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Essays on Microeconomic Theory

Geyu Yang

Washington University in St. Louis

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WASHINGTON UNIVERSITY IN ST. LOUIS

Department of Economics

Dissertation Examination Committee:

Brian Rogers, Chair
Mariagiovanna Baccara
Marcus Berliant
SangMok Lee
Jonathan Weinstein

Essays on Microeconomic Theory

by
Geyu Yang

A dissertation presented to
The Graduate School
of Washington University in
partial fulfillment of the
requirements for the degree
of Doctor of Philosophy

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Geyu Yang

Washington University in St. Louis

May 2019

Dedicated to my parents and my girlfriend.

ABSTRACT OF THE DISSERTATION

Essays on Microeconomic Theory

by

Geyu Yang

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Professor Brian Rogers, Chair

This dissertation consists of three chapters on topics in microeconomic theory. In Chapter 1, I study reputation effects under uncertain monitoring. I examine a repeated game between a long-run player and a series of short-run opponents. The long-run player can either be a strategic type or a commitment type that plays the same action in every period. The modeling innovation is that the short-run player is unsure about the monitoring structure. The uncertainty about the monitoring structure introduces new challenges to reputation building because there may not be a direct relationship between the distribution of signals and the long-run player's strategy. Thus the long-run player may not have the ability to establish a reputation for commitment. I show that, when the short-run players cannot statistically distinguish commitment action from a bad action, the standard reputation results break down. I also provide sufficient conditions under which reputation effects on long-run player's payoffs can be extended to the current framework. When the commitment payoff is the highest payoff he can get, the conditions can be relaxed. In Chapter 2, I study a bounded rationality model of opinion formation in which there are two different types of agents: naive agents and sophisticated agents. All agents update their opinions by taking

weighted averages of neighbors' opinions. Naive agents truthfully report their opinions, but sophisticated agents can strategically report opinions to manipulate naive agents. I show that the limiting opinions are completely determined by sophisticated agents' biases and the structure of the network and that, generically, there is no consensus. I analyze how disagreement is affected by the intensity of lying cost, diverging interests, and the structure of the social network. I also show that naive agents do not have any social influence and sophisticated agents' social influence can be decomposed into two separate factors: direct influence and indirect influence. In Chapter 3, which is co-authored with Pinar Yildirim, we investigate the impact of informal lending on the types and terms of contracts offered by formal banks, considering factors that facilitate informal lending activity such as social ties among consumers. The density of the connections among consumers represents the degree to which those with and without wealth mix, indirectly capturing the degree of inequality in a society. We develop a model which relates the density of social connections to the availability of informal lending activity. We show that a low to moderate degree of informal activity in a market can help poor entrepreneurs because it motivates the bank to compete by cutting down the interest rate of unsecured loans offered to these consumers. In turn, the bank faces an overinvestment problem when financial inclusion is higher. As informal borrowing opportunities increase further, the bank's benefit from increased access to credit diminishes. It earns higher rents by increasing the rates on wealthy low-risk consumers who can informally lend to their social contacts. As a consequence, the overinvestment problem is replaced by an underinvestment problem, and creditworthy entrepreneurs are deprived of loans from the bank. We argue that although the entrepreneurial investment shrinks, only those projects with the best return are awarded financing, implying that the average

investment in the market is now more attractive.

Chapter 1: Robustness of Reputation Effects under Uncertain Monitoring

1.1 Introduction

It has long been recognized that reputation is an important dimension of long-run relationships. In the canonical reputation model (e.g., Fudenberg and Levine 1989, 1992; Gossner 2011), a long-run player (Player 1, he) faces a sequence of short-run players (Player 2, she). There is incomplete information about the type of the long-run player. He can either be of a simple commitment type who plays a fixed (possibly mixed) action at every period or of a normal type who acts strategically. The introduction of incomplete information affects dramatically the equilibrium payoffs and behaviors in the game because the long-run player has the option to build a reputation of commitment by mimicking the commitment type. The differences between the equilibria in the games of incomplete and complete information are known as the “reputations effects”.

In this paper, the reputation effect I focus on is the lower bound of the equilibrium payoffs to the normal type long-run player. In the canonical reputation model, when the long-run player is sufficiently patient, he can be assured of at least a payoff arbitrarily close to his commitment payoff at every equilibrium. Intuitively, normal type Player 1 can always deviate to mimic the commitment type player. Eventually, he can build a reputation for always playing the commitment action. Therefore, Player 2 eventually plays a best response to it, and Player 1 can obtain the commitment payoff. This reputation effect applies even when Player 2 can only observe some signals generated by Player 1’s actions instead of his

actions.

Despite the vast literature on reputation effects, the studies in this area all focus on the case in which the monitoring structure is common knowledge. In some cases, this assumption may seem too strong. Consider the following example. An individual is planning a trip to a new city and want to book a hotel. Before doing so, he/she may read the reviews on TripAdvisor or Expedia. The assumption that the monitoring structure is common knowledge means that consumers know the exact probability of a good review given that the hotel tries to provide good service.

However, there are two issues here. First, numerous studies have documented the existence of fake reviews (e.g., Luca and Zervas 2016; Mayzlin et al. 2014). Mayzlin et al. (2014) examine differences in the reviews posted on TripAdvisor and Expedia for different types of hotels and show that different types of hotels have different incentives to create fake reviews. Therefore, two hotels with very different qualities may have the same review score. Second, different people may have different tastes. Even if the reviews are good and accurate, it does not necessarily mean that the individual will like it. As a result, consumers cannot accurately monitor the actions of the hotels; they may still know that, when the hotel exerts high effort, the outcomes are more likely to be good but nothing more. Then it is much harder for the hotels to build a good reputation. In this situation, does the hotel still try to build a good reputation? Are reputation effects still robust in the presence of this uncertainty?

This paper addresses these questions. I study a repeated game of incomplete information between a long-run player (Player 1, he) and an infinite sequence of short-run players (Player 2, she). Player 1 is either a normal type who maximizes his discounted payoff or is committed

to playing a fixed action at every stage of the game. Player 2 observes only past signals and updates her belief. The key modeling innovation is that there is incomplete information about the monitoring structure. Formally, at the beginning of the game, a state of the world is realized. The state consists of two parts: the type of Player 1 and the monitoring structure. A monitoring structure is a mapping from the set of Player 1's action to the set of distributions of signals. Only Player 1 knows the true state. Player 2 has a prior belief about the state. I assume that the prior belief distribution has full support. To focus on the uncertainty about monitoring structure, I assume that there is only one commitment type. I relax this assumption in Section 1.4.

One new challenge introduced by uncertain monitoring is that Player 2's prediction of the distribution of signals is no longer directly linked to her prediction of Player 1's strategy. Then even if we show that with high probability, Player 2's prediction of the distribution of signals is almost correct in most periods, we still know little about her prediction of Player 1's strategy, not to mention her strategy.

I focus on the case in which the set of monitoring structures is finite, while an infinite set is discussed in Section 1.5. I first provide sufficient conditions under which the reputation effects on Player 1's payoffs can be extended to the current framework. A key step is to introduce the set of all the "well-behaved" monitoring structures with respect to the realized monitoring structure. The commitment strategy under the realized monitoring structure generates a certain distribution of signals. I call a monitoring structure "well-behaved" if it satisfies the following: for any strategy under this monitoring structure that can generate the same distribution, Player 2's best response to it is the same as her best response to the commitment action. In Theorem 1.1 and Proposition 1.1, I show that if all monitoring

structures are “well-behaved” or if Player 2’s prior belief about Player 1 being normal and the monitoring structure not “well-behaved” is sufficiently small, then normal type Player 1 can still be assured of his commitment payoff. The assumptions may seem restrictive, but I give an example to show that they actually cover most cases. I also examine the case in which the commitment payoff is the highest payoff Player 1 can get from the stage game. I show in Proposition 1.2 that if for any monitoring structure, all monitoring structures with higher subscripts are “well-behaved” then normal type Player 1 under all monitoring structures can be assured of his commitment payoff. The conditions are significantly more mild than the conditions in Proposition 1.1, especially for monitoring structures with large subscripts. I give an example to show that this proposition is extremely useful when the cardinality of the set of Player 1’s actions is two.

I then provide sufficient conditions for the existence of “bad” equilibria in which the payoff of Player 1 is strictly lower than the commitment payoff. In Proposition 1.3, I show that if there is a “bad” equilibrium of complete information stage game, and Player 2 cannot statistically distinguish commitment strategy from the “bad” strategy forming the “bad” equilibrium, then there exists a Bayes Nash equilibrium (BNE) in which Player 1’s payoff in the equilibrium is strictly lower than the commitment payoff. The intuition is as follows. Suppose normal type Player 1 always plays the “bad” strategy under all monitoring structures. Because Player 2 cannot statistically distinguish it from the commitment strategy, Player 2’s posterior belief about Player 1 being normal is bounded by her prior belief. Then if Player 2’s prior belief about Player 1 being normal is sufficiently strong, no matter the history, she always chooses a best response to the “bad” strategy, so Player 1’s payoff is strictly lower than his commitment payoff. When the “bad” equilibrium is a mixed-strategy Nash

equilibrium, reputation building may fail under different conditions, as shown in Proposition 1.4. The intuition is similar. The difference is that the normal type Player 1 can play different strategies under different monitoring structures. I give an example to show that Proposition 1.4 may relax dramatically the requirements on monitoring structures in Proposition 1.3, and in some cases the conditions in Proposition 1.4 are (almost) sufficient and necessary.

I next examine the case in which there is more than one commitment type. In Proposition 1.5, I show that if Player 2 can statistically distinguish different commitment strategies then the main results can be extended. The intuition is as follows. Suppose Player 1 deviates to mimic one of the commitment types. With high probability, Player 2 can learn eventually that Player 1 is either the normal type or the commitment type that he is mimicking. Therefore, introducing other commitment types does not affect Player 2's strategy and also does not affect Player 1's payoff.

Finally, I investigate the case in which the set of monitoring structures is a continuum. The model becomes much more complicated, so I focus on the "product-choice" game. There are only two actions and the commitment strategy is pure. Negative results can be extended easily. In contrast, positive results are much harder to extend because the proofs rely on the assumption that the set of monitoring structures is finite. I can only prove a weaker version here.

My paper relates to an extensive literature on the adverse selection approach to reputation effects. The idea was introduced by Kreps and Wilson (1982) and Milgrom and Roberts (1982) in the context of finitely repeated games. Fudenberg and Levine (1989, 1992) discussed reputation effects in infinitely repeated games. They showed that a suffi-

ciently patient long-run player can obtain a payoff that is at least as much as the payoff he could obtain by committing publicly to playing any of the commitment strategies. Gossner (2011) introduced entropy techniques to this literature and greatly simplified the proof of the above.

Some interesting variations of the canonical reputation model have been studied. Wiseman (2008) and Ekmekci et al. (2012) studied models with unobservable stochastic replacements for the long-run player and showed that the long-run player could establish a reputation for commitment. Mailath and Samuelson (2001) studied a similar model, but they focused on a different kind of reputation. They provide conditions under which the long-run player could establish a reputation for not being the commitment type.

Liu (2011) investigated a model in which short-run players must pay a cost to observe past signals. Liu and Skrzypacz (2014) discussed the case in which short-run players have limited records. They both focus on the reputation dynamics instead of payoffs.

Jehiel and Samuelson (2012) studied a model of boundedly rational short-run players. They provided a characterization of both equilibrium payoffs and equilibrium behavior. They showed that the payoff bound for the rational long-run player can be strictly larger than the commitment payoff.

Pei (2018) investigated a model in which Player 1 has persistent private information that matters for Player 2's payoffs and provided a sufficient and (almost) necessary condition under which reputation effects can be extended.

The model most similar to this paper is by Deb and Ishii (2018). They also studied a variation of the canonical reputation model in which the long-run player is privately informed about the monitoring structure. They also investigated the robustness of reputation effects.

The main difference is that they assumed that there is a set of nonstationary commitment types who switch infinitely often between “signaling actions” that help the consumer learn the unknown monitoring state and “collection actions” that are desirable for payoffs, while I maintain standard assumptions.

1.2 Model

One long-lived Player 1 (with discount factor $\delta \in (0, 1)$) interacts with a sequence of short-lived Player 2. In each period, two players play a simultaneous move game. Player 1 chooses a (possibly mixed) action $\alpha_1^t \in \Delta(A_1)$ and Player 2 chooses a (possibly mixed) action $\alpha_2^t \in \Delta(A_2)$. Both A_1 and A_2 are finite sets. A signal y_t is drawn from a finite set Y . The probability that y is realized depends on Player 1’s action $a_1^t \in A_1$ and the monitoring structure. Let D be the set of monitoring structures, with $d : A_1 \rightarrow \Delta(Y)$ a typical element. Denote the distribution of signals by $\rho_{d, a_1^t}(\cdot) \in \Delta(Y)$.

Player 1 is either a normal type who maximizes his discounted payoff or is committed to playing a fixed action at every stage of the game. For expositional clarity, in most of the paper, I assume that there is only one commitment type. Let $\Xi = \{\xi_0, \xi(\hat{\alpha}_1)\}$ be the set of types, in which ξ_0 is a normal type and $\xi(\hat{\alpha}_1)$ is a commitment type who always plays $\hat{\alpha}_1 \in \Delta(A_1)$. Let $\Theta = \Xi \times D$ be the set of states, with a typical element $\theta = (\xi, d) \in \Theta$. At the beginning of the game, a state is realized and only Player 1 observes the state. Player 2 has a prior belief of the states which is given by $\pi^0(\cdot) \in \Delta(\Theta)$. I assume that π^0 has full support. Abusing notation, denote Player 2’s prior belief about Player 1 being normal and monitoring structure being d by $\pi^0(\xi_0)$ and $\pi^0(d)$, respectively.

I assume that Player 2 only observes past signals, so the set of histories for Player 2 in period t is $\mathcal{H}_2^t = (Y)^{t-1}$ and $\mathcal{H}_2 = \cup_{t=1}^{\infty} \mathcal{H}_2^t$ is the set of all histories for Player 2. Player 1, on the other hand, knows everything. Let $\mathcal{H}_1^t = (A_1 \times A_2 \times Y)^{t-1}$, then the set of histories for Player 1 in period t is $\Theta \times \mathcal{H}_1^t$ and $\mathcal{H}_1 = \cup_{t=1}^{\infty} (\Theta \times \mathcal{H}_1^t) = \Theta \times \cup_{t=1}^{\infty} \mathcal{H}_1^t$ is the set of all histories for Player 1.

A strategy for Player 1 is a mapping

$$\sigma_1 : \mathcal{H}_1 \rightarrow \Delta(A_1)$$

which satisfies that for any $h_1^t \in \mathcal{H}_1^t$ and any d , $\sigma_1((\xi(\hat{\alpha}_1), d), h_1^t) = \hat{\alpha}_1$. A strategy for Player 2 is a mapping

$$\sigma_2 : \mathcal{H}_2 \rightarrow \Delta(A_2)$$

Let $\Omega = \Theta \times (A_1, A_2, Y)^\infty$ be the set of outcomes and $P \in \Delta(\Omega)$ be the probability measure induced by the strategy profile $\sigma = (\sigma_1, \sigma_2)$ and the probability distribution of states π^0 . Denote the event that the state is θ by $\{\theta\}$. Let $Q_\theta(\cdot) = P(\cdot | \{\theta\})$ be the probability measure on the set of outcomes given that the state is θ , then

$$P(\cdot) = \sum_{\theta} \pi^0(\theta) Q_\theta(\cdot)$$

Let \mathcal{F}_t be the σ -algebra generated by \mathcal{H}_1^t , $\hat{\mathcal{F}}_t$ be the σ -algebra generated by \mathcal{H}_2^t , and \mathcal{F} be the σ -algebra generated by Ω . Then (Ω, \mathcal{F}, P) is a probability space, both $\{\mathcal{F}_t : t \geq 0\}$ and $\{\hat{\mathcal{F}}_t : t \geq 0\}$ are filtrations on this probability space, and $\hat{\mathcal{F}}_t \subseteq \mathcal{F}_t$. Player 2's posterior belief in period t that the state is θ is the $\hat{\mathcal{F}}_t$ -measurable random variable $\pi^t(\theta) = P(\{\theta\} | \hat{\mathcal{F}}_t)$.

The stage-game payoff functions are u_1 for normal type player 1, and u_2 for player 2.

The expected payoff for Player 2 in period t is

$$E_P \left(u_2(a_1^t, a_2^t) \mid \hat{\mathcal{F}}_t \right)$$

and the expected payoff for normal Player 1 in state $\theta = (\xi_0, d)$ is

$$E_{Q_\theta} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) \right)$$

Let

$$u_{\hat{\alpha}_1} = \min_{\alpha_2 \in BR_2(\hat{\alpha}_1)} u_1(\alpha_1, \alpha_2)$$

It is the least payoff Player 1 obtains when choosing $\hat{\alpha}_1$, and Player 2 always plays a best response which is called the “commitment payoff.”

The solution concept is Bayes Nash equilibrium (BNE).

1.3 Main Results

In this section, I analyze how uncertain monitoring affects the standard reputation effects of the long-run player’s payoffs. I provide sufficient conditions under which reputation effects hold, and I also characterize the conditions under which reputation effects break down. I focus on the case in which the cardinality of Θ is finite, while an infinite states set is discussed in Section 1.5. Let $|D| = N$ and $D = \{d_1, \dots, d_N\}$, then $|\Theta| = 2N$.

1.3.1 Robustness of Reputation Effects

I first provide sufficient conditions for the robustness of the reputation effects. I extend the notion of ε -confirming best responses of Fudenberg and Levine (1992) to this framework. Define the d -weak- ε -confirming best response as follows.

Definition 1.1. $\alpha_2 \in \Delta(A_2)$ is a d -weak- ε -confirming best response to $\hat{\alpha}_1 \in \Delta(A_1)$ if there exist $(\alpha'_{1,1}, \dots, \alpha'_{1,N}) \in (\Delta(A_1))^N$ and $(\pi'(d_1), \dots, \pi'(d_N)) \in \mathbb{R}_+^N$ such that

1. $\sum_{i=1}^N \pi'(d_i) = 1$;
2. α_2 is a best response to $\sum_{i=1}^N \pi'(d_i) \alpha'_{1,i}$;
3. $\pi'(d_i) \|\rho_{d_i, \alpha'_{1,i}} - \rho_{d, \hat{\alpha}_1}\|_\infty < \varepsilon, \forall d_i \in D$.

The set of d -weak- ε -confirming best responses to α_1 is denoted by $B_\varepsilon^{w,d}(\alpha_1)$. Let

$$v_{\alpha_1}^{w,d}(\varepsilon) = \min_{\alpha_2 \in B_\varepsilon^{w,d}(\alpha_1)} u_1(\alpha_1, \alpha_2).$$

Then the following proposition provides a lower bound for the payoff of normal type of Player 1.

Theorem 1.1. *For any realized monitoring structure d and every $\varepsilon > 0$, there is an M such that, for all δ , normal type Player 1's payoff under monitoring structure d is greater than or equal to*

$$(1 - \varepsilon) \delta^M v_{\hat{\alpha}_1}^{w,d}(\varepsilon) + (1 - (1 - \varepsilon) \delta^M) \min_{(\alpha_1, \alpha_2)} u_1(\alpha_1, \alpha_2)$$

in any BNE.

The main idea of the proof of Theorem 1.1 follows the classical argument of Fudenberg and Levine (1992) and Sorin (1999). I also calculate normal type Player 1's payoff when mimicking the commitment type. One new challenge introduced by uncertain monitoring is that Player 2's prediction of the distribution of signals is no longer linked directly to her prediction of Player 1's strategy. Suppose that we do not know Player 2's posterior beliefs on monitoring structures. Then even if we show that with high probability that Player 2's prediction of the distribution of signals is almost correct in most periods, we still know little about her prediction of Player 1's strategy. To overcome this challenge, I provide a more detailed analysis of Player 2's posterior beliefs. I show that if the realized monitoring structure is d and normal type Player 1 mimics commitment type, with high probability, Player 2's strategy is a d -weak- ε -confirming best response to $\hat{\alpha}_1$, except for a fixed number of periods. The idea is that Player 2 learns eventually that the real distribution is $\rho_{d, \hat{\alpha}_1}$ and Player 1 always plays some strategy that generates the same distribution as $\rho_{d, \hat{\alpha}_1}$.

Theorem 1.1 provides a lower bound ($v_{\hat{\alpha}_1}^{w,d}(\varepsilon)$) to the payoff of normal type Player 1, the problem is that $v_{\hat{\alpha}_1}^{w,d}(\varepsilon)$ may not be a good lower bound. Consider the following example.

Example 1.1. Consider the following “product-choice” game. It involves one long-lived Player 1 and a sequence of short-lived Player 2. Player 1 plays either H or L and Player 2 plays either c or s. The payoffs are shown in the following payoff matrix:

		Player 2	
		c	s
Player 1	H	2,3	0,2
	L	3,0	1,1

Suppose $\hat{\alpha}_1 = H$. Player 1's actions are monitored via a public signal: $y \in \{\bar{y}, \underline{y}\}$. Then

$d = d_i$ represents the following monitoring structure:

$$\rho_{d_i, H}(y) = \begin{cases} \rho_1^{d_i} & y = \bar{y} \\ 1 - \rho_1^{d_i} & y = \underline{y} \end{cases}$$

$$\rho_{d_i, L}(y) = \begin{cases} \rho_2^{d_i} & y = \bar{y} \\ 1 - \rho_2^{d_i} & y = \underline{y} \end{cases}$$

Suppose $\rho_2^{d_2} = \rho_1^{d_1}$. Consider the case in which the realized monitoring structure is d_1 .

Then, for any $\varepsilon > 0$ and any $0 < \hat{\varepsilon} < \varepsilon$, in the definition of the d -weak- ε -confirming best response, let $\pi'(d_2) = 1 - \hat{\varepsilon}$, $\pi'(d_i) = \frac{\hat{\varepsilon}}{N-1} \forall i \neq 2$, $\alpha'_{1,2} = L$, and $\alpha'_{1,i} = H \forall i \neq 2$. When $\hat{\varepsilon}$ is sufficiently small, s is a best response to $\sum_{i=1}^N \pi'(d_i) \alpha'_{1,i}$, so $s \in B_\varepsilon^{w,d_1}(H)$ and $v_H^{w,d_1}(\varepsilon) = 0$.

The next question is: when will $v_{\hat{\alpha}_1}^{w,d}(\varepsilon)$ be a good lower bound? Let $\tilde{D}_i^1 = \{d \in D : \text{for any } \alpha_1 \in \Delta(A_1), \text{ if } \rho_{d_i, \hat{\alpha}_1} = \rho_{d, \alpha_1}, \text{ then } BR_2(\alpha_1) = BR_2(\hat{\alpha}_1)\}$, $\tilde{D}_{-i}^1 = D \setminus \tilde{D}_i^1$, $\tilde{\Theta}_i^1 = \{\theta \in \Theta : d \in \tilde{D}_i^1, \xi = \xi_0\}$, $\hat{\Theta} = \{\theta \in \Theta : \xi = \xi(\hat{\alpha}_1)\}$, and $\tilde{\Theta}_{-i}^1 = \{\theta \in \Theta : d \in \tilde{D}_{-i}^1, \xi = \xi_0\}$. The following Proposition shows the sufficient conditions under which $v_{\hat{\alpha}_1}^{w,d}(\varepsilon)$ is a good lower bound.

Proposition 1.1.

1. For any realized monitoring structure d_i , if $D \subseteq \tilde{D}_i^1$, then for any $\varepsilon > 0$, there exists δ^* such that for any $\delta > \delta^*$, normal type Player 1's payoff under the monitoring structure d_i is greater than or equal to $u_{\hat{\alpha}_1} - \varepsilon$ in any BNE.
2. For any realized monitoring structure d_i and any $\varepsilon > 0$, there exists δ^* and η^* such that

if for any $\theta \in \tilde{\Theta}_{-i}^1$, $\pi^0(\theta) < \eta^*$, and $\delta > \delta^*$, then normal type Player 1's payoff under the monitoring structure d_i is greater than or equal to $u_{\hat{\alpha}_1} - \varepsilon$ in any BNE.

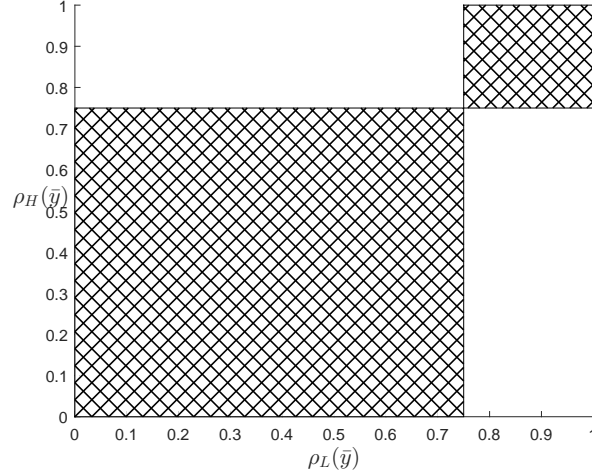
The intuition is as follows. Suppose normal type Player 1 mimics the commitment type. As I discussed above, Player 2 learns eventually that the real distribution is $\rho_{d, \hat{\alpha}_1}$, and Player 1 always plays some strategy that generates the same distribution as $\rho_{d, \hat{\alpha}_1}$. If $D \subseteq \tilde{D}_i^1$, then Player 2's best response to any convex combination of such strategies is the same as her best response to $\hat{\alpha}_1$. Then, eventually, normal type Player 1 receives $u_{\hat{\alpha}_1}$ in every round of the game. In this sense, for any realized monitoring structure d_i , \tilde{D}_i^1 can be viewed as all of the “well-behaved” monitoring structures with respect to d_i .

If $D \not\subseteq \tilde{D}_i^1$, we do not have a clear prediction for Player 2's strategy. If Player 2's posterior belief about the state belonging to $\tilde{\Theta}_{-i}^1$ is large, she may think that Player 1's action is completely different from $\hat{\alpha}_1$, and then her best response may also be different. Fortunately, Player 2's posterior belief is bounded by her prior belief in some sense. Therefore, when Player 2's prior belief about the state belonging to $\tilde{\Theta}_{-i}^1$ is sufficiently small, we can still obtain a good lower bound. The assumptions in Proposition 1.1 may seem restrictive, but \tilde{D}_i^1 actually cover most cases, as shown in the following example.

Example 1.2. Consider the same “product-choice” game as in Example 1.1. I still assume $\hat{\alpha}_1 = H$, and Player 1's actions are monitored via a public signal: $y \in \{\bar{y}, \underline{y}\}$. Then, $d = d_i$ represents the following monitoring structure:

$$\rho_{d_i, H}(y) = \begin{cases} \rho_1^{d_i} & y = \bar{y} \\ 1 - \rho_1^{d_i} & y = \underline{y} \end{cases}$$

Figure 1: "Well-behaved" monitoring structures example 1



$$\rho_{d_i, L}(y) = \begin{cases} \rho_2^{d_i} & y = \bar{y} \\ 1 - \rho_2^{d_i} & y = \underline{y} \end{cases}$$

Suppose that

$$\rho_{d_1, H}(y) = \begin{cases} \frac{3}{4} & y = \bar{y} \\ \frac{1}{4} & y = \underline{y} \end{cases}$$

and $\rho_{d_1, L}(\bar{y}) \neq \frac{3}{4}$.

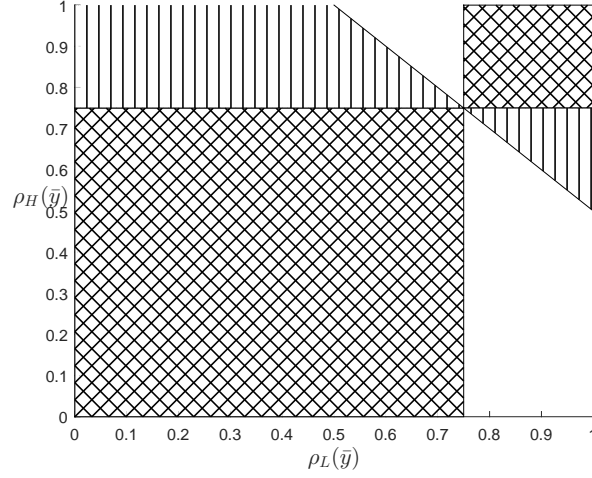
What is \tilde{D}_1^1 in this example? Let us first consider the set of monitoring structures in which either $\rho_{d, H}(\bar{y}), \rho_{d, L}(\bar{y}) > \frac{3}{4}$ or $\rho_{d, H}(\bar{y}), \rho_{d, L}(\bar{y}) < \frac{3}{4}$, as illustrated in Figure 1.

For any monitoring structure d belonging to this set and any $\alpha_1 \in \Delta(A_1)$, it is easy to check that $\rho_{d, \alpha_1}(\bar{y}) \neq \rho_{d_1, H}(\bar{y})$. According to the definition of \tilde{D}_i^1 , this set is a subset of \tilde{D}_1^1 .

What about a monitoring structure d outside the shaded region? It is possible that there exists $\alpha_1 = (\gamma H, (1 - \gamma)L) \neq H$, such that $\rho_{d_1, H} = \rho_{d, \alpha_1}$. Then

$$\gamma \rho_{d, H}(\bar{y}) + (1 - \gamma) \rho_{d, L}(\bar{y}) = \frac{3}{4}$$

Figure 2: "Well-behaved" monitoring structures example 2



so

$$\gamma = \frac{\frac{3}{4} - \rho_{d,L}(\bar{y})}{\rho_{d,H}(\bar{y}) - \rho_{d,L}(\bar{y})}$$

and

$$\begin{aligned} BR_2(\alpha_1) &= \{c\} \\ \iff \frac{1}{2} < \gamma \leq 1 \\ \iff \rho_{d,H}(\bar{y}) + \rho_{d,L}(\bar{y}) < \frac{3}{2}, \rho_{d,L}(\bar{y}) < \frac{3}{4} \text{ and } \rho_{d,H}(\bar{y}) \geq \frac{3}{4} \\ \text{or } \rho_{d,H}(\bar{y}) + \rho_{d,L}(\bar{y}) > \frac{3}{2}, \rho_{d,L}(\bar{y}) > \frac{3}{4} \text{ and } \rho_{d,H}(\bar{y}) \leq \frac{3}{4}. \end{aligned}$$

Taking the union of this set with the set discussed above, we obtain \tilde{D}_1^1 , as illustrated in Figure 2.

The conditions in Proposition 1.1 can be relaxed if $u_{\hat{\alpha}_1}$ is the highest payoff Player 1 can obtain from the stage game. Let $\bar{u}_1 = \max_{a_1, a_2} u_1(a_1, a_2)$ and $\tilde{D}_i^2 = \{d \in D : \text{for any } \alpha_1 \in \Delta(A_1), \text{ if } \rho_{d_i, \hat{\alpha}_1} = \rho_{d_i, \alpha_1}, \text{ then for any } a_2 \in A_2, \bar{u}_1 > u_1(\alpha_1, a_2)\}$.

Proposition 1.2. *If the stage game payoff satisfies $u_{\hat{\alpha}_1} = \bar{u}_1$ and, for any i , $\{d_i, \dots, d_N\} \subseteq \tilde{D}_i^1$ and $\{d_1, \dots, d_{i-1}\} \subseteq \tilde{D}_i^1 \cup \tilde{D}_i^2$, then for any realized monitoring structure d_i and any $\varepsilon > 0$, there exists δ^* such that, for any $\delta > \delta^*$, normal type Player 1's payoff under the monitoring structure d_i is greater than or equal to $u_{\hat{\alpha}_1} - \varepsilon$ in any BNE.*

It is easy to check that if there exists a permutation of $\{1, \dots, N\}$, denoted by μ , such that for any i , $\{d_{\mu(i)}, \dots, d_{\mu(N)}\} \subseteq \tilde{D}_{\mu(i)}^1$ and $\{d_{\mu(1)}, \dots, d_{\mu(i-1)}\} \subseteq \tilde{D}_{\mu(i)}^1 \cup \tilde{D}_{\mu(i)}^2$, then the results in Proposition 1.2 still hold. Because $\bar{u}_1 = \max_{a_1, a_2} u_1(a_1, a_2)$, $\{d_1, \dots, d_{i-1}\} \subseteq \tilde{D}_i^2$ is a very mild condition. Therefore the conditions in this proposition basically are that for any i , any monitoring structure with a higher subscript is “well-behaved.” The conditions are significantly more mild than the conditions in Proposition 1.1, especially for monitoring structures with large subscripts.

The intuition is as follows. For $i = 1$, $\{d_i, \dots, d_N\} \subseteq \tilde{D}_i^1 \implies D \subseteq \tilde{D}_i^1$, I can apply Proposition 1.1. Now let us consider $i = 2$. I still consider the deviation that normal type of Player 1 always plays $\hat{\alpha}_1$. As discussed above, Player 2 learns eventually that the real distribution is $\rho_{d_2, \hat{\alpha}_1}$ and Player 1 always plays some strategy that generates the same distribution as $\rho_{d_2, \hat{\alpha}_1}$. Since $\{d_2, \dots, d_N\} \subseteq \tilde{D}_2^1$, the only possibility for Player 2 not choosing an action belonging to $BR_2(\hat{\alpha}_1)$ is that her posterior belief about the state being (ξ_0, d_1) is sufficiently large and that $d_1 \notin \tilde{D}_2^1$, then $d_1 \in \tilde{D}_2^2$. This also implies that, in equilibrium, normal type Player 1, under the monitoring structure d_1 , almost always plays $\alpha_{1,1} \in \Delta(A_1)$, such that $\rho_{d_2, \hat{\alpha}_1} = \rho_{d, \alpha_{1,1}}$. When Player 1 follows this strategy, Player 1's payoff under the monitoring structure d_1 is strictly below \bar{u}_1 . However, when the monitoring structure is d_1 , Player 1 can always obtain \bar{u}_1 by mimicking $\xi(\hat{\alpha}_1)$, which leads to a contradiction. The same

logic applies when we consider $i > 2$.

This proposition is extremely useful when $|A_1| = 2$ as shown in the following example.

Example 1.3. Consider the following “battle of the sexes” game. It involves one long-lived Player 1 and a sequence of short-lived Player 2. Player 1 plays either H or L, and Player 2 plays either c or s. The payoffs are shown in the following payoff matrix:

		Player 2	
		c	s
Player 1	H	3,1	0,0
	L	0,0	1,3

Suppose $\hat{\alpha}_1 = H$ and that Player 1’s actions are monitored via a public signal: $y \in \{\bar{y}, \underline{y}\}$.

Then, $d = d_i$ represents the following monitoring structure:

$$\rho_{d_i, H}(y) = \begin{cases} \rho_1^{d_i} & y = \bar{y} \\ 1 - \rho_1^{d_i} & y = \underline{y} \end{cases}$$

$$\rho_{d_i, L}(y) = \begin{cases} \rho_2^{d_i} & y = \bar{y} \\ 1 - \rho_2^{d_i} & y = \underline{y} \end{cases}$$

As proved in the Appendix, if for any $d_i \in D$, $\rho_1^{d_i} > \rho_2^{d_i}$, then for any $\varepsilon > 0$, and any $d_i \in D$, there exists δ^* such that for any $\delta > \delta^*$, normal type Player 1’s payoff under the monitoring structure d_i in any BNE is greater than or equal to $u_{\hat{\alpha}_1} - \varepsilon$.

1.3.2 Failure of Reputation Building

I next provide sufficient conditions under which reputation effects break down. Let $\Gamma_\alpha = \{\rho_{d,\alpha}(\cdot); d \in D\}$ and let $D_{\rho,\alpha} = \{d \in D : \rho_{d,\alpha}(\cdot) = \rho\}$. Γ_α is the set of distribution of signals induced by α and $D_{\rho,\alpha}$ is the set of monitoring structures under which the distribution of signals induced by α is ρ . The following proposition shows that uncertain monitoring may cause a failure of reputation building.

Proposition 1.3. *Suppose there exist $\tilde{\alpha}_1 \in \Delta(A_1)$ and $\tilde{a}_2 \in A_2$ such that $(\tilde{\alpha}_1, \tilde{a}_2)$ is a Nash equilibrium of complete information stage game, $u_2(\tilde{\alpha}_1, \tilde{a}_2) > u_2(\tilde{\alpha}_1, a_2)$ for all $a_2 \neq \tilde{a}_2$, $u_1(\tilde{\alpha}_1, \tilde{a}_2) < u_{\hat{\alpha}}$, and $\Gamma_{\hat{\alpha}_1} \subseteq \Gamma_{\tilde{\alpha}_1}$. There exists a $\hat{L} > 0$, such that if Player 2's prior belief $\pi^0(\cdot)$ satisfies that*

$$\frac{\min_{\rho \in \Gamma_{\tilde{\alpha}_1}} \sum_{d \in D_{\rho, \tilde{\alpha}_1}} \pi^0(\xi_0, d)}{\max_{\rho \in \Gamma_{\hat{\alpha}_1}} \sum_{d \in D_{\rho, \hat{\alpha}_1}} \pi^0(\xi(\hat{\alpha}_1), d)} \geq \hat{L},$$

then for any δ , there exists a BNE such that normal type Player 1 always plays $\tilde{\alpha}_1$ and Player 2 always plays \tilde{a}_2 .

It is easy to check that Player 1's payoff in the equilibrium described in Proposition 1.3 is strictly lower than the commitment payoff, so this proposition shows that reputation effects are not fully robust when there is uncertainty about the monitoring structure. The proof of Proposition 1.3, as well as all proofs of the following results, are in the Appendix. The intuition behind Proposition 1.3 is as follows. In equilibrium, since $\Gamma_{\hat{\alpha}_1} \subseteq \Gamma_{\tilde{\alpha}_1}$, Player 2 cannot statistically distinguish commitment type from normal type, so Player 2's posterior belief about Player 1 being normal is bounded by her prior belief. Then, if Player 2's prior belief about Player 1 being normal is sufficiently strong, no matter the history, she always

chooses a best response to $\tilde{\alpha}_1$; therefore, Player 1 also has no incentive to deviate. The following example illustrates the conditions in Proposition 1.3.

Example 1.4. Consider the same “product-choice” game as in Examples 1.1 and 1.2.

Suppose $\hat{\alpha}_1 = H$, then $BR_2(\hat{\alpha}_1) = \{c\}$ and $u_{\hat{\alpha}_1} = 2$. Player 1’s actions are monitored via a public signal: $y \in \{\bar{y}, \underline{y}\}$. Then $d = d_i$ represents the following monitoring structure:

$$\rho_{d_i, H}(y) = \begin{cases} \rho_1^{d_i} & y = \bar{y} \\ 1 - \rho_1^{d_i} & y = \underline{y} \end{cases}$$

$$\rho_{d_i, L}(y) = \begin{cases} \rho_2^{d_i} & y = \bar{y} \\ 1 - \rho_2^{d_i} & y = \underline{y} \end{cases}$$

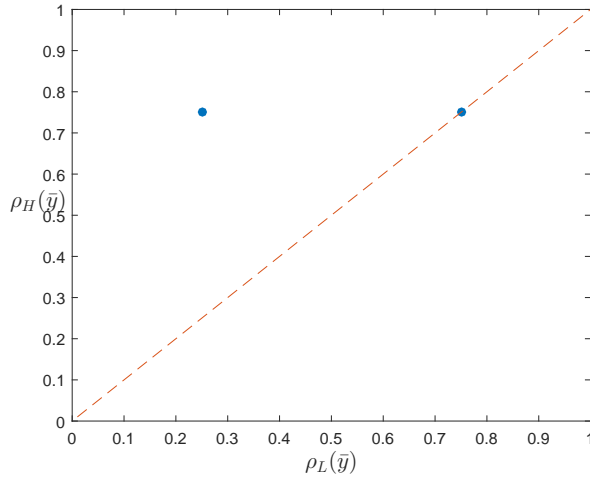
$\Gamma_H = \{\rho_{d_1, H}, \dots, \rho_{d_N, H}\}$ and $\Gamma_L = \{\rho_{d_1, L}, \dots, \rho_{d_N, L}\}$. I further assume that monitoring structure and the type of Player 1 are independently distributed. Then according to Proposition 1.3, if $\Gamma_H \subseteq \Gamma_L$, there exists a $0 < \hat{\mu} < 1$, such that for any $\pi^0(\xi_0) \geq \hat{\mu}$ and any δ , there exists a BNE such that normal type Player 1 always plays L and Player 2 always plays s . Figure 3 shows three specific examples; the first example is also discussed by Deb and Ishii (2018).

When the bad equilibrium of the complete information stage game is a mixed strategy equilibrium, reputation building may fail under different conditions. Let $\hat{\theta}_i = (\xi(\hat{\alpha}_1), d_i)$ and $\theta_i^0 = (\xi_0, d_i)$.

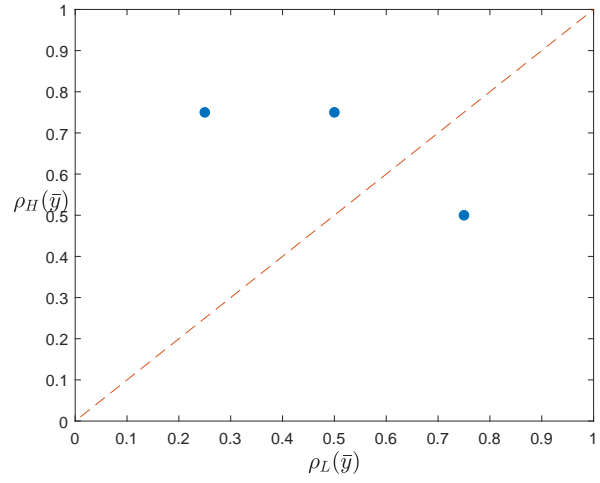
Proposition 1.4. *Suppose there exist $\tilde{\alpha}_1 \in \Delta(A_1)$ and $\tilde{\alpha}_2 \in \Delta(A_2)$ such that $(\tilde{\alpha}_1, \tilde{\alpha}_2)$ is a Nash equilibrium of the complete information stage game and there exists a permutation of*

Figure 3: Failure of reputation building examples part 1

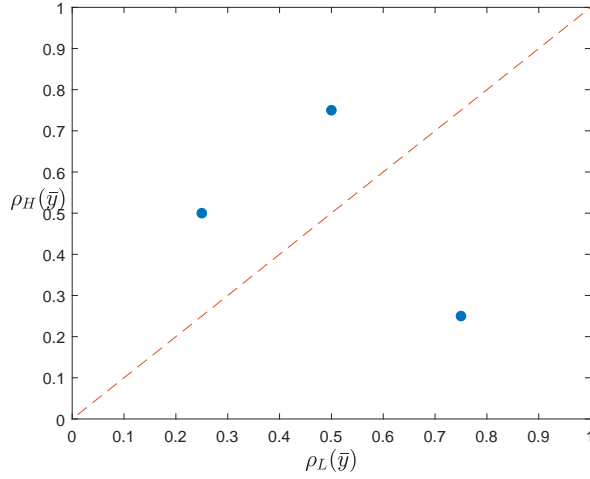
(a) $(\rho_1^{d_1}, \rho_2^{d_1}) = (\frac{3}{4}, \frac{1}{4})$ and $(\rho_1^{d_2}, \rho_2^{d_2}) = (\frac{3}{4}, \frac{3}{4})$



(b) $(\rho_1^{d_1}, \rho_2^{d_1}) = (\frac{3}{4}, \frac{1}{4})$, $(\rho_1^{d_2}, \rho_2^{d_2}) = (\frac{3}{4}, \frac{1}{2})$ and $(\rho_1^{d_3}, \rho_2^{d_3}) = (\frac{1}{2}, \frac{3}{4})$



(c) $(\rho_1^{d_1}, \rho_2^{d_1}) = (\frac{1}{2}, \frac{1}{4})$, $(\rho_1^{d_2}, \rho_2^{d_2}) = (\frac{3}{4}, \frac{1}{2})$ and $(\rho_1^{d_3}, \rho_2^{d_3}) = (\frac{1}{4}, \frac{3}{4})$



$\{1, \dots, N\}$, denoted by μ , such that, for any i :

1. there exist $\tilde{\alpha}_{1,i} \in \Delta(A_1)$ and $e_i \in (0, 1)$, such that $\rho_{d_{\mu(i)}, \tilde{\alpha}_{1,i}} = \rho_{d_i, \hat{\alpha}_1}$, $\tilde{\alpha}_1 = e_i \hat{\alpha}_1 + (1 - e_i) \tilde{\alpha}_{1,i}$ and $\tilde{\alpha}_{1,i} \in BR_1(\tilde{\alpha}_2)$,
2. $\frac{\pi^0(\hat{\theta}_i)}{\pi^0(\theta_{\mu(i)}^0)} = \frac{e_i}{1 - e_i}$.

Then, for any δ , there exists a BNE such that normal type Player 1 always plays $\tilde{\alpha}_{1,i}$ under monitoring structure $d_{\mu(i)}$ and Player 2 always plays $\tilde{\alpha}_2$.

The idea is similar to Proposition 1.3. The difference is that normal type Player 1 can play different actions under different monitoring structures. In some cases this flexibility can relax dramatically the requirements on monitoring structures, as shown in the following example.

Example 1.5. Consider the same “battle of the sexes” game as in Example 1.3. Suppose $\hat{\alpha}_1 = H$ and that Player 1’s actions are monitored via a public signal: $y \in \{\bar{y}, \underline{y}\}$. Then $d = d_i$ represents the following monitoring structure:

$$\rho_{d_i, H}(y) = \begin{cases} \rho_1^{d_i} & y = \bar{y} \\ 1 - \rho_1^{d_i} & y = \underline{y} \end{cases}$$

$$\rho_{d_i, L}(y) = \begin{cases} \rho_2^{d_i} & y = \bar{y} \\ 1 - \rho_2^{d_i} & y = \underline{y} \end{cases}$$

Suppose $D = \{d_1, d_2\}$ and $(\rho_1^{d_1}, \rho_2^{d_1}) = (\frac{3}{4}, \frac{1}{4})$. The commitment payoff is 3. Let $(\rho_1^{d_2}, \rho_2^{d_2}) = (\gamma_1, \gamma_2)$. Proposition 1.3 can be applied only if $\Gamma_H \subseteq \Gamma_L \iff \gamma_2 = \frac{3}{4}$, and $\gamma_1 \in \{\frac{1}{4}, \frac{3}{4}\}$. With

Proposition 1.4, this set can be dramatically extended. As shown in the Appendix, if γ_1, γ_2 satisfy

$$\begin{aligned}\frac{3}{4} &\leq \gamma_2 < 1 \\ \frac{1}{4} &\leq \gamma_1 < \frac{5}{8}\end{aligned}$$

and Player 2's prior belief $\pi^0(\cdot)$ satisfies

$$\begin{aligned}\frac{\pi^0((\xi(H), d_2))}{\pi^0((\xi_0, d_1))} &= 5 - 8\gamma_1 \\ \frac{\pi^0((\xi(H), d_1))}{\pi^0((\xi_0, d_2))} &= \frac{3 - \gamma_2 - 3\gamma_1}{\gamma_2 - \gamma_1}\end{aligned}$$

for all $\delta \in (0, 1)$, there exists a BNE such that normal type Player 1 plays $((2\gamma_1 - \frac{1}{2})H, (\frac{3}{2} - 2\gamma_1)L)$ when the monitoring structure is d_1 and plays $(\frac{\gamma_2 - \frac{3}{4}}{\gamma_2 - \gamma_1}H, \frac{\frac{3}{4} - \gamma_1}{\gamma_2 - \gamma_1}L)$ when the monitoring structure is d_2 , while Player 2 always plays $(\frac{1}{4}c, \frac{3}{4}s)$. It is easy to check that normal type Player 1's payoff in equilibrium is smaller than the commitment payoff. Figure 4 shows the region of monitoring structures discussed in this example.

With Proposition 1.2, I can show that the conditions here are (almost) sufficient and necessary. To be more specific, I can show that:

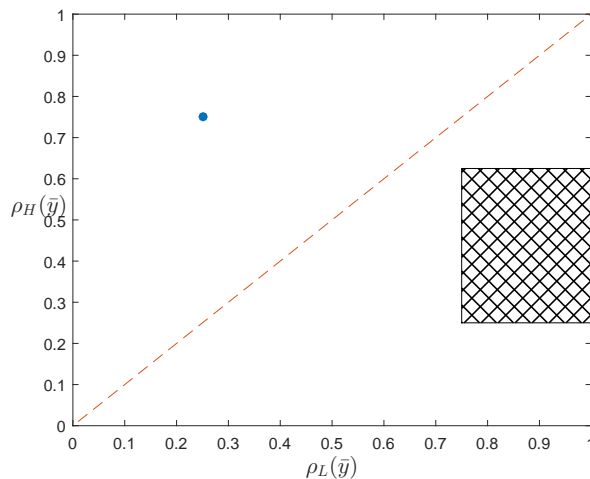
If $\gamma_1 \neq \gamma_2$ and either

$$\gamma_2 \in \left(0, \frac{3}{4}\right)$$

or

$$\gamma_1 \in \left(0, \frac{1}{4}\right) \cup \left(\frac{5}{8}, 1\right),$$

Figure 4: Failure of reputation building examples part 2



then for any monitoring structure d_i and any $\varepsilon > 0$, there exists δ^* such that for any $\delta > \delta^*$, normal type Player 1's payoff under monitoring structure d_i is greater than or equal to $u_{\hat{\alpha}_1} - \varepsilon$ in any BNE.

1.4 Many Commitment Types

In this section, I consider the case in which there is more than one commitment type. Let $\Upsilon = \{\xi(\hat{\alpha}_{1,1}), \xi(\hat{\alpha}_{1,2}), \dots, \xi(\hat{\alpha}_{1,\hat{N}})\}$ be the set of commitment types. Then $\Xi = \{\xi_0\} \cup \Upsilon$ is the set of types. Further, let $\xi(\hat{\alpha}_{1,i})$ represent a commitment type who always plays $\hat{\alpha}_{1,i} \in \Delta(A_1)$. Next, let $\hat{\theta}_{i,k} = (\xi(\hat{\alpha}_{1,i}), d_k)$. Other settings and notations are the same as before.

I consider the case in which commitment types satisfy the following assumption.

Assumption 1.1. For any $i \neq j$, $\Gamma_{\hat{\alpha}_{1,i}} \cap \Gamma_{\hat{\alpha}_{1,j}} = \emptyset$.

Γ_α is the set of distribution of signals induced by α , so this assumption means that Player 2 can statistically distinguish different commitment types. To deal with many commitment types, I start with the following proposition. Let $\tilde{G}(\varepsilon, \theta) = \{t \in \mathbb{N}; \pi^t(\theta) < \varepsilon\}$.

Proposition 1.5. *For any game satisfying Assumption 1.1, there exist an M and e such that, for any $i \neq j$, any $k, l, \varepsilon > 0$,*

$$Q_{\hat{\theta}_{i,k}} \left(\left\{ |\mathbb{N} \setminus \tilde{G}(\varepsilon, \hat{\theta}_{j,l})| \geq M \right\} \right) \leq e\varepsilon$$

in any BNE.

The idea follows the intuition I discussed in relation to Proposition 1.1. When normal type Player 1's mimics commitment type $\xi(\hat{\alpha}_{1,i})$ under the monitoring structure d_k , Player 2 eventually learns that the true distribution is $\rho_{d_k, \hat{\alpha}_{1,i}}$. Since $\Gamma_{\hat{\alpha}_{1,i}} \cap \Gamma_{\hat{\alpha}_{1,j}} = \emptyset$, Player 2 can statistically distinguish different commitment types; hence Player 2 eventually learns that she is not facing a commitment type $\hat{\alpha}_{1,j}$.

This proposition implies that Player 2's posterior belief eventually will be almost the same with or without these additional commitment types. Player 2's behavior, therefore, will also be the same eventually, as will be Player 1's payoff.

1.5 Infinite Set of Monitoring Structures

So far, I have only analyzed the case in which the set of monitoring structures is finite. In this section I consider the case in which the set is a continuum.

Since the game becomes much more complicated, I only focus on the “product-choice” game discussed in Examples 1.3 and 1.2. The payoffs are shown in the following payoff matrix:

		Player 2	
		c	s
Player 1	H	2,3	0,2
	L	3,0	1,1

Suppose that $\hat{\alpha}_1 = H$, then $BR_2(\hat{\alpha}_1) = \{c\}$ and $u_{\hat{\alpha}_1} = 2$. Player 1's actions are monitored via a public signal: $y \in \{\bar{y}, \underline{y}\}$. Any $d \in D$ represents the following monitoring structure:

$$\rho_{d,H}(y) = \begin{cases} \rho_1^d & y = \bar{y} \\ 1 - \rho_1^d & y = \underline{y} \end{cases}$$

$$\rho_{d,L}(y) = \begin{cases} \rho_2^d & y = \bar{y} \\ 1 - \rho_2^d & y = \underline{y} \end{cases}$$

Then there is a bijection from D to $(0, 1)^2$:

$$f(d) = (\rho_1^d, \rho_2^d).$$

Abusing notation, in this section I will sometimes use (ρ_1^d, ρ_2^d) to represent d . For expositional clarity, I further assume that the monitoring structure and Player 1's type are independently distributed.

Suppose that the prior density for the monitoring structures (ρ_1, ρ_2) is given by $\pi^0(\rho_1, \rho_2)$. Assume that the marginal distributions of ρ_1 and ρ_2 both exist and are denoted by $\pi_1^0(\rho_1)$ and $\pi_2^0(\rho_2)$. The following proposition discusses the conditions under which the classical

results do not hold.

Proposition 1.6. *If $\text{supp}(\pi_1^0) \subseteq \text{supp}(\pi_2^0)$ and $\sup_{\rho} \frac{\pi_1^0(\rho)}{\pi_2^0(\rho)} < \infty$, then, there is a $1 > \hat{\mu} > 0$, such that for any $\pi^0(\xi_0) \geq \hat{\mu}$ there exists a Nash equilibrium such that normal type Player 1 always chooses L and Player 2 always chooses s .*

The proof duplicates that of Proposition 1.3. The intuition is also similar. In equilibrium, since $\text{supp}(\pi_1^0) \subseteq \text{supp}(\pi_2^0)$, Player 2 cannot statistically distinguish a commitment type from a normal type. When Player 2's prior belief about Player 1 being normal is sufficiently strong, she will always believe that Player 1 is a normal type with high probability no matter the history. Therefore she always chooses a best response to L , and Player 1 also has no incentive to deviate.

Figure 5 illustrates some possible sets that satisfy the conditions in Proposition 1.6. The distributions in Figure 5 are all uniforms.

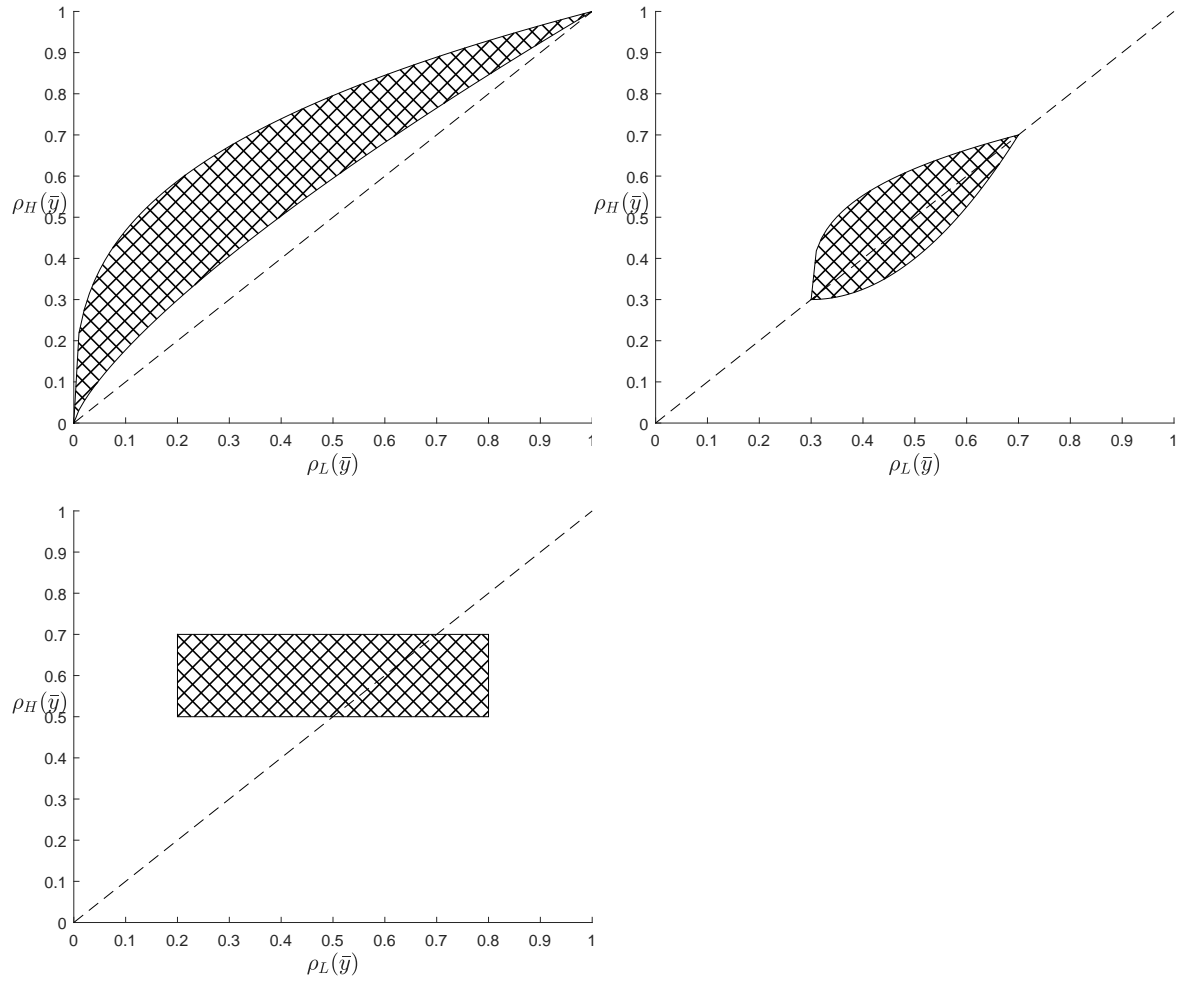
I now discuss the conditions under which the classical results can be extended. The techniques I used before cannot be applied here. For example, in Proposition 1.1, if the set of monitoring structure is infinite, then M approaches infinity. Since

$$\begin{aligned} & \lim_{M \rightarrow \infty} \left[(1 - \varepsilon) \delta^M v_{\hat{\alpha}_1}^{w,d}(\varepsilon) + (1 - (1 - \varepsilon) \delta^M) \min_{(\alpha_1, \alpha_2)} u_1(\alpha_1, \alpha_2) \right] \\ &= \min_{(\alpha_1, \alpha_2)} u_1(\alpha_1, \alpha_2), \end{aligned}$$

it is no longer a good lower bound. To overcome this challenge, I use techniques similar to those of Gossner (2011).

Given D , define the d -weak- ε -entropy-confirming best response as follows.

Figure 5: Infinite set of monitoring structure example 1



Definition 1.2. $\alpha_2 \in \Delta(A_2)$ is a d -weak- ε -entropy-confirming best response to $\alpha_1 \in \Delta(A_1)$

if there exist $\{\alpha'_1(d')\}_{d' \in D} \subseteq \Delta(A_1)$ and $\{\pi'(d')\}_{d' \in D} \subseteq \mathbb{R}_+$ such that

1. $\pi'(d') > 0$ if and only if $d' \in D$;
2. $\iint_D \pi'(d') d\rho_1^{d'} d\rho_2^{d'} = 1$;
3. α_2 is a best response to $\iint_D \alpha'_1(d') \pi'(d') d\rho_1^{d'} d\rho_2^{d'}$;
4. $d(\rho_{d, \alpha_1} \parallel \iint_D \rho_{d'', \alpha'_1(d'')} \pi'(d') d\rho_1^{d'} d\rho_2^{d'}) < \varepsilon$.

The set of d -weak- ε -entropy confirming best responses to α_1 is denoted by $\hat{B}_\varepsilon^{w,d}(\alpha_1)$. Let

$$\hat{v}_{\alpha_1}^{w,d}(\varepsilon) = \min_{\alpha_2 \in \hat{B}_\varepsilon^{w,d}(\alpha_1)} u_1(\alpha_1, \alpha_2)$$

$\hat{w}_{\alpha_1}^{w,d}$ be the largest convex function below $\hat{v}_{\alpha_1}^{w,d}$. Further, let $D_{\rho_1^d, \varepsilon_1} = \{(\rho_1, \rho_2) \mid \rho_1^d - \varepsilon_1 < \rho_1 < \rho_1^d + \varepsilon_1\}$. I start with the following proposition.

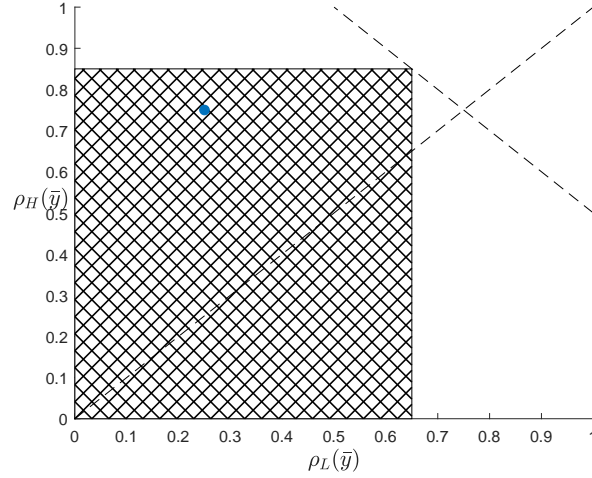
Proposition 1.7. *Suppose $0 < \mu < 1$. If d satisfies*

1. $d \in D$,
2. for any $\varepsilon_1 > 0$, $\varepsilon_2 = \iint_{D \cap D_{\rho_1^d, \varepsilon_1}} \pi^0(d') d\rho_1^{d'} d\rho_2^{d'} > 0$,

then, for any $\varepsilon_1 > 0$, normal type of Player 1's payoff in any BNE under the monitoring structure d is greater than or equal to

$$\hat{w}_H^{w,d} \left(-(1 - \delta) \log [\pi^0(\xi(H)) \varepsilon_2] + \frac{\varepsilon_1^2}{\min\{\rho_1^d - \varepsilon_1, 1 - \rho_1^d - \varepsilon_1\}} \right)$$

Figure 6: Infinite set of monitoring structure example 2



According to the definition of $\hat{B}_\varepsilon^{w,d}(\alpha_1)$, as proven in this proposition, I can still show that when normal type Player 1 mimics the commitment type, Player 2 eventually learns that the real distribution is $\rho_{d,H}$ but nothing further. In some cases, this is enough to guarantee that there is a good lower bound for normal type Player 1's payoff, as shown in the following proposition.

Proposition 1.8. *Suppose $0 < \mu < 1$. If d satisfies that*

1. $d \in D$,
2. for any $\varepsilon_1 > 0$, $\varepsilon_2 = \iint_{D \cap D_{\rho_1^d, \varepsilon_1}} \pi^0(d') d\rho_1^{d'} d\rho_2^{d'} > 0$,
3. $\rho_1^d > \frac{1}{2} \sup_{d' \in D} \rho_1^{d'} + \frac{1}{2} \sup_{d' \in D} \rho_2^{d'}$,

then, for any $\varepsilon > 0$, there exists a $\delta^ > 0$ such that for any $\delta > \delta^*$ normal type Player 1's payoff under the monitoring structure d is greater than or equal to $u_H - \varepsilon$ in any BNE.*

Let $d = (\frac{3}{4}, \frac{1}{4})$. Figure 6 illustrates one possible D that satisfies the conditions in Proposition 1.8.

This is a subset of the set discussed in Example 1.2 which still covers a lot of cases.

1.6 Conclusion

I examine how the information structure affects reputation effects in long-run relationships. Formally, I study a repeated game between a long-run player and a series of short-run opponents. There is incomplete information about the type of long-run player playing the game. He can either be of a simple commitment type who plays a fixed (possibly mixed) action at every period or of a normal type who acts strategically. The reputation effect that I focus on is the lower bound of the equilibrium payoffs to the normal type long-run player.

The modeling innovation is that the short-run player is unsure about the monitoring structure. The uncertainty about the monitoring structure introduces new challenges to reputation building because there may not be a direct relationship between the distribution of signals and the long-run player's strategy. Thus, the long-run player may not have the ability to establish a reputation for commitment.

I first provide sufficient conditions under which reputation effects on long-run player's payoffs can be extended to the current framework. I introduce the set of "well-behaved" monitoring structures. I show that, if the short-run players' prior belief about the monitoring structures belonging to this set is sufficiently large, then reputation effects are robust. I give an example to show that these conditions cover most cases. I also show that when the commitment payoff is the highest payoff Player 1 can get, the conditions can be relaxed.

I also provide sufficient conditions for the existence of "bad" equilibria in which the payoff of the long-run player is strictly lower than the commitment payoff. I show that, when the short-run players cannot statistically distinguish commitment action from a bad action, the standard reputation results break down.

Last, I investigate two variations of this model. First, I relax the assumptions to allow more than one commitment type and show that this does not affect the main results. Second, I investigate a case in which the set of monitoring structures is a continuum. I only consider a very simple setting and show that Negative results can be extended easily, in contrast, positive results are much harder to extend.

There are two interesting directions for future research. First, I only consider a specific game when I investigate the case in which the set of monitoring structures is a continuum. It would be interesting to analyze whether the results can be extended to other games. Second, it would also be interesting to combine this paper with Pei (2018) to investigate a model in which there is uncertainty about both the monitoring structure and Player 2's payoffs.

Chapter 2: Opinion Manipulation and Disagreement in Social Networks

2.1 Introduction

In many cases, individuals form opinions through a process of social learning, in which they update their opinions based on both their own experiences and the information they receive from others, including friends, family, and coworkers, as well as media sources. One interesting and puzzling fact about this process is that individuals remain divided in their opinions even when they consistently communicate with each other. Disagreements are ubiquitous on economic and political phenomena such as whether a politician is competent, whether gun control is a solution to gun violence and other crimes, climate change and its causes, and so on. A survey conducted by Gallup in 2015 found that 40 percent of conservative Republicans thought that global warming would never happen, compared with only 3 percent of liberal Democrats (Dugan 2015). Given this, it is essential to understand why disagreement among individuals in a society can persist and how disagreement is affected by the structure of social networks.

I base my study on the model of DeGroot (1974). In this classical framework, there are N agents in the society. Agent i has an initial opinion $x_i^0 \in R$. A weighted directed network describes the social structure of the society in which the weight represents the degree to which an agent trusts another agent. At each date, the agents receive information from their friends/neighbors in the social network and update their opinions. An agent's new opinion is simply the weighted average of the information he/she receives. In DeGroot

(1974), the agents report their opinions truthfully. If the network is strongly connected and aperiodic, agents will eventually reach consensus. However, as some researchers (Acemoglu et al. 2013, Bindel et al. 2015) have argued, this emphasis on consensus can only describe a particular type of opinion dynamics and is unable to explain the persistent disagreement among individuals in a society.

I study a model of opinion dynamics in which, generically, there is no consensus. I analyze how the extent of disagreement is affected by the network structure and derive a new measure of social influence. The crucial difference between this model and that of DeGroot (1974) is that some agents can lie. More specifically, there are two types of agents in society: sophisticated and naive. Sophisticated agents have different personal biases and are willing to persuade others to believe in their biases. These biases are exogenous. They can lie to naive agents, but there is a cost to lying, scaled by the parameter $\alpha > 0$, which represent the intensity of lying cost. To ensure tractability, I assume that the sophisticated agents are myopic and the utility is quadratic. Naive agents behave the same as the agents in DeGroot (1974), and they can not infer the accuracy of the signals.

To illustrate this approach, consider, for example, the 2016 presidential election. We can think of naive agents as the public and sophisticated agents as media sources. During the campaign, people regularly updated their opinions about which candidate is better. They received information from the media, and they also communicated with their family and friends. The media may have been biased and may have sought to advance candidates they favored by strategically disseminating information. In fact, recent evidence shows that some news providers created fake news to advance the candidates they preferred (Allcott and Gentzkow 2017). When news providers don't report their opinions truthfully, there is a cost

to lying, whether a psychological cost or a real one.

Since the sophisticated agents are myopic, the equilibrium can be calculated for each period separately. First, I show that, in each period, there exists a unique Nash equilibrium in which the signals sent by sophisticated agents can be represented by linear combinations of the opinions of all agents in the previous period and the biases of the sophisticated agents. When the intensity of lying cost (α) is large, the results are more clear and intuitive. The signals are perturbation of the real opinions of the sophisticated agents. The perturbation has the order $\frac{1}{\alpha}$. It is negatively associated with the opinions of other agents in the previous period and positively correlated with the agents' biases.

Second, I investigate the opinion dynamics. I show that, when the network is strongly connected and aperiodic, and the intensity of lying cost is larger than a constant, the opinions converge but, generically, do not converge to a consensus. The asymptotic opinions are proved to be determined by the sophisticated agents' bias and the structure of the network and are not affected by the initial opinions. Then I provide a more detailed analysis of the asymptotic opinions. I find that the asymptotic opinion vector equals to a vector of same value \hat{c} plus the vector representing disagreement. Disagreement also has the order of $\frac{1}{\alpha}$. It is positively correlated with the inverse of the spectral gap (the difference between the two largest eigenvalues) of the network. There is a vast literature of probability theory devoted to studying the relation between the spectral gap and other properties of the social network. Details can be found in Jackson (2008), Levin et al. (2009), Golub and Sadler (2016). The primary takeaway is that the spectral gap is small if the society is segregated. Therefore, the result indicates that, when the society is more segregated, disagreement is greater.

Third, I study the social influence. As the intensity of lying cost approaches infinity,

disagreement vanishes and the asymptotic opinions are all close to \hat{c} , which is a weighted average of the bias of the sophisticated agents. This weight can be interpreted as a measure of social influence. It can be decomposed into two separate factors: direct influence and indirect influence. The direct influence is simply a summation of the weights that naive agents place on the sophisticated agent. The indirect influence is a summation of these weights multiplied by the eigenvector centrality of respective naive agents.

I next examine the opinion dynamics in a slightly different environment, in which the sophisticated agents assign varying degrees of importance to different naive agents. The results are almost the same as before. The only significant difference is that the direct influence now is the summation of the weights that naive agents place on the sophisticated agent multiplied by the respective degree of importance.

Finally, I investigate the case in which the sophisticated agents are forward-looking. The model then becomes so complicated that I can only solve a particular case in which there are only two stubborn sophisticated agents, who never change their opinions, and one naive agent who places weight only on these two sophisticated agents. I show that, in comparison with the case of myopic agents, the agent who has more influence gains even more influence, and the utility of both sophisticated agents is larger due to a reduction in lying cost. I also examine comparative statics on the discount factor. I prove that these two effects become more significant when the discount factor is greater.

My analysis shows that one reason why disagreement persists is that individuals with diverging interests misrepresent opinions for their own benefit. Moreover, I demonstrate that, when the cost of lying is low, and the spectral gap is small, the opinion variance is high. This may provide one explanation for the polarization of public opinions: the growth of

social media. Since the last decade, social media use has risen sharply. A recent survey shows that 62 percent of U.S. adults get their news on social media (Gottfried and Shearer 2016). There are two reasons to think that this change in media technology increases the divergence in public opinions. First, social media providers face a much lower cost of a bad reputation than mass media, which means a lower intensity of lying cost. Second, the exposure to ideologically diverse news and opinions on social media is limited by both individual choices and algorithmic recommendation systems (Baskshy, Messing and Adamic 2015). This leads to a small spectral gap of the social networks.

My paper is related to an extensive body of literature on non-Bayesian learning.¹ In this literature, the classical result is that, if the network is strongly connected and aperiodic, agents will eventually reach consensus (DeGroot 1974, DeMarzo et al. 2003, Golub and Jackson 2010). Besides this paper, several other models have been proposed to explain persistent disagreements. One approach is provided by models incorporating some homophily, which means that opinions that are too far from one's own are given little or no weight. Opinions in this kind of model eventually converge to a limit opinion profile, in which agents are partitioned into several groups and agents have the same limit opinion if and only if they are in the same group. A survey of this approach is provided by Lorenz (2007). Another method is provided by models in which some agents are stubborn, which means that they always put some weight on their initial opinion. This updating rule was first proposed by Fiedlin and Johnsen (1990). Bindel et al. (2015) used this framework to quantify the inherent social cost of this lack of consensus. Acemoglu et al. (2013) also investigated a model with stubborn

¹There is also a significant body of literature on Bayesian learning, see Golub and Sadler (2016) for a survey.

agents. They used an inhomogeneous stochastic gossip model and showed that the presence of stubborn agents leads to persistent opinion fluctuations and disagreement.

The model most similar in structure to this paper is by Buechel et al. (2015). They also studied a dynamic model of opinion formation in social networks in which agents can lie (misrepresent their opinion). We both use a quadratic utility function. The difference is that, in their paper, an agent's motive for distorting a true opinion is conformity or counter-conformity, which means that the agent states an opinion either closely aligned with or far from the group opinion. The characteristics of their model are also entirely different. In their paper, when the opinion dynamics converge, agents reach consensus. They focus on how the long-run group opinion is affected by conformity and whether information aggregation is undermined by misrepresentation of opinions.

2.2 Model

2.2.1 Description of the Environment

A finite set of agents, indexed by $i \in \{1, 2, \dots, N\}$, interact in a social network. The social network structure is captured through a non-negative matrix V , where $v_{ij} \geq 0$ represents the trust that agent i places in agent j . The matrix V is row stochastic, so that $\sum_{j=1}^N v_{ij} = 1$ for any i . I study a discrete-time dynamic model. Each agent i starts with an initial opinion $x_i^0 \in \mathbb{R}$ and then exchanges information with their neighbors at $t = 1, 2, 3, \dots$. At time t , the opinions of all agents are collected in $X^t = (x_1^t, x_2^t, \dots, x_N^t)' \in \mathbb{R}^N$. In the classical DeGroot

(1974) framework, agents report their opinions truthfully and the updating rule is:

$$X^{t+1} = VX^t$$

DeMarzo et al. (2003) provide one microfoundation for this updating rule: agents are boundedly rational in the sense that they treat all information they receive as new and fail to adjust for possible repetition.

In this model, I assume that some agents can state an opinion different from their true opinion. There are two kinds of agents: naive and sophisticated. Without loss of generality, the first d agents are sophisticated. I denote the set of sophisticated agents by $S(d) = \{1, 2, \dots, d\}$ and the set of naive agents by $A(d) = \{d + 1, \dots, N\}$. Naive agents report their opinions truthfully; in contrast, sophisticated agent i can express an opinion s_i^t that can be different from his/her true opinion x_i^t . Naive agents only observe stated opinions. On the other hand, I assume that sophisticated agents know the true opinions of others. This assumption is not important for the results but simplifies the analysis.

Therefore, the opinions of sophisticated agents are updated according to:

$$x_i^t = \sum_{j=1}^N v_{ij} x_j^{t-1}$$

and the opinions of naive agents are updated according to:

$$x_i^t = \sum_{j \in S(d)} v_{ij} s_j^t + \sum_{j \in A(d)} v_{ij} x_j^{t-1}$$

Let $X_1^{t-1} = \begin{pmatrix} x_1^{t-1} \\ x_2^{t-1} \\ \vdots \\ x_d^{t-1} \end{pmatrix}$ and $X_2^{t-1} = \begin{pmatrix} x_{d+1}^{t-1} \\ x_{d+2}^{t-1} \\ \vdots \\ x_N^{t-1} \end{pmatrix}$. Let me partition matrix V into 4 blocks as follows:

$$V = \begin{matrix} & \begin{matrix} d & N-d \end{matrix} \\ \begin{matrix} d \\ N-d \end{matrix} & \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \end{matrix} \quad (1)$$

in which, V_1 and V_2 represent how sophisticated agents trust sophisticated agents and naive agents; V_3 and V_4 represent how naive agents are influenced by sophisticated agents and naive agents. The updating rules can be written in matrix form:

$$X_1^t = V_1 X_1^{t-1} + V_2 X_2^{t-1} \quad (2)$$

$$X_2^t = V_3 X_1^{t-1} + V_4 X_2^{t-1} \quad (3)$$

Sophisticated agent i has an exogenous ideal point $b_i \in \mathbb{R}$ and tries to convince naive agents to believe in b_i . For simplicity, I use quadratic preference and assume that sophisticated agents are myopic. A sophisticated agent i 's utility function at time t is:

$$u_i(X^{t-1}, s_i^t, b_i) = - \sum_{j \in A(d)} (x_j^t - b_i)^2 - \alpha (s_i^t - x_i^{t-1})^2$$

$-\alpha (s_i^t - x_i^{t-1})^2$ represents the cost of lying, where $\alpha > 0$ displays the intensity of lying cost.

The costs may arise from probabilistic ex post facto verification that results in penalties if

misreporting is detected. They can also be purely physiological. Numerous experimental studies have provided evidence that people have an intrinsic aversion to lying even if lying cannot be detected at all (e.g, Abeler et al. 2014).

2.2.2 Equilibrium

Since sophisticated agents are all myopic, I can analyze the equilibrium for each period separately. In each period, sophisticated agents play a static game, in which they simultaneously choose s_i^t to maximize their utility. Given other sophisticated agents' strategies, agent $i \in S(d)$ solves:

$$\max_{s_i^t} u_i(x^{t-1}, s_i^t, b_i) = - \sum_{j \in A(d)} (x_j^t - b_i)^2 - \alpha (s_i^t - x_i^{t-1})^2$$

in which $x_j^t = \sum_{k \in S(d)} v_{jk} s_k^t + \sum_{k \in A(d)} v_{jk} x_k^{t-1}$. The FOC is:

$$\begin{aligned} & -2 \sum_{j \in A(d)} v_{ji} (x_j^t - b_i) - 2\alpha (s_i^t - x_i^{t-1}) = 0 \\ \iff & \sum_{j \in A(d)} v_{ji} (\sum_{k \in S(d)} v_{jk} s_k^t) + \alpha s_i^t = \alpha x_i^{t-1} - \sum_{j \in A(d)} v_{ji} (\sum_{k \in A(d)} v_{jk} x_k^{t-1}) + \sum_{j \in A(d)} v_{ji} b_i \end{aligned}$$

Then the equilibrium strategies of all sophisticated agents are the solutions to the following simultaneous equations:

$$\begin{aligned} \sum_{j \in A(d)} v_{j1} (\sum_{k \in S(d)} v_{jk} s_k^t) + \alpha s_1^t &= \alpha x_1^{t-1} - \sum_{j \in A(d)} v_{j1} (\sum_{k \in A(d)} v_{jk} x_k^{t-1}) + \sum_{j \in A(d)} v_{j1} b_1 \\ \sum_{j \in A(d)} v_{j2} (\sum_{k \in S(d)} v_{jk} s_k^t) + \alpha s_2^t &= \alpha x_2^{t-1} - \sum_{j \in A(d)} v_{j2} (\sum_{k \in A(d)} v_{jk} x_k^{t-1}) + \sum_{j \in A(d)} v_{j2} b_2 \\ & \vdots \\ \sum_{j \in A(d)} v_{jd} (\sum_{k \in S(d)} v_{jk} s_k^t) + \alpha s_d^t &= \alpha x_d^{t-1} - \sum_{j \in A(d)} v_{jd} (\sum_{k \in A(d)} v_{jk} x_k^{t-1}) + \sum_{j \in A(d)} v_{jd} b_d \end{aligned}$$

Let

$$B = \begin{pmatrix} \sum_{j \in A(d)} v_{j1} b_1 \\ \sum_{j \in A(d)} v_{j2} b_2 \\ \vdots \\ \sum_{j \in A(d)} v_{jd} b_d \end{pmatrix} \text{ and } S^t = \begin{pmatrix} s_1^t \\ s_2^t \\ \vdots \\ s_d^t \end{pmatrix}. \text{ Then, this system of equations can be written}$$

as:

$$(\alpha I_d + V_3^T V_3) S^t = \alpha X_1^{t-1} - V_3^T V_4 X_2^{t-1} + B$$

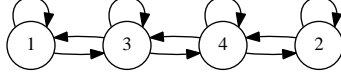
Proposition 2.9. *For any $\alpha > 0$, and any social network, there exists a unique Nash equilibrium in each period. The signals sent by sophisticated agents, in equilibrium, satisfy that*

$$S^t = \alpha(\alpha I_d + V_3^T V_3)^{-1} X_1^{t-1} - (\alpha I_d + V_3^T V_3)^{-1} V_3^T V_4 X_2^{t-1} + (\alpha I_d + V_3^T V_3)^{-1} B \quad (4)$$

The proof of Proposition 2.9 as well as all proofs of the following results are in the Appendix. The proof works by showing that $\alpha I_d + V_3^T V_3$ is positive definite, thus, invertible. Proposition 2.9 shows that in each period there exists a unique Nash equilibrium such that the sophisticated agents report their opinions as a linear combination of the true opinions, the opinions of the naive agents and their biases. When α is large, I can rewrite S^t as follows:

$$\begin{aligned} S^t &= \alpha(\alpha I_d + V_3^T V_3)^{-1} X_1^{t-1} - (\alpha I_d + V_3^T V_3)^{-1} V_3^T V_4 X_2^{t-1} + (\alpha I_d + V_3^T V_3)^{-1} B \\ &= (I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} X_1^{t-1} - \frac{1}{\alpha} (I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T V_4 X_2^{t-1} + \frac{1}{\alpha} (I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} B \\ &= X_1^{t-1} + \frac{1}{\alpha} (-V_3^T V_3 X_1^{t-1} - V_3^T V_4 X_2^{t-1} + B) + O(\frac{1}{\alpha^2}) \end{aligned}$$

Figure 7: Network example 1



The opinions expressed by sophisticated agents are perturbation of their real opinions. The perturbation has the order $\frac{1}{\alpha}$. When the intensity of lying cost (α) is large, the difference between stated opinions and true opinions is small. This deviation is positively associated with the agent's bias and negatively correlated with the opinions of other agents in the previous period. The intuition is straightforward. First, when sophisticated agent i has a higher ideal point (b_i), he/she express a higher opinion to make sure that the naive agents also have higher opinions. Second, $V_4 X_2^{t-1}$ represents how naive agents are affected by other naive agents and $V_3 X_1^{t-1}$ represents approximately how naive agents are affected by sophisticated agents. When these two parts are high, agent i either has no need to express a high opinion or may even have to express a low opinion to dilute the influence of other agents. The following example also illustrates this expression.

Example 2.6. Consider a network consisting of four agents. Agents 1 and 2 are both sophisticated agents while the others are naive. The network is shown in Figure 1. Then

$$V_3^T = V_3 = \begin{pmatrix} v_{31} & 0 \\ 0 & v_{42} \end{pmatrix}$$

and

$$(\alpha I_d + V_3^T V_3)^{-1} = \begin{pmatrix} \frac{1}{\alpha + v_{31}^2} & 0 \\ 0 & \frac{1}{\alpha + v_{42}^2} \end{pmatrix}$$

so

$$\begin{aligned}
\begin{pmatrix} s_1^t \\ s_2^t \end{pmatrix} &= \begin{pmatrix} \frac{\alpha}{\alpha+v_{31}^2} & 0 \\ 0 & \frac{\alpha}{\alpha+v_{42}^2} \end{pmatrix} \begin{pmatrix} x_1^{t-1} \\ x_2^{t-1} \end{pmatrix} - \begin{pmatrix} \frac{1}{\alpha+v_{31}^2} & 0 \\ 0 & \frac{1}{\alpha+v_{42}^2} \end{pmatrix} \begin{pmatrix} v_{31} & 0 \\ 0 & v_{42} \end{pmatrix} \begin{pmatrix} v_{33} & v_{34} \\ v_{43} & v_{44} \end{pmatrix} \begin{pmatrix} x_3^{t-1} \\ x_4^{t-1} \end{pmatrix} \\
&+ \begin{pmatrix} \frac{1}{\alpha+v_{31}^2} & 0 \\ 0 & \frac{1}{\alpha+v_{42}^2} \end{pmatrix} \begin{pmatrix} v_{31}b_1 \\ v_{42}b_2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\alpha}{\alpha+v_{31}^2}x_1^{t-1} \\ \frac{\alpha}{\alpha+v_{42}^2}x_2^{t-1} \end{pmatrix} + \begin{pmatrix} \frac{v_{31}}{\alpha+v_{31}^2}(b_1 - v_{33}x_3^{t-1} - v_{34}x_4^{t-1}) \\ \frac{v_{42}}{\alpha+v_{42}^2}(b_2 - v_{43}x_3^{t-1} - v_{44}x_4^{t-1}) \end{pmatrix}
\end{aligned}$$

s_i^t is decreasing in x_3^{t-1} and x_4^{t-1} and increasing in x_i^{t-1} and b_i .

2.3 Opinion Dynamics

2.3.1 Convergence

opinions in a given period as a function of true opinions and the ideal points of sophisticated agents. By combining Proposition 2.9 and updating rules 2, 3, I can derive the dynamics of true opinions. Let

$$U = \begin{pmatrix} V_1 & V_2 \\ \alpha V_3(\alpha I_d + V_3^T V_3)^{-1} & (I_{N-d} - V_3(\alpha I_d + V_3^T V_3)^{-1} V_3^T) V_4 \end{pmatrix}$$

$$\hat{B} = \begin{pmatrix} 0_d \\ V_3(\alpha I_d + V_3^T V_3)^{-1} B \end{pmatrix}$$

and $X^t = \begin{pmatrix} X_1^t \\ X_2^t \end{pmatrix}$, then the transformation from X^{t-1} to X^t is represented by the following matrix equation:

$$X^t = UX^{t-1} + \hat{B}$$

Then,

$$\begin{aligned} X^t &= UX^{t-1} + \hat{B} \\ &= U^2X^{t-2} + U\hat{B} + \hat{B} \\ &\quad \vdots \\ &= U^tX^0 + \sum_{i=0}^{t-1} U^i\hat{B} \end{aligned}$$

in which U^tX^0 displays the influences of initial opinions and $\sum_{i=0}^{t-1} U^i\hat{B}$ represents the effect of biases.

Now I can discuss the condition for convergence of opinions. In order to describe the results, I need to introduce some standard graph-theoretic definitions. In this paper, I use the same definitions employed by Golub and Jackson (2010).

Definition 2.3. A walk in V is a sequence of nodes i_1, i_2, \dots, i_K , not necessarily distinct, such that $V_{i_k i_{k+1}} > 0$ for each $k \in 1, \dots, K - 1$. The length of the walk is defined as $K - 1$.

Definition 2.4. A path in V is a walk consisting of distinct nodes.

Definition 2.5. The matrix V is strongly connected if there is path in V from any node to any other node.

Definition 2.6. A cycle is a walk i_1, i_2, \dots, i_K such that $i_1 = i_K$. The length of a cycle with K (not necessarily distinct) entries is defined as $K - 1$. A cycle is simple if the only node appearing twice in the sequence is the starting (and ending) node.

Definition 2.7. The matrix V is aperiodic if the greatest common divisor of the lengths of its simple cycles is 1.

Let me begin with the simple case where all sophisticated agents have the same ideal point:

$$b_1 = b_2 = \dots = b_d = b$$

Lemma 2.1 shows that, without loss of generality, I can assume that $b = 0$.

Lemma 2.1. *When all sophisticated agents have the same ideal point b , let $Z^t =$*

$$\begin{pmatrix} x_1^t - b \\ x_2^t - b \\ \vdots \\ x_N^t - b \end{pmatrix},$$

then $Z^t = UZ^{t-1}$.

Then

$$\begin{aligned} X^t &= UX^{t-1} \\ &\vdots \\ &= U^t X^0 \end{aligned}$$

In DeGroot's model, if V is aperiodic and strongly connected, then for any X^0 , $\left(\lim_{t \rightarrow \infty} V^t X^0\right)_i = lX^0$, where l is the unique left eigenvector of V corresponding to eigenvalue 1, whose entries

sum to 1. Proposition 2.10 shows that the opinion dynamics here are quite different from those in the DeGroot model, even though the expressions look the same and $\lim_{\alpha \rightarrow \infty} U = V$.

Proposition 2.10. *If V is strongly connected and aperiodic, then there exists an $\hat{\alpha} > 0$ such that for any $\alpha > \hat{\alpha}$ and any vector $X^0 \in \mathbb{R}^N$, $\lim_{t \rightarrow \infty} U^t X^0 = 0$*

In the proof of this Proposition, I show first that there exists an $\hat{\alpha} > 0$ and $M \in \mathbb{N}^+$ such that, for any $\alpha > \hat{\alpha}$, U^M is a positive matrix. Then I demonstrate that $\|U^M\|_\infty < 1$ by calculating $U^M \mathbf{1}_N$. Therefore, $\lim_{k \rightarrow \infty} \|U^{M*k}\|_\infty = 0$ and $\lim_{t \rightarrow \infty} U^t X^0 = 0$. The condition presented here is fairly weak. In addition to the standard requirements regarding network structure, I only need α to be large. Since the sophisticated agents are myopic, if α is small, they may send some extreme signals, which leads to divergence of opinions.

Proposition 2.10 shows that, if all sophisticated agents have the same ideal point, sophisticated agents can perfectly manipulate the asymptotic opinions. The sophisticated agents do not need to be at special locations in the network or have substantial direct influence. A special case occurs when there is only one sophisticated agent. Agents eventually believe what that agent wants them to believe. .

I now turn to the general setting in which sophisticated agents have different ideal points. Then $X^t = U^t X^0 + \sum_{i=0}^{t-1} U^i \hat{B}$. Now Proposition 2.10 can be interpreted in a different way: the influences of initial opinions vanish in the long run. If $\lim_{t \rightarrow \infty} \sum_{i=0}^{t-1} U^i \hat{B}$ exists, then $\lim_{t \rightarrow \infty} X^t = \lim_{t \rightarrow \infty} \sum_{i=0}^{t-1} U^i \hat{B}$; the asymptotic opinion vector is completely characterized by the second term. Therefore, whether the opinions converge is determined by this Neumann series $\sum_{i=0}^{t-1} U^i$. It turns out that the condition for convergence is the same as Proposition 2.10.

Proposition 2.11. *If V is strongly connected and aperiodic, then there exists an $\hat{\alpha} > 0$ such*

that for any $\alpha > \hat{\alpha}$ and any vector $X^0 \in R^N$, $\lim_{t \rightarrow \infty} X^t = (I - U)^{-1} \hat{B}$.

The proof of Proposition 2.11 follows from Proposition 2.10. Since there exists an $\hat{\alpha} > 0$ and $M \in N^+$ such that for any $\alpha > \hat{\alpha}$ $\|U^M\|_\infty < 1$, the Neumann series $\sum_{i=0}^{t-1} U^i$ converges to $(I - U)^{-1}$. Therefore, X^t converges to $(I - U)^{-1} \hat{B}$.

2.3.2 Disagreement

In this subsection, I study the properties of the long-term opinions. Proposition 2.11 shows that the long-term opinion vector is $(I - U)^{-1} \hat{B}$. It is determined by α (the intensity of lying cost), the structure of the network, and the biases of sophisticated agents. Since there are two inverses in $(I - U)^{-1}$, it is not easy to analyze what this indicates. The following result shows that, generically, there is no consensus.

Proposition 2.12. *If the opinions converge, $\text{rank}(V_3) = d$ and sophisticated agents don't have the same ideal point, then there is no consensus.*

The intuition behind Proposition 2.12 is as follows. If there is a consensus c , then

$$X_2^t = V_3 S^t + V_4 X_2^{t-1} \implies \sum_{j \in S(d)} v_{ij} c = (1 - \sum_{j \in A(d)} v_{ij}) c = \sum_{j \in S(d)} v_{ij} s_j \text{ for any } i \in A(d)$$

Since $\text{rank}(V_3) = d$, the stated opinions must be the same as the consensus c . However, sophisticated agents have different ideal points, so they should not express the same opinion.

Proposition 2.12 shows that, since sophisticated agents with diverging interests misrepresent opinions to influence the opinions of naive agents, disagreement persists in the society. The condition $\text{rank}(V_3) = d$ in this Proposition represents that the weights that naive agents

assign to sophisticated agents are linearly independent. It also indicates that all sophisticated agents can gain the attention of some naive agents. The other result that can be derived from this formula $(I - U)^{-1}\hat{B}$ is that, when the difference between ideal points of sophisticated agents is small, the long-run opinion variance is also small.

Proposition 2.13.

$$\max_i \left((I - U)^{-1}\hat{B} \right)_i - \min_i \left((I - U)^{-1}\hat{B} \right)_i \leq c_1(V, \alpha) (b_{max} - b_{min})$$

The proof of Proposition 2.13 is directly derived from Lemma 2.1. This Proposition shows that, given the network structure and the intensity of lying cost, the range of long-run opinions is bounded by the range of ideal points of sophisticated agents multiplied by a constant. When the network is very simple, I can derive a simple expression for $c_1(V, \alpha)$, as shown in the following example.

Example 2.7. Consider the same network as in Example 2.6. Then

$$I - U = \begin{pmatrix} 1 - v_{11} & 0 & -v_{13} & 0 \\ 0 & 1 - v_{22} & 0 & -v_{24} \\ -\frac{\alpha v_{31}}{\alpha + v_{31}^2} & 0 & 1 - \frac{\alpha v_{33}}{\alpha + v_{31}^2} & -\frac{\alpha v_{34}}{\alpha + v_{31}^2} \\ 0 & -\frac{\alpha v_{42}}{\alpha + v_{42}^2} & -\frac{\alpha v_{43}}{\alpha + v_{42}^2} & 1 - \frac{\alpha v_{44}}{\alpha + v_{42}^2} \end{pmatrix}$$

and

$$\hat{B} = \begin{pmatrix} 0 \\ 0 \\ \frac{v_{31}^2 b_1}{\alpha + v_{31}^2} \\ \frac{v_{42}^2 b_2}{\alpha + v_{42}^2} \end{pmatrix}$$

$$\begin{pmatrix} 1 - v_{11} & 0 & -v_{13} & 0 \\ 0 & 1 - v_{22} & 0 & -v_{24} \\ -\frac{\alpha v_{31}}{\alpha + v_{31}^2} & 0 & 1 - \frac{\alpha v_{33}}{\alpha + v_{31}^2} & -\frac{\alpha v_{34}}{\alpha + v_{31}^2} \\ 0 & -\frac{\alpha v_{42}}{\alpha + v_{42}^2} & -\frac{\alpha v_{43}}{\alpha + v_{42}^2} & 1 - \frac{\alpha v_{44}}{\alpha + v_{42}^2} \end{pmatrix} \begin{pmatrix} x_1^\infty \\ x_2^\infty \\ x_3^\infty \\ x_4^\infty \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{v_{31}^2 b_1}{\alpha + v_{31}^2} \\ \frac{v_{42}^2 b_2}{\alpha + v_{42}^2} \end{pmatrix}$$

so

$$x_1^\infty = x_3^\infty \text{ and } x_2^\infty = x_4^\infty$$

then

$$\begin{pmatrix} v_{31}^2 + \alpha v_{34} & -\alpha v_{34} \\ -\alpha v_{43} & v_{42}^2 + \alpha v_{43} \end{pmatrix} \begin{pmatrix} x_1^\infty \\ x_2^\infty \end{pmatrix} = \begin{pmatrix} v_{31}^2 b_1 \\ v_{42}^2 b_2 \end{pmatrix}$$

and

$$\begin{aligned}
\begin{pmatrix} x_1^\infty \\ x_2^\infty \end{pmatrix} &= \frac{1}{v_{31}^2 + v_{42}^2 + \alpha(v_{34}v_{42}^2 + v_{43}v_{31}^2)} \begin{pmatrix} v_{42}^2 + \alpha v_{43} & \alpha v_{34} \\ \alpha v_{43} & v_{31}^2 + \alpha v_{34} \end{pmatrix} \begin{pmatrix} v_{31}^2 b_1 \\ v_{42}^2 b_2 \end{pmatrix} \\
&= \frac{v_{42}^2 v_{31}^2}{v_{31}^2 + v_{42}^2 + \alpha(v_{34}v_{42}^2 + v_{43}v_{31}^2)} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\
&\quad + \frac{\alpha(v_{43}v_{31}^2 b_1 + v_{34}v_{42}^2 b_2)}{v_{31}^2 + v_{42}^2 + \alpha(v_{34}v_{42}^2 + v_{43}v_{31}^2)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}$$

so

$$c_1(V, \alpha) = \frac{v_{42}^2 v_{31}^2}{v_{31}^2 + v_{42}^2 + \alpha(v_{34}v_{42}^2 + v_{43}v_{31}^2)}$$

c_1 is decreasing in α , v_{34} and v_{43} .

When the network is more complicated, the expression of $c_1(V, \alpha)$ becomes messy and difficult to interpret. However, when α is large, I can rewrite $(I - U)^{-1} \hat{B}$ to obtain some insights. There is a need for a further assumption: V is diagonalizable. This assumption is fairly weak. One sufficient condition is that V has n distinct eigenvalues. Since a randomly chosen stochastic matrix almost certainly has n distinct eigenvalues, V is generically diagonalizable. Suppose $V = G \Lambda G^{-1}$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}$$

$$G^{-1} = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_N \end{pmatrix}$$

and

$$G = \begin{pmatrix} r_1 & r_2 & \cdots & r_N \end{pmatrix}$$

$\{\lambda_i\}$ are the eigenvalues listed in descending order by modulus, l_i and r_i are the respective left and right eigenvectors. Since V is a row stochastic matrix, $\lambda_1 = 1, l_1 = l, r_1 = 1_N$. Now it is possible to rewrite the long-run opinion vector $(I - U)^{-1}\hat{B}$.

Proposition 2.14. *If V is strongly connected, aperiodic and diagonalizable, then there exists an $\hat{\alpha} > 0$ such that, for any $\alpha > \hat{\alpha}$, there exists a constant \check{c} such that*

$$\hat{X} = \check{c}1_N + \frac{1}{\alpha} \left(\sum_{i=2}^N \frac{r_i l_i}{1 - \lambda_i} \right) \hat{B} + O\left(\frac{1}{\alpha^2}\right)$$

in which

$$\hat{B} = \left[I - \frac{1}{l \begin{pmatrix} 0 & 0 \\ 0 & V_3 V_3^T \end{pmatrix} 1_N} \begin{pmatrix} 0 & 0 \\ 0 & V_3 V_3^T \end{pmatrix} 1_N l \right] \begin{pmatrix} 0_d \\ V_3 B \end{pmatrix}$$

For the proof, I consider the inverse of $G^{-1}(I - U)G$. Since

$$\begin{aligned}
& G^{-1}(I - U)G \\
&= \begin{pmatrix} 0 & 0 \\ 0 & I_{N-1} - A_{22} \end{pmatrix} + \frac{1}{\alpha} G^{-1} \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T \end{pmatrix} VG \\
&= \begin{pmatrix} 0 & 0 \\ 0 & I_{N-1} - A_{22} \end{pmatrix} + \frac{1}{\alpha} G^{-1} \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T \end{pmatrix} G\Lambda \\
&= \begin{pmatrix} 0 & 0 \\ 0 & I_{N-1} - A_{22} \end{pmatrix} + \frac{1}{\alpha} G^{-1} \begin{pmatrix} 0 & 0 \\ 0 & V_3 V_3^T \end{pmatrix} G\Lambda \\
&\quad - \frac{1}{\alpha^2} G^{-1} \begin{pmatrix} 0 & 0 \\ 0 & V_3 V_3^T V_3 V_3^T \end{pmatrix} G\Lambda + O\left(\frac{1}{\alpha^3}\right)
\end{aligned}$$

It is then possible to calculate the inverse of $G^{-1}(I - U)G$ by applying the block matrix inverse formula. The rest is just algebra.

This Proposition shows that, given any strongly connected, aperiodic and diagonalizable network, when α is large, the long-run opinion vector can be rewritten as a vector with the same value plus a vector representing disagreement among agents. The disagreement vector is $O(\frac{1}{\alpha})$, so disagreement vanishes as α approaches infinity, as shown in the following example.

Example 2.8. Consider a network consisting of six agents. Agents 1 and 2 are both sophis-

ticated agents, while the others are naive, and

$$V = \begin{pmatrix} 0.2 & 0 & 0.4 & 0.4 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0.4 & 0.4 \\ 0.4 & 0 & 0.2 & 0.4 & 0 & 0 \\ 0.4 & 0 & 0.2 & 0 & 0 & 0.4 \\ 0 & 0.4 & 0 & 0 & 0.2 & 0.4 \\ 0 & 0.4 & 0 & 0.4 & 0.2 & 0 \end{pmatrix}$$

The network is shown in Figure 8. The ideal point of agent 1 is 1, the ideal point of agent 2 is -1 . The relation between the range of $(I - U)^{-1} \hat{B}$ and α is presented in Figure 9. The range of long-term opinions decreases from 1.241 to 0.002 as α increases from 1 to 1000.

Proposition 2.14 also shows that the variance of long-term opinions is affected by $1 - \lambda_i$, which represents the difference between the largest eigenvalue and the i th largest eigenvalues. The variance of long-term opinions is likely to be large when at least one of them is small. This does not mean that disagreement approaches infinity as the difference goes to 0 because this approximation works only when α is the dominant factor. If λ_i ($i \neq 1$) is close to 1, then it must be true that the spectral gap (the difference between the module of the two

Figure 8: Network example 2

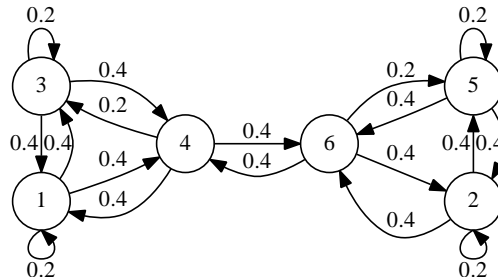


Figure 9: Range of Long-run Opinions vs Intensity of Lying Cost

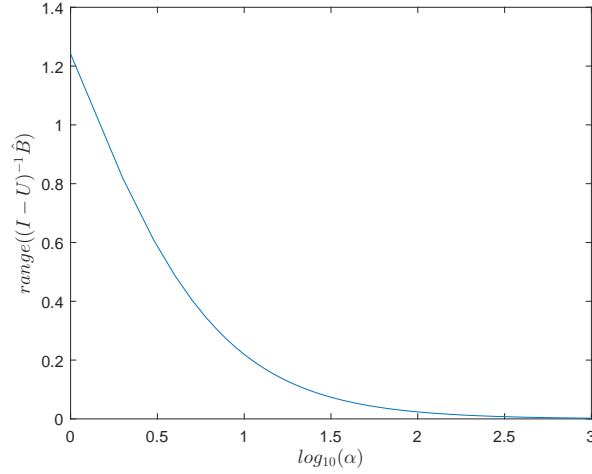
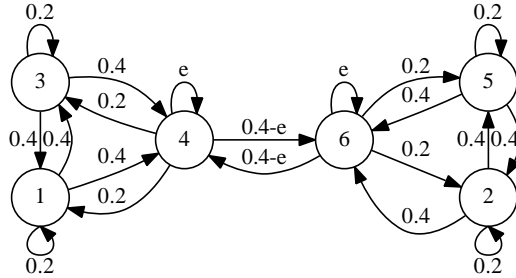


Figure 10: Network example 3



largest eigenvalues) is also small. In fact, when $\lambda_2 \in R^+$, the spectral gap is $1 - \lambda_2$.

There is a large literature in both applied mathematics and economics devoted to studying the spectral gap (Jackson 2008, Levin et al. 2009, Golub and Sadler 2016). In the framework of DeGroot, researchers have proven that this gap determines the speed of convergence. The relation between the spectral gap and the geometric structure of the graph is also investigated. The basic idea is that the spectral gap is small when the society is segregated, for instance, there exist some bottlenecks in the graph. The following example illustrates this insight.

Example 2.9. Consider a network similar to the network in Example 3. The only difference

is how much Agent 4 trusts agent 6, and vice versa. The matrix is

$$V = \begin{pmatrix} 0.2 & 0 & 0.4 & 0.4 & 0 & 0 \\ 0 & 0.2 & 0 & 0 & 0.4 & 0.4 \\ 0.4 & 0 & 0.2 & 0.4 & 0 & 0 \\ 0.4 & 0 & 0.2 & e & 0 & 0.4 - e \\ 0 & 0.4 & 0 & 0 & 0.2 & 0.4 \\ 0 & 0.4 & 0 & 0.4 - e & 0.2 & e \end{pmatrix}$$

It is shown in Figure 10. If $e = 0$, then it is the same as the network in Example 3.

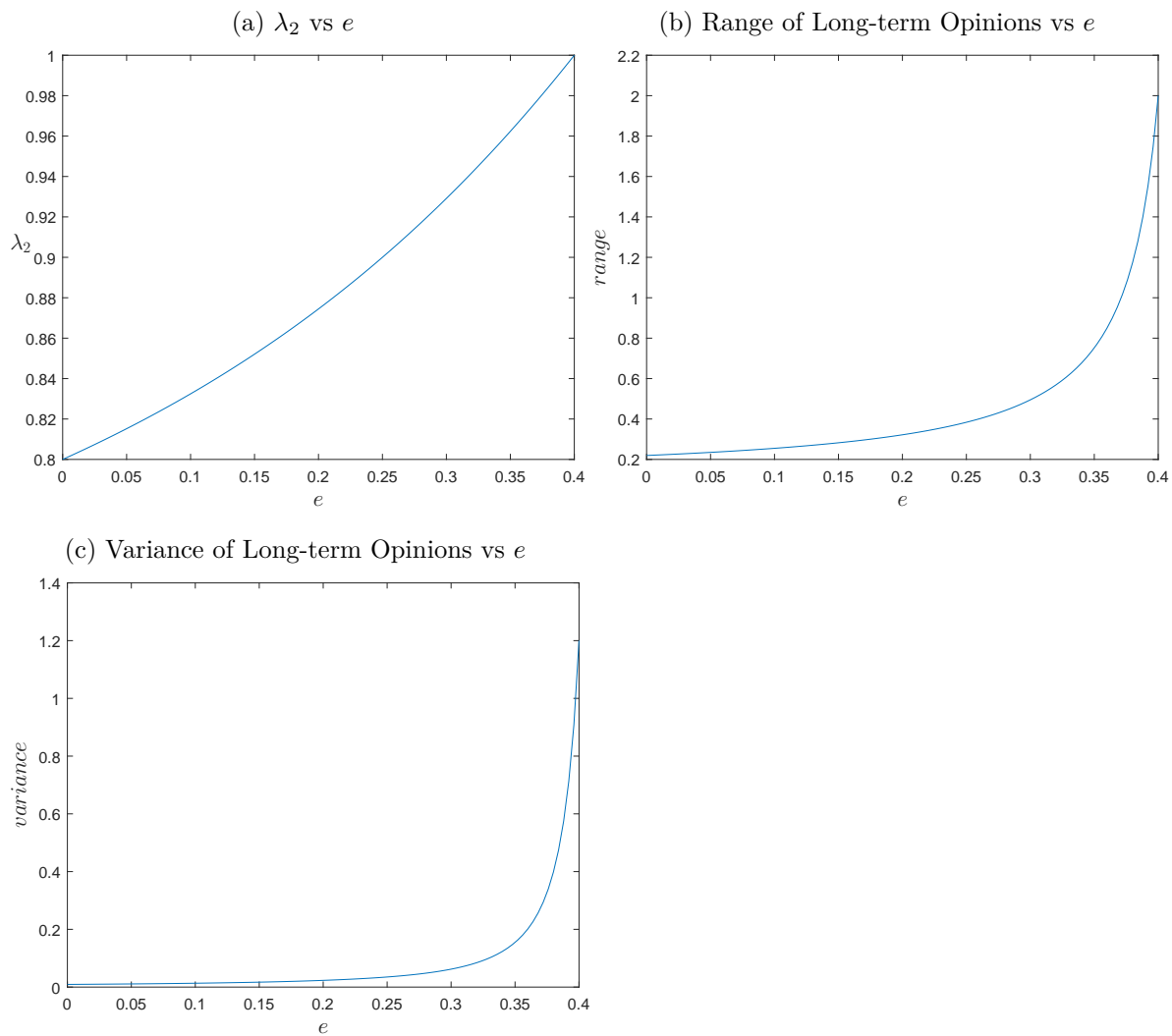
Suppose that $\alpha = 10$. The relation among the range and variance of long-term opinions, λ_2 and e is presented in Figure 11. As e increases from 0 to 0.4, the bottleneck gets tighter and tighter, so λ_2 increases from 0.8 to 1, which means that the society is more and more segregated. As a result, both the range and the variance of long-term opinions increase.

2.3.3 Social Influence

I am now left to address the influence of each agent on the long-term opinions of other agents given his/her network position. As discussed above, the long-term opinion vector is $(I - U)^{-1}\hat{B}$ and disagreement generically persists. Therefore, it is much harder to analyze the influence, in comparison to the DeGroot model. To simplify the discussion, let us restrict our attention to the case where α is large. Let $l = \left(l(1) \quad l(2) \quad \dots \quad l(N) \right)$ and

$$\hat{l}(i) = \frac{\left(\sum_{j \in A(d)} v_{ji} \right) \left(\sum_{j \in A(d)} v_{ji} l(j) \right)}{\sum_{k \in S(d)} \left(\sum_{j \in A(d)} v_{jk} \right) \left(\sum_{j \in A(d)} v_{jk} l(j) \right)}$$

Figure 11: Long-term Opinions



for every $i \in S(d)$. The following result shows how opinion leadership is determined by the structure of the social network when α is large.

Proposition 2.15. *For any i , $\left(\lim_{\alpha \rightarrow \infty} (I - U)^{-1} \hat{B}\right)_i = \hat{c}$, in which $\hat{c} = \sum_{i \in S(d)} b_i \hat{l}(i)$.*

When V is diagonalizable, the proof of Proposition 2.15 is directly derived from the proof of Proposition 2.14. If V is not diagonalizable, it is much harder to prove this result. In the proof, I first show that

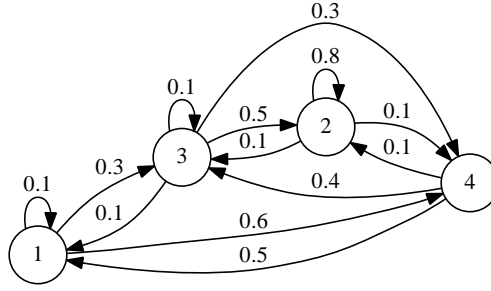
$$\lim_{\alpha \rightarrow \infty} \left[\max_i \left((I - U)^{-1} \hat{B} \right)_i - \min_i \left((I - U)^{-1} \hat{B} \right)_i \right] = 0$$

Then I multiply both sides of the equation $(I - U)X = \hat{B}$ by l to calculate \hat{c} .

This Proposition shows that, when α is large, long-term opinions are all close to one point, which is a weighted average of the ideal points of sophisticated agents. Let $\hat{l} = \left(\hat{l}(1) \ \hat{l}(2) \ \dots \ \hat{l}(d) \right)$; then \hat{l} is the new centrality vector. $\hat{l}(i)$ is determined by $\left(\sum_{j \in A(d)} v_{ji} \right) \left(\sum_{j \in A(d)} v_{ji} l(j) \right)$. The first part $\left(\sum_{j \in A(d)} v_{ji} \right)$ is the total trusts that all naive agents assign to i , thus, this can be viewed as the direct influence. On the other hand, the second part $\left(\sum_{j \in A(d)} v_{ji} l(j) \right)$ is the summation of the product of how much a naive agent trust i and how much influence this naive agent has on other agents. The second part can be viewed as the indirect influence. The following example shows that this new centrality measure can be very different from the eigenvector centrality.

Example 2.10. Consider a network consisting of four agents. Agents 1 and 2 are both

Figure 12: Network example 4



sophisticated agents, while the others are naive, and

$$V = \begin{pmatrix} 0.1 & 0 & 0.3 & 0.6 \\ 0 & 0.8 & 0.1 & 0.1 \\ 0.1 & 0.5 & 0.1 & 0.3 \\ 0.5 & 0.1 & 0.4 & 0 \end{pmatrix}$$

The network is shown in Figure 12. The eigenvector centrality vector is

$$l = \begin{pmatrix} 0.117 & 0.529 & 0.177 & 0.177 \end{pmatrix}$$

and the new centrality vector is

$$\hat{l} = \begin{pmatrix} 0.500 & 0.500 \end{pmatrix}$$

In the DeGroot model, the influence of Agent 2 is much bigger than that of Agent 1. However, the new centralities of these two agents are the same.

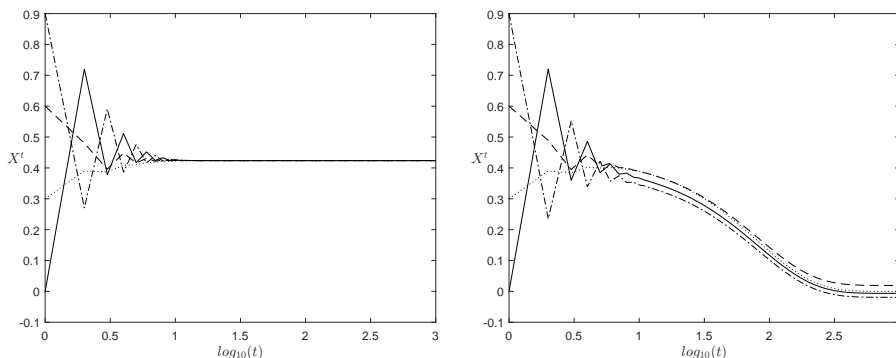
Proposition 2.15 can also be written as: $\left(\lim_{\alpha \rightarrow \infty} \lim_{t \rightarrow \infty} X^t\right)_i = \hat{c}$. Since $\lim_{\alpha \rightarrow \infty} U = V$, it is easy to check that $\left(\lim_{t \rightarrow \infty} \lim_{\alpha \rightarrow \infty} X^t\right)_i = lX^0$. Generically, $\hat{c} \neq lX^0$, which means that, even though

disagreement vanishes as α approaches infinity, the long-term opinions are not close to the consensus in the DeGroot model. Mathematically, this is because different orders of limiting operations lead to different results. These two different limits actually determine the opinion dynamics when α is large. The opinions first approach lX^0 , then move together from lX^0 to \hat{c} , as illustrated in the following example.

Example 2.11. The network here is the same as the network in Example 3. Suppose $X^0 = (0, 0.3, 0.6, 0.9)$, $b_1 = -0.5$, $b_2 = 0.5$ and $\alpha = 10$. Then $lX^0 = 0.424$ and $\hat{c} = 0$.

Figure 13 compares the dynamics of opinions in the DeGroot Model with those in my model. As shown in the graph, in the first several periods, these two dynamics are almost identical. That is because, when the cost of lying is large, the influences of initial opinions dominate the first several periods. After that, these influences gradually disappear and the influence of bias slowly becomes the dominant factor. The opinions eventually approach \hat{c} .

Figure 13: Opinion dynamics



2.4 Extension

2.4.1 Different Degrees of Importance

In the previous section, I considered the case in which sophisticated agents care equally about the naive agents. Now, I try to extend the model to a more general setup where the sophisticated agents assign different degrees of importance to different naive agents. I assume that the utility function of sophisticated agent i at time t is:

$$u_i(x^{t-1}, s_i^t, b_i) = - \sum_{j \in A(d)} \beta_{ji} (x_j^t - b_i)^2 - \alpha (s_i^t - x_i^{t-1})^2 \quad (5)$$

in which $\beta_{ji} > 0$ represents how important naive agent j is to sophisticated agent i . β_{ji} satisfies that, for any i , $\sum_{j \in A(d)} \beta_{ji} = 1$. The FOCs now are

$$\sum_{j \in A(d)} \beta_{ji} v_{ji} (\sum_{k \in S(d)} v_{jk} s_k^t) + \alpha s_i^t = \alpha x_i^{t-1} - \sum_{j \in A(d)} \beta_{ji} v_{ji} (\sum_{j \in A(d)} v_{jk} x_k^{t-1}) + \sum_{j \in A(d)} \beta_{ji} v_{ji} b_i \quad (6)$$

let

$$\tilde{V}_3 = \begin{pmatrix} \beta_{d+1,1} v_{d+1,1} & \cdots & \beta_{d+1,d} v_{d+1,d} \\ \vdots & \ddots & \vdots \\ \beta_{N,1} v_{N,1} & \cdots & \beta_{N,d} v_{N,d} \end{pmatrix}$$

$$\tilde{B} = \begin{pmatrix} \sum_{j \in A(d)} \beta_{j1} v_{j1} b_1 \\ \sum_{j \in A(d)} \beta_{j2} v_{j2} b_2 \\ \vdots \\ \sum_{j \in A(d)} \beta_{jd} v_{jd} b_d \end{pmatrix}$$

$$\tilde{U}_3 = \alpha V_3(\alpha I_d + \tilde{V}_3^T V_3)^{-1}, \tilde{U}_4 = \left[I_{N-d} - V_3(\alpha I_d + \tilde{V}_3^T V_3)^{-1} \tilde{V}_3^T \right] V_4, \tilde{U} = \begin{pmatrix} U_1 & U_2 \\ \tilde{U}_3 & \tilde{U}_4 \end{pmatrix} \text{ and}$$

$$\tilde{B}_1 = \begin{pmatrix} 0_d \\ V_3(\alpha I_d + \tilde{V}_3^T V_3)^{-1} \tilde{B} \end{pmatrix}. \text{ Following the same steps as set out in the previous sections,}$$

I can show that the opinion dynamics are characterized by the following matrix equation:

$$X^t = \tilde{U} X^{t-1} + \tilde{B}_1 \quad (7)$$

The results are almost the same as before, except that there is a different centrality vector. Let

$$\tilde{l}(i) = \frac{\left(\sum_{j \in A(d)} \beta_{ji} v_{ji} \right) \left(\sum_{j \in A(d)} v_{ji} l(j) \right)}{\sum_{k \in S(d)} \left(\sum_{j \in A(d)} \beta_{jk} v_{jk} \right) \left(\sum_{j \in A(d)} v_{jk} l(j) \right)}$$

for every $i \in S(d)$, and $\tilde{l} = \begin{pmatrix} \tilde{l}(1) & \tilde{l}(2) & \dots & \tilde{l}(d) \end{pmatrix}$. As I illustrate below, \tilde{l} is the new centrality vector.

Proposition 2.16. *For any i , $\left(\lim_{\alpha \rightarrow \infty} \lim_{t \rightarrow \infty} X^t \right)_i = \tilde{c}$, in which $\tilde{c} = \sum_{i \in S(d)} b_i \tilde{l}(i)$.*

Proposition 2.16 shows that, when the sophisticated agents assign different degrees of importance to different naive agents, the centrality vector is different. The indirect influence remains the same as before but the direct influence is now $\sum_{j \in A(d)} \beta_{ji} v_{ji}$. The trust that all naive agents place in i is weighted according to the degrees of importance.

2.4.2 Farsighted Sophisticated Agents

So far, I have examined the case where the sophisticated agents are myopic. In this section, I attempt to extend the model to a case in which the sophisticated agents are forward-looking.

I only consider the simplest case due to tractability considerations.

Suppose there are two sophisticated agents and one naive agent. The sophisticated agents are stubborn, which means that they never change their minds ($v_{11} = v_{22} = 1$). Without loss of generality, I assume that the initial opinions of the sophisticated agents are -1 (Agent 1) and 1 (Agent 2). The sophisticated agents try to convince the naive agent to believe what they believe. ($b_1 = -1, b_2 = 1$). I also assume that $v_{33} = 0$. Then the stage payoffs are given by

$$\Pi_1 = -(v_{31}x_t + v_{32}y_t + 1)^2 - \alpha(x_t + 1)^2$$

$$\Pi_2 = -(v_{31}x_t + v_{32}y_t - 1)^2 - \alpha(y_t - 1)^2$$

in which x_t and y_t are signals sent by sophisticated agents.

I follow the method proposed by Maskin and Tirole (1987) to solve this model. I also assume that in odd period $2t + 1$, Agent 1 chooses x_{2t+1} , which will remain unchanged until period $2t + 3$; and in even period $2t + 2$, Agent 1 chooses y_{2t+2} , which will remain unchanged until period $2t + 4$. I also focus on Markov perfect equilibrium. Since $v_{33} = 0$, the state in each period is the action of the other sophisticated agent in the last period. Denote the pairs of dynamic reaction functions by (R_1, R_2) . (R_1, R_2) is an MPE if and only if there exist valuation functions $\{(V_1, W_1), (V_2, W_2)\}$ such that for any $\{\hat{x}, \hat{y}\}$:

$$V_1(\hat{y}) = \max_x \{\Pi_1(x, \hat{y}) + \delta W_1(x)\}$$

$$R_1(\hat{y}) \in \operatorname{argmax}_x \{\Pi_1(x, \hat{y}) + \delta W_1(x)\}$$

$$W_1(\hat{x}) = \Pi_1(\hat{x}, R_2(\hat{x})) + \delta V_1(R_2(\hat{x}))$$

As in Maskin and Tirole (1987), the solution should satisfy that

$$\frac{dR_2}{dx}(\hat{x}) = \frac{-\Pi_1^1(\hat{x}, R_1^{-1}(\hat{x})) - \delta\Pi_1^1(\hat{x}, R_2(\hat{x}))}{\delta\Pi_1^2(\hat{x}, R_2(\hat{x})) + \delta^2\Pi_1^2(R_1(R_2(\hat{x})), R_2(\hat{x}))}$$

$$\frac{dR_1}{dy}(\hat{y}) = \frac{-\Pi_2^2(R_2^{-1}(\hat{y}), \hat{y}) - \delta\Pi_2^2(R_1(\hat{y}), \hat{y})}{\delta\Pi_2^1(R_1(\hat{y}), \hat{y}) + \delta^2\Pi_2^1(R_1(\hat{y}), R_2(R_1(\hat{y})))}$$

I also assume that R_i are linear functions:

$$R_1(y) = a_1 + \gamma_1 y$$

$$R_2(x) = a_2 + \gamma_2 x$$

Then

$$\gamma_2 = -\frac{\alpha(x+1) + v_{31}(v_{31}x + v_{32}R_1^{-1}(x) + 1) + \alpha\delta(x+1) + \delta v_{31}(v_{31}x + v_{32}R_2(x) + 1)}{\delta v_{32}(v_{31}x + v_{32}R_2(x) + 1) + \delta^2 v_{32}(v_{31}R_1(R_2(x)) + v_{32}R_2(x) + 1)}$$

$$\gamma_1 = -\frac{\alpha(y-1) + v_{32}(v_{31}R_2^{-1}(y) + v_{32}y - 1) + \alpha\delta(y-1) + \delta v_{32}(v_{31}R_1(y) + v_{32}y - 1)}{\delta v_{31}(v_{31}R_1(y) + v_{32}y - 1) + \delta^2 v_{31}(v_{31}R_1(y) + v_{32}R_2(R_1(y)) - 1)}$$

Then it must be true that

$$(1 + \delta)(\alpha + v_{31}^2) + \frac{1}{\gamma_1}v_{31}v_{32} + \delta v_{31}v_{32}\gamma_2 + \delta v_{31}v_{32}\gamma_2 + \delta v_{32}^2\gamma_2^2 + \delta^2 v_{31}v_{32}\gamma_1\gamma_2^2 + \delta^2 v_{32}^2\gamma_2^2 = 0$$

$$\iff \delta^2 v_{31}v_{32}\gamma_1^2\gamma_2^2 + \delta(1 + \delta)v_{32}^2\gamma_1\gamma_2^2 + 2\delta v_{31}v_{32}\gamma_1\gamma_2 + (1 + \delta)(\alpha + v_{31}^2)\gamma_1 + v_{31}v_{32} = 0 \quad (8)$$

$$\delta^2 v_{31}v_{32}\gamma_1^2\gamma_2^2 + \delta(1 + \delta)v_{31}^2\gamma_1^2\gamma_2 + 2\delta v_{31}v_{32}\gamma_1\gamma_2 + (1 + \delta)(\alpha + v_{32}^2)\gamma_2 + v_{31}v_{32} = 0 \quad (9)$$

and

$$\begin{aligned}
& -(\delta v_{32}\gamma_2 + v_{31} + \alpha)(1 + \delta) + v_{31}v_{32}\frac{a_1}{\gamma_1} \\
& = \delta v_{31}v_{32}a_2 + \delta(1 + \delta)v_{32}^2a_2\gamma_2 + \delta^2v_{31}v_{32}a_1\gamma_2 + \delta^2v_{31}v_{32}a_2\gamma_1\gamma_2
\end{aligned} \tag{10}$$

$$\begin{aligned}
& (\delta v_{31}\gamma_1 + v_{32} + \alpha)(1 + \delta) + v_{31}v_{32}\frac{a_2}{\gamma_2} \\
& = \delta v_{31}v_{32}a_1 + \delta(1 + \delta)v_{31}^2a_1\gamma_1 + \delta^2v_{31}v_{32}a_2\gamma_1 + \delta^2v_{31}v_{32}a_1\gamma_1\gamma_2
\end{aligned} \tag{11}$$

Let $p = v_{31}$, then $v_{32} = 1 - p$. I first prove the existence of real roots:

Lemma 2.2. *Given $0 < p < 1$ and $0 < \delta \leq 1$, there exists an $\alpha^* > 0$, such that for any $\alpha > \alpha^*$,*

1. *there exist real solutions to the system of equations (3.7) and (3.8), denoted by γ_1^* and*

γ_2^ ,*

2.

$$\gamma_1^* = -\frac{p(1-p)}{(1+\delta)(\alpha+p^2)} + O\left(\frac{1}{\alpha^3}\right)$$

$$\gamma_2^* = -\frac{p(1-p)}{(1+\delta)(\alpha+(1-p)^2)} + O\left(\frac{1}{\alpha^3}\right)$$

Lemma 2.2 shows that, when α is large, there exists real solutions and the solutions are represented by simple formulas. We can then begin to calculate a_1 and a_2 .

Lemma 2.3. *Given $0 < p < 1$ and $0 < \delta \leq 1$, there exists an $\alpha^* > 0$, such that for any $\alpha > \alpha^*$, if $\gamma_i = \gamma_i^*$, then $a_1^* = \frac{\Delta_2}{\Delta_1}$ and $a_2^* = \frac{\Delta_3}{\Delta_1}$, where*

$$\Delta_1 = 1 - \frac{3\delta^2 p^2 (1-p)^2}{(1+\delta)^2 (\alpha+p^2) (\alpha+(1-p)^2)} + O\left(\frac{1}{\alpha^3}\right)$$

$$\begin{aligned} \Delta_2 = & -1 - \frac{p(1-p)}{\alpha+p^2} - \frac{\delta p(1-p)\alpha}{(1+\delta)(\alpha+p^2)(\alpha+(1-p)^2)} \\ & + [-3p+1+\delta] \frac{\delta p(1-p)^2}{(1+\delta)^2 (\alpha+p^2) (\alpha+(1-p)^2)} + O\left(\frac{1}{\alpha^3}\right) \end{aligned}$$

$$\begin{aligned} \Delta_3 = & 1 + \frac{p(1-p)}{\alpha+(1-p)^2} + \frac{\delta p(1-p)\alpha}{(1+\delta)(\alpha+p^2)(\alpha+(1-p)^2)} \\ & + [3(1-p) - (1+\delta)] \frac{\delta p^2(1-p)}{(1+\delta)^2 (\alpha+p^2) (\alpha+(1-p)^2)} + O\left(\frac{1}{\alpha^3}\right) \end{aligned}$$

By lemmas 2.2 and 2.3, I can show that, when α is large, there are always real solutions to the system of equations such that γ_1^* , γ_2^* are both negative numbers close to 0 and $a_1^* \approx -a_2^* \approx -1$. It is easy to check that $\Pi_1(x, \hat{y}) + \delta W_1(x)$ is concave, so there exists an MPE where $\gamma_i = \gamma_i^*$, $a_i = a_i^*$, and the strategies are

$$R_1(y) = a_1^* + \gamma_1^* y$$

$$R_2(x) = a_2^* + \gamma_2^* x$$

It is also easy to check that the linear system is stable.

When α is large, the state will quickly converge to the steady state:

$$(x^*, y^*) = \left(\frac{a_1^* + \gamma_1^* a_2^*}{1 - \gamma_1^* \gamma_2^*}, \frac{a_2^* + \gamma_2^* a_1^*}{1 - \gamma_1^* \gamma_2^*} \right)$$

What follows are some properties of the steady state. The steady state will be compared with the case where the sophisticated agents are myopic. Denote the opinion of the naive agent in a steady state and in a myopic case by z^* and \hat{z} , respectively.

Proposition 2.17. *There exists an $\alpha^* > 0$, such that for any $\alpha > \alpha^*$, $z^* > \hat{z}$ if and only if $1 - p > p$*

Proposition 2.17 shows that, when the sophisticated agents are forward-looking instead of myopic, the agent who has more influence when all agents are naive gains even more influence. The stage payoff in these two different cases can also be compared. Denote the stage payoffs of the sophisticated agents in a steady state and in a myopic case by Π_i^* and $\hat{\Pi}_i$, respectively.

Proposition 2.18. *Given $0 < p < 1$ and $0 < \delta \leq 1$, there exists an $\alpha^* > 0$, such that for any $\alpha > \alpha^*$, $\Pi_i^* > \hat{\Pi}_i$.*

Proposition 2.18 shows that, when the sophisticated agents are forward-looking instead of myopic, the stage payoffs of both sophisticated agents increase, even though one has more influence while the other has less. This is because the cost of lying is smaller.

The last proposition discusses some comparative statics with respect to δ .

Proposition 2.19. *Given $0 < p < 1$ and $0 < \delta_1 < \delta_2 \leq 1$, there exists an $\alpha^* > 0$, such that for any $\alpha > \alpha^*$,*

1. $\Pi_i^*(\delta_2) > \Pi_i^*(\delta_1)$
2. $z^*(\delta_2) > z^*(\delta_1)$ if and only if $1 - p > p$

Proposition 2.19 shows that, when the sophisticated agents are more patient, the effects I discuss in Propositions 2.17 and 2.18 become more significant.

2.5 Conclusion

This paper provides a possible explanation for the persistent disagreement among individuals in a society and studies the relation between the extent of disagreement and the structure of social networks. It considers a bounded rationality model of opinion formation. I show that, when the network is strongly connected and aperiodic, and the intensity of lying cost is sufficiently large, the opinions converge but, generically, do not converge to a consensus. The paper further characterizes the asymptotic opinions. These are determined by the sophisticated agents' biases and the structure of the network and are not affected by initial opinions. The extent of disagreement is negatively correlated with the intensity of lying cost and the spectral gap, which represents the degree of segregation. I then study social influence. I find that social influence can differ significantly from eigenvector centrality and that it can be decomposed into two separate factors: direct influence and indirect influence. Finally, I investigate two variations of this model. First, I relax the assumptions to allow the sophisticated agents to assign different degrees of importance to various naive agents. I show that this only changes the direct influence. Second, I investigate a case in which the sophisticated agents are forward-looking. I only consider a very simple network and show that, in comparison with the case of myopic agents, wherein the agent who has more

influence gains even more influence, the utility of both sophisticated agents is larger. I also examine comparative statics on the discount factor. I prove that these two effects grow in significance as the discount factor enlarges.

To ensure tractability, I focus on the case in which the sophisticated agents are myopic. Moreover, when the sophisticated agents are forward-looking, I only analyze the simplest network. An interesting question is whether we can extend these results to a more general setting in which agents are forward-looking and communicate with each other through an arbitrary network. It would also be interesting to see whether these results remain valid when the network is endogenously formed.

Chapter 3: Informal Lending and Entrepreneurship

3.1 Introduction

Entrepreneurship can yield a path to economic independence, boost job creation, raise incomes, and help consumers to make a living. Unfortunately, it is stifled by restricted access to credit, equity, or payment services. A World Bank survey² found that most entrepreneurs faced difficulty accessing capital from banks and ended up resorting to borrowing from informal resources; an outcome that concerns many because it is believed to negatively correlate with entrepreneurial growth and performance. For reasons such as lack of capital, excessive red taping, and inability to perfectly monitor applicants, loans which could spur entrepreneurial activity cannot be allocated to consumers (Field et al., 2013). While financial access is a pressing issue in many developing parts of the world, it is also a critical issue in the United States. The Obama Administration, in a June 2016 briefing³, reported that financial inclusion is a problem for 20% of the US population. These consumers cannot obtain the approval to use reliable and affordable financial products and use alternate services, such as same day loans, borrowing at pawn shops, and borrowing informally.⁴

Given the impact of financial inclusion on consumers, it is startling to find that the literature in entrepreneurship largely ignored the reasons behind lack of access. In this paper, we address this gap by studying the impact of formal and informal borrowing opportunities on

²<http://www.worldbank.org/en/results/2013/04/05/msme-finance-expanding-opportunities-and-creating-jobs>

³<https://obamawhitehouse.archives.gov/blog/2016/06/10/financial-inclusion-united-states>

⁴Consequentially, the U.S. Department of the Treasury initiated a program to improve access to financial services for all, particularly for the low and moderate income households. Among the initiatives set up by the Obama administration were Financial Inclusion Forum, the Financial Empowerment Innovation Fund, and participation in the G-20's Global Partnership for Financial Inclusion. The Administration has also proposed in its 2017 Budget pilots for new approaches that provide shorter and longer term financial assistance and savings tools, to help workers build up "rainy day" reserve funds.

consumer entrepreneurship. Most consumer entrepreneurship happens in the form of investment into small and medium micro enterprises (SMEs). SMEs are particularly important for emerging economies as they account for the largest share of employment and GDP⁵. They help families to pay bills, build wealth, and attain social mobility in the long term (Demetriades and Hussein, 1996). We investigate how terms of borrowing alter entrepreneurial investment in a market. Specifically, our paper answers the following research questions: How does the availability of informal borrowing influence financial inclusion (access to credit for all consumers) and terms of borrowing from formal resources? What is the impact of informal lending on the volume and innovativeness of entrepreneurial output?

We build a model that considers how financial inclusion (or, the lack of it) endogenously rises as opportunities of informal borrowing emerge in a market. We focus on those who are in need of a loan to create an SME, and assume that borrowers are heterogeneous with respect to their existing wealth and the lucrativeness of their entrepreneurial idea. While the bank would like to screen for an applicant's risk of default and prefers to fund more innovative ideas, it can assess risk only imperfectly. To hedge its risk, it can ask consumers for a non-cash collateral. As a result, consumers with productive ideas and enough wealth to pledge a collateral can signal their low risk and gain access to financial products. Consumers without wealth are either excluded of financing opportunities, or are extended a loan without a collateral at a high interest rate. As a result, some of these consumers may have an incentive to borrow in the informal market. Informal borrowing markets suffer less from the information asymmetry problem that a bank faces because those lending to friends, family, or other social contacts are usually better informed about the nonpayment risk associated

⁵According to World Bank statistics.

with these individuals. Thus, they have a better assessment of each other's risk compared to the bank (de Meza et al., 1987).

Our model focuses on two problems that emerge in entrepreneurial markets due to imperfect ability to screen consumer risk of default. First, consumers with productive ideas but no wealth are pooled with others with unproductive ideas because neither can put down a collateral. This results in an underinvestment problem such that the creditworthy entrepreneurs are deprived of loans that would yield positive expected returns for them and the bank. On the other hand, if the bank provides loans that do not require a collateral, it assumes the risk of losses from lending to those with unproductive ideas; thus faces an overinvestment problem.⁶ Banks try to balance the overinvestment and underinvestment problems in designing their loan contracts and the contract terms determine who gets to borrow, and which entrepreneurial projects end up being realized.

We find that in a market where there are few informal borrowing opportunities, the lender offers unsecured loans at a more attractive (lower) rate to entrepreneurs without wealth. Therefore some informal activity can be beneficial for the entrepreneurs. But if there are plenty of opportunities to borrow informally, this motivates banks to take advantage of the informal activity in the market. The bank finds it more attractive to limit credit access and increases the rates of secured loans so that it can motivate these consumers to lend informally and earn higher profits that way. The bank is financially better off in markets with high informal economy if it restricts access and offers only secured loans at high rates. This results in a severe underinvestment problem where deserving productive entrepreneurs

⁶Existence of informal lending leads to two types of underinvestment problems. The first is exclusion of the consumers with productive ideas but no wealth to pledge for a collateral. The second underinvestment problem is the reduced investment by the wealthy consumers who would otherwise invest into a project but find it more profitable to lend in the informal lending market.

are forced to borrow informally because there are no unsecured loans offered. Moreover, instead of borrowing and investing into their ideas, some consumers find it more attractive to lend to contacts and some other consumers receive no access to credit and cannot realize their entrepreneurial project. So low levels of informal borrowing opportunities catalyze entrepreneurship, but high levels of it incentivize banks to restrict access to credit and earn rents from the high rates charged to a small group of the wealthy who have the ability to earn money through informal loans.

In the extensions, we first consider how introduction of joint liability contracts influences entrepreneurial activity. Because lenders cannot perfectly assess the risk associated with a consumer, they hedge against risk by joint loan contracts. In joint contracts, borrowers who belong to the same group can co-sign a loan together by assuming partial liability on the loan of a friend. When one defaults on the loan, the co-signer has to cover for his debt.⁷ Consumers who are friends know each others' risk types better than the bank (Ghatak, 2000), and individuals self-select to form loan groups with low risk friends. So lender's risk of non-payment is reduced in joint lending contracts. We find that joint lending can mitigate the under and overinvestment problems while improving the volume of entrepreneurial investment. Therefore they are desirable from managerial and policy perspectives.

Second, we extend the model to consider heterogeneity in ties, and additional motivations for lending such as helping friends or investing in personal relationships, rather than earning rent. Consumers may borrow from others who they have weaker ties with, who may want to help them to succeed financially. We show that these additional factors which can increase

⁷The attractiveness of the joint lending programs can come from various number of factors, including social pressure to keep up with their payments because of the consequences associated with default, whose severity may vary from humiliation to ostracism by the group Wei et al. (2016)

the strength of relationships and outside borrowing opportunities can limit financial access further.

Our study makes several distinct contributions to several strands of the literature. First, we contribute to the consumer finance literature by studying how informal borrowing opportunities influence financial inclusion and terms of borrowing for consumers. A large number of banks, fintech startups, and consulting firms in the US are interested in developing financial products to improve the lives of the low and middle income consumers. While there is a rich literature focusing on a variety of issues in the consumer finance domain, including behavioral responses to the characteristics of financial products(e.g., Feinberg et al., 1986; Soman, 2003; Soman and Cheema, 2002; Christen and Morgan, 2005; Gourville and Soman, 1998), attitudes towards saving(Benartzi and Thaler, 1999; Hershfield et al., 2011; McKenzie and Liersch, 2011; Karlan et al., 2016), and financial literacy (e.g., Bolton et al., 2011; Gaurav et al., 2011; Hadar et al., 2013), to our knowledge no study analyzed the conditions that create inequality in consumers' access to financing.

Second, we contribute to the studies focusing on new product introduction and entrepreneurship. While there is a significant research body in diffusion of new products and innovation (e.g., Mahajan et al., 1990; Mahajan and Muller, 1998; Hauser et al., 2006; Sood and Tellis, 2009), there are very few studies on entrepreneurship of low-income consumers who has to resort to informal borrowing. This study is the first to focus on the relationship between borrowing opportunities and entrepreneurial outcomes. It is also the first to study this relationship from the perspective of social ties.

Third, we make a contribution to the literature on emerging markets (e.g., Anderson-Macdonald et al., 2015; Anderson-Macdonald and Thomson, 2015; Bollinger and Yao, 2016).

In business and economics, there is a growing interest in emerging markets (Sancheti and Sudhir, 2014; Sudhir and Talukdar, 2015; Sudhir et al., 2015; Anderson-Macdonald et al., 2015; Anderson-Macdonald and Thomson, 2015; Kishore et al., 2015). Nevertheless the number of studies focusing on lack of finance and its reasons is small. Wei et al. (2016) list some of the reasons for the difficulty of accessing credit as lack of capital, insufficient consumer history to demonstrate creditworthiness and imperfect screening of consumer applications by financial institutions. We contribute to this literature by showing how these challenges may also result in informal borrowing.

Fouth, we contribute to the literature on the coexistence of formal and informal finance (Bose, 1998; Hoff and Stiglitz, 1998; Jain, 1999; Karaivanov and Kessler, 2018; Lee and Persson, 2016; Madestam, 2014, among others). While these papers have discussed the cases in which the informal lenders are either bank competitors or channels of bank funds, to our knowledge no study analyzed how the density of social connections affect the interaction between formal and informal finance. Finally, the paper relates to the literature on development and incomplete financial markets (e.g., Ahlin and Jiang 2008; Banerjee and Newman 1998; also see Buera et al., 2015 for a survey of studies on macro side), and informal transfers in social networks (e.g., Ambrus et al., 2014; Karlan et al., 2009).

The rest of the paper is organized as follows. In Section 3.2, we develop a model to study the financial inclusion in the context of formal and informal lending and summarize the results. We present several extensions in Section 3.3. In Section 3.4, we discuss our conclusions along with managerial and policy insights.

3.2 Model

We consider a market with a bank (also referred to as the “lender” throughout the text) and a community with N consumers (referred to as the “borrowers”). The bank provides individual or group loans⁸ and assumes limited liability: It can only collect assets pledged as collateral for a loan. Borrowers are endowed with one unit of labor⁹ and have the opportunity to earn a return from an investment into a risky project¹⁰. This investment represents the idea of developing a micro-enterprise (i.e., entrepreneurial investment).

We assume that entrepreneurs are interested in borrowing a loan and they are heterogeneous with respect to their risk of defaulting on a loan. Conditional on their risk level, each applicant belongs to one of three risk segments: Ω_1 , Ω_2 , and Ω_3 . Without loss of generality, let borrowers in Ω_3 have high risk, and borrowers in segments Ω_1 and Ω_2 have low risk of default. Let the number of consumers in Ω_i be N_i , with $\sum_i^3 N_i = N$. For simplicity, we denote the entrepreneurial idea consumers in Ω_i can invest in by project i and the likelihood of success of the project with p_i . Conditional on being successful, project i generates a return of R_i . We will assume that the innovativeness of ideas is correlated with their average financial return, that is, a more innovative project yields a higher expected return (R_i) on average.¹¹ If individuals choose not to invest into an entrepreneurial idea, they can keep an outside option of earnings \bar{u} .¹²

The risk of a consumer is unobservable to the lender. The inability to screen consumer

⁸Group lending is the practice of lending to multiple consumers through a single, joint contract where the risks and returns of borrowers are linked. Group loans are common practice in developing economies.

⁹Equivalently, this implies that the investment required by each project is 1 unit of a capital resource.

¹⁰The project represents an idea or a business.

¹¹Otherwise there is little incentive for any entrepreneur to invest into the project.

¹²This income, for instance, may correspond to income from farming for an individual who is considering opening a small store in a village. We assume that all consumers and the lender are risk neutral and care only about the expected profit.

risk is a defining issue in developing markets due to lack of data on the financial history of thin-file or no-file loan applicants.¹³ As a screening device for high risk borrowers, the lender can ask the consumers to pledge a collateral before giving any loans. The collateral serves as an upfront payment and we assume the lender takes it in non-cash forms. Examples of collateral in developed economies include household goods, or livestock and land in emerging economies. It is lost if the borrower defaults.

The bank offers a menu of contracts with (i) a low interest loan with a collateral, or (ii) a high interest loan with no collateral. Let the interest rates the lender sets for loans with and without collateral be r_1 and r_2 , respectively, where $r_2 > r_1$ must be maintained for the contracts to be functional in screening. Consumers with low risk ideas and wealth self-select into loans with collateral to signal low risk. Those who lack sufficient wealth have to take a loan without collateral even though they may be creditworthy. They borrow at a higher interest rate since they are pooled with the high risk applicants and are at a disadvantage in obtaining credit. We assume, without loss of generality, that only consumers in Ω_1 have the resources to put down a collateral, and normalize the price of the collateral to 1. To distinguish between these consumers, we will refer to this group as the wealthy and the remaining consumers as the unwealthy consumers. Here, the term wealth should be read as relative wealth since in reality most of these consumers may be in need of capital.

A borrower's risk is jointly represented by the probability of success and the rate of return associated with their projects. First, without loss of generality, we assume that the entrepreneurial projects of consumers in the first two segments have a higher likelihood of

¹³Many countries today still lack credit history or other financial transactions data on consumers either due to limited number of non-cash financial transactions taking place or due to high cost of recording data from consumers (Wei et al., 2016).

Table 1: Consumer Segments

Segment	Description
Ω_1	Low risk segment with some wealth and productive ideas. These borrowers can signal their low risk type by pledging a collateral and borrow at a lower rate. If they invest in their idea, the expected return is nonnegative.
Ω_2	Low risk segment without wealth and productive ideas. These borrowers take on loans without collateral and at the higher rate. If they invest in their idea, the expected return is nonnegative.
Ω_3	High risk consumers without wealth and with non-productive ideas. These borrowers borrow loans without collateral and at the higher rate. If they invest in their idea, the expected return is negative.

success compared to those in the third segment:

$$p_1 = p_2 \equiv p > p_3 \equiv \hat{p}. \quad (\text{A1})$$

Second, we assume that the ideas of each segment vary in their innovativeness and thus their financial return. While those in segments Ω_1 , Ω_2 have productive ideas (i.e., ideas which yield a nonnegative expected return when invested in), the ideas of the consumers in Ω_3 are unproductive and yield an expected return that is lower than their outside option:

$$pR_1, pR_2 - 1 > \bar{u}$$

$$\hat{p}R_3 - 1 < \bar{u}. \quad (\text{A2})$$

The characteristics of the three borrower segments are summarized in Table 1. The segments allow us to study the issues relevant to financial inclusion, entrepreneurship and social networks. Notice that a lender would like to fund productive borrowers, namely, consumers in Ω_1 and Ω_2 . Ω_3 consists of borrowers who should not receive a loan, since they have high risk and unproductive ideas. The challenge is distinguishing consumers in

Ω_2 from those in Ω_3 , since none of these consumers can pledge a collateral and the lender cannot effectively differentiate between them. When loans are extended to the high risk, non-productive borrowers (Ω_3), this leads to an *overinvestment problem*: consumers who should not obtain loans receive loans and generate economic losses on average from their borrowing. If the bank acts conservatively to avoid the overinvestment problem, it can run into the opposite problem of *underinvestment*, where productive borrowers (in Ω_2 , and under some conditions in Ω_1) are excluded from financing. This second outcome is equally undesirable since it reduces the innovative output and may prevent capital-constrained consumers from growing economically.

Faced with difficulties to access credit, borrowers in emerging economies often turn to informal sources, friends, family, or other lenders. We will consider a market where due to their ability to borrow at low rates, consumers in Ω_1 lend to their contacts with productive ideas in Ω_2 . In this scenario, the borrowers in Ω_1 lend to friends instead of investing in their ideas. The magnitude of this “leak” will depend on the social structure and the degree of mixing among the segments: if the individuals who are unwealthy have more contacts in Ω_1 , they have more informal borrowing opportunities. By extending credit to the low risk segment Ω_1 , the lender creates a competition for itself in the informal lending market. We will explore how the presence of this outside informal market influences the innovativeness of ideas realized.

3.2.1 Individual Loan Contracts

First we consider lending via individual contracts without consideration of leaks (i.e., assuming no monetary transfers between consumer segments). The expected payoff for borrowers

in segment Ω_1 from borrowing and investing in a microenterprise respectively is:

$$\underbrace{p(R_1 - r_1)}_{\text{Exp. return if successful}} + \underbrace{(1-p)(-1)}_{\text{Exp. loss if unsuccessful}} - \underbrace{\bar{u}}_{\text{Loss of other income}} \quad (12)$$

The first term in the expression is the expected payoff if the project is successful. The second term indicates the loss of the collateral if the project fails. Additionally, investment into an enterprise implies the loss of the outside option.

For consumers in Ω_2 , a similar expression for the expected payoff can be written by plugging in the corresponding rate terms and expected loss of outside income:

$$p(R_2 - r_2) - \bar{u}. \quad (13)$$

The lender is profit maximizing by setting the interest rates to extract consumer surplus.

The rates for the two segments are obtained by solving the following two equations:

$$p(R_1 - r_1) + (1-p)(-1) = \bar{u} \quad (14)$$

$$p(R_2 - r_2) = \bar{u}$$

This defines the optimal interest rates:

$$r_1 = R_1 - \frac{1}{p} - \frac{\bar{u}}{p} + 1, \quad (15)$$

$$r_2 = R_2 - \frac{\bar{u}}{p}. \quad (16)$$

The rates depend on the innovativeness of ideas, probability of success, and the outside

option of the borrower. The lender incentivizes borrowers to put down a collateral if the return on borrowing with a collateral is higher than borrowing without it. Formally, the incentive compatibility condition should satisfy

$$\underbrace{p(R_1 - r_1) + (1 - p)(-1)}_{\text{Exp. return from the secured contract}} > \underbrace{p(R_1 - r_2)}_{\text{Exp. return from the unsecured contract}} \quad (17)$$

The LHS in this expression is the expected return on borrowing a loan with and the RHS is the expected return on borrowing a loan without a collateral. Solving the inequality yields that for secured loans to be offered, $r_2 - r_1 > \frac{1}{p} - 1$ must hold. Otherwise, consumers in group Ω_1 would benefit from not revealing their low risk-type and would borrow the zero-collateral loan, pooling with other borrowers. The financial firm would be worse off in this condition since it would lose its ability to detect low-risk consumers. Proposition 3.20 formally states the condition for the lender to effectively screen applicants, i.e., to satisfy $r_2 - r_1 > \frac{1}{p} - 1$.

Proposition 3.20. (*Condition for Screening Consumer Risk*) *For financial firms to effectively screen consumer risk and offer secured contracts, the productive but unwealthy consumers' ideas must be more innovative than those of the wealthy. Formally,*

$$R_2 > R_1 \quad (A3)$$

should hold. When this condition holds, the interest rates for the secured and unsecured contracts are $r_1 = R_1 - \frac{1}{p} - \frac{\bar{u}}{p} + 1$ and $r_2 = R_2 - \frac{\bar{u}}{p}$, respectively.

We have a separating equilibrium when the low risk entrepreneurs can signal their type by putting down a collateral. When $R_2 > R_1$ holds, consumers in Ω_1 have an incentive to

reveal their low-risk type. While consumers in Ω_2 would benefit from pretending to be a borrower in segment Ω_1 , they cannot do so without putting down a collateral. To realize their investment, they need banks to offer unsecured loans which do not require a collateral. To offer an unsecured loan, a bank's expected return must be higher when offering both contracts. Proposition 3.21 below describes the conditions under which a profit-maximizing lender offers secured and unsecured loans.

Proposition 3.21. (*Full Financial Inclusion*) *When the lender is offering secured loans, it will also offer unsecured loans if ideas of the middle segment are innovative such that:*

$$R_2 \geq \frac{\bar{u}}{p} + \frac{1}{\bar{p}} \tag{A4}$$

where $\bar{p} \equiv \frac{N_2}{N_2+N_3}p + \frac{N_3}{N_2+N_3}\hat{p}$. Under this case, full financial inclusion can be achieved since all the consumers in the market can find a product offering that is targeted at them.

Proposal 3.21 shows a condition that results in full financial inclusion, or the condition for the bank to offer both types of contracts. Key insight is that financial access relies on the innovativeness of the ideas of those in relative poverty who cannot signal their credit-worthiness by pledging a collateral, and are indistinguishable from those with high risk. The term \bar{p} on the RHS of the condition (A4) is a weighted average of the success rates of these two segments, p and \hat{p} , where the weights are their proportion in the market. The higher the weighted average success, the more likely a market is to reach full financial inclusion. This is desirable from a policy maker's perspective. If the productive segment has ideas that are financially more attractive (higher R_2) or of lower risk (higher p), this compensates for the

expected losses from the unproductive segment. Markets in which the ratio of productive unwealthy is higher (higher $\frac{N_2}{N_2+N_3}$) or in which the productive ideas have a higher likelihood of success (higher p) are more likely to have unsecured loans.

Full financial inclusion, as condition (A4) demonstrates, implies a loss on average from the unproductive consumers in segment Ω_3 and is resulting in an overinvestment problem. The losses associated with lending to Ω_3 are cross-subsidized by the return from the productive entrepreneurs. In particular, whenever the rates associated with unsecured loans are low, this may contribute to the adverse selection problem where high risk consumers have an incentive to borrow because

$$\hat{p}(R_3 - r_2) \geq \bar{u} \tag{18}$$

is more likely to hold (Ghatak, 2000; de Meza et al., 1987). Corollary 3.1 formalizes the conditions that feed into the overinvestment problem with individual loan contracts.

Corollary 3.1. (*Overinvestment Problem*) *The overinvestment problem arises when high risk consumer's ideas are sufficiently more innovative compared to that of the unwealthy low risk consumers:*

$$R_3 - R_2 \geq \bar{u} \left(\frac{1}{\hat{p}} - \frac{1}{p} \right). \tag{A5}$$

The corollary emphasizes a trade-off. On the surface, it may seem like the bank would prefer all entrepreneurs in the market to have highly innovative ideas. But if the unproductive borrowers have innovative projects, they are incentivized to borrow and cannot be

distinguished from those in Ω_2 . The return from their investment increases their income although it wastes the resources of the bank. This creates undesirable results for the lender, although may be preferable from the perspective of a policy maker.

3.2.2 Impact of Informal Lending on Entrepreneurial Activity

When unsecured loans are not available or lending rates are very high, borrowers may use alternate financial products. A common alternative is informal lending, where wealthy consumers borrow and lend to others. The emergence of informal lending is endogenous to the conditions of formal lending. In this section, we study what conditions motivate informal lending and how it shapes entrepreneurial output in a market.

Consider a consumer who can borrow from a friend instead of the bank. We assume that if two individuals are friends, they know each other's risk type, in line with Wei et al. (2016). Informal lending takes place when the consumers in segment Ω_1 (i.e., those who borrow at a lower rate) lend to their friends in segment Ω_2 instead of investing in their project. A necessary condition for exchange is that the middle segment's projects must be more innovative than those of the first segment (i.e, $R_2 \geq R_1$). In this case, consumers in Ω_1 can expect higher returns from lending to Ω_2 compared to that from their investment. If $R_2 < R_1$ holds, the highest interest rate that borrowers in Ω_2 are willing to borrow at is $R_2 - \frac{\bar{u}}{p}$. And even if the bank offers the same interest rate (as in the case with no leaks), those in segment Ω_1 will not lend to segment Ω_2 .¹⁴

Via informal lending, consumers in Ω_1 earn rent from the ideas of the productive un-

¹⁴In reality, the reasons for lending to friends may not be financial. For example, individuals may lend to their friends for reasons such as altruism, desire to help a friend or a family member, or due to utility from betterment of friendship. In the current section, we make the conservative assumption that the only reason for lending to others is financial gains, but we consider alternate motivations in Section 3.3.2.

wealthy. So there is some social arbitrage taking place, where consumers who know their friends better than the bank knows them take advantage of this informational friction. For banks, consequences of informal lending is ambiguous. On the surface, it generates losses from to consumers borrow from friends. At the same time, it can charge the wealthy higher rates, since they can earn higher returns by lending to friends. A consequence of informal lending is that it results in a decline in the entrepreneurial activity of the first segment (Ω_1).

Recall that informal transfers can only happen if $R_2 \geq R_1$. So the bank (indirectly) funds more innovative borrowers. As some of these borrowers turn to earn rents from informal lending, the number of individuals who are investing in a microenterprise is lower. So the average funded project is more innovative (have a higher return associated with it when there is informal activity). This may explain why venture capital investment is flooding to emerging economies, which typically hold informal activity (Kho, 2011).¹⁵

To model informal lending, we denote the probability that a consumer in Ω_1 is friends with a consumer in Ω_2 by q .¹⁶ This parameter also represents the degree of mixing between the segments or the social classes according to wealth levels, and is common knowledge. We assume that each borrower in Ω_2 can be friends with one consumer in Ω_1 , but this is not a restricting assumption.¹⁷ Let's denote the borrowers in Ω_1 (Ω_2) who have friends in Ω_2 (Ω_1) by Ω_1^* (Ω_2^*) and their friends by $\Omega_2^*(\Omega_1^*)$. Let's represent the remaining consumers in $\Omega_1(\Omega_2)$ by $\Omega_1^{**}(\Omega_2^{**})$. Formally, $\Omega_1 = \Omega_1^* \cup \Omega_1^{**}$ and $\Omega_2 = \Omega_2^* \cup \Omega_2^{**}$. The connections of the consumers in Ω_3 do not influence our analysis qualitatively. This is because if a consumer $i \in \Omega_3$ is

¹⁵According to a report by World Economic Forum, in 2015 the countries that made a big jump in attracting venture capital were the emerging economies, doubling their total investment in a year (Vanham, 2015).

¹⁶Although we use the term friend, a broader category of social ties is considered.

¹⁷With more than one social contacts, the key results would still hold qualitatively.

connected to another consumer, then her friend knows that i has a high default risk and an unproductive idea, so he will not lend to her. So borrowers in Ω_3 can only obtain loans from the bank.

The lender can follow various strategies each of which is associated with a different payoff: (1) offer only the secured loan, (2) offer only the unsecured loan, and (3) offer both the secured and the unsecured loans. We will analyze each one of these strategies next. We will compare when the lender earns higher profits after studying all strategies.

Lender Offers Only Secured Loans. If the lender chooses to offer only the secured loan, it can only serve wealthy consumers (Ω_1). But it can adjust the terms to (1) serve all consumers in Ω_1 or to (2) only serve consumers in Ω_1^* . By choosing the latter, the lender is capitalizing on the relationships between these consumers and sets the borrowing rate higher.

If the lender wants to serve all consumers in Ω_1 , he can set the interest rate to $r_1 = R_1 + 1 - \frac{1}{p} - \frac{\bar{u}}{p}$. Its expected payoff is

$$\Pi_{11} = N_1 p (r_1 - 1) = N_1 (p R_1 - \bar{u} - 1).$$

Alternatively, the lender can only lend to Ω_1^* at a higher rate, and earn rent through informal lending. In this case, it sets the interest at $r_1 = R_2 + 1 - \frac{1}{p} - \frac{\bar{u}}{p}$. Thus only borrowers in Ω_1^* find it worthwhile to take a loan. Then the expected payoff is

$$\Pi_{12} = q N_1 p (r_1 - 1) = q N_1 (p R_2 - \bar{u} - 1).$$

Lender Offers Only Unsecured Loans. If the lender chooses to offer only unsecured loans, he also has two options, (1) serve all consumers, or (2) only serve those in Ω_2 and Ω_3 . (As we will show, it is not profitable to serve only Ω_1). If the lender chooses to serve all consumers, it would set the interest rate low at $r_2 = R_1 - \frac{\bar{u}}{p}$ to attract the entire market. In this case, the expected profit is:

$$\Pi_{21} = (N_1 + N_2)(p(R_1 - \frac{\bar{u}}{p}) - 1) + N_3(\hat{p}(R_1 - \frac{\bar{u}}{p}) - 1).$$

If it chooses to exclude Ω_1 , it can set the interest rate higher at $r_2 = R_2 - \frac{\bar{u}}{p}$. In this case, it is not profitable for consumers in Ω_1 to borrow because $p(R_1 - r_2) = p(R_1 - R_2 + \frac{\bar{u}}{p}) < \bar{u}$.

So the profit of the lender is:

$$\Pi_{22} = N_2(p(R_2 - \frac{\bar{u}}{p}) - 1) + N_3(\hat{p}(R_2 - \frac{\bar{u}}{p}) - 1).$$

Lender Offers Both Secured and Unsecured Loans. If the lender chooses to offer both secured and unsecured loans, he has three options: (1) serve all consumers, (2) serve all consumers in Ω_1 , or (3) to serve a portion of consumers in Ω_1 . Before we can calculate the expected profits of these three options, we analyze how the lender sets interest rates.

As long as the borrowers in Ω_1 lend to their friends at a rate lower than the bank's rate, their contacts in Ω_2 will borrow informally. To eliminate informal lending, the bank must set its rates such that customers in Ω_1 prefer not to lend to their friends. If the bank chooses to offer secured and unsecured contracts together, to incentivize borrowing of both groups,

it should ensure a nonnegative expected return:

$$r_1 \leq R_1 + 1 - \frac{1}{p} - \frac{\bar{u}}{p}, \quad (19)$$

$$r_2 \leq R_2 - \frac{\bar{u}}{p}. \quad (20)$$

Under these conditions, if borrowers in Ω_1 expect lower returns from lending to their contacts compared to that from investing, there will not be an informal market. Since the highest rate a borrower in segment Ω_1 can charge to a borrower in Ω_2 is r_2 (otherwise the borrower would opt for the unsecured loan offered by the bank), consumers in Ω_1 would prefer not to lend to friends if:

$$\underbrace{p(r_2 - r_1) - (1 - p) + \bar{u}}_{\text{Exp. return from lending to a friend}} \leq \underbrace{p(R_1 - r_1) - (1 - p)}_{\text{Exp. return from entrepreneurial activity}} \quad (21)$$

If the inequality holds, informal lending will not exist. The left hand side of the inequality is the expected return from lending to social contacts taking into consideration the collateral and the ability to keep outside income. The right hand side is the expected return from investing in the entrepreneurial idea taking into account the loss of outside income. Combining the constraints in Equations (19)–(21), the interest rates which maximize the lender's payoff while eliminating informal lending are:

$$r_1 = R_1 + 1 - \frac{1}{p} - \frac{\bar{u}}{p}, \quad (22)$$

$$r_2 = R_1 - \frac{\bar{u}}{p}. \quad (23)$$

In this case, the expected payoff becomes

$$\Pi_{31} = N_1(p(R_1 - \frac{\bar{u}}{p}) - 1) + N_2(p(R_1 - \frac{\bar{u}}{p}) - 1) + N_3(\hat{p}(R_1 - \frac{\bar{u}}{p}) - 1).$$

If the lender does not want to compete with informal market, he can raise the interest rate of unsecured loans to $r_2 = R_2 - \frac{\bar{u}}{p}$. Then consumers in Ω_2^* will borrow from their social contacts and the expected payoff for the lender is

$$\Pi_{32} = N_1(p(R_1 - \frac{\bar{u}}{p}) - 1) + (N_2 - qN_1)(p(R_2 - \frac{\bar{u}}{p}) - 1) + N_3(\hat{p}(R_2 - \frac{\bar{u}}{p}) - 1)$$

If the lender further does not try to serve customers in Ω_1^{**} , he can raise the interest rate of secured loans to $r_1 = R_2 + 1 - \frac{1}{p} - \frac{\bar{u}}{p}$ so that he can get all surplus from Ω_1^* . The expected payoff is

$$\begin{aligned} \Pi_{33} &= qN_1(p(R_2 - \frac{\bar{u}}{p}) - 1) + (N_2 - qN_1)(p(R_2 - \frac{\bar{u}}{p}) - 1) + N_3(\hat{p}(R_2 - \frac{\bar{u}}{p}) - 1) \\ &= N_2(p(R_2 - \frac{\bar{u}}{p}) - 1) + N_3(\hat{p}(R_2 - \frac{\bar{u}}{p}) - 1) \end{aligned}$$

Let $q_1^* \equiv \frac{N_2+N_3}{N_1} \frac{\bar{p}(R_2-\frac{\bar{u}}{p})-1}{p(R_2-\frac{\bar{u}}{p})-1}$, $q_2^* \equiv \frac{pR_1-\bar{u}-1}{pR_2-\bar{u}-1}$, $q_3^* \equiv \frac{pR_1-\bar{u}-1}{pR_2-\bar{u}-1} + \frac{N_2+N_3}{N_1} \frac{\bar{p}(R_1-\frac{\bar{u}}{p})-1}{p(R_2-\frac{\bar{u}}{p})-1}$ and $q_4^* \equiv \frac{N_2+N_3}{N_1} \frac{\bar{p}(R_2-R_1)}{p(R_2-\frac{\bar{u}}{p})-1}$.

Then the lender's optimal strategy can be summarized in the following proposition:

Proposition 3.22. *The density of social connections determines the degree of financial inclusion and the segments served by the lender. Specifically, when conditions (A1)-(A5) hold, the contracts offered and the coverage in the market are laid out in Table 2.*

Table 2: Borrowing Conditions vs. Density of Social Connections

Innovativeness of Projects	Density of Connections	Segments with Credit Access	Contracts Offered	Contract Terms
$R_1 < \frac{\bar{u}}{p} + \frac{1}{p}$	$q < q_1^*$ and $q < q_2^*$	Ω_1, Ω_2^* and Ω_3	Secured and unsecured	$r_1 = R_1 - \frac{1}{p} - \frac{\bar{u}}{p} + 1$ $r_2 = R_2 - \frac{\bar{u}}{p}$
	$q < q_1^*$ and $q \geq q_2^*$	Ω_1^*, Ω_2^* and Ω_3 or Ω_2 and Ω_3	Secured and unsecured	$r_1 = R_2 - \frac{1}{p} - \frac{\bar{u}}{p} + 1$ $r_2 = R_2 - \frac{\bar{u}}{p}$
$R_1 \geq \frac{\bar{u}}{p} + \frac{1}{p}$ and $R_2 - R_1 \geq \frac{N_1}{(N_2+N_3)\bar{p}} [p(R_1 - \frac{\bar{u}}{p}) - 1]$	$q \geq q_1^*$ and $q < q_2^*$	Ω_1	Secured	$r_1 = R_1 - \frac{1}{p} - \frac{\bar{u}}{p} + 1$
	$q \geq q_1^*$ and $q \geq q_2^*$	Ω_1^*	Secured	$r_1 = R_2 - \frac{1}{p} - \frac{\bar{u}}{p} + 1$
	$q < q_2^* \leq q_1^*$	Ω_1, Ω_2^* and Ω_3	Secured and unsecured	$r_1 = R_1 - \frac{1}{p} - \frac{\bar{u}}{p} + 1$ $r_2 = R_2 - \frac{\bar{u}}{p}$
	$q_2^* \leq q < q_1^*$	Ω_1^*, Ω_2^* and Ω_3 or Ω_2 and Ω_3	Secured and unsecured	$r_1 = R_2 - \frac{1}{p} - \frac{\bar{u}}{p} + 1$ $r_2 = R_2 - \frac{\bar{u}}{p}$
$R_1 \geq \frac{\bar{u}}{p} + \frac{1}{p}$ and $R_2 - R_1 < \frac{N_1}{(N_2+N_3)\bar{p}} [p(R_1 - \frac{\bar{u}}{p}) - 1]$	$q_2^* \leq q_1^* \leq q$	Ω_1^*	Secured	$r_1 = R_2 - \frac{1}{p} - \frac{\bar{u}}{p} + 1$
	$q < q_4^* < q_3^*$	Ω_1, Ω_2^* and Ω_3	Secured and unsecured	$r_1 = R_1 - \frac{1}{p} - \frac{\bar{u}}{p} + 1$ $r_2 = R_2 - \frac{\bar{u}}{p}$
	$q_4^* \leq q < q_3^*$	Ω_1, Ω_2 and Ω_3	Secured and unsecured	$r_1 = R_1 - \frac{1}{p} - \frac{\bar{u}}{p} + 1$ $r_2 = R_1 - \frac{\bar{u}}{p}$
	$q_4^* < q_3^* \leq q$	Ω_1^*	Secured	$r_1 = R_2 - \frac{1}{p} - \frac{\bar{u}}{p} + 1$

Proposition 3.22 shows that the density of social connections will influence informal activity, which in turn influences financial inclusion, types of loan contracts offered, and the rates at which individuals can borrow. In Figure 14 we demonstrate the relationship between the innovativeness of projects, density of social connections and the lender's optimal strategy.

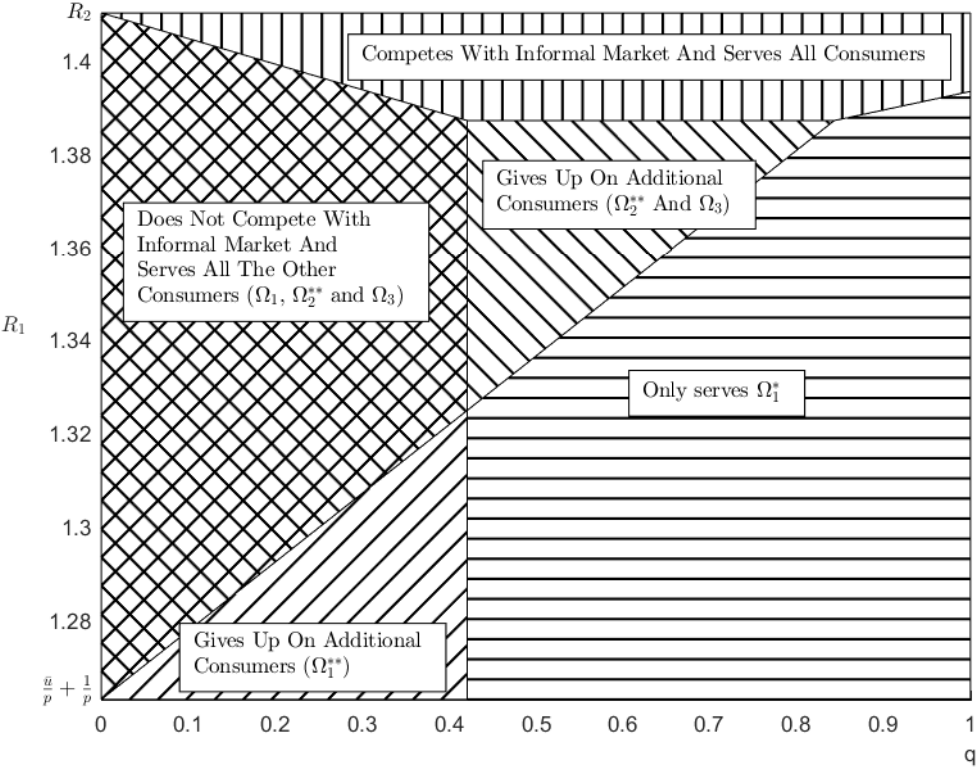
Proposition 3.23. (*Informal Lending and Volume of Entrepreneurial Activity*)

In a market with informal lending, the volume of entrepreneurial activity is lower compared to that in a market without informal lending. The average innovativeness of the entrepreneurial ideas, however, is higher.

With informal lending, the overall volume of entrepreneurial activity of the first segment is reduced. On the upside, informal lending corrects for the inefficiencies in screening of innovative ideas. Borrowers in Ω_1 provide loans to those in segment Ω_2 only if the return on their own ideas is lower than their contacts. As a consequence, the projects which are funded are, on average, more innovative. Thus informal lending facilitates transfers from the less innovative to the more innovative ideas. Higher innovation may result in higher quality services and products to be offered to consumers. We next highlight the conditions for the lender to eliminate the informal market in Proposition 3.24.

Proposition 3.24. (*Preventing Informal Lending*) *If $R_1 \geq \frac{\bar{u}}{p} + \frac{1}{p}$, $R_2 - R_1 < \frac{N_1}{(N_2 + N_3)\bar{p}} \left[p(R_1 - \frac{\bar{u}}{p}) - 1 \right]$ and $q_4^* \leq q < q_3^*$, the lender can offer both secured and unsecured contracts at rates $r_1 = R_1 + 1 - \frac{1}{p} - \frac{\bar{u}}{p}$ and $r_2 = R_1 - \frac{\bar{u}}{p}$ and prevent informal lending.*

Figure 14: Lender's Optimal Strategy vs. Density of Social Connections



$$R_2 = 1.41, R_3 = 1.425, p = 0.95, \hat{p} = 0.8, \bar{u} = 0.2, N_1 = N_2 = 1, N_3 = 2$$

If $R_1 < \frac{\bar{u}}{p} + \frac{1}{\bar{p}}$, or $R_1 \geq \frac{\bar{u}}{p} + \frac{1}{\bar{p}}$ and $R_2 - R_1 \geq \frac{N_1}{(N_2+N_3)\bar{p}} \left[p(R_1 - \frac{\bar{u}}{p}) - 1 \right]$, the relation between financial inclusion and the density of social ties can be summarized in the following proposition.

Proposition 3.25. (i) If $R_1 < \frac{\bar{u}}{p} + \frac{1}{\bar{p}}$, or $R_1 \geq \frac{\bar{u}}{p} + \frac{1}{\bar{p}}$ and $R_2 - R_1 \geq \frac{N_1}{(N_2+N_3)\bar{p}} \left[p(R_1 - \frac{\bar{u}}{p}) - 1 \right]$,

financial inclusion becomes more restricted as the density of social ties, q , increases.

(ii) If $R_1 \geq \frac{\bar{u}}{p} + \frac{1}{\bar{p}}$ and $R_2 - R_1 < \frac{N_1}{(N_2+N_3)\bar{p}} \left[p(R_1 - \frac{\bar{u}}{p}) - 1 \right]$, *the relation between financial inclusion and the density of social ties is ambiguous.*

Part (i) of Proposition 3.25 states that as the density of social connections (q) increases, access to financial services from formal lenders can become more restricted in an economy. In particular, if $q \leq q_1^*, q_2^*$, only the middle class (i.e., unwealthy productive consumers) with social connections to productive unwealthy (Ω_2^*) are not served by the formal lender. When the density increases such that $q_2^* \leq q < q_1^*$ or $q_1^* \leq q < q_2^*$, the lender gives up on more consumers (Ω_1^{**} , Ω_2^{**} , and Ω_3 , respectively). If $q \geq q_1^*$ and $q \geq q_2^*$, only Ω_1^* obtains financial products. So in markets with higher density of social connections, formal financial access is granted to a smaller segment of the society. This implies a loss on entrepreneurial volume, since a total investment of $((1 - q)N_1 + N_2)$ is wasted.

Restricting access to financial services results in an underinvestment problem, for two reasons. Firstly, some productive unwealthy consumers (Ω_2^{**}) are deprived of loans. These are the individuals without social connections to obtain informal loans. If the conditions in Proposition 3.25 hold, then the lender does not want to compete with the informal market and offers a pooling contract to Ω_2^{**} and Ω_3 . When q is large, the number of consumers in Ω_2^{**} is small, so the profit from lending to Ω_2^{**} is not sufficient to compensate for the loss

from lending to Ω_3 . The lender then gives up on both segments. Second, consumers in Ω_1^* lose their funding altogether, since the lender is fully aware that it can charge a higher interest if it gives up on these consumers. When the density of social connections q is high, it is profitable to choose this strategy. In both cases, the volume of innovative activity goes down, because there are fewer individuals who borrow to invest in ideas.

It is important to clarify the role of the consumers in group Ω_3 and how they influence the entrepreneurial investment opportunities of the other two segments. Recall that Ω_3 represents the consumers with unproductive ideas. These are the consumers whom the lender would want to avoid, if it could, in lending. However, the lender cannot distinguish between the consumers in Ω_2 and Ω_3 . When some individuals in Ω_1 lend to their contacts in Ω_2 , the lender attains a lower profit from offering unsecured loans. As a result, it is not willing to offer unsecured contracts when there is strong informal lending activity. A model that omits the existence of this group would falsely conclude that the bank's payoff from offering unsecured loans would not change in the existence of informal lending.

Part (ii) of the Proposition 3.25 states that when R_1 is large but $R_2 - R_1$ is small, the relationship between financial inclusion and the density of social ties is ambiguous. If the density is small such that $q < q_4^* < q_3^*$, only the productive unwealthy with social connections (Ω_2^*) are not served by the formal lender. When q is larger ($q_4^* \leq q < q_3^*$), the lender will compete and prevent informal lending. When q is even larger ($q_4^* < q_3^* \leq q$), only Ω_1^* obtains service.

Why is the relationship between density and financial inclusion different between parts (i) and (ii)? It is because the subjects of comparison are different in the two corollaries. When R_1 is large, serving all borrowers is more attractive than serving only Ω_1 , even at a

low interest rate ($R_1 - \frac{\bar{u}}{p}$). When the difference between R_2 and R_1 is small, lender prefers to serve all borrowers, charging a lower interest rate rather than giving up on N_1 borrowers. Thus an increase in the density of social connections results in two effects. First, the cost of giving up on consumers in Ω_2^* increases. Second, the profit from only serving Ω_1^* at a higher interest increases. When q is small, the first effect dominates, but when q is large, the second effect takes over. So, as q increases, the bank first serves more consumers (Ω_2^*) but then serves only Ω_1^* .

3.3 Extensions

In this section, we consider other characteristics of markets in which access to credit is a problem. First we consider the use of joint liability contracts and how it alters financial inclusion as well as the over and underinvestment problems, compared to individualized contracts. Second, we consider heterogeneity in the strength of ties and motivations for lending to see how robust our qualitative findings are to these factors. To focus on markets with challenges to credit access, we will make the following assumption:

$$R_1 < \frac{\bar{u}}{p} + \frac{1}{\bar{p}} \tag{A6}$$

This assumption eliminates the trivial cases where the lender serves all consumers charging a low interest rate.

3.3.1 Joint Liability Contracts and Entrepreneurial Activity

As an alternative to individual contracts, banks may offer joint liability contracts, which are also known as “group loans”. These contracts ask borrowers to form a group such that they are jointly liable for each others’ loans. As such, when individuals’ borrowing risks are joined through the terms of the loan, these contracts alleviate risk by addressing adverse selection and moral hazard issues introduced by informal lending. With joint liability contracts, the task of screening applicants is delegated to borrowers who have private information about their contacts’ risk types. They screen candidates with low risk and prefer to borrow with them. We represent a joint liability contract with (r, c) . The term r represents the individual liability or the interest which the borrower must pay back to the bank. The contract is structured such that if the project of the borrower succeeds, he agrees to pay an additional joint liability fee of c per member of his group whose project has failed. If his own project fails, then he pays nothing.

Subsequently, we study the conditions for when joint liability contracts can improve consumers’ access to loans and can improve entrepreneurial activity.

We first consider a market in which the lender offers joint loans instead of an unsecured loan to screen for risk, and assume no informal lending. The analysis we carry out will hold the conditions the same with those assumed in Section 3.2.1 ((A1)-(A5)). Joint loans screen the risk the consumer risk through friends selection of choosing to undersign the contract with another person. Since consumers hold private information about their social contacts and their financial health, they would rationally choose to borrow with others who have at worst the same risk with them (Ghatak, 2000). Since a low risk consumer would not want

to group with a high risk consumer, groups cannot be formed unless two borrowers of the same risk come together. Otherwise, one party will not be willing to undersign the loan. So consumers in Ω_2 will form groups with only consumers in Ω_2 and not with the consumers in Ω_3 . As such, joint liability contracts lead to positive assortative matching in borrowing groups and help the lender to indirectly sort the market in risk groups.

Similar to our benchmark model, the terms of the optimal joint liability contract is determined such that consumers in Ω_2 can be served while those in Ω_3 are driven out of the market. We will solve the model for a group of two consumers, but the intuition and modeling approach hold for groups of more than two people as well. For consumers in Ω_2 , the participation constraint to undersign a joint liability contract implies that their expected return should exceed their outside option:

$$\underbrace{p^2(R_2 - r_2)}_{\text{exp. return when both borrowers are successful}} + \underbrace{p(1-p)(R_2 - r_2 - c)}_{\text{exp. return when co-borrower fails}} \geq \underbrace{\bar{u}}_{\text{outside option}} \quad (24)$$

To reduce the overinvestment problem, the lender sets the rates such that the unproductive borrowers are discouraged. For the entrepreneurs in Ω_3 , the expected return falls short of outside option if:

$$\hat{p}^2(R_3 - r_2) + \hat{p}(1 - \hat{p})(R_3 - r_2 - c) < \bar{u}. \quad (25)$$

Moreover, limited-liability constraints require return from an investment to be nonnegative:

$$R_2 - r_2 - c \geq 0,$$

$$R_3 - r_2 - c \geq 0.$$

When (A5) holds, $R_3 > R_2$ and the constraint $R_3 - r_2 - c \geq 0$ is satisfied anytime $R_2 - r_2 - c \geq 0$ is satisfied.

Compared to individual liability contracts, joint liability contracts work in two additional ways for the lender to screen out those with high risk. First, a borrower's expected return from borrowing under joint liability depends not only on his, but also on his peer's success (i.e., jointly p^2 or \hat{p}^2). Altogether, the probability of success is lower. Second, if the project of the co-borrower is not successful, a borrower incurs a cost of c . These increase the lender's ability to screen applicants since only those with sufficiently high return and probability of success choose to borrow. Thus joint lending is more stringent compared to individual liability contracts. With joint lending, the conditions to resolve overinvestment problem is given in the following proposition.

Proposition 3.26. *(i) Joint liability contracts prevent overinvestment problem iff*

$$R_3 - R_2 < \hat{p}\bar{u}\left(\frac{1}{\hat{p}^2} - \frac{1}{p^2}\right). \quad (\text{A7})$$

(ii) Joint liability contracts are more effective than individual liability contracts in preventing the overinvestment problem if

$$R_2 + \hat{p}\bar{u}\left(\frac{1}{\hat{p}^2} - \frac{1}{p^2}\right) > R_3 \geq R_2 + \bar{u}\left(\frac{1}{\hat{p}} - \frac{1}{p}\right).$$

Proposition 3.26 sets a condition comparable to Corollary 3.1 about preventing the overinvestment problem. Similar to the condition defined for individual contracts, the difference in the returns of the ideas of the productive and unproductive unwealthy applicants should

be sufficiently small. Suppose conditions (A1)-(A4) hold and that all parameters, except for R_3 , are given. Even though the risky borrowers' projects are unproductive, they can be sufficiently innovative, implying that R_3 can still be high. When R_3 is small (i.e., when it is smaller than $R_2 + \bar{u}(\frac{1}{\hat{p}} - \frac{1}{p})$), these projects are unattractive, and individual or joint liability contracts are not useful for the individuals in Ω_3 because neither contract is profitable. When R_3 is greater than or equal to $(R_2 + \hat{p}\bar{u}(\frac{1}{\hat{p}^2} - \frac{1}{p^2}))$, contracts are not useful for the bank to prevent overinvestment problem, because neither joint nor individual contracts can drive the consumers in Ω_3 out of market. When the return R_3 is intermediate (i.e., in the range $R_2 + \bar{u}(\frac{1}{\hat{p}} - \frac{1}{p})$ and $R_2 + \hat{p}\bar{u}(\frac{1}{\hat{p}^2} - \frac{1}{p^2})$), joint liability contracts are more efficient than the individual liability contracts in preventing the overinvestment problem due to the additional benefits in screening.

How does joint liability change the borrowing rates, compared to individual liability contracts? When there are no leaks, for borrowers in Ω_1 , the rates remain identical in joint and individual contracts. But for borrowers in Ω_2 and Ω_3 , the rates may vary. When $R_3 \geq R_2 + \hat{p}\bar{u}(\frac{1}{\hat{p}^2} - \frac{1}{p^2})$, the bank cannot prevent the overinvestment problem and therefore these consumers borrow at the same rate they did with individual contracts, $r_2 = R_2 - \frac{\bar{u}}{p}$. If $R_3 < R_2 + \hat{p}\bar{u}(\frac{1}{\hat{p}^2} - \frac{1}{p^2})$, the bank can drive the applicants in Ω_3 out of the market with a joint liability contract with the following terms:

$$r_2 = R_2 - \frac{\bar{u}}{p^2},$$

$$c = \frac{\bar{u}}{p^2}.$$

So when $R_3 < R_2 + \hat{p}\bar{u}(\frac{1}{\hat{p}^2} - \frac{1}{p^2})$, the borrowing rate r_2 for the unwealthy consumers is lower.

The bank can still extract all of the surplus of Ω_2 , and they do not need to compensate for those in Ω_3 .

3.3.2 Strength of Ties and Other Motivations for Lending

In our benchmark model, we assumed that all informal lenders are identical, independent of the type of social relationship with the borrower. In reality, the strength of ties of social connections may matter and change the formal and informal lending environment. Our main model focuses on the density of social connections (the degree of mixing between classes) when studying informal loans. It is natural to consider that informal borrowing ability is determined not only by the proportion of other segment consumers, but also how one knows these consumers. For instance, close family and friends may extend loans with zero interest. Alternatively, some may lend for reasons like altruism or investing into a personal relationship. In this section, we explore how the strength of ties influences informal lending. We also formulate individuals' additional utility from social capital when their friends and family gain wealth.

For formal lending only the strength of ties between the consumers in Ω_1 and Ω_2 matter. Suppose $i \in \Omega_1$ is linked to $j \in \Omega_2$. Let the returns from lending be joint for these two individuals, that is, i cares not only about his personal payoff but also about the expected payoff of his friend, j . We denote the strength of tie ij by $t_{ij} \in \{0, 1\}$, where $t_{ij} = 1$ implies that the tie between i and j is strong and otherwise the tie is weak. Consequently, connected consumer subsegments can be separated into two groups conditional on the strength of their tie. Ω_1^* is divided into $\hat{\Omega}_1^*$ and $\tilde{\Omega}_1^*$; and Ω_2^* is divided into $\hat{\Omega}_2^*$ and $\tilde{\Omega}_2^*$ where borrowers $\hat{\Omega}_1^*$ ($\tilde{\Omega}_1^*$) have strong (weak) ties with friends in $\hat{\Omega}_2^*$ ($\tilde{\Omega}_2^*$). Let α be the probability that a tie is

strong. Then, the probability that a consumer in Ω_1 is strongly (weakly) linked to another consumer in Ω_2 is $\alpha q ((1 - \alpha)q)$.

We assume that borrowers with weak ties still only care about the expected return and we represent the utility of borrower i from lending to a friend j at rate \hat{r} with strong ties as a composite term of the expected return from lending to his friend (E_i) and the return of his friend on her investment $f(E_j)$:

$$U_i(\hat{r}) = E_i + f(E_j).$$

Similar to our benchmark model, the individual return of customer i depends on the probability of success of the friend's investment, keeping the outside option:

$$E_i = p(\hat{r} - r_1) - (1 - p) + \bar{u}$$

or,

$$E_i = p(\hat{r} - r_2) + \bar{u}$$

This expression represents the case in which customer i borrows the unsecured loans and then lends to his friend. $p(\hat{r} - r_2)$ is the expected return if the project of his friend succeeds.

The return of j from her entrepreneurial investment is:

$$E_j = p(R_2 - \hat{r}).$$

For simplicity, we assume that $\frac{dU_i(\hat{r})}{d\hat{r}} < 0$ and $f(E_j)$ is large enough, so the borrowers with

strong ties are always willing to lend to their friends with the lowest interest rate. It is easy to check that the lowest interest rate is $r_1 - \bar{u}$ or $r_2 - \bar{u}$ because customer i can use other income \bar{u} to help his friend pay interest.

The lender still has three different kinds of strategies: (1) to offer only the secured loan, (2) to offer only the unsecured loan, and (3) to offer both the secured and the unsecured loans. We discuss the lender's optimal strategy in Proposition 3.27.

Proposition 3.27. *Let $\alpha_1^* \equiv \frac{pR_1 - \bar{u} - 1}{p\bar{u} + pR_2 - \bar{u} - p}$, $\alpha_2^* \equiv \frac{pR_2 - \bar{u} - 1}{p\bar{u} + pR_2 - \bar{u} - p}$, $\alpha_3^* \equiv \frac{pR_1 - \bar{u} - 1}{p\bar{u} + 1 - p}$ and $\alpha_4^* \equiv \frac{pR_2 - \bar{u} - 1}{p\bar{u} + 1 - p}$.*

When conditions (A1)-(A7) hold and consumers with strong ties are willing to offer the lowest interest rate, the optimal strategies of the lender and the resulting financial inclusion outcomes are summarized in Table 3.

Table 3: Density of Social Connections and Financial Inclusion with non-financial Utility

	Density of Social Connections	Segments Served
$q \geq q_1^*, q \geq q_2^*$	$\alpha \geq \alpha_2^*$ and $\alpha \geq \frac{1}{q}q_1^*$	$\hat{\Omega}_1^*$
	$\alpha \geq \alpha_2^*$ and $\alpha < \frac{1}{q}q_1^*$, or $\alpha < \alpha_2^*$ and $\alpha \geq (1 - \frac{1}{q}q_1^*)\alpha_3^*$	$\hat{\Omega}_1^*, \Omega_2/\hat{\Omega}_2^*$ and Ω_3
	$\alpha < \alpha_2^*$ and $\alpha < (1 - \frac{1}{q}q_1^*)\alpha_3^*$	Ω_1^*
$q \geq q_1^*, q < q_2^*$	$\alpha \geq \frac{1}{q}q_1^*$ and $\alpha \geq \frac{1}{q}\alpha_1^*$	$\hat{\Omega}_1^*$
	$\alpha \geq \frac{1}{q}\alpha_1^*$ and $\alpha < \frac{1}{q}q_1^*$, or $\alpha < \frac{1}{q}\alpha_1^*$ and $\alpha \geq \frac{1}{q}\alpha_4^*(1 - q_1^*)$	$\hat{\Omega}_1^*, \Omega_2/\hat{\Omega}_2^*$ and Ω_3
	$\alpha < \frac{1}{q}\alpha_1^*$ and $\alpha < \frac{1}{q}\alpha_4^*(1 - q_1^*)$,	Ω_1
$q < q_1^*$	$\alpha \geq \frac{1}{q}\alpha_3^* - \alpha_4^*$	$\hat{\Omega}_1^*, \Omega_2/\hat{\Omega}_2^*$ and Ω_3
	$\alpha < \frac{1}{q}\alpha_3^* - \alpha_4^*$	Ω_1, Ω_2^* and Ω_3

We can see that when the proportion of consumers with strong ties ($\hat{\Omega}_1^*$) are large enough ($\alpha q \geq q_1^*$ and $\alpha q \geq \alpha_1^* = q_2^*\alpha_2^*$), the lender will choose to give up on all other consumers and only serve $\hat{\Omega}_1^*$. When the density of social connections decreases (through either α or q), the lender will choose to lend to more consumers. Eventually, when $\alpha < \frac{1}{q}\alpha_3^* - \alpha_4^*$ and $q < q_1^*$, the lender will only give up on Ω_2^* .

So how do alternative motives influence financial inclusion? Motives such as helping a friend create additional competition for the lender, creating the pressure at first to cut down on rates, and thereby helps consumers to access credit. If the utility from other motives increases further, it reduces the attractiveness of investing into income-constrained consumers, resulting in the bank pulling out of these markets and an overall limitation in the access to financial products. As a consequence, the entrepreneurial opportunities decrease in this market. Strength of ties, on the other hand, creates an additional condition which influences the availability of funds. For the same level of mixing between the segments, if individuals generally hold stronger ties, financial access will be more limited with the same intuition.

3.4 Conclusions and Key Insights

Few sectors serve as vital a purpose as financial services do for the well-being of consumers; enabling their everyday transactions as well as life-long investments. Unfortunately, a significant number of consumers around the world lack access to financial services to realize any significant investments into their future. The World Bank estimates this number to be around 2 billion¹⁸. These consumers lack the ability to obtain loans for an entrepreneurial investment, even though some of them are perfectly creditworthy. Among top reasons for the lack of credit is imperfect information of lenders about consumers risk and inability to screen applicants. As these consumers cannot borrow from banks, they end up borrowing informally from other resources, and it is not how it influences access to financing from formal sources and entrepreneurial output. Informal credit is easy to access since it can be provided

¹⁸Source: <http://www.worldbank.org/en/topic/financialinclusion/overview>

by anyone without any formal screening process, or having to comply with any regulations. There is often no legal process to be a loan provider. Moreover informal loans do not require a collateral.

In this paper, we study how the availability of informal opportunities influences the type of contracts offered in a market as well as the borrowing terms for consumers. Studying the reasons behind the success of entrepreneurial investment is particularly important when consumers face stagnant poverty. As the available contracts change, the volume and innovativeness of the entrepreneurial activity also change in a market. A number of earlier studies pointed out to the problems resulting from the existence of informal markets (Aleem, 1990; Karlan and Zinman, 2008; Ayyagari et al., 2010). Our findings show that while informal borrowing can reduce the volume of entrepreneurial activity, it also increases the quality of funded ideas. This is because informal lending takes place only when the wealthier consumers find it worthwhile to lend to others if they find arbitrage opportunities. As a result, surviving and funded ideas are more innovative and are associated with higher returns.

To study how financial inclusion changes with informal lending opportunities, our model uses a parameter indicating the density of connections in a society, or the degree of mixing between the (relatively) wealthy and the unwealthy consumers. When formal lending fuels informal exchanges as opposed to investment in microenterprises, the volume of entrepreneurial activity by the wealthy is reduced, but indirectly, these informal funds support the entrepreneurial activity of those with lesser wealth. Through informal lending, consumers with productive ideas can obtain loans. So banks indirectly fund the entrepreneurial ideas associated with higher returns. When the volume of informal lending is small to moderate, it helps to reduce the cost of entrepreneurship since lenders are in competition with

the informal market. As the informal borrowing opportunities increase further, however, the lender pulls out of the unsecured lending market and restricts credit access. This is because it can earn higher profits if it lends to the wealthy at a higher rate. By restricting the credit access of the unwealthy, the bank creates opportunities for the wealthy to lend to these consumers. So some of the wealthy low risk consumers choose not to invest in an SME and only a portion of the unwealthy who can borrow informally can pursue their ideas. Entrepreneurial activity is reduced. Despite the low volume of entrepreneurial activity, in this case, the average investment is more innovative and is associated with a higher return.

To our knowledge, our paper is the first to study financial inclusion and the impact of informal lending. Our results in part explain the low financial inclusion observed in emerging countries with collectivist cultures where informal lending is common practice. It predicts that a more active informal market will discourage banks from extending loans to the overall population. As a result, small business investment in these countries remain low and many individuals face permanent poverty.

Policy Relevance. Expansion of financial inclusion is a priority for many NGOs. The Obama White House also declared that access to safe and affordable financial instruments is a high priority. Governments such as the one in India battles through their Central Bank to prevent informal money lending to spread access to the formal banking system (Parussini, 2015).

One important consequence of lack of credit access in both United States and around the world is the reduced investment into activities which can help low income consumers to move permanently out of poverty, such as investment into a small family owned business.

While there has been some improvement in the past decades, financial inclusion is still low in many countries. We show that the underlying reasons behind low financial access have to do with the outside borrowing options. In a tight community, the informal opportunities created can incentivize firms to pull out of markets.

Surveys from developing countries show that informal lenders generally charge interest rates that are higher than the interest rates charged by institutional lenders. This difference raised interest from the managers and policy makers. Our study explains precisely why it is rational to expect such a gap in the interest rates. It is because the banks close their doors to those who cannot borrow without a collateral, and further increase the interest rates on the wealthy who can. As some of the wealthy choose to lend to others at a higher interest rate (than their borrowing rate), interest rates are elevated for all consumers.

One consideration for managers and policy makers to reduce informal lending activity is to improve the ability of the banks to screen candidates, and thereby reduce the information asymmetry problem. Alternate contracts (such as group borrowing opportunities) and use of new and big data methods to evaluate consumer risk more precisely (Wei et al., 2016) can reduce the adverse consequences of information asymmetry. We saw in the analysis that because of the imperfections in screening, the lender did not always have an incentive to cut interest rates even though it could increase its market share. But if the lender could increase its screening power, it could try to serve a larger portion of the market and can cut down on the interest rates without having to worry about the overinvestment problem.

Moreover, on the demand side, borrowers are generally not well-informed about the problems about screening applicants. By voluntarily providing more information about themselves - through opening of savings accounts, borrowing and repaying small amounts

- consumers can help to resolve some of the information asymmetry problems and improve their chance of attaining credit when they need it.

Future Research. Since they have such wide consequences for a consumer's quality of living, issues related to access to finance are at the heart of entrepreneurship. Considering the importance of both consumer finance and innovation, it is surprising that in literature, the studies focusing on these topics, particularly the relationship between them are relatively rare (Hauser et al., 2006). Recent studies are focusing on expanding credit access and its outcomes (Wei et al., 2016), but given the importance of improving financial access, there is a need for additional analytical and empirical studies which can expand the scope of the literature. Future studies can focus on other key aspects of the developing economies and their challenges, focusing on stagnant gender or racial discrimination issues. Difficulty in accessing credit could well be the reason for the lower innovative output observed in developing economies. There is a need for future research which can address these additional problems. Moreover, while our study captures the benefits of joint lending about overcoming information asymmetry, the literature (e.g., Ghatak, 2000) argued that there are some other benefits such as monitoring or enforcement of repayment. We do not study these issues in this study and future studies may also consider these benefits.

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Appendix

5.1 Chapter 1

5.1.1 Proof of Theorem 1.1.

The proof follows the classical argument of Fudenberg and Levine (1992) and Sorin (1999).

I start with several lemmas.

Lemma 5.4. *Fix σ and π^0 . For any $\varepsilon, \psi > 0$ and $\theta \in \Theta$, there exists a positive integer K , such that*

$$Q_\theta (|\{t \geq 1 : d^t(P, Q_\theta) \geq \psi\}| \geq K) \leq \varepsilon$$

where

$$d^t(P, Q_\theta) = \sup_{C \in \hat{\mathcal{F}}_{t+1}} |P(C|\hat{\mathcal{F}}_t)(\omega) - Q_\theta(C|\hat{\mathcal{F}}_t)(\omega)|$$

Lemma 5.5. *Fix σ and π^0 . For any $\eta > 0$, $\theta \in \Theta$ and $K \geq 1$,*

$$P (|\{t \geq 1 : V_\theta^t \geq \eta\}| \geq K) \leq \frac{1}{K\eta^2}$$

where $V_\theta^t = E_P(|\pi^{t+1}(\theta) - \pi^t(\theta)||\hat{\mathcal{F}}_t)(\omega)$

Lemma 5.6. *For any $\theta \in \Theta$, $V_\theta^t \geq \pi^t(\theta)d^t(P, Q_\theta)$.*

These three lemmas are very similar to Lemma 15.4.3-15.4.5 in Mailath and Samuelson (2006) so I omit the proofs here.

Let $G_{\varepsilon, \hat{\theta}}(\omega) = \{t \in \mathbb{N}; \exists \theta \text{ s.t. } \pi^t(\theta) d^t(Q_\theta, Q_{\hat{\theta}})(\omega) \geq \varepsilon\}$.

Lemma 5.7. *For any $\varepsilon > 0$, and any $\hat{\theta} \in \Theta$, there exists $M > 0$ such that $Q_{\hat{\theta}} \left(\left| \left\{ \left| G_{\frac{\varepsilon}{2}, \hat{\theta}} \right| \geq M \right\} \right| \geq \varepsilon \right) \leq$*

Proof. Let

$$d^t(Q_{\hat{\theta}}, Q_\theta) = \sup_{C \in \hat{\mathcal{F}}_t} |Q_{\hat{\theta}}(C | \hat{\mathcal{F}}_{t-1})(\omega) - Q_\theta(C | \hat{\mathcal{F}}_{t-1})(\omega)|$$

According to Lemma 5.4, for any $\varepsilon > 0$, there exists a positive integer K^* , such that for any $K \geq K^*$,

$$Q_{\hat{\theta}} \left(\left| \left\{ t \geq 1 : d^t(P, Q_{\hat{\theta}}) \geq \frac{\varepsilon}{4} \right\} \right| \geq K \right) \leq \frac{\varepsilon}{2}$$

According to Lemma 5.5 and Lemma 5.6, for any $\eta > 0$ and $\theta \in \Theta$,

$$P \left(\left| \{t \geq 1 : \pi^t(\theta) d^t(P, Q_\theta) \geq \eta\} \right| \geq K \right) \leq P \left(\left| \{t \geq 1 : V_\theta^t \geq \eta\} \right| \geq K \right) \leq \frac{1}{K\eta^2}$$

Since $|\Theta| = 2N$,

$$\begin{aligned} & \{\omega : |\{t \geq 1 : \exists \theta \text{ s.t. } \pi^t(\theta) d^t(P, Q_\theta) \geq \eta\}| \geq 2NK\} \\ & \subseteq \{\omega : \exists \theta \text{ s.t. } |\{t \geq 1 : \pi^t(\theta) d^t(P, Q_\theta) \geq \eta\}| \geq K\} \end{aligned}$$

so

$$\begin{aligned}
& P(|\{t \geq 1 : \exists \theta \text{ s.t. } \pi^t(\theta) d^t(P, Q_\theta) \geq \eta\}| \geq 2NK) \\
& \leq P(|\{t \geq 1 : \pi^t(\theta) d^t(P, Q_\theta) \geq \eta\}| \geq K) \\
& \leq \frac{1}{K\eta^2}
\end{aligned}$$

Since $P = \sum_{\theta} \pi^0(\theta) Q_\theta \geq \pi^0(\hat{\theta}) Q_{\hat{\theta}}$,

$$\begin{aligned}
& Q_{\hat{\theta}}(|\{t \geq 1 : \exists \theta \text{ s.t. } \pi^t(\theta) d^t(P, Q_\theta) \geq \eta\}| \geq 2NK) \\
& \leq \frac{1}{\pi^0(\hat{\theta})} P(|\{t \geq 1 : \exists \theta \text{ s.t. } \pi^t(\theta) d^t(P, Q_\theta) \geq \eta\}| \geq 2NK) \\
& \leq \frac{1}{K\eta^2 \pi^0(\hat{\theta})}
\end{aligned}$$

For any t , $\pi^t(\theta) d^t(Q_{\hat{\theta}}, Q_\theta) \leq \pi^t(\theta) d^t(P, Q_{\hat{\theta}}) + \pi^t(\theta) d^t(P, Q_\theta)$, so

$$\begin{aligned}
& \pi^t(\theta) d^t(Q_{\hat{\theta}}, Q_\theta) \geq \frac{\varepsilon}{2} \\
& \implies \pi^t(\theta) d^t(P, Q_{\hat{\theta}}) \geq \frac{\varepsilon}{4} \text{ or } \pi^t(\theta) d^t(P, Q_\theta) \geq \frac{\varepsilon}{4} \\
& \implies d^t(P, Q_{\hat{\theta}}) \geq \frac{\varepsilon}{4} \text{ or } \pi^t(\theta) d^t(P, Q_\theta) \geq \frac{\varepsilon}{4}
\end{aligned}$$

Therefore, let $M = (2N + 1)K$ and $K = \max \left\{ \left\lceil \frac{32}{\varepsilon^3 \pi^0(\hat{\theta})} \right\rceil, K^* \right\}$

$$\begin{aligned}
& Q_{\hat{\theta}} \left(\left\{ \left| G_{\frac{\varepsilon}{2}, \hat{\theta}} \right| \geq M \right\} \right) \\
&= Q_{\hat{\theta}} \left(\left\{ \left| G_{\frac{\varepsilon}{2}, \hat{\theta}} \right| \geq M \right\} \cap \left\{ \left| \left\{ t \geq 1 : d^t(P, Q_{\hat{\theta}}) \geq \frac{\varepsilon}{4} \right\} \right| \geq K \right\} \right) \\
&\quad + Q_{\hat{\theta}} \left(\left\{ \left| G_{\frac{\varepsilon}{2}, \hat{\theta}} \right| \geq M \right\} \cap \left\{ \left| \left\{ t \geq 1 : d^t(P, Q_{\hat{\theta}}) \geq \frac{\varepsilon}{4} \right\} \right| < K \right\} \right) \\
&\leq Q_{\hat{\theta}} \left(\left\{ \left| \left\{ t \geq 1 : d^t(P, Q_{\hat{\theta}}) \geq \frac{\varepsilon}{4} \right\} \right| \geq K \right\} \right) \\
&\quad + Q_{\hat{\theta}} \left(\left\{ \left| \left\{ t \geq 1 : d^t(P, Q_{\hat{\theta}}) < \frac{\varepsilon}{4} \text{ and } \exists \theta \text{ s.t. } \pi^t(\theta) d^t(Q_{\theta}, Q_{\hat{\theta}})(\omega) \geq \frac{\varepsilon}{2} \right\} \right| \geq M - K \right\} \right) \\
&\leq Q_{\hat{\theta}} \left(\left\{ \left| \left\{ t \geq 1 : d^t(P, Q_{\hat{\theta}}) \geq \frac{\varepsilon}{4} \right\} \right| \geq K \right\} \right) \\
&\quad + Q_{\hat{\theta}} \left(\left\{ \left| \left\{ t \geq 1 : \exists \theta \text{ s.t. } \pi^t(\theta) d^t(P, Q_{\theta}) \geq \frac{\varepsilon}{4} \right\} \right| \geq 2NK \right\} \right) \\
&\leq \frac{\varepsilon}{2} + \frac{16}{K \varepsilon^2 \pi^0(\hat{\theta})} \\
&\leq \varepsilon
\end{aligned}$$

□

I can now prove this Proposition.

Proof. Fix $\hat{\theta} = (\xi(\hat{\alpha}_1), d)$. I also try to provide a lower bound on the payoff to Player 1 of normal type mimicking commitment type. Since the true monitoring structure is d , the true distribution is $Q_{\hat{\theta}}$. The payoff of Player 1 of normal type mimicking commitment type is

$$E_{Q_{\hat{\theta}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) \right)$$

According to Lemma 5.7,

$$Q_{\hat{\theta}} \left(\left\{ \left| G_{\frac{\varepsilon}{2}, \hat{\theta}} \right| < M \right\} \right) \geq 1 - \varepsilon$$

which means when the true monitoring structure is d and Player 1 mimics commitment type, with probability greater than or equal to $1 - \frac{\varepsilon}{2}$, the number of periods that $\exists \theta$ s.t. $\pi^t(\theta) d^t(Q_{\theta}, Q_{\hat{\theta}})(\omega) \geq \frac{\varepsilon}{2}$ is at most M . For the rest of the time, for any θ , $\pi^t(\theta) d^t(Q_{\theta}, Q_{\hat{\theta}})(\omega) < \frac{\varepsilon}{2}$.

The next thing I want to prove is that, if for any θ , $\pi^t(\theta) d^t(Q_{\theta}, Q_{\hat{\theta}})(\omega) < \frac{\varepsilon}{2}$, then Player 2's strategy is a d -weak- ε -confirming best response to $\hat{\alpha}_1$. For any d_i ,

$$\pi^t(d_i) = \pi^t((\xi_0, d_i)) + \pi^t((\xi(\hat{\alpha}_1), d_i))$$

$$d^t(Q_{(\xi_0, d_i)}, Q_{\hat{\theta}}) = \|\rho_{d_i, \alpha_1^t} - \rho_{d, \hat{\alpha}_1}\|_{\infty}$$

and

$$d^t(Q_{(\xi(\hat{\alpha}_1), d_i)}, Q_{\hat{\theta}}) = \|\rho_{d_i, \hat{\alpha}_1} - \rho_{d, \hat{\alpha}_1}\|_{\infty}$$

Then

$$\begin{aligned}
& \pi^t(d_i) \left\| \frac{\pi^t((\xi_0, d_i))}{\pi^t(d_i)} \rho_{d_i, \alpha_1^t} + \frac{\pi^t((\xi(\hat{\alpha}_1), d_i))}{\pi^t(d_i)} \rho_{d_i, \hat{\alpha}_1} - \rho_{d, \hat{\alpha}_1} \right\|_\infty \\
& \leq \pi^t((\xi_0, d_i)) \left\| \rho_{d_i, \alpha_1^t} - \rho_{d, \hat{\alpha}_1} \right\|_\infty + \pi^t((\xi(\hat{\alpha}_1), d_i)) \left\| \rho_{d_i, \hat{\alpha}_1} - \rho_{d, \hat{\alpha}_1} \right\|_\infty \\
& \leq \varepsilon
\end{aligned}$$

so Player 2's strategy is a d -weak- ε -confirming best response to $\hat{\alpha}_1$. Therefore,

$$\begin{aligned}
& E_{Q_{\hat{\theta}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) \right) \\
& = Q_{\hat{\theta}} \left(\left\{ \left| G_{\varepsilon, \hat{\theta}} \right| < M \right\} \right) E_{Q_{\hat{\theta}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) \middle| \left\{ \left| G_{\varepsilon, \hat{\theta}} \right| < M \right\} \right) \\
& \quad + \left(1 - Q_{\hat{\theta}} \left(\left\{ \left| G_{\varepsilon, \hat{\theta}} \right| < M \right\} \right) \right) E_{Q_{\hat{\theta}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) \middle| \left\{ \left| G_{\varepsilon, \hat{\theta}} \right| \geq M \right\} \right) \\
& \geq Q_{\hat{\theta}} \left(\left\{ \left| G_{\varepsilon, \hat{\theta}} \right| < M \right\} \right) \left(\delta^M v_{\alpha_1}^{w,d}(\varepsilon) + (1 - \delta^M) \min_{(\alpha_1, \alpha_2)} u_1(\alpha_1, \alpha_2) \right) \\
& \quad + \left(1 - Q_{\hat{\theta}} \left(\left\{ \left| G_{\varepsilon, \hat{\theta}} \right| < M \right\} \right) \right) \min_{(\alpha_1, \alpha_2)} u_1(\alpha_1, \alpha_2)
\end{aligned}$$

Since $v_{\alpha_1}^{w,d}(\varepsilon) \geq \min_{(\alpha_1, \alpha_2)} u_1(\alpha_1, \alpha_2)$,

$$E_{Q_{\hat{\theta}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) \right) \geq (1 - \varepsilon) \delta^M v_{\alpha_1}^{w,d}(\varepsilon) + (1 - (1 - \varepsilon) \delta^M) \min_{(\alpha_1, \alpha_2)} u_1(\alpha_1, \alpha_2)$$

□

5.1.2 Proof of Proposition 1.1 part 1.

Proof. Let $A_1^{d_i}(\hat{\alpha}_1) = \{\alpha_1 \in \Delta(A_1) : \exists d \in D \text{ s.t. } \rho_{d_i, \hat{\alpha}_1} = \rho_{d, \alpha_1}\}$. Then

$$D \subseteq \tilde{D}_i^1 \iff \text{for any } \alpha_1 \in A_1^{d_i}(\hat{\alpha}_1), BR_2(\alpha_1) = BR_2(\hat{\alpha}_1)$$

It is easy to check that $BR_2(\hat{\alpha}_1) \subseteq B_\varepsilon^{w,d}(\hat{\alpha}_1)$. I only need to show that there exists ε^* , such that for any $\varepsilon < \varepsilon^*$, $B_\varepsilon^{w,d}(\hat{\alpha}_1) \subseteq BR_2(\hat{\alpha}_1)$.

Fix any $a_2 \notin BR_2(\hat{\alpha}_1)$, and $\hat{a}_2 \in BR_2(\hat{\alpha}_1)$. Let $\beta = \inf_{\alpha_1 \in A_1^{d_i}(\hat{\alpha}_1)} [u_2(\alpha_1, \hat{a}_2) - u_2(\alpha_1, a_2)]$. Since $A_1^{d_i}(\hat{\alpha}_1)$ is compact, $\beta > 0$. For any $(\alpha'_{1,1}, \dots, \alpha'_{1,N})$ and $(\pi'(d_1), \dots, \pi'(d_N))$ satisfying the conditions in the definition of d_i -weak- ε -confirming best response and any k , if $\|\rho_{d, \alpha'_{1,k}} - \rho_{d_i, \hat{\alpha}_1}\|_\infty > \sqrt{\varepsilon}$, then $\pi'(d) < \sqrt{\varepsilon}$. According to the definition of $A_1^{d_i}(\hat{\alpha}_1)$, when ε is small enough,

$$\|\rho_{d, \alpha'_{1,k}} - \rho_{d_i, \hat{\alpha}_1}\|_\infty \leq \sqrt{\varepsilon} \implies \exists \hat{\alpha}_{1,k} \in A_1^{d_i}(\hat{\alpha}_1) \text{ s.t. } \|\alpha'_{1,k} - \hat{\alpha}_{1,k}\|_\infty = O(\sqrt{\varepsilon})$$

Let $D_{\sqrt{\varepsilon}} = \{d_k \in D : \|\rho_{d, \alpha'_{1,k}} - \rho_{d_i, \hat{\alpha}_1}\|_\infty \leq \sqrt{\varepsilon}\}$. Since $\hat{a}_2 \in BR_2(\hat{\alpha}_{1,k}) = BR_2(\hat{\alpha}_1)$ and $a_2 \notin BR_2(\hat{\alpha}_1) = BR_2(\hat{\alpha}_{1,k})$, $u_2(\hat{\alpha}_{1,k}, \hat{a}_2) > u_2(\hat{\alpha}_{1,k}, a_2)$, and, for any k such that $d_k \in D_{\sqrt{\varepsilon}}$, $u_2(\hat{\alpha}_{1,k}, \hat{a}_2) > u_2(\hat{\alpha}_{1,k}, a_2)$. $\sum_{d_k \in D_{\sqrt{\varepsilon}}} \pi'(d_k) [u_2(\hat{\alpha}_{1,k}, \hat{a}_2) - u_2(\hat{\alpha}_{1,k}, a_2)] \geq \beta(1 - \sum_{d_k \in D_{\sqrt{\varepsilon}}} \pi(d_k)) >$

0. Since u_2 is continuous, $\sum_{d_k \in D_{\sqrt{\varepsilon}}} \pi'(d_k) [u_2(\alpha'_{1,k}, \hat{a}_2) - u_2(\alpha'_{1,k}, a_2)] \geq \beta + O(\sqrt{\varepsilon})$. Then

$$\begin{aligned}
& u_2\left(\sum_{d_k \in D} \pi'(d_k) \alpha'_{1,k}, a_2\right) \\
&= \sum_{d_k \in D} \pi'(d_k) u_2(\alpha'_{1,k}, a_2) \\
&= \sum_{d_k \in D_{\sqrt{\varepsilon}}} \pi'(d_k) u_2(\alpha'_{1,k}, a_2) + O(\sqrt{\varepsilon}) \\
&\leq \sum_{d_k \in D_{\sqrt{\varepsilon}}} \pi'(d_k) u_2(\alpha'_{1,k}, \hat{a}_2) + \beta + O(\sqrt{\varepsilon}) \\
&= \sum_{d_k \in D} \pi'(d_k) u_2(\alpha'_{1,k}, \hat{a}_2) + \beta + O(\sqrt{\varepsilon}) \\
&= u_2\left(\sum_{d_k \in D} \pi'(d_k) \alpha'_{1,k}, \hat{a}_2\right) + \beta + O(\sqrt{\varepsilon})
\end{aligned}$$

β does not depend on the choice of $(\alpha'_{1,1}, \dots, \alpha'_{1,N})$ and $(\pi'(d_1), \dots, \pi'(d_N))$. Therefore, there exists ε^* , such that for any $\varepsilon < \varepsilon^*$, $a_2 \notin B_\varepsilon^{d_i}(\hat{\alpha}_1)$. Then $B_\varepsilon^{w, d_i}(\hat{\alpha}_1) \subseteq BR_2(\hat{\alpha}_1)$. \square

5.1.3 Proof of Proposition 1.1 part 2.

Let $\tau_i(\theta, \varepsilon)(\omega) = \inf\{t \geq 0 : \frac{\pi^t(\theta)}{\pi^t(\hat{\theta}_i)} > \varepsilon\}$, $C_i^t(\theta, \varepsilon) = \{\omega : \tau_i(\theta, \varepsilon) \leq t\}$, $C_i^\infty(\varepsilon) = \{\omega : \text{there exists } \theta \in \tilde{\Theta}_{-i}^1 \text{ and } t, \text{ such that } \frac{\pi^t(\theta)(h_2^t)}{\pi^t(\hat{\theta}_i)(h_2^t)} > \varepsilon\}$ and $\hat{C}_i^\infty(\varepsilon) = \{\omega : \text{there exists } \theta \in \tilde{\Theta}_{-i}^1 \text{ and } t, \text{ such that } \pi^t(\theta) > \varepsilon\}$. I start with several lemmas.

Lemma 5.8. $Q_{\hat{\theta}_i}(\hat{C}_i^\infty(\varepsilon)) < \sum_{\theta \in \tilde{\Theta}_{-i}^1} \frac{\pi^0(\theta)}{\varepsilon \pi^0(\hat{\theta}_i)}$.

Proof. For any $\theta \in \tilde{\Theta}_{-i}^1$, using the observation that $\frac{\pi^t(\theta)(h_2^t)}{\pi^t(\hat{\theta}_i)(h_2^t)}$ is a martingale under the probability measure $Q_{\hat{\theta}_i}$, according to the Optional Stopping Theorem,

$$\begin{aligned}
& \frac{\pi^0(\theta)}{\pi^0(\hat{\theta}_i)} \\
& \geq E_{Q_{\hat{\theta}_i}} \left(\frac{\pi^{\min\{\tau_i(\theta, \varepsilon), t\}}(\theta)(h_2^{\min\{\tau_i(\theta, \varepsilon), t\}})}{\pi^{\min\{\tau_i(\theta, \varepsilon), t\}}(\hat{\theta}_i)(h_2^{\min\{\tau_i(\theta, \varepsilon), t\}})} \right) \\
& \geq Q_{\hat{\theta}_i}(C_i^t(\theta, \varepsilon)) E_{Q_{\hat{\theta}_i}} \left(\frac{\pi^{\min\{\tau_i(\theta, \varepsilon), t\}}(\theta)(h_2^{\min\{\tau_i(\theta, \varepsilon), t\}})}{\pi^{\min\{\tau_i(\theta, \varepsilon), t\}}(\hat{\theta}_i)(h_2^{\min\{\tau_i(\theta, \varepsilon), t\}})} \middle| C_i^t(\theta, \varepsilon) \right) \\
& = Q_{\hat{\theta}_i}(C_i^t(\theta, \varepsilon)) E_{Q_{\hat{\theta}_i}} \left(\frac{\pi^{\tau_i(\theta, \varepsilon)}(\theta)(h_2^{\tau_i(\theta, \varepsilon)})}{\pi^{\tau_i(\theta, \varepsilon)}(\hat{\theta}_i)(h_2^{\tau_i(\theta, \varepsilon)})} \middle| C_i^t(\theta, \varepsilon) \right) \\
& > \varepsilon Q_{\hat{\theta}_i}(C_i^t(\theta, \varepsilon))
\end{aligned}$$

Since $C_i^t(\theta, \varepsilon) \subseteq C_i^{t+1}(\theta, \varepsilon)$, $Q_{\hat{\theta}_i}(C_i^\infty(\theta, \varepsilon)) < \frac{\pi^0(\theta)}{\varepsilon \pi^0(\hat{\theta}_i)}$. Then

$$Q_{\hat{\theta}_i}(\hat{C}_i^\infty(\varepsilon)) \leq Q_{\hat{\theta}_i}(C_i^\infty(\varepsilon)) \leq \sum_{\theta \in \tilde{\Theta}_{-i}^1} Q_{\hat{\theta}_i}(C_i^\infty(\theta, \varepsilon)) < \sum_{\theta \in \tilde{\Theta}_{-i}^1} \frac{\pi^0(\theta)}{\varepsilon \pi^0(\hat{\theta}_i)}$$

□

Lemma 5.9. *Fix any $d_i \in D$. There exists $\varepsilon^* > 0$ such that for any t , if for any $\theta \in \tilde{\Theta}_{-i}^1$, $\pi^t(\theta) < \varepsilon^*$, and for any $\theta \in \Theta$, $\pi^t(\theta)d^t(Q_\theta, Q_{\hat{\theta}_i}) < \varepsilon^*$, then $a_2^t \in BR_2(\hat{a}_1)$.*

Proof. In period t , Player 2's utility is

$$\begin{aligned}
& E_P(u_2(a_1^t, a_2^t) \mid h_{2,t}) \\
& = \sum_{\theta \in \Theta} P(\{\theta\} \mid h_{2,t}) E_P(u_2(a_1^t, a_2^t) \mid h_{2,t}, \theta) \\
& = \sum_{\theta \in \Theta} \pi^t(\theta) E_P(u_2(a_1^t, a_2^t) \mid h_{2,t}, \theta)
\end{aligned}$$

For any $\theta \in \Theta$, $\pi^t(\theta)d^t(Q_\theta, Q_{\hat{\theta}_i}) < \varepsilon$, so either $\pi^t(\theta) < \sqrt{\varepsilon}$ or $d^t(Q_\theta, Q_{\hat{\theta}_i}) < \sqrt{\varepsilon}$. Let $\Theta_{\sqrt{\varepsilon}}^t = \{\theta \in \Theta : \pi^t(\theta) < \sqrt{\varepsilon}\}$. Then

$$\begin{aligned} & E_P(u_2(a_1^t, a_2^t) \mid h_{2,t}) \\ &= \sum_{\theta \in \Theta_{\sqrt{\varepsilon}}^t} \pi^t(\theta) E_P(u_2(a_1^t, a_2^t) \mid h_{2,t}, \theta) + \sum_{\theta \in \Theta \setminus \Theta_{\sqrt{\varepsilon}}^t} \pi^t(\theta) E_P(u_2(a_1^t, a_2^t) \mid h_{2,t}, \theta) \end{aligned}$$

For any $\theta \in \tilde{\Theta}_{-i}^1$, $\pi^t(\theta) < \varepsilon < \sqrt{\varepsilon}$, so $\tilde{\Theta}_{-i}^1 \subseteq \Theta_{\sqrt{\varepsilon}}^t$. For any $\theta \in \Theta \setminus \Theta_{\sqrt{\varepsilon}}^t$, either $\theta \in \hat{\Theta}$ or $\theta \in \tilde{\Theta}_i^1$. If $\theta \in \hat{\Theta}$, then $E_P(\alpha_1^t \mid h_{2,t}, \theta) = \hat{\alpha}_1$. According to the definition of \tilde{D}_i^1 , if $d \in \tilde{D}_i^1$ and $\rho_{d_i, \hat{\alpha}_1} = \rho_{d, \alpha_1}$, then $BR_2(\alpha_1) = BR_2(\hat{\alpha}_1)$. Therefore, if $\theta \in \tilde{\Theta}_i^1$ and $d^t(Q_\theta, Q_{\hat{\theta}_i}) < \sqrt{\varepsilon}$, when ε is small enough, there exists $\alpha_\theta \in \Delta(A_1)$ such that $\|E_P(\alpha_1^t \mid h_{2,t}, \theta) - \alpha_\theta\|_\infty = O(\sqrt{\varepsilon})$, $\rho_{d_i, \hat{\alpha}_1} = \rho_{d, \alpha_\theta}$ and $BR_2(\alpha_\theta) = BR_2(\hat{\alpha}_1)$.

Fix any $a_2 \notin BR_2(\hat{\alpha}_1)$, and $\hat{a}_2 \in BR_2(\hat{\alpha}_1)$. Let $\beta = \inf_{\alpha_1 \in \{\alpha_1 \in \Delta(A_1) : \exists d \in \tilde{D}_i^1, s.t. \rho_{d_i, \hat{\alpha}_1} = \rho_{d, \alpha_1}\}} [u_2(\alpha_1, \hat{a}_2) - u_2(\alpha_1, a_2)]$. Since $\{\alpha_1 \in \Delta(A_1) : \exists d \in \tilde{D}_i^1, s.t. \rho_{d_i, \hat{\alpha}_1} = \rho_{d, \alpha_1}\}$ is compact, $\beta > 0$. Thus, for any $\theta \in \Theta \setminus \Theta_{\sqrt{\varepsilon}}^t$, there exists $\alpha_\theta \in \Delta(A_1)$ such that $\|E_P(\alpha_1^t \mid h_{2,t}, \theta) - \alpha_\theta\|_\infty = O(\sqrt{\varepsilon})$ and $u_2(\alpha_\theta, \hat{a}_2) > u_2(\alpha_\theta, a_2) + \beta$.

Therefore,

$$\begin{aligned}
& E_P(u_2(a_1^t, \hat{a}_2) | h_{2,t}) - E_P(u_2(a_1^t, \tilde{a}_2) | h_{2,t}) \\
&= \sum_{\theta \in \Theta_{\sqrt{\varepsilon}}^t} \pi^t(\theta) E_P(u_2(a_1^t, \hat{a}_2) | h_{2,t}, \theta) + \sum_{\theta \in \Theta \setminus \Theta_{\sqrt{\varepsilon}}^t} \pi^t(\theta) E_P(u_2(a_1^t, \hat{a}_2) | h_{2,t}, \theta) \\
&\quad - \sum_{\theta \in \Theta_{\sqrt{\varepsilon}}^t} \pi^t(\theta) E_P(u_2(a_1^t, \tilde{a}_2) | h_{2,t}, \theta) - \sum_{\theta \in \Theta \setminus \Theta_{\sqrt{\varepsilon}}^t} \pi^t(\theta) E_P(u_2(a_1^t, \tilde{a}_2) | h_{2,t}, \theta) \\
&= O(\sqrt{\varepsilon}) + \sum_{\theta \in \Theta \setminus \Theta_{\sqrt{\varepsilon}}^t} \pi^t(\theta) E_P(u_2(a_1^t, \hat{a}_2) - u_2(a_1^t, \tilde{a}_2) | h_{2,t}, \theta) \\
&= O(\sqrt{\varepsilon}) + \sum_{\theta \in \Theta \setminus \Theta_{\sqrt{\varepsilon}}^t} \pi^t(\theta) (u_2(E_P(\alpha_1^t | h_{2,t}, \theta), \hat{a}_2) - u_2(E_P(\alpha_1^t | h_{2,t}, \theta), \tilde{a}_2)) \\
&= O(\sqrt{\varepsilon}) + \sum_{\theta \in \Theta \setminus \Theta_{\sqrt{\varepsilon}}^t} \pi^t(\theta) (u_2(\alpha_\theta, \hat{a}_2) - u_2(\alpha_\theta, \tilde{a}_2)) \\
&> O(\sqrt{\varepsilon}) + \sum_{\theta \in \Theta \setminus \Theta_{\sqrt{\varepsilon}}^t} \pi^t(\theta) \beta
\end{aligned}$$

so there exists $\varepsilon^* > 0$, such that ,

$$E_P(u_2(a_1^t, \hat{a}_2) | h_{2,t}) > E_P(u_2(a_1^t, \tilde{a}_2) | h_{2,t})$$

then $a_2^t \in BR_2(\hat{a}_1)$. □

Proof. Now I can prove the proposition.

Fix a monitoring structure d_i and an $\varepsilon > 0$. According to Lemma 5.7, for any $\varepsilon_1 > 0$, there exists $M > 0$ such that $Q_{\hat{\theta}_i} \left(\left\{ \left| G_{\varepsilon_1, \hat{\theta}_i} \right| \geq M \right\} \right) \leq 2\varepsilon_1$. According to Lemma 5.8, for

any ε_2 , $Q_{\hat{\theta}_i}(\hat{C}_i^\infty(\varepsilon_2)) < \sum_{\theta \in \tilde{\Theta}_{-i}^1} \frac{\pi^0(\theta)}{\varepsilon_2 \pi^0(\hat{\theta}_i)}$. Then,

$$Q_{\hat{\theta}_i} \left(\left\{ \left| G_{\varepsilon_1, \hat{\theta}_i} \right| < M \right\} \cap \left[\Omega \setminus \hat{C}_i^\infty(\varepsilon_2) \right] \right) \geq 1 - 2\varepsilon_1 - \sum_{\theta \in \tilde{\Theta}_{-i}^1} \frac{\pi^0(\theta)}{\varepsilon_2 \pi^0(\hat{\theta}_i)}$$

Therefore,

$$\begin{aligned} & E_{Q_{\theta_i^0}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) \right) \\ & \geq E_{Q_{\hat{\theta}_i}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(\hat{\alpha}_1, a_2^t) \right) \\ & \geq Q_{\hat{\theta}_i} \left(\left\{ \left| G_{\varepsilon_1, \hat{\theta}_i} \right| < M \right\} \cap \left[\Omega \setminus \hat{C}_i^\infty(\varepsilon_2) \right] \right) \\ & \quad * E_{Q_{\hat{\theta}_i}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(\hat{\alpha}_1, a_2^t) \middle| \left\{ \left| G_{\varepsilon_1, \hat{\theta}_i} \right| < M \right\} \cap \left[\Omega \setminus \hat{C}_i^\infty(\varepsilon_2) \right] \right) \\ & \geq \left[1 - 2\varepsilon_1 - \sum_{\theta \in \tilde{\Theta}_{-i}^1} \frac{\pi^0(\theta)}{\varepsilon_2 \pi^0(\hat{\theta}_i)} \right] \\ & \quad * E_{Q_{\hat{\theta}_i}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(\hat{\alpha}_1, a_2^t) \middle| \left\{ \left| G_{\varepsilon_1, \hat{\theta}_i} \right| < M \right\} \cap \left[\Omega \setminus \hat{C}_i^\infty(\varepsilon_2) \right] \right) \end{aligned}$$

Fix any $d_i \in D$. There exists $\varepsilon^* > 0$ such that for any t , if for any $\theta \in \tilde{\Theta}_{-i}^1$, $\pi^t(\theta) < \varepsilon$, and for any $\theta \in \Theta$, $\pi^t(\theta) d^t(Q_\theta, Q_{\hat{\theta}_i}) < \varepsilon$, then $a_2^t \in BR_2(\hat{\alpha}_1)$.

According to Lemma 5.9, there exist $\varepsilon_1^* > 0$ such that for any $\varepsilon_1 < \varepsilon_1^*$,

$$\begin{aligned} & E_{Q_{\hat{\theta}_i}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(\hat{\alpha}_1, a_2^t) \middle| \left\{ \left| G_{\varepsilon_1, \hat{\theta}_i} \right| < M \right\} \cap \left[\Omega \setminus C_i^\infty(\varepsilon_1^*) \right] \right) \\ & \geq \delta^M u_{\hat{\alpha}_1} + (1 - \delta^M) \min_{(\alpha_1, \alpha_2)} u_1(\alpha_1, \alpha_2) \end{aligned}$$

If $\pi^0(\theta) < \eta$, then

$$\begin{aligned}
& E_{Q_{\theta_i^0}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) \right) \\
& \geq \left[1 - 2\varepsilon_1 - \sum_{\theta \in \tilde{\Theta}_{-i}^1} \frac{\pi^0(\theta)}{\varepsilon_1^* \pi^0(\hat{\theta}_i)} \right] * \left[\delta^M u_{\hat{\alpha}_1} + (1 - \delta^M) \min_{(\alpha_1, \alpha_2)} u_1(\alpha_1, \alpha_2) \right] \\
& \geq \left[1 - 2\varepsilon_1 - \sum_{\theta \in \tilde{\Theta}_{-i}^1} \frac{\eta}{\varepsilon_1^* \min_j \{\pi^0(\hat{\theta}_j)\}} \right] * \left[\delta^M u_{\hat{\alpha}_1} + (1 - \delta^M) \min_{(\alpha_1, \alpha_2)} u_1(\alpha_1, \alpha_2) \right]
\end{aligned}$$

Since

$$\lim_{\substack{\delta \rightarrow 1 \\ \varepsilon_1, \eta \rightarrow 0}} \left[1 - 2\varepsilon_1 - \sum_{\theta \in \tilde{\Theta}_{-i}^1} \frac{\eta}{\varepsilon_1^* \min_j \{\pi^0(\hat{\theta}_j)\}} \right] * \left[\delta^M u_{\hat{\alpha}_1} + (1 - \delta^M) \min_{(\alpha_1, \alpha_2)} u_1(\alpha_1, \alpha_2) \right] = u_{\hat{\alpha}_1}$$

there exists δ^* and η^* such that if for any $\theta \in \tilde{\Theta}_{-i}^1$, $\pi^0(\theta) < \eta^*$, and $\delta > \delta^*$, then normal type of Player 1's payoff in any BNE under monitoring structure d_i is greater than or equal to $u_{\hat{\alpha}_1} - \varepsilon$. □

5.1.4 Proof of Proposition 1.2.

Let $\tilde{G}(\varepsilon, \theta) = \{t \in \mathbb{N}; \pi^t(\theta) < \varepsilon\}$. I start with several lemmas.

Lemma 5.10. *For any $\gamma \in (0, 1]$ and $\theta \in \Theta$, $Q_\theta(\cup_{t \geq 1} \{\pi^t(\theta) \leq \gamma \pi^0(\theta)\}) \leq \gamma P(\cup_{t \geq 1} \{\pi^t(\theta) \leq \gamma \pi^0(\theta)\})$*

This lemma is very similar to Lemma 15.4.6 in Mailath and Samuelson (2006) so I omit the proofs here.

Lemma 5.11. *For any $\theta, \hat{\theta} \in \Theta$, and any t ,*

$$Q_{\hat{\theta}} \left(\{\pi^t(\hat{\theta}) \geq \varepsilon_2 \text{ and } \pi^t(\theta) \geq \varepsilon_1\} \right) \geq \varepsilon_2 Q_\theta \left(\{\pi^t(\hat{\theta}) \geq \varepsilon_2 \text{ and } \pi^t(\theta) \geq \varepsilon_1\} \right)$$

Proof. Let $\hat{\mathcal{H}}_2^t = \{h_2^t : \pi^t(\hat{\theta})(h_2^t) \geq \varepsilon_2 \text{ and } \pi^t(\theta)(h_2^t) \geq \varepsilon_1\}$. For any $h_2^t \in \hat{\mathcal{H}}_2^t$, by definition,

$$\pi^t(\hat{\theta})(h_2^t)Q_\theta(h_2^t) = \pi^t(\theta)(h_2^t)Q_{\hat{\theta}}(h_2^t)$$

then

$$\sum_{h_2^t \in \hat{\mathcal{H}}_2^t} \pi^t(\hat{\theta})(h_2^t)Q_\theta(h_2^t) = \sum_{h_2^t \in \hat{\mathcal{H}}_2^t} \pi^t(\theta)(h_2^t)Q_{\hat{\theta}}(h_2^t)$$

so

$$\begin{aligned} & Q_{\hat{\theta}} \left(\{ \pi^t(\hat{\theta}) \geq \varepsilon_2 \text{ and } \pi^t(\theta) \geq \varepsilon_1 \} \right) \\ &= \sum_{h_2^t \in \hat{\mathcal{H}}_2^t} Q_{\hat{\theta}}(h_2^t) \\ &\geq \sum_{h_2^t \in \hat{\mathcal{H}}_2^t} \pi^t(\theta)(h_2^t)Q_{\hat{\theta}}(h_2^t) \\ &= \sum_{h_2^t \in \hat{\mathcal{H}}_2^t} \pi^t(\hat{\theta})(h_2^t)Q_\theta(h_2^t) \\ &\geq \varepsilon_2 \sum_{h_2^t \in \hat{\mathcal{H}}_2^t} Q_\theta(h_2^t) \\ &= \varepsilon_2 Q_\theta \left(\{ \pi^t(\hat{\theta}) \geq \varepsilon_2 \text{ and } \pi^t(\theta) \geq \varepsilon_1 \} \right) \end{aligned}$$

□

Now I can prove the proposition.

Proof. The proof can be separated into two parts.

Part 1

Suppose, for the sake of contradiction, that this proposition is false. And let i^* be the

smallest i that the statement is false. Then there exists $\varepsilon^* > 0$ and an increasing sequence $\{\delta_m^*\}_\infty$ such that $\lim_{m \rightarrow \infty} \delta_m^* = 1$ and given any δ_m^* , there exists a BNE such that normal type of Player 1's payoff in this equilibrium under monitoring structure d_{i^*} is strictly smaller than $u_{\hat{\alpha}_1} - \varepsilon^*$:

$$E_{Q_{\theta_{i^*}^0}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) \right) < u_{\hat{\alpha}_1} - \varepsilon^*$$

Consider the deviation in which Player 1 always plays $\hat{\alpha}_1$. Then

$$\begin{aligned} \varepsilon^* &< u_{\hat{\alpha}_1} - E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(\hat{\alpha}_1, a_2^t) \right) \\ &= E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} [u_{\hat{\alpha}_1} - u_1(\hat{\alpha}_1, a_2^t)] \right) \end{aligned}$$

Then for any $\varepsilon_1, \varepsilon_2 > 0$

$$\begin{aligned} \varepsilon^* &< E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} [u_{\hat{\alpha}_1} - u_1(\hat{\alpha}_1, a_2^t)] \right) \\ &= E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} [u_{\hat{\alpha}_1} - u_1(\hat{\alpha}_1, a_2^t)] 1_{\tilde{G}(\varepsilon_1, \hat{\theta}_{i^*})}(t) \right) \\ &\quad + E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} [u_{\hat{\alpha}_1} - u_1(\hat{\alpha}_1, a_2^t)] 1_{(\mathbb{N} \setminus \tilde{G}(\varepsilon_1, \hat{\theta}_{i^*})) \cap (\cap_{\theta \in \hat{\Theta}_{-i^*}^1} \tilde{G}(\varepsilon_2, \theta))}(t) \right) \\ &\quad + E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} [u_{\hat{\alpha}_1} - u_1(\hat{\alpha}_1, a_2^t)] 1_{(\mathbb{N} \setminus \tilde{G}(\varepsilon_1, \hat{\theta}_{i^*})) \cap [\mathbb{N} \setminus (\cap_{\theta \in \hat{\Theta}_{-i^*}^1} \tilde{G}(\varepsilon_2, \theta))]}(t) \right) \end{aligned}$$

According to Lemma 5.10, $Q_{\hat{\theta}_{i^*}}(\cup_{t \geq 1} \{\pi^t(\hat{\theta}_{i^*}) \leq \gamma \pi^0(\hat{\theta}_{i^*})\}) \leq \gamma P(\cup_{t \geq 1} \{\pi^t(\hat{\theta}_{i^*}) \leq \gamma \pi^0(\hat{\theta}_{i^*})\}) \leq$

γ , so

$$\begin{aligned}
& E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} [u_{\hat{\alpha}_1} - u_1(\hat{\alpha}_1, a_2^t)] 1_{\tilde{G}(\varepsilon_1, \hat{\theta}_{i^*})}(t) \right) \\
& \leq u_{\hat{\alpha}_1} E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} 1_{\tilde{G}(\varepsilon_1, \hat{\theta}_{i^*})}(t) \right) \\
& \leq u_{\hat{\alpha}_1} Q_{\hat{\theta}_{i^*}}(\cup_{t \geq 1} \{\pi^t(\hat{\theta}_{i^*}) \leq \varepsilon_1\}) \\
& \leq \frac{u_{\hat{\alpha}_1}}{\pi^0(\hat{\theta}_{i^*})} \varepsilon_1
\end{aligned}$$

The second term:

$$\begin{aligned}
& E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} [u_{\hat{\alpha}_1} - u_1(\hat{\alpha}_1, a_2^t)] 1_{(\mathbb{N} \setminus \tilde{G}(\varepsilon_1, \hat{\theta}_{i^*})) \cap (\cap_{\theta \in \tilde{\Theta}_{-i^*}^1} \tilde{G}(\varepsilon_2, \theta))}(t) \right) \\
& \leq E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} [u_{\hat{\alpha}_1} - u_1(\hat{\alpha}_1, a_2^t)] 1_{(\cap_{\theta \in \tilde{\Theta}_{-i^*}^1} \tilde{G}(\varepsilon_2, \theta))}(t) \right)
\end{aligned}$$

According to Lemma 5.7, for any $\varepsilon_2 > 0$ there exists M such that

$$Q_{\hat{\theta}_{i^*}} \left(\left\{ \left| G_{\varepsilon_2, \hat{\theta}_{i^*}} \right| \geq M \right\} \right) \leq 2\varepsilon_2$$

so

$$\begin{aligned}
& E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} [u_{\hat{\alpha}_1} - u_1(\hat{\alpha}_1, a_2^t)] 1_{\left(\cap_{\theta \in \tilde{\Theta}_{-i^*}^1} \tilde{G}(\varepsilon_2, \theta)\right)}(t) \right) \\
&= E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} [u_{\hat{\alpha}_1} - u_1(\hat{\alpha}_1, a_2^t)] 1_{\left(\cap_{\theta \in \tilde{\Theta}_{-i^*}^1} \tilde{G}(\varepsilon_2, \theta)\right) \cap G_{\varepsilon_2, \hat{\theta}_{i^*}}}(t) \right) \\
&\quad + E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} [u_{\hat{\alpha}_1} - u_1(\hat{\alpha}_1, a_2^t)] 1_{\left(\cap_{\theta \in \tilde{\Theta}_{-i^*}^1} \tilde{G}(\varepsilon_2, \theta)\right) \cap (\mathbb{N} \setminus G_{\varepsilon_2, \hat{\theta}_{i^*}})}(t) \right) \\
&\leq 2\varepsilon_2 u_{\hat{\alpha}_1} + (1 - \delta^M) u_{\hat{\alpha}_1} \\
&\quad + E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} [u_{\hat{\alpha}_1} - u_1(\hat{\alpha}_1, a_2^t)] 1_{\left(\cap_{\theta \in \tilde{\Theta}_{-i^*}^1} \tilde{G}(\varepsilon_2, \theta)\right) \cap (\mathbb{N} \setminus G_{\varepsilon_2, \hat{\theta}_{i^*}})}(t) \right)
\end{aligned}$$

in which

$$\begin{aligned}
& 1_{\left(\cap_{\theta \in \tilde{\Theta}_{-i^*}^1} \tilde{G}(\varepsilon_2, \theta)\right) \cap (\mathbb{N} \setminus G_{\varepsilon_2, \hat{\theta}_{i^*}})} = 1 \\
& \iff \text{for any } \theta \in \tilde{\Theta}_{-i^*}^1, \pi^t(\theta) < \varepsilon_2, \text{ and for any } \theta \in \Theta \pi^t(\theta) d^t(Q_\theta, Q_{\hat{\theta}_{i^*}}) < \varepsilon_2
\end{aligned}$$

According to Lemma 5.9 there exists $\varepsilon_2^* > 0$ such that for any $\varepsilon_2 < \varepsilon_2^*$,

$$\begin{aligned}
& 1_{\left(\cap_{\theta \in \tilde{\Theta}_{-i^*}^1} \tilde{G}(\varepsilon_2, \theta)\right) \cap (\mathbb{N} \setminus G_{\varepsilon_2, \hat{\theta}_{i^*}})} = 1 \\
& \implies a_2^t \in BR_2(\hat{\alpha}_1).
\end{aligned}$$

then

$$\begin{aligned}
& E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} [u_{\hat{\alpha}_1} - u_1(\hat{\alpha}_1, a_2^t)] 1_{\left(\cap_{\theta \in \hat{\Theta}_{-i^*}^1} \tilde{G}(\varepsilon_2, \theta)\right)}(t) \right) \\
& \leq 2\varepsilon_2 u_{\hat{\alpha}_1} + (1 - \delta^M) u_{\hat{\alpha}_1}
\end{aligned}$$

Then

$$\begin{aligned}
& E_{Q_{\hat{\theta}_{i^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} [u_{\hat{\alpha}_1} - u_1(\hat{\alpha}_1, a_2^t)] 1_{\left(\mathbb{N} \setminus \tilde{G}(\varepsilon_1, \hat{\theta}_{i^*})\right) \cap \left[\mathbb{N} \setminus \left(\cap_{\theta \in \hat{\Theta}_{-i^*}^1} \tilde{G}(\varepsilon_2, \theta)\right)\right]}(t) \right) \\
& > \varepsilon^* - \frac{u_{\hat{\alpha}_1}}{\pi^0(\hat{\theta}_{i^*})} \varepsilon_1 - 2\varepsilon_2 u_{\hat{\alpha}_1} - (1 - \delta^M) u_{\hat{\alpha}_1}
\end{aligned}$$

There exists $\tilde{\varepsilon}_1 = \frac{\varepsilon^* \pi^0(\hat{\theta}_{i^*})}{8u_{\hat{\alpha}_1}} > 0$, $\tilde{\varepsilon}_2 = \min\{\varepsilon_2^*, \frac{\varepsilon^*}{8u_{\hat{\alpha}_1}}\} > 0$, such that for any $\varepsilon_1 < \tilde{\varepsilon}_1$, and any $\varepsilon_2 < \tilde{\varepsilon}_2$, $\frac{u_{\hat{\alpha}_1}}{\pi^0(\hat{\theta}_{i^*})} \varepsilon_1 + 2\varepsilon_2 u_{\hat{\alpha}_1} < \frac{\varepsilon^*}{4}$. There also exists $\hat{\delta}$ such that for any $\delta > \hat{\delta}$, $(1 - \delta^M) u_{\hat{\alpha}_1} < \frac{\varepsilon^*}{4}$.

Then

$$\varepsilon^* - \frac{u_{\hat{\alpha}_1}}{\pi^0(\hat{\theta}_{i^*})} \varepsilon_1 - 2\varepsilon_2 u_{\hat{\alpha}_1} - (1 - \delta^M) u_{\hat{\alpha}_1} > \frac{\varepsilon^*}{2}$$

Since

$$\begin{aligned}
& \left(\mathbb{N} \setminus \tilde{G}(\varepsilon_1, \hat{\theta}_{i^*})\right) \cap \left[\mathbb{N} \setminus \left(\cap_{\theta \in \hat{\Theta}_{-i^*}^1} \tilde{G}(\varepsilon_2, \theta)\right)\right] \\
& = \mathbb{N} \setminus \left(\tilde{G}(\varepsilon_1, \hat{\theta}_{i^*}) \cup \left(\cap_{\theta \in \hat{\Theta}_{-i^*}^1} \tilde{G}(\varepsilon_2, \theta)\right)\right) \\
& = \mathbb{N} \setminus \left(\cap_{\theta \in \hat{\Theta}_{-i^*}^1} \left[\tilde{G}(\varepsilon_1, \hat{\theta}_{i^*}) \cup \tilde{G}(\varepsilon_2, \theta)\right]\right) \\
& = \cup_{\theta \in \hat{\Theta}_{-i^*}^1} \left(\mathbb{N} \setminus \left[\tilde{G}(\varepsilon_1, \hat{\theta}_{i^*}) \cup \tilde{G}(\varepsilon_2, \theta)\right]\right)
\end{aligned}$$

Since $D_{-i} \subseteq \tilde{D}_i^1$ and $D_i \subseteq \tilde{D}_i^1 \cup \tilde{D}_i^2$, $\tilde{D}_{-i}^1 \subseteq \tilde{D}_i^2$. Therefore there exists $\hat{i} < i^*$ such that $d_{\hat{i}} \in \tilde{D}_{\hat{i}^*}^2$ and

$$E_{Q_{\hat{\theta}_{\hat{i}^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} [u_{\hat{\alpha}_1} - u_1(\hat{\alpha}_1, a_2^t)] 1_{\mathbb{N} \setminus [\tilde{G}(\varepsilon_1, \hat{\theta}_{\hat{i}^*}) \cup \tilde{G}(\varepsilon_2, \theta_{\hat{i}}^0)]}(t) \right) > \frac{\varepsilon^*}{2N}$$

so

$$\begin{aligned} & (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} Q_{\hat{\theta}_{\hat{i}^*}} \left(\{ \pi^t(\theta_{\hat{i}}^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{\hat{i}^*}) \geq \varepsilon_1 \} \right) \\ &= (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} E_{Q_{\hat{\theta}_{\hat{i}^*}}} \left(1_{\mathbb{N} \setminus [\tilde{G}(\varepsilon_1, \hat{\theta}_{\hat{i}^*}) \cup \tilde{G}(\varepsilon_2, \theta_{\hat{i}}^0)]}(t) \right) \\ &= E_{Q_{\hat{\theta}_{\hat{i}^*}}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} 1_{\mathbb{N} \setminus [\tilde{G}(\varepsilon_1, \hat{\theta}_{\hat{i}^*}) \cup \tilde{G}(\varepsilon_2, \theta_{\hat{i}}^0)]}(t) \right) \\ &> \frac{\varepsilon^*}{2N u_{\hat{\alpha}_1}} \end{aligned}$$

in which

$$\mathbb{N} \setminus \left[\tilde{G}(\varepsilon_1, \hat{\theta}_{\hat{i}^*}) \cup \tilde{G}(\varepsilon_2, \theta_{\hat{i}}^0) \right] = \{ t \in \mathbb{N}; \pi^t(\theta_{\hat{i}}^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{\hat{i}^*}) \geq \varepsilon_1 \}$$

To summarize, in part 1, I have shown that, if i^* is the smallest i that the statement is false, then there exists $\varepsilon^* > 0$, $\hat{\delta} > 0$, and an increasing sequence $\{\delta_m^*\}_\infty$, such that $\lim_{m \rightarrow \infty} \delta_m^* = 1$ and given any $\delta_m^* > \hat{\delta}$, there exists $\tilde{\varepsilon}_1 > 0$, $\tilde{\varepsilon}_2 > 0$, $\hat{i} < i^*$, and a BNE, such that $d_{\hat{i}} \in \tilde{D}_{\hat{i}^*}^2$ and in equilibrium for any $\varepsilon_1 < \tilde{\varepsilon}_1$, and $\varepsilon_2 < \tilde{\varepsilon}_2$,

$$(1 - \delta_m^*) \sum_{t=1}^{\infty} \delta_m^{*t-1} Q_{\hat{\theta}_{\hat{i}^*}} \left(\{ \pi^t(\theta_{\hat{i}}^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{\hat{i}^*}) \geq \varepsilon_1 \} \right) > \frac{\varepsilon^*}{2N u_{\hat{\alpha}_1}}$$

Part 2

$$\begin{aligned}
& E_{Q_{\theta_i^0}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) \right) \\
&= E_{Q_{\theta_i^0}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) * 1_{G_{\varepsilon_3, \theta_i^0}}(t) \right) \\
&+ E_{Q_{\theta_i^0}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) * 1_{\{\pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1\}} * 1_{\mathbb{N} \setminus G_{\varepsilon_3, \theta_i^0}}(t) \right) \\
&+ E_{Q_{\theta_i^0}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) * 1_{\mathbb{N} \setminus \{\pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1\}} * 1_{\mathbb{N} \setminus G_{\varepsilon_3, \theta_i^0}}(t) \right)
\end{aligned}$$

According to Lemma 5.7, for any $\varepsilon_3 > 0$ there exists \hat{M} such that

$$Q_{\theta_i^0} \left(\left\{ \left| G_{\varepsilon_3, \theta_i^0} \right| \geq \hat{M} \right\} \right) \leq 2\varepsilon_3$$

Then

$$\begin{aligned}
& E_{Q_{\theta_i^0}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) * 1_{G_{\varepsilon_3, \theta_i^0}}(t) \right) \\
&\leq \left[2\varepsilon_3 + (1 - \delta^{\hat{M}}) \right] \bar{u}_1 \\
&= \left[2\varepsilon_3 + (1 - \delta^{\hat{M}}) \right] u_{\hat{\alpha}_1}
\end{aligned}$$

According to Lemma 5.11,

$$\begin{aligned}
& (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} Q_{\theta_i^0} \left(\{ \pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1 \} \right) \\
& \geq \varepsilon_2 (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} Q_{\hat{\theta}_{i^*}} \left(\{ \pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1 \} \right) \\
& > \frac{\varepsilon_2 \varepsilon^*}{2Nu_{\hat{\alpha}_1}}
\end{aligned}$$

$$E_{Q_{\theta_i^0}} \left(\left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) * 1_{\{ \pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1 \}} * 1_{\mathbb{N} \setminus G_{\varepsilon_3, \theta_i^0}}(t) \right) \right)$$

Let's consider all the periods such that

$$\begin{aligned}
& 1_{\{ \pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1 \}} * 1_{\mathbb{N} \setminus G_{\varepsilon_3, \theta_i^0}}(t) = 1 \\
& \iff \pi^t(\theta_i^0) \geq \varepsilon_2, \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1, \text{ and for any } \theta \in \Theta \pi^t(\theta) d^t(Q_\theta, Q_{\theta_i^0}) < \varepsilon_3
\end{aligned}$$

Then $d^t(Q_{\hat{\theta}_{i^*}}, Q_{\theta_i^0}) < \frac{\varepsilon_3}{\varepsilon_1}$. According to the definition of \tilde{D}_i^2 , for any $d \in \tilde{D}_i^2$ and any $\alpha_1 \in \Delta(A_1)$, if $\rho_{d_{i^*}, \hat{\alpha}_1} = \rho_{d, \alpha_1}$, then for any $a_2 \in A_2$, $u_{\hat{\alpha}_1} > u_1(\alpha_1, a_2)$. When $\frac{\varepsilon_3}{\varepsilon_1}$ is small enough,

$$d^t(Q_{\hat{\theta}_{i^*}}, Q_{\theta_i^0}) < \frac{\varepsilon_3}{\varepsilon_1} \implies \exists \alpha_{\theta_i^0} \text{ s.t. } \| E_{Q_{\theta_i^0}} \left(a_1^t | \{ \pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1 \} \cap \mathbb{N} \setminus G_{\varepsilon_3, \theta_i^0} \right) - \alpha_{\theta_i^0} \|_{\infty} = O\left(\frac{\varepsilon_3}{\varepsilon_1}\right) \text{ and}$$

Since $\{ \alpha_1 \in \Delta(A_1) : \exists d \in \tilde{D}_i^2, \text{ s.t. } \rho_{d_{i^*}, \hat{\alpha}_1} = \rho_{d, \alpha_1} \}$ is compact and A_2 is finite, $\beta =$

inf $[u_{\hat{\alpha}_1} - u_1(\alpha_1, \alpha_2)] > 0$ and $\beta < u_{\hat{\alpha}_1}$ does

$\alpha_1 \in \{\alpha_1 \in \Delta(A_1) : \exists d \in \tilde{D}_i^2, \text{ s.t. } \rho_{d_{i^*}, \hat{\alpha}_1} = \rho_{d, \alpha_1}\}$

$$\alpha_2 \in \Delta(A_2)$$

not depend on the choice of \hat{i} . Therefore, there exists an $\tilde{\varepsilon}_4 > 0$ such that for any $\frac{\varepsilon_3}{\varepsilon_1} \leq \tilde{\varepsilon}_4$, if

$$d^t(Q_{\hat{\theta}_{i^*}}, Q_{\theta_i^0}) < \frac{\varepsilon_3}{\varepsilon_1}, \text{ then } E_{Q_{\theta_i^0}} \left(u_1(a_1^t, a_2^t) | \{\pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1\} \cap \mathbb{N} \setminus G_{\varepsilon_3, \theta_i^0} \right) \leq u_{\hat{\alpha}_1} -$$

$$\frac{\beta}{2}$$

$$\begin{aligned} & E_{Q_{\theta_i^0}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) * 1_{\{\pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1\}} * 1_{\mathbb{N} \setminus G_{\varepsilon_3, \theta_i^0}}(t) \right) \\ &= (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} Q_{\theta_i^0} \left(\{\pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1\} \cap \mathbb{N} \setminus G_{\varepsilon_3, \theta_i^0} \right) E_{Q_{\theta_i^0}} \left(u_1(a_1^t, a_2^t) | \{\pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1\} \right) \\ &\leq (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} Q_{\theta_i^0} \left(\{\pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1\} \right) \left(u_{\hat{\alpha}_1} - \frac{\beta}{2} \right) \end{aligned}$$

The last term is

$$\begin{aligned} & E_{Q_{\theta_i^0}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) * 1_{\mathbb{N} \setminus \{\pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1\}} * 1_{\mathbb{N} \setminus G_{\varepsilon_3, \theta_i^0}}(t) \right) \\ &\leq E_{Q_{\theta_i^0}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} \bar{u}_1 * 1_{\mathbb{N} \setminus \{\pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1\}} \right) \\ &= (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_{\hat{\alpha}_1} E_{Q_{\theta_i^0}} \left[1 - 1_{\{\pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1\}}(t) \right] \end{aligned}$$

Then

$$\begin{aligned}
& E_{Q_{\theta_i^0}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) \right) \\
& \leq \left[2\varepsilon_3 + (1 - \delta^{\hat{M}}) \right] u_{\hat{\alpha}_1} \\
& + (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} Q_{\theta_i^0} \left(\{ \pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1 \} \right) \left(u_{\hat{\alpha}_1} - \frac{\beta}{2} \right) \\
& + (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_{\hat{\alpha}_1} E_{Q_{\theta_i^0}} \left[1 - 1_{\{ \pi^t(\theta_i^0) \geq \varepsilon_2 \text{ and } \pi^t(\hat{\theta}_{i^*}) \geq \varepsilon_1 \}}(t) \right] \\
& \leq \left[2\varepsilon_3 + (1 - \delta^{\hat{M}}) \right] u_{\hat{\alpha}_1} \\
& + u_{\hat{\alpha}_1} - \frac{\beta}{2} (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} E_{Q_{\theta_i^0}} \left(1_{\{ \pi^t(\hat{\theta}) \geq \varepsilon_2 \text{ and } \pi^t(\theta^*) \geq \varepsilon_1 \}}(t) \right) \\
& < u_{\hat{\alpha}_1} + 2\varepsilon_3 u_{\hat{\alpha}_1} - \frac{\beta \varepsilon_2 \varepsilon^*}{4N u_{\hat{\alpha}_1}} + (1 - \delta^{\hat{M}}) u_{\hat{\alpha}_1}
\end{aligned}$$

Let $\varepsilon_1 = \frac{\tilde{\varepsilon}_1}{2}$, $\varepsilon_2 = \frac{\tilde{\varepsilon}_2}{2}$, $\varepsilon_3 = \min \left\{ \frac{\tilde{\varepsilon}_1 \tilde{\varepsilon}_4}{2}, \frac{\varepsilon^* \tilde{\varepsilon}_2 \beta}{32N [u_{\hat{\alpha}_1}]^2} \right\}$, then there exists $\tilde{\delta} > \hat{\delta} > 0$ such that for any $\delta > \tilde{\delta}$,

$$\begin{aligned}
& E_{Q_{\theta_i^0}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) \right) \\
& < u_{\hat{\alpha}_1} - \frac{\varepsilon^* \tilde{\varepsilon}_2 \beta}{32N u_{\hat{\alpha}_1}}
\end{aligned}$$

To summarize, I have shown that, if i^* is the smallest i that the statement is false, then there exists $\varepsilon^* > 0$, $\tilde{\delta} > 0$, and an increasing sequence $\{\delta_m^*\}_\infty$, such that $\lim_{m \rightarrow \infty} \delta_m^* = 1$ and given any $\delta_m^* > \tilde{\delta}$, there exists $\tilde{\varepsilon}_2, \beta > 0$, $\hat{i} < i^*$, and a BNE, such that $d_i \in \tilde{D}_i^2$ and in

equilibrium

$$E_{Q_{\theta_i^0}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) \right) < u_{\hat{\alpha}_1} - \frac{\varepsilon^* \tilde{\varepsilon}_2 \beta}{32N u_{\hat{\alpha}_1}}$$

in which $\tilde{\varepsilon}_2$, β , $\tilde{\delta}$ do not depend on the choice of BNE and \hat{i} .

However, i^* is the smallest i that the statement is false, so there exists $\bar{\delta}$ such that for any $\hat{i} < i^*$ and any $\delta > \bar{\delta}$,

$$E_{Q_{\theta_i^0}} \left((1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_1(a_1^t, a_2^t) \right) \geq u_{\hat{\alpha}_1} - \frac{\varepsilon^* \tilde{\varepsilon}_2 \beta}{32N u_{\hat{\alpha}_1}}$$

a contradiction. Therefore, this proposition is true. \square

5.1.5 Proof of Example 1.3.

Proof. Without loss of generality, suppose that, for any $i > j$, $\rho_1^{d_i} \leq \rho_1^{d_j}$. It is easy to verify that the stage game payoff satisfies $u_{\hat{\alpha}_1} = \bar{u}_1$. I only need to check that for any i , $\{d_i, \dots, d_N\} \subseteq \tilde{D}_i^1$ and $\{d_1, \dots, d_{i-1}\} \subseteq \tilde{D}_i^1 \cup \tilde{D}_i^2$.

Fix an i^* , for any $j < i^*$, $\rho_1^{d_{i^*}} \leq \rho_1^{d_j}$. For any $\alpha_1 = (\alpha_1(H), 1 - \alpha_1(H)) \in \Delta(A_1)$, if $\rho_{d_{i^*}, H} = \rho_{d_j, \alpha_1}$, then $\rho_1^{d_{i^*}} = \rho_1^{d_j} * \alpha_1(H) + \rho_2^{d_j} * (1 - \alpha_1(H))$. Since $\rho_1^{d_j} > \rho_2^{d_j}$ and $\rho_1^{d_j} \geq \rho_1^{d_{i^*}}$, either $\rho_1^{d_j} = \rho_1^{d_{i^*}}$ and $\alpha_1(H) = 1$, or $\rho_1^{d_j} > \rho_1^{d_{i^*}}$ and $\alpha_1(H) < 1$. It is also possible that for any $\alpha_1 \in \Delta(A_1)$, $\rho_{d_{i^*}, H} \neq \rho_{d_j, \alpha_1}$. In all cases, $d_j \in \tilde{D}_{i^*}^1 \cup \tilde{D}_{i^*}^2$, so $\{d_1, \dots, d_{i^*-1}\} \subseteq \tilde{D}_{i^*}^1 \cup \tilde{D}_{i^*}^2$.

For any $j \geq i^*$, $\rho_1^{d_{i^*}} \geq \rho_1^{d_j}$. If $\rho_2^{d_j} < \rho_1^{d_j} < \rho_1^{d_{i^*}}$, then for any $\alpha_1 \in \Delta(A_1)$, $\rho_{d_{i^*}, H} \neq \rho_{d_j, \alpha_1}$.

If $\rho_2^{d_j} < \rho_1^{d_j} = \rho_1^{d_{i^*}}$, then for any $\alpha_1 \in \Delta(A_1)$ such that $\rho_{d_{i^*}, H} = \rho_{d_j, \alpha_1}$, $\alpha_1 = H$. In both cases, $d_j \in \tilde{D}_{i^*}^1$, so $\{d_{i^*}, \dots, d_N\} \subseteq \tilde{D}_{i^*}^1$. \square

5.1.6 Proof of Proposition 1.3.

Proof. Player 2's posterior belief in period t that Player 1 is normal type is

$$\pi^t(\xi_0) = \sum_{\xi=\xi_0} P(\{\theta\}|\hat{\mathcal{F}}_t)$$

Let

$$L^t = \frac{\sum_{\xi=\xi_0} P(\{\theta\}, h_2^t)}{\sum_{\xi=\xi(\hat{\alpha})} P(\{\theta\}, h_2^t)} = \frac{\sum_{\xi=\xi_0} \pi^0(\theta) P(h_2^t|\{\theta\})}{\sum_{\xi=\xi(\hat{\alpha})} \pi^0(\theta) P(h_2^t|\{\theta\})}$$

Then given any history of Player 2 $h_2^t \in \mathcal{H}_2^t$,

$$\begin{aligned} & \pi^t(\xi_0)(h_2^t) \\ &= \sum_{\xi=\xi_0} P(\{\theta\}|h_2^t) \\ &= \frac{\sum_{\xi=\xi_0} P(\{\theta\}, h_2^t)}{\sum P(\{\theta\}, h_2^t)} \\ &= \frac{L^t}{L^t + 1} \end{aligned}$$

Since Y is a finite set, given any θ , $P(h_2^t|\{\theta\})$ is a probability mass function of a multi-

nomial distribution. Denote these functions by $f(h_2^t, \rho_{d,\alpha})$. Then

$$\begin{aligned}
L^t &= \frac{\sum_d \pi^0(\xi_0, d) f(h_2^t, \rho_{d,\hat{\alpha}})}{\sum_d \pi^0(\xi(\hat{\alpha}), d) f(h_2^t, \rho_{d,\hat{\alpha}})} \\
&= \frac{\sum_{\rho \in \Gamma_{\hat{\alpha}_1}} \sum_{d \in D_{\rho, \hat{\alpha}_1}} \pi^0(\xi_0, d) f(h_2^t, \rho)}{\sum_{\rho \in \Gamma_{\hat{\alpha}_1}} \sum_{d \in D_{\rho, \hat{\alpha}_1}} \pi^0(\xi(\hat{\alpha}_1), d) f(h_2^t, \rho)}
\end{aligned}$$

Since $\Gamma_{\hat{\alpha}_1} \subseteq \Gamma_{\tilde{\alpha}_1}$,

$$\begin{aligned}
L^t &\geq \frac{\sum_{\rho \in \Gamma_{\hat{\alpha}_1}} \sum_{d \in D_{\rho, \hat{\alpha}_1}} \pi^0(\xi_0, d) f(h_2^t, \rho)}{\sum_{\rho \in \Gamma_{\hat{\alpha}_1}} \sum_{d \in D_{\rho, \hat{\alpha}_1}} \pi^0(\xi(\hat{\alpha}_1), d) f(h_2^t, \rho)} \\
&\geq \frac{\sum_{\rho \in \Gamma_{\hat{\alpha}_1}} \left[\min_{\rho \in \Gamma_{\hat{\alpha}_1}} \sum_{d \in D_{\rho, \hat{\alpha}_1}} \pi^0(\xi_0, d) \right] f(h_2^t, \rho)}{\sum_{\rho \in \Gamma_{\hat{\alpha}_1}} \left[\max_{\rho \in \Gamma_{\hat{\alpha}_1}} \sum_{d \in D_{\rho, \hat{\alpha}_1}} \pi^0(\xi(\hat{\alpha}_1), d) \right] f(h_2^t, \rho)} \\
&= \frac{\min_{\rho \in \Gamma_{\hat{\alpha}_1}} \sum_{d \in D_{\rho, \hat{\alpha}_1}} \pi^0(\xi_0, d)}{\max_{\rho \in \Gamma_{\hat{\alpha}_1}} \sum_{d \in D_{\rho, \hat{\alpha}_1}} \pi^0(\xi(\hat{\alpha}_1), d)}
\end{aligned}$$

Since $u_2(\tilde{\alpha}_1, \tilde{a}_2) > u_2(\tilde{\alpha}_1, a_2)$ for all $a_2 \neq \tilde{a}_2$, there exists a $0 < \hat{\varepsilon} < 1$ such that if $\pi^t(\xi_0)(h_2^t) \geq$

$\hat{\varepsilon}$ Player 2 has no incentive to deviate. Let $\hat{L} = \frac{\hat{\varepsilon}}{1-\hat{\varepsilon}}$. Then

$$\frac{\min_{\rho \in \Gamma_{\hat{\alpha}_1}} \sum_{d \in D_{\rho, \hat{\alpha}_1}} \pi^0(\xi_0, d)}{\max_{\rho \in \Gamma_{\hat{\alpha}_1}} \sum_{d \in D_{\rho, \hat{\alpha}_1}} \pi^0(\xi(\hat{\alpha}_1), d)} \geq \hat{L} \Rightarrow \pi^t(\xi_0)(h_2^t) \geq \hat{\varepsilon}, \forall h_2^t \in \mathcal{H}_2^t$$

so Player 2 does not deviate from equilibrium strategy. Player 1 also has no incentive to

deviate. □

5.1.7 Proof of Proposition 1.4.

Proof. Since $\tilde{\alpha}_{1,i} \in BR_2(\tilde{\alpha}_2)$, Player 1 has no incentive to deviate.

Given Player 1's strategy, in any period t , Player 2 play a best response to

$$\sum_{\theta} \pi^t(\theta) E(\sigma_1 | \theta, h_2^t)$$

Since $\rho_{d_{\mu(i)}, \tilde{\alpha}_{1,i}} = \rho_{d_i, \hat{\alpha}_1}$, for any t , $P(h_2^t | \{\hat{\theta}_i\}) = P(h_2^t | \{\theta_{\mu(i)}^0\})$. Therefore,

$$\begin{aligned} & \frac{\pi^t(\hat{\theta}_i)}{\pi^t(\theta_{\mu(i)}^0)} \\ &= \frac{P(\{\hat{\theta}_i\} | h_2^t)}{P(\{\theta_{\mu(i)}^0\} | h_2^t)} \\ &= \frac{P(\{\hat{\theta}_i\}, h_2^t)}{P(\{\theta_{\mu(i)}^0\}, h_2^t)} \\ &= \frac{\pi^0(\hat{\theta}_i)}{\pi^0(\theta_{\mu(i)}^0)} \frac{P(h_2^t | \{\hat{\theta}_i\})}{P(h_2^t | \{\theta_{\mu(i)}^0\})} = \frac{\pi^0(\hat{\theta}_i)}{\pi^0(\theta_{\mu(i)}^0)} \end{aligned}$$

Then

$$\begin{aligned}
& \sum_{\theta} \pi^t(\theta) E(\sigma_1 | \theta, h_2^t) \\
&= \sum_i \left[\pi^t(\hat{\theta}_i) \hat{\alpha}_1 + \pi^t(\theta_{\mu(i)}^0) \tilde{\alpha}_{1,i} \right] \\
&= \sum_i \left[\pi^t(\hat{\theta}_i) + \pi^t(\theta_{\mu(i)}^0) \right] \left[\frac{\pi^t(\hat{\theta}_i) \hat{\alpha}_1}{\pi^t(\hat{\theta}_i) + \pi^t(\theta_{\mu(i)}^0)} + \frac{\pi^t(\theta_{\mu(i)}^0)}{\pi^t(\hat{\theta}_i) + \pi^t(\theta_{\mu(i)}^0)} \tilde{\alpha}_{1,i} \right] \\
&= \sum_i \left[\pi^t(\hat{\theta}_i) + \pi^t(\theta_{\mu(i)}^0) \right] \left[\frac{\pi^0(\hat{\theta}_i) \hat{\alpha}_1}{\pi^0(\hat{\theta}_i) + \pi^0(\theta_{\mu(i)}^0)} + \frac{\pi^0(\theta_{\mu(i)}^0)}{\pi^0(\hat{\theta}_i) + \pi^0(\theta_{\mu(i)}^0)} \tilde{\alpha}_{1,i} \right] \\
&= \sum_i \left[\pi^t(\hat{\theta}_i) + \pi^t(\theta_{\mu(i)}^0) \right] (e_i \hat{\alpha}_1 + (1 - e_i) \tilde{\alpha}_{1,i}) \\
&= \tilde{\alpha}_1
\end{aligned}$$

Since $(\tilde{\alpha}_1, \tilde{\alpha}_2)$ is a Nash equilibrium of the complete information stage game, Player 2 has no incentive to deviate. \square

5.1.8 Proof of Example 1.5 part 1.

Proof. $((\frac{3}{4}H, \frac{1}{4}L), (\frac{1}{4}c, \frac{3}{4}s))$ is the mixed strategy NE of the complete information stage game. According to Proposition 1.4, I need to find $\tilde{\alpha}_{1,i} \in \Delta(A_1)$ and $e_i \in [0, 1]$, such that

$$\rho_{d_{\mu(i)}, \tilde{\alpha}_{1,i}} = \rho_{d_i, H} \text{ and } (\frac{3}{4}H, \frac{1}{4}L) = e_i H + (1 - e_i) \tilde{\alpha}_{1,i}.$$

$$\rho_{d_2, \tilde{\alpha}_{1,1}} = \rho_{d_1, H} \implies \tilde{\alpha}_{1,1} = \left(\frac{\gamma_2 - \frac{3}{4}}{\gamma_2 - \gamma_1} H, \frac{\frac{3}{4} - \gamma_1}{\gamma_2 - \gamma_1} L \right)$$

$$\rho_{d_1, \tilde{\alpha}_{1,2}} = \rho_{d_2, H} \implies \tilde{\alpha}_{1,2} = \left(\left(2\gamma_1 - \frac{1}{2} \right) H, \left(\frac{3}{2} - 2\gamma_1 \right) L \right)$$

then

$$e_1 = \frac{3 - \gamma_2 - 3\gamma_1}{3 - 4\gamma_1}$$

$$e_2 = \frac{5 - 8\gamma_1}{6 - 8\gamma_1}$$

In order to make $e_i \in (0, 1)$, $\frac{\gamma_2 - \frac{3}{4}}{\gamma_2 - \gamma_1} \in [0, 1]$ and $(2\gamma_1 - \frac{1}{2}) \in [0, 1]$, it must be true that

$$\frac{3}{4} \leq \gamma_2 < 1$$

$$\frac{1}{4} \leq \gamma_1 < \frac{5}{8}$$

□

5.1.9 Proof of Example 1.5 part 2.

Proof. Follow the same steps as in Example 1.2, I can show that

$$\begin{aligned} & \tilde{D}_1^1 \\ &= \left\{ (\rho_1^d, \rho_2^d) : \rho_2^d < \frac{3}{4} \text{ and } 3\rho_1^d + \rho_2^d < 3 \right\} \\ & \cup \left\{ (\rho_1^d, \rho_2^d) : \rho_2^d > \frac{3}{4} \text{ and } 3\rho_1^d + \rho_2^d > 3 \right\} \end{aligned}$$

and

$$\begin{aligned} & \tilde{D}_2^1 \\ &= \{(\rho_1^d, \rho_2^d) : \rho_2^d < \gamma_1 \text{ and } 3\rho_1^d + \rho_2^d < 4\gamma_1\} \\ & \cup \{(\rho_1^d, \rho_2^d) : \rho_2^d > \gamma_1 \text{ and } 3\rho_1^d + \rho_2^d > 4\gamma_1\} \end{aligned}$$

Since $\bar{u}_1 = u_1(a_1, a_2)$ if and only if $(a_1, a_2) = (H, c)$, $(\rho_1^d, \rho_2^d) \notin \tilde{D}_1^1 \cup \tilde{D}_1^2$ if and only if $(\rho_1^d, \rho_2^d) = (\frac{3}{4}, \frac{3}{4})$, $(\rho_1^d, \rho_2^d) \notin \tilde{D}_2^1 \cup \tilde{D}_2^2$ if and only if $(\rho_1^d, \rho_2^d) = (\gamma_1, \gamma_1)$.

Then

$$d_1 \in \tilde{D}_1^1, d_2 \in \tilde{D}_2^1$$

$$\begin{aligned} & d_2 \in \tilde{D}_1^1 \\ & \iff \left(\gamma_2 - \frac{3}{4}\right) (3\gamma_1 + \gamma_2 - 3) > 0 \end{aligned}$$

and

$$\begin{aligned} & d_1 \in \tilde{D}_2^1 \\ & \iff \left(\frac{1}{4} - \gamma_1\right) \left(3 * \frac{3}{4} + \frac{1}{4} - 4\gamma_1\right) > 0 \\ & \iff \left(\gamma_1 - \frac{1}{4}\right) \left(\gamma_1 - \frac{5}{8}\right) > 0 \\ & \iff \gamma_1 \in \left(0, \frac{1}{4}\right) \cup \left(\frac{5}{8}, 1\right) \end{aligned}$$

If $\gamma_1 \in \left(0, \frac{1}{4}\right) \cup \left(\frac{5}{8}, 1\right)$, then $\{d_1, d_2\} \subseteq \tilde{D}_2^1$ and $d_1 \in \tilde{D}_1^1$. Since $\gamma_1 \neq \gamma_2$, $d_2 \in \tilde{D}_1^1 \cup \tilde{D}_1^2$. I can

apply Proposition 1.2. If $\gamma_1 \in [\frac{1}{4}, \frac{5}{8}]$ and $\gamma_2 \in (0, \frac{3}{4})$, then $3\gamma_1 + \gamma_2 - 3 < 0$, so $\{d_1, d_2\} \subseteq \tilde{D}_1^1$. Since $d_1 \in \tilde{D}_2^1 \cup \tilde{D}_2^2$ and $d_2 \in \tilde{D}_2^1$, I can still apply Proposition 1.2. \square

5.1.10 Proof of Lemma 1.5.

Proof. Since for any $i \neq j$ $\Gamma_{\hat{\alpha}_{1,i}} \cap \Gamma_{\hat{\alpha}_{1,j}} = \emptyset$, there exists a constant $\hat{\varepsilon}$ such that for any $i \neq j$, any k, l and any t

$$d^t(Q_{\hat{\theta}_{j,l}}, Q_{\hat{\theta}_{i,k}}) \geq \hat{\varepsilon}$$

Then

$$\pi^t(\hat{\theta}_{j,l}) \geq \varepsilon \implies \pi^t(\hat{\theta}_{j,l}) d^t(Q_{\hat{\theta}_{j,l}}, Q_{\hat{\theta}_{i,k}}) \geq \hat{\varepsilon} \varepsilon$$

According to Lemma 5.7, for any $\varepsilon > 0$, and any $\hat{\theta}_{i,k} \in \Theta$, there exists $M > 0$ such that

$$Q_{\hat{\theta}_{i,k}} \left(\left\{ \left| G_{\hat{\varepsilon}\varepsilon, \hat{\theta}_{i,k}} \right| \geq M \right\} \right) \leq 2\hat{\varepsilon}\varepsilon$$

Then

$$\mathbb{N} \setminus \tilde{G}(\varepsilon, \hat{\theta}_{j,l}) \subseteq G_{\hat{\varepsilon}\varepsilon, \hat{\theta}_{i,k}}$$

and

$$\left\{ |\mathbb{N} \setminus \tilde{G}(\varepsilon, \hat{\theta}_{j,l})| \geq M \right\} \subseteq \left\{ \left| G_{\hat{\varepsilon}\varepsilon, \hat{\theta}_{i,k}} \right| \geq M \right\}$$

so

$$\begin{aligned} & Q_{\hat{\theta}_{i,k}} \left(\left\{ |\mathbb{N} \setminus \tilde{G}(\varepsilon, \hat{\theta}_{j,l})| \geq M \right\} \right) \\ & \leq Q_{\hat{\theta}_{i,k}} \left(\left\{ \left| G_{\hat{\varepsilon}\varepsilon, \hat{\theta}_{i,k}} \right| \geq M \right\} \right) \\ & \leq 2\hat{\varepsilon}\varepsilon \end{aligned}$$

Let $e = \hat{e}$. □

5.1.11 Proof of Proposition 1.6.

Proof. Suppose in the first $M + N$ periods, there are M \bar{y} and N \underline{y} . Then

$$\pi^t(\xi(H)) = \frac{\pi^0(\xi(H))P(\{\xi(H)\}, h_2^t)}{\pi^0(\xi(H))P(\{\xi(H)\}, h_2^t) + \pi^0(\xi_0)P(\{\xi_0\}, h_2^t)}$$

in which

$$P(\{\xi(H)\}, h_2^t) = \iint_D \rho_1^M (1 - \rho_1)^N \pi^0(\rho_1, \rho_2) d\rho_1 d\rho_2$$

and

$$P(\{\xi_0\}, h_2^t) = \iint_D \rho_2^M (1 - \rho_2)^N \pi^0(\rho_1, \rho_2) d\rho_1 d\rho_2$$

It is sufficient to show that if $\text{supp}(\pi_1^0) \subseteq \text{supp}(\pi_2^0)$ then $\frac{\phi_H(M,N)}{\phi_L(M,N)} \leq \sup_{\rho} \frac{\pi_1^0(\rho)}{\pi_2^0(\rho)}$.

It is easy to calculate that

$$\begin{aligned} & \iint_D \rho_1^M (1 - \rho_1)^N \pi^0(\rho_1, \rho_2) d\rho_1 d\rho_2 \\ &= \int_{\text{supp}(\pi_1^0)} \rho_1^M (1 - \rho_1)^N \pi_1^0(\rho_1) d\rho_1 \\ &= \int_{\text{supp}(\pi_1^0)} \rho^M (1 - \rho)^N \pi_1^0(\rho) d\rho \end{aligned}$$

and

$$\begin{aligned}
& \iint_D \rho_2^M (1 - \rho_2)^N \pi^0(\rho_1, \rho_2) d\rho_1 d\rho_2 \\
&= \int_{\text{supp}(\pi_2^0)} \rho_2^M (1 - \rho_2)^N \pi_2^0(\rho_2) d\rho_2 \\
&= \int_{\text{supp}(\pi_2^0)} \rho^M (1 - \rho)^N \pi_2^0(\rho) d\rho
\end{aligned}$$

When $\text{supp}(\pi_1^0) \subseteq \text{supp}(\pi_2^0)$,

$$\begin{aligned}
& \frac{\phi_H(M, N)}{\phi_L(M, N)} \\
&= \frac{\int_{\text{supp}(\pi_1^0)} \rho^M (1 - \rho)^N \pi_1^0(\rho) d\rho}{\int_{\text{supp}(\pi_2^0)} \rho^M (1 - \rho)^N \pi_2^0(\rho) d\rho} \\
&\leq \frac{\int_{\text{supp}(\pi_1^0)} \rho^M (1 - \rho)^N \pi_1^0(\rho) d\rho}{\int_{\text{supp}(\pi_1^0)} \rho^M (1 - \rho)^N \pi_2^0(\rho) d\rho} \\
&= \frac{\int_{\text{supp}(\pi_1^0)} \frac{\pi_1^0(\rho)}{\pi_2^0(\rho)} \rho^M (1 - \rho)^N \pi_2^0(\rho) d\rho}{\int_{\text{supp}(\pi_1^0)} \rho^M (1 - \rho)^N \pi_2^0(\rho) d\rho} \\
&\leq \sup_{\rho} \frac{\pi_1^0(\rho)}{\pi_2^0(\rho)}
\end{aligned}$$

□

5.1.12 Proof of Proposition 1.7.

Proof. Fix a monitoring structure $d \in D$. For any $\varepsilon_1 > 0$, according to the definition of $D_{\rho_1^d, \varepsilon_1}$, $d \in D \cap D_{\rho_1^d, \varepsilon_1}$. Let $Q_{(\xi, d)}^{2, t}$, $P^{2, t}$ be the marginal distribution of $Q_{(\xi, d)}$, P on Player 2's histories $h_{2, t} \in \mathcal{H}_2^t$, respectively. Under P , Player 2's beliefs on the next stage's signals are given by $p_{2, t}(y) = P(y | \hat{\mathcal{F}}_t)$ and if Player 2 knew that Player 1 is of type H , and the

true monitoring structure is d , Player 2's beliefs on the next stage's signals are given by

$$q_{2,t}^{\xi(H),d}(y) = Q_{(\xi,d)}(y|\hat{\mathcal{F}}_t).$$

Then

$$\begin{aligned} & d(Q_{(\xi(H),d)}^{2,t}||P^{2,t}) \\ &= \sum Q_{(\xi(H),d)}^{2,t} \log \frac{Q_{(\xi(H),d)}^{2,t}}{\sum_{\xi} \iint_D Q_{(\xi,d')}^{2,t} \pi^0(\xi) \pi^0(d') d\rho_1^{d'} d\rho_2^{d'}} \\ &\leq \sum Q_{(\xi(H),d)}^{2,t} \log \frac{Q_{(\xi(H),d)}^{2,t}}{\pi^0(\xi(H)) \iint_D Q_{(\xi(H),d')}^{2,t} \pi^0(d') d\rho_1^{d'} d\rho_2^{d'}} \\ &\leq \sum Q_{(\xi(H),d)}^{2,t} \log \frac{Q_{(\xi(H),d)}^{2,t}}{\pi^0(\xi(H)) \varepsilon_2 \iint_{D \cap D_{\rho_1^d, \varepsilon_1}} Q_{(\xi(H),d')}^{2,t} \frac{\pi^0(d')}{\varepsilon_2} d\rho_1^{d'} d\rho_2^{d'}} \\ &\leq -\log [\pi^0(\xi(H)) \varepsilon_2] + \sum Q_{(\xi(H),d)}^{2,t} \log \frac{Q_{(\xi(H),d)}^{2,t}}{\iint_{D \cap D_{\rho_1^d, \varepsilon_1}} Q_{(\xi(H),d')}^{2,t} \frac{\pi^0(d')}{\varepsilon_2} d\rho_1^{d'} d\rho_2^{d'}} \end{aligned}$$

Since relative entropy is jointly convex,

$$\begin{aligned} & d(Q_{(\xi(H),d)}^{2,t}||P^{2,t}) \\ &\leq -\log [\pi^0(\xi(H)) \varepsilon_2] + \iint_{D \cap D_{\rho_1^d, \varepsilon_1}} \left(d(Q_{(\xi(H),d)}^{2,t}||Q_{(\xi(H),d')}^{2,t}) \right) \frac{\pi^0(d')}{\varepsilon_2} d\rho_1^{d'} d\rho_2^{d'} \end{aligned}$$

Since $d \in D \cap D_{\rho_1^d, \varepsilon_1}$, for any $d' \in D \cap D_{\rho_1^d, \varepsilon_1}$, $|\rho_1^d - \rho_1^{d'}| < \varepsilon_1$. Apply the chain rule to

$$d(Q_{(\xi(H),d)}^{2,t}||Q_{(\xi(H),d')}^{2,t}).$$

$$d(Q_{(\xi(H),d)}^{2,t}||Q_{(\xi(H),d')}^{2,t}) = td((\rho_1^d, 1 - \rho_1^d)||(\rho_1^{d'}, 1 - \rho_1^{d'})) \leq t \frac{\varepsilon_1^2}{\min\{\rho_1^{d'}, 1 - \rho_1^{d'}\}}$$

so

$$\begin{aligned}
& d(Q_{(\xi(H),d)}^{2,t} || P^{2,t}) \\
& \leq -\log [\pi^0(\xi(H))\varepsilon_2] + \iint_{D \cap D_{\rho_1^d, \varepsilon_1}} t \frac{\varepsilon_1^2}{\min\{\rho_1^{d'}, 1 - \rho_1^{d'}\}} \frac{\pi^0(d')}{\varepsilon_2} d\rho_1^{d'} d\rho_2^{d'} \\
& \leq -\log [\pi^0(\xi(H))\varepsilon_2] + \frac{t\varepsilon_1^2}{\min\{\rho_1^d - \varepsilon_1, 1 - \rho_1^d - \varepsilon_1\}}
\end{aligned}$$

Conditional on $h_{2,t}$, Player 2 chooses a best response to $\iint_D \pi^t(d') E_P(\sigma_1 | h_{2,t}, d') d\rho_1^{d'} d\rho_2^{d'}$, which is a d -weak- $d(q_{2,t}^{\xi(H),d} || p_{2,t})$ -entropy-confirming best response to H , so given (ρ_1^d, ρ_2^d) ,

$$\begin{aligned}
& (1 - \delta) \sum_{t \geq 1} \delta^{t-1} E_{Q_{(\xi_0, d)}} u_1 \\
& \geq (1 - \delta) \sum_{t \geq 1} \delta^{t-1} E_{Q_{(\xi(H), d)}} u_1 \\
& \geq (1 - \delta) \sum_{t \geq 1} \delta^{t-1} E_{Q_{(\xi(H), d)}} \hat{v}_H^{w,d}(d(q_{2,t}^{\xi(H),d} || p_{2,t})) \\
& \geq (1 - \delta) \sum_{t \geq 1} \delta^{t-1} E_{Q_{(\xi(H), d)}} \hat{w}_H^{w,d}(d(q_{2,t}^{\xi(H),d} || p_{2,t})) \\
& \geq \hat{w}_H^{w,d} \left((1 - \delta) \sum_{t \geq 1} \delta^{t-1} E_{Q_{(\xi(H), d)}} d(q_{2,t}^{\xi(H),d} || p_{2,t}) \right) \\
& = \hat{w}_H^{w,d} \left((1 - \delta)^2 \sum_{n \geq 1} \delta^{n-1} \sum_{t=1}^n E_{Q_{(\xi(H), d)}} d(q_{2,t}^{\xi(H),d} || p_{2,t}) \right) \\
& = \hat{w}_H^{w,d} \left((1 - \delta)^2 \sum_{n \geq 1} \delta^{n-1} d(Q_{(\xi(H), d)}^{2,n} || P^{2,n}) \right) \\
& \geq \hat{w}_H^{w,d} \left((1 - \delta)^2 \sum_{n \geq 1} \delta^{n-1} \left[-\log [\pi^0(\xi(H))\varepsilon_2] + \frac{n\varepsilon_1^2}{\min\{\rho_1^d - \varepsilon_1, 1 - \rho_1^d - \varepsilon_1\}} \right] \right) \\
& = \hat{w}_H^{w,d} \left(-(1 - \delta) \log [\pi^0(\xi(H))\varepsilon_2] + \frac{\varepsilon_1^2}{\min\{\rho_1^d - \varepsilon_1, 1 - \rho_1^d - \varepsilon_1\}} \right)
\end{aligned}$$

□

5.1.13 Proof of Proposition 1.8.

Proof. According to Proposition 1.7, for any $\varepsilon_1 > 0$, normal type of Player 1's payoff in any BNE under monitoring structure d is greater than or equal to

$$\hat{w}_H^{w,d} \left(-(1 - \delta) \log [\pi^0(\xi(H))\varepsilon_2] + \frac{\varepsilon_1^2}{\min\{\rho_1^d - \varepsilon_1, 1 - \rho_1^d - \varepsilon_1\}} \right)$$

Since for any $\varepsilon > 0$, there exists $\delta^* > 0$ and $\varepsilon_1 > 0$, such that for any $\delta > \delta^*$,

$$-(1 - \delta) \log [\pi^0(\xi(H))\varepsilon_2] + \frac{\varepsilon_1^2}{\min\{\rho_1^d - \varepsilon_1, 1 - \rho_1^d - \varepsilon_1\}} \leq \varepsilon$$

Then it is sufficient to show that there is a $\hat{\varepsilon} > 0$ such that for any $\varepsilon < \hat{\varepsilon}$, $\hat{B}_\varepsilon^{w,d}(H) = \{c\}$.

$$\begin{aligned} & d \left(\rho_{d,H} \parallel \iint_D \rho_{d'', \alpha'_1(d'')} \pi'(d') d\rho_1^{d'} d\rho_2^{d'} \right) \\ &= \rho_1^d \log \frac{\rho_1^d}{\iint_D [\rho_1^{d'} \alpha'_1(d')(H) + \rho_2^{d'} \alpha'_1(d')(L)] \pi'(d') d\rho_1^{d'} d\rho_2^{d'}} \\ &+ (1 - \rho_1^d) \log \frac{1 - \rho_1^d}{1 - \iint_D [\rho_1^{d'} \alpha'_1(d')(H) + \rho_2^{d'} \alpha'_1(d')(L)] \pi'(d') d\rho_1^{d'} d\rho_2^{d'}} \end{aligned}$$

$$\begin{aligned}
& \iint_D [\rho_1^{d'} \alpha'_1(d')(H) + \rho_2^{d'} \alpha'_1(d')(L)] \pi'(d') d\rho_1^{d'} d\rho_2^{d'} \\
&= \iint_D \rho_1^{d'} \alpha'_1(d')(H) \pi'(d') d\rho_1^{d'} d\rho_2^{d'} + \iint_D \rho_2^{d'} \alpha'_1(d')(L) \pi'(d') d\rho_1^{d'} d\rho_2^{d'} \\
&\leq \alpha'_1(d')(H) \pi'(d') d\rho_1^{d'} d\rho_2^{d'} \sup_{d' \in D} \rho_1^{d'} + (1 - \rho_1^{d'} \alpha'_1(d')(H) \pi'(d') d\rho_1^{d'} d\rho_2^{d'}) \sup_{d' \in D} \rho_2^{d'} \\
&\leq \frac{1}{2} \sup_{d' \in D} \rho_1^{d'} + \frac{1}{2} \sup_{d' \in D} \rho_2^{d'}
\end{aligned}$$

Since $\rho_1^d \log \frac{\rho_1^d}{x} + (1 - \rho_1^d) \log \frac{1 - \rho_1^d}{1 - x}$ is monotonically decreasing when $x < \rho_1^d$ and $\rho_1^d > \frac{1}{2} \sup_{d' \in D} \rho_1^{d'} +$

$$\frac{1}{2} \sup_{d' \in D} \rho_2^{d'},$$

$$\begin{aligned}
& d \left(\rho_{d, H} \parallel \iint_D \rho_{d'', \alpha'_1(d'')} \pi'(d'') d\rho_1^{d''} d\rho_2^{d''} \right) \\
&\geq \rho_1^d \log \frac{\rho_1^d}{\frac{1}{2} \sup_{d' \in D} \rho_1^{d'} + \frac{1}{2} \sup_{d' \in D} \rho_2^{d'}} + (1 - \rho_1^d) \log \frac{1 - \rho_1^d}{1 - \left(\frac{1}{2} \sup_{d' \in D} \rho_1^{d'} + \frac{1}{2} \sup_{d' \in D} \rho_2^{d'} \right)} \\
&> 0
\end{aligned}$$

Let $\hat{\varepsilon} = \rho_1^d \log \frac{\rho_1^d}{\frac{1}{2} \sup_{d' \in D} \rho_1^{d'} + \frac{1}{2} \sup_{d' \in D} \rho_2^{d'}} + (1 - \rho_1^d) \log \frac{1 - \rho_1^d}{1 - \left(\frac{1}{2} \sup_{d' \in D} \rho_1^{d'} + \frac{1}{2} \sup_{d' \in D} \rho_2^{d'} \right)}$. According to the definition of d -weak- ε -entropy-confirming best response, $\hat{B}_\varepsilon^{w,d}(H) = \{c\}$.

□

5.2 Chapter 2

5.2.1 Proof of Proposition 2.9

Proof. There exists a unique NE if and only if $\alpha I_d + V_3^T V_3$ is invertible. For every non-zero

column vector z of d real numbers, denoted by $\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_d \end{pmatrix}$, $z^T(\alpha I_d + V_3^T V_3)z = \alpha z^T z + z^T V_3^T V_3 z$

. It is easy to check that:

$$\alpha z^T z = \alpha \left(\sum_{i=1}^d z_i^2 \right)$$

and

$$z^T V_3^T V_3 z = \sum_{i=d+1}^N \left(\sum_{j=1}^d v_{ij} z_j \right)^2$$

Since z is non-zero, $\alpha \left(\sum_{i=1}^d z_i^2 \right) > 0$ and $\sum_{i=d+1}^N \left(\sum_{j=1}^d v_{ij} z_j \right)^2 \geq 0$. Then $z^T(\alpha I_d + V_3^T V_3)z = \alpha z^T z + z^T V_3^T V_3 z > 0$. By definition, $\alpha I_d + V_3^T V_3$ is positive definite, thus invertible \square

5.2.2 Proof of Lemma 2.1

Proof. Denote $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ of n elements by 1_n , then: \square

$$\begin{pmatrix} x_1^t - b \\ x_2^t - b \\ \vdots \\ x_N^t - b \end{pmatrix} - U \begin{pmatrix} x_1^{t-1} - b \\ x_2^{t-1} - b \\ \vdots \\ x_N^{t-1} - b \end{pmatrix} = X^t - UX^{t-1} - b1_N + bU1_N$$

Since V is row stochastic, $V_11_d + V_21_{N-d} = 1_d$ and $V_31_d + V_41_{N-d} = 1_{N-d}$

$$-b1_N + bU1_N = b \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1_{N-d} + \alpha V_3(\alpha I_d + V_3^T V_3)^{-1}1_d + (I_{N-d} - V_3(\alpha I_d + V_3^T V_3)^{-1}V_3^T)V_41_{N-d} \end{pmatrix}$$

$$\begin{aligned} & -1_{N-d} + \alpha V_3(\alpha I_d + V_3^T V_3)^{-1}1_d + (I_{N-d} - V_3(\alpha I_d + V_3^T V_3)^{-1}V_3^T)V_41_{N-d} \\ = & -1_{N-d} + \alpha V_3(\alpha I_d + V_3^T V_3)^{-1}1_d + (I_{N-d} - V_3(\alpha I_d + V_3^T V_3)^{-1}V_3^T)(1_{N-d} - V_31_d) \\ = & \alpha V_3(\alpha I_d + V_3^T V_3)^{-1}1_d + V_3(\alpha I_d + V_3^T V_3)^{-1}V_3^T V_31_d - V_3(\alpha I_d + V_3^T V_3)^{-1}V_3^T 1_{N-d} - V_31_d \\ = & V_3(\alpha I_d + V_3^T V_3)^{-1}(\alpha I_d + V_3^T V_3)1_d - V_31_d - V_3(\alpha I_d + V_3^T V_3)^{-1}V_3^T 1_{N-d} \\ = & -V_3(\alpha I_d + V_3^T V_3)^{-1}V_3^T 1_{N-d} \end{aligned}$$

Then

$$-b1_N + bU1_N = b \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -V_3(\alpha I_d + V_3^T V_3)^{-1}V_3^T 1_{N-d} \end{pmatrix} = -\hat{B}$$

so

$$\begin{pmatrix} x_1^t - b \\ x_2^t - b \\ \vdots \\ x_N^t - b \end{pmatrix} - U \begin{pmatrix} x_1^{t-1} - b \\ x_2^{t-1} - b \\ \vdots \\ x_N^{t-1} - b \end{pmatrix} = X^t - UX^{t-1} - b1_N + bU1_N = X^t - UX^{t-1} - \hat{B} = 0$$

5.2.3 Proof of proposition 2.10

Proof.

$$U = \begin{pmatrix} V_1 & V_2 \\ \alpha V_3(\alpha I_d + V_3^T V_3)^{-1} & (I_{N-d} - V_3(\alpha I_d + V_3^T V_3)^{-1} V_3^T) V_4 \end{pmatrix}$$

so

$$\begin{aligned} U - V &= \begin{pmatrix} 0 & 0 \\ V_3((I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} - I_d) & (-V_3(\alpha I_d + V_3^T V_3)^{-1} V_3^T) V_4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ -\frac{1}{\alpha} V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T & -\frac{1}{\alpha} V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T V_4 \end{pmatrix} \\ &= -\frac{1}{\alpha} \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T \end{pmatrix} V \end{aligned}$$

Then

$$U = \left[I_N - \frac{1}{\alpha} \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T \end{pmatrix} \right] V$$

Since V is strongly connected and aperiodic, there exists a M such that for any $m \geq M-1$ every element of matrix V^m is positive. Denote $\frac{1}{\alpha} \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}V_3^T \end{pmatrix}$ by P . Then

$$U^M = [V - PV]^M = V^M - V^{M-1}PV + \dots + (-PV)^M$$

Since M is finite and $\lim_{\alpha \rightarrow \infty} P = 0$, there exists a $\alpha_1 > 0$ such that for any $\alpha > \alpha_1$, every element of matrix U^M is positive.

Now let me prove that $\|U^M\|_\infty < 1$.

$$U^M \mathbf{1}_N = [V - PV]^M \mathbf{1}_N = \mathbf{1}_N - V^{M-1}P\mathbf{1}_N + \dots + (-PV)^M \mathbf{1}_N = \mathbf{1}_N - V^{M-1}P\mathbf{1}_N + O\left(\frac{1}{\alpha^2}\mathbf{1}_N\right)$$

in which

$$P\mathbf{1}_N = \frac{1}{\alpha} \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}V_3^T \end{pmatrix} \mathbf{1}_N = \frac{1}{\alpha} \begin{pmatrix} 0 \\ V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}V_3^T \mathbf{1}_{N-d} \end{pmatrix}$$

Then

$$U^M \mathbf{1}_N = \mathbf{1}_N - \frac{1}{\alpha} V^{M-1} \begin{pmatrix} 0 \\ V_3 V_3^T \mathbf{1}_{N-d} \end{pmatrix} + O\left(\frac{1}{\alpha^2} \mathbf{1}_N\right) = \mathbf{1}_N - \frac{1}{\alpha} \left[V^{M-1} \begin{pmatrix} 0 \\ V_3 V_3^T \mathbf{1}_{N-d} \end{pmatrix} + O\left(\frac{1}{\alpha} \mathbf{1}_N\right) \right]$$

Since V is strongly connected, $V_3 \neq 0$, then all elements of matrix $V_3 V_3^T$ are non-negative and at least one of them is positive. Then all elements of vector $\begin{pmatrix} 0 \\ V_3 V_3^T \mathbf{1}_{N-d} \end{pmatrix}$ are also non-negative and that at least one of them is positive. As proved above, every element of V^{M-1}

is positive, thus every element of $V^{M-1} \begin{pmatrix} 0 \\ V_3 V_3^T 1_{N-d} \end{pmatrix}$ is positive. Let r be the minimum

of elements of $V^{M-1} \begin{pmatrix} 0 \\ V_3 V_3^T 1_{N-d} \end{pmatrix}$. Then there exists a $\alpha_2 > 0$, such that for any $\alpha > \alpha_2$

, every element of matrix $V^{M-1} \begin{pmatrix} 0 \\ V_3 V_3^T 1_{N-d} \end{pmatrix} + O(\frac{1}{\alpha} 1_N)$ is larger than $\frac{r}{2}$. Then for any

$\alpha > \hat{\alpha} = \max\{\frac{r}{2}, \alpha_1, \alpha_2\}$, every element of U^M is positive and every element of $U^M 1_N$ is no larger than $1 - \frac{r}{2\alpha}$, so $\|U^M\|_\infty = \max_i \sum_j |u_{ij}| \leq 1 - \frac{r}{2\alpha}$.

For any initial opinion vector X^0

$$\|U^M X^0\|_\infty \leq \|U^M\|_\infty \|X^0\|_\infty \leq (1 - \frac{r}{2\alpha}) \|X^0\|_\infty$$

Thus, $\lim_{k \rightarrow \infty} U^{M*k} X^0 = 0$. Since M is a finite constant, for any $1 \leq k_1 \leq M-1$, $\lim_{k \rightarrow \infty} U^{M*k+k_1} X^0 =$

$U^{k_1} \lim_{l \rightarrow \infty} U^{M*k} X^0 = 0$, therefore $\lim_{t \rightarrow \infty} U^t X^0 = 0$. □

5.2.4 Proof of proposition 2.11

Proof. Since V is strongly connected and aperiodic, then, as I proved in proposition 2.10, there exist a $\hat{\alpha} > 0$ and $M \in \mathbb{N}^+$ such that for any $\alpha > \hat{\alpha}$, $\|U^M\|_\infty < 1$. Then the Neumann series $\sum_{i=1}^k U^i$ converge and $(I - U)^{-1} = \lim_{k \rightarrow \infty} \sum_{i=1}^k U^i$. By proposition 2.10, $\lim_{t \rightarrow \infty} U^t X^0 = 0$,

so $\lim_{t \rightarrow \infty} X^t$ exists and

$$\begin{aligned} \lim_{t \rightarrow \infty} X^t &= \lim_{t \rightarrow \infty} U^t X^0 + \lim_{t \rightarrow \infty} \sum_{i=0}^{t-1} U^i \hat{B} \\ &= (I - U)^{-1} \hat{B} \end{aligned}$$

□

5.2.5 Proof of proposition 2.12

Proof. Since $\text{rank}(V_3) = d$, I can rearrange the code name of naive agents such that V_3 can be partitioned into 2 blocks as follows:

$$V_3 = \begin{pmatrix} V_{31} \\ V_{32} \end{pmatrix}$$

in which V_{31} is a square matrix ($d * d$) and $\text{rank}(V_{31}) = d$. This will not change the network structure, thus does not affect the result of proposition 2.11.

Suppose that there is consensus, denoted by c . Then it must be true that

$$\begin{aligned}
\hat{B} &= (I - U)1_N c \\
&= \left\{ 1_N - \left[I_N - \frac{1}{\alpha} \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T \end{pmatrix} \right] V 1_N \right\} c \\
&= \left\{ 1_N - \left[I_N - \frac{1}{\alpha} \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T \end{pmatrix} \right] 1_N \right\} c \\
&= \frac{c}{\alpha} \begin{pmatrix} 0_d \\ V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T 1_{N-d} \end{pmatrix}
\end{aligned}$$

Since

$$\hat{B} = \frac{1}{\alpha} \begin{pmatrix} 0_d \\ V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} B \end{pmatrix}$$

the equation above is equivalent to

$$\begin{aligned}
V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T 1_{N-d} c &= V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} B \\
\iff V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} (V_3^T 1_{N-d} c - B) &= 0 \\
\iff \begin{pmatrix} V_{31}(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} (V_3^T 1_{N-d} c - B) \\ V_{32}(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} (V_3^T 1_{N-d} c - B) \end{pmatrix} &= 0
\end{aligned}$$

Then it must be true that

$$V_{31}(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} (1_{N-d} c - B) = 0$$

Since V_{31} and $(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}$ are both invertible, $(V_3^T1_{N-d}c - B) = 0$.

$$V_3^T1_{N-d}c = \begin{pmatrix} \sum_{j \in A(d)} v_{j1} \\ \sum_{j \in A(d)} v_{j2} \\ \vdots \\ \sum_{j \in A(d)} v_{jd} \end{pmatrix} c$$

and

$$B = \begin{pmatrix} \sum_{j \in A(d)} v_{j1}b_1 \\ \sum_{j \in A(d)} v_{j2}b_2 \\ \vdots \\ \sum_{j \in A(d)} v_{jd}b_d \end{pmatrix}$$

Since $\text{rank}(V_3) = d$, for any $1 \leq i \leq d$, $\sum_{j \in A(d)} v_{ji} > 0$, then

$$\begin{aligned} (V_3^T1_{N-d}c - B) &= 0 \\ \implies b_1 = b_2 = \dots = b_d &= c \end{aligned}$$

Contradiction. □

5.2.6 Proof of Proposition 2.13

Proof. According to the proof of Lemma 2.1,

$$1_N = (I - U)^{-1} \begin{pmatrix} 0_d \\ V_3(\alpha I_d + V_3^TV_3)^{-1}V_3^T1_{N-d} \end{pmatrix}$$

so

$$\begin{aligned}
& (I - U)^{-1} \hat{B} - b_{min} \mathbf{1}_N \\
&= (I - U)^{-1} \begin{pmatrix} 0_d \\ V_3(\alpha I_d + V_3^T V_3)^{-1} \begin{pmatrix} \sum_{j \in A(d)} v_{j1}(b_1 - b_{min}) \\ \sum_{j \in A(d)} v_{j2}(b_2 - b_{min}) \\ \vdots \\ \sum_{j \in A(d)} v_{jd}(b_d - b_{min}) \end{pmatrix} \end{pmatrix} \\
&= (I - U)^{-1} \begin{pmatrix} 0_{d \times d} \\ V_3(\alpha I_d + V_3^T V_3)^{-1} \end{pmatrix} \begin{pmatrix} \sum_{j \in A(d)} v_{j1} & 0 & \cdots & 0 \\ 0 & \sum_{j \in A(d)} v_{j2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{j \in A(d)} v_{jd} \end{pmatrix} \begin{pmatrix} b_1 - b_{min} \\ b_2 - b_{min} \\ \vdots \\ b_d - b_{min} \end{pmatrix}
\end{aligned}$$

then every element of $(I - U)^{-1} \hat{B} - b_{min} \mathbf{1}_N$ is a linear combination of $b_i - b_{min}$. Let

$$W = (I - U)^{-1} \begin{pmatrix} 0_{d \times d} \\ V_3(\alpha I_d + V_3^T V_3)^{-1} \end{pmatrix} \begin{pmatrix} \sum_{j \in A(d)} v_{j1} & 0 & \cdots & 0 \\ 0 & \sum_{j \in A(d)} v_{j2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{j \in A(d)} v_{jd} \end{pmatrix}$$

then

$$(I - U)^{-1}\hat{B} - b_{min}1_N = W \begin{pmatrix} b_1 - b_{min} \\ b_2 - b_{min} \\ \vdots \\ b_d - b_{min} \end{pmatrix}$$

so

$$\begin{aligned} & \max_i |((I - U)^{-1}\hat{B})_i - b_{min}| \\ &= \left\| (I - U)^{-1}\hat{B} - b_{min}1_N \right\|_{\infty} \\ &\leq \left\| W \right\|_{\infty} \max_i |b_i - b_{min}| \\ &= \left\| W \right\|_{\infty} (b_{max} - b_{min}) \end{aligned}$$

Let $c_1 = 2\|W\|_{\infty}$, then,

$$\begin{aligned} & \max_i \left((I - U)^{-1}\hat{B} \right)_i - \min_i \left((I - U)^{-1}\hat{B} \right)_i \\ &\leq 2 \max_i |((I - U)^{-1}\hat{B})_i - b_{min}| \\ &\leq c_1 (b_{max} - b_{min}) \end{aligned}$$

□

5.2.7 Proof of Proposition 2.14

Proof.

$$\begin{aligned}
& G^{-1}(I - U)G \\
&= G^{-1}(I - V)G + \frac{1}{\alpha}G^{-1} \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}V_3^T \end{pmatrix} VG \\
&= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 - \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 - \lambda_N \end{pmatrix} + \frac{1}{\alpha}H - \frac{1}{\alpha^2}\hat{H} + O\left(\frac{1}{\alpha^3}\right) \\
&= \begin{pmatrix} 0 & 0 \\ 0 & I_{N-1} - \Lambda_{22} \end{pmatrix} + \frac{1}{\alpha}H - \frac{1}{\alpha^2}\hat{H} + O\left(\frac{1}{\alpha^3}\right)
\end{aligned}$$

in which $h_{ij} = l_i \begin{pmatrix} 0 & 0 \\ 0 & V_3V_3^T \end{pmatrix} Vr_j = \lambda_j l_i \begin{pmatrix} 0 & 0 \\ 0 & V_3V_3^T \end{pmatrix} r_j$ and $\hat{h}_{ij} = \lambda_j l_i \begin{pmatrix} 0 & 0 \\ 0 & V_3V_3^TV_3V_3^T \end{pmatrix} r_j$,

then

$$G^{-1}(I - U)G = \begin{pmatrix} \frac{1}{\alpha}h_{11} - \frac{1}{\alpha^2}\hat{h}_{11} + O\left(\frac{1}{\alpha^3}\right) & \frac{1}{\alpha}H_{12} - \frac{1}{\alpha^2}\hat{H}_{12} + O\left(\frac{1}{\alpha^3}\right) \\ \frac{1}{\alpha}H_{21} - \frac{1}{\alpha^2}\hat{H}_{21} + O\left(\frac{1}{\alpha^3}\right) & I_{N-1} - \Lambda_{22} + \frac{1}{\alpha}H_{22} - \frac{1}{\alpha^2}\hat{H}_{22} + O\left(\frac{1}{\alpha^3}\right) \end{pmatrix}$$

Since $h_{11} > 0$ and for any $k \neq 1$, $1 - \lambda_k \neq 0$, there exists a $\hat{\alpha} > 0$ such that for any $\alpha > \hat{\alpha}$,

$\frac{1}{\alpha}h_{11} + O(\frac{1}{\alpha^2}) > 0$ and

$$\begin{aligned}\Sigma &= I_{N-1} - \Lambda_{22} + \frac{1}{\alpha}H_{22} - \frac{1}{\alpha^2}\hat{H}_{22} + O(\frac{1}{\alpha^3}) \\ &\quad - \frac{1}{\frac{1}{\alpha}h_{11} + O(\frac{1}{\alpha^2})} \left(\frac{1}{\alpha}H_{12} - \frac{1}{\alpha^2}\hat{H}_{12} + O(\frac{1}{\alpha^3}) \right) \left(\frac{1}{\alpha}H_{21} - \frac{1}{\alpha^2}\hat{H}_{21} + O(\frac{1}{\alpha^3}) \right) \\ &= I_{N-1} - \Lambda_{22} + \frac{1}{\alpha}H_{22} - \frac{1}{\alpha h_{11}}H_{21}H_{12} + O(\frac{1}{\alpha^2})\end{aligned}$$

is invertible. Then

$$\frac{1}{\alpha}(I - U)^{-1} = \frac{1}{\alpha}G (G^{-1}(I - U)G)^{-1} G^{-1}$$

in which

$$\begin{aligned}& \frac{1}{\alpha} (G^{-1}(I - U)G)^{-1} \\ &= \frac{1}{\alpha} \left(\begin{array}{cc} \frac{1}{\alpha}h_{11} - \frac{1}{\alpha^2}\hat{h}_{11} + O(\frac{1}{\alpha^3}) & \frac{1}{\alpha}H_{12} - \frac{1}{\alpha^2}\hat{H}_{12} + O(\frac{1}{\alpha^3}) \\ \frac{1}{\alpha}H_{21} - \frac{1}{\alpha^2}\hat{H}_{21} + O(\frac{1}{\alpha^3}) & I_{N-1} - \Lambda_{22} + \frac{1}{\alpha}H_{22} - \frac{1}{\alpha^2}\hat{H}_{22} + O(\frac{1}{\alpha^3}) \end{array} \right)^{-1} \\ &= \left(\begin{array}{cc} \frac{\alpha}{\alpha h_{11} - \hat{h}_{11}} + \frac{\alpha}{(\alpha h_{11} - \hat{h}_{11})^2} H_{12} \Sigma^{-1} H_{21} + O(\frac{1}{\alpha^2}) & -\frac{1}{\alpha h_{11} - \hat{h}_{11}} H_{12} \Sigma^{-1} + O(\frac{1}{\alpha^2}) \\ \frac{1}{\alpha h_{11} - \hat{h}_{11}} \Sigma^{-1} H_{21} + O(\frac{1}{\alpha^2}) & \frac{1}{\alpha} \Sigma^{-1} \end{array} \right) \\ &= \left(\begin{array}{cc} \frac{1}{h_{11} - \frac{1}{\alpha}\hat{h}_{11}} + \frac{1}{\alpha} \frac{1}{(h_{11} - \frac{1}{\alpha}\hat{h}_{11})^2} H_{12} \Sigma^{-1} H_{21} + O(\frac{1}{\alpha^2}) & -\frac{1}{\alpha} \frac{1}{h_{11} - \frac{1}{\alpha}\hat{h}_{11}} H_{12} \Sigma^{-1} + O(\frac{1}{\alpha^2}) \\ -\frac{1}{\alpha} \frac{1}{h_{11} - \frac{1}{\alpha}\hat{h}_{11}} \Sigma^{-1} H_{21} + O(\frac{1}{\alpha^2}) & \frac{1}{\alpha} \Sigma^{-1} \end{array} \right)\end{aligned}$$

Since

$$\frac{1}{h_{11} - \frac{1}{\alpha}\hat{h}_{11}} = \frac{1}{h_{11}} + \frac{1}{\alpha} \frac{\hat{h}_{11}}{h_{11}^2} + O(\frac{1}{\alpha^2})$$

and

$$\frac{1}{\left(h_{11} - \frac{1}{\alpha}\hat{h}_{11}\right)^2} = \frac{1}{h_{11}^2} + \frac{2}{\alpha} \frac{\hat{h}_{11}}{h_{11}^3} + O\left(\frac{1}{\alpha^2}\right)$$

$$\begin{aligned} & \frac{1}{\alpha} (G^{-1}(I - U)G)^{-1} \\ &= \begin{pmatrix} \frac{1}{h_{11}} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\alpha} \begin{pmatrix} \frac{\hat{h}_{11}}{h_{11}^2} + \frac{1}{h_{11}^2} H_{12} \Sigma^{-1} H_{21} & -\frac{1}{h_{11}} H_{12} \Sigma^{-1} \\ -\frac{1}{h_{11}} \Sigma^{-1} H_{21} & \Sigma^{-1} \end{pmatrix} + O\left(\frac{1}{\alpha^2}\right) \end{aligned}$$

Since

$$\Sigma^{-1} = (I_{N-1} - \Lambda_{22})^{-1} + O\left(\frac{1}{\alpha}\right)$$

$$\begin{aligned} & \frac{1}{\alpha} (G^{-1}(I - U)G)^{-1} \\ &= \begin{pmatrix} \frac{1}{h_{11}} & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\alpha} \begin{pmatrix} \frac{\hat{h}_{11}}{h_{11}^2} + \frac{1}{h_{11}^2} H_{12} (I_{N-1} - \Lambda_{22})^{-1} H_{21} & -\frac{1}{h_{11}} H_{12} (I_{N-1} - \Lambda_{22})^{-1} \\ -\frac{1}{h_{11}} (I_{N-1} - \Lambda_{22})^{-1} H_{21} & (I_{N-1} - \Lambda_{22})^{-1} \end{pmatrix} \\ &+ O\left(\frac{1}{\alpha^2}\right) \end{aligned}$$

$$\begin{aligned}
& \frac{1}{\alpha}(I - U)^{-1} \\
&= \frac{1}{\alpha}G(G^{-1}(I - U)G)^{-1}G^{-1} \\
&= G \begin{pmatrix} \frac{1}{h_{11}} & 0 \\ 0 & 0 \end{pmatrix} G^{-1} \\
&+ \frac{1}{\alpha}G \begin{pmatrix} \frac{\hat{h}_{11}}{h_{11}^2} + \frac{1}{h_{11}^2}H_{12}(I_{N-1} - \Lambda_{22})^{-1}H_{21} & -\frac{1}{h_{11}}H_{12}(I_{N-1} - \Lambda_{22})^{-1} \\ -\frac{1}{h_{11}}(I_{N-1} - \Lambda_{22})^{-1}H_{21} & (I_{N-1} - \Lambda_{22})^{-1} \end{pmatrix} G^{-1} \\
&+ O\left(\frac{1}{\alpha^2}\right)
\end{aligned}$$

Since

$$G = \begin{pmatrix} 1_N & r_2 & \cdots & r_N \end{pmatrix}$$

there exists a vector \tilde{c}_1 , such that

$$\begin{aligned}
& G \begin{pmatrix} \frac{\hat{h}_{11}}{h_{11}^2} + \frac{1}{h_{11}^2}H_{12}(I_{N-1} - \Lambda_{22})^{-1}H_{21} & -\frac{1}{h_{11}}H_{12}(I_{N-1} - \Lambda_{22})^{-1} \\ 0 & 0 \end{pmatrix} G^{-1} \\
&= 1_N \tilde{c}_1
\end{aligned}$$

so

$$\begin{aligned}
& \frac{1}{\alpha}(I-U)^{-1} \\
&= \frac{1_N l}{h_{11}} + \frac{1}{\alpha} 1_N \check{c}_1 \\
&+ \frac{1}{\alpha} G \begin{pmatrix} 0 & 0 \\ -\frac{1}{h_{11}}(I_{N-1} - \Lambda_{22})^{-1} H_{21} & (I_{N-1} - \Lambda_{22})^{-1} \end{pmatrix} G^{-1} + O\left(\frac{1}{\alpha^2}\right)
\end{aligned}$$

$$\begin{aligned}
& G \begin{pmatrix} 0 & 0 \\ -\frac{1}{h_{11}}(I_{N-1} - \Lambda_{22})^{-1} H_{21} & 0 \end{pmatrix} G^{-1} \\
&= G \begin{pmatrix} 0 & 0 \\ -\frac{1}{h_{11}}(I_{N-1} - \Lambda_{22})^{-1} H_{21} & 0 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_N \end{pmatrix} \\
&= G \begin{pmatrix} 0 \\ -\frac{1}{h_{11}}(I_{N-1} - \Lambda_{22})^{-1} H_{21} l_1 \end{pmatrix} \\
&= -\frac{1}{h_{11}} \begin{pmatrix} r_2 & \cdots & r_N \end{pmatrix} (I_{N-1} - \Lambda_{22})^{-1} H_{21} l_1 \\
&= -\frac{1}{h_{11}} \begin{pmatrix} \frac{r_2}{1-\lambda_2} & \cdots & \frac{r_N}{1-\lambda_N} \end{pmatrix} \begin{pmatrix} l_2 \\ \vdots \\ l_N \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & V_3 V_3^T \end{pmatrix} r_1 l_1 \\
&= -\frac{1}{h_{11}} \sum_{i=2}^N \frac{r_i l_i}{1-\lambda_i} \begin{pmatrix} 0 & 0 \\ 0 & V_3 V_3^T \end{pmatrix} r_1 l_1
\end{aligned}$$

and

$$G \begin{pmatrix} 0 & 0 \\ 0 & (I_{N-1} - \Lambda_{22})^{-1} \end{pmatrix} G^{-1} \\ = \sum_{i=2}^N \frac{r_i l_i}{1 - \lambda_i}$$

so

$$\begin{aligned} & \frac{1}{\alpha}(I - U)^{-1} \\ &= \frac{1_N l}{h_{11}} + \frac{1}{\alpha} 1_N \check{c}_1 \\ & \quad + \frac{1}{\alpha} G \begin{pmatrix} 0 & 0 \\ -\frac{1}{h_{11}} (I_{N-1} - \Lambda_{22})^{-1} H_{21} & (I_{N-1} - \Lambda_{22})^{-1} \end{pmatrix} G^{-1} + O\left(\frac{1}{\alpha^2}\right) \\ &= \frac{1_N l}{h_{11}} + \frac{1}{\alpha} 1_N \check{c}_1 + \frac{1}{\alpha} \sum_{i=2}^N \frac{r_i l_i}{1 - \lambda_i} \left[I - \frac{1}{h_{11}} \begin{pmatrix} 0 & 0 \\ 0 & V_3 V_3^T \end{pmatrix} r_1 l_1 \right] + O\left(\frac{1}{\alpha^2}\right) \end{aligned}$$

then

$$\begin{aligned}
\hat{X} &= (I - U)^{-1} \hat{B} \\
&= \left[\frac{1_N l}{h_{11}} + \frac{1}{\alpha} 1_N \check{c}_1 + \frac{1}{\alpha} \sum_{i=2}^N \frac{r_i l_i}{1 - \lambda_i} \left[I - \frac{1}{h_{11}} \begin{pmatrix} 0 & 0 \\ 0 & V_3 V_3^T \end{pmatrix} r_1 l_1 \right] + O\left(\frac{1}{\alpha^2}\right) \right] \\
&\quad \left[\begin{pmatrix} 0_d \\ V_3 B \end{pmatrix} - \frac{1}{\alpha} \begin{pmatrix} 0_d \\ V_3 V_3^T V_3 B \end{pmatrix} + O\left(\frac{1}{\alpha^2}\right) \right] \\
&= \check{c}_1 1_N + \frac{1}{\alpha} \sum_{i=2}^N \frac{r_i l_i}{1 - \lambda_i} \left[I - \frac{1}{h_{11}} \begin{pmatrix} 0 & 0 \\ 0 & V_3 V_3^T \end{pmatrix} 1_N l \right] \begin{pmatrix} 0_d \\ V_3 B \end{pmatrix} + O\left(\frac{1}{\alpha^2}\right)
\end{aligned}$$

□

5.2.8 Proof of proposition 2.15

Proof. First let me prove that when α is large enough, all limiting opinions are close to each other. I have proved that

$$\hat{X} = \lim_{t \rightarrow \infty} X^t = (I - U)^{-1} \hat{B}$$

Then as I proved in proposition 2.10 there exists a $\hat{\alpha} > 0$ such that for any $\alpha > \hat{\alpha}$, there exists $M > 0$, such that U^M is a positive matrix and $\|U^M\|_\infty < 1$, then

$$\begin{aligned}
\hat{X} &= U \hat{X} + \hat{B} \\
&= U^M \hat{X} + \hat{B} + \sum_{k=1}^{M-1} U^k \hat{B} \\
&= U^M \hat{X} + O\left(\frac{1}{\alpha} 1_N\right)
\end{aligned}$$

According to Lemma 2.1, without loss of generality, I can assume that $b_i > 0$. then

$$l \begin{pmatrix} 0_d \\ V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}B \end{pmatrix} = l \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}V_3^T \end{pmatrix} V \hat{X}$$

Denote $l \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}V_3^T \end{pmatrix} V$ by $w = (w_1, \dots, w_N)$. Since

$$\lim_{\alpha \rightarrow \infty} l \begin{pmatrix} 0_d \\ V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}B \end{pmatrix} = l \begin{pmatrix} 0_d \\ V_3B \end{pmatrix} > 0$$

and

$$\lim_{\alpha \rightarrow \infty} \sum_{i=1}^N w_i = \lim_{\alpha \rightarrow \infty} l \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}V_3^T \end{pmatrix} V \mathbf{1}_N = l \begin{pmatrix} 0_d \\ V_3V_3^T \end{pmatrix} \mathbf{1}_N > 0$$

so there exists a $\tilde{\alpha} > 0$ such that for any $\alpha > \tilde{\alpha}$, $\sum_{i=1}^N w_i > 0$ and $l \begin{pmatrix} 0_d \\ V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}B \end{pmatrix} >$

0. Let $\bar{x} = \max_i \{\hat{x}_i\}$ and $\underline{x} = \min_i \{\hat{x}_i\}$, then for any $\alpha > \tilde{\alpha}$

$$\sum_{i=1}^N w_i \bar{x} \geq l \begin{pmatrix} 0_d \\ V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}B \end{pmatrix}$$

and

$$\sum_{i=1}^N w_i \underline{x} \leq l \begin{pmatrix} 0_d \\ V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}B \end{pmatrix}$$

so $\bar{x} > 0$ and there exists a constant $e > 0$ such that $\underline{x} < e$. Let $U^M = (u_{ij}^M)$ and $V^M = (v_{ij}^M)$

, then $\bar{x} = \sum_j u_{ij}^M \hat{x}_j + O(\frac{1}{\alpha})$ so

$$\begin{aligned} O(\frac{1}{\alpha}) &= \bar{x} - \sum_j u_{ij}^M \hat{x}_j \\ &= \left(1 - \sum_j u_{ij}^M\right) \bar{x} + \sum_j u_{ij}^M (\bar{x} - \hat{x}_j) \\ &> \sum_j u_{ij}^M (\bar{x} - \hat{x}_j) = \sum_j \left[v_{ij}^M + O(\frac{1}{\alpha})\right] (\bar{x} - \hat{x}_j) > 0 \end{aligned}$$

Then it must be true that for any $1 \leq k \leq N$, $\hat{x}_j = \bar{x} + O(\frac{1}{\alpha})$. Since \underline{x} is bounded above, all

\hat{x}_j are bounded. Then

$$l \begin{pmatrix} 0_d \\ V_3 B \end{pmatrix} + O(\frac{1}{\alpha}) = l \begin{pmatrix} 0 & 0 \\ 0 & V_3 V_3^T \end{pmatrix} V \mathbf{1}_N \underline{x} + O(\frac{1}{\alpha})$$

Take limits of both sides of this equation, then $\lim_{\alpha \rightarrow \infty} \hat{X}$ exist and can be denoted by $\mathbf{1}_N \hat{c}$ in

which

$$\hat{c} = \frac{l \begin{pmatrix} 0_d \\ V_3 B \end{pmatrix}}{l \begin{pmatrix} 0 \\ V_3 V_3^T \mathbf{1}_{N-d} \end{pmatrix}} = \frac{\sum_{i \in S(d)} b_i \left[\left(\sum_{j \in A(d)} v_{ji} \right) \left(\sum_{j \in A(d)} v_{ji} l(j) \right) \right]}{\sum_{k \in S(d)} \left[\left(\sum_{j \in A(d)} v_{jk} \right) \left(\sum_{j \in A(d)} v_{jk} l(j) \right) \right]}$$

□

5.2.9 Proof of proposition 2.16

Proof. The proof is almost the same as the proof of proposition 2.15. The only two differences are

$$\tilde{U} = \left[I_N - \frac{1}{\alpha} \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha}\tilde{V}_3^T V_3)^{-1}\tilde{V}_3^T \end{pmatrix} \right] V$$

and

$$\tilde{B}_1 = \begin{pmatrix} 0_d \\ V_3(\alpha I_d + \tilde{V}_3^T V_3)^{-1}\tilde{B} \end{pmatrix}$$

so

$$\begin{aligned} l \begin{pmatrix} 0_d \\ V_3(I_d + \frac{1}{\alpha}\tilde{V}_3^T V_3)^{-1}\tilde{B} \end{pmatrix} &= l \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha}\tilde{V}_3^T V_3)^{-1}\tilde{V}_3^T \end{pmatrix} V \hat{X} \\ l \begin{pmatrix} 0_d \\ V_3\tilde{B} \end{pmatrix} + O\left(\frac{1}{\alpha}\right) &= l \begin{pmatrix} 0 & 0 \\ 0 & V_3\tilde{V}_3^T \end{pmatrix} V \mathbf{1}_{N-d} + O\left(\frac{1}{\alpha}\right) \end{aligned}$$

Take limits of both sides of this equation, then $\lim_{\alpha \rightarrow \infty} \hat{X}$ exist and can be denoted by $\mathbf{1}_{N-d}\tilde{c}$ in

which

$$\begin{aligned} \tilde{l}(i) &= \frac{\left(\sum_{j \in A(d)} \beta_{ji} v_{ji}\right) \left(\sum_{j \in A(d)} v_{ji} l(j)\right)}{\sum_{k \in S(d)} \left(\sum_{j \in A(d)} \beta_{jk} v_{jk}\right) \left(\sum_{j \in A(d)} v_{jk} l(j)\right)} \\ \tilde{c} &= \frac{l \begin{pmatrix} 0_d \\ V_3\tilde{B} \end{pmatrix}}{l \begin{pmatrix} 0 \\ V_3\tilde{V}_3^T \mathbf{1}_{N-d} \end{pmatrix}} = \frac{\sum_{i \in S(d)} b_i \left(\sum_{j \in A(d)} \beta_{ji} v_{ji}\right) \left(\sum_{j \in A(d)} v_{ji} l(j)\right)}{\sum_{k \in S(d)} \left(\sum_{j \in A(d)} \beta_{jk} v_{jk}\right) \left(\sum_{j \in A(d)} v_{jk} l(j)\right)} \end{aligned}$$

□

5.2.10 Proof of lemma 2.2

Proof. I will first prove that there exist α^* , $\varepsilon > 0$, such that for any $\alpha > \alpha^*$, there always exist real roots of the system of equations above and $0 > \gamma_1^*, \gamma_2^* > -\frac{\varepsilon}{\alpha}$

Denote v_{31} by p , then $v_{32} = 1 - p$. Then

$$\begin{aligned} & \delta^2 v_{31} v_{32} \gamma_1^2 \gamma_2^2 + \delta(1 + \delta) v_{31}^2 \gamma_1^2 \gamma_2 + 2\delta v_{31} v_{32} \gamma_1 \gamma_2 + (1 + \delta)(\alpha + v_{32}^2) \gamma_2 + v_{31} v_{32} = 0 \\ \iff & \delta^2 p(1 - p) \gamma_1^2 \gamma_2^2 + \delta(1 + \delta) p^2 \gamma_1^2 \gamma_2 + 2\delta p(1 - p) \gamma_1 \gamma_2 + (1 + \delta)(\alpha + (1 - p)^2) \gamma_2 + p(1 - p) = 0 \end{aligned}$$

Suppose this equation is a quadratic function of γ_2 , then the roots are

$$\begin{aligned} \gamma_2^* &= \frac{-\mu \pm \sqrt{\mu^2 - 8\delta^2 p^2 (1 - p)^2 \gamma_1^2}}{2\delta^2 p(1 - p) \gamma_1^2} \\ &= \frac{2p(1 - p)}{-\mu \mp \sqrt{\mu^2 - 8\delta^2 p^2 (1 - p)^2 \gamma_1^2}} \end{aligned}$$

where

$$\mu = \delta(1 + \delta) p^2 \gamma_1^2 + 2\delta p(1 - p) \gamma_1 + (1 + \delta)(\alpha + (1 - p)^2)$$

let $f(\gamma_1) = \frac{2p(1-p)}{-\mu - \sqrt{\mu^2 - 8\delta^2 p^2 (1-p)^2 \gamma_1^2}}$, substitute γ_2 for $f(\gamma_1)$ in the other equation:

$$\delta^2 p(1-p) \gamma_1^2 f^2(\gamma_1) + \delta(1+\delta)(1-p)^2 \gamma_1 f^2(\gamma_1) + 2\delta p(1-p) \gamma_1 f(\gamma_1) + (1+\delta)(\alpha + p^2) \gamma_1 + p(1-p) = 0$$

If there exists a real root of this equation, then there also exists real roots of the system of

these two equations. Denote the left-hand side by $g(\gamma_1)$. It is easy to check that

$$g(0) > 0$$

Then the only thing that I need to check is that there exist $\varepsilon, \alpha^* > 0$ such that for any $\alpha > \alpha^*$, $g(\frac{-\varepsilon}{\alpha}) < 0$. When α is large enough,

$$\begin{aligned} f\left(\frac{-\varepsilon}{\alpha}\right) &= \frac{2p(1-p)}{- (1+\delta)(\alpha + (1-p)^2) + O\left(\frac{1}{\alpha}\right) - \sqrt{\left((1+\delta)(\alpha + (1-p)^2) + O\left(\frac{1}{\alpha}\right)\right)^2 - O\left(\frac{1}{\alpha^2}\right)}} \\ &= -\frac{p(1-p)}{(1+\delta)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^3}\right) \end{aligned}$$

Then

$$\begin{aligned} g\left(\frac{-\varepsilon}{\alpha}\right) &= O\left(\frac{1}{\alpha^2}\right) + (1+\delta)(\alpha + p^2)\frac{-\varepsilon}{\alpha} + p(1-p) \\ &= O\left(\frac{1}{\alpha}\right) - \varepsilon(1+\delta) + p(1-p) \end{aligned}$$

let $\varepsilon = \frac{p(1-p)+1}{1+\delta}$, then $g(\frac{-\varepsilon}{\alpha}) < 0$. Since $g(\cdot)$ is a continuous function, there exists a real root $0 > \gamma_1^* > \frac{-\varepsilon}{\alpha}$.

$$\text{Since } 0 > \gamma_1^* > \frac{-\varepsilon}{\alpha}, \gamma_2^* = f(\gamma_1^*) = \frac{2p(1-p)}{-\mu - \sqrt{\mu^2 - 8\delta^2 p^2 (1-p)^2 \gamma_1^{*2}}} = -\frac{p(1-p)}{(1+\delta)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^3}\right).$$

Following the same logic, I can also prove that

$$\gamma_1^* = -\frac{p(1-p)}{(1+\delta)(\alpha + p^2)} + O\left(\frac{1}{\alpha^3}\right)$$

□

5.2.11 Proof of lemma 2.3

Proof. By lemma 2.2

$$\gamma_1^* = -\frac{p(1-p)}{(1+\delta)(\alpha+p^2)} + O\left(\frac{1}{\alpha^3}\right)$$

and

$$\begin{aligned} (1+\delta)(\alpha+p^2)\gamma_1 &= O\left(\frac{1}{\alpha^3}\right) - p(1-p) - 2\delta p(1-p)\gamma_1\gamma_2 \\ &= O\left(\frac{1}{\alpha^3}\right) - p(1-p) - 2\frac{\delta p^3(1-p)^3}{(1+\delta)^2(\alpha+p^2)(\alpha+(1-p)^2)} \end{aligned}$$

so

$$\begin{aligned} & -(\delta v_{32}\gamma_2 + v_{31} + \alpha)(1+\delta) + v_{31}v_{32}\frac{a_1}{\gamma_1} \\ &= \delta v_{31}v_{32}a_2 + \delta(1+\delta)v_{32}^2a_2\gamma_2 + \delta^2v_{31}v_{32}a_1\gamma_2 + \delta^2v_{31}v_{32}a_2\gamma_1\gamma_2 \\ \iff & -\gamma_1(\delta(1-p)\gamma_2 + p + \alpha)(1+\delta) \\ &= [\delta p(1-p)\gamma_1 + \delta(1+\delta)(1-p)^2\gamma_1\gamma_2 + \delta^2p(1-p)\gamma_1^2\gamma_2] a_2 - (1 - \delta^2\gamma_1\gamma_2)p(1-p)a_1 \\ \iff & -\delta(1+\delta)(1-p)\gamma_1\gamma_2 - (1+\delta)p(1-p)\gamma_1 - (1+\delta)(\alpha+p^2)\gamma_1 \\ &= \left[-\frac{\delta p^2(1-p)^2\alpha}{(1+\delta)(\alpha+p^2)(\alpha+(1-p)^2)} + O\left(\frac{1}{\alpha^3}\right) \right] a_2 \\ & - \left[\left(1 - \delta^2\frac{p^2(1-p)^2}{(1+\delta)^2(\alpha+p^2)(\alpha+(1-p)^2)}\right)p(1-p) + O\left(\frac{1}{\alpha^3}\right) \right] a_1 \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow 1 + \frac{p(1-p)}{\alpha + p^2} + [2p - (1 + \delta)] \frac{\delta p(1-p)^2}{(1 + \delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^3}\right) \\
&= \left[-1 + \delta^2 \frac{p^2(1-p)^2}{(1 + \delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^3}\right) \right] a_1 \\
&\quad + \left[-\frac{\delta p(1-p)\alpha}{(1 + \delta)(\alpha + p^2)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^3}\right) \right] a_2
\end{aligned}$$

Follow the same steps as above, equation 4.7 is equivalent to

$$\begin{aligned}
&-1 - \frac{p(1-p)}{\alpha + (1-p)^2} - [2(1-p) - (1 + \delta)] \frac{\delta p^2(1-p)}{(1 + \delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^3}\right) \\
&= \left[-\frac{\delta p(1-p)\alpha}{(1 + \delta)(\alpha + p^2)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^3}\right) \right] a_1 \\
&\quad + \left[-1 + \delta^2 \frac{p^2(1-p)^2}{(1 + \delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^3}\right) \right] a_2
\end{aligned}$$

It is a system of linear equations. It is easy to calculate that $a_1^* = \frac{\Delta_2}{\Delta_1}$ and $a_2^* = \frac{\Delta_3}{\Delta_1}$. □

5.2.12 Proof of proposition 2.17

Proof. By lemma 2.2 and 2.3

$$1 - \gamma_1^* \gamma_2^* = 1 - \frac{p^2(1-p)^2}{(1 + \delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^3}\right)$$

$$\begin{aligned}
& a_1^* + \gamma_1^* a_2^* \\
&= \frac{1}{\Delta_1} [\Delta_2 + \gamma_1^* \Delta_3] \\
&= \frac{1}{\Delta_1} \left[-1 - \frac{p(1-p)}{\alpha + p^2} - \frac{\delta p(1-p)\alpha}{(1+\delta)(\alpha + p^2)(\alpha + (1-p)^2)} \right. \\
&\quad + [-3p + 1 + \delta] \frac{\delta p(1-p)^2}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} \\
&\quad - \frac{p(1-p)}{(1+\delta)(\alpha + p^2)} - \frac{p^2(1-p)^2}{(1+\delta)(\alpha + p^2)(\alpha + (1-p)^2)} \\
&\quad \left. - \frac{\delta p^2(1-p)^2\alpha}{(1+\delta)^2(\alpha + p^2)^2(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^3}\right) \right] \\
&= \frac{1}{\Delta_1} \left[-1 - \frac{2p(1-p)\alpha}{(\alpha + p^2)(\alpha + (1-p)^2)} \right. \\
&\quad \left. + \frac{p(1-p)^2(p(1-\delta)^2 - 2(1+\delta))}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^3}\right) \right]
\end{aligned}$$

and

$$\begin{aligned}
& a_2^* + \gamma_2^* a_1^* \\
&= \frac{1}{\Delta_1} \left[1 + \frac{2p(1-p)\alpha}{(\alpha + p^2)(\alpha + (1-p)^2)} \right. \\
&\quad \left. - \frac{p^2(1-p)((1-p)(1-\delta)^2 - 2(1+\delta))}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^3}\right) \right]
\end{aligned}$$

then

$$\begin{aligned}
x^* &= \frac{a_1^* + \gamma_1^* a_2^*}{1 - \gamma_1^* \gamma_2^*} \\
&= \frac{1}{(1 - \gamma_1^* \gamma_2^*)} \frac{1}{\Delta_1} \left[-1 - \frac{2p(1-p)\alpha}{(\alpha + p^2)(\alpha + (1-p)^2)} + \frac{p(1-p)^2(p(1-\delta)^2 - 2(1+\delta))}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^3}\right) \right] \\
&= \frac{1}{(1+\delta)^2 [\alpha^2 + (p^2 + (1-p)^2)\alpha] + 2\delta(1-\delta)p^2(1-p)^2 + O\left(\frac{1}{\alpha}\right)} \\
&\quad \left[-(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2) - 2(1+\delta)^2 p(1-p)\alpha + p(1-p)^2(p(1-\delta)^2 - 2(1+\delta)) + O\left(\frac{1}{\alpha^3}\right) \right] \\
&= \frac{-(1+\delta)^2(\alpha^2 + \alpha) - 4\delta p^2(1-p)^2 - 2(1+\delta)p(1-p)^2 + O\left(\frac{1}{\alpha}\right)}{(1+\delta)^2 [\alpha^2 + (p^2 + (1-p)^2)\alpha] + 2\delta(1-\delta)p^2(1-p)^2 + O\left(\frac{1}{\alpha}\right)}
\end{aligned}$$

and

$$\begin{aligned}
y^* &= \frac{a_2^* + \gamma_2^* a_1^*}{1 - \gamma_1^* \gamma_2^*} \\
&= \frac{(1+\delta)^2(\alpha^2 + \alpha) + 4\delta p^2(1-p)^2 + 2(1+\delta)p^2(1-p) + O\left(\frac{1}{\alpha}\right)}{(1+\delta)^2 [\alpha^2 + (p^2 + (1-p)^2)\alpha] + 2\delta(1-\delta)p^2(1-p)^2 + O\left(\frac{1}{\alpha}\right)}
\end{aligned}$$

the opinion of naive agent in steady state is

$$\begin{aligned}
&px^* + (1-p)y^* \\
&= \frac{(1-2p)}{(1+\delta)^2 [\alpha^2 + (p^2 + (1-p)^2)\alpha] + 2\delta(1-\delta)p^2(1-p)^2 + O\left(\frac{1}{\alpha}\right)} \\
&\quad \left[(1+\delta)^2(\alpha^2 + \alpha) + 4\delta p^2(1-p)^2 + O\left(\frac{1}{\alpha}\right) \right]
\end{aligned}$$

When the sophisticated agents are myopic, the opinion of the naive agent in steady state is

$$p\hat{x} + (1-p)\hat{y} = \frac{(1-2p)(\alpha^2 + \alpha)}{\alpha^2 + (p^2 + (1-p)^2)\alpha}$$

then

$$\begin{aligned}
& px^* + (1-p)y^* > p\hat{x} + (1-p)\hat{y} \\
\iff & \frac{(1-2p) \left[(1+\delta)^2(\alpha^2 + \alpha) + 4\delta p^2(1-p)^2 + O\left(\frac{1}{\alpha}\right) \right]}{(1+\delta)^2 [\alpha^2 + (p^2 + (1-p)^2)\alpha] + 2\delta(1-\delta)p^2(1-p)^2 + O\left(\frac{1}{\alpha}\right)} > \frac{(1-2p)(\alpha^2 + \alpha)}{\alpha^2 + (p^2 + (1-p)^2)\alpha}
\end{aligned}$$

it is easy to check that, when α is large enough

$$px^* + (1-p)y^* \geq p\hat{x} + (1-p)\hat{y} \iff p < \frac{1}{2}$$

□

5.2.13 Proof of proposition 2.18

Proof. In steady state, the stage payoff for agent 1 is

$$\Pi_1^* = -(px^* + (1-p)y^* + 1)^2 - \alpha(x^* + 1)^2$$

Since

$$\begin{aligned}
& px^* + (1-p)y^* \\
= & \frac{(1-2p)}{(1+\delta)^2 [\alpha^2 + (p^2 + (1-p)^2)\alpha] + 2\delta(1-\delta)p^2(1-p)^2 + O\left(\frac{1}{\alpha}\right)} \\
& \left[(1+\delta)^2(\alpha^2 + \alpha) + 4\delta p^2(1-p)^2 + O\left(\frac{1}{\alpha}\right) \right]
\end{aligned}$$

and

$$x^* = \frac{-(1+\delta)^2(\alpha^2 + \alpha) - 4\delta p^2(1-p)^2 - 2(1+\delta)p(1-p)^2 + O(\frac{1}{\alpha})}{(1+\delta)^2[\alpha^2 + (p^2 + (1-p)^2)\alpha] + 2\delta(1-\delta)p^2(1-p)^2 + O(\frac{1}{\alpha})}$$

$$\begin{aligned} & \Pi_1^* - \hat{\Pi}_1 \\ &= -(px^* + (1-p)y^* + 1)^2 - \alpha(x^* + 1)^2 + (p\hat{x} + (1-p)\hat{y} + 1)^2 + \alpha(\hat{x} + 1)^2 \\ &= (z^* + \hat{z} + 2)(\hat{z} - z^*) + \alpha(x^* + \hat{x} + 2)(\hat{x} - x^*) \end{aligned}$$

Let

$$\Gamma = [\alpha^2 + (p^2 + (1-p)^2)\alpha] \left[(1+\delta)^2 [\alpha^2 + (p^2 + (1-p)^2)\alpha] + 2\delta(1-\delta)p^2(1-p)^2 + O(\frac{1}{\alpha}) \right]$$

then

$$\begin{aligned} \hat{x} - x^* &= \frac{-2\delta(1+\delta)p(1-p)^3\alpha^2 + O(\alpha)}{\Gamma} \\ \hat{z} - z^* &= \frac{-2\delta(1+\delta)(1-2p)p^2(1-p)^2\alpha^2 + O(\alpha)}{\Gamma} \end{aligned}$$

$$\begin{aligned} & z^* + \hat{z} + 2 \\ &= \frac{4(1+\delta)^2(1-p)\alpha^4 + O(\alpha^3)}{\Gamma} \end{aligned}$$

$$\begin{aligned}
& x^* + \hat{x} + 2 \\
&= \frac{-4(1 + \delta)^2 p(1 - p)\alpha^3 + O(\alpha^2)}{\Gamma}
\end{aligned}$$

then

$$\begin{aligned}
& \Pi_1^* - \hat{\Pi}_1 \\
&= \frac{4(1 + \delta)^2(1 - p)\alpha^4 + O(\alpha^3)}{\Gamma} - \frac{2\delta(1 + \delta)(1 - 2p)p^2(1 - p)^2\alpha^2 + O(\alpha)}{\Gamma} \\
&\quad + \alpha \frac{-4(1 + \delta)^2 p(1 - p)\alpha^3 + O(\alpha^2)}{\Gamma} - \frac{2\delta(1 + \delta)p(1 - p)^3\alpha^2 + O(\alpha)}{\Gamma} \\
&= \frac{-8\delta(1 + \delta)^3(1 - 2p)p^2(1 - p)^3\alpha^6 + 8\delta(1 + \delta)^3 p^2(1 - p)^4\alpha^6 + O(\alpha^5)}{\Gamma^2} \\
&= \frac{8\delta(1 + \delta)^3 p^3(1 - p)^3\alpha^6 + O(\alpha^5)}{\Gamma^2} > 0
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \Pi_2^* - \hat{\Pi}_2 \\
&= -(px^* + (1 - p)y^* - 1)^2 - \alpha(y^* - 1)^2 + (p\hat{x} + (1 - p)\hat{y} - 1)^2 + \alpha(\hat{y} - 1)^2 \\
&= (z^* + \hat{z} - 2)(\hat{z} - z^*) + \alpha(y^* + \hat{y} - 2)(\hat{y} - y^*)
\end{aligned}$$

$$\begin{aligned}
& z^* + \hat{z} - 2 \\
&= \frac{-4(1 + \delta)^2 p\alpha^4 + O(\alpha^3)}{\Gamma}
\end{aligned}$$

$$\hat{y} - y^* = \frac{2\delta(1 + \delta)p^3(1 - p)\alpha^2 + O(\alpha)}{\Gamma}$$

$$\begin{aligned}
& y^* + \hat{y} - 2 \\
&= \frac{4(1 + \delta)^2 p(1 - p)\alpha^3 + O(\alpha^2)}{\Gamma}
\end{aligned}$$

then

$$\begin{aligned}
& \Pi_2^* - \hat{\Pi}_2 \\
&= \frac{-4(1 + \delta)^2 p\alpha^4 + O(\alpha^3)}{\Gamma} - \frac{2\delta(1 + \delta)(1 - 2p)p^2(1 - p)^2\alpha^2 + O(\alpha)}{\Gamma} \\
&\quad + \alpha \frac{4(1 + \delta)^2 p(1 - p)\alpha^3 + O(\alpha^2)}{\Gamma} - \frac{2\delta(1 + \delta)p^3(1 - p)\alpha^2 + O(\alpha)}{\Gamma} \\
&= \frac{8\delta(1 + \delta)^3(1 - 2p)p^3(1 - p)^2\alpha^6 + 8\delta(1 + \delta)^3p^4(1 - p)^2\alpha^6 + O(\alpha^5)}{\Gamma^2} \\
&= \frac{8\delta(1 + \delta)^3p^3(1 - p)^3\alpha^6 + O(\alpha^5)}{\Gamma^2} > 0
\end{aligned}$$

□

5.2.14 Proof of proposition 2.19

Proof.

$$\begin{aligned}
& z^*(\delta_1) - z^*(\delta_2) \\
&= \frac{(1-2p) \left[(1+\delta_1)^2(\alpha^2 + \alpha) + 4\delta_1 p^2(1-p)^2 + O\left(\frac{1}{\alpha}\right) \right]}{(1+\delta_1)^2 [\alpha^2 + (p^2 + (1-p)^2)\alpha] + 2\delta_1(1-\delta_1)p^2(1-p)^2 + O\left(\frac{1}{\alpha}\right)} \\
&\quad - \frac{(1-2p) \left[(1+\delta_2)^2(\alpha^2 + \alpha) + 4\delta_2 p^2(1-p)^2 + O\left(\frac{1}{\alpha}\right) \right]}{(1+\delta_2)^2 [\alpha^2 + (p^2 + (1-p)^2)\alpha] + 2\delta_2(1-\delta_2)p^2(1-p)^2 + O\left(\frac{1}{\alpha}\right)} \\
&= \frac{1-2p}{\Omega(\delta_1)\Omega(\delta_2)} \left[2(1+\delta_1)^2\delta_2(1-\delta_2)p^2(1-p)^2\alpha^2 \right. \\
&\quad + 4\delta_1(1+\delta_2)^2p^2(1-p)^2\alpha^2 - 2\delta_1(1-\delta_1)(1+\delta_2)^2p^2(1-p)^2\alpha^2 \\
&\quad \left. - 4(1+\delta_1)^2\delta_2p^2(1-p)^2\alpha^2 + O(\alpha) \right] \\
&= \frac{1-2p}{\Omega(\delta_1)\Omega(\delta_2)} \left[4(1+\delta_1)(1+\delta_2)(\delta_1-\delta_2)p^2(1-p)^2\alpha^2 + O(\alpha) \right]
\end{aligned}$$

Since $\delta_1 < \delta_2$, $z^*(\delta_1) < z^*(\delta_2)$ if and only if $1 - 2p > 0$

$$\begin{aligned}
& \Pi_1^*(\delta_1) - \Pi_1^*(\delta_2) \\
&= \frac{8\delta_1(1+\delta_1)^3 p^3(1-p)^3 \alpha^6 + O(\alpha^5)}{\Gamma^2(\delta_1)} - \frac{8\delta_2(1+\delta_2)^3 p^3(1-p)^3 \alpha^6 + O(\alpha^5)}{\Gamma^2(\delta_2)} \\
&= \frac{1}{\Gamma^2(\delta_1)\Gamma^2(\delta_2)} \left\{ [8\delta_1(1+\delta_1)^3 p^3(1-p)^3 \alpha^6 + O(\alpha^5)] \Gamma^2(\delta_2) \right. \\
&\quad \left. - [8\delta_2(1+\delta_2)^3 p^3(1-p)^3 \alpha^6 + O(\alpha^5)] \Gamma^2(\delta_1) \right\} \\
&= \frac{1}{\Gamma^2(\delta_1)\Gamma^2(\delta_2)} \left[8\delta_1(1+\delta_1)^3(1+\delta_2)^4 p^3(1-p)^3 \alpha^{14} \right. \\
&\quad \left. - 8(1+\delta_1)^4 \delta_2(1+\delta_2)^3 p^3(1-p)^3 \alpha^{14} + O(\alpha^{13}) \right] \\
&= \frac{8(1+\delta_1)^3(1+\delta_2)^3 p^3(1-p)^3 (\delta_1 - \delta_2) \alpha^{14} + O(\alpha^{13})}{\Gamma^2(\delta_1)\Gamma^2(\delta_2)} < 0
\end{aligned}$$

□

5.3 Chapter 3

5.3.1 Proof of Proposition 3.20

The condition

$$r_2 - r_1 > \frac{1}{p} - 1$$

simply follows from solving the inequality (17). Then plugging into this condition the rates expressed in Equation 14 yields:

$$\left(R_2 - \frac{\bar{u}}{p}\right) - \left(R_1 - \frac{1}{p} - \frac{\bar{u}}{p} + 1\right) > \frac{1}{p} - 1$$

which implies that $R_2 > R_1$ must hold for the condition to be satisfied. QED.

5.3.2 Proof of Proposition 3.21

If the lender extends credit to entrepreneurs in Ω_1 , then for each dollar lent, it can earn a return of $p(r_1 - 1)$. So it is willing to offer secured loans iff

$$r_1 \geq 1 \iff pR_1 \geq 1 + \bar{u}$$

which is satisfied due to condition (A2). Since the lender cannot distinguish between consumers in Ω_2 and Ω_3 , it is willing to offer unsecured loans iff its overall return from offering the contract is non-negative. Formally, the condition is:

$$N_2(pr_2 - 1) + N_3(\hat{p}r_2 - 1) \geq 0,$$

implying that $R_2 \geq \frac{\bar{u}}{p} + \frac{1}{\bar{p}}$ must hold, where $\bar{p} \equiv \frac{N_2}{N_2+N_3}p + \frac{N_3}{N_2+N_3}\hat{p}$. QED.

5.3.3 Proof of Corollary 3.1

Consumers in the high risk segment invest in their ideas only when $\hat{p}(R_3 - r_2) \geq \bar{u}$. Substituting $r_2 = R_2 - \bar{u}/p$ into this expression yields the necessary condition $R_3 - R_2 \geq \bar{u} \left(\frac{1}{\hat{p}} - \frac{1}{p} \right)$. QED.

5.3.4 Proof of Proposition 3.22

We can derive the lender's optimal strategy by comparing the seven payoffs laid out in the document. Notice that $\Pi_{31} = \Pi_{21}$ and $\Pi_{33} = \Pi_{22}$. Calculating the differences between the

profit functions gives:

$$\Pi_{33} - \Pi_{12} = \Pi_{32} - \Pi_{11} = N_2(p(R_2 - \frac{\bar{u}}{p}) - 1) + N_3(\hat{p}(R_2 - \frac{\bar{u}}{p}) - 1) - qN_1(pR_2 - \bar{u} - 1)$$

$$\Pi_{11} - \Pi_{12} = \Pi_{32} - \Pi_{33} = N_1(pR_1 - \bar{u} - 1) - qN_1(pR_2 - \bar{u} - 1)$$

$$\Pi_{31} - \Pi_{11} = N_2(p(R_1 - \frac{\bar{u}}{p}) - 1) + N_3(\hat{p}(R_1 - \frac{\bar{u}}{p}) - 1)$$

$$\begin{aligned} \Pi_{31} - \Pi_{12} &= N_1(p(R_1 - \frac{\bar{u}}{p}) - 1) + N_2(p(R_1 - \frac{\bar{u}}{p}) - 1) \\ &\quad + N_3(\hat{p}(R_1 - \frac{\bar{u}}{p}) - 1) - qN_1(pR_2 - \bar{u} - 1) \end{aligned}$$

$$\Pi_{31} - \Pi_{32} = -N_2p(R_2 - R_1) - N_3\hat{p}(R_2 - R_1) + qN_1(p(R_2 - \frac{\bar{u}}{p}) - 1)$$

$$\Pi_{32} - \Pi_{12} = N_1(p(R_1 - \frac{\bar{u}}{p}) - 1) + (N_2 - 2qN_1)(p(R_2 - \frac{\bar{u}}{p}) - 1) + N_3(\hat{p}(R_2 - \frac{\bar{u}}{p}) - 1)$$

$$\Pi_{31} - \Pi_{33} = N_1(p(R_1 - \frac{\bar{u}}{p}) - 1) - N_2p(R_2 - R_1) - N_3\hat{p}(R_2 - R_1)$$

$$\Pi_{31} - \Pi_{32} = N_2(p(R_1 - \frac{\bar{u}}{p}) - 1) + N_3(\hat{p}(R_1 - \frac{\bar{u}}{p}) - 1)$$

When we compare the expressions

1. If $R_1 < \frac{\bar{u}}{p} + \frac{1}{p}$, then $\Pi_{31} < \Pi_{11}$ holds. And

$$\Pi_{33} > \Pi_{12} \iff \Pi_{32} > \Pi_{11} \iff q < q_1^*$$

- (a) When $q < q_1^*$ and $q < q_2^*$, $\Pi_{32} > \Pi_{11} > \Pi_{31}$, Π_{12} and $\Pi_{32} > \Pi_{33}$. Lender's optimal strategy is not to compete with informal market and serve all consumers in Ω_1 , and the interest rates are

$$r_1 = R_1 + 1 - \frac{1}{p} - \frac{\bar{u}}{p}$$

$$r_2 = R_2 - \frac{\bar{u}}{p}.$$

Consumers in Ω_2^* borrow through their social contacts. So segments with access to financing are Ω_1 , Ω_2^{**} and Ω_3 .

- (b) When $q < q_1^*$ and $q \geq q_2^*$, $\Pi_{33} = \Pi_{22} \geq \Pi_{32} > \Pi_{11} > \Pi_{31}$ and $\Pi_{33} > \Pi_{12}$. Lender's optimal strategy is either to only offer an unsecured loan and set $r_2 = R_2 - \frac{\bar{u}}{p}$ or to offer both secured and unsecured loans and set $r_1 = R_2 + 1 - \frac{1}{p} - \frac{\bar{u}}{p}$, $r_2 = R_2 - \frac{\bar{u}}{p}$. In the first case, consumers in Ω_1 do not borrow from the lender, so segments with access to financing are Ω_2 and Ω_3 . In the latter case, only consumers in Ω_1^* with social contacts find it profitable to borrow from the lender, and consumers in Ω_2^* borrow from social contacts. So segments with access to financing are Ω_1^* , Ω_2^{**} and Ω_3 .
- (c) When $q \geq q_1^*$ and $q < q_2^*$, $\Pi_{11} > \Pi_{12} \geq \Pi_{33}$ and $\Pi_{11} \geq \Pi_{32}, \Pi_{31}$. Lender's optimal strategy is to only offer secured loans and set $r_1 = R_1 + 1 - \frac{1}{p} - \frac{\bar{u}}{p}$. Only Ω_1 is served by the bank.
- (d) When $q \geq q_1^*$ and $q \geq q_2^*$, $\Pi_{12} \geq \Pi_{11} \geq \Pi_{32} \Pi_{31}$ and $\Pi_{12} \geq \Pi_{33}$. Lender's optimal strategy is to only serve secured loans and set $r_1 = R_2 + 1 - \frac{1}{p} - \frac{\bar{u}}{p}$. Only Ω_1^* is served by the bank.

2. If $R_1 \geq \frac{\bar{u}}{p} + \frac{1}{p}$ and $R_2 - R_1 \geq \frac{N_1}{(N_2 + N_3)\bar{p}} \left[p(R_1 - \frac{\bar{u}}{p}) - 1 \right]$, then $\Pi_{33} \geq \Pi_{31} \geq \Pi_{11}$. We only

need to compare Π_{33} , Π_{32} and Π_{12} . Notice that

$$\begin{aligned} q_1^* &= \frac{N_2 + N_3}{N_1} \frac{\bar{p}(R_2 - \frac{\bar{u}}{p}) - 1}{p(R_2 - \frac{\bar{u}}{p}) - 1} \\ &= \frac{N_2 + N_3}{N_1} \frac{\bar{p}(R_2 - R_1)}{p(R_2 - \frac{\bar{u}}{p}) - 1} + \frac{N_2 + N_3}{N_1} \frac{\bar{p}(R_1 - \frac{\bar{u}}{p}) - 1}{p(R_2 - \frac{\bar{u}}{p}) - 1} \\ &\geq \frac{pR_1 - \bar{u} - 1}{pR_2 - \bar{u} - 1} = q_2^* \end{aligned}$$

Then the results are the same as the case where $R_1 < \frac{\bar{u}}{p} + \frac{1}{p}$, with the only difference being that $q_2^* > q \geq q_1^*$ can not hold.

3. If $R_1 \geq \frac{\bar{u}}{p} + \frac{1}{p}$ and $R_2 - R_1 < \frac{N_1}{(N_2 + N_3)\bar{p}} \left[p(R_1 - \frac{\bar{u}}{p}) - 1 \right]$, then $\Pi_{31} \geq \Pi_{11}$ and $\Pi_{31} > \Pi_{33}$. We compare Π_{31} , Π_{32} and Π_{12} . Notice

$$\begin{aligned} q_4^* &= \frac{N_2 + N_3}{N_1} \frac{\bar{p}(R_2 - R_1)}{p(R_2 - \frac{\bar{u}}{p}) - 1} \\ &< \frac{pR_1 - \bar{u} - 1}{pR_2 - \bar{u} - 1} < q_3^* \end{aligned}$$

- (a) If $q \geq q_3^* > q_4^*$, then $\Pi_{12} \geq \Pi_{31} > \Pi_{32}$. Lender's optimal strategy is to only offer secured loans and set $r_1 = R_2 + 1 - \frac{1}{p} - \frac{\bar{u}}{p}$. Only Ω_1^* is served by the bank.
- (b) If $q_4^* \leq q < q_3^*$, then $\Pi_{31} \geq \Pi_{32}$, Π_{12} . Lender's optimal strategy is to compete with the informal market and serve all consumers. It either only serves unsecured loans, or serves both secured and unsecured loans. The interest rates are $r_1 = R_1 + 1 - \frac{1}{p} - \frac{\bar{u}}{p}$, $r_2 = R_1 - \frac{\bar{u}}{p}$.
- (c) If $q < q_4^* < q_3^*$, then $\Pi_{32} > \Pi_{31} > \Pi_{12}$. Lender's optimal strategy is not to compete with informal market and serve all consumers in Ω_1 , and the interest rates are $r_1 = R_1 + 1 - \frac{1}{p} - \frac{\bar{u}}{p}$, $r_2 = R_2 - \frac{\bar{u}}{p}$. Consumers in Ω_2^* borrow from social contacts, so segments

with access to financing are Ω_1 , Ω_2^{**} and Ω_3 .

5.3.5 Proof of Proposition 3.23

When (A1)-(A5) hold and there is no informal market, all borrowers are served by the bank and all entrepreneurial activity is funded. The average innovativeness of entrepreneurial ideas is

$$\frac{N_1 (pR_1 - 1) + N_2 (pR_2 - 1) + N_2 (\hat{p}R_3 - 1)}{N_1 + N_2 + N_3}.$$

When there is an informal market, as shown in Proposition 3.22, some borrowers can't access to credit. For instance, when $R_1 < \frac{\bar{u}}{p} + \frac{1}{p}$ and $q > q_1^*, q_2^*$, the bank only lends to Ω_1^* and only the activity of Ω_2^* is funded. The average innovativeness of entrepreneurial ideas is $pR_2 - 1$. Due to (A2) and (A3):

$$\begin{aligned} & pR_2 - 1 \\ &= \frac{N_1 (pR_2 - 1) + N_2 (pR_2 - 1) + N_2 (pR_2 - 1)}{N_1 + N_2 + N_3} \\ &> \frac{N_1 (pR_1 - 1) + N_2 (pR_2 - 1) + N_2 (\hat{p}R_3 - 1)}{N_1 + N_2 + N_3} \end{aligned}$$

The average innovativeness of entrepreneurial ideas in the latter case is higher.

5.3.6 Proof of Proposition 3.24 and 3.25

The proofs follow from the Proof of Proposition 3.22.

5.3.7 Proof of Proposition 3.26

Part (i): First let's prove sufficiency. Suppose that

$$R_3 - R_2 < \hat{p}\bar{u}\left(\frac{1}{\hat{p}^2} - \frac{1}{p^2}\right),$$

and let the bank set the joint liability contract terms to

$$r_2 = R_2 - \frac{\bar{u}}{p^2}, \quad c = \frac{\bar{u}}{p^2}.$$

It is easy to check that

$$R_2 - r_2 - c = 0$$

$$p^2(R_2 - r_2) + p(1 - p)(R_2 - r_2 - c) = \bar{u}$$

and

$$\begin{aligned} & \hat{p}^2(R_3 - r_2) + \hat{p}(1 - \hat{p})(R_3 - r_2 - c) \\ &= \hat{p}\left(R_3 - R_2 + \frac{\bar{u}}{p^2}\right) - \hat{p}(1 - \hat{p})\frac{\bar{u}}{p^2} \\ &< \hat{p}\left[\hat{p}\bar{u}\left(\frac{1}{\hat{p}^2} - \frac{1}{p^2}\right) + \frac{\bar{u}}{p^2}\right] - \hat{p}(1 - \hat{p})\frac{\bar{u}}{p^2} \\ &= \bar{u} \end{aligned}$$

Both equations (24) and (25) are satisfied. The bank gets all surplus of consumers in Ω_2 . So the

problem is to find the largest r_2 and c satisfying:

$$\begin{cases} p(R_2 - r_2) - p(1 - p)c \geq \bar{u} & (1) \\ \hat{p}(R_3 - r_2) - \hat{p}(1 - \hat{p})c < \bar{u} & (2) \\ R_2 - r_2 - c \geq 0 & (3) \end{cases}$$

Notice that either condition (1) or condition (3) should be binding, otherwise the bank can always increase r_2 to charge more from consumers in Ω_2 . If condition (3) is binding, then $R_2 - r_2 = c$, substituting this expression into the other two conditions yields:

$$\frac{R_3 - R_2}{\hat{p}} - \frac{\bar{u}}{\hat{p}^2} + R_2 < r_2 \leq R_2 - \frac{\bar{u}}{p^2} \Rightarrow R_3 - R_2 < \hat{p}\left(\frac{\bar{u}}{\hat{p}^2} - \frac{\bar{u}}{p^2}\right)$$

This gives the condition in the proof.

Part(ii): Recall that the overinvestment problem is prevented with individual contracts if $R_3 - R_2 < \bar{u}\left(\frac{1}{\hat{p}} - \frac{1}{p}\right)$. From Part (i), the condition for joint liability is $R_3 - R_2 < \hat{p}\bar{u}\left(\frac{1}{\hat{p}^2} - \frac{1}{p^2}\right)$. Comparing these two regions, $\bar{u}\left(\frac{1}{\hat{p}} - \frac{1}{p}\right) \leq R_3 - R_2 < \hat{p}\bar{u}\left(\frac{1}{\hat{p}^2} - \frac{1}{p^2}\right)$. QED.

5.3.8 Proof of Proposition 3.27

We solve for the lender payoff under each strategy and compare the payoffs.

- **Lender Offers Only Secured Loans.** If the lender chooses to offer only the secured loan, besides the strategies we discuss in Section 2, it can also choose to forego $\tilde{\Omega}_1^*$. As we write in the main text, for simplicity we assume that the the borrowers with strong ties are always willing to lend to their friends with the lowest interest rate. Besides investments, the only

income of the consumers in this model is the outside option \bar{u} , so the lowest interest rate the borrower can offer to his friend is $r_1 - \bar{u}$. Then their friends are willing to borrow from them if and only if $r_1 - \bar{u} \leq R_2 - \frac{\bar{u}}{p}$, because otherwise their friend will prefer outside option \bar{u} . The lender can raise the interest rate to $\bar{u} + R_2 - \frac{\bar{u}}{p}$ and the expected payoff is

$$\hat{\Pi}_{12} = \alpha q N_1 p (\bar{u} + R_2 - \frac{\bar{u}}{p} - 1) = \alpha q N_1 (p\bar{u} + pR_2 - \bar{u} - p)$$

- **Lender Offers Both Secured and Unsecured Loans.** If the lender chooses to offer both secured and unsecured loans, the strategies are different from those in Section 2. We first show that the lender can no longer serve all consumers. Suppose that the lender is trying to serve all consumers, then it must be true that

$$\hat{r} = r_1 - \bar{u} \geq r_2 \implies r_1 > r_2.$$

However, if $r_1 > r_2$ holds, consumers in Ω_1 are always willing to pretend to be consumers in Ω_2 , and we have a contradiction. The bank then has to give up on $\hat{\Omega}_2^*$. If the bank chooses to serve all other consumers then the interest rates are $r_1 = R_1 + 1 - \frac{1}{p} - \frac{\bar{u}}{p}$ and $r_2 = R_1 - \frac{\bar{u}}{p}$ and the expected payoff is

$$\hat{\Pi}_{31} = N_1 (p(R_1 - \frac{\bar{u}}{p}) - 1) + (N_2 - \alpha q N_1) (p(R_1 - \frac{\bar{u}}{p}) - 1) + N_3 (\hat{p}(R_1 - \frac{\bar{u}}{p}) - 1).$$

If the bank does not want to compete with the informal market, it can raise the interest rate of unsecured loans to $r_2 = R_2 - \frac{\bar{u}}{p}$. Then consumers in $\tilde{\Omega}_2^*$ will also borrow from their social contacts and the expected payoff for the lender is $\hat{\Pi}_{32} = \Pi_{32}$. If the lender further

does not even try to serve customers in Ω_1^{**} , he can raise the interest rate of secured loans to $r_1 = R_2 + 1 - \frac{1}{p} - \frac{\bar{u}}{p}$. The expected payoff is $\hat{\Pi}_{33} = \Pi_{33}$. The lender now can even give up on $\tilde{\Omega}_1^*$ and rise the interest rate of secured loans even higher to $r_1 = \bar{u} + R_2 - \frac{\bar{u}}{p}$, the expected payoff then is

$$\begin{aligned}\hat{\Pi}_{34} &= \alpha q N_1 p (\bar{u} + R_2 - \frac{\bar{u}}{p} - 1) + (N_2 - \alpha q N_1) (p(R_2 - \frac{\bar{u}}{p}) - 1) + N_3 (\hat{p}(R_2 - \frac{\bar{u}}{p}) - 1) \\ &= \alpha q N_1 (p\bar{u} + 1 - p) + N_2 (p(R_2 - \frac{\bar{u}}{p}) - 1) + N_3 (\hat{p}(R_2 - \frac{\bar{u}}{p}) - 1)\end{aligned}$$

- **Lender Offers Only Unsecured Loans.** It is easy to check that if the lender chooses to offer only the unsecured loans, he will get either the same or lower expected payoff comparing with other strategies, so, for simplicity, we assume that the lender always prefers not to offer only unsecured loans. Since (A6) holds, $\Pi_{11} > \Pi_{31} > \hat{\Pi}_{31}$. It can also be show that $\hat{\Pi}_{34} > \hat{\Pi}_{32}$, so we only need to compare Π_{11} , Π_{12} , $\hat{\Pi}_{12}$, Π_{32} and $\hat{\Pi}_{34}$. Notice that

$$\Pi_{11} > \Pi_{12} \iff q < q_2^*$$

$$\Pi_{11} > \hat{\Pi}_{12} \iff \alpha < \frac{1}{q} \alpha_1^*$$

$$\Pi_{11} > \Pi_{32} \iff q > q_1^*$$

$$\Pi_{11} > \hat{\Pi}_{34} \iff \alpha < \frac{1}{q} \alpha_4^* (1 - q_1^*)$$

$$\Pi_{12} > \hat{\Pi}_{12} \iff \alpha < \alpha_2^*$$

$$\Pi_{12} > \Pi_{32} \iff q < \frac{1}{2} (q_1^* + q_2^*)$$

$$\Pi_{12} > \hat{\Pi}_{34} \iff \alpha < (1 - \frac{1}{q}q_1^*)\alpha_3^*$$

$$\hat{\Pi}_{12} > \Pi_{32} \iff \alpha > (\frac{q_1^*}{q} - 1)\alpha_2^* + \frac{1}{q}\alpha_1^*$$

$$\hat{\Pi}_{12} > \hat{\Pi}_{34} \iff \alpha > \frac{1}{q}q_1^*$$

Comparing these expressions:

1. If $q \geq q_1^*$, $q \geq q_2^*$, then $\Pi_{12} \geq \Pi_{11} \geq \Pi_{32}$.

- (a) In this case, if $\alpha \geq \alpha_2^*$ and $\alpha \geq \frac{1}{q}q_1^*$, $\Pi_{12} \leq \hat{\Pi}_{12}$ and $\hat{\Pi}_{12} \geq \hat{\Pi}_{34}$. The optimal strategy is only offering secured loans to $\hat{\Omega}_1^*$.
- (b) If $\alpha \geq \alpha_2^*$ and $\alpha < \frac{1}{q}q_1^*$, or $\alpha < \alpha_2^*$ and $\alpha \geq (1 - \frac{1}{q}q_1^*)\alpha_3^*$, then $\hat{\Pi}_{12} \leq \hat{\Pi}_{34}$ and $\Pi_{12} \leq \hat{\Pi}_{34}$, the optimal strategy is serving $\hat{\Omega}_1^*$, $\Omega_2/\hat{\Omega}_2^*$ and Ω_3 . If $\alpha < \alpha_2^*$ and $\alpha < (1 - \frac{1}{q}q_1^*)\alpha_3^*$, then $\Pi_{12} > \hat{\Pi}_{34}$ and $\Pi_{12} > \hat{\Pi}_{12}$, the optimal strategy is only offering secured loans to Ω_1^* .

2. If $q \geq q_1^*$, $q < q_2^*$, then $\Pi_{11} > \Pi_{12}$ and $\Pi_{11} \geq \Pi_{32}$.

- (a) In this case, if $\alpha \geq \frac{1}{q}q_1^*$ and $\alpha \geq \frac{1}{q}\alpha_1^*$, $\Pi_{11} \leq \hat{\Pi}_{12}$ and $\hat{\Pi}_{12} \geq \hat{\Pi}_{34}$. The optimal strategy is only offering secured loans to $\hat{\Omega}_1^*$.
- (b) If $\alpha \geq \frac{1}{q}\alpha_1^*$ and $\alpha < \frac{1}{q}q_1^*$, or $\alpha < \frac{1}{q}\alpha_1^*$ and $\alpha \geq \frac{1}{q}\alpha_4^*(1 - q_1^*)$, then $\Pi_{11} \leq \hat{\Pi}_{12}$ and $\hat{\Pi}_{12} < \hat{\Pi}_{34}$, the optimal strategy is serving $\hat{\Omega}_1^*$, $\Omega_2/\hat{\Omega}_2^*$ and Ω_3 .
- (c) If $\alpha < \frac{1}{q}\alpha_1^*$ and $\alpha < \frac{1}{q}\alpha_4^*(1 - q_1^*)$, then $\Pi_{11} > \hat{\Pi}_{12}$ and $\Pi_{11} > \hat{\Pi}_{34}$, the optimal strategy is only offering secured loans to Ω_1 .

3. If $q < q_1^*$, then $1 - \frac{1}{q}q_1^* < 0$, so $\Pi_{12} < \hat{\Pi}_{34}$, $\hat{\Pi}_{12} < \hat{\Pi}_{34}$ and $\Pi_{11} < \Pi_{32}$.

- (a) In this case, if $\alpha \geq \frac{1}{q}\alpha_3^* - \alpha_4^*$, then $\Pi_{32} \leq \hat{\Pi}_{34}$, the optimal strategy is the optimal strategy is serving $\hat{\Omega}_1^*$, $\Omega_2/\hat{\Omega}_2^*$ and Ω_3 .

(b) If $\alpha < \frac{1}{q}\alpha_3^* - \alpha_4^*$, then $\Pi_{32} > \hat{\Pi}_{34}$, the optimal strategy is serving Ω_1 , Ω_2^{**} and Ω_3 .