A Q-analogue and a Symmetric Function Analogue of a Result by Carlitz, Scoville and Vaughan

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A Q-analogue and a Symmetric Function Analogue of a Result by Carlitz, Scoville and Vaughan

by

Yifei Li

A dissertation presented to
The Graduate School of Washington University in partial fulfillment of the requirements for the degree of Doctor of Philosophy

May 2019
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Yifei Li
Washington University
May 2019
Dedicated to My Family.
Abstract of the Dissertation

A Q-analogue and a Symmetric Function Analogue of a Result by Carlitz, Scoville and Vaughan

by

Li, Yifei

Doctor of Philosophy in Mathematics,
Washington University in St. Louis, 2019.
Professor John Shareshian, Chair

We derive an equation that is analogous to a well-known symmetric function identity:
\[ \sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0. \]
Here the elementary symmetric function \( e_i \) is the Frobenius characteristic of the representation of \( S_i \) on the top homology of the subset lattice \( B_i \), whereas our identity involves the representation of \( S_n \times S_n \) on the top homology of Segre product of \( B_n \) with itself. We then obtain a q-analogue of a polynomial identity given by Carlitz, Scoville and Vaughan through examining the Segre product of the subspace lattice \( B_n(q) \) with itself. We recognize the connection between the Euler characteristic of the Segre product of \( B_n(q) \) with itself and the representation on the homology of Segre product of \( B_n \) with itself by recovering our polynomial identity from specializing the identity on the representation of \( S_n \times S_n \).
Chapter 1

Introduction

1.1 Motivation and Introduction

Poset (partially ordered set) topology is not only fundamental to many aspects of combinatorics, but it also brings together other branches of mathematics such as commutative algebra, geometry, group theory, representation theory, and topology. This thesis studies poset topology and related representation theory. In particular, we study the representation of the symmetric group on the homology of certain posets that admit edge-lexicographical labelings (EL-labeling in short). Those posets are called EL-shellable posets.

The theory of shellability was first introduced by Schläfli in the nineteenth century for computing the Euler characteristic of a convex polytope [10], and was then widely used in enumerative combinatorics in the late twentieth century (Björner [1], Stanley [15]). A shelling of a simplicial complex is a methodical way of gluing maximal faces together in a well-behaved manner. A poset whose order complex admits a shelling is called a shellable poset and an EL-labeling gives a shelling of the poset. A shelling of $P$ gives useful information about its combinatorial, algebraic and topological properties. Björner’s work [1] gives an understanding of the homotopy type of a shellable poset, which is a wedge of spheres. Later, Björner and Wachs’ work [3] on edge-lexicographical labeling identifies a set of maximal chains of such a poset with those spheres.
The symmetric group $\mathcal{S}_n$ is the group of permutations on the set of $n$ numbers $[n] := \{1, 2, ..., n\}$. We study the EL-shellable posets that have an $\mathcal{S}_n$ action. This action induces a representation of $\mathcal{S}_n$ on the reduced homology group of the poset. Then the Frobenius characteristic maps a representation of $\mathcal{S}_n$ to a homogenous degree $n$ symmetric function. Symmetric functions provide a very convenient way to describe representations of the symmetric groups, hence are very helpful in studying representations.

The introductory Chapter 1 also provides basic definitions and simple examples of a few fundamental concepts that are required to understand this thesis.

In Chapter 2, we define the product Frobenius characteristic map, which takes a representation of $\mathcal{S}_n \times \mathcal{S}_n$ to a symmetric function in two sets of variables. This map serves as a useful tool in studying representations of $\mathcal{S}_n \times \mathcal{S}_n$. We prove that this map is a homomorphism of rings, which is also a key feature of the usual Frobenius characteristic map. Then we derive an analogue of a well-known symmetric function identity, which involves the representation of $\mathcal{S}_n$ on top homology of the Boolean algebra. Our analogue involves the representation of $\mathcal{S}_n \times \mathcal{S}_n$ on the homology of the Segre product of the Boolean algebra with itself.

In Chapter 3, we present our initial finding, Theorem 3.2.8, a $q$-analogue of a result given by Carlitz, Scoville, and Vaughan. Their result gives a combinatorial interpretation of the coefficients of the reciprocal $J_0$ Bessel function. Carlitz, Scoville and Vaughan proved that those coefficients count the number of pairs of permutations of $\mathcal{S}_n$ with no common ascent. Bessel functions are solutions to Bessel differential equations. Our $q$-analogue provides a combinatorial interpretation of the coefficients of the reciprocal $J_{0}^{(1)}$ $q$-Bessel function. They count the number of maximal chains whose labels are pairs of permutations of $\mathcal{S}_n$ with no common ascent in the Segre product of the subspace
lattice with itself. The Segre product of of subspace lattice with itself contains pairs of subspaces with the same dimension. Those coefficients are the Euler characteristic of this Segre product poset.

Lastly, we have a short Chapter 4 demonstrating the relation of the two analogues we obtained in Chapter 2 and 3.

1.2 Preliminaries

1.2.1 Poset and Hasse diagram

A partially ordered set (poset) is a set $P$ together with a binary relation $\leq$ satisfying the following axioms:

- For all $x \in P$, $x \leq x$ (reflexivity).

- If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry).

- If $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

Different from totally ordered set, two elements in a poset may be incomparable. A simple but interesting example, which we will use to illustrate various concepts, is the Boolean algebra. The Boolean algebra $B_n$ is the collection of all subsets of $[n] := \{1, 2, ..., n\}$ ordered by containment. So $\{1, 3\} \leq \{1, 2, 3\}$, but $\{1, 3\}$ and $\{1, 2\}$ are not comparable.

Two elements $x, y \in P$ have an upper bound $u \in P$ if $u$ satisfies $x \leq u$ and $y \leq u$. We call $u$ a least upper bound (or join) of $x$ and $y$, if $u$ is an upper bound of $x$ and $y$, and every upper bound $v$ of $x$ and $y$ satisfies $u \leq v$. The join of $x$ and $y$ (if exist) is unique and is denoted $x \lor y$. The greatest lower bound (or meet) of $x$ and $y$ is defined dually and denoted $x \land y$. 3
A poset is called a **lattice** if every pair of its elements has a join and a meet. A finite lattice clearly has a bottom (smallest) element and a top (largest) element, which are usually written as \( \hat{0} \) and \( \hat{1} \) respectively. The Boolean algebra \( B_n \) is a lattice with \( \hat{0} = \emptyset \) and \( \hat{1} = [n] \). We will also refer to \( B_n \) as the subset lattice. A poset with a top element \( \hat{1} \) and a bottom element \( \hat{0} \) are said to be **bounded**. We define the bounded extension of \( P \) as \( \hat{P} := P \cup \{\hat{0}, \hat{1}\} \). Note that \( \hat{0} \) and \( \hat{1} \) are added even if \( P \) already has a bottom or a top element.

A finite poset \( P \) can be represented by a **Hasse diagram**. Each element of \( P \) is a vertex in the Hasse diagram of \( P \). Let \( x, y \in P \) and \( x < y \). If no element \( z \in P \) satisfies \( x < z < y \), then we say that \( x \) is **covered by** \( y \) and write \( x \preceq y \). This cover relation is represented in the Hasse diagram by an edge connecting vertices \( x \) and \( y \). Figure 1.1 is the Hasse diagram of a small Boolean algebra \( B_3 \). This thesis concerns only finite posets. All posets appearing here after will be assumed to be finite.

Totally ordered subsets of a poset \( P \) are called **chains**. In figure 1.1, the chain \( \emptyset \preceq 1 \preceq 12 \preceq 123 \) is an example of a chain that is **maximal** in \( B_3 \). The **length** \( l(c) \) of a finite chain \( c \) is defined to be \( \#c - 1 \). The **rank** of a finite poset \( P \) is \( \max\{l(c) :
$c$ is a maximal chain of $P$. When every maximal chain of $P$ is of the same length $n$, we call $P$ a **graded** poset of rank $n$. The subset lattice $B_3$ is then graded of rank 3. For a graded poset of rank $n$, we can define a unique **rank function** $\rho : P \to [n] \cup \{0\}$ that satisfies:

- $\rho(\hat{0}) = 0$,
- for $x, y \in P$ and $x < y$, $\rho(y) = \rho(x) + 1$.

The Boolean algebra $B_n$ has a natural rank function that tells us the cardinality of each subset of $[n]$. A good place to learn more about poset structures is R. Stanley’s book [16].

### 1.2.2 The order complex and (co)homology of a poset

To study the topology of a poset, we study the topology of a certain simplicial complex associated with the poset. An **abstract simplicial complex** $\Delta$ [21] on a finite vertex set $V$ is a nonempty collection of subsets (each subset is called a **face** or a **simplex**) of $V$ such that

- every vertex in $V$ is an element of $\Delta$
- if $G \in \Delta$ then every subset of $G$ is also an element of $\Delta$.

For every poset $P$, the **order complex** of $P$, denoted by $\Delta(P)$, can be realized in $\mathbb{R}^n$ using a geometric simplicial complex whose vertices are elements of $P$ and faces are chains of $P$. It is important to note that every geometric simplicial complex is homeomorphic to the geometric realization of the order complex of some poset, thus studying the order complexes does not restrict us to a small subclass of topological spaces. The vertices and
faces in the order complex can be seen geometrically, hence we can study its topological properties. More details on simplicial complexes can be found in [21] and [12]. Let $\bar{B}_n$ denote $B_n - \{\emptyset, [n]\}$. The geometric realization of $\Delta(\bar{B}_n)$ is in fact homeomorphic to a simple and beautiful geometric object, the $(n - 2)$-sphere $S^{n-2}$. See figure 1.2.

![Figure 1.2. Order complex of $\bar{B}_3$, $\Delta(\bar{B}_3) \cong S^1$](image)

Given a face $F \in \Delta$, the **dimension** of $F$, denoted $\dim F$, is $|F| - 1$. The maximal faces are called **facets**. In the case of a order complex of a poset, the dimension of a face in the complex is the length of the associated chain. When the poset is graded of rank, all facets of its order complex are of the same dimension. Such simplicial complexes are said to be **pure**.

The **homology** of a poset $P$ is the simplicial homology of the topological space $\Delta(P)$. Usually when studying poset topology, we are interested in the *reduced* simplicial homology of a poset. In this thesis, we will only deal with reduced homology groups and will just call them homology groups for convenience. Now let us introduce those concepts in terms of chains of the poset. For a more in depth understanding of simplicial homology, please refer to *Algebraic Topology* by Allen Hatcher [7].

Let $k$ be a field or the ring of integers. The $j$th **chain space** of a poset $P$ is defined as follows:

$$C_j(P; k) := \text{k-module freely generated by } j\text{-chains (chains of length } j) \text{ of } P.$$
Elements of $C_j(P; k)$ are of form $\sum_i \alpha_i c_i$, where $\alpha \in k$ and $c_i$'s are $j$-chains of $P$. For a $j$-chain $(x_1 < ... < x_{j+1})$, define the boundary map $\partial_j : C_j(P; k) \rightarrow C_{j-1}(P; k)$ by

$$\partial_j(x_1 < ... < x_{j+1}) = \sum_{i=1}^{j+1} (-1)^i (x_1 < ... \hat{x}_i < ... < x_{j+1}),$$

where the hat $\hat{x}_i$ means omitting the vertex $x_i$. Then we extend $\partial_j$ by linearity. We can easily check that $\partial_{j-1} \partial_j = 0$. The cycle space $Z_j(P; k)$ is defined to be kernel of $\partial_j$ and the boundary space $B_j(P; k)$ is the image of $\partial_{j+1}$. Define the homology of the poset $P$ in the $j$th dimension by

$$\tilde{H}_j(P; k) := Z_j(P; k)/B_j(P; k).$$

An open interval $(s, t)$ of $P$ is the set of all element $u \in P$ such that $s < u < t$. The cohomology of a poset is defined using a coboundary map $\delta_j : C_j(P; k) \rightarrow C_{j+1}(P; k)$ such that for all chains $x_1 < ... < x_j$, $\delta_j(x_1 < ... < x_j) = \sum_{i=1}^{j+1} (-1)^i \sum_{x \in (x_i, x_{i+1})} (x_1 < ... x_{i-1} < x < x_i < ... < x_j)$, where $x_0 = \hat{0} \in \hat{P}$, $x_{j+1} = \hat{1} \in \hat{P}$, and $(x_{i-1}, x_i)$ is an open interval of $P$. The co-cycyle space is defined as $Z^j(P; k) := \ker\delta_j$ and the coboundary space is defined as $B^j(P; k) := \im\delta_{j-1}$. Similar to the homology group, the cohomology of the poset $P$ in the $j$th dimension is

$$\tilde{H}^j(P; k) := Z^j(P; k)/B^j(P; k).$$

Given a simplicial complex $\triangle$ and $F \in \triangle$, let $\langle F \rangle := \{G : G \subseteq F\}$. Then $\triangle$ is said to be shellable if its facets can be arranged in a linear order $F_1, F_2, ..., F_t$ such that $\left( \bigcup_{i=1}^{k-1} \langle F_i \rangle \right) \cap \langle F_k \rangle$ is a pure and $(\dim F_k - 1)$-dimensional for all $k = 2, ..., t$. This ordering of facets is a shelling of $\triangle$. If $\triangle(P)$ is shellable, we say $P$ is shellable.

Shellable posets have the homotopy type of wedges of spheres (See Björner and Wachs [3]). In this case, its homology $\tilde{H}_j(\triangle; \mathbb{Z})$ and cohomology $\tilde{H}^j(\triangle; \mathbb{Z})$ are the same and
both are isomorphic to $\mathbb{Z}^{r_i}$, where $r_i$ is the number of spheres of dimension $i$ [21]. The homology $\tilde{H}_{n-2}(\bar{B}_n)$ is therefore $\mathbb{Z}$. The rich and interesting topological properties of $\triangle(P)$ provide strong motivation for studies on poset topology.

### 1.2.3 Segre product posets

New posets can be formed using existing posets. Given posets $P$ and $Q$, the **direct product** of $P$ and $Q$ is the poset $P \times Q$ on the set $(x, y) : x \in P$ and $y \in Q$ with the poset relation $(x, y) \leq (u, v)$ if only if $x \leq u$ in $P$ and $y \leq v$ in $Q$. Posets studied in this thesis are a form of product poset, called the **Segre product** of posets.

**Definition 1.2.1** (A full definition can be found in [4]) Let $P$ be a pure poset with a rank function $\rho$, then the **Segre product poset** of $P$ with itself, denoted by $P \circ_\rho P$, is defined to be the induced subposet of the product poset $P \times P$ consisting of the pairs $(x, y) \in P \times P$ such that $\rho(x) = \rho(y)$.

One important concept in algebra and poset topology is Cohen-Macaulayness. Cohen-Macaulay posets have very nice structures. In particular, Cohen-Macaulay posets have reduced homology groups concentrated in the top dimension as they are homotopic to a wedge of spheres [1]. The next result follows from Theorem 1 in Björner and Welker [4].

**Proposition 1.2.2** Let $P$ be a pure poset. Let $\rho : P \rightarrow \mathbb{N}$ be the rank function of $P$. If $P$ is Cohen-Macaulay over the field $k$, then the Segre product poset $P \circ_\rho P$ is Cohen-Macaulay over $k$.

In this thesis, $P$ is either the subset lattice $B_n$ or the subspace lattice $B_n(q)$. Both poset are pure with a rank function. The subspace lattice $B_n(q)$ will be introduced later. Because of the Cohen-Macaulayness of $B_n \circ_\rho B_n$ and $B_n(q) \circ_\rho B_n(q)$, those posets are well
behaved, which motivated us to investigate the representation of $\mathcal{S}_n \times \mathcal{S}_n$ on the reduced homology of $B_n \circ \rho B_n$ and related Whitney homology groups.

1.2.4 The symmetric group and group representations

The symmetric group $\mathcal{S}_n$ is the group of permutations on a set of $n$ objects. It is customary to use the set $[n] := \{1, 2, ..., n\}$ of $n$ numbers. A common way to write out a permutation is the one line notation. That is, for $\sigma \in \mathcal{S}_4$, $\sigma = 2314$ means $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$, and $\sigma(4) = 4$. We will only use the one line notation in this thesis for its simplicity and correspondence with labelings of poset chains, which we will introduce later.

A matrix representation of an abstract group can give us better understanding of the group. Let $\text{Mat}_d$ denote the set of all $d \times d$ matrices with entries in $\mathbb{C}$. Let $\text{GL}_d$ be the complex general linear group of degree $d$, which is the group of all invertible (with respect to multiplication) matrices in $\text{Mat}_d$.

**Definition 1.2.3** [13] A matrix representation of a group $G$ is a group homomorphism

$$X : G \rightarrow \text{GL}_d.$$ 

Equivalently, to each $g \in G$ is assigned $X(g) \in \text{Mat}_d$ such that

1. $X(e) = I$ the identity matrix, and

2. $X(gh) = X(g)X(h)$ for all $g, h \in G$.

The representation has **degree**, or **dimension** $d$.

For a matrix representation $X$ of a group $G$, we define the **character** $\chi$ of $X$ by

$$\chi(g) = \text{tr}X(g),$$
where tr denotes the trace of a matrix and \( g \in G \). The character is a key information of a representation. The following are some properties of characters (see Proposition 1.8.5 in Sagan[13]):

1. Let \( d = \text{dim}X \), then \( \chi(e) = d \).

2. Elements in the same conjugacy class of \( G \) have the same character value.

Let us look at a simple example, the defining representation of the symmetric group \( S_n \). Let \( \sigma \in S_n \), define \( X(\sigma) = (x_{i,j})_{n \times n} \) such that

\[
x_{i,j} = \begin{cases} 
1 & \text{if } \sigma(j) = i, \\
0 & \text{otherwise.}
\end{cases}
\]

It can be checked easily that the character value of \( \sigma \) is the number of points fixed by \( \sigma \). Permutations of the same cycle type are in the same conjugacy class of \( S_n \) and they clearly have the same number of fixed points.

Now expand on the idea of Matrix representations. Matrices are essentially linear transformations. We can also represent elements of a group \( G \) by linear transformations of some vector space. For a vector space \( V \), let \( GL(V) \) denote the general linear group of \( V \), i.e. the set of all invertible linear transformations of \( V \) to itself. The group \( GL(V) \) is in fact isomorphic to \( GL_d \) for \( d = \text{dim}V \).

**Definition 1.2.4** (Sagan [13]) Let \( V \) be a vector space and \( G \) be a group. Then \( V \) is a \textbf{G-module} if there is a group homomorphism

\[
\rho : G \rightarrow GL(V).
\]

Equivalently, \( V \) is a \( G \)-module if there is a multiplication, \( g \cdot v \), of elements of \( V \) by elements of \( G \) such that
1. $gv \in V$, 

2. $g(cv + dw) = c(gv) + d(gw)$, 

3. $(gh)v = g(hv)$, and 

4. $ev = v$

for all $g, h \in G; v, w \in V$; and scalars $c, d \in \mathbb{C}$. 

Each group homomorphism gives a $G$-module, which is a representation of $G$. When the group $G$ is clear in the context, we will often omit $G$ and just use “module” for short. The character of a $G$-module $V$ is the character of a matrix representation obtained by choosing a basis for $V$. Though many matrix representations can correspond to one $G$-module $V$, the matrices representing the same element $g$ will be conjugates of each other, hence their trace will be the same. The character of a $G$-module is then well-defined. 

A key result of group representations is that we can break up large representations into smaller representations. The ones that cannot be broken up further are called irreducible representations. Let $V$ be a $G$-module. A submodule of $V$ is a subspace $W$ that is closed under the action of $G$, i.e., $w \in W \Rightarrow gw \in W$ for all $g \in G$. If $W$ is a submodule of $V$, we write $W \leq V$.

**Theorem 1.2.5** (Maschke’s Theorem, see Chapter 8 in James and Liebeck[9] and Theorem 1.5.3 in Sagan[13]) Let $G$ be a finite group and let $V$ be a nonzero $G$-module. Then

$$V = W^{(1)} \oplus W^{(2)} \oplus \cdots \oplus W^{(k)},$$

where each $W^{(i)}$ is an irreducible $G$-submodule of $V$. 

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Maschke’s Theorem is a fundamental result in representation theory, which signifies that every nonzero $G$-module is a direct sum of irreducible $G$-submodules. There are far-reaching consequences of Maschke’s Theorem. For an in depth study on representations of the symmetric group, please see the two texts from B. Sagan[13], and G. James and A. Kerber[8].
Chapter 2

A Symmetric Function Analogue

2.1 The Space of Symmetric Functions and characteristic map

Essentially, symmetric functions are power series invariant under the action of all symmetric groups. In this section, we will provide the definition and some well-known bases of the space of symmetric functions.

Consider an infinite set of variables $x = \{x_1, x_2, x_3, \ldots\}$. Define the formal power series ring to be

$$\mathbb{C}[[x]] := \{ \sum_{n \geq 0} a_n x^n : a_n \in \mathbb{C} \text{ for all } n \}.$$ 

It is a ring with the usual addition and multiplication. We are not concerned with the convergence of those series because we will never substitute a value for $x_i$. The term *formal* is used to indicate that fact. The terms of a power series in $\mathbb{C}[[x]]$ are monomials of forms $x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} \cdots x_i^{\lambda_i}$. The degree of such a monomial is $n$ given $n = \sum_{i} \lambda_i$. For a formal power series $f(x) \in \mathbb{C}[[x]]$, if every monomial in $f(x)$ has degree $n$, we say that $f(x)$ is homogeneous of degree $n$.

Given a permutation $\sigma$ of $\mathbb{N}$, let $\sigma$ act on $f(x) \in \mathbb{C}[[x]]$ by permutating the indices of the variables. That is

$$\sigma f(x_1, x_2, x_3, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, \ldots).$$
A function that is invariant under such action of all permutations of \( N \) is called a \textbf{symmetric function}.

The expression \( n = \sum_i \lambda_i \), in fact, gives a partition of \( n \). A \textbf{partition} is any sequence

\[
\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots)
\]

of non-negative integers and finitely many non-zero terms satisfying:

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \ldots
\]

We do not distinguish two sequences that only differ by the number of zeros in the end.

The positive terms \( \lambda_i \) of the sequence are called \textbf{parts} of a partition \( \lambda \). The \textbf{length} of \( \lambda \), denoted by \( l(\lambda) \), is the number of parts, and the \textbf{weight} of \( \lambda \), denoted by \( |\lambda| \), is the sum \( \sum_i \lambda_i \). If \( |\lambda| = n \), we say \( \lambda \) is a \textbf{partition} of \( n \) or \( \lambda \) partitions \( n \), and we denote this by \( \lambda \vdash n \).

Sometimes we need to know the number of parts of the same size, then it is convenient to write a partition \( \lambda \) in the following way:

\[
\lambda = (1^{m_1}2^{m_2}\ldots r^{m_r}\ldots),
\]

where \( m_i \) is the number of parts of \( \lambda \) that equal \( i \). The number \( m_i \) is called the \textbf{multiplicity} of \( i \) in \( \lambda \). For example \((2, 1, 1)\) is a partition of 4, and we write \((2, 1, 1)\) as \((211)\) or \((1^22)\) for simplicity.

Given a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \), the \textbf{monomial symmetric function} corresponding to \( \lambda \) is

\[
m_\lambda = m_\lambda(x) = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \ldots x_{i_l}^{\lambda_l},
\]

summed over all distinct monomials with exponents \( \lambda_1, \ldots, \lambda_l \). For example,

\[
m_{(21)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_2 x_3^2 + \ldots.
\]
When $\lambda \vdash n$, $m_{\lambda}$ is homogeneous of degree $n$. Clearly, the monomials $m_{\lambda}$ are fixed under the action of any symmetric group.

**Definition 2.1.1** The ring of symmetric functions is

$$\Lambda = \Lambda(x) = \mathbb{C}m_{\lambda},$$

which is the vector space spanned by all the $m_{\lambda}$.

It is easy to verify that $\Lambda$ is in fact a subring of $\mathbb{C}[[x]]$. We can decompose $\Lambda$ as

$$\Lambda = \bigoplus_{n \geq 0} \Lambda^n,$$

where $\Lambda^n$ is the space spanned by degree $n$ monomial symmetric functions $m_{\lambda}$.

**Proposition 2.1.2** The set $\{m_{\lambda} : \lambda \vdash n\}$ forms a basis for $\Lambda^n$, the space of homogeneous degree $n$ symmetric functions. The dimension of $\Lambda^n$ is the number of partitions of $n$.

**Proof**  The $m_{\lambda}$ are independent. \hfill \blacksquare

We will now introduce three more bases for $\Lambda^n$.

**Definition 2.1.3** For an integer $r \geq 1$, the $r$-th elementary symmetric functions $e_r$ is the sum of all square-free monomials of degree $r$. That is

$$e_r = m_{(1^r)} = \sum_{i_1 < i_2 < \cdots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

The $r$-th complete homogeneous symmetric functions is the sum of all monomials of degree $r$. That is

$$h_r = \sum_{\lambda \vdash r} m_{\lambda} = \sum_{i_1 \leq i_2 \leq \cdots \leq i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

The $r$-th power sum symmetric function is the sum of $r$-th powers of all variables. That is

$$p_r = m_{(r)} = \sum_{i \geq 1} x_i^r.$$

We define $e_0 = h_0 = p_0 = 1$. 

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For example, when $r = 2$,

\[ e_2 = x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + \ldots, \]

\[ h_2 = x_1^2 + x_2^2 + x_1x_2 + x_3^2 + x_1x_3 + x_2x_3 + x_4^2 + \ldots, \]

\[ p_2 = x_1^2 + x_2^2 + x_3^2 + \ldots. \]

**Definition 2.1.4** For each partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, define

\[ e_\lambda = e_{\lambda_1}e_{\lambda_2} \ldots, \]

\[ h_\lambda = h_{\lambda_1}h_{\lambda_2} \ldots, \]

\[ p_\lambda = p_{\lambda_1}p_{\lambda_2} \ldots. \]

**Theorem 2.1.5** (Theorem 4.3.7 in Sagan[13], and for a more detailed treatment see Macdonald[11] and Stanley [17] chapter 7.) The following sets are bases for $\Lambda^n$ the space of homogeneous degree $n$ symmetric functions.

1. $\{e_\lambda : \lambda \vdash n\}$.
2. $\{h_\lambda : \lambda \vdash n\}$.
3. $\{p_\lambda : \lambda \vdash n\}$.

Symmetric functions provide a convenient way of describing representations of the symmetric group. One of the properties of representations is that their character values are constant on each conjugacy class of the group. We call functions, that are constant on conjugacy classes, **class functions**. The conjugacy classes of a symmetric group $S_n$ are determined by cycle types of its elements. Elements having the same cycle type are in the same class. Each cycle type corresponds to a partition of $n$. For instance, given
\( \sigma = (12)(34)(5) \in S_5 \) written in cycle notation, the cycle type of \( \sigma \) is \((221)\), which is a partition of 5. For an element \( \sigma \in S_n \) with cycle type \( \mu \vdash n \), we write \( \text{type}(\sigma) = \mu \). We now have that each partition of \( n \) corresponds to a conjugacy class of \( S_n \).

Let \( R^n \) be the space of class functions on \( S_n \). Given a partition \( \mu = (1^{m_1}2^{m_2} \ldots) \) of \( n \), \( z_\mu := \prod_{i=1}^{n} i^{m_i} m_i! \).

**Definition 2.1.6** \( \text{The (Frobenius) characteristic map} \) is \( ch^n : R^n \to \Lambda^n \) defined by

\[
ch^n(\chi) = \sum_{\mu \vdash n} z_{\mu}^{-1} \chi_\mu p_\mu,
\]

where \( \chi_\mu \) is the value of \( \chi \) on the class \( \mu \) and \( p_\mu \) is the power sum symmetric function.

A very important basis for \( \Lambda^n \) is the **Schur functions**. Schur functions \( s_\lambda \) are the Frobenius characteristics of irreducible \( S_n \)-modules (see Sagan [13] and Stanley [17] chapter 7). Here we will not give a definition of the Schur functions as there are many ways to define them using different approaches, and Schur functions are not used in the main theorems of this thesis.

Now consider \( R = \bigoplus_n R^n \). Let \( ch = \bigoplus ch^n \). Let \( V \) and \( W \) be representations of \( S_m \) and \( S_n \) with characters \( f \) and \( g \). Then \( f \otimes g \) is the character of \( V \otimes W \). Define the **induction product** \( f \circ g \) as

\[
f \circ g = f \otimes g \uparrow_{S_m \times S_n}^{S_{m+n}}.
\]

A fundamental property of the characteristic map is the following:

**Proposition 2.1.7** (Stanley [17, Proposition 7.18.2]) \( \text{The Frobenius characteristic map} \)

\( ch : R \to \Lambda \) is a bijective ring homomorphism, i.e., \( ch \) is one-to-one and onto, and satisfies

\[
ch(f \circ g) = ch(f)ch(g).
\]
The significance of $ch$ being a ring isomorphism is that now we can have products of characters. The induction product is the ring operation. Notice that while the tensor product $f \otimes g$ is a character of $S_m \times S_n$, the induction product $f \circ g$ is a character of $S_{m+n}$.

The Frobenius characteristic map is often used to study a representation of the symmetric group. For instance, let $S_n$ act on the top homology of the proper part of the boolean algebra, $\bar{B}_n := B_n - \{\hat{0}, \hat{1}\}$. This action induces a representation of $S_n$. Let $\chi$ be the character of this representation. Then

$$ch(\chi) = e_n,$$

where $e_n$ is the elementary symmetric function (See Example 1.3 in Sundaram [19]).

### 2.2 The product Frobenius characteristic map

In this section we define a product Frobenius characteristic map to help understand representations of $S_m \times S_n$. Let us consider two sets of variables $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$.

We use $\Lambda(x) = \oplus_n \Lambda^n(x)$ and $\Lambda(y) = \oplus_n \Lambda^n(y)$ to denote the rings of symmetric functions in variables $(x_1, x_2, ...)$ and $(y_1, y_2, ...)$. respectively.

**Definition 2.2.1** Let $\chi$ be a class function on $S_m \times S_n$. The product Frobenius characteristic map $ch : R \times R \rightarrow \Lambda(x) \times \Lambda(y)$ is defined as:

$$ch(\chi) = \sum_{(\mu, \lambda) \vdash (m,n)} z_{\mu}^{-1} z_{\lambda}^{-1} \chi(\mu, \lambda) p_\mu(x) p_\lambda(y),$$

where $\chi(\mu, \lambda)$ is the value of $\chi$ on the class $(\mu, \lambda)$ and $p_\mu, p_\lambda$ are power sum symmetric functions. The class $(\mu, \lambda)$ is indexed by a partition $\mu$ of $m$ and a partition $\lambda$ of $n$ that tell us the cycle shapes of elements of $S_m$ and $S_n$ respectively.
The irreducible representations of $S_m \times S_n$ are of the form $A^{(i)} \otimes B^{(j)}$, where $A^{(i)}$ and $B^{(j)}$ are each irreducibles of $S_m$ and $S_n$ respectively (Sagan [13, Theorem 1.11.3]). A representation $V$ of $S_m \times S_n$ can then be decomposed into a sum of irreducibles of $S_m \times S_n$.

**Proposition 2.2.2** Let $V$, a representation of $S_m \times S_n$, have the following decomposition:

$$V = \bigoplus_{i,j} c_{ij} A^{(i)} \otimes B^{(j)},$$

where $A^{(i)}$'s and $B^{(j)}$'s are irreducible representations of $S_m$ and $S_n$ respectively and $c_{ij}$ are non-negative integers. Then the product Frobenius characteristic of $V$ is

$$ch(V) = \sum_{i,j} c_{ij} ch^m(A^{(i)})(x)ch^n(B^{(j)})(y).$$

Here $ch^m(A^{(i)})(x)$ is the usual Frobenius characteristic of $A^{(i)}$ in the variable $x$ and $ch^n(B^{(j)})(y)$ is defined similarly.

**Proof** Let $\chi$ denote the character of $V$. Let $\chi^{Ai}$ and $\chi^{Bj}$ be the characters of $A^{(i)}$ and $B^{(j)}$ respectively. Then $\chi = \sum_{i,j} c_{ij} \chi^{Ai} \otimes \chi^{Bj}$. Definition 2.2.1 gives us

$$ch(V) = \sum_{(\mu,\lambda) \vdash (m,n)} z_\mu^{-1} z_\lambda^{-1} \chi(\mu, \lambda)p_\mu(x)p_\lambda(y).$$

The character $\chi(\mu, \lambda) = \sum_{i,j} c_{ij} (\chi^{Ai}(\mu) \otimes \chi^{Bj}(\mu, \lambda))$ by [13, Corollary 1.9.4]. And $\chi^{Ai} \otimes \chi^{Bj}(\mu, \lambda) = \chi^{Ai\mu} \chi^{Bj\lambda}$ due to [13, Theorem 1.11.2]. Then

$$ch(V) = \sum_{(\mu,\lambda) \vdash (m,n)} z_\mu^{-1} z_\lambda^{-1} \sum_{i,j} c_{ij} \chi^{Ai\mu} \chi^{Bj\lambda} p_\mu(x)p_\lambda(y)$$

$$= \sum_{i,j} c_{ij} \left( \sum_{\mu \vdash m} z_\mu^{-1} \chi^{Ai\mu}(x) \right) \left( \sum_{\lambda \vdash n} z_\lambda^{-1} \chi^{Bj\lambda}(y) \right)$$

$$= \sum_{i,j} c_{ij} ch^m(A^{(i)})(x)ch^n(B^{(j)})(y).$$
Because the product Frobenius characteristic map is basically an extension of the usual (Frobenius) characteristic map, we keep the notation $ch$ for product Frobenius characteristic map even though $ch$ was previously defined to be $\oplus_n ch^n$ in various literature (Sagan [13], Stanley [17]). The meaning of $ch$ will be clear in the given context.

**Remark 2.2.3** Given $V$ a representation of $S_m$ with character $f$ and $W$ a representation of $S_n$ with character $g$, let $V = \oplus_i a_i A^{(i)}$ and $W = \oplus_j b_j B^{(j)}$ be their decompositions into irreducibles. It can be easily verified that the product Frobenius characteristic $ch(f \otimes g) = ch(f)(x)ch(g)(y)$. It is a symmetric function in $\Lambda^m \times \Lambda^n$, while the usual Frobenius characteristic $ch(f \circ g) = ch(f)(x)ch(g)(x)$ is a symmetric function in $\Lambda^{m+n}$.

We would like the product Frobenius characteristic map to be a homomorphism of rings as well. Given a class function $\psi$ on $S_k \times S_l$ and a class function $\phi$ on $S_m \times S_n$, $\psi \otimes \phi$ is a class function on $(S_k \times S_l) \times (S_m \times S_n)$. We want to produce a class function on $S_{k+m} \times S_{l+n}$.

**Definition 2.2.4** For $\psi$ and $\phi$ as given above, we define the induction product $\psi \circ \phi$ to be $\psi \otimes \phi \uparrow^{S_{k+m} \times S_{l+n}}_{(S_k \times S_l) \times (S_m \times S_n)}$.

The following proposition will show that this induction product is a ring operation that makes $R \times R$ into an algebra.

**Proposition 2.2.5** Assume given $\psi$ a class function on $S_k \times S_l$, and $\phi$ a class function on $S_m \times S_n$. The product Frobenius characteristic map $ch : R \times R \rightarrow \Lambda(x) \times \Lambda(y)$ is an algebra isomorphism, i.e., $ch$ is one-to-one and onto, and satisfies

$$ch(\psi \circ \phi) = ch(\psi)ch(\phi).$$

Before proving this proposition, we need to first establish a lemma:
Lemma 2.2.6 If $f$ is a class function on $S_k \times S_m$ and $g$ is a class function on $S_l \times S_n$, then

$$f \otimes g \uparrow_{(S_k \times S_m) \times (S_l \times S_n)}^{S_{k+m} \times S_{l+n}} = f \uparrow_{S_k \times S_m}^{S_{k+m}} \otimes g \uparrow_{S_l \times S_n}^{S_{l+n}}.$$ 

Proof Suppose $S_k \times S_m < S_{k+m}$ has coset representatives $\{s_1, s_2, ..., s_q\}$, $q = \binom{k+m}{k}$, and $S_l \times S_n < S_{l+n}$ has coset representatives $\{t_1, t_2, ..., t_r\}$, $r = \binom{l+n}{l}$. Then $\{(s_i, t_j)\}$, $i \in [q]$, $j \in [r]$, is a set of coset representatives for $(S_k \times S_m) \times (S_l \times S_n) < S_{k+m} \times S_{l+n}$. Given $H \leq G$ and a class function $\phi$ on $H$, define $\phi^\circ$ to be the class function on $G$ such that $\phi^\circ(\sigma) = \phi(\sigma)$ if $\sigma \in H$ and $\phi^\circ(\sigma) = 0$ if $\sigma \notin H$. For the class functions $f$ and $g$ given in the lemma, we have $(f \otimes g)^\circ = f^\circ \otimes g^\circ$. For $(\sigma_{k+m}, \sigma_{l+n}) \in S_{k+m} \times S_{l+n},$

$$f \otimes g \uparrow_{(S_k \times S_m) \times (S_l \times S_n)}^{S_{k+m} \times S_{l+n}} ((\sigma_{k+m}, \sigma_{l+n})) = \sum_{i,j} (f \otimes g)^\circ ((s_i^{-1}, t_j^{-1}) (\sigma_{k+m}, \sigma_{l+n}) (s_i, t_j))$$

$$= \sum_i f^\circ (s_i^{-1} \sigma_{k+m} s_i) \sum_j g^\circ (t_j^{-1} \sigma_{l+n} t_j)$$

$$= f \uparrow_{S_k \times S_m}^{S_{k+m}} (\sigma_{k+m}) g \uparrow_{S_l \times S_n}^{S_{l+n}} (\sigma_{l+n})$$

$$= f \uparrow_{S_k \times S_m}^{S_{k+m}} \otimes g \uparrow_{S_l \times S_n}^{S_{l+n}} ((\sigma_{k+m}, \sigma_{l+n})).$$

The second and fourth equalities come from [13, Theorem 1.11.2].

Proof of Proposition 2.2.5

Since the Frobenius characteristic map is bijective, so is the product Frobenius characteristic map. It is sufficient to show that the map is a homomorphism. Suppose $\psi = \sum a_{ij} \psi_k^{(i)} \otimes \psi_l^{(j)}$ with $\psi_k^{(i)}$’s and $\psi_l^{(j)}$’s are irreducible characters of representations...
of \( S_k \) and \( S_l \) respectively. Similarly, \( \phi = \sum_{u,v} b_{uv} \phi_m^{(u)} \otimes \phi_n^{(v)} \). For any \( \sigma_k \in S_k, \sigma_l \in S_l, \omega_m \in S_m, \) and \( \omega_n \in S_n, \) as above, we have

\[
\psi \otimes \phi((\sigma_k, \sigma_l), (\omega_m, \omega_n)) = \left( \sum_{i,j} a_{ij} \psi_{k}^{(i)}(\sigma_k) \psi_{l}^{(j)}(\sigma_l) \right) \left( \sum_{u,v} b_{uv} \phi_{m}^{(u)}(\omega_m) \phi_{n}^{(v)}(\omega_n) \right)
\]

\[
= \sum_{i,j,u,v} a_{ij} b_{uv} \psi_{k}^{(i)}(\sigma_k) \phi_{m}^{(u)}(\omega_m) \psi_{l}^{(j)}(\sigma_l) \phi_{n}^{(v)}(\omega_n)
\]

\[
= \sum_{i,j,u,v} a_{ij} b_{uv} \left( \psi_{k}^{(i)} \otimes \phi_{m}^{(u)} \otimes \psi_{l}^{(j)} \otimes \phi_{n}^{(v)} \right)(\sigma_k, \omega_m, \sigma_l, \omega_n).
\]

Thus, \( \psi \otimes \phi = \sum_{i,j,u,v} a_{ij} b_{uv} \left( \psi_{k}^{(i)} \otimes \phi_{m}^{(u)} \otimes \psi_{l}^{(j)} \otimes \phi_{n}^{(v)} \right). \) So

\[
\psi \circ \phi = \psi \otimes \phi \uparrow_{S_k \times S_l \times S_m \times S_n} \uparrow_{(S_k \times S_l) \times (S_m \times S_n)}
\]

\[
= \sum_{i,j,u,v} a_{ij} b_{uv} \left( \psi_{k}^{(i)} \otimes \phi_{m}^{(u)} \otimes \psi_{l}^{(j)} \otimes \phi_{n}^{(v)} \right) \uparrow_{S_k \times S_m \times S_l \times S_n}
\]

\[
= \sum_{i,j,u,v} a_{ij} b_{uv} \left( \psi_{k}^{(i)} \otimes \phi_{m}^{(u)} \right) \uparrow_{S_k \times S_m} \otimes \left( \psi_{l}^{(j)} \otimes \phi_{n}^{(v)} \right) \uparrow_{S_l \times S_n}
\]

\[
= \sum_{i,j,u,v} a_{ij} b_{uv} \left( \psi_{k}^{(i)} \circ \phi_{m}^{(u)} \right) \otimes \left( \psi_{l}^{(j)} \circ \phi_{n}^{(v)} \right).
\]
by Lemma 2.2.6. Now take the product Frobenius characteristic of both sides of the above equation. For clarity, we keep track of variables $x$ and $y$. By Remark 2.2.3 and then Proposition 2.1.7 we get

$$
ch(\psi \circ \phi)(x, y) = \sum_{i,j,u,v} a_{ij}b_{uv}ch(\psi_k^{(i)} \circ \phi_m^{(u)})(x)ch(\psi_l^{(j)} \circ \phi_n^{(v)})(y)
$$

$$
= \sum_{i,j,u,v} a_{ij}b_{uv}ch(\psi_k^{(i)})(x)ch(\phi_m^{(u)})(x)ch(\psi_l^{(j)})(y)ch(\phi_n^{(v)})(y)
$$

$$
= \sum_{i,j} a_{ij}ch(\psi_k^{(i)})(x)ch(\psi_l^{(j)})(y)\sum_{u,v} b_{uv}ch(\phi_m^{(u)})(x)ch(\phi_n^{(v)})(y)
$$

$$
= ch(\psi)(x, y)ch(\phi)(x, y)
$$

\[\square\]

2.3 A symmetric function analogue

Using the product Frobenius characteristic map, we derive an equation that is analogous to a well-known symmetric function identity (see Stanley [17, equation (7.13)]): for $n \geq 1$,

$$
\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0.
$$

The thing to note is that the elementary symmetric function $e_i$ is the Frobenius characteristic of the representation of $S_i$ on the top homology of $\tilde{B}_i$ (see the example in the end of section 2.1). Our analogue will involve the representation of $S_n \times S_n$ on the top homology of the proper part of Segre product poset $B_n \circ_{\rho_n} B_n$.

In the proof of the following theorem, we used Whitney homology technique, which was introduced by Sundaram [19] for pure posets and then generalized by Wachs [20] for semipure posets.
Theorem 2.3.1 For the subset lattice $B_n$ with rank function $\rho_n$, let $P_n$ be the proper part of the Segre product poset $B_n \circ_{\rho_n} B_n$. Write $S_n$ for the symmetric group on $[n]$. The action of $S_n \times S_n$ induces a representation on the reduced top homology of $P_n$. Let $\text{ch}(\widetilde{H}_{n-2}(P_n))$ be the product Frobenious characteristic of this representation. Then

$$
\sum_{i=0}^{n} (-1)^i h_{n-i}(x) h_{n-i}(y) \text{ch}(\widetilde{H}_{i-2}(P_i)) = 0, 
$$

(2.3.1)

where $h_k$’s are the complete homogeneous symmetric functions.

Proof Let $Q$ be $P_n \cup \hat{0}$, which is Cohen-Macaulay. We consider the Whitney homology of $Q$ as discussed in Sundaram [19]. The action of $S_n \times S_n$ on $Q$ induces a representation of $S_n \times S_n$ on the reduced top homology of $Q$ and its Whitney homology groups. From the work of Sundaram on Whitney homology (Sundaram [18, 19], Wachs [21]), we know that

$$
\widetilde{H}_{n-2}(P_n) \cong \bigoplus_{r=0}^{n-1} (-1)^{n-1+r} \text{WH}_r(Q).
$$

Let $x$ be a rank $r$ element of $Q$. Then the stabilizer of $x$ is the young subgroup $(S_r \times S_{n-r}) \times (S_r \times S_{n-r})$. We can view the Whitney homology groups as representations. The poset $Q$ is Cohen-Macaulay and has a bottom element $\hat{0}$, the Whitney homology of $Q$ is defined to be

$$
\text{WH}_r(Q) = \bigoplus_{x \in Q_r / (S_n \times S_n)} \widetilde{H}_{r-2}(\hat{0}, x) \uparrow_{(S_r \times S_{n-r}) \times (S_r \times S_{n-r})}^{S_n \times S_n},
$$

where $Q_r$ is the set of rank $r$ elements in $Q$ and $Q_r / (S_n \times S_n)$ is a set of orbit representatives (see Lecture 4.4 in Wachs’ Poset Topology [21]). The action of $S_n \times S_n$ on $Q_r$ is transitive. So the contribution of the $r$th Whitney homology to $\widetilde{H}_{n-2}(P_n)$ is the induced representation $\widetilde{H}_{r-2}(\hat{0}, x) \uparrow^{S_n \times S_n}_{(S_r \times S_{n-r}) \times (S_r \times S_{n-r})}$ for any $x$ in $Q_r$. The open interval $(\hat{0}, x)$ is isomorphic to the poset $P_r$. We then have
\( \tilde{H}_{n-2}(P_n) \cong \bigoplus_{r=0}^{n-1} (-1)^{n-1+r} \tilde{H}_{r-2}(P_r) \uparrow_{(S_r \times S_{n-r}) \times (S_r \times S_{n-r})}^{S_n \times S_n} \).

Taking the product Frobenius characteristic of both sides of the above equation,

\[
ch(\tilde{H}_{n-2}(P_n)) = \sum_{r=0}^{n-1} (-1)^{n-1+r} ch(\tilde{H}_{r-2}(P_r) \uparrow_{(S_r \times S_{n-r}) \times (S_r \times S_{n-r})}^{S_n \times S_n}). \tag{2.3.2}
\]

Now we would like to relate \( ch(\tilde{H}_{r-2}(P_r)) \) with the product Frobenius characteristic of the representation induced to \( S_n \times S_n \). Let \( \psi_r \) be the character of the \((S_r \times S_r)\)-module \( \tilde{H}_{r-2}(P_r) \). Write \( 1_{S_{n-r} \times S_{n-r}} \) for the character of the trivial representation of \( S_{n-r} \times S_{n-r} \).

When viewing \( \tilde{H}_{r-2}(P_r) \) as a \((S_r \times S_{n-r}) \times (S_r \times S_{n-r})\)-module, its character equals \( \psi_r \otimes 1_{S_{n-r} \times S_{n-r}} \) (Sagan, [13, Theorem 1.11.2]). Let \( \psi_r \otimes 1_{S_{n-r} \times S_{n-r}} \) denote the induction product of \( \psi_r \) and \( 1_{S_{n-r} \times S_{n-r}} \). Then

\[
\tilde{H}_{r-2}(P_r) \uparrow_{(S_r \times S_{n-r}) \times (S_r \times S_{n-r})}^{S_n \times S_n} = \psi_r \otimes 1_{S_{n-r} \times S_{n-r}} \uparrow_{(S_r \times S_{n-r}) \times (S_r \times S_{n-r})}^{S_n \times S_n}
= \psi_r \circ 1_{S_{n-r} \times S_{n-r}}.
\]

It follows from Proposition 2.2.5 that the product Frobenius characteristic

\[
ch(\psi_r \circ 1_{S_{n-r} \times S_{n-r}}) = ch(\psi_r)ch(1_{S_{n-r} \times S_{n-r}}).
\]

Thus, equation (2.3.2) becomes

\[
ch(\tilde{H}_{n-2}(P_n)) = \sum_{r=0}^{n-1} (-1)^{n-1+r} ch(\tilde{H}_{r-2}(P_r))ch(1_{S_{n-r} \times S_{n-r}})
= \sum_{r=0}^{n-1} (-1)^{n-1+r} ch(\tilde{H}_{r-2}(P_r))(x)ch(1_{S_{n-r}})(y). \tag{2.3.3}
\]

It is known that the Frobenius characteristic of the trivial representation of \( S_n \) is \( h_n \) (See Equation (7.85) in Stanley [17]). Multiplying both sides of equation (2.3.3) by \((-1)^{n-1}\), we get
\((-1)^{n-1} ch(\tilde{H}_{n-2}(P_n)) = \sum_{r=0}^{n-1} (-1)^r ch(\tilde{H}_{r-2}(P_r)) h_{n-r}(x) h_{n-r}(y)\).

Finally, we conclude that
\[
\sum_{i=0}^{n} (-1)^i h_{n-i}(x) h_{n-i}(y) ch(\tilde{H}_{i-2}(P_i)) = 0.
\]

Theorem 2.3.1 was motivated by our initial finding, which we will present in the next chapter. Our initial finding can be seen as a specialized case of equation (2.3.1), suggesting the truth of Theorem 2.3.1.
Chapter 3

$q$-analogue of a result by Carlitz, Scoville, and Vaughan

3.1 Introduction — Carlitz, Scoville, and Vaughan’s result

Consider the power series $f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!n!}$ and define the numbers $\omega_0, \omega_1, \omega_2, \ldots$ by

$$\frac{1}{f(z)} = \sum_{n=0}^{\infty} \omega_n \frac{z^n}{n!n!}.$$  

It follows quickly from the definition that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 \omega_k = 0. \quad (3.1.1)$$

Given $\sigma \in \mathcal{S}_n$ a permutation of $[n] := \{1, 2, \ldots, n\}$. We call $i \in [n-1]$ an ascent of $\sigma$ if $\sigma(i) < \sigma(i + 1)$. Carlitz, Scoville and Vaughan [5] proved that the number $\omega_k$ in equation (3.1.1) is the number of pairs of permutations of $\mathcal{S}_k$ with no common ascent. For example, $\omega_2 = 3$. The three pairs of permutations of $[2]$ with no common ascent written in one-line notation are $(12, 21), (21, 12), (21, 21)$. The Bessel function $J_0(z)$ is essentially $f(z^2)$ (See Section 1.2 in Gasper and Rahman [6]). Carlitz, Scoville and Vaughan’s result provided a combinatorial interpretation of the coefficients $\omega_k$ in the reciprocal Bessel function, which in turn, gives a method to compute those coefficients.
Recall that $[n]_q := q^{n-1} + q^{n-2} + ... + 1$ is the $q$-analogue of the natural number $n$ and $[n]_q! := \prod_{i=1}^{n} [i]_q$. Then the $q$-analogue of $\binom{n}{k}$ is $\binom{n}{k}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}$. For a permutation $\sigma \in \mathcal{S}_n$, the **inversion statistic** is defined by

$$inv(\sigma) := |\{(i, j) : 1 \leq i < j \leq n \text{ and } \sigma(i) > \sigma(j)\}|.$$ 

In this chapter we will prove the following $q$-analogue of Carlitz, Scoville and Vaughan’s result. Let $\mathcal{D}_n$ denote the set $\{(\sigma, \omega) \in \mathcal{S}_n \times \mathcal{S}_n \mid \sigma \text{ and } \omega \text{ have no common ascent}\}$. Define $W_i(q) = \sum_{(\sigma, \omega) \in \mathcal{D}_i} q^{inv(\sigma) + inv(\omega)}$, then

$$\sum_{i=0}^{n} \binom{n}{i}^2 (-1)^i W_i(q) = 0. \quad (3.1.2)$$

Put $F(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{[n]_q! [n]_q!}$. The function $F\left(\frac{z^2}{2(1-q)}\right)$ is the $q$-Bessel function $J_0^{(1)}(z; q)$. The $q$-Bessel functions were first introduced by F. H. Jackson in 1905 and can be found in later literature (see Gasper and Rahman [6]). By Equation (3.1.2), the polynomials $W_n(q)$ satisfy $\frac{1}{F(z)} = \sum_{n=0}^{\infty} W_n(q) \frac{z^n}{[n]_q! [n]_q!}$. We have thus found a combinatorial meaning for coefficients of the reciprocal $q$-Bessel function $1/J_0^{(1)}(z; q)$, and a formula for computing those coefficients. The coefficient $W_n(q)$ is in fact the reduced Euler characteristic of the Segre product poset $B_n(q) \circ \rho B_n(q)$.

### 3.2 A $q$-analogue of Carlitz, Scoville, and Vaughan’s result

Recall the definition of $B_n(q)$. Let $q$ be a prime power and $\mathbb{F}_q$ be the finite field of $q$ elements. Consider the $n$-dimensional vector space $\mathbb{F}_q^n$ and its subspaces, then $B_n(q)$ is the lattice of those subspaces ordered by inclusion. The poset $B_n(q)$ is a geometric lattice whose every subspace is a span of its atoms (Stanley [16, Example 3.10.2]). It is graded with a rank function $\rho(W) :=$ the dimension of the subspace $W$. 
An edge labeling of a bounded poset $P$ is a map $\lambda : \mathcal{E}(P) \rightarrow \Sigma$, where $\mathcal{E}(P)$ is the set of edges of the Hasse diagram of $P$ and $\Sigma$ is some poset. A maximal chain $c = (\hat{0} \prec x_1 \prec \cdots \prec x_t \prec \hat{1})$ is increasing if $\lambda(\hat{0}, x_1) < \lambda(x_1, x_2) < \cdots < \lambda(x_t, \hat{1})$ in $\Sigma$, and decreasing if $\lambda(\hat{0}, x_1) \geq \lambda(x_1, x_2) \geq \cdots \geq \lambda(x_t, \hat{1})$ in $\Sigma$. A chain $c$ is associated with a word

$$\lambda(c) = \lambda(\hat{0}, x_1)\lambda(x_1, x_2)\cdots\lambda(x_t, \hat{1}).$$

If $\lambda(c_1)$ lexicographically precedes $\lambda(c_2)$, we say that $c_1$ lexicographically precedes $c_2$ and we denote this by $c_1 <_L c_2$. Let us review the definition of an EL-labeling of a poset.

**Definition 3.2.1** (Björner and Wachs [2, Definition 2.1]) An edge labeling is called an EL-labeling (edge lexicographical labeling) if for every interval $[x, y]$ in $P$,

1. there is a unique increasing maximal chain $c$ in $[x, y]$, and
2. $c <_L c'$ for all other maximal chains $c'$ in $[x, y]$.

It is well known that $B_n(q)$ admits an EL-labeling (See [21, Exercise 3.4.7]). The lexicographic ordering of all maximal chains in an EL-labeling gives a shelling order of all facets of the order complex. A poset that admits an EL-labeling is shellable. Here we describe a specific EL-labeling of $B_n(q)$, which we will use to prove our results, in the following steps:

1. For a 1-dimensional subspace $X$ of $\mathbb{F}_q^n$ (an atom of $B_n(q)$), let $x$, a row vector, be a basis element of $X$. Let $A$ denote the set of all atoms of the subspace lattice $B_n(q)$. We define a map $f : A \rightarrow [n]$, $f(X) =$ the index of the right most non-zero coordinate of $x$. For example, in $B_3(3)$, if $X = \text{span of } \{<1,0,1>\}$, $Y = \text{span of } \{<2,1,0>\}$, $f(X) = 3$ and $f(Y) = 2$.

2. For $X$ any subspace of $\mathbb{F}_q^n$, let $A(X)$ denote the set of atoms whose span is $X$. Let $Y$ be an element of $B_n(q)$ that covers $X$, then $A(Y) \supset A(X)$. Denote the set
Figure 3.1. An EL-labeling of $B_2(2)$

$f(A(Y)) \setminus f(A(X))$ by $\mathcal{L}$. Let $\rho$ be the rank function of $B_n(q)$, which is defined by the dimensions of the subspaces. Because $\rho(Y) - \rho(X) = 1$, the set $\mathcal{L}$, which is a subset of $[n]$, must have exactly one element. This number is the new dimension added to $Y$. And that will be the label of the edge $(X, Y)$.

Example 3.2.2 Figure 3.1 is an edge labeling of the subspace lattice $B_2(2)$ following the rules described above. We will show in the following proposition that this labeling is an EL-labeling.

Proposition 3.2.3 The edge labeling described above is an EL-labeling on the subspace lattice $B_n(q)$.

Proof Edges in the same chain cannot take duplicate labels since $\mathbb{F}_q^n$ is $n$-dimensional and any maximal chain must take all labels in $\{1, 2, \ldots, n\}$. Let $[X, Y]$ be a closed interval in $B_n(q)$. All maximal chains of $[X, Y]$ will take labels from the set $\mathcal{L} := f(A(Y)) \setminus f(A(X))$. Let $0 < a_1 < a_2 < \cdots < a_l \leq n$ be all the elements of $\mathcal{L}$. For each $i$, $1 \leq i \leq l$, there is a 1-dimensional subspace $V_i$ of $\mathbb{F}_q^n$ with $f(V_i) = a_i$ and $V_i \lor X$, the join of $V_i$ and $X$, is in $[X, Y]$. The chain $c = (X \ll X \lor V_1 \ll \cdots \ll X \lor V_1 \lor V_2 \lor \cdots \lor V_l = Y)$ is an increasing maximal chain of $[X, Y]$. Any other 1-dimensional subspace $V'_i$ satisfying
\[ f(V'_i) = a_i \] and \[ X \lor V_1 \lor \cdots \lor V_{i-1} \lor V'_i \in [X,Y] \] must equal \[ X \lor V_1 \lor \cdots \lor V_i. \] Since there is only one way to arrange the \( a_i \)'s increasingly, \( c \) satisfies definition 3.2.1 condition (1).

Suppose there is another maximal chain \( c' = (X = W_0 \prec W_1 \prec \cdots \prec W_l = Y) \). Let

\[ f(A(W_i)) \setminus f(A(W_{i-1})) = b_i \] for all \( i \in [l] \). Let \( k, 1 \leq k \leq l \), be the smallest integer such that \( b_k \neq a_k \). We know that \( b_k \) must be in \( \mathcal{L} \) and \( b_k \neq a_1, a_2, \ldots, a_k \). Also \( a_1, a_2, \ldots, a_k \) are the smallest \( k \) elements of \( \mathcal{L} \) arranged increasingly. It follows immediately that \( b_k > a_k \). Therefore condition (2) in the definition of EL-labeling is also satisfied.

Under this EL-labeling, each maximal chain of the subspace lattice \( B_n(q) \) can then be identified with a permutation \( \sigma \) of \( S_n \). See section 3.1 for the definition of inversion statistic \( \text{inv}(\sigma) \).

**Lemma 3.2.4** The number of maximal chains of \( B_n(q) \) assigned label \( \sigma \in S_n \) is \( q^{\text{inv}(\sigma)} \).

**Proof** Let \( \sigma \in S_n \), for each 1-dimensional subspace of \( \mathbb{F}_q^n \), we can take the vector whose right most non-zero coordinate is 1 as its basis element. For each \( i \in [n-1] \), let \( \text{inv}(\sigma(i)) \) denote the number of pairs \( (i,j) \) such that \( 1 \leq i < j \leq n \) and \( \sigma(i) > \sigma(j) \). The number of ways to choose an atom \( W_1 \) such that the edge \((0,W_1)\) takes label \( \sigma(1) \) is \( q^{\text{inv}(\sigma(1))} \).

Let \( k \in [n] \), assume the chain \( 0 \prec W_1 \prec \cdots \prec W_{k-1} \), has label \( \sigma(1)\sigma(2)\cdots\sigma(k-1) \). For each \( i \in [k-1] \), pick an atom \( V_i \in A(W_{k-1}) \) with \( f(V_i) = \sigma(i) \) and \( v_i \) the basis element of \( V_i \). The vectors \( v_1, v_2, \ldots, v_{k-1} \) are linearly independent hence form a basis of \( W_{k-1} \).

In order for the edge \((W_{k-1},W_k)\) to take label \( \sigma(k) \), \( W_k \) needs to be the join of \( W_{k-1} \) and an atom whose basis element, call it \( v_k \), has 1 on the \( \sigma(k) \)th coordinate and all 0’s after the \( \sigma(k) \)th coordinate. Then \( v_1, v_2, \ldots, v_k \) will form a basis for \( W_k \). So we need to find the number of ways to choose a \( v_k \) that each results in a distinct \( W_k \).
The vector $e_{\sigma(k)} = < 0, ..., 0, 1, 0, ... 0 >$ who has 1 on the $\sigma(k)$th coordinate and 0 everywhere else certainly is a choice for $v_k$. For each $j$, such that $1 \leq k < j \leq n$ and $\sigma(k) > \sigma(j)$, the edge label $\sigma(j)$ comes after $\sigma(k)$ in this chain label $\sigma$. So $W_{k-1}$ has no vectors whose right most non-zero coordinate is the $\sigma(j)$th. And the $\sigma(j)$th coordinate appears before the $\sigma(k)$th in a vector. So varying the $\sigma(j)$th coordinate of $e_{\sigma(k)}$ will produce new vectors that are not in the span of $\{v_1, ..., v_{k-1}, e_{\sigma(k)}\}$. There are $\text{inv}(\sigma(k))$ choices for $j$, and for each $j$, there are $q$ choices for the value of the $j$th coordinate. Each choice will produce a distinct $v_k$ that is linearly independent of $v_1, v_2, ..., v_{k-1}$, thus a distinct $W_k$. Therefore for any given chain $0 < W_1 < ... < W_{k-1}$ assigned label $\sigma(1)\sigma(2)...\sigma(k-1)$, there are $q^{\text{inv}(\sigma(k))}$ choices for $W_k$ such that the edge $(W_{k-1}, W_k)$ takes label $\sigma(k)$. Hence the number of maximal chains assigned label $\sigma$ is

$$\prod_{i=1}^{i=n} q^{\text{inv}(\sigma(i))} = q^{\sum_{i=1}^{i=n} \text{inv}(\sigma(i))} = q^{\text{inv}(\sigma)}.$$ 

The following theorem from Björner and Wachs is essential to connecting the permutations of $S_n$ with the Segre product poset $B_n(q) \circ_\rho B_n(q)$:

**Theorem 3.2.5** (Björner and Wachs [3, Theorem 4.1], see also Wachs [21, Theorem 3.2.4]). Suppose $P$ is a poset for which $\hat{P}$ admits an EL-labeling. Then $P$ has the homotopy type of a wedge of spheres, where the number of $i$-spheres is the number of decreasing maximal $(i+2)$-chains of $\hat{P}$.

Now consider the Segre product of $B_n(q)$ with itself. Denote the proper part of this Segre product by $P_n(q)$. Using the EL-labeling of $B_n(q)$ described right after definition 3.2.1, the Segre product poset $B_n(q) \circ_\rho B_n(q)$ admits an edge-labeling in which the labels are ordered pairs from the poset $[n] \times [n]$. A label $(i, j) \in [n] \times [n] \leq (k, l)$ if and only if $i \leq k$ and $j \leq l$. It is easy to verify that this labeling of $B_n(q) \circ_\rho B_n(q)$ is an EL-labeling. The decreasing chains are labeled with pairs of permutations with no common ascent.
Given a pair of permutations \((\sigma, \omega)\), the number of decreasing maximal chains assigned label \((\sigma, \omega)\) is \(q^{\text{inv}(\sigma)} \cdot q^{\text{inv}(\omega)}\) from Lemma 3.2.4. Recall that \(D_n\) denotes the set of pairs of permutations \((\sigma, \omega)\) \(\in S_n \times S_n\) with no common ascent. We immediately arrive at the following proposition:

**Proposition 3.2.6** Let \(W_n(q)\) be the total number of decreasing maximal chains of \(B_n(q) \circ \rho B_n(q)\). Then

\[
W_n(q) = \sum_{(\sigma, \omega) \in D_n} q^{\text{inv}(\sigma) + \text{inv}(\omega)}.
\]

**Remark 3.2.7** The Segre product poset \(B_n(q) \circ B_n(q)\) is the \(q\)-analogue of the Segre product poset \(B_n \circ B_n\), agreeing with the formal definition of a \(q\)-analogue in R. Simion’s paper [14]. She showed that the \(q\)-analogue of an EL-shellable poset is also EL-shellable. This particular EL-labeling of \(B_n(q) \circ B_n(q)\) provided intuition and a combinatorial interpretation for \(W_n(q)\).

**Theorem 3.2.8** Let \(P_n(q)\) be the proper part of the Segre product poset \(B_n(q) \circ \rho B_n(q)\). Let \([\begin{array}{c} n \\ i \end{array}]_q\) be the \(q\)-analogue of \(\binom{n}{i}\) and \(W_n(q)\) be the total number of decreasing maximal chains of \(B_n(q) \circ \rho B_n(q)\). Then

\[
\sum_{i=0}^{n} \left[ \begin{array}{c} n \\ i \end{array} \right]_q^2 (-1)^i W_i(q) = 0. \tag{3.2.1}
\]

**Proof** The poset \(P_n(q)\) is pure. By Theorem 3.2.5, \(P_n(q)\) has the homotopy type of a wedge of \((n - 2)\)-spheres, and the number of decreasing maximal \((n - 2)\)-chains is the number of spheres. Then following from Proposition 3.2.6, \(W_n(q)\) is the number of \(n - 2\)-dimensional faces of \(P_n(q)\) and the reduced Euler characteristic of \(\Delta(P_n(q))\) is

\[
\tilde{\chi}(\Delta(P_n(q))) = (-1)^n W_n(q). \tag{3.2.2}
\]
We know that the Möbius number of a poset is the same as its reduced Euler Characteristic by Philip Hall’s theorem (Stanley [16, Proposition 3.8.6]), which gives us
\[
\mu_{\tilde{P}_n(q)}(\hat{0}, \hat{1}) = (-1)^n W_n(q) = \tilde{\chi}(\Delta(P_n(q))). \tag{3.2.3}
\]

On the other hand, by the definition of the möbius function,
\[
\mu(\hat{0}, \hat{1}) = - \sum_{\hat{0} \leq x < \hat{1}} \mu(\hat{0}, x).
\]
Each \( x \) in \( P_n(q) \) is a subspace of \( \mathbb{F}_q^n \times \mathbb{F}_q^n \), which is the product of two \( k \)-dimensional subspaces \( X_1, X_2 \) of \( \mathbb{F}_q^n \) for some \( k \) with \( 0 \leq k < n \). But the intervals \( [\hat{0}, X_i] \) and \( [\hat{0}, X_2] \) are isomorphic to the poset \( B_k(q) \), hence \( \mu(\hat{0}, x) \) is just \( \mu_{\tilde{P}_k(q)}(\hat{0}, \hat{1}) \), where \( P_k(q) = B_k(q) \circ \rho B_k(q) \setminus \{\hat{0}, \hat{1}\} \). The number of \( k \)-dimensional subspaces of \( \mathbb{F}_q^n \) is \( \left[ \frac{n}{k} \right]_q \) (Stanley [16, Proposition 1.7.2]), the \( q \)-analogue of \( \binom{n}{k} \). So the number of distinct \( x = (X_1, X_2) \) where \( X_1 \) and \( X_2 \) are \( k \)-dimensional subspaces is \( \left[ \frac{n}{k} \right]^2_q \). Therefore we have
\[
\mu_{\tilde{P}_n(q)}(\hat{0}, \hat{1}) = - \sum_{i=0}^{n-1} \left[ \frac{n}{i} \right]^2_q \mu_{\tilde{P}_i(q)}(\hat{0}, \hat{1}) = - \sum_{i=0}^{n-1} \left[ \frac{n}{i} \right]^2_q (-1)^i W_i(q).
\]
Consequently,
\[
\sum_{i=0}^{n} \left[ \frac{n}{i} \right]^2_q (-1)^i W_i(q) = 0.
\]
By Proposition 3.2.6, \( W_i(q) = \sum_{(\sigma, \omega) \in \mathcal{D}_i} q^{(\text{inv}(\sigma) + \text{inv}(\omega))} \) is the number of decreasing maximal chains of \( P_i(q) \), where \( \mathcal{D}_i \) denotes the set of pairs of permutations \( (\sigma, \omega) \in S_i \times S_i \) with no common ascent.

**Corollary 3.2.9** The Euler characteristic of the Segre product of the subspace lattice \( B_n(q) \circ \rho B_n(q) \) is \( (-1)^n W_n(q) \).

**Proof** See equation (3.2.2) in the proof of theorem 3.2.8. 

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3.3 An alternative proof of Carlitz, Scoville, and Vaughan’s result

In [5], Carlitz, Scoville and Vaughan gave the coefficients $\omega_k$ of the reciprocal of the Bessel function $J_0(z)$ a combinatorial explanation. They showed that $\omega_k$ is the number of pairs of $k$-permutations with no common ascent. When letting $q = 1$ in our $q$-analogue (3.2.1), the subspaces of $\mathbb{F}_q^n$ become subsets of $\{1, 2, \ldots, n\}$. The value $W_n(1) = \sum_{(\sigma, \omega) \in D_n} 1^{\text{inv}(\sigma) + \text{inv}(\omega)}$ simply counts the number of pairs of permutations of $[n]$ with no common ascent, i.e. $\omega_n$. We then obtain the above result from Carlitz, Scoville and Vaughan.

The proof of theorem 3.2.8 can also be easily adapted to an alternative proof of Carlitz, Scoville and Vaughan’s result (3.1.1) by changing $B_n(q)$ to $B_n$, using $P_n$ instead of $P_n(q)$ to denote the Segre product, and recognizing that the intervals in the alternating sum for the Möbius number of $\hat{P}_n$ are isomorphic to smaller subset lattices $B_i$’s. Carlitz, Scoville and Vaughan’s proof in [5] included general cases where occurrences of common ascent are allowed. Our proof provides a less technical approach by utilizing Björner and Wachs’ work on shellability and poset homology [3].
Chapter 4

The Connection of Two Analogues

4.1 Specialization of Symmetric Functions

Let \( ps : \Lambda \rightarrow \mathbb{Q}[q] \) be the \textbf{stable principal specialization}. For a symmetric function \( f(x_1, x_2, x_3, \ldots) \), \( ps(f) \) is defined to be \( f(1, q, q^2, \ldots) \). A summary of the specializations of different bases for the symmetric functions can be found in Stanley’s \textit{Enumerative Combinatorics vol. 2} [17, proposition 7.8.3]. Consider a symmetric function \( f \) in two sets of variables \((x_1, x_2, \ldots)\) and \((y_1, y_2, \ldots)\). We take the stable principal specialization of \( f \) in each set of variables, that is substituting \((1, q, q^2, \ldots)\) for both \((x_1, x_2, \ldots)\) and \((y_1, y_2, \ldots)\).

The product Frobenius characteristic of the \( S_n \times S_n \)-module \( \tilde{H}_{n-2}(P_n) \) is a symmetric function in two sets of variables. Then it is natural to ask what we can say about its specialization. It turns out that \( ps(ch(\tilde{H}_{n-2}(P_n))) \) has an interesting relation with the Euler characteristic of the Segre product poset \( B_n(q) \circ \rho \circ B_n(q) \).

4.2 The Connection

Recall that \( P_n \) is the proper part of the Segre product of the subset lattice \( B_n \) with itself. The product Frobenius characteristic of the \( S_n \times S_n \)-module \( \tilde{H}_{n-2}(P_n) \) has an innate connection with \( W_n(q) \). The following theorem provides an equation that connects the stable principal specialization of \( ch(\tilde{H}_{n-2}(P_n)) \) and the Euler characteristic \( W_n(q) \).
Theorem 4.2.1 Let $P_n$ be the proper part of Segre product $B_n \circ \rho B_n$ and $S_n$ the symmetric group. The action of $S_n \times S_n$ on $B_n \circ B_n$ induces a representation on the reduced top homology of $P_n$. Let $(-1)^n W_n(q)$ be the Euler characteristic of the Segre product $B_n(q) \circ \rho B_n(q)$. For a symmetric function $f$ in two sets of variables $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$, the stable principal specialization $ps(f)$ specializes both $x_i$ and $y_i$ to $q^{i-1}$.

Then

$$ps(ch(\tilde{H}_{n-2}(P_n))) = \frac{W_n(q)}{\prod_{i=1}^{n} (1 - q^i)^2},$$

where $ch(V)$ is the product Frobenius characteristic of a $S_n \times S_n$-module $V$.

Proof We will use induction. The base cases $n = 2$ and $n = 3$ can be verified by hand.

$$ps(ch(\tilde{H}_0(P_2))) = \frac{q^2 + 2q}{(1 - q)^2(1 - q^2)^2} = \frac{W_2(q)}{(1 - q)^2(1 - q^2)^2}$$

and

$$ps(ch(\tilde{H}_1(P_3))) = \frac{q^6 + 4q^5 + 6q^4 + 6q^3 + 2q^2}{(1 - q)^2(1 - q^2)^2(1 - q^3)^2} = \frac{W_3(q)}{(1 - q)^2(1 - q^2)^2(1 - q^3)^2}.$$  

Assume that the statement is true for $P_i$, $i = 1, \ldots, n - 1$. Now let us consider the reduced top homology of $P_n$. Equation (2.3.1) gives us a way to express $ch(\tilde{H}_{n-2}(P_n))$ in terms of the product Frobenius characteristic of smaller posets. That is

$$ch(\tilde{H}_{n-2}(P_n)) = \sum_{i=0}^{n-1} (-1)^{n-1+i} h_{n-i}(x) h_{n-i}(y) ch(\tilde{H}_{i-2}(P_i))$$

Then we take the stable principal specialization of both sides of equation (4.2.2). We know from Stanley’s Enumerative Combinatorics vol. 2 that $ps(h_n) = \prod_{i=1}^{n} \frac{1}{1 - q^i}$ [17]. It follows from our induction hypothesis that
\[
\text{ps}(ch(\tilde{H}_{n-2}(P_n))) = \sum_{i=0}^{n-1} (-1)^{n-1+i} \text{ps}(ch(\tilde{H}_{i-2}(P_i))) \prod_{j=1}^{n-i} \frac{1}{(1 - q^j)^2}
\]

\[
= \sum_{i=0}^{n-1} (-1)^{n-1+i} \frac{W_i(q)}{\prod_{k=1}^{n} (1 - q^k)^2} \prod_{j=1}^{n-i} \frac{1}{(1 - q^j)^2}
\]

\[
= \frac{1}{\prod_{k=1}^{n} (1 - q^k)^2} \sum_{i=0}^{n-1} (-1)^{n-1+i} W_i(q) \prod_{j=1}^{n} \frac{1}{(1 - q^j)^2}
\]

\[
= \frac{1}{\prod_{k=1}^{n} (1 - q^k)^2} \sum_{i=0}^{n-1} (-1)^{n-1+i} W_i(q) \left[ \frac{n}{i} \right]_q^2.
\]

Finally, using the identity involving the Euler characteristic \( W_n(q) \) given in Theorem 3.2.8, we obtain

\[
\text{ps}(ch(\tilde{H}_n(P_n))) = \frac{W_n(q)}{\prod_{j=1}^{n} (1 - q^j)^2}.
\]

When we take the stable principal specialization of Equation (2.3.1) in Theorem 2.3.1, we obtain Equation (3.2.1) in Theorem 3.2.8 and Equation (4.2.1) in Theorem 4.2.1. Therefore we can view the symmetric function analogue identity (2.3.1) as a generalization of our \( q \)-analogue identity (3.2.1) to a symmetric group representation result.
Bibliography


