively:

$$T_r := (2\pi i)^r \delta_{[-\infty,0]^r}$$

$$\Omega_r := \int_{\square^r} \wedge_{k=1}^r d\log z_k$$

$$R_r := \int_{\square^r} \log z_1 \wedge_{k=2}^r d\log z_k - (2\pi i)\int_{[-\infty,0]\times\square^{r-1}} \log z_2 \wedge_{k=3}^r d\log z_k + \ldots + (-2\pi i)^r \int_{[-\infty,0]^{r-1}\times\square^1} d\log z_r.$$

When $T_Z = \partial\Gamma$ and $\Omega_Z = d\Xi$, by adding the differential $D((2\pi i)^n\Gamma, \Xi, 0) = (-(2\pi i)^n T_Z, -\Omega_Z, -\Xi + (2\pi i)^n\delta_\Gamma)$ we can simplify the formula. Especially when $d \coloneqq \dim X \leq p$ or $p \leq n$, since $F^pD^{2p-n}(X)$ vanishes and hence Ω_Z is trivial, we obtain

$$\begin{aligned} AJ^{p,n}(Z)(\omega) &= (-2\pi i)^{p-n}(R_Z + (2\pi i)^n\delta_\Gamma)(\omega) \\ &= \frac{1}{(-2\pi i)^{n-p}}\left(\int_X R_Z \wedge \omega + (2\pi i)^n \int_\Gamma \omega\right) \end{aligned}$$

for each closed test form ω in $F^{d-p+1}\Omega^{2d-2p+n+1}(X)$, yielding a class in $J^{p,n}(X) \cong \{F^{d-p+1}H^{2d-2p+n+1}(X, \mathbb{C})\}^\vee / H_{2d-2p+n+1}(X, \mathbb{Q}(p))$.

More generally, the KLM formula holds for a smooth quasi-projective U, and even for a normal crossing divisor Y on X by changing each complex appearing in the formula to the simple complex associated to a certain double complex (See Section 5.3). Especially it defines the cycle map

$$cl_{\mathcal{D}}^{p-1,n-1} \colon CH^{p-1}(Y, n-1) \to H_{\mathcal{D},Y}^{2p-n+1}(X, \mathbb{Q}(p))$$

and hence

$$cl^{p-1,n-1} \colon CH^{p-1}(Y, n-1) \to \mathrm{Hdg}(H_Y^{2p-n+1}(X, \mathbb{Q}(p)))$$

for the cohomologies with support on Y. For the detail of the construction, see Section 5.9 of [11] and Section 3 of [9].

3. Examples of Indecomposable cycles

3.1 Indecomposable cycles on Jacobians of curves

We introduce A. Collino's construction of the indecomposable cycles on Jacobians of hyperelliptic curves. The construction is natural, and has an analogy with the Ceresa cycles.

For a very general genus g curve C with a fixed point $x \in C$, we define a map $\imath_x \colon C \to J(C) \coloneqq J^1(C)$ to its Jacobian defined by

$$i_x(y) = AJ(y - x).$$

With the push forward $i_x^- \coloneqq \iota_*(i_x)$ by the involution ι, the Ceresa cycle is defined by

$$i_x(C) - i_x^-(C) \in CH^{g-1}(J(C)).$$

Since two points $x, x' \in C$ are connected by C itself, we obtain the unique class in $\mathrm{Griff}^{g-1}(J(C))$ which is independent from the choice of x. Ceresa showed that this cycle is actually not algebraically trivial when $g \geq 3$, though obviously it is homologically trivial ([12]).

Collino constructed a natural higher cycle in $CH^g(J(C),1)$ for a general hyperelliptic curve based on the same idea. For a given hyperelliptic curve C of genus g, let $h\colon C \to \mathbb{P}^1$ be the double cover and take two different ramification points $p, q \in C$. By changing the parameter, we may assume that $h(p) = 0$ and $h(q) = \infty$. Then we obtain a precycle $(h_p, i_p(C)) \in \mathcal{Z}^g(J(C),1)$ with the rational function $h_p\colon i_p(C) \to \mathbb{P}^1$ induced by h via i_p. Similarly the point q defines another precycle $(h_q, i_q(C))$. Note that $i_p(C) \cup i_q(C) = \{0, AJ(p-q)\}$. Since $\mathrm{div}(h_p) = -\mathrm{div}(h_q)$, $((h_p, i_p(C)) + (h_q, i_q(C))$ defines a higher cycle in $CH^g(J(C),1)$. For the simplicity with respect to the symmetry we shift p and q with a point $\xi = \frac{AJ(p-q)}{2} \in J(C)$, and then we obtain the required higher cycle

$$\begin{aligned} Z(C) &= ((h_p)_\xi, i_p(C) - \xi) + ((h_q)_\xi, i_q(C) - \xi) \\ &= ((h_p)_\xi, i_p(C) - \xi) - (((h_q)_\xi)^{-1}, i_q(C) - \xi). \end{aligned}$$

Here, $(h_p)_\xi, (h_q)_\xi$ mean the translations of the rational functions. When we consider the Abel-Jacobi values in $J^{2g,1}(J(C)) \cong H_{\mathcal{D}}^{2g-1}(J(C), \mathbb{Q}(g))$, the decomposable cycles define the subgroup W as the image of

$$CH^{g-1}(J(C)) \otimes \mathbb{C}^* \to H_{\mathcal{D}}^{2g-2}(J(C), \mathbb{Q}(1)) \otimes \mathbb{C}^* \to J^{2g,1}(J(C)).$$

Changing the parameter by the translation is contained in W, so that we obtain the well-defined regulator class

$$\nu(C,p,q) \coloneqq AJ^{2,1}(Z(C)) \in J^{2g,1}(J(C))/W,$$

which depends on C, p, q and independent from the choice of ξ. By considering the moduli space of hyperelliptic curves with two Weierstrass points p, q, hence finally we obtain an admissible normal function ν (we introduce the general property of admissible normal functions and its invariant in Chapter 4).

Recall that a given variation of mixed Hodge structures $\mathcal{V}$ over S defines the complex

$$C^\bullet := (\mathcal{V} \to \Omega^1_S \otimes \mathcal{V} \to \Omega^2_S \otimes \mathcal{V} \to \dots)$$

with the connection ∇ as the diffenrential. The hodge filtration $\mathcal{F}^\bullet$ on $\mathcal{V}$ also defines a filtration

$$F^p C^\cdot := (\mathcal{F}^p\mathcal{V} \to \Omega^1_S \otimes \mathcal{F}^{p-1}\mathcal{V} \to \Omega^2_S \otimes \mathcal{F}^{p-2}\mathcal{V} \to \dots)$$

by the Griffiths' transversality condition, so that we obtain a short exact sequence

$$0 \to H^1(S, \mathcal{H}^0(F^0C^\bullet)) \to \mathbb{H}^1(S, F^0C^\bullet) \to H^0(S, \mathcal{H}^1(F^0C^\bullet)) \to 0$$

from the hypercohomology spectral sequence. Since a normal function can be defined as an element of $\mathbb{H}^0(S, C^\bullet/(F^0C^\bullet \oplus \mathbb{V}))$ with the underlying (rational) local system $\mathbb{V}$, we obtain two invariants of admissible normal functions in $H^0(S, \mathcal{H}^1(F^0C^\bullet))$ via $\mathbb{H}^1(S, F^0C^\bullet)$ and $H^1(S, \mathbb{V})$ via $\mathcal{H}^1(S, F^0C^\bullet \oplus \mathbb{V})$ respectively. The former one is called the infinitesimal invariant of the given normal function, and the latter is called the topological invariant.

Proposition 1 ([13]) *When we consider the deformation of (C, p, q) to a nodal curve defined by attaching p and q, the associated infinitesimal invariant $\delta\nu$ is non trivial.*

We may construct ν' as a normal function on the primitive Jacobian and can show that this invariant $\delta\nu$ is independent from the choice of the lifting ν. Hence $Z(C)$ is indecomposable for a general choice of C. Collino and Fakhruddin also extended this result to the Jacobian of a smooth projective curve C ([14]). They showed that the decomposability of a given higher cycle is preserved by the specialization, and construct a higher cycle on $J(C)$ which is supported on the four embedded copies of C such that they can be specialized to the difference of the above natural cycle on a hiperelliptic curve and its translation.

3.2 Hodge-$\mathcal{D}$-Conjecture for general $K3$ surfaces

In this section we introduce the proof of the Hodge-$\mathcal{D}$-Conjecture for general $K3$ surfaces and abelian surfaces by X. Chen and J. Lewis. Here the meaning of general is not algebraic, but analytic. Specifically, a projective variety X_s is general in a given family over an algebraic variety S (parametrized by $s \in S$) when s is in the complement of a countable union of real analytic subvarieties in S.

Theorem 1 ([10]) *With the above terminology, the Hodge-$\mathcal{D}$-Conjecture holds for general polarized $K3$, Abelian, and Kummer surfaces.*

Note that the result for general $K3$ surfaces immediately deduces that for Abelian and Kummer surfaces. In fact, special Kummer surfaces (which arise from reduced Abelian surfaces) is dense in the period domain of marked $K3$ surfaces ([15]). For a given Abelian variety A, we obtain the corresponding special Kummer Surface X by taking the quotient by the involution and then blow-up along 16 double points. The induced correspondence defines a surjection $H^2(X, \mathbb{R}(1)) \to H^2(A, \mathbb{R}(1))$, which is compatible with the real regulator maps.

The main idea of the proof is to construct sufficiently many higher Chow cycles from rational curves, which deforms to specific divisors in a BL $K3$ surfaces. A $K3$ surface is called a BL $K3$ surface when it has the Picard lattice

$$\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix},$$

hence the Picard group is generated by two effective divisors F and C such that $C^2 = -2, C \cdot F = 1$, and $F^2 = 0$. One can construct such a surface as an elliptic fibration over $\mathbb{P}^1$ with its fiber F and the unique section C. The following proposition shows some essential properties of BL $K3$ surfaces:

Proposition 2 *(i) A general $K3$ surface can be degenerated to a BL $K3$ surface X with the primitive class degenerated to $C + gF$.*

(ii) Let $D \subset X$ be the limit of a family of rational curves and assume that $D \in |C+gF|$. Then D has the form of

$$D = C \cup F_1 \cup F_2 \cup \ldots \cup F_g$$

with rational curves $\{F_i \in |F|\}$.

We consider a *BL* K3 surface S with the elliptic fibration such that there is exactly 6 singular fibers $F_1, F_2, \ldots, F_6$. Then each F_i consists of the union of four rational curves and S has the maximum Picard number 20.

Recall that one can obtain a precycle in $\mathcal{Z}^2(X,1)$ on a smooth surface X as a combination of curves $\{C_i\}$ and rational functions $\{f_i\}$ over them such that $\sum \mathrm{div}(f_i) = 0$. Especially, when each C_i is a rational curve and $\cup(C_i)$ is normal crossing, we may choose the isomorphism between C_i and $\mathbb{P}^1$ such that each of $0, \infty \in \mathbb{P}^1$ is an intersection with another rational curve C_j. By finding a combination of $\{C_i\}$ such that these boundaries 0 and ∞ cancel out as the divisor class, we obtain an element of $CH^2(X,1)$. To follow this construction, we shall take a family $\mathcal{X}/\Delta$ of $K3$ surfaces of genus g over the unit disk $\Delta \subset \mathbb{C}$ such that $X_0 = S$, and find the limiting curves on X_0 of rational curves on the general fiber X_t $(t \in \Delta)$. Here, a problem is that any two limiting curves D_1 and D_2 in $|C + gF|$ do not intersect properly by Prop 2. Hence we firstly need to take the blow up $\widetilde{\mathcal{X}}$ of $\mathcal{X}$ along $F_1, \ldots, F_6$. Take one of the singular fibers F_i and denote it by the union of four rational curves $E = E_0 \cup E_1 \cup E_2 \cup E_3$ (We assume that E_0 is the unique curve which intersects with C). Then the exceptional divisor R over E is in the central fiber $\widetilde{X_0}$. The inverse image $R_i = p^{-1}(E_i)$ by the projection $p\colon R \to E$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and each of $\{R_i\}$ intersects along the fiber $Q_i = p^{-1}(E_{i-1} \cap E_i)$. Note that each Q_i contains a rational double point r_i of $\widetilde{\mathcal{X}}$.

Now, Choose a map $\phi_{i,i+1}\colon Q_i \to Q_{i+1}$ (we denote just ϕ for the simplicity) with a point $x \in Q_i$ such that the curve connecting x and $\phi(x)$ is in the linear system $|E_i|$. We denote $r_i^{(k)} = \phi^k(r_i)$ and take the curve $\overline{r_i^{(j)} r_i^{(j+1)}}$ which connects $r_i^{(j)}$ and $r_i^{(j+1)}$ in R_{i+j} (mod 4). Now we can construct an example of limiting rational curves:

Proposition 3 ([10] Section 8, 9) *Take the unique curve $N_\Gamma \subset R_0$ in $|E_0+Q_0|$ which contains $r_0^{(-3)}, r_0, q$, and $N_\Sigma \subset R_0$ in $|E_0+2Q_0|$ which contains $r_1, r_1^{(3)}, r_2^{(-1)}, r_2^{(2)}, q$. Let Γ and Σ be reduced curves in R defined by*

$$\begin{cases} \Gamma \quad := N_\Gamma + \sum_{j=-3}^{-1} \overline{r_0^{(j)} r_0^{(j+1)}} \\ \Sigma \quad := N_\Sigma + \sum_{j=0}^{2} \overline{r_1^{(j)} r_1^{(j+1)}} + \sum_{j=-1}^{1} \overline{r_2^{(j)} r_2^{(j+1)}} \end{cases}$$

Then both Γ and Σ are limits of rational curves. More precisely, there exists a family of rational maps $\mathcal{Y}_\Gamma, \mathcal{Y}_\Sigma \to \widetilde{\mathcal{X}}$ which is compatible with the projection to Δ such that the images of $Y_{\Gamma,0} = (\mathcal{Y}_\Gamma)_0$ and $Y_{\Sigma,0} = (\mathcal{Y}_\Sigma)_0$ are Γ and Σ respectively.

Since Γ and Σ intersects at exactly three points $u, v, q \in N_\Gamma \cap N_\Sigma$ by putting the boundaries 0 and ∞ on u and v respectively, we obtain a family of higher cycles ϵ in $CH^2(\mathcal{X}, 1)$. (Precisely, this procedure constructs an element of the higher Chow group $\widetilde{CH}^2(\mathcal{X}, 1)$ of prestable maps, since Γ and Σ themselves are reducible. However, the restriction to each component defines a natural projection $\widetilde{CH}^2(\mathcal{X}, 1) \to CH^2(\mathcal{X}, 1)$. See [CL, Section 7].) We may change the choices of r_i as the following pairs of Γ and Σ;

$$\begin{cases} \Gamma := N_\Gamma + \sum_{j=0}^{2} \overline{r_1^{(j)} r_1^{(j+1)}} \\ \Sigma := N_\Sigma + \sum_{j=-3}^{-1} \overline{r_0^{(j)} r_0^{(j+1)}} + \sum_{j=-1}^{1} \overline{r_2^{(j)} r_2^{(j+1)}} \\ (r_1, r_1^{(3)}, q \in N_\Gamma,\ r_0, r_0^{(-3)}, r_2^{(-1)}, r_2^{(2)}, q \in N_\Sigma) \end{cases}$$

$$\begin{cases} \Gamma := N_\Gamma + \sum_{j=-1}^{1} \overline{r_2^{(j)} r_2^{(j+1)}} \\ \Sigma := N_\Sigma + \sum_{j=-3}^{-1} \overline{r_0^{(j)} r_0^{(j+1)}} + \sum_{j=0}^{2} \overline{r_1^{(j)} r_1^{(j+1)}} \\ (r_2^{(-1)}, r_1^{(2)}, q \in N_\Gamma,\ r_0, r_0^{(-3)}, r_1, r_1^{(3)}, q \in N_\Sigma) \end{cases}$$

then each choice defines other higher cycles ϵ' and ϵ'' respectively. By the local computation of the Abel Jacobi map on the singular fiber $\widetilde{X_0}$, one can show the following

Lemma 1 *The subspace in $H^{1,1}_{\mathbb{R}}(\widetilde{X_0})$ spanned by the images of $\epsilon_0, \epsilon'_0, \epsilon''_0$ by the real regulator map $r^{2,1}_{\mathcal{D},\mathbb{R}}$ contains the cycle class $c_1(E_0), c_1(E_1), c_1(E_1 + E_2 + E_3)$.*

By changing the role of r_2 to r_4 in the above construction, we also obtain the similar higher cycles and can show that the linear combinations of their real regulator values contains $c_1(E_3)$. Applying this argument to each choice of the singular fiber E from $F_1, \dots F_6$, we finally obtain the proof of Theorem 2.

4. Singularities and limits of Normal Functions

4.1 Invariants of admissible normal functions

In this chapter, we introduce two invariants of normal functions which are called the singularity and limit. When we obtain a family of higher cycles over a family of projective varieties, the Abel-Jacobi map defines the corresponding admissible normal function. When the family is a semistable degeneration we can observe its singularity by using the Clemens-Schmid exact sequence. If the normal function has the trivial singularity, then we obtain its limit value in the limiting Jacobian.

Let $\overline{S}$ be a complex manifold and $j\colon S \to \overline{S}$ be an open immersion of a Zariski open subset S. For a variation of Hodge structure $\mathcal{H}$, its generalized Jacobian bundle is defined by

$$J(\mathcal{H}) := \frac{\mathcal{H}}{\mathcal{F}^0\mathcal{H} + \mathbb{H}_{\mathbb{Q}}}.$$

A holomorphic horizontal section of $J(\mathcal{H})$ is called a $J(\mathcal{H})$-*valued normal function* over S. The group of $J(\mathcal{H})$-valued normal functions $NF(S, \mathcal{H})$ is canonically isomorphic to $\mathrm{Ext}^1_{\mathrm{VMHS}(S)}(\mathbb{Z}, \mathcal{H})$ with the category $\mathrm{VMHS}(S)$ of variations of mixed Hodge structures over S. Moreover, $\mathrm{VMHS}(S)$ contains the subcategory $\mathrm{VMHS}(S)^{\mathrm{ad}}_{\overline{S}}$ of admissible variations of Hodge structures ([16]). An element of the subgroup $\mathrm{Ext}^1_{\mathrm{VMHS}(S)^{\mathrm{ad}}_{\overline{S}}}(\mathbb{Z}, \mathcal{H})$ of $\mathrm{Ext}^1_{\mathrm{VMHS}(S)}(\mathbb{Z}, \mathcal{H})$ is called an *admissible normal function* with respect to $\overline{S}$. By Section 2 of [17], the group of admissible normal functions $NF(S, \mathcal{H})^{\mathrm{ad}}_{\overline{S}} \otimes \mathbb{Q}$ with rational coefficients is isomorphic to $\mathrm{Ext}^1_{\mathrm{MHM}(S)^{ps}_{\overline{S}}}(\mathbb{Q}, \mathcal{H})$. Here, $\mathrm{MHM}(S)^{ps}_{\overline{S}}$ is the category of smooth polarizable mixed Hodge modules over S.

Let ν be an admissible normal function over S and ι_s be the embedding of a point $s \in \overline{S}$. We define a map $sing_s$ by the composition

$$\begin{aligned} NF(S, \mathcal{H})^{\mathrm{ad}}_{\overline{S}} \otimes \mathbb{Q} \cong & \mathrm{Ext}^1_{\mathrm{MHM}(S)^{ps}_{\overline{S}}}(\mathbb{Q}, \mathcal{H}) \\ \xrightarrow{(\iota_s^* Rj_*)^{\mathrm{Hdg}}} & \mathrm{Ext}^1_{D^b\mathrm{MHM}(\{s\})}(\mathbb{Q}, \iota_s^* Rj_* \mathcal{H}) \\ \cong & \mathrm{Ext}^1_{D^b\mathrm{MHS}}(\mathbb{Q}, \iota_s^* Rj_* \mathcal{H}) \\ \to & \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^1((\iota_s^* Rj_*)\mathcal{H})). \end{aligned}$$

The invariant $sing_s(\nu)$ is called the *singularity of the normal function* ν at s. From the spectral sequence for the cohomology functor and $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, -)$, we also obtain a natural map $lim_s\colon \mathrm{Ker}(sing_s) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H^0(\iota_s^* Rj_*\mathcal{H}))$ which makes the following commutative diagram:

$$
\begin{array}{ccc}
 & & 0 \\
 & & \uparrow \\
 & & \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^1((\iota_s^* Rj_*)\mathcal{H})) \\
 & \overset{sing_s}{\nearrow} & \uparrow \\
NF(S,\mathcal{H})^{\mathrm{ad}}_{S} \otimes \mathbb{Q} & \xrightarrow{(\iota_s^* Rj_*)^{\mathrm{Hdg}}} & \mathrm{Ext}^1_{D^b\mathrm{MHS}}(\mathbb{Q}, \iota_s^* Rj_*\mathcal{H}) \\
\uparrow & & \uparrow \\
\mathrm{Ker}(sing_s) & \xrightarrow{lim_s} & \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H^0(\iota_s^* Rj_*\mathcal{H})) \\
 & & \uparrow \\
 & & 0
\end{array}
$$

We can apply the above theory of admissible normal functions to a family of higher cycles on smooth projective varieties, because of the following result of Brylinski and Zucker: Let $f\colon \mathfrak{X}^* \to S$ be a smooth proper family of quasi-projective varieties. A higher cycle

$$\mathfrak{Z}^* \in CH^p(\mathfrak{X}^*, n)_{prim} := \bigcap_{x \in S} \mathrm{Ker}(CH^p(\mathfrak{X}^*, n) \to CH^p(X_x, n) \to \mathrm{Hdg}^{p,n}(X_x))$$

defines a holomorphic section $\nu_{\mathfrak{Z}}$ of $J(\mathcal{H}^{p,n})$ for $\mathcal{H}^{p,n} := R^{2p-n-1}\pi_*\mathbb{Q}(p) \otimes \mathcal{O}_S$ by taking the fiberwise Abel-Jacobi values.

Theorem 2 *[18] $\nu_{\mathfrak{Z}^*}$ is an admissible normal function.*

If $\mathfrak{X}^*$ is the restriction of a proper family $\mathfrak{X}$ over $\overline{S}$ to S and $\mathfrak{Z}^*$ is that of a family of higher cycle $\mathfrak{Z} \in CH^p(\mathfrak{X}, n)$, with the complement $\mathfrak{X}_{sing} := \mathfrak{X} \setminus \mathfrak{X}^*$, we obtain the localization exact sequence

$$\cdots \to CH^p(\mathfrak{X}_{sing}, n) \to CH^p(\mathfrak{X}, n) \to CH^p(\mathfrak{X}^*, n) \xrightarrow{res} CH^{p-1}(\mathfrak{X}_{sing}, n-1) \to \cdots .$$

Here, the morphism res is defined by Bloch's moving lemma. In fact, for each $\gamma \in CH^p(\mathfrak{X}^*, n)$ this lemma guarantees that there exists a precycle $\Gamma \in \mathcal{Z}^p(\mathfrak{X}, n)$ such that its restriction to $\mathfrak{X}^*$ is a higher cycle with the same class to γ. We can see that $res(\gamma) := \partial\Gamma$ is actually in $\mathfrak{X}_{sing}$.

4.2 Singularities and Limits for semistable degenerations

Now we consider the special case that $\mathfrak{X}$ is a one-parameter semistable degeneration. It means that S is a projective curve and each singular fiber $X_{s_0} \subset \mathfrak{X}_{sing}$ is a reduced simple normal crossing divisor. Take a point in the discriminant locus $s_0 \in \overline{S} \setminus S$ and let $\Delta \subset \overline{S}$ be the unit disk in a local coordinate of $\overline{S}$ with the origin s_0. By changing the coordinate of S if we need, we may assume that X_{s_0} is the unique singular fiber in the restriction $\mathfrak{X}|_\Delta$.

The upper half plane $\mathfrak{H}$ can be considered the universal cover of $\Delta^* := \Delta \setminus \{0\}$. With the base change $\mathfrak{X}_\mathfrak{H} := \mathfrak{X}|_\Delta \times_{\Delta^*} \mathfrak{H}$, we obtain the commutative specialization diagram

$$\begin{array}{ccccc} \mathfrak{X}_\mathfrak{H} & \xrightarrow{k} & \mathfrak{X}|_\Delta & \xleftarrow{i} & X_{s_0} \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{H} & \longrightarrow & \Delta & \longleftarrow & \{0\}. \end{array}$$

Since $\mathfrak{X}_{\mathfrak{H}}$ is homotopic to any general fiber X_t ($t \neq 0$), we can define the specialization map

$$sp\colon H^k(X_{s_0}, \mathbb{Q}) \to H^k(\mathfrak{X}_{\mathfrak{H}}, \mathbb{Q}) \cong H^k(X_t, \mathbb{Q})$$

induced by the adjoint morphism $\mathbb{Q}_{X_{s_0}} \to i^* Rk_* k^* \mathbb{Q}_{\mathfrak{X}|_\Delta}$. We remark that originally this map is defined analytically by Clemens' retraction $\mathfrak{X}|_\Delta \to X_{s_0}$, but generally this retraction is not holomorphic ([19]. Also, since the local monodromy T around s_0 is unipotent, it defines the log monodromy action

$$N_{s_0} := \sum_{l=1}^{k} \frac{(-1)^{l-1}}{l}(T-I)^l \colon H^k(X_t, \mathbb{Q}) \to H^k(X_t, \mathbb{Q}).$$

In this setting, there exists a mixed Hodge structure on $H^k(X_t, \mathbb{Q})$ such that sp and N_{s_0} are morphisms of mixed Hodge structures of weight 0 and -1 respectively (with the usual mixed hodge structure on $H^k(X_{s_0}, \mathbb{Q})$). See [19] and Chapter 11 of [20]. This is called the limiting mixed Hodge structure (LMHS) and we denote $H^k_{lim}(X_t, \mathbb{Q})$ for $H^k(X_t, \mathbb{Q})$ equipped with LMHS. With this mixed Hodge structure, we obtain the Clemens-Schmid exact sequence:

Theorem 3 *[19] There is a long exact sequence of mixed Hodge structures*

$$\cdots \to H^k(X_0, \mathbb{Q}) \xrightarrow{sp} H^k_{lim}(X_t, \mathbb{Q}) \xrightarrow{N_{s_0}} H^k_{lim}(X_t, \mathbb{Q}(-1))$$
$$\xrightarrow{\alpha} H_{2(d-1)-k}(X_{s_0}, \mathbb{Q}(-d)) \xrightarrow{\phi} H^{k+2}(X_{s_0}, \mathbb{Q}) \to \cdots .$$

Here, ϕ is the composition of the Poincaré-Lefschetz duality $H_{2(d-1)-k}(X_{s_0}, \mathbb{Q}(-d)) \cong H^{k+2}(\mathfrak{X}|_\Delta, \mathfrak{X}|_{\Delta^*}; \mathbb{Q})$, the natural morphism $H^{k+2}(\mathfrak{X}|_\Delta, \mathfrak{X}|_{\Delta^*}; \mathbb{Q}) \to H^{k+2}(\mathfrak{X}|_\Delta, \mathbb{Q})$ and the isomorphism $H^{k+2}(\mathfrak{X}|_\Delta, \mathbb{Q}) \cong H^{k+2}(X_{s_0}, \mathbb{Q})$. Moreover α factors through $H^{k+1}(\mathfrak{X}|_{\Delta^*}, \mathbb{Q})$.

Since $\dim S = 1$, $\iota_s^* Rj_* \mathcal{H}^{p,n}$ is quasi-isomorphic to the complex

$$\{H_{lim}^{2p-n-1}(X_t, \mathbb{Q}(p)) \xrightarrow{N_s} H_{lim}^{2p-n-1}(X_t, \mathbb{Q}(p-1))\}$$

([21]). Hence in this case, we may consider the singularities and limits as the invariants in $\mathrm{Hdg}(\mathrm{Coker} N_s) := \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, \mathrm{Coker} N_s)$ and $J_{lim,s} := \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, \mathrm{Ker}(N_s)) \cong J(H_{lim}^{2p-n-1}(X_t, \mathbb{Q}(p)))$ respectively. As an extension class, we can represent the admissible normal function ν by a short exact sequence

$$0 \to \mathcal{V} \to \mathcal{E}_\nu \to \mathbb{Q}_S \to 0$$

of variations of mixed Hodge structures with the underlying local systems

$$0 \to \mathbb{V} \to \mathbb{E}_\nu \to \mathbb{Q}_S \to 0.$$

Deligne's extension $\tilde{\mathbb{E}}_\nu := e^{-\frac{1}{2\pi i}\log s N_s}\mathbb{E}_\nu$ defines the extension $\mathcal{E}_{\nu,e} := \tilde{\mathbb{E}}_\nu \otimes \mathcal{O}_\Delta$ and the admissibility of ν means that $\nu = \nu_F - \nu_{\mathbb{Q}}$ in the Jacobian bundle with a lift ν_F and $\nu_{\mathbb{Q}}$ of 1 to $\mathcal{E}_{\nu,e}$ and $\tilde{\mathbb{E}}_{\nu,0}$ respectively such that $\nu_F|_{\Delta^*}$ is in the Hodge filtration $\mathcal{F}^0(\mathcal{E}_\nu)$ and $N\nu_{\mathbb{Q}}$ is in the monodromy weight filtration $W_{-2}\tilde{\mathbb{V}}_0$. With these notations, specifically the singularity at $s_0 = 0 \in \Delta$ is given by

$$sing_{s_0}(\nu) = [N\nu_{\mathbb{Q}}](\equiv [N\nu_F(0)]).$$

Let $\mathfrak{Z}^*$ be a higher cycle over $\mathfrak{X}^*$. If the general fiber X_t is a smooth projective variety, $\mathfrak{Z}^*$ is in $CH^p(\mathfrak{X}^*, n)_{prim}$ since the generalized Hodge classes vanish. By Theorem 2, it defines the admissible normal function $\nu_{\mathfrak{Z}^*}$. Hence we obtain a map $\mathcal{AJ}^{p,n}\colon CH^p(\mathfrak{X}^*, n) \to$

$NF(S, \mathcal{H}^{p,n})^{\mathrm{ad}}_{\overline{S}} \otimes \mathbb{Q}$. On the other hand, since S is a curve, the codomain of singularities is $\mathrm{Hdg}(\mathrm{Coker}(N_{s_0}))$ for $N_{s_0} \colon H^{2p-n-1}_{lim}(X_t, \mathbb{Q}(p)) \to H^{2p-n-1}_{lim}(X_t, \mathbb{Q}(p-1))$ as we have seen above . By Theorem 3, this group can be regarded as a subgroup of $H_{2d-2p+n-1}(X_{s_o}, \mathbb{Q}(p-d))$. We denote

$$\mathrm{Hdg}_{p-1,n-1}(X_{s_0}) \coloneqq \mathrm{Hdg}(H_{2d-2p+n-1}(X_{s_o}, \mathbb{Q}(p-d))).$$

Note that the Poincaré-Lefschetz duality isomorphism induces the natural map

$$\beta \colon \mathrm{Hdg}(H^{2p-n+1}_{X_{s_0}}(\mathfrak{X}, \mathbb{Q}(p))) \to \mathrm{Hdg}_{p-1,n-1}(X_{s_0})$$

since the isomorphism is a morphism of mixed Hodge structures.

Since each singular fiber is a simple normal crossing divisor, we can consider the cycle map from $CH^{p-1}(X_{s_0}, n-1)$ as the end of the previous section. With this map, we obtain a relation of $res(\mathfrak{Z})$ and $sing_{s_0}(\nu_{\mathfrak{Z}^*})$:

Proposition 4 *Suppose $\mathfrak{X} \to \overline{S}$ be a semistable degeneration of smooth projective varieties and $n \geq p$ or $p \geq d$. Then for each $s_0 \in \overline{S} \setminus S$, there is a commutative diagram*

$$\begin{array}{ccccc}
CH^p(\mathfrak{X}^*, n) & \xrightarrow{\mathcal{AJ}^{p,n}} & NF(S, \mathcal{H}^{p,n}_f)^{\mathrm{ad}}_{\overline{S}} \otimes \mathbb{Q} & \xrightarrow{sing_{s_0}} & \mathrm{Hdg}(\mathrm{Coker}(N_{s_0})) \\
\downarrow{\scriptstyle res} & \searrow^{cl^{p,n}} & & & \\
CH^{p-1}(\mathfrak{X}_{sing}, n-1) & & \mathrm{Hdg}^{p,n}(\mathfrak{X}^*) & & \downarrow{\scriptstyle \mathrm{Hdg}(\alpha)} \\
\downarrow{\scriptstyle i^*_{s_0}} & & & \searrow^{\mathrm{Hdg}(r)} & \\
CH^{p-1}(X_{s_0}, n-1) & \xrightarrow{cl^{p-1,n-1}} & \mathrm{Hdg}(H^{2p-n+1}_{X_{s_0}}(\mathfrak{X}, \mathbb{Q}(p))) & \xrightarrow{\beta} & \mathrm{Hdg}_{p-1,n-1}(X_{s_0}).
\end{array}$$

Here, i_{s_0} is the inclusion $X_{s_0} \subset \mathfrak{X}_{sing}$ and r is defined as the composition of natural maps $H^{2p-n}(\mathfrak{X}^*, \mathbb{Q}(p)) \to H^{2p-n}(\mathfrak{X}^*|_{\Delta^*}, \mathbb{Q}(p)) \xrightarrow{res_{s_0}} H^{2p-n+1}(\mathfrak{X}|_\Delta, \mathfrak{X}|_\Delta \setminus X_{s_0}; \mathbb{Q}(p))$ and the Poincaré-Lefschetz duality isomorphism.

Proof The commutativity of the lower triangular diagram follows from the functoriality of the cycle maps for the pull back and residue maps. To see that of the upper triangular diagram, recall that the image of the cycle map $cl^{p,n}(\mathfrak{Z}^*)$ for a given familly $\mathfrak{Z}^*$ in $CH^p(\mathfrak{X}^*, n)$ is obtained by the class of a topological cycle $[(2\pi i)^p T_{\mathfrak{Z}'^*}]$ via the KLM formula (for the complement of a normal crossing divisor) $cl^{p,n}_{\mathcal{D}}(\mathfrak{Z}^*) = [(2\pi i)^{p-n}((2\pi i)^n T_{\mathfrak{Z}'^*}, \Omega_{\mathfrak{Z}'^*}, R_{\mathfrak{Z}'^*})]$ with a representative $\mathfrak{Z}'^*$ in $\mathcal{Z}^p_{\mathbb{R}}(\mathfrak{X}^*, n)$ of $\mathfrak{Z}^*$. Hence $\mathrm{Hdg}(r) \circ cl^{p,n}(\mathfrak{Z}^*)$ coincides with the dual of $(2\pi i)^p res_{s_0}([T_{\mathfrak{Z}'^*}])$.

On the other hand, we can take a chain Γ_t with $\partial \Gamma_t = T_{Z_t}$ since $[T_{Z_t}] = 0$ on each general fiber X_t. From the assumption $n \geq p$ or $p \geq d$, $\Omega_{\mathfrak{Z}^*} = 0$, and hence we may simplify the triple for $cl^{p,n}_{\mathcal{D}}(Z_t)$ to $(0, 0, R'_{Z_t} := (2\pi i)^{p-n} R_{Z_t} + (2\pi i)^p \delta_{\Gamma_t})$ by adding $D((2\pi i)^p \Gamma_t, 0, 0) = (0, 0, (2\pi i)^p \delta_{\Gamma_t})$. Therefore $\nu(t) := AJ^{p,n}(Z_t) = \nu_{\mathbb{Q}}(t) - \nu_F(t)$ can be represented by the family of currents $\{R'_{Z_t}\}$, on whose class $[R'_{Z_t}]$ the Gauss-Manin connection ∇ is computed by locally, lifting the $\{R'_{Z_t}\}$ to $R'_{\mathfrak{Z}^*_U}$ and applying d to get $\Omega_{\mathfrak{Z}^*_U}$. Hence $\nabla \nu = [\Omega_{\mathfrak{Z}^*}] = cl^{p,n}(\mathfrak{Z}^*)$. It is well-known that $res_{s_0}(\nabla) = -2\pi i N$, therefore

$$\begin{aligned} sing_{s_0} \circ \mathcal{AJ}^{p,n}(\mathfrak{Z}^*) = N\nu_F(0) &= (-2\pi i)^{-1} res_{s_0}(\nabla)(\nu_F(0)) \\ &= (2\pi i)^{-1} res_{s_0}(\nabla)(\nu) \\ &= (2\pi i)^{-1} res_{s_0}(\nabla \nu) = (2\pi i)^{-1}(res_{s_0} \circ cl^{p,n})(\mathfrak{Z}^*). \end{aligned}$$

Since α is a morphism of type (1-d,1-d) it coincides with the above computation.

■

By Theorem 3, Coker(N_{s_0}) is isomorphic to Ker(ϕ). Hence we obtain the composition

$$\widetilde{sing}_{s_0} := \mathrm{Hdg}(\beta) \circ cl^{p-1,n-1} \circ i_{s_0}^* \circ res \colon CH^p(\mathfrak{X}^*, n) \to \mathrm{Hdg}(\mathrm{Ker}(\phi)).$$

Corollary 1 *In the situation of the above proposition, $sing_{s_0}(\nu_{\mathfrak{Z}^*}) \neq 0$ if and only if $\widetilde{sing}_{s_0}(\mathfrak{Z}^*) \neq 0$.*

We also can describe the limit invariant $lim_{s_0}(\nu_{\mathfrak{Z}^*})$ as follows under the assumption that $res(\mathfrak{Z}^*)$ vanishes. Since the specialization map $sp\colon H^{2p-n-1}(X_{s_0}, \mathbb{Q}(p)) \to H^{2p-n-1}_{lim}(X_t, \mathbb{Q}(p))$ is a morphism of MHS, it induces a map $J(sp)\colon J(X_{s_0}) \to J_{lim,s_0}$. Now $lim_{s_0}(\nu_{\mathfrak{Z}^*})$ is an invariant in the right hand side, but we also can extend $\mathfrak{Z}^*$ to a higher cycle $\mathfrak{Z}$ in $CH^p(\mathfrak{X}, n)$. It defines the pullback Z_0 in $H^{2p-n}_{\mathcal{M}}(X_{s_0}, \mathbb{Q}(p))$. Then

Theorem 4 *[4]*

$$lim_{s_0}(\nu_{\mathfrak{Z}^*}) = J(sp)(AJ_{X_{s_0}}(Z_0)).$$

Especially, when we have a family of Hodge classes $\omega(s)$ in $\mathrm{Hdg}(H_{2p-n-1}(X_t, \mathbb{Q}(-p)))$ such that it lifts to a class on $\mathfrak{X}^*$ with non-trivial residue on X_0, dually it induces a splitting

$$\eta\colon H^{2p-n-1}(X_0, \mathbb{Q}(p)) \twoheadrightarrow \mathbb{Q}(p)$$

of the morphism of MHS. The analytic limit of the paring $\langle \nu_{\mathfrak{Z}}(s), \omega(s) \rangle$ can be obtained as the period

$$\lim_{s \to s_0} \langle \nu_{\mathfrak{Z}}(s), \omega(s) \rangle \equiv J(\eta)(AJ_{X_0}(Z_0))$$

in $J(\mathbb{Q}(p)) \cong \mathbb{C}/\mathbb{Q}(p)$.

Remark 2 *More generally, the above theorem does not require the SSD condition. ([4], Section 5.3).*

5. Higher Cycles on a certain types of $K3$ surface

5.1 Construction of Families of Higher Cycles

In this chapter, we consider a certain family of degree d surfaces $\mathcal{X}$ in $\mathbb{P}^3$ of a general form. Over this type of family, we can construct a family of higher cycles in $CH^2(X_t, 1)$ for the general fiber X_t. We classify these elements into families of decomposable cycles $\mathscr{D}$ and other two types of families of higher cycles $\mathscr{I}_0, \mathscr{I}_\infty$, which are $\mathbb{R}$-regulator indecomposable when $d = 4$. In the case of quartic surfaces, in later sections we will prove the Hodge-$\mathcal{D}$-conjecture for a general fiber X_t of $\mathcal{X}$ by showing that the images of $\mathscr{I}_0$ and $\mathscr{I}_\infty$ by $r^{2,1}_{\mathcal{D},\mathbb{R}}$ span the regulator indecomposable cycles $\mathrm{Coker}(\mu) \cong H^{1,1}_{tr}(X, \mathbb{R}(1))$.

Let L_i $(1 \leq i \leq d)$ and M_l $(1 \leq l \leq d)$ be linear forms in $\mathbb{P}^3$ in general position. Define a flat family of degree d surfaces $\mathcal{X}$ over $\mathbb{P}^1$ by

$$\mathcal{X} := \{X_t \colon L_1 L_2 \cdots L_d + t M_1 M_2 \cdots M_d = 0\}_{t \in \mathbb{P}^1} \subset \mathbb{P}^3 \times \mathbb{P}^1.$$

The base locus B of this family is obtained by

$$B = \bigcup_{1 \leq i \leq d} B_i \qquad (B_i := \bigcup_{1 \leq l \leq d} L_i \cap M_l).$$

Its general fiber X_t is smooth, and $X_0 = (L_1 L_2 \cdots L_d = 0)$ is a simple normal crossing divisor on $\mathcal{X}$. In fact, each point of X_0 has an analytic neighborhood with coordinates such that X_t is defined by the equation $xy + tz = 0$ or simpler (the same holds for X_∞). Hence the base locus B includes no singular points on X_t. Near X_0, this local equation also shows that the singular loci of the total family $\mathcal{X}$ are given by $d\binom{d}{2}$ nodes defined by $p_l^{ij} := L_i \cap L_j \cap M_l$. We denote the projection to the parameter t by $\pi\colon \mathcal{X} \to \mathbb{P}^1$ and also define $S := \mathbb{P}^1 \setminus (\text{discriminant locus})$ and $\mathcal{X}^* := \pi^{-1}(S)$. We write $\mathcal{L}_i, \mathcal{M}_l \subset \mathcal{X}$ for the constant families of planes defined by L_i and M_l respectively.

We start by constructing some decomposable cycles. Recall that each element of $CH^2(X,1)$ can be represented by a formal sum of pairs of divisors and rational functions over them such that the sum of their zeros and poles vanishes. Take a constant family of lines $\mathcal{L}_i \cap \mathcal{M}_l$ as a divisor of $\mathcal{X}^*$. Since $\mathcal{X}^*$ does not include either X_0 or X_∞, the projection π is an invertible function over $\mathcal{X}^*$. Hence its restriction $\pi|_{\mathcal{L}_i \cap \mathcal{M}_l}$ defines an element of $\mathbb{C}^*$ via the identification $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) \cong \mathbb{C}^*$. Thus the pair $(\pi|_{\mathcal{L}_i \cap \mathcal{M}_l}, \mathcal{L}_i \cap \mathcal{M}_l)$ defines a family $\lambda_{il} \in CH^2(\mathcal{X}^*, 1)$ of decomposable cycles via the map (2.1). We define

$$\mathscr{D} := \{\lambda_{il} \mid 1 \le i, l \le d\}.$$

Next we define $\mathscr{I}_0$. Take three planes $L_i, L_j, L_k (1 \le i < j < k \le d)$ and another one M_l. For each $\alpha \in \{i, j, k\}$, again we take the divisor $\mathcal{L}_\alpha \cap \mathcal{M}_l$, but for the rational function we take an isomorphism $\phi_{\alpha l}\colon L_\alpha \cap M_l \xrightarrow{\cong} \mathbb{P}^1$ on each $t \in \mathbb{P}^1$ defined by

$$\phi_{\alpha l} = \frac{L_{\sigma(\alpha)}}{L_{\sigma^2(\alpha)}}.$$

Here, $\sigma \in \mathfrak{S}_3$ is the cyclic permutation defined by $\begin{pmatrix} i & j & k \\ j & k & i \end{pmatrix}$. Hence $\phi_{\alpha l}^{-1}(0) = L_\alpha \cap L_{\sigma(\alpha)} \cap M_l$ and $\phi_{\alpha l}^{-1}(\infty) = L_\alpha \cap L_{\sigma^2(\alpha)} \cap M_l$. We use the same notation $\phi_{\alpha l}$ for the rational function over $\mathcal{X}$ defined by $\phi_{\alpha l}$ constantly with respect to t. Then we obtain a precycle

$$\Gamma_{\alpha l} \coloneqq (\phi_{\alpha l}, \mathcal{L}_\alpha \cap \mathcal{M}_l) \in \mathcal{Z}^2(\mathcal{X}^*, 1).$$

By the definition of $\phi_{\alpha l}$, $\partial(\Gamma_{\alpha l})$ is the divisor $[\mathcal{L}_i \cap \mathcal{L}_{\sigma(\alpha)}] - [\mathcal{L}_i \cap \mathcal{L}_{\sigma^2(\alpha)}]$. Hence the precycle

$$\gamma_{ijk,l} \coloneqq \Gamma_{il} + \Gamma_{jl} + \Gamma_{kl}$$

satisfies $\partial(\gamma_{ijk,l}) = 0$. Thus we obtain a higher cycle $\gamma_{ijk,l} \in CH^2(\mathcal{X}^*, 1)$. We also use the same notation $\gamma_{ijk,l} \in CH^2(X_t, 1)$ for each fiber at $t = 0$ (Figure 1). We define

$$\mathscr{I}_0 \coloneqq \{\gamma_{ijk,l} \mid 1 \le i < j < k \le d, 1 \le l \le d\}.$$

Finally, $\mathscr{I}_\infty$ is defined by changing L and M in the above construction of $\mathscr{I}_0$. Specifically, for three planes M_l, M_m, M_n and L_i and for each $\beta \in \{l, m, n\}$, we take an isomorphism $\psi_{i\beta}\colon L_i \cap M_\beta \xrightarrow{\cong} \mathbb{P}^1$ such that $\psi_{i\beta}^{-1}(0) = L_i \cap M_\beta \cap M_{\sigma(\beta)}$ and $\psi_{i\beta}^{-1}(\infty) = L_i \cap M_\beta \cap M_{\sigma^2(\beta)}$. Then it defines a precycle $\Gamma'_{i\beta} \coloneqq (\psi_{i\beta}, \mathcal{L}_i \cap \mathcal{M}_\beta)$ and we can see that

$$\delta_{i,lmn} \coloneqq \Gamma'_{il} + \Gamma'_{im} + \Gamma'_{in}$$

is also an element of $CH^2(\mathcal{X}^*, 1)$. We define

$$\mathscr{I}_\infty \coloneqq \{\delta_{i,lmn} \mid 1 \le i \le d, 1 \le l < m < n \le d\}.$$

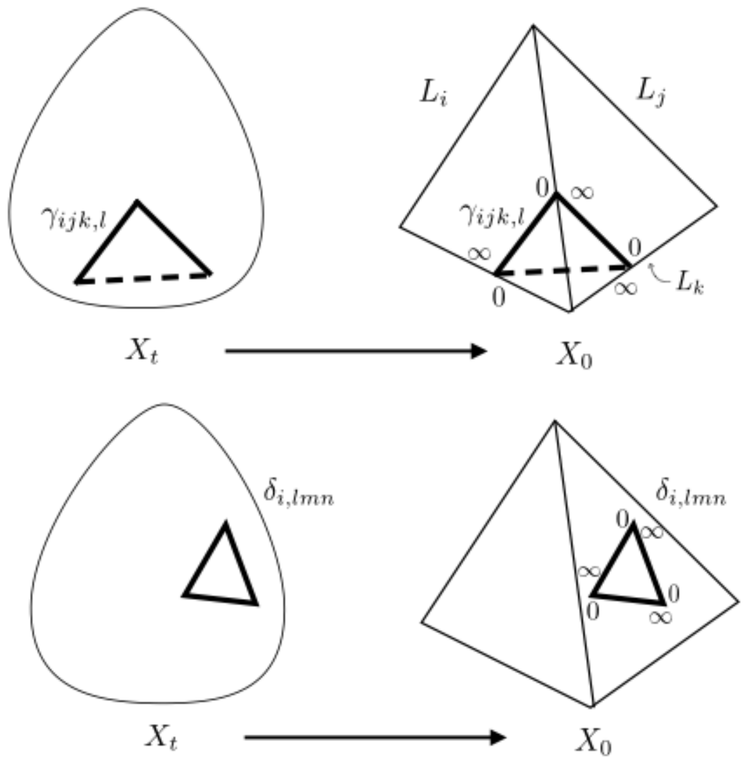

Figure 5.1. Higher Chow cycle $\gamma_{ijk,l}$ and $\delta_{i,lmn}$

5.2 Higher Chow Cycles with Non-trivial Singularities

We use the same notations as Section 5.1. In this section we shall prove the following statement.

Theorem 5 *Let* $*$ *be* 0 *or* ∞*. For general choices of* $\{L_i\}$ *and* $\{M_l\}$*,* $sing_*\circ\mathcal{AJ}^{2,1}(\mathscr{I}_*\cup\mathscr{D})$ *spans* $\mathrm{Hdg}(\mathrm{Coker}(N_*))\subset\mathrm{Hdg}_{1,0}(X_*)$.

To apply the discussions in the previous section, first of all, we resolve the singularities of $\mathcal{X}$ to obtain a semistable degeneration family $\tilde{\mathcal{X}}$. Recall that $\mathcal{X}$ has $d\binom{d}{2}$ nodal singularities $\{p_l^{ij}\}_{1\le i<j\le d, 1\le l\le d}$ which are included in the base locus $B = \bigcup B_i$. If we denote $\mathbb{P}^3 = P^0$ and define a successive blow up P^i of $\mathbb{P}^3$ inductively by the blow up $b_i\colon P^i \to P^{i-1}$ of P^{i-1} along the strict transformation of B_i in P^{i-1}, then the composition $b_i \circ b_{i-1} \circ \cdots \circ b_1$ defines a strict transformation $\mathcal{X}^i \to \mathcal{X}$. Since each $p_l^{ij} \in B_i$ is a node, this strict transformation resolves p_l^{ij}. We define a smooth family $\tilde{\mathcal{X}}$ by taking a resolution of the remaining singularities in $\mathcal{X}^d$. Denote the composition of these resolutions by $b\colon \tilde{\mathcal{X}} \to \mathcal{X}$. Though a singular fiber $\tilde{X}_{t_0}$ of $\tilde{\mathfrak{X}}$ may not be a simple normal crossing unless $t_0 = 0$ or ∞, by the semistable reduction theorem ([22]), we may assume that $\tilde{\pi}\colon \tilde{\mathcal{X}} \to S$ with a finite cover $S \to \mathbb{P}^1$ is a semistable degeneration family after repeating base changes and desingularizations. Note that $\tilde{X}_t \cong X_t$ and $\tilde{X}_0 \cong (\mathcal{X}^d)_0$ is given by adding the exceptional curve $E_l^{ij} := b^{-1}(p_{ij,l})$ to $L_j \subset X_0$. More precisely, for the strict transformation $\tilde{L_j}$ of L_j,

$$\mathrm{Pic}(\tilde{L_j}) \cong b^*\mathrm{Pic}(L_j) \oplus (\bigoplus_{i<j,l} \mathbb{Z}[E_l^{ij}]) \cong (\mathbb{Z}l_j) \oplus (\bigoplus_{i<j,l} \mathbb{Z}e_l^{ij}).$$

Here, l_j is the divisor class of the general line in $L_j \cong \mathbb{P}^2$ and e_l^{ij} is that of $E_l^{ij} \cong \mathbb{P}^1$.

Now, we have the invariant $\widetilde{sing}_0\colon CH^2(\tilde{\mathcal{X}}, 1) \to \mathrm{Hdg}(\mathrm{Ker}(\phi))$ by the discussion in the previous section. Since both X_* and $\mathscr{I}_*$ ($* = 0, \infty$) have the symmetry by replacing each linear form L_i with M_i and M_l with L_l, we also can consider $\widetilde{sing}_\infty$ with another blow up $b'\colon \tilde{\mathcal{X}}' \to \mathcal{X}$ defined by replacing $B_i = \bigcup(L_i \cap M_l)$ by $B_i' := \bigcup(M_i \cap L_l)$ in the above construction of $\tilde{\mathcal{X}}$. From now on we consider only $\widetilde{sing}_0$, but one can obtain exactly the same result for $\widetilde{sing}_\infty$ by replacing linear forms.

Lemma 2 *Consider* l_i *and* e_l^{ij} *as elements in* $H_2(\tilde{X}_0, \mathbb{Q}(-1))$. *Then a basis of the* $\mathbb{Q}$-*vector space* $\mathrm{Hdg}(\mathrm{Ker}(\phi)) \subset \mathrm{Hdg}(H_2(\tilde{X}_0, \mathbb{Q}(-1)))$ *is given by*

$$\mathcal{B} := \left\{ \sum_{1 \le i \le d} l_i \right\} \cup \left\{ \sum_{1 \le l' \le d} (e_l^{ij} - e_{l'}^{ij}) \right\}_{1 \le i < j \le d, 1 \le l \le (d-1)}$$

In particular,

$$\dim(\mathrm{Hdg}(\mathrm{Coker}(N_0))) = \mathrm{Hdg}(\mathrm{Ker}(\phi)) = 1 + (d-1)\binom{d}{2}$$

.

Proof We firstly find a basis of $\mathrm{Hdg}(H_2(\tilde{X}_0, \mathbb{Q}(-1))) = H_2(\tilde{X}_0, \mathbb{Q})^{(-1,-1)}$, and then find that of $\mathrm{Hdg}(\mathrm{Ker}(\phi))$. For the simplicity we denote $Y := \tilde{X}_0$, $Y_I := \bigcap_{i \in I} \tilde{L}_i$ and $Y^{[k]} := \coprod_{|I|=k+1} Y_I$.

The weight spectral sequence in this case is given by dualizing that for cohomology groups ([23]):

$$E^1_{p,q} = H_q(Y^{[p]}, \mathbb{Q}) \Rightarrow H_{p+q}(Y, \mathbb{Q}).$$

Since it degenerates at E^2 and differentials for cohomology groups are compatible with the Gysin morphisms,

$$\mathrm{Gr}^W_{-2}(H_2(X_0, \mathbb{Q})) = E^\infty_{0,2} \cong \mathrm{Coker}(d_1 \colon H_2(Y^{[1]}, \mathbb{Q}) \to H_2(Y^{[0]}, \mathbb{Q}))$$

and the differential d_1 is given by the natural morphism.

Since the strict transformation $\widetilde{L_i \cap L_j}$ is isomorphic to the original line $L_i \cap L_j$, $H_2(\widetilde{L_i \cap L_j}, \mathbb{Q})^{(-1,-1)}$ $(i < j)$ is generated by the unique class l_{ij}. Via $d_1 \colon H_2(Y^{[1]}, \mathbb{Q}) \to$

$H_2(Y^{[0]}, \mathbb{Q})$, each of (l_{ij}) induces a relation in $H_2(Y^{[0]}, \mathbb{Q})$. To see this relation, we should represent l_{ij} in each of $\mathrm{Pic}(\tilde{L}_i)$ and $\mathrm{Pic}(\tilde{L}_j)$ with respect to the above basis. The intersection products for each $i < j$ are given by

$$
\begin{aligned}
&(l_i \cdot l_i)_{\tilde{L}_i} = 1 \\
&(l_j \cdot e_l^{ij})_{\tilde{L}_j} = 0 \\
&(e_l^{ij} \cdot e_{l'}^{i'j})_{\tilde{L}_j} = \begin{cases} -1 & ((i,l) = (i',l')) \\ 0 & ((i,l) \neq (i',l')), \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&(l_{ij} \cdot l_i)_{\tilde{L}_i} = (l_{ij} \cdot l_j)_{\tilde{L}_j} = 1 \\
&(l_{ij} \cdot e_l^{ij})_{\tilde{L}_j} = 1 \\
&(l_{ij} \cdot e_l^{i'j})_{\tilde{L}_j} = 0 \quad (i \neq i').
\end{aligned}
$$

Hence we can see that

$$
\begin{aligned}
&l_{ij} = l_i \quad \text{in} \quad \mathrm{Pic}(\tilde{L}_i) \\
&l_{ij} = l_j - (\sum_l e_l^{ij}) \quad \text{in} \quad \mathrm{Pic}(\tilde{L}_j).
\end{aligned}
$$

Therefore

$$
H_2(Y, \mathbb{Q})^{(-1,-1)} \cong \langle\ \{l_i\}_{1 \leq i \leq d}, \{e_l^{ij}\}_{1 \leq j < i \leq d, 1 \leq l \leq d}\ \rangle \ /\ \{l_j - l_i = \sum_{1 \leq l \leq d} e_l^{ij}\}_{1 \leq i < j \leq d}.
$$

Since each relation is independent from others, it also shows that

$$
\dim(H_2(Y, \mathbb{Q})^{(-1,-1)}) = d + d\binom{d}{2} - \binom{d}{2} = d\big(1 + \frac{(d-1)^2}{2}\big).
$$

To find the required basis of the Hdg(Kerϕ), recall that ϕ is induced by the Poincaré-Lefschetz duality, and hence is defined by taking the intersection products:

$$\phi\colon \alpha \mapsto (\alpha \cdot \_)_{\tilde{\mathcal{X}}} \quad (\alpha \in H_2(Y, \mathbb{Q}(-1))).$$

By the transversality, we obtain

$$(l_i \cdot [\tilde{L}_{i'}])_{\tilde{\mathcal{X}}} = 1 \quad (i' \neq i)$$

$$(e_l^{ij} \cdot [\tilde{L}_{i'}])_{\tilde{\mathcal{X}}} = \begin{cases} 1 & (i' = i) \\ \\ 0 & (i' \neq i, j). \end{cases}$$

Moreover, since the graph of the map $\tilde{X} \to \mathbb{P}^1$ defines an algebraic cycle in $Z^1(\tilde{\mathcal{X}} \times \mathbb{P}^1)$, $Y = \tilde{X}_0$ and $\tilde{X}_t$ are rationally equivalent. Since $\tilde{X}_t \cap V = \emptyset$ for any subvariety V of Y, $(\alpha \cdot Y)_{\tilde{\mathcal{X}}} = (\alpha \cdot \sum \tilde{L}_i)_{\tilde{\mathcal{X}}} = 0$ for any $\alpha \in H_2(Y, \mathbb{Q})$. Thus we also can see that

$$(l_i \cdot [\tilde{L}_i])_{\tilde{\mathcal{X}}} = (l_i \cdot [Y] - [\sum_{i' \neq i} \tilde{L}_{i'}])_{\tilde{\mathcal{X}}} = -(d-1)$$

$$(e_l^{ij} \cdot [\tilde{L}_j])_{\tilde{\mathcal{X}}} = (e_l^{ij} \cdot [Y] - [\sum_{j' \neq j} \tilde{L}_{j'}])_{\tilde{\mathcal{X}}} = -1.$$

Summarizing the computation, we obtain the following intersection matrix:

$$\left(\begin{array}{c|cccccccccccc}
 & l_1 & l_2 & \dots & l_d & e_1^{12} & e_2^{12} & \dots & e_{d-1}^{12} & e_1^{13} & e_2^{13} & \dots & e_{d-1}^{d-1,d} \\
\hline
\tilde{L}_1 & -(d-1) & 1 & \dots & 1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 0 \\
\tilde{L}_2 & 1 & -(d-1) & \dots & 1 & -1 & -1 & \dots & -1 & 0 & 0 & \dots & 0 \\
\tilde{L}_3 & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & -1 & -1 & \dots & 0 \\
\vdots & & & & & & \vdots & & & & & & \\
\tilde{L}_{d-1} & 1 & 1 & \dots & 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\
\tilde{L}_d & 1 & 1 & \dots & -(d-1) & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -1
\end{array}\right)$$

Note that we do not need to consider e_d^{ij}, since it is generated by the above other classes via the relation $l_j - l_i = \sum_l e_l^{ij}$. It is easy to check that $\mathcal{B}$ in the statement is a basis of the kernel of this matrix. In fact, by the intersection products we can see that the $1+(d-1)\binom{d}{2}$ elements of $\mathcal{B}$ are linearly independent. On the other hand, since the columns for $(e_1^{1j})_j$ generate any other columns, $\dim(\mathrm{Hdg}(\mathrm{Ker}(\phi))) = d\big(1+\frac{(d-1)^2}{2}\big) - (d-1) = 1+(d-1)\binom{d}{2}$.

■

We use the following lemma later to prove Lemma 4.

Lemma 3 *the equation* $L_{\sigma(\alpha)} = 0$ *in* $\mathcal{N}_{il} := \widetilde{\mathcal{L}_i \cap \mathcal{M}}_l \subset \tilde{\mathcal{X}}$ *defines the divisor class*

$$e_l^{\alpha,\sigma(\alpha)} \times \{0\} + [p_l^{\alpha,\sigma(\alpha)} \times \mathbb{P}^1] \quad \text{if} \quad \alpha < \sigma(\alpha)$$

$$[p_l^{\sigma(\alpha),\alpha} \times \mathbb{P}^1] \quad \text{if} \quad \alpha > \sigma(\alpha).$$

Proof Since the statement is local, we take the local coordinates $(x, y, z; t) \in \mathbb{A}^3 \times \mathbb{P}^1$ on an analytic open set $\mathcal{U} \subset \mathcal{X}$ about $p^{\alpha,\sigma(\alpha)}$. By taking a sufficiently small $\mathcal{U}$, we may assume that

$$x = L_\alpha, \quad y = L_{\sigma(\alpha)}, \quad z = M_l$$

and

$$\mathcal{U} = \{xy + tz = 0\} \subset \mathbb{A}^3 \times \mathbb{P}^1.$$

In particular, $p^{\alpha,\sigma(\alpha)}$ is the unique singular point in $\mathcal{U}$.

If we assume $\alpha < \sigma(\alpha)$, then only the α-th blow up b_α changes $\mathcal{U}$. Note that $b_{\sigma(\alpha)}$ is isomorphic over $\mathcal{U}$ since the node $p^{\alpha,\sigma(\alpha)}$ has already been resolved. Therefore, the strict transformation $\tilde{\mathcal{U}} \subset \tilde{\mathcal{X}}$ of $\mathcal{U}$ is isomorphic to the strict transformation $Xy + tZ = 0$ via the blow up of $\mathbb{A}^3 \times \mathbb{P}^1$ along $x = z = 0$. Here, $[X : Z] \in \mathbb{P}^1$ is the blow up coordinate. Then $\mathcal{N}_{il} = \{x = z = Xy + tZ = 0\} \subset \mathbb{A}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$ defines a smooth curve for each $t \neq 0$, but it degenerates to two lines $\{x = y = z = 0\}$ and $\{x = z = X = 0\}$ as t goes to 0. Hence the function $L_{\sigma(\alpha)} = y = 0$ defines two lines $\{x = y = z = t = 0\}$ and $\{x = y = z = X = 0\}$. The former one is $E_l^{\alpha,\sigma(\alpha)} \times \{0\}$ and the latter is $p_l^{\alpha,\sigma(\alpha)} \times \mathbb{P}^1$.

If $\alpha > \sigma(\alpha)$, we should change x and y in the above discussion. Specifically, we may assume that $\mathcal{N}_{il}$ is defined by the closure of $\{x = z = xY + tZ = Yz - Zy = 0\} \cap \big(\mathbb{A}^3 \times \mathbb{P}^1 \times \mathbb{P}^1 \setminus \{y = z = 0\}\big) = \{x = z = Z = 0\} \cap \{y \neq 0\}$. Hence $\mathcal{N}_{il} = \{x = z = Z = 0\}$ and the function $L_{\sigma(\alpha)} = y = 0$ defines only one line $\{x = y = z = Z = 0\} = p_l^{\sigma(\alpha),\alpha} \times \mathbb{P}^1$. ∎

By taking the strict transformation of each subvariety in $\mathcal{X}^* \times \mathbb{P}^n$ by the blow up $\tilde{\mathcal{X}} \times \mathbb{P}^n \to \mathcal{X} \times \mathbb{P}^n$, we obtain higher cycles $\tilde{\mathfrak{Z}} \in CH^p(\tilde{\mathcal{X}}^*, n)$ from each $\mathfrak{Z} \in CH^p(\mathcal{X}^*, n)$. We denote $\widetilde{sing}_0(\mathfrak{Z}) \coloneqq \widetilde{sing}_0(\tilde{\mathfrak{Z}})$

Remark 3 *Since $\tilde{\mathcal{X}}^*$ is isomorphic to $\mathcal{X}^*$, $\mathcal{H}_{\pi}^{p,n} \cong \mathcal{H}_{\tilde{\pi}}^{p,n}$. Moreover, $\nu_{\tilde{\mathfrak{Z}}} = \nu_{\mathfrak{Z}}$ via this isomorphism by the functoriality of the Abel-Jacobi map. Hence we can identify their singular invariants.*

Lemma 4 *For each $1 \leq i < j < k \leq d$ and $1 \leq l \leq d$,*

(i) $\widetilde{sing}_0(\gamma_{ijk,l}) = e_l^{ij} + e_l^{jk} - e_l^{ik}$

(ii) $\widetilde{sing}_0(\lambda_{il}) = l_i - \left(\sum_{i'<i} e_l^{i'i}\right) + (\sum_{i'>i} e_l^{ii'})$.

Proof (i) Recall that the higher cycle $\gamma_{ijk,l}$ is consisted by the graph in $\mathcal{X}^* \times \mathbb{P}^1$ of $\phi_{\alpha l}$. This rational function also defines a graph $\Gamma_{\phi_{\alpha l}}$ on $\mathcal{X} \times \mathbb{P}^1$. Let $\tilde{\Gamma}_{\phi_{\alpha l}}$ be the strict transformation of $\Gamma_{\phi_{\alpha l}}$, so that $\tilde{\gamma}_{ijk,l} = (\tilde{\Gamma}_{\phi_{il}} + \tilde{\Gamma}_{\phi_{jl}} + \tilde{\Gamma}_{\phi_{kl}})|_{\mathcal{X}^*}$. By the definition of $res\colon CH^2(\tilde{\mathcal{X}}^*, 1) \to CH^1(\tilde{\mathcal{X}}_{sing})$, $res(\tilde{\gamma}_{ijk,l}) = \partial(\tilde{\Gamma}_{\phi_{il}} + \tilde{\Gamma}_{\phi_{jl}} + \tilde{\Gamma}_{\phi_{kl}})$.

We now compute $\partial(\tilde{\Gamma}_{\phi_{\alpha l}})$. For $\bullet = 0, \infty$, let $\rho_\bullet\colon \tilde{\mathcal{X}} \hookrightarrow \tilde{\mathcal{X}} \times \mathbb{P}^1$ be the natural embedding at 0 or ∞. Then, by the definition of $\phi_{\alpha l}$, the pullback $\rho_\bullet^*(\tilde{\Gamma}_{\phi_{\alpha l}}) = (\pi_{\tilde{\mathcal{X}}})_*(\tilde{\Gamma}_{\phi_{\alpha l}} \cdot \tilde{\mathcal{X}} \times \{\bullet\})_{\tilde{\mathcal{X}} \times \mathbb{P}^1}$ is given by the equation

$$L_{\sigma(\alpha)} = 0 \quad \text{for} \quad \bullet = 0$$

$$L_{\sigma^2(\alpha)} = 0 \quad \text{for} \quad \bullet = \infty$$

over $\mathcal{N}_{il} \subset \tilde{\mathcal{X}}$. Therefore, by applying Lemma 3 with $\alpha = i, j, k$ respectively, we obtain

$$\partial(\tilde{\Gamma}_{\phi_{il}}) = [e_l^{ij} \times \{0\}] + [p_l^{ij} \times \mathbb{P}^1] - [e_l^{ik} \times \{0\}] - [p_l^{ik} \times \mathbb{P}^1]$$

$$\partial(\tilde{\Gamma}_{\phi_{jl}}) = [e_l^{jk} \times \{0\}] + [p_l^{jk} \times \mathbb{P}^1] - [p_l^{ij} \times \mathbb{P}^1]$$

$$\partial(\tilde{\Gamma}_{\phi_{kl}}) = [e_l^{ik} \times \{0\}] - [e_l^{jk} \times \{0\}] - [p_l^{jk} \times \mathbb{P}^1]$$

and hence

$$\partial(\tilde{\Gamma}_{\phi_{il}} + \tilde{\Gamma}_{\phi_{jl}} + \tilde{\Gamma}_{\phi_{kl}}) = e_l^{ij} + e_l^{jk} - e_l^{ik}$$

via the identification of $X_0 \times \{0\}$ with X_0.

(ii) Since λ_{il} is defined with the rational function $\pi\colon \mathcal{X} \to \mathbb{P}^1$, $\partial(\tilde{\lambda}_{il})|_{X_0} = \widetilde{L_i \cap M_j} \subset \tilde{X}_0$. In the proof of Lemma 3, we have seen that $\mathcal{N}_{il}$ degenerates to $(d-i)$ exceptional curves $\{E_{i'l}\}_{i'>i}$ and the other one $C_{il} \subset \tilde{L}_i$, which is isomorphic to the strict transformation of $L_i \cap M_l$ by the blow up of L_i at $d(i-1)$ points $\{p_l^{i'i}\}_{i'<i}$. By the intersection products

$$(l_i \cdot [C_{il}])_{\tilde{L}_i} = 1$$

$$(e_l^{i'i} \cdot [C_{il}])_{\tilde{L}_i} = 1,$$

we can see that $[C_{il}] = l_i - \left(\sum_{i'<i} e_l^{i'i}\right)$ in $\mathrm{Pic}(\tilde{L}_i)$. Therefore

$$[\widetilde{L_i \cap M_j}] = [C_{il}] + [\bigoplus_{i'>i} E_{i'l}] = l_i - \big(\sum_{i'<i} e_l^{i'i}\big) + \big(\sum_{i'>i} e_l^{ii'}\big)$$

in $\mathrm{Pic}(\tilde{X}_0)$. ■

Proof [Proof of Theorem 5] From Corollary 1 and the remark above Lemma 4, we should show that the basis in Lemma 2 is in $\widetilde{sing}_0(\mathscr{I}_* \cup \mathscr{D})$. As mentioned before Theorem 5, the discussions for $\widetilde{sing}_0$ and $\widetilde{sing}_\infty$ are exactly the same. Hence we prove only for the case $* = 0$.

Firstly, we can see that

$$\widetilde{sing}_0(\sum_{1\le i\le d}\lambda_{il}) = \sum_{1\le i\le d} l_i$$

for each l by Lemma 4 and direct computation. Hence it suffices to show that $\sum_{1\le l'\le d}(e^{ij}_l - e^{ij}_{l'})$ is in the span of $\widetilde{sing}_0(\mathscr{I}_* \cup \mathscr{D})$ for each $1 \le i < j \le d$ and $1 \le l, l' \le d$. Since $l_j - l_i = \sum_{1\le l'\le d} e^{ij}_{l'}$ in $H_2(Y,\mathbb{Q})^{(-1,-1)}$,

$$\begin{aligned}
\widetilde{sing}_0(\lambda_{il}-\lambda_{jl}) &= (l_i - l_j) + 2e^{ij}_l - (\sum_{k<i} e^{ki}_l - e^{kj}_l) + (\sum_{i<k<j} e^{ik}_l + e^{jk}_l) + (\sum_{j<k} e^{ik}_l - e^{jk}_l)\\
&= -(\sum_{1\le l'\le d} e^{ij}_{l'}) + de^{ij}_l - (\sum_{k<i} e^{ki}_l - e^{ij}_l - e^{kj}_l)\\
&\quad + (\sum_{i<k<j} e^{ik}_l + e^{jk}_l - e^{ij}_l) + (\sum_{j<k} -e^{ij}_l + e^{ik}_l - e^{jk}_l)\\
&= -(\sum_{1\le l'\le d} e^{ij}_l - e^{ij}_{l'}) - \sum_{k<i}\gamma_{kij,l} + \sum_{i<k<j}\gamma_{ikj,l} - \sum_{j<k}\gamma_{ijk,l}.
\end{aligned}$$

Therefore

$$\sum_{1\le l'\le d}(e^{ij}_l - e^{ij}_{l'}) = \widetilde{sing}_0(\lambda_{jl} - \lambda_{il} - \sum_{k<i}\gamma_{kij,l} + \sum_{i<k<j}\gamma_{ikj,l} - \sum_{j<k}\gamma_{ijk,l}).$$

■

From the above computations, specifically we obtain the expression of the singularities relative to the basis $\mathcal{B}$:

$$\widetilde{sing}_0(\gamma_{ijk,l}) = \frac{1}{d}\left(\sum_{l'}(e^{ij}_l - e^{ij}_{l'}) + \sum_{l'}(e^{jk}_l - e^{jk}_{l'}) - \sum_{l'}(e^{ik}_l - e^{ik}_{l'})\right)$$

$$\widetilde{sing}_0(\lambda_{il}) = \frac{1}{d}\left(\sum_i l_i - \sum_{i'<i}\left(\sum_{l'}(e_l^{ii'} - e_{l'}^{ii'})\right) + \sum_{i'>i}\left(\sum_{l'}(e_l^{ii'} - e_{l'}^{ii'})\right)\right)$$

Hence the singularity of $\gamma_{ijk,l}$ is linearly independent from the singularities of $\mathscr{D}$. This implies

Corollary 2 *Each of $\gamma_{ijk,l}$ is an $\mathbb{R}$-regulator indecomposable cycle.*

5.3 Higher Chow Cycles with Non-trivial Limits

Theorem 2 shows that $\mathscr{I}_0$ and $\mathscr{I}_\infty$ have non-trivial singularities, but at different singular fibers X_0 and X_∞ respectively. From the construction of each higher cycle in $\delta_{i,lmn}$ in $\mathscr{I}_\infty$, it is clear that its singularity at X_0 is trivial. To show the linearly independence of $\mathscr{I}_0 \cup \{\delta_{i,lmn}\}$, we compute the limit invariant of $\delta_{i,lmn}$ at X_0.

We use the same notation $Y = \tilde{X}_0$, $Y_I = \bigcap_{i\in I}\tilde{L}_i$ as before and also denote $Y^I \coloneqq \bigcup_{j\notin I} Y_{I\cup\{j\}}$ Recall that the motivic cohomology of the simple normal crossing divisor $Y = \tilde{X}_0$ is obtained by

$$H^{2p-n}_{\mathcal{M}}(Y, \mathbb{Q}(p)) = H^{-n}(Z^\bullet_Y(p)).$$

Here, we take a subgroup $Z^p_\#(Y_I, \bullet) \coloneqq Z^p_{\mathbb{R}}(Y_I, \bullet)_{Y^I} \subset Z^p(Y_I, \bullet)$ which consists of the precycles in good position with respect to Y^I ([9], Section 8) and $Z^\bullet_Y(p)$ is the associated simple complex to the double complex

$$Z^{k,m}_Y(p) = \bigoplus_{|I|=k+1} Z^p_\#(Y_I, -m)$$

with Bloch's differential $\partial_{\mathcal{B}}$ and the alternating sum $\partial_{\mathcal{I}}$ of the pullbacks by the inclusion $Y_{I\cup\{j\}} \hookrightarrow Y_I$. Similarly, the normal currents $D^{\bullet}_{\#}(Y_I)$ and integral currents $C^{\bullet}_{\#}(Y_I, \mathbb{Q}(p))$ denotes the associated simple complex $K^{\bullet}_Y(p)$ of a double complex

$$K_Y^{k,m} := \bigoplus_{|I|=k+1} \{C_{\#}^{2p+m}(Y_I, \mathbb{Q}(p)) \oplus F^p D_{\#}^{2p+m}(Y_I) \oplus D_{\#}^{2p+m-1}\}$$

and the Deligne cohomology can be obtained by

$$H_{\mathcal{D}}^{2p-n}(Y, \mathbb{Q}(p)) = H^{-n}(K_Y^{\bullet}(p)).$$

When the class of a higher cycle Z in $H_{\mathcal{M}}^{2p-n}(Y, \mathbb{Q}(p))$ can be represented by

$$\{Z_I^{[k]} \in Z_Y^{k,-k-n}(p)\}_{k,|I|=k+1},$$

the componentwise KLM formula

$$\{(2\pi i)^{p-k}((2\pi i)^k T_{Z_I^{[k]}}, \Omega_{Z_I^{[k]}}, R_{Z_I^{[k]}})\}$$

in $K_Y^{-n}(p)$ induces $AJ_Y^{p,n}(Z)$.

Now, the strict transformation $\widetilde{\delta}_{i,lmn}$ in Y_I of $\delta_{i,lmn}$ satisfies this condition and hence we can show

Lemma 5 *When $d = 4$, $AJ_Y^{2,1}(\widetilde{\delta}_{i,lmn})$ is non-trivial in $H_{\mathcal{D}}^3(Y, \mathbb{Q}(1))$.*

Proof Consider the moduli of the families $\mathcal{X}$. Since the choices of the linear forms are general, it suffices to show the non-triviality of $AJ_Y^{2,1}(\widetilde{\delta}_{i,lmn})$ for a particular family in this moduli space. For the simplicity we assume that $i = 4, l = 1, m = 2, n = 3$, and we choose the linear forms

$$
\begin{aligned}
&L_1\colon X = 0 && M_1\colon X + \mu Y - Z + W = 0\\
&L_2\colon Y = 0 && M_2\colon \mu X - Y + Z + W = 0\\
&L_3\colon Z = 0 && M_3\colon -X + Y + \mu Z + W = 0\\
&L_4\colon W = 0 && M_4\colon X + Y + Z - \mu W = 0.
\end{aligned}
$$

Here, μ is the primitive 6th root of unity $\frac{1+\sqrt{3}i}{2}$. This family is an example of a *tempered* family, a notion which is defined in Section 3 of [3] for more general toric hypersurfaces. For our case, this condition is equivalent to each p_i^{lm} having the root of unity coordinates with respect to $[X : Y : Z : W]$. A crucial point of the smooth tempered families of toric hypersurafaces which is defined by a reflexive Newton polytope is that the natural Hodge class

$$\frac{1}{(2\pi i)^n} d\log x_1 \wedge d\log x_2 \wedge \dots d\log x_n \in H^n((\mathbb{C}^*)^n, \mathbb{Q}(n))$$

defined by the toric coordinate symbol $\{x_1, x_2, \dots, x_n\} \in H^n_{\mathcal{M}}((\mathbb{C}^*)^n, \mathbb{Q}(n))$ can be extended to the Hodge class on the family itself. Therefore if we take 2-form

$$\omega \coloneqq \left(\frac{1}{2\pi i}\right)^2 \frac{dx}{x} \wedge \frac{dy}{y}$$

with $(x, y) \coloneqq (X/Z, Y/Z)$ for each general fiber X_t, dually it defines a family $\{\omega(t) \in \mathrm{Hdg}(H_2(X_t, \mathbb{Q}(-2)))\}$. By the KLM formula, we shall compute the membrane integral of this test 2-form ω on the triangle Γ whose edges are the strict transformations of the three lines

$$L_4 \cap M_1 \colon X + \mu Y - Z = 0$$

$$L_4 \cap M_2 \colon \mu X + Y + Z = 0$$

$$L_4 \cap M_3 \colon -X + Y + \mu Z = 0$$

coming from $\widetilde{\delta}_{i,lmn}$. Then

$$\begin{aligned} AJ_{i,lmn} \coloneqq AJ_Y^{2,1}(\widetilde{\delta}_{i,lmn})(\omega) &= (-2\pi i)\left(\int_{L_i} R_{\widetilde{\delta}_{i,lmn}} \wedge \omega + (2\pi i)\int_\Gamma \omega\right) \\ &= \left((-2\pi i)\int_{\widetilde{\delta}_{i,lmn}} \log(t)\omega\right) - \left((2\pi i)^2 \int_\Gamma \omega\right). \end{aligned}$$

However, its first term vanishes since dx and dy are linearly dependent on $\widetilde{\delta}_{i,lmn}$. Since the vertices of Γ with respect to the coordinates (x, y) are given by

$$p_i^{lm} = (-\mu, 2-\mu), \quad p_i^{ln} = (i\sqrt{3}, \mu^2), \quad p_i^{mn} = (\tfrac{1}{3}(1+\mu), -\tfrac{1}{\sqrt{3}}i),$$

we obtain

$$\begin{aligned} -AJ_{i,lmn} &= \int_\Gamma \frac{dx}{x} \wedge \frac{dy}{y} \\ &= \int_{-\frac{1}{\sqrt{3}}i}^{2-\mu} \left(\int_{y+\mu}^{-\mu y+1} \frac{dx}{x}\right)\frac{dy}{y} + \int_{2-\mu}^{\mu^2} \left(\int_{y+\mu}^{\frac{1}{\mu}y-\frac{1}{\mu}} \frac{dx}{x}\right)\frac{dy}{y} \\ &= \left(\int_{-\frac{1}{\sqrt{3}}i}^{2-\mu} \frac{\log(-\mu y+1)}{y} dy\right) + \left(\int_{2-\mu}^{\mu^2} \frac{\log(\frac{1}{\mu}y - \frac{1}{\mu})}{y} dy\right) - \left(\int_{-\frac{1}{\sqrt{3}}i}^{\mu^2} \frac{\log(y+\mu)}{y} dy\right). \end{aligned}$$

Generally, the integral of a multivalued function $\frac{\log(a+bz)}{z}$ $(a, b \in \mathbb{C})$ is

$$\int \frac{\log(a+bz)}{z} dz = -\mathrm{Li}_2(-\frac{b}{a}z) + \log(z)\left(\log(a+bz) - \log(1+\frac{b}{a}z)\right).$$

with the dilogarithm function Li_2. By applying this integral to each term of $-AJ_{i,lmn}$, we can see that

$$\int_{-\frac{1}{\sqrt{3}}i}^{2-\mu} \frac{\log(-\mu y+1)}{y}dy = -\mathrm{Li}_2\left(1+\mu\right) + \mathrm{Li}_2\left(\frac{1}{1+\mu}\right)$$
$$\int_{2-\mu}^{\mu^2} \frac{\log(\frac{1}{\mu}y-\frac{1}{\mu})}{y}dy = \left(-\mathrm{Li}_2\left(\frac{1}{-\mu}\right) + \mathrm{Li}_2(1-\mu^2)\right) + \left(-\frac{2}{9}\pi^2 + \frac{2}{3}i\pi\log 3\right)$$
$$-\int_{-\frac{1}{\sqrt{3}}i}^{\mu^2} \frac{\log(y+\mu)}{y}dy = \left(\mathrm{Li}_2(-\mu) - \mathrm{Li}_2\left(\frac{1}{1-\mu^2}\right)\right) + \left(\frac{7}{18}\pi^2 - \frac{1}{6}i\pi\log 3\right).$$

and hence

$$\begin{aligned}-AJ_{i,lmn} =& \mathrm{Li}_2(-\mu) - \mathrm{Li}_2\left(\frac{1}{-\mu}\right) + \mathrm{Li}_2\left(\frac{1}{1+\mu}\right) - \mathrm{Li}_2\left(1+\mu\right)\\ &+ \mathrm{Li}_2(1-\mu^2) - \mathrm{Li}_2\left(\frac{1}{1-\mu^2}\right) + \left(\frac{1}{6}\pi^2 + \frac{1}{2}i\pi\log(3)\right).\end{aligned}$$

To compute the dilogarithm terms, we also use functional equations

$$\mathrm{Li}_2\left(\frac{z-1}{z}\right) - \mathrm{Li}_2\left(z\right) = -\frac{1}{6}\pi^2 + \log(z)\log(1-z) - \frac{1}{2}\log(z)^2$$
$$\mathrm{Li}_2\left(\frac{1}{1-z}\right) - \mathrm{Li}_2\left(z\right) = \frac{1}{6}\pi^2 + \log(-z)\log(1-z) - \frac{1}{2}\log(1-z)^2,$$

for z which is not on the branch cuts. Note that μ satisfies the equations

$$\frac{1}{1+\mu} = \frac{1}{1-(-\mu)},\ 1+\mu = \frac{(-\frac{1}{\mu})-1}{-\frac{1}{\mu}},\ 1-\mu^2 = \frac{(-\mu)-1}{-\mu},\ \frac{1}{1-\mu^2} = \frac{1}{1-(-\frac{1}{\mu})}.$$

Hence we can show

$$\begin{aligned}
\mathrm{Li}_2\left(\frac{1}{1+\mu}\right) - \mathrm{Li}_2\left(1+\mu\right) =& \mathrm{Li}_2(-\mu) - \mathrm{Li}_2\left(\frac{1}{-\mu}\right) \\
&+ \frac{1}{3}\pi^2 - \frac{3}{8}\pi^2 - \frac{\log(3)^2}{8} - \frac{1}{4}i\pi\log(3) \\
\mathrm{Li}_2(1-\mu^2) - \mathrm{Li}_2\left(\frac{1}{1-\mu^2}\right) =& \mathrm{Li}_2(-\mu) - \mathrm{Li}_2\left(\frac{1}{-\mu}\right) - \frac{1}{3}\pi^2 \\
&+ \frac{3}{8}\pi^2 + \frac{\log(3)^2}{8} - \frac{1}{4}i\pi\log(3).
\end{aligned}$$

With $-\frac{1}{\mu} = \overline{-\mu}$, finally we obtain

$$\begin{aligned}
-AJ_{i,lmn} &= 3(\mathrm{Li}_2(-\mu) - \mathrm{Li}_2(\overline{-\mu})) + \left(-\frac{1}{2}i\pi\log(3)\right) + \left(\frac{1}{6}\pi^2 + \frac{1}{2}i\pi\log(3)\right) \\
&= 3(\mathrm{Li}_2(-\mu) - \overline{\mathrm{Li}_2(-\mu)}) + \zeta(2).
\end{aligned}$$

Since the first term is purely imaginary and non zero, it shows that $AJ_{i,lmn}$ is non-trivial in $\mathbb{C}/\mathbb{Q}(2)$. ■

Since $\{\omega(t)\}$ is the family of Hodge classes, as we see at the end of Section 5, we have

$$\lim_{t\to 0}\langle \nu_{\delta_{\mathrm{i.lmn}}}(t), \omega(t)\rangle \equiv AJ_{i,lmn} \in \mathbb{C}/\mathbb{Q}(2).$$

From the above lemma, the right hand side is non-trivial and hence we have proven

Theorem 6 *Suppose $d = 4$. For general choices of $\{L_i\}$ and $\{M_l\}$, $\nu_{\delta_{i.lmn}}$ has non-trivial limit. Especially the higher cycles $\{\delta_{i.lmn}\}\cup\mathscr{I}_0\cup\mathscr{D}$ are linearly independent in $CH^2(X_t, 1)$.*

5.4 Hodge-$\mathcal{D}$-Conjecture for a certain type of $K3$ surfaces

In this section we consider the case that $d = 4$. Hence X_t is a $K3$ surface with the form

$$X_t \colon L_1L_2L_3L_4 + tM_1M_2M_3M_4 = 0,$$

and $H^3_{\mathcal{D}}(X_t, \mathbb{R}(2)) \cong H^{1,1}_{\mathbb{R}}(X_t)(1)$ is 20-dimensional. Though the real regulator map is generally not injective, by computing the limit of real regulator values we can see that the image of 20 higher cycles $\{\delta_{i.lmn}\} \cup \mathscr{I}_0 \cup \mathscr{D}$ actually spans this vector space.

Theorem 7 *When $d = 4$ and $\{L_i\}, \{M_l\}$ are very general, $r^{2,1}_{\mathcal{D},\mathbb{R}}(\{\delta_{i.lmn}\} \cup \mathscr{I}_0 \cup \mathscr{D})$ are linearly independent in $H^{1,1}_{\mathbb{R}}(X_t)(1)$, explicitly validating the Hodge-$\mathcal{D}$-Conjecture this case.*

Proof Since Hdg(Coker(N)) is 19 dimensional, Hdg(Ker(N)) is also 19 dimensional. For a fixed X_t, take a basis $d_1, \ldots, d_{19}$ of Hdg(Ker(N)). By Theorem 5, the images of linear combinations of higher cycles in $\mathscr{I}_0 \cup \mathscr{D}$ give these classes. We also take an element $\gamma_2 \in H_{lim} := H^2_{lim}(X_t, \mathbb{Q}(2))$ which does not vanish in $\mathrm{Gr}^W_4 H_{lim}$. Denote $\gamma_1 := N\gamma_2$, $\gamma_0 := N^2\gamma_2$. Though each $\{\gamma_i\}$ defines a multivalued section of the cohomology sheaf $\mathcal{H}^2$, from them we can define single valued sections by

$$e'_i := e^{-l(t)N}\gamma_i(t).$$

with $l(t) := \frac{log(t)}{2\pi i}$. Specifically $e'_0 = \gamma_0$, $e'_1 = \gamma_1 - l(t)\gamma_0$, and $e'_2 = \gamma_2 - l(t)\gamma_1 + \frac{l^2(t)}{2}\gamma_0$. Hence $\{e'_0, e'_1, e'_1, d_1, \ldots, d_{19}\}$ is a single valued frame of the extension $\mathcal{H}^2_e$. Since d_i is already a single valued section (in other word, $d_i = e^{-l(t)N} d_i(t)$), we obtain a single valued frame

$$\{e'_0, e'_1, e'_1, d_1, \ldots, d_{19}\}$$

of the cohomology sheaf $\mathcal{H}^2_e$.

Take a holomorphic section $\omega(t) \in F^2(H^2_{lim,\mathbb{C}})$ such that $\omega \neq 0$ in Gr^W_4. Generally $\omega(t)$ can be written as

$$\omega(t) = e'_2 + f(t)e'_1 + g(t)e'_0 + \sum_{i=1}^{19} h_i(t)d_i$$

with holomorphic functions $f(t), g(t), h_i(t)$ by normalizing ω with respect to the coefficient of e'_2. By changing t to the new coordinate $t' := te^{-2\pi i f(t)}$ (hence $l(t') = l(t) - f(t)$), we define e_i from e'_i:

$$e_i(t') := e^{f(t)N} e_i(t) = e^{-l(t')N}\gamma_i(t).$$

Note that this shift of the parameter does not change d_i. Hence

$$\begin{aligned}
\omega(t) &= (\gamma_2 - l(t)\gamma_1 + \frac{l^2(t)}{2}\gamma_0) + f(t)(\gamma_1 - l(t)\gamma_0) + g(t)\gamma_0 + \sum h_i(t)d_i \\
&= (\gamma_2 - l(t')\gamma_1 + + \frac{l^2(t')}{2}\gamma_0) + (g(t) - \frac{f^2(t)}{2})\gamma_0 + g(t)\gamma_0 + \sum h_i(t)d_i \\
&= e_2 + (g(t) - \frac{f(t)^2}{2})e_0 + \sum h_i(t)d_i.
\end{aligned}$$

For the simplicity, we use the notation t for t' instead of the original coordinate from here. Then, by changing f, g, h_i to new functions, we can write ω as

$$\omega(t) = e_2 + g(t)e_0 + \sum h_i(t)d_i = e_2 + \kappa(t)$$

with $\kappa(t) \coloneqq g(t)e_0 + \sum h_i(t)d_i \in \mathrm{Ker}N$. Though e_2 may not be in $F^2(H^2_{lim,\mathbb{C}})$, we obtain

$$e_1 = Ne_2 = N\omega$$

since $\kappa(t) \in \mathrm{Ker}N$. Hence $e_1 \in F^1 \cap W_2(H^2_{lim,\mathbb{C}})$ and $e_0 = Ne_1 \in F^0 \cap W_0(H^2_{lim,\mathbb{C}})$.

By the definition, it is easy to check that the quadratic form $Q(e_i, e_j)$ for the polarization is given by the matrix

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

after a normalization of γ_2. Also note that the conjugates satisfy the equalities

$$\overline{e_0} = e_0$$

$$\overline{e_1} = e_1 + 2i\Im(l)e_0$$

$$\overline{e_2} = e_2 + 2i\Im(l)e_1 - 2(\Im(l))^2 e_0$$

$$\overline{d_i} = d_i.$$

with the imaginary part $\Im(l) = -\frac{\log|t|}{2\pi}$ of $l(t)$.

Take a non zero element $\eta \in H^{1,1}_{lim,\mathbb{R}}$ which is linearly independent from $d_1, \ldots, d_{19} \in H^{1,1}_{lim,\mathbb{R}}$. We shall express η by using e_0, e_1, e_2. Since $\eta \in F^1$, there exists a C^∞ function $\phi(t)$ such that

$$\eta = \omega + \phi(t)e_1 = e_2 + \phi(t)e_1 + \kappa(t).$$

η is also a real form, hence

$$\eta = \overline{\eta} = (e_2 + 2i\Im(l)e_1 - 2(\Im(l))^2 e_0) + \overline{\phi(t)}(e_1 + 2i\Im(l)e_0) + \overline{\kappa(t)} \in F^1(H^2_{lim}).$$

Specifically $\overline{\kappa(t)} = \overline{g(t)}e_0 + \sum \overline{h_i(t)}d_i$, hence this term does not include any e_1 term. Thus we can compare the e_1 terms of η and $\overline{\eta}$ to obtain $\Im(\phi(t)) = \Im(l)$. Also, the coefficient of the e_0 term of $2\eta = \eta + \overline{\eta}$ is given by $2\Re(g(t)) - 2(\Im(l))^2 + 2i\overline{\phi(t)}\Im(l)$, which must be a real number. This implies that $\phi(t)$ is pure imaginary. Therefore

$$\phi(t) = i\Im(l)$$

and hence

$$\eta = e_2 + i\Im(l)e_1 + \kappa(t).$$

Now, we compute our real regulator value $R(t)$ of $\delta_{i,lmn}$. We know that the singular invariant of $\delta_{i,lmn}$ is trivial and its limit invarinat is a pure imaginary number $iL \coloneqq AJ_{i,lmn} \in \mathbb{C}/\mathbb{Q}(2)$. Hence we may write

$$R(t) = iLe_0 + t\left(\alpha_0(t)e_0 + \alpha_1(t)e_1 + \alpha_2(t)e_2 + \sum_{j=1}^{19} \beta_j(t)d_j\right)$$

with holomorphic functions $\alpha_i(t), \beta_j(t)$. Hence

$$\begin{aligned}
\Im(R(t)) &= -\frac{i}{2}(R(t) - \overline{R(t)}) \\
&= -\frac{i}{2}\left(iLe_0 + t\left(\alpha_0(t)e_0 + \alpha_1(t)e_1 + \alpha_2(t)e_2 + \sum \beta_j(t)d_j\right)\right. \\
&\quad - (-iLe_0 + \bar{t}\left(\overline{\alpha_0(t)}e_0 + \overline{\alpha_1(t)}(e_1 + 2i\Im(l)e_0)\right. \\
&\quad \left.\left. + \overline{\alpha_2(t)}(e_2 + 2i\Im(l)e_1 - 2(\Im(l))^2e_0) + \sum \overline{\beta_j(t)}d_j\right)\right) \\
&= \left(L + \Im(t\alpha_0(t)) - \overline{t\alpha_1(t)}\Im(l) - i\overline{t\alpha_2(t)}(\Im(l))^2\right)e_0 + \\
&\quad \left(\Im(t\alpha_1(t)) - \overline{t\alpha_2(t)}\Im(l)\right)e_1 + \Im(t\alpha_2(t))e_2 + \sum \Im(t\beta_j(t))d_j.
\end{aligned}$$

Finally we consider the limit of $Q(\Im(R(t)), \eta)$ as $t \to 0$. Note that $d_i \in \mathrm{Hdg}(\mathrm{Ker}(N))$ is orthogonal to each of e_0, e_1, e_2. With the notation $q_{ij} := Q(d_i, d_j)$, hence we obtain

$$\begin{aligned}
Q(\Im(R(t)), \eta) = Q\Big(&\Big(L + \Im(t\alpha_0(t)) - \overline{t\alpha_1(t)}\Im(l) - i\overline{t\alpha_2(t)}(\Im(l))^2\Big) e_0 + \\
&\Big(\Im(t\alpha_1(t)) - \overline{t\alpha_2(t)}\Im(l)\Big) e_1 + \Im(t\alpha_2(t))e_2, e_2 + i\Im(l)e_1 + g(t)e_0\Big) \\
&+ Q\left(\sum \Im(t\beta_j(t))d_j, \sum h_i(t)d_i\right) \\
&= -(L + \Im(t\alpha_0(t)) - \overline{t\alpha_1(t)}\Im(l)) + i\Im(t\alpha_1(t))\Im(l) - g(t)\Im(t\alpha_2) \\
&+ \sum_{i,j} h_i(t)\Im(t\beta_j(t))q_{ij} \\
&= -L - \Im(t\alpha_0(t)) + \Re(t\alpha_1(t))\Im(l) - g(t)\Im(t\alpha_2) + \sum_{i,j} h_i(t)\Im(t\beta_j(t))q_{ij}.
\end{aligned}$$

This value goes to $-L$ as $t \to 0$. On the other hand, for each d_i,

$$Q(\Im(R(t)), d_i) = Q(\sum_j \Im(t\beta_j(t))d_j, d_i) = \sum_j \Im(t\beta_j(t))q_{ij}$$

goes to 0 as $t \to 0$. Hence we conclude that

$$\lim_{t\to 0} Q(\Im(R(t)), \eta) = -L$$
$$\lim_{t\to 0} Q(\Im(R(t)), d_j) = 0.$$

This shows that $r^{2,1}_{\mathcal{D},\mathbb{R}}(\{\delta_{i.lmn}\} \cup \mathscr{I}_0 \cup \mathscr{D})$ are linearly independent. In fact, a linear combination of $\mathscr{I}_0 \cup \mathscr{D}$ defines an admissible normal function $R_i(t)$ for each i $(1 \leq i \leq 19)$ with $sing_0(R_i(t)) = d_i$. This function has a form

$$R_i(t) := \alpha_0(t)e_0 + \alpha_1(t)e_1 + \alpha_2(t)e_2 + i\log(t)d_i + \sum_{j\neq i} \beta_j(t)d_j.$$

and the admissibility implies that each $\alpha_i(t)$ and $\beta_j(t)$ is a holomorphic function ([24], Proposition 5.28). Therefore

$$\lim_{t\to 0} Q(\Im(\frac{1}{\log(t)}R_i(t)), d_j) = \begin{cases} 1 & (i=j) \\ \\ 0 & (i\neq j) \end{cases}$$

and moreover

$$\lim_{t\to 0} Q(\Im(\frac{1}{\log(t)}R_i(t)), \eta) \text{ is a finite number } C_i.$$

In fact,

$$\begin{aligned} \Im\left(\frac{R_i(t)}{\log(t)}\right) = &\left(\Im(\frac{\alpha_0}{\log(t)}) - \Im(l)\overline{\frac{\alpha_1}{\log(t)}} - i(\Im(l))^2\overline{\frac{\alpha_2}{\log(t)}}\right) e_0 \\ &+ \left(\Im(\frac{\alpha_1}{\log(t)}) - \Im(l)\overline{\frac{\alpha_2}{\log(t)}}\right) e_1 \\ &+ \Im(\frac{\alpha_2}{\log(t)})e_2 + d_i + \sum_{j\neq i}\Im(\frac{\beta_2}{\log(t)})d_j, \end{aligned}$$

hence the only non-vanishing term of $\Im\left(\frac{1}{\log(t)}R_i(t)\right)$ as $t\to 0$ is

$$-\left(\Im(l)\overline{\frac{\alpha_1}{\log(t)}}e_0 + \Im(l)\overline{\frac{\alpha_2}{\log(t)}}e_1 + d_i\right)$$

and its paring with $\eta = e_2 + i\Im(l)e_1 + \kappa(t)$ is finite. Summarizing them, the matrix defined by paring $\lim_{t\to 0} Q(\_,\_)$ is given as

$$\left(\begin{array}{c|ccccc} & \eta & d_1 & d_2 & \dots & d_{19} \\ \hline \Im(R(t)) & -L & 0 & 0 & \dots & 0 \\ \Im(\frac{1}{\log(t)}R_1(t)) & C_1 & 1 & 0 & \dots & 0 \\ \Im(\frac{1}{\log(t)}R_2(t)) & C_2 & 0 & 1 & \dots & 0 \\ \vdots & & & & \ddots & \\ \Im(\frac{1}{\log(t)}R_{19}(t)) & C_{19} & 0 & 0 & \dots & 1 \end{array}\right).$$

Hence its determinant is non-trivial and it implies that the real regulator value $\Im(R(t)), \Im(R_1(t)), \dots, \Im(R_{19}$ are linearly independent. ■

The above computation of the real regulator values especially show that

Corollary 3 *When $d = 4$, each of $\delta_{i,lmn}$ is an $\mathbb{R}$-regulator indecomposable cycle.*

Remark 4 *(1) If we assume the given VMHS is a nilpotent orbit and the family is tempered, the above computation in the proof is much simpler. In fact these conditions imply $\omega = e_2$ and hence $\eta = e_2 + i\Im(l)e_1$. To compute the pairing with this η and each d_j as $t \to 0$, we may assume that*

$$R(t) = iLe_0$$

$$\frac{R_i(t)}{\log(t)} = \frac{\alpha_1(t)}{\log(t)}e_1 + id_i.$$

Hence we obtain exactly the same matrix as above.

(2) Though we can construct the family of higher cycles $\delta_{i,lmn}$ even for general $d \geq 5$, there are two problems to apply the similar discussion to prove the Hodge-$\mathcal{D}$-Conjecture. Firstly, by applying an action of PGL_3 which maps the plane L_i and three lines $L_i \cap M_l$, $L_i \cap M_m$, $L_i \cap M_n$ to the special ones in the proof of Lemma 5, we may take exactly the same family of test forms $\{\omega(t)\}$. However, this 2-form may not be a Hodge class unless $\mathcal{X}$ is a tempered family after applying the action. Another issue is that generally $\dim(H^{1,1}_{\mathbb{R}}(X_t)) - \dim(\mathrm{Hdg}(\mathrm{Coker}N)) > 1$, *hence we need to show not only the non-triviality of $\delta_{i,lmn}$, but the linearly independence of some of $\{r^{2,1}_{\mathcal{D},\mathbb{R}}(\delta_{ijk,l})\}$ (for example, we need four linearly independent classes when $d = 5$). One possible approach to solve this point is to find an enough number of test Hodge classes such that the matrix of the paring* $\lim_{t\to 0} Q(\_,\_)$ *is regular. Finally here, we just state*

Conjecture 2 *For general d, $r^{2,1}_{\mathcal{D},\mathbb{R}}(\mathscr{I}_0 \cup \mathscr{I}_\infty \cup \mathscr{D})$ spans $H^{1,1}_{\mathbb{R}}(X_t)(1)$. Hence the Hodge-$\mathcal{D}$-Conjecture holds this case.*

6. Threefold with Non-Trivial Griffiths Groups

6.1 Construction of Non-Trivial Elements of Griffiths Groups

As an application of our main result in Chapter 5 , we construct non-trivial elements of the Griffiths group of a certain threefold which is constructed from X_t.

When we consider a proper smooth family $\mathcal{Y}$ over a quasi-projective curve S such that its completion $\overline{\mathcal{Y}} \to \overline{S}$ is also proper and $\overline{\mathcal{Y}}$ is smooth, a given family of cycles $\mathcal{Z}$ in $CH^p(\overline{\mathcal{Y}})$ defines the class of Z_t in $\mathrm{Griff}^p(Y_t)$ on each fiber Y_t ($t \in \overline{S}$). Since $AJ(CH^p_{alg}(Y_s)) \subset H^{2p-1}_{\mathrm{Hdg}}(Y_s, \mathbb{Q}(p))$, where $H^{2p-1}_{\mathrm{Hdg}}(Y_s, \mathbb{Q}(p))$ denotes the largest sub-Hodge structure of $H^{2p-1}(Y_s, \mathbb{Q}(p))$ in $H^{0,-1} \oplus H^{-1,0}$, the Abel-Jacobi map induces a map

$$\mathrm{Griff}^p(Y_s) \to J\left(H^{2p-1}(Y_s, \mathbb{Q}(p))/H^{2p-1}_{\mathrm{Hdg}}(Y_s, \mathbb{Q}(p))\right).$$

On the other hand, for each discriminant locus $0 \in \overline{S} \setminus S$, Z_0 is an element of the motivic cohomology $H^{2p}_{\mathcal{M}}(Y_0, \mathbb{Q}(p))$. Suppose that Y_0 is a SNCD with the strata $Y_0^{[k]}$, then we also obtain the induced map

$$CH^p_{\mathrm{ind}}(Y_0^{[1]}, 1) \to J(H^{2p-2}_{tr}(Y_0, \mathbb{Q}(p))) \cong J(H^{2p-2}(Y_0, \mathbb{Q}(p))/N^1 H^{2p-2}(Y_0, \mathbb{Q}(p))$$

by the Abel-Jacobi map. Note that the element $Z_0 \in W^{-1}H^{2p}_{\mathcal{M}}(Y_0, \mathbb{Q}(p)) := \mathrm{Ker}(H^{2p}_{\mathcal{M}}(Y_0, \mathbb{Q}(p)) \to H^{2p}_{\mathcal{M}}(Y_0^{[0]}, \mathbb{Q}(p)))$ also defines an element of $CH^p_{\mathrm{ind}}(Y_0^{[1]}, 1)$, since the degree 0 term of $Z^{\bullet}_{Y_0}(p)$ is the direct sum of the following boxed components:

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \boxed{Z^p_{\#}(Y_0^{[1]}, 1)} & \xrightarrow{\partial_{\mathcal{B}}} & Z^p_{\#}(Y_0^{[1]}) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \partial_{\mathcal{I}}} & & \uparrow{\scriptstyle \partial_{\mathcal{I}}} & & \\
\cdots & \longrightarrow & Z^p_{\#}(Y_0^{[0]}, 1) & \xrightarrow{\partial_{\mathcal{B}}} & \boxed{Z^p_{\#}(Y_0^{[0]})} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}
$$

With the analytic limit of the Abel-Jacobi value, hence we obtain the diagram

$$
\begin{array}{ccccc}
 & & \mathrm{Griff}^p(Y_s) & \xrightarrow{AJ} & J\left(\frac{H^{2p-1}(Y_s, \mathbb{Q}(p))}{H^{2p-1}_{\mathrm{Hdg}}(Y_s, \mathbb{Q}(p))}\right) \\
 & \nearrow^{\iota_s^*} & & & \\
W^{-1}H^{2p}_{\mathcal{M}}(\mathcal{Y}, \mathbb{Q}(p)) & & & & \downarrow{\scriptstyle \lim_{s \to s_0}} \\
 & \searrow_{\iota_0^*} & & & \\
 & & CH^p_{\mathrm{ind}}(Y_0^{[1]}, 1) & \xrightarrow{AJ} & J\left(\frac{H^{2p-2}(Y_0, \mathbb{Q}(p))}{N^1H^{2p-2}(Y_0, \mathbb{Q}(p))}\right).
\end{array}
$$

Since this diagram is commutative (Theorem 2.2 of [4]), if Z_0 defines an $\mathbb{R}$-regulator indecomposable cycle, it implies that Z_s is non-trivial in $\mathrm{Griff}^k(Y_s)$ for a general s.

Now, as the proper smooth family $\mathcal{Y}$, we take a resolution of the singular family $\mathcal{Y}'$ defined as follows: Firstly take a general $t_0 \in \mathbb{P}^1$ near 0. Then X_{t_0} is a smooth degree d surface which is discussed in the previous sections. For simplicity, we denote its defining function by $L + t_0 M = 0$. With a new parameter $u \in \mathbb{P}^1$, hence we obtain a family of degree $d+1$ threefolds

$$\mathcal{Y}' := \{Y'_s \colon (L + t_0 M)(\frac{u}{u^2-1}) + s(L + t_0 M(\frac{u}{u^2-1})) = 0\}$$

in $\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$. Clearly Y'_0 is a singular fiber consisting of the union of the constant family $X_0 \times \mathbb{P}^1$ along u and two copies of $\mathbb{P}^3$ at $u = 0, \infty$. The singular loci of $\mathcal{Y}'$ must be on the base locus $L + t_0 M = L + t_0 M(\frac{u}{u^2-1}) = 0$, and hence we can compute the Jacobian by taking the local coordinates such as $x = L_i, y = L_j, z = M_s$ for $[x : y : z] \in \mathbb{P}^3$. Thus we can see that the singular locus of $\mathcal{Y}'$ near $s = 0$ comprises the lines

$$Y'_0 \cap L_i \cap M_l \cap \{u = 0, \infty\}$$

on Y_0 and points

$$Y'_t \cap L_i \cap L_j \cap M_l \cap \{u = 0, \infty\}$$

on every fiber Y_s. We resolve them by the successive blow ups of $\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$ along the constant family of lines $L_i \cap M_l$ for each combination of i, l and then the family turns to be a semistable degeneration toward $s = 0$. After resolving the other singularities of $\mathcal{Y}'$, we obtain a smooth family of degree $d+1$ threefolds $\mathcal{Y}$ over $\overline{S} = \mathbb{P}^1$ which degenerates to a SNCD Y_0.

Remark 5 *In particular this is a degenerating family of Calabi-Yau threefolds when $d = 4$. A motivation of this construction comes from the toric geometry. X_t is defined as* *a Laurent polynomial of a 3 dimensional reflexive Newton polytope Δ. By changing the coordinate u to $w = \frac{u-1}{u+1}$ adjusting the coefficients, we obtain the equation*

$$(L + t_0 M)(w - \frac{1}{w}) + s(L + t_0 M(w - \frac{1}{w}) = 0.$$

This is a Laurent Polynomial with support contained in the Minkowski sum of the interval $[-1, 1]$ as a polytope and Δ, and is an example of the construction of Tyurin degenerations of Calabi-Yau threefolds from a nef-partition of a reflexive polytope ([6], Chapter 3.1). It will be a future work to extend this construction to more general nef-partitions.

Since a component of Y_0' is the constant family $X_{t_0} \times \mathbb{P}^1$, each $\gamma_{ijk,l}$ and $\delta_{i,lmn}$ is also on Y_0'. We denote their pullback to Y_0 by $\widetilde{\gamma_{ijk,l}}$ and $\widetilde{\delta_{i,lmn}}$ respectively.

Theorem 8 *For each higher Chow cycle $\gamma_{ijk,l}$ ($\delta_{i,lmn}$) on X_{t_0}, there exists a family of algebraic 1-cycles $\mathcal{C}_{ijk,l}$ (resp. $\mathcal{D}_{i,lmn}$) on $\mathcal{Y}$ such that the fiber $(\mathcal{C}_{ijk,l})_0$ in Y_0 defines the same class with $\widetilde{\gamma_{ijk,l}}$ (resp. $\widetilde{\delta_{i,lmn}}$) in $H^4_{\mathcal{M}}(Y_0, \mathbb{Q}(2))$.*

Since the above successive blow up is isomorphic over the component $X_{t_0} \times \mathbb{P}^1$, Corollary 3 implies that their pullbacks $\widetilde{\gamma_{ijk,l}}$ are also $\mathbb{R}$-regulator indecomposable. When $d = 4$, similarly Corollary 2 implies the same indecomposability of $\widetilde{\delta_{i,lmn}}$. Thus, by the discussion at the beginning of this section,

Corollary 4 *The class of each algebraic cycle $(\mathcal{C}_{ijk,l})_s$ in $\mathrm{Griff}^2(Y_s)$ is non-trivial for a general s. When $d = 4$, it also holds for $(\mathcal{D}_{i,lmn})_s$.*

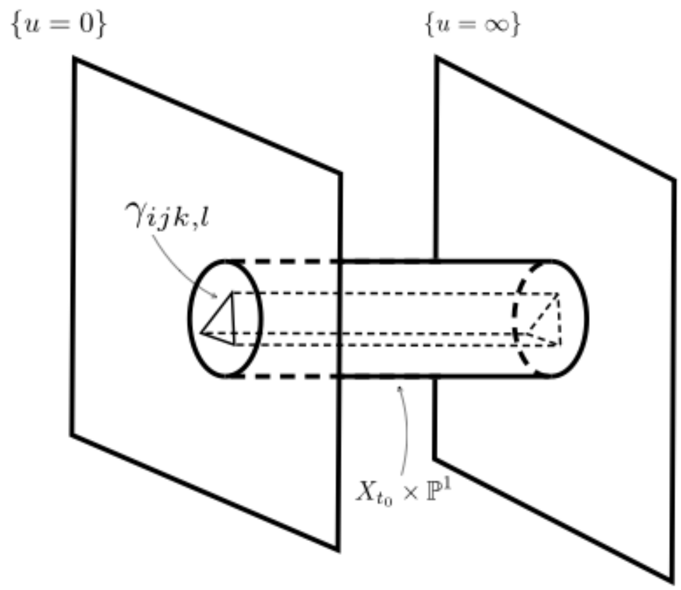

Figure 6.1. Central fiber Y_0' of the family $\mathcal{Y}'$

Proof [Proof of Theorem 8] Since the construction of $\mathcal{C}_{ijk,l}$ and $\mathcal{D}_{i,lmn}$ are exactly the same after changing L with M, we only consider $\mathcal{C}_{ijk,l}$. Recall that $\gamma_{ijk,l}$ is the sum of precycles with the form

$$\Gamma_{\alpha l} = (\frac{L_{\sigma(\alpha)}}{L_{\sigma^2(\alpha)}}, \mathcal{L}_\alpha \cap \mathcal{M}_l).$$

For each of these precycles, we define a precycle $\Delta_{\alpha l}$ on $\mathcal{Y}$ as follows: Since we are taking the successive blow up along $\mathcal{L}_i \cap \mathcal{M}_l$, $\mathcal{L}_j \cap \mathcal{M}_l$, and then $\mathcal{L}_k \cap \mathcal{M}_l$, the zero locus $\mathcal{L}_\alpha \cap \mathcal{M}_l \cap \mathcal{Y} \cap \{u = 0\}$ with the original equation of L_i and M_j defines the unique irreducible component for $\alpha = i$, but two components for $\alpha = j, k$. In fact, each intersection $\mathcal{L}_\alpha \cap \mathcal{L}_{\sigma(\alpha)}$ in the above zero locus defines an exceptional curve after the strict transformation. (See the local explanation after this proof). Hence the equation

$\mathcal{L}_\alpha \cap \mathcal{M}_l \cap \mathcal{Y} \cap \{u = 0\}$ on $\mathcal{Y}$ is generally given by (a strict transformation of) $\mathbb{P}^1 \times \mathbb{P}^1$, and the exceptional curve $\mathbb{P}^1$ if $\alpha = j, k$. We denote the former irreducible component by $Z_{\alpha l}$, which can be considered a reduced algebraic cycle. Therefore we can define a precycle

$$\Delta_{\alpha l} \coloneqq (\frac{L_{\sigma(\alpha)}}{L_{\sigma^2(\alpha)}}, Z_{\alpha l}).$$

Then the restriction of this precycle on the component $X_{t_0} \times \mathbb{P}^1 \subset Y_0$ is the strict transformation $\widetilde{\Gamma_{\alpha l}}$ of the original $\Gamma_{\alpha l}$ on $X_{t_0} \times \{u = 0\}$. Hence the fiber $(\Delta_{ijk,l})_0$ of $\Delta_{ijk,l} \coloneqq \Delta_{il} + \Delta_{jl} + \Delta_{kl}$ at $s = 0$ is an element of $Z^2_\#(Y_0^{[0]}, 1)$ such that $\partial_\mathcal{I}(\Delta_{ijk,l})_0 = \widetilde{\Gamma_{il}} + \widetilde{\Gamma_{jl}} + \widetilde{\Gamma_{kl}} = \widetilde{\gamma_{ijk,l}}$. Therefore we should define $\mathcal{C}_{ijk,l}$ by

$$\mathcal{C}_{ijk,l} \coloneqq \partial_\mathcal{B}(\Delta_{ijk,l})_0.$$

■

We remark locally what the algebraic cycle $\mathcal{C}_{ijk,l}$ is. By changing the coordinates of $\mathbb{P}^3$, assume $x = L_i, y = L_j, z = M_s$. Then, with an invertible function f, locally $\mathcal{Y}$ is given by the strict transformation of $(xy + t_0 f z)u + s(xy(u^2 - 1) + t_0 f z u) = 0$ for the blow up along $L_i = M_s = 0$ and then $L_j = \widetilde{M_s} = 0$. Denoting the blow up coordinates by$[X : Z]$ for the former one and $[Y : \widetilde{Z}]$ for the latter, specifically $\mathcal{Y}$ is given by the system of equations.

$$\begin{cases} (XY + t_0 f\widetilde{Z})u + s(XY(u^2-1) + t_0 f\widetilde{Z}u) = 0 \\ xZ = zX \\ y\widetilde{Z} = ZY \end{cases}$$

Over these equations, $\Delta_{ijk,l}$ is given by Δ_{il} and Δ_{jl}, which are defined by

$$\Delta_{il} = (y, \{x = z = u = 0\}),\ \Delta_{jl} = (\frac{1}{x}, \{y = Z = u = 0\}).$$

Note that their support cycles are the (blow up of) $\mathbb{P}^1 \times \mathbb{P}^1$ only when $s = 0$. If $s \neq 0$, we need to take the intersection with $sXY = 0$ additionally. In these local coordinates, the boundary $\mathcal{C}_{ijk,l} \coloneqq \partial_{\mathcal{B}}(\Delta_{ijk,l})_0$ is given by the algebraic cycle

$$[\{x = y = z = u = 0\}] - [\{x = y = Z = u = 0\}],$$

which is exactly the exceptional curve $\mathbb{P}^1$ parametrized by $[X : Z]$. Denoting this $\mathbb{P}^1$ by P_{ij}, therefore globally we obtain

$$\mathcal{C}_{ijk,l} = P_{ij} + P_{jk} - P_{ik}$$

as Figure 4. The boundary $\mathcal{C}_{ijk,l}$ itself is on each fiber Y_s, but the precycle $\Delta_{ijk,l}$ is only on Y_0. Hence the class of the higher cycle $\gamma_{ijk,l}$ "goes down" to $\mathcal{C}_{ijk,l}$ by the K-theory elevator on the singular fiber.

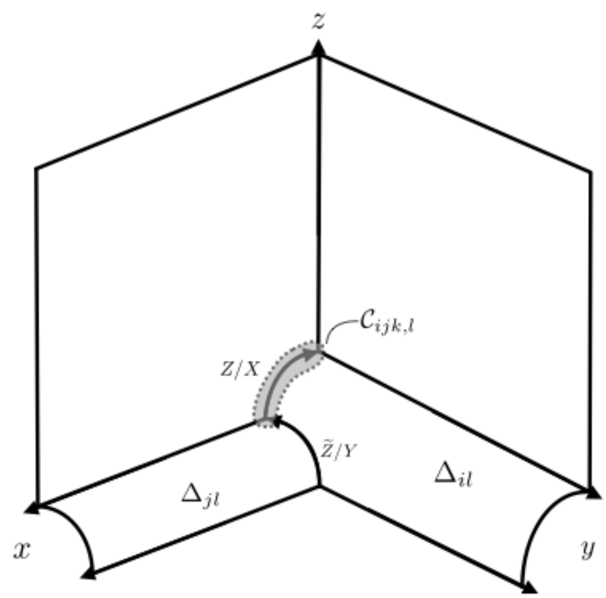

Figure 6.2. Local figure of $\Delta_{ijk,l}$

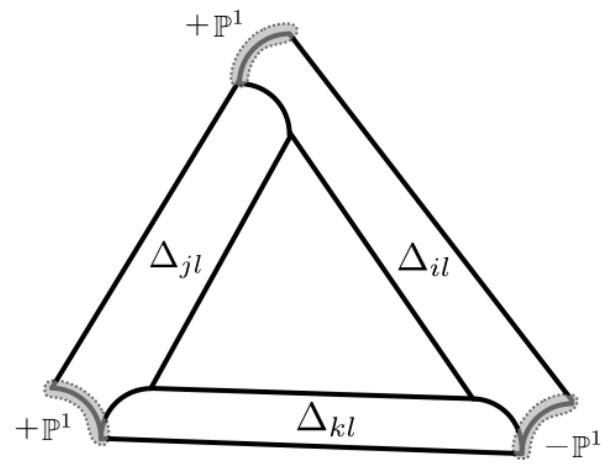

Figure 6.3. Algebraic Cycle $\mathcal{C}_{ijk,l}$

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