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Index Theory for Invariant Elliptic Operators on Manifolds with Proper Cocompact Group Actions

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by

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ABSTRACT OF THE DISSERTATION

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In this thesis, we study G-invariant elliptic operators, and in particular Dirac operators, on the space of invariant sections of a Hermitian bundle over a (non-compact) manifold with a proper and cocompact Lie group action. We provide a canonical way to define the Hilbert space of invariant sections for proper and cocompact actions, and prove that the G-invariant Dirac operators, and more generally, elliptic operators, are Fredholm for the Hilbert space we constructed. Using the framework developed in this thesis, we give a new proof of a generalized Lichnerowicz Vanishing Theorem for proper cocompact group actions as an application.

Chapter 1

Introduction

1.1 Summary

A fundamental result for elliptic operators on closed (compact and without boundary) manifolds is that they are Fredholm operators. In 1963, M. F. Atiyah and I. M. Singer proved their famous theorem using the method of K-theory and pseudo-differential operators. The Atiyah-Singer index theorem connects the analytical and topological invariants of an elliptic operator and gives a topological formula for the index of an elliptic operator on any closed oriented smooth manifold [AS63; AS68].

It is then very natural to ask what one could say for elliptic operators on non-compact manifolds. Many works has been done when the manifold admits a proper cocompact G-action. For example, Mathai and Zhang in [MZ10] defined an invariant index (the Mathai-Zhang index) for G-equivariant elliptic differential operators on Sobolev spaces of invariant sections. Tang, Yao and Zhang showed a generalized de Rham Laplace-Beltrami operator on manifolds with a proper cocompact action is elliptic [TYZ13]. Ma and Zhang, in their paper which solves the

Vergne conjecture [MZ14], studied the index problem for Dirac operators on manifolds with boundaries. Wang in 2014 proved an L^2-index theorem for elliptic pseudodifferential operators invariant under proper cocompact actions of unimodular locally compact groups [Wan14]. [PPT15] provides a unification of several well-known equivariant index theorems for proper compact actions of Lie groupoids. In Hochs and Mathai's paper [HM17], they showed that the invariant, transversally L^2-index of deformed Dirac operator on twisted spinor bundles over a Spin^c-manifold can be well-defined.

In this thesis, we study G-invariant elliptic operators, and in particular Dirac operators, on the space of invariant sections of a Hermitian bundle over a (non-compact) manifold with a proper and cocompact Lie group action. Instead of using local cut-off functions to define Sobolev space as in [MZ10], we show a canonical way of defining the Hilbert space of invariant sections for proper and cocompact actions, and prove that the G-invariant Dirac operators, and more generally, elliptic operators, are Fredholm for the Hilbert space. The thesis is constructed as follows. In Chapter 1 we recall some basic definitions and results of Lie groups, vector bundles and Dirac operators. Their detailed proofs and other relavent discussions can be found in [LM89] and [BGV04]. In Chapter 2 we will explain how to define the Hilbert spaces in a canonical way using the proper and cocompact action when the manifold is not necessarily compact. We will show that self-adjointness still holds for G-invariant Dirac operators in the new space we construct. In the second half of Chapter 2 we state and prove our main theorem by assuming the existence of parametrix. In Chapter 3 and 4 we show the existence of parametrix by reviewing the theory of pseudo-differential operators first. Finally in Chapter 5 we providede a new proof of a generalized Lichnerowicz vanishing theorem as an application.

1.2 Lie Group Actions

In this section we recall some definitions of Lie group and Lie group action on manifolds and bundles. Throughout this thesis, G is assumed to be a Lie group and M a smooth manifold of dimension n.

Definition 1.1. A measure μ on G is called left (or right) invariant if $\mu(S) = \mu(gS)$ (or $\mu(S) = \mu(Sg)$ for right invariance) for all measurable sets $S \subseteq G$. According to Haar's theorem, a left (or right) measure always exists for any Lie group and is unique up to multiplication by a positive constant. Such an invariant measure is called a Haar measure.

For a left Haar measure μ on M, $\nu(S) := \mu(Sg)$ is also left invariant. So from uniqueness we know

$$\mu(Sg) = \Delta(g)\mu(S) \tag{1.1}$$

for any measurable $S \subseteq G$ with $\Delta(g) > 0$.

Definition 1.2. The function $\Delta : G \to \mathbb{R}_+$ defined in (1.1) is called the *modular function* of G. It is a group homomorphism from G to the multiplicative group $\mathbb{R}_+$. A Lie group is called *unimodular* if Δ is identically equal to 1.

Next let us recall some definitions of Lie group actions on manifolds and vector bundles.

Definition 1.3. Let $G \curvearrowright M$ be a (left) Lie group G-action on a smooth manifold M. The action is called *proper* if the map $\rho : G \times M \to M \times M$ defined by $(g, x) \mapsto (gx, x)$ is proper (the inverse of a compact set is compact); and *cocompact* if the quotient space M/G is compact, or equivalently, if there is a compact subset $K \subseteq M$ such that the image of K under the G-action covers M.

Definition 1.4. Let $\mathcal{E}$ be a (complex) vector bundle endowed with a (left) G-action. The bundle is called *G-equivariant* if

(i) It is compatible with the G-action on the base manifold M:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{g} & \mathcal{E} \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi} \\ M & \xrightarrow{g} & M \end{array}$$

for any $g \in G$. If $x \in M$ and $v \in \mathcal{E}_x$, we can write the G-action as

$$g(x, v) = (gx, \gamma_g \cdot v),$$

where $\gamma_g : \mathcal{E}_x \to \mathcal{E}_{gx}$ is the map on fibers induced by the G-action on $\mathcal{E}$.

(ii) The induced map $\gamma_g : \mathcal{E}_x \to \mathcal{E}_{gx}$ is linear.

The above definition of G-equivariant bundles gives rise to the G-action on sections of $\mathcal{E}$.

Definition 1.5. Assume $\mathcal{E}$ is a G-equivariant vector bundle and let $s \in \Gamma(\mathcal{E})$ be a section of $\mathcal{E}$. The G-action on $\mathcal{E}$ induces a G-action on $\Gamma(\mathcal{E})$:

$$gs(x) := \gamma_g \cdot s(g^{-1}x).$$

To avoid any confusion, for the rest part of this thesis, $gs(x)$ will always denote the evaluation of the section gs at x; and we will use the induced fiber map γ_g to specify the group action on the bundle.

Suppose $\mathcal{E}$ is a G-equivariant (complex) vector bundle, and $\mathcal{E}^*$ the dual bundle of $\mathcal{E}$. Then $\mathcal{E}^*$ also carries a G-action and is G-equivariant under such action. More precisely, for $(x, \xi) \in \mathcal{E}^*$, the group action is given by $g(x,\xi) := (gx, \tilde{\gamma}_g \cdot \xi)$, where $\tilde{\gamma}_g : \mathcal{E}_x^* \to \mathcal{E}_{gx}^*$ satisfies

$$\langle \gamma_g \cdot v, \tilde{\gamma}_g \cdot \xi \rangle_{gx} = \langle v, \xi \rangle_x, \quad \text{for all } v \in \mathcal{E}_x. \tag{1.2}$$

The wedge bracket $\langle \cdot, \cdot \rangle$ is the pairing on the fiber.

In the end of this section, let us recall the definitions of principal bundle and associated bundle.

Definition 1.6. A *principal G-bundle $P \xrightarrow{\pi} M$* is a fiber bundle P with a right G-action on the fibers satisfying

$$\pi(p \cdot g) = \pi(p)$$

for all $p \in P$ and $g \in G$, and such that the G-action is free and transitive on the fibers. Therefore each fiber of P is diffeomorphic to G itself, and its base $M \cong P/G$.

Definition 1.7. If P is a principal G-bundle and E is a left G-space. The *associated bundle* $P \times_G E$ is the fiber bundle $(P \times E)/\sim$, where the equivalence relation is defined by

$$(p \cdot g, f) \sim (p, g \cdot f)$$

for all $p \in P$, $g \in G$ and $f \in E$. In particular, if E is a vector space which carries a linear representation of G, then $P \times_G E$ is a vector bundle over M.

1.3 Differential Operators

Definition 1.8. Let $\mathcal{E}$ be a complex vector bundles over M. A *differential operator of order* k is a linear map $D : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$, where $\Gamma(\mathcal{E})$ denotes the space of smooth section of $\mathcal{E}$, satisfies the following property. In any coordinate neighborhood U of M and local trivialization

$\mathcal{E}|_U \cong U \times \mathbb{C}^p$, the operator D has the form:

$$D = \sum_{|\alpha| \leqslant k} A_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

where each $A^\alpha(x)$ is a smooth matrix-valued function.

Next we define the principal symbol of a differential operator. A detailed and more general discussion about the symbols of pseudo-differential operators can be found in Chapter 3.

Definition 1.9. Suppose D is a differential operator of order k. Let ξ be a covector in the cotangent plane T_x^*M and in local coordinates $\xi = \sum_i \xi_i \, \mathrm{d}x^i$. The principal symbol σ'_D : $T^*M \to \mathrm{End}(\mathcal{E})$ assigns an endomorphism of $\mathcal{E}_x$ to each point $(x, \xi) \in T_x^*M$:

$$\sigma'_D(x, \xi) := \mathbf{i}^k \sum_{|\alpha|=k} A_\alpha \xi^\alpha. \tag{1.3}$$

Definition 1.10. A differential operator D is called *elliptic* if its principal symbol $\sigma'_D(x, \xi)$ is invertible for all $\xi \neq 0$.

1.4 Dirac Operators

In this section we introduce a specific class of elliptic operators: the Dirac operators. We will first recall the definition of connections and Clifford bundles.

1.4.1 Connections

Definition 1.11. Let $\mathcal{E}$ be a vector bundle over M. Recall that a *connection* on $\mathcal{E}$ is a differential operator

$$\nabla : \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E})$$

such that $\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s$ for any $f \in C^\infty(M)$ and $s \in \Gamma(\mathcal{E})$. Moreover, if $\mathcal{E}$ is equipped with an inner product (or Hermitian product for complex vector bundles), then we call a connection *Riemannian* if

$$\mathrm{d}(s,t) = (\nabla s, t) + (s, \nabla t)$$

for any $s, t \in \Gamma(\mathcal{E})$.

In particular, for a connection ∇ on the tangent bundle, the *torsion tensor* is a vector-valued 2-form:

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$$

for two vector fields X, Y. A connection is called *torsion-free* if the torsion tensor $T = 0$. For a Riemannian manifold, there exists a Riemannian, torsion-free connection on the tangent bundle, called the *Levi-Civita* connection, and will be denoted by $\boldsymbol{\nabla}$.

1.4.2 Clifford Algebra and Clifford Bundle

Suppose that V is a real vector space with a quadratic form $q(\cdot,\cdot)$. Let

$$\mathcal{T}(V) := \sum_{r=0}^{\infty} \Big(\bigotimes^r V \Big)$$

be the tensor algebra (over $\mathbb{R}$) of V and $\mathcal{I}_q(V)$ the ideal in $\mathcal{T}(V)$ generated by elements of the form

$$\mathcal{I}_q(V) := \langle v \otimes v + q(v,v)\,\mathbb{1} \,:\, v \in V \rangle.$$

Definition 1.12. The Clifford algebra of V, denoted by $C\ell(V,q)$, is defined to be the quotient

$$C\ell(V,q) := \mathcal{T}(V)/\mathcal{I}_q(V). \tag{1.4}$$

Alternatively, $C\ell(V,q)$ is the unital algebra generated by the vector space V subject to the following relations

$$v \cdot w + w \cdot v = -2q(v,w)\,\mathbb{1}.$$

We denote $\iota : V = \bigotimes^1 V \hookrightarrow \mathcal{T}(V) \xrightarrow{\pi} C\ell(V,q)$ the natural embedding of V into $C\ell(V,q)$.

The Clifford algebra plays an essential role in studying spin geometry and Dirac operators. Next we summarize some of its important properties. A more comprehensive discussion can be found in [LM89, Chap. 1].

Proposition 1.13 (Universal Property, [LM89, Chap. 1, Proposition 1.1])**.** *Given an associative unital algebra $\mathcal{A}$ (over $\mathbb{R}$) and a linear map $f : V \to \mathcal{A}$ such that $f(v) \cdot f(v) + q(v,v)\,\mathbb{1} = 0$ for all $v \in V$, there exists a unique algebra homomorphism $\tilde{f} : C\ell(V,q) \to \mathcal{A}$ such that f factors through $\tilde{f}$, i.e., $f = \tilde{f} \circ \iota$.*

A direct consequence of Proposition 1.13 is that any transformation

$$\alpha \in \mathrm{O}(V,q) := \{f \in \mathrm{GL}(V) : f^*q = q\}$$

of V extends to an automorphism of $C\ell(V,q)$. Take $\alpha(v) := -v$ and we still denote the extended automorphism by $\alpha : C\ell(V,q) \to C\ell(V,q)$. Since $\alpha^2 = I$, we can decompose $C\ell(V,q)$ into

$$C\ell(V,q) = C\ell^0(V,q) \oplus C\ell^1(V,q) \tag{1.5}$$

where $C\ell^i(V,q) = \{\psi \in C\ell(V,q) : \alpha(\psi) = (-1)^i\psi\}$ are the eigenspaces of α satisfying

$$C\ell^i(V,q) \cdot C\ell^j(V,q) \subseteq C\ell^{(i+j)}(V,q), \tag{1.6}$$

where the indices are taken modulo 2. A decomposition (1.5) of an algebra satisfying (1.6) is called a $\mathbb{Z}_2$-*grading* and the algebra is called a $\mathbb{Z}_2$-*graded algebra*. So $C\ell(V,q)$ is a $\mathbb{Z}_2$-graded algebra. The subalgebra $C\ell^0(V,q)$ is called the *even part* and the subspace $C\ell^1(V,q)$ is called the *odd part*.

The next proposition shows that the Clifford algebra as vector space can be identified with the exterior algebra:

Proposition 1.14 (see [LM89, Chap. 1, Proposition 1.2 & 1.3]). *Suppose $\{e_1, \dots, e_n\}$ is an orthogonal basis of (V,q), then $C\ell(V,q)$ is a real vector space with a basis*

$$\{e_0 := \mathbb{1}, e_{i_1}e_{i_2}\cdots e_{i_k} : 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n\}.$$

Compared to the exterior algebra $\bigwedge^* V$, we conclude that there is a canonical vector space isomorphism

$$\begin{array}{ccc} \bigwedge^* V & \xrightarrow{\cong} & C\ell(V,q), \\ e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} & \mapsto & e_{i_1}e_{i_2}\cdots e_{i_k}. \end{array}$$

Note that the exterior algebra is also $\mathbb{Z}_2$-graded: $\bigwedge^* V = \bigwedge^{\text{even}} V \oplus \bigwedge^{\text{odd}} V$. So the canonical isomorphism above also preserves the gradings.

Using Clifford algebras, we can consider the Clifford bundles and Clifford modules:

Definition 1.15. Let M be a Riemannian manifold of dimension n, the *Clifford bundle* $C\ell(M)$ is the fiber bundle of Clifford algebras over M such that each fiber $C\ell_x(M)$ is the Clifford algebra $C\ell(T_x^*M)$ of the Euclidean space T_x^*M.

Alternatively, from Proposition 1.13 we know that $O(n)$ acts on $C\ell(T_x^*M)$, so the Clifford module can be represented as an associated bundle: $C\ell(M) = O(M) \times_{O(n)} C\ell(\mathbb{R}^n)$, where $O(M)$ is the orthogonal frame bundle, a principal $O(n)$-bundle over M. Using the associated bundle definition, it is clear that the Levi-Civita connection on M (or equivalently, $O(M)$) is extended to $C\ell(M)$.

Definition 1.16. Let M be a Riemannian manifold and $\mathcal{E}$ a $\mathbb{Z}_2$-graded vector bundle (i.e. $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$) over M with a (real or complex) metric such that $\mathcal{E}_0$ and $\mathcal{E}_1$ are orthogonal. We call $\mathcal{E}$ a *Clifford module* if there is an action $C\ell(M) \otimes \mathcal{E} \to \mathcal{E}$ which makes each fiber $\mathcal{E}_x$ a module over the algebra $C\ell(T_x^*M)$ and that the action respects the $\mathbb{Z}_2$-grading:

$$C\ell^i(M) \cdot \mathcal{E}_j \subseteq \mathcal{E}_{i+j \bmod 2}.$$

In the rest of this thesis, we will denote the action of $C\ell(M)$ on $\mathcal{E}$ by $\mathbf{c}(a)s$ for $a \in \Gamma(C\ell(M))$ and $s \in \Gamma(\mathcal{E})$. The actions $\mathbf{c}$ is called the *Clifford multiplication*.

Remark 1. From now on, we will denote $\mathcal{E}_0$ by $\mathcal{E}^+$ and $\mathcal{E}_1$ by $\mathcal{E}^-$.

1.4.3 Definition of Dirac Operators

Definition 1.17. Assume $\mathcal{E}$ is a Clifford module equipped with a Rimannian connection ∇. The *Dirac operator* $\not{D} : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$, is a first-order differential operator defined by

$$\not{D}s := \sum \mathbf{c}(e^i)\nabla_{e_i}s, \tag{1.7}$$

for $s \in \Gamma(\mathcal{E})$, where $\{e_i\}$ is a local orthonormal frame and $\{e^j\}$ is the dual coframe.

Formula (1.7) is well-defined (i.e., independent of the choice of local frames) because it can be viewed as a composition:

$$\not{D} : \Gamma(\mathcal{E}) \xrightarrow{\nabla} \Gamma(T^*M \otimes \mathcal{E}) \hookrightarrow \Gamma(C\ell(M) \otimes \mathcal{E}) \to \Gamma(\mathcal{E}).$$

A well-known property of the Dirac operator is that it is elliptic:

Theorem 1.18 ([LM89, Chap. 2, Lemma 5.1]). *Let $\not{D}$ be the Dirac operator on the bundle $\mathcal{E}$ defined above. Then for any $(x, \xi) \in T^*M$ we have that*

$$\sigma'_{\not{D}}(x, \xi) = i\,\mathbf{c}(\xi), \tag{1.8}$$

$$\sigma'_{\not{D}^2}(x, \xi) = \|\xi\|^2. \tag{1.9}$$

In particular, both $\not{D}$ and $\not{D}^2$ are elliptic operators.

We usually impose two more conditions on the Clifford module $\mathcal{E}$ when studying Dirac operators. We first require the Clifford multiplication of covectors to be skew symmetric:

$$(\mathbf{c}(\alpha)s_1, s_2)_x + (s_1, \mathbf{c}(\alpha)s_2)_x = 0 \tag{1.10}$$

for $s_1, s_2 \in \Gamma(\mathcal{E})$ and $\alpha \in T_x^*M$. We also require the connection ∇ on $\mathcal{E}$ to be a module derivation:

$$\nabla(\mathbf{c}(a)s) = \mathbf{c}(\boldsymbol{\nabla} a) + \mathbf{c}(a)\nabla s \tag{1.11}$$

for $a \in C\ell(M)$ and $s \in \Gamma(\mathcal{E})$, where $\boldsymbol{\nabla}$ is the Levi-Civita connection on $C\ell(M)$.

Definition 1.19. A Clifford module $\mathcal{E}$ endowed with a Riemannian connection ∇ satisfies condition (1.10) and (1.11) is called a *Dirac bundle*.

Lemma 1.20 ([LM89, Chap. 2, Proposition 5.3]). *The Dirac operator $\not{D}$ of any Dirac bundle over a Riemannian manifold M is formally self-adjoint, i.e.,*

$$\int_M (\not{D}s_1, s_2)\,\Omega = \int_M (s_1, \not{D}s_2)\,\Omega$$

for any sections $s_1, s_2 \in \Gamma(\mathcal{E})$ with compact supports, where Ω is the Riemannian volume form.

Due to the self-adjointness property, the kernel and cokernel of $\not{D}$ are isomorphic. However, since Clifford multiplication respects grading, it is obvious that the Dirac operator is odd: i.e., $\Gamma(\mathcal{E}^+) \xrightarrow{\not{D}} \Gamma(\mathcal{E}^-)$ and $\Gamma(\mathcal{E}^-) \xrightarrow{\not{D}} \Gamma(\mathcal{E}^+)$. So we can restrict the Dirac operator on subbundles $\mathcal{E}^+$ and $\mathcal{E}^-$ and have:

$$\not{D}_+ : \Gamma(\mathcal{E}^+) \to \Gamma(\mathcal{E}^-) \quad \text{and} \quad \not{D}_- : \Gamma(\mathcal{E}^-) \to \Gamma(\mathcal{E}^+). \tag{1.12}$$

The self-adjointness of $\not{D}$ implies that $\not{D}_-$ is the adjoint of $\not{D}_+$, so $\ker \not{D}_- \cong \operatorname{coker} \not{D}_+$.

Definition 1.21. Let $\not{D}$ be a Dirac operator on a Dirac bundle $\mathcal{E}$ and let $\not{D}_+$ and $\not{D}_-$ be operators in (1.12). Suppose $\dim \ker \not{D}_+$ and $\dim \operatorname{coker} \not{D}_+$ are both finite, then the *index* of $\not{D}_+$ is

$$\operatorname{index} \not{D}_+ = \dim \ker \not{D}_+ - \dim \operatorname{coker} \not{D}_+ = \dim \ker \not{D}_+ - \dim \ker \not{D}_-. \tag{1.13}$$

Definition 1.22 (Fredholm Operator)**.** We call a bounded operator *Fredholm* if it has a finite-dimensional kernel and cokernel. We call a closed unbounded operator *Fredholm* if it has a closed range and a finite-dimensional kernel and cokernel.

It is well-known (see for example [LM89, Chap. 3, Theorem 5.2]) that elliptic operators on a compact manifold are Fredholm. So by Theorem 1.18, the index (1.13) is well-defined for Dirac operators on compact manifolds.

Chapter 2

Manifolds with Proper and Cocompact Group Actions

In this chapter, we first introduce some useful results of proper and cocompact Lie group actions, and then we will look at the properties of G-invariant elliptic operators and define a Hilbert space of invariant sections. In the end of this chapter, we present an index theorem for invariant elliptic operators.

2.1 Proper and Cocompact Actions

Let G be a Lie group and M a spin manifold with an orientation preserving proper and cocompact (left) Lie group G-action. One important result of such action is the existence of a cut-off function on M (see [Bou04, Chap. VII,§2.4]):

Lemma 2.1. *There is a compactly-supported non-negative smooth function $c(x)$ on M such that*

$$\int_G c(g^{-1}x)\,\mathrm{d}g = 1, \quad \forall x \in M. \tag{2.1}$$

The measure d*g in* (2.1) *is the (left) Haar measure on G and we shall call such function $c(x)$ a cut-off function.*

Proof (see [Tu99, Proposition 6.11]). Take a collection of precompact sets $\{U_\lambda\}_{\lambda=1}^{\infty}$ covering M and for each U_λ we find some $f_\lambda \in C_c(M)$ such that $f_\lambda = 1$ on U_λ. Since $\bigcup_{\lambda=1}^{\infty} \pi(U_\lambda)$ covers the compact space M/G, where $\pi : M \to M/G$ denotes the projection map, we can choose a finite subcovering $\bigcup_{i=1}^{n} \pi(U_{\lambda_i})$ that covers M/G. Define

$$f = \sum_{i=1}^{n} f_{\lambda_i} \quad \text{and} \quad h(x) = \int_{g\in G} f(g^{-1}x)\,\mathrm{d}g. \tag{2.2}$$

Obviously, $h(x)$ is (left) G-invariant. Since $\bigcup_{\lambda=1}^{n} \pi(U_{\lambda_i})$ covers M/G, $h(x)$ vanishes nowhere on M. Let $c(x) := f(x)/h(x)$, so that

$$\int_G c(g^{-1}x)\,\mathrm{d}g = \int_G \frac{f(g^{-1}x)}{h(g^{-1}x)}\,\mathrm{d}g = \frac{\int_G f(g^{-1}x)\,\mathrm{d}g}{h(x)} = 1. \qquad \blacksquare$$

Corollary 2.2. *Let $G(x)$ denote the orbit of $x \in M$ under G-action. Then the support of the cut-off function $c(x)$ intersects with each orbit $G(x)$ for any $x \in M$.*

Proof. The support of $c(x)$ is the same as the support of $f = \sum_{i=1}^{n} f_{\lambda_i}$ in (2.2). For any $x \in M$, its equivalent class $[x] \in M/G$ is covered by some $\pi(U_{\lambda_j})$, so we conclude that $G(x)$ intersects with U_{λ_j}, thus $\operatorname{supp} c$, non-trivially for any $x \in M$. $\blacksquare$

Using the cut-off function $c(x)$ we can construct a G-invariant Riemannian metric g (see [MZ10, formula (2.3)]). Let Ω be a (left) G-invariant volume form on M and $C^\infty(M)^G$ the space of smooth (left) G-invariant functions on M. For any two smooth (left) G-invariant functions ϕ and ψ on M we define a Hermitian product using $c(x)$:

$$(\phi,\psi)_c := \int_M c(x)\phi(x)\overline{\psi(x)}\,\Omega. \tag{2.3}$$

Lemma 2.3. *The sesquilinear form* $(\cdot,\cdot)_c$ *in* (2.3) *is non-degenerate, and thus defines a Hermitian product on* $C^\infty(M)^G$.

Proof. Assume there exists a smooth function ϕ such that $\int_M c(x)|\phi(x)|^2\,\Omega = 0$, then we must have $c(x)|\phi(x)|^2 = 0$, thus $\phi = 0$ within the support of c. Moreover, since the support of c intersects with each orbit $G(x)$ non-trivially, for any y outside $\operatorname{supp} c$ we can find some $x \in \operatorname{supp} c$ and $g \in G$ satisfying $g\cdot x = y$. So we have $0 = c(x)|\phi(x)|^2 = c(x)|\phi(y)|^2$, which shows ϕ also vanishes anywhere outside $\operatorname{supp} c$. ■

It is worth noting that the Hermitian product in (2.3) is independent of the choices of cut-off functions if the Lie group is *unimodular*. Recall that a Lie group is unimodular if its Haar measure is both left and right invariant.

Corollary 2.4. *Let* $c_1, c_2 \in C_c(M)$ *be two non-negative functions satisfying* (2.1)*. If* G *is a unimodular Lie group, then* $||\cdot||_{c_1} = ||\cdot||_{c_2}$.

Proof. For a unimodular group G, we can rewrite (2.1) into $\int_G c_2(gx)\ \mathrm{d}g = 1$. For any $\phi \in C^\infty(M)^G$, we have

$$||\phi||^2_{c_1} = \int_M \Big(\int_G c_2(gx)\ \mathrm{d}g\Big)\, c_1(x)|\phi(x)|^2\,\Omega. \tag{2.4}$$

Since both c_1 and c_2 are compactly-supported, Fubini's theorem applies and we have

$$\begin{aligned}\int_M \Big(\int_G c_2(gx)\ \mathrm{d}g\Big)\, c_1(x)|\phi(x)|^2\,\Omega &= \int_G \mathrm{d}g \int_M c_2(gx)c_1(x)|\phi(x)|^2\,\Omega \\ &= \int_G \mathrm{d}g \int_M c_2(x)c_1(g^{-1}x)|\phi(g^{-1}x)|^2\,\Omega \\ &= \int_G c_1(g^{-1}x)\ \mathrm{d}g \int_M c_2(x)|\phi(x)|^2\,\Omega \\ &= \|\phi\|_{c_2}^2.\end{aligned} \tag{2.5}$$

Therefore the Hermitian product in (2.3) is canonical for unimodular Lie groups. ■

Definition 2.5. We denote $L^2_G(M)$ to be the completion of $C^\infty(M)^G$ with respect to $(\cdot,\cdot)_c$.

2.2 Dirac Operators on the Space of Invariant Sections

Definition 2.6. Suppose that $\mathcal{E}$ is a $\mathbb{Z}_2$-graded G-equivariant Hermitian bundle on M and let $\Gamma(\mathcal{E})^G$ denote the space of G-invariant sections of $\mathcal{E}$. For any $s_1, s_2 \in \Gamma(\mathcal{E})^G$, we define a Hermitian product similar to (2.3):

$$(s_1, s_2)_c := \int_M c(x)(s_1, s_2)_x\ \mathrm{d}x,$$

where $(s_1, s_2)_x$ is the Hermitian product on fiber $\mathcal{E}_x$. We denote the completion of $\Gamma(\mathcal{E})^G$ by $L^2_G(\mathcal{E})$. If we restrict the Hermitian product on sections of even/odd subbundles $\mathcal{E}^\pm$, then the completion of $\Gamma(\mathcal{E}^\pm)^G$ is denoted by $L^2_G(\mathcal{E}^\pm)$.

2.2.1 Unimodular Group

In this section we present a theorem about the duality of Dirac operators in the case of unimodular groups. The Lie group G throughout this section is assumed to be unimodular.

Theorem 2.7. *If* $\not{D}_+ : \Gamma(\mathcal{E}^+) \to \Gamma(\mathcal{E}^-)$ *is a G-invariant Dirac operator on M, then*

$$(\not{D}_+ s_1, s_2)_c = (s_1, \not{D}_- s_2)_c \tag{2.6}$$

for any $s_1 \in \Gamma(\mathcal{E}^+)^G, s_2 \in \Gamma(\mathcal{E}^-)^G$, *where* $\not{D}_-$ *is the odd part of* $\not{D}$. *In other words,* $\not{D}_-$ *is the adjoint of* $\not{D}_+ : L^2_G(\mathcal{E}^+) \to L^2_G(\mathcal{E}^-)$.

To prove Theorem 2.7, we first need a modified version of the divergence theorem:

Lemma 2.8. *For any G-invariant vector field X on M we have*

$$\int_M c(x)\,\mathrm{tr}(\boldsymbol{\nabla} X)\,\Omega = 0, \tag{2.7}$$

where $\boldsymbol{\nabla}$ *is the Levi-Civita connection.*

Proof. The Lie derivative $\mathcal{L}_X\big(c(x)\Omega\big) = c(x)\mathcal{L}_X\Omega + X(c)(x)\Omega$ and since $\mathcal{L}_X\Omega = \mathrm{tr}(\boldsymbol{\nabla} X)\Omega$, we have

$$\int_M c(x)\mathrm{tr}(\boldsymbol{\nabla} X)\Omega = \int_M \mathcal{L}_X\big(c(x)\Omega\big) - \int_M X(c)(x)\,\Omega. \tag{2.8}$$

Notice that $\mathcal{L}_X\big(c(x)\Omega\big)$ is exact due to Cartan's magic formula. It suffices to show that

$$\int_M X(c)(x)\,\Omega = 0. \tag{2.9}$$

Given the fact that Ω is G-invariant and by (2.1) we know

$$\Omega = \int_G c(g^{-1}x)(g^{-1})^*\Omega\;\mathrm{d}g.$$

Hence

$$\int_M X(c)(x)\,\Omega = \int_M X(c)(x)\int_G c(g^{-1}x)(g^{-1})^*\Omega\;\mathrm{d}g. \tag{2.10}$$

Let us denotes the support of c by W, which is compact. The function $X(c)(x)c(g^{-1}x)$ is supported on the pre-image of $W \times W$ through the map $\rho : G \times M \to M \times M, \rho(g, x) = (g^{-1}x, x)$. The map ρ is proper since the G-action is proper. Therefore, $\operatorname{supp} X(c)(x)c(g^{-1}x)$ is compact and we can use Fubini's theorem to write (2.10) into

$$\begin{aligned} &\int_G \mathrm{d}g \int_M X(c)(x)c(g^{-1}x)(g^{-1})^*\Omega \\ &= \int_G \mathrm{d}g \int_M (g^{-1})^*\big(X(c)(gx)c(x)\Omega\big) \\ &= \int_G \mathrm{d}g \int_M X(c)(gx)c(x)\Omega \\ &= \int_M c(x)\Big(\int_G X(c)(gx)\,\mathrm{d}g\Big)\Omega. \end{aligned} \tag{2.11}$$

The modular function $\Delta = 1$ for unimodular Lie groups, so (2.11) becomes

$$\int_M c(x)\Big(\int_G X(c)(gx)\,\mathrm{d}g\Big)\Omega = \int_M c(x)\Big(\int_G X(c)(g^{-1}x)\,\mathrm{d}g\Big)\Omega. \tag{2.12}$$

If we let $\alpha = \mathrm{d}c$, thanks to the G-invariance of X, we have

$$X(c)(g^{-1}x) = \langle \alpha, X\rangle_{g^{-1}x} = \langle \alpha(g^{-1}x), (\mathrm{d}g^{-1})_x X\rangle = \langle (g^{-1})^*\alpha, X\rangle_x, \tag{2.13}$$

and $(g^{-1})^*\alpha = (g^{-1})^*\mathrm{d}c = \mathrm{d}((g^{-1})^*c) = \mathrm{d}(c(g^{-1}x))$. Because the pairing $\langle\,\cdot,\cdot\,\rangle$ is continuous and that $c(g^{-1}x)$ is compactly-supported in $G \times M$, we conclude

$$\begin{aligned} \int_G X(c)(g^{-1}x)\,\mathrm{d}g &= \int_G \langle\, \mathrm{d}c(g^{-1}x), X\,\rangle_x\,\mathrm{d}g \\ &= \langle \int_G \mathrm{d}c(g^{-1}x)\,\mathrm{d}g, X\,\rangle_x \\ &= \langle\, \mathrm{d}\Big(\int_G c(g^{-1}x)\,\mathrm{d}g\Big), X\,\rangle_x = 0, \end{aligned} \tag{2.14}$$

where in the last equation, we have used (2.1). ■

Now we proceed to prove Theorem 2.7 using Lemma 2.8:

Proof of Theorem 2.7. Locally we can write the Dirac operator as:

$$\not{D} = \sum_i \mathbf{c}\,(\mathrm{d}x^i)\nabla_{\partial_i}$$

where $\mathbf{c}\,(\mathrm{d}x^i)$ is the Clifford multiplication of $\mathrm{d}x^i$.

For any $s_1, s_2 \in \Gamma(\mathcal{E})^G$, let X be a vector filed on M given by $\langle \alpha, X\rangle = (s_1, \mathbf{c}\,(\alpha)s_2)$ for any 1-form α, where $(\cdot,\cdot)$ is the Hermitian product. There is a relation between $\not{D}_+$ and $\not{D}_-$ (see [BGV04, Proposition 3.44]):

$$(\not{D}_+ s_1, s_2)_x = (s_1, \not{D}_- s_2)_x - \mathrm{tr}(\boldsymbol{\nabla} X)_x, \tag{2.15}$$

where the connection in $\boldsymbol{\nabla} X$ is the Levi-Civita connection of the Riemannian metric.

We prove that X is G-invariant, i.e., $\langle \alpha, X\rangle_{gx} = \langle \alpha, g_*(X)\rangle_{gx}$ for any 1-form α and $g \in G$, where $g_*(X)$ is the pushforward of X by $g : M \to M$. By definition we have

$$\langle \alpha, X\rangle_{gx} = \big(s_1, \mathbf{c}(\alpha(gx))s_2\big)_{gx},$$
$$\langle \alpha, g_*(X)\rangle_{gx} = \langle g^*\alpha, X\rangle_x = \big(s_1, \mathbf{c}(g^*\alpha)s_2\big)_x.$$

Since s_1 and s_2 are both G-invariant and $\mathcal{E}$ is a G-equivariant bundle, we have that $s_i(gx) = \gamma_g \cdot g^{-1}s_i(x) = \gamma_g \cdot s_i(x)$, where $\gamma_g : \mathcal{E}_x \to \mathcal{E}_{gx}$ is the G-action on the fibers. Notice $\mathcal{E}$ is also an equivariant Clifford module, which implies

$$\mathbf{c}(\alpha(gx))s_2(gx) = \mathbf{c}(\alpha(gx))\big(\gamma_g \cdot s_2(x)\big) = \gamma_g \cdot \big(\mathbf{c}(g^*\alpha(x))s_2(x)\big).$$

Therefore we compute using the above formulae of $s_i(gx)$ and $\mathbf{c}(g^*\alpha)s_2(x)$ that

$$\begin{aligned}\langle \alpha, X\rangle_{gx} &= \big(s_1(gx), \mathbf{c}(\alpha(gx))s_2(gx)\big)_{gx} = \big(\gamma_g \cdot s_1(x), \gamma_g \cdot (\mathbf{c}(g^*\alpha(x))s_2(x))\big)_{gx} \\ &= \big(s_1(x), \mathbf{c}(g^*\alpha(x))s_2(x)\big)_x = \langle g^*\alpha, X\rangle_x = \langle \alpha, g_*(X)\rangle_{gx},\end{aligned} \tag{2.16}$$

where between the two lines we have used the G-invariance of the Hermitian product. Formula (2.16) holds for all α, so $X = g_*(X)$ for all g, which shows that X is G-invariant. Now if we multiply both sides of (2.15) with the cut-off function c and integrating over M we have

$$\begin{aligned}(\not{D}_+ s_1, s_2)_c &= (s_1, \not{D}_- s_2)_c - \int_M c(x)\mathrm{tr}(\nabla X)\Omega, \\ &= (s_1, \not{D}_+ s_2)_c\end{aligned}$$

for any $s_1, s_2 \in \Gamma(\mathcal{E})^G$. ∎

2.2.2 Non-unimodular Group

When G is non-unimodular, we need to modify our definition of the Hilbert space. Let Δ be the modular function on G and $\lambda = \Delta^{1/2}$. A section $s \in \Gamma(\mathcal{E})$ is called *λ-invariant* if

$$gs = \lambda(g)s. \tag{2.17}$$

By definition, $g^{-1}s(x) = \gamma_{g^{-1}} \cdot s(gx)$. For a λ-invariant section s, $g^{-1}s = \lambda(g^{-1})s$. We have

$$s(gx) = \gamma_g \cdot g^{-1}s(x) = \lambda(g^{-1})\,\gamma_g \cdot s(x). \tag{2.18}$$

Let X be a vector field such that $\langle \alpha, X\rangle := (s_1, \mathbf{c}(\alpha)s_2)$ for sections s_1, s_2 satisfying (2.17), we have

$$
\begin{aligned}
\langle \alpha, X\rangle_{gx} &= \big(s_1(gx), \mathbf{c}(\alpha(gx))s_2(gx)\big) \\
&= \big(\lambda(\mathrm{g}^{-1})\,\gamma_{\mathrm{g}}\cdot s_1(x), \lambda(\mathrm{g}^{-1})\,\gamma_{\mathrm{g}}\cdot(\mathbf{c}(\mathrm{g}^*\alpha)s_2(x))\big) \\
&= \Delta^{-1}(\mathrm{g})\big(s_1(x), \mathbf{c}(\mathrm{g}^*\alpha)s_2(x)\big) \\
&= \Delta^{-1}(\mathrm{g})\langle \alpha, \mathrm{g}_*(X)\rangle_{gx}
\end{aligned}
\tag{2.19}
$$

for any 1-form α. We conclude

$$
X(gx) = \Delta^{-1}(\mathrm{g})\,\mathrm{g}_*(X). \tag{2.20}
$$

In the case when G is not unimodular we should change (2.12) into

$$
\int_G X(c)(gx)\,\mathrm{dg} = \int_G \Delta^{-1}(\mathrm{g})\,X(c)(\mathrm{g}^{-1}x)\,\mathrm{dg}, \tag{2.21}
$$

Imitating our calculation in (2.13), we get

$$
\begin{aligned}
\Delta^{-1}(\mathrm{g})\,X(c)(\mathrm{g}^{-1}x) &= \Delta^{-1}(\mathrm{g})\,\langle \mathrm{d}c, X\rangle_{\mathrm{g}^{-1}x} \\
&= \Delta^{-1}(\mathrm{g})\,\langle \mathrm{d}c(\mathrm{g}^{-1}x), \lambda^2(\mathrm{g})\mathrm{g}_*^{-1}(X)\rangle \\
&= \Delta^{-1}(\mathrm{g})\,\lambda^2(\mathrm{g})\,\langle (\mathrm{g}^{-1})^*\mathrm{d}c, X\rangle_x \\
&= \langle (\mathrm{g}^{-1})^*\mathrm{d}c, X\rangle_x.
\end{aligned}
$$

Together with (2.14) we have

$$
\int_G X(c)(gx)\,\mathrm{dg} = \int_G \langle (\mathrm{g}^{-1})^*\mathrm{d}c, X\rangle\,\mathrm{dg} = 0.
$$

To summarize, we have the following lemma:

Lemma 2.9. *For a smooth vector field X on M satisfying $X(gx) = \Delta^{-1}(g)g_*(X)$, we have*

$$\int_M c(x)\mathrm{tr}(\nabla X)\,\Omega = 0. \tag{2.22}$$

Definition 2.10. For all sections of $\mathcal{E}$ satisfying (2.17), we define a Hermitian product

$$(s_1, s_2)_\lambda := \int_M c(x)(s_1, s_2)_x\,\Omega \tag{2.23}$$

on the space of these sections denoted as $\Gamma_\lambda(\mathcal{E})^G$. Using the Hermitian product $(\cdot,\cdot)_\lambda$, we can complete $\Gamma_\lambda(\mathcal{E})^G$ into a Hilbert space $L^2_\lambda(\mathcal{E})$, and obviously $L^2_\lambda(\mathcal{E}) = L^2_G(\mathcal{E})$ for unimodular groups.

The following theorem is a generalization of Theorem 2.7 for non-unimodular groups.

Theorem 2.11. *Let $\not{D}$ be a G-invariant Dirac operator on $\mathcal{E}$. For two sections $s_1 \in \Gamma_\lambda(\mathcal{E}^+)$ and $s_2 \in \Gamma_\lambda(\mathcal{E}^-)$ we have*

$$(\not{D}_+ s_1, s_2)_\lambda = (s_1, \not{D}_- s_2)_\lambda. \tag{2.24}$$

Proof. First we notice $\not{D}$ does act on $\Gamma_\lambda(\mathcal{E})^\pm$: $g(\not{D}s) = \not{D}(gs) = \not{D}(\lambda(g)s) = \lambda(g)\not{D}s$ for all $s \in \Gamma(\mathcal{E}), g \in G$. To show equation (2.24), we integrate both sides of (2.15), which still holds true for $s_1 \in \Gamma_\lambda(\mathcal{E}^+)$ and $s_2 \in \Gamma_\lambda(\mathcal{E}^-)$, and get

$$(\not{D}_+ s_1, s_2)_\lambda - (s_1, \not{D}_- s_2)_\lambda = -\int_M c(x)\mathrm{tr}(\nabla X)\,\Omega.$$

We conclude our proof by noting that X satisfies formula (2.20), so by Lemma 2.9

$$\int_M c(x)\mathrm{tr}(\nabla X)\,\Omega = 0. \qquad \blacksquare$$

2.3 Fredholmness

In this section we will present our main theorem: a G-invariant Dirac operator $\not{D}$ on $L^2_\lambda(\mathcal{E})$ is Fredholm. Note that $\not{D}$ is odd and formally self-adjoint, so we can represent $\not{D}$ as $\not{D} = \begin{pmatrix} & \not{D}_+^* \\ \not{D}_+ & \end{pmatrix}$. We will prove the Fredholmness of $\not{D}$ by showing that the kernel and cokernel of $\not{D}_+ : L^2_\lambda(\mathcal{E}^+) \to L^2_\lambda(\mathcal{E}^-)$ are finite-dimensional, and the range of $\not{D}_+$ is closed.

2.3.1 Integral Kernels

Definition 2.12. Let $\pi_1, \pi_2 : M \times M \to M, \pi_1(x, y) = x, \pi_2(x, y) = y$ be two projections. We define a vector bundle $\mathcal{E} \boxtimes \mathcal{E}^*$ over $M \times M$ (see [BGV04], Chap. 2) as

$$\mathcal{E} \boxtimes \mathcal{E}^* := \pi_1^* \mathcal{E} \otimes \pi_2^* \mathcal{E}^*. \tag{2.25}$$

A smooth section $k(x, y) \in \Gamma(\mathcal{E} \boxtimes \mathcal{E}^*)$ is called an integral kernel.

Recall from formula (1.2) that the G-action on $\mathcal{E}$ naturally induces an action on $\mathcal{E}^*$, which is also G-equivariant. Since $g\sigma(x) = \tilde{\gamma}_g \cdot \sigma(g^{-1}x)$ by definition for $\sigma \in \Gamma(\mathcal{E}^*)$, we have that $\langle gs, g\sigma \rangle_{gx} = \langle s, \sigma \rangle_x$ for sections $s \in \Gamma(\mathcal{E})$ and $\sigma \in \Gamma(\mathcal{E}^*)$.

The next lemma shows a G-invariant integral kernel supported near the diagonal defines a compact operator on $L^2_\lambda(\mathcal{E})$.

Lemma 2.13. *Assume $k(x, y)$ is a G-invariant section of $\mathcal{E} \boxtimes \mathcal{E}^*$, which is supported in an ϵ-neighborhood of the diagonal in $M \times M$, then the integral operator $\hat{K} : L^2_\lambda(\mathcal{E}) \to L^2_\lambda(\mathcal{E})$,*

$$\hat{K}s(x) = \int_M k(x, y)s(y)\, \mathrm{d}y, \tag{2.26}$$

is compact, where $\mathrm{d}y$ *is the G-invariant measure on M. The term $k(x,y)s(y)$ above is the pairing of* $s(\cdot)\in\Gamma(\mathcal{E})$ *with* $k(x,\cdot)\in\Gamma(\mathcal{E}^*)$.

Proof. We first show the section $\hat{K}s(x)$ is λ-invariant:

$$\begin{aligned} g(\hat{K}s)(x) &= \gamma_g\cdot\hat{K}s(g^{-1}x) = \gamma_g\cdot\int_M k(g^{-1}x,y)s(y)\,\mathrm{d}y \\ &= \gamma_g\cdot\int_M k(g^{-1}x,g^{-1}y)\,s(g^{-1}y)\,\mathrm{d}y. \end{aligned} \tag{2.27}$$

Since s is λ-invariant, by equation (2.18) we have

$$s(g^{-1}y) = \lambda(g)\,\gamma_{g^{-1}}\cdot s(y).$$

Similarly, since $k\in\mathcal{E}\boxtimes\mathcal{E}^*$ is G-invariant,

$$k(g^{-1}x,g^{-1}y) = (\gamma_g\boxtimes\tilde{\gamma}_g)^{-1}\cdot k(x,y) = (\gamma_{g^{-1}}\boxtimes\tilde{\gamma}_{g^{-1}})\cdot k(x,y).$$

Putting the above two equations together, by (2.27) we have

$$g(\hat{K}s)(x) = \lambda(g)\,\gamma_g\cdot\int_M(\gamma_{g^{-1}}\boxtimes\tilde{\gamma}_{g^{-1}})\cdot k(x,y)\,\gamma_{g^{-1}}\cdot s(y)\,\mathrm{d}y. \tag{2.28}$$

For a fixed x, since the G-actions on $\mathcal{E}$ and $\mathcal{E}^*$ are compatible, we have

$$\langle\tilde{\gamma}_{g^{-1}}\cdot k(x,y),\gamma_{g^{-1}}\cdot s(y)\rangle_{g^{-1}y} = \langle k(x,y),s(y)\rangle_y. \tag{2.29}$$

So we can simplify (2.28) as

$$
\begin{aligned}
g(\hat{K}s)(x) &= \lambda(g)\,\gamma_g \cdot \int_M (\gamma_{g^{-1}} \boxtimes \tilde{\mathrm{id}}) \cdot k(x,y)s(y)\,\mathrm{d}y \\
&= \lambda(g) \int_M k(x,y)s(y)\,\mathrm{d}y \\
&= \lambda(g)\hat{K}s(x).
\end{aligned}
\tag{2.30}
$$

Now we proceed to show that $\hat{K}$ is bounded. Let W denote the support of c and $\widetilde{W}$ be a compact set containing an ϵ-neighborhood of W. Recall from the proof of Lemma 2.1 that there is a collection of open subsets $\{U_{\lambda_i} \subseteq M\}$ and bump functions $\{f_{\lambda_i}\}$ such that $\bigcup_{i=1}^n \pi(U_{\lambda_i}) = M/G$ and $f_{\lambda_i}|_{U_{\lambda_i}} = 1$. Let $U_0 = \bigcup_{i=1}^n U_{\lambda_i}$ so that $f = \sum f_{\lambda_i} \geqslant 1$ on U_0. We know from Corollary 2.2 that U_0 intersects with each orbit $G(x)$ non-trivially, so the collection $\{gU_0\}_{g\in G}$ covers M and therefore $\widetilde{W}$. Choose a finite covering $\{g_1U_0, \dots, g_kU_0\}$ of $\widetilde{W}$ and let

$$
C_i = \big(\sup_{x\in W} h(x)\big) \cdot \big(\sup_{y\in W} \|\gamma_{g_i}(y)\|^2\big)
$$

for $h(x)$ defined in (2.2), where $\|\cdot\|$ denotes the operator norm.

Recall that for any λ-invariant section $s(x)$ of $\mathcal{E}$, we have by equation (2.18):

$$
s(gx) = \lambda(g^{-1})\,\gamma_g \cdot s(x).
$$

So if we set $z_i := g_i^{-1}x$, then $s(x) = s(g_i z_i) = \lambda(g_i^{-1})\gamma_{g_i} \cdot s(z_i)$. Therefore

$$\begin{aligned}
\|s\|^2_{L^2(g_i U_0)} &= \int_{g_i U_0} |s(x)|^2 \, \mathrm{d}x \\
&= \Delta(g_i^{-1}) \int_{g_i U_0} |\gamma_{g_i} \cdot s(z_i)|^2 \, \mathrm{d}x \\
&\leqslant \Delta(g_i^{-1}) \big(\sup_{z_i \in W} \|\gamma_{g_i}(z_i)\|^2 \big) \int_{x \in g_i U_0} |s(z_i)|^2 \, \mathrm{d}x \\
&= \Delta(g_i^{-1}) \big(\sup_{z_i \in W} \|\gamma_{g_i}(z_i)\|^2 \big) \int_{z_i \in U_0} |s(z_i)|^2 \, \mathrm{d}z_i.
\end{aligned} \tag{2.31}$$

Since $f \geqslant 1$ on U_0, $c(z_i)|s(z_i)|^2 = f(z_i)|s(z_i)|^2/h(x) \geqslant |s(z_i)|^2/h(z_i)$ for any $z_i \in U_0$, so on U_0 we have $|s(z_i)|^2 \leqslant h(z_i)c(z_i)|s(z_i)|^2$. Therefore (2.31) gives

$$\begin{aligned}
\|s\|^2_{L^2(g_i U_0)} &\leqslant \Delta(g_i^{-1}) \big(\sup_{z_i \in W} \|\gamma_{g_i}(z_i)\|^2 \big) \int_{U_0} |s(z_i)|^2 \, \mathrm{d}z_i \\
&\leqslant \Delta(g_i^{-1}) \big(\sup_{z_i \in W} \|\gamma_{g_i}(z_i)\|^2 \big) \int_{U_0} h(z_i)c(z_i)|s(z_i)|^2 \, \mathrm{d}z_i \\
&\leqslant \Delta(g_i^{-1}) \big(\sup_{z_i \in W} \|\gamma_{g_i}(z_i)\|^2 \big) \cdot \big(\sup_{Z_i \in W} h(z_i) \big) \int_M c(z_i)|s(z_i)|^2 \, \mathrm{d}z_i \\
&= \frac{C_i}{\Delta(g_i)} \|s\|^2_\lambda.
\end{aligned} \tag{2.32}$$

Recall that $\{g_1 U_0, \dots, g_k U_0\}$ covers $\widetilde{W}$, so

$$\|s\|^2_{L^2(\widetilde{W})} \leqslant \Big(\sum_{i=1}^k \frac{C_i}{\Delta(g_i)} \Big) \|s\|^2_\lambda. \tag{2.33}$$

For a fixed $(x,y) \in M \times M$, let $\|k(x,y)\|$ denote the operator norm of $k(x,y) \in \mathrm{End}(\mathcal{E}_y, \mathcal{E}_x)$. If s is a λ-invariant section of $\mathcal{E}$, then

$$\begin{aligned} \|\hat{K}s\|_\lambda^2 &= \int_M c(x)|\hat{K}s(x)|^2 \,\mathrm{d}x \\ &= \int_M c(x)\Big|\int_M k(x,y)s(y)\,\mathrm{d}y\Big|^2 \,\mathrm{d}x \\ &\leqslant \int_W c(x)\Big(\int_{\widetilde{W}} |k(x,y)s(y)|^2 \,\mathrm{d}y\Big) \,\mathrm{d}x \\ &\leqslant \int_W c(x)\Big(\int_{\widetilde{W}} \|k(x,y)\|^2|s(y)|^2 \,\mathrm{d}y\Big) \,\mathrm{d}x \\ &= \int_{\widetilde{W}}\Big(\int_W c(x)\|k(x,y)\|^2 \,\mathrm{d}x\Big)|s(y)|^2 \,\mathrm{d}y. \end{aligned} \tag{2.34}$$

Let $h(x,y) := c(x)\|k(x,y)\|^2$ be a smooth function defined on $(x,y) \in W \times \widetilde{W}$. So

$$\begin{aligned} \int_W c(x)\|k(x,y)\|^2 \,\mathrm{d}x = \int_W h(x,y)\,\mathrm{d}x &\leqslant \int_W h(y,y)\,\mathrm{d}x + \int_W |h(x,y)-h(y,y)|\,\mathrm{d}x \\ &= \int_W c(y)\|k(y,y)\|^2 \,\mathrm{d}x + \int_W |h(x,y)-h(y,y)|\,\mathrm{d}x \\ &\leqslant \varkappa c(y) + \eta, \end{aligned} \tag{2.35}$$

for constants

$$\varkappa := \mathrm{measure}(W) \cdot \sup_{y\in\widetilde{W}} \|k(y,y)\|^2$$

and

$$\eta := \mathrm{measure}(W) \cdot \sup_{(x,y)\in W\times\widetilde{W}} |h(x,y)-h(y,y)|.$$

Therefore we conclude from equation (2.34) and (2.35) that $\hat{K}$ is bounded:

$$\begin{aligned}
\|\hat{K}s\|_\lambda^2 &\leqslant \int_{\widetilde{W}} \Big(\int_W c(x)\|k(x,y)\|^2 \, \mathrm{d}x\Big)|s(y)|^2 \, \mathrm{d}y \\
&\leqslant \int_{\widetilde{W}} \big(\kappa c(y) + \eta\big)|s(y)|^2 \, \mathrm{d}y \\
&\leqslant \kappa \int_{\widetilde{W}} c(y)|s(y)|^2 \, \mathrm{d}y + \eta \int_{\widetilde{W}} |s(y)|^2 \, \mathrm{d}y = \kappa\|s\|_\lambda^2 + \eta\|s\|_{L^2(\widetilde{W})}^2 \\
&\leqslant (\kappa + \eta \sum \frac{C_i}{\Delta(g_i)})\|s\|_\lambda^2.
\end{aligned} \tag{2.36}$$

To prove compactness, we need to show for any bounded sequence $\{s_\mu\}$ in $L^2_\lambda(\mathcal{E})$, there is a Cauchy subsequence of $\{\hat{K}s_\mu\}$. According to (2.33) if $\{s_\mu\}$ is bounded in $L^2_\lambda(\mathcal{E})$, it is also bounded in $L^2(W)$. Since the restriction of $\hat{K}$ on a compact set W,

$$\hat{K}|_W s(x) := \int_W k(x,y)s(y) \, \mathrm{d}y, \quad x \in W,$$

is a compact operator on $L^2(W)$, we can find a subsequence $\{\hat{K}s_j\}$ converges to some (locally-defined) section t in $L^2(W)$. Since each $\hat{K}s_j$ is G-invariant, the limit t is locally G-invariant, that is, if $x, y \in W$ and $y = gx$ for som $g \in G$, then we must have $\gamma_g \cdot t(x) = t(y)$. Hence we can extend t to a G-invariant section on $\mathcal{E}$, which we will still denote by t, using the G-action: $\forall x \in M$, there is some $g \in G$ such that $g^{-1}x \in W$ and

$$t(x) := \gamma_g \cdot t(g^{-1}x).$$

It is easy to check the subsequence $\{\hat{K}s_j\}$ also converges to t in $L^2_\lambda(\mathcal{E})$:

$$\begin{aligned}||\hat{K}s_j - t||^2_\lambda &= \int_W c(x)|\hat{K}s_j(x) - t(x)|^2 \,\mathrm{d}x \\ &\leqslant (\sup c(x)) \cdot ||\hat{K}s_j(x) - t(x)||^2_{L^2(W)} \to 0.\end{aligned}$$

■

2.3.2 Parametrix

Next we will show that $\not{D}_+ : L^2_\lambda(\mathcal{E}^+) \to L^2_\lambda(\mathcal{E}^-)$ is of closed range and has a (co)kernel of finite rank by looking at the properties of its paramatrix Q:

Theorem 2.14. *There is a densely defined closed operator $Q : L^2_\lambda(\mathcal{E}) \to L^2_\lambda(\mathcal{E})$ such that the Schwartz kernels of both $(Q\not{D}_+ - I)$ and $(\not{D}_+ Q - I)$ are smooth functions on $M \times M$ that are supported near the diagonal of $M \times M$ and invariant with respect to the diagonal (left) G-action.*

According to Lemma 2.13, $(Q\not{D}_+ - I)$ and $(\not{D}_+ Q - I)$ defined above extend to compact operators, which implies that $\not{D}_+ : L^2_\lambda(\mathcal{S}^+) \to L^2_\lambda(\mathcal{E}^-)$ as a densely defined operator which has a closed range and a (co)kernel of finite rank. Therefore, we conclude:

Corollary 2.15. *The Dirac operator $\not{D} = \begin{pmatrix} & \not{D}_+^* \\ \not{D}_+ & \end{pmatrix} : L^2_\lambda(\mathcal{E}) \to L^2_\lambda(\mathcal{E})$ is Fredholm.*

The proof of Theorem 2.14 works in the exactly same way for G-invariant elliptic operators, and we have:

Theorem 2.16. *Let D be a G-invariant elliptic operator on $\mathcal{E}$, then the induced operator $D : L^2_\lambda(\mathcal{E}) \to L^2_\lambda(\mathcal{E})$ is Fredholm.*

In the next chapter, we will recall the theory of pseudo-differential operators on manifolds and prove Theorem 2.14.

Chapter 3

Pseudo-differential Operators

In this chapter we will define pseudo-differential operators (ψDOs for short) on manifolds through a coordinate-free approach and develop the corresponding symbol calculus for our definition. We will then look at symbols for differential operators, and construct an "inverse" symbol for elliptic differential operators. In the next chapter, we will use the "inverse" symbol to construct a parametrix we need in Theorem 2.14.

The study of ψDOs started in mid 1960s by many mathematicians, most notably Lars Hörmander. In $\mathbb{R}^n$, it can be viewed as a generalization of Fourier transformation; while the calculus of ψDOs on manifolds is traditionally defined using local coordinates. However, in late 1970s, H. Widom suggested a method of defining full symbols of ψDOs using an affine connection and developed a version of symbol calculus with local coordinates and standard local phase function [Wid78; Wid80]. In early 1990s, Yu. Safarov gave a new definition of ψDOs in a coordinate-free way, by using invariant oscillatory integrals over the cotangent bundle [Saf97; MS11]. We will introduce Safarov's definition of ψDOs after a brief review of the case of $\mathbb{R}^n$.

Throughout the chapter, $\mathbf{D}_x^\alpha = (-\mathbf{i})^{|\alpha|}(\partial^{|\alpha|}/\partial x^\alpha)$ are partial derivatives with respect to local coordinates $\{x^k\}$, and C_c^∞ denotes the space of smooth functions with compact support and $\mathcal{D}'$ is the space of distributions.

3.1 ψDOs on $\mathbb{R}^n$

Let $P = \sum_{|\alpha|\leqslant m} A^\alpha(x)\mathbf{D}^\alpha$ be a differential operator of order m acting on functions in $C_c^\infty(\mathbb{R}^n)$. Using the Fourier inversion formula, we have

$$Pu(x) = (2\pi)^{-n/2}\int e^{\mathbf{i}\langle x,\xi\rangle}\,\widehat{Pu}(\xi)\,\mathrm{d}\xi = (2\pi)^{-n/2}\int e^{\mathbf{i}\langle x,\xi\rangle}\,\sigma_P(x,\xi)\,\hat{u}(\xi)\,\mathrm{d}\xi,$$

where

$$\sigma_P(x,\xi) = \sum_{|\alpha|\leqslant m} A^\alpha(x)\xi^\alpha \tag{3.1}$$

is called the *full symbol* of P, and the leading homogeneous term (in ξ) of σ_P is called the *principal symbol*.

If we replace the symbol by a larger class of smooth functions $p(x,\xi)$ which satisfies that for any index α,α', there is a constant $C_{\alpha,\alpha'} > 0$ such that for all x,ξ,

$$|\mathbf{D}_x^\alpha \mathbf{D}_\xi^{\alpha'} p(x,\xi)| \leqslant C_{\alpha\alpha'}(1+|\xi|)^{m-|\alpha'|}, \tag{3.2}$$

we can define an integral operator $\tilde{P} : C_c^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^n)$:

$$\tilde{P}u(x) = (2\pi)^{-n/2}\int e^{\mathbf{i}\langle x,\xi\rangle}\,p(x,\xi)\,\hat{u}(\xi)\,\mathrm{d}\xi,$$

which is called a pseudo-differential operator on $\mathbb{R}^n$.

Remark 2. The symbol class defined in (3.2) is denoted by $S^m_{1,0}(\mathbb{R}^n)$ in [Hör07] by Hörmander, and it is the only class we need throughout the thesis.

It is worth noting here, that for a general manifold we cannot have full symbols like (3.1) directly because such calculation depends on the coordinate system. However, the principal symbol can be correctly defined on the cotangent bundle, which is invariant under change of coordinates.

A more comprehensive discussion of ψDOs on $\mathbb{R}^n$ can be found in [Hör07; Tay81], along with many other books.

3.2 ψDOs on Manifolds

In this section we introduce Safarov's definition of ψDOs on manifolds and summarize a few important properties. Detailed proofs and discussions can be found in [Saf97] and [MS11].

Let M be a smooth n-dimensional Riemannian manifold with a torsion-free connection Γ, x a point of M, and ξ a covector in $T^*_x M$. Given a coordinate system $\{x^k\}$ and a vector field $v = \sum_k v^k(x)\,\partial_{x^k}$ on M, the *horizontal lift*

$$\nabla_v = \sum_k v^k(x)\,\partial_{x^k} + \sum_{i,j,k} \Gamma^i_{jk}(x) v^k(x) \xi_i\,\partial_{\xi_j} \tag{3.3}$$

is a vector field on T^*M, where Γ^i_{jk} are Christoffel symbols of Γ. We denote by ∇_i the horizontal lift of the vector field ∂_{x^i}.

3.2.1 Classes of Symbols

Definition 3.1. The symbol class $S^m(\Gamma)$ consists of all smooth functions $a(x,\xi) \in C^\infty(T^*M)$ satisfying

$$|\partial_\xi^\alpha \nabla_{i_1} \cdots \nabla_{i_q} a(x,\xi)| \leqslant C_{K,\alpha,i_1,\dots,i_q}(1+|\xi|)^{m-|\alpha|} \tag{3.4}$$

using any coordinates $\{x^k\}$ and for all α and $i_1,\dots,i_q$, when x runs over a compact set K. We call $a(x,\xi) \in S^m(\Gamma)$ a symbol of order m and define $S^{-\infty} = \cap_m S^m(\Gamma)$ to be the class of smoothing symbols, which consists of all functions with all their derivatives vanishing faster than any power of $|\xi|$ as $|\xi| \to \infty$.

Analogously, functions $a(y;x,\xi) \in C^\infty(M \times T^*M)$ are called amplitudes of order m if in any coordinate systems $\{x^k\},\{y^k\}$ and for all α,β and $i_i,\dots,i_q$,

$$|\partial_y^\beta \partial_\xi^\alpha \nabla_{i_1} \cdots \nabla_{i_q} a(y;x,\xi)| \leqslant C_{K,\alpha,\beta,i_1,\dots,i_q}(1+|\xi|)^{m-|\alpha|} \tag{3.5}$$

when (x,y) runs over a compact $K \subseteq M \times M$. Traditionally, the class of m-th order amplitudes is also denoted by $S^m(\Gamma)$.

Remark 3. In fact, a Riemannian structure is not necessary to define symbol classes. It suffices to replace the Riemannian metric $|\cdot|$ in (3.4) and (3.5) by a positive function homogeneous in ξ of degree 1. Thus we can define the symbol and amplitude class for general smooth manifolds.

The following lemma lists some basic properties of symbols and amplitudes.

Lemma 3.2. *If $a \in S^{m_1}(\Gamma)$, $b \in S^{m_2}(\Gamma)$ and let $m = \max\{m_1, m_2\}$, then*

$$ab \in S^{m_1+m_2}(\Gamma), \quad \partial_\xi^\alpha a \in S^{m-|\alpha|}(\Gamma), \quad \nabla_{v_1} \cdots \nabla_{v_q} a \in S^m(\Gamma)$$

for any vector fields $v_1,\dots,v_q$.

Moreover, for a multi-index α *with* $|\alpha| = q$*, define* $\nabla_x^\alpha = \frac{1}{q!}\sum \nabla_{i_1}\cdots\nabla_{i_q}$ *where the sum is taken over all ordered collections of indices* $i_1, \ldots, i_q$ *corresponding to* α*. With this notation we have*

$$\nabla_x^\alpha a(x,\xi) \in \mathrm{S}^m(\Gamma).$$

Sometimes it is convenient to write a symbol into a formal series:

Definition 3.3. Let $a(x,\xi) \in \mathrm{S}^m(\Gamma)$ and $a_k(x,\xi) \in \mathrm{S}^{m_k}(\Gamma)$, where $m_k \searrow -\infty$ as $k \to \infty$. We shall use the notation

$$a \sim \sum_k a_k, \quad \text{as } |\xi| \to \infty,$$

if $a - \sum_{j=0}^k a_j \in \mathrm{S}^{m_{k+1}}$ for all k. Such formal series of symbols is called asymptotic.

The next lemma ([Saf97, Lemma 3.2]) allows us to construct new symbols from asymptotic series.

Lemma 3.4. *Let* $a_k \in \mathrm{S}^{m_k}(\Gamma)$ *where* $m_k \searrow -\infty$ *as* $k \to \infty$ *and let* $m = \max\{m_k\}$*. Then there exists a symbol* $a \in \mathrm{S}^m(\Gamma)$ *such that* $a \sim \sum_k a_k$*, and such a is unique modulo* $\mathrm{S}^{-\infty}$.

3.2.2 Definition of ψDOs

By the Schwartz kernel theorem, for any linear operator $A : C_c^\infty(M) \to \mathcal{D}'(M)$, there exists a distribution $\mathcal{A}(x,y) \in \mathcal{D}'(M \times M)$ such that

$$\langle Au, v\rangle = \langle \mathcal{A}(x,y), u(y)v(x)\rangle$$

for all $u, v \in C_c^\infty(M)$. Such distribution $\mathcal{A}(x,y)$ is called the Schwartz kernel of A.

Definition 3.5. A linear operator $A : C_c^\infty(M) \to C^\infty(M)$ with the Schwartz kernel $\mathcal{A}(x,y)$ is called a ψDO of order m if $\mathcal{A}(x,y)$ is smooth outside the diagonal in $M \times M$ and in each coordinate chart $U \times U \subseteq M \times M$, $\mathcal{A}(x,y)$ modulo a smooth function can be represented by

$$\mathcal{A}(x,y) = \int_{\mathbb{R}^d} e^{\mathbf{i}(x-y)\cdot\xi}\, a(y;x,\xi)\, \mathrm{d}\xi$$

for some m-amplitude $a(y;x,\xi)$.

A ψDO is called *properly supported* if both projections $\operatorname{supp}\mathcal{A} \to M$ are proper, and in particular, differential operators are properly supported ψDOs. The class of operators with smooth kernels are called smoothing operators and is denoted by $\Psi^{-\infty}$. Clearly, any ψDO is a sum of a properly supported ψDO and a smoothing operator.

Next we will focus on a more special class of ψDOs, but first let us recall the definition of densities on manifolds: a κ-density μ on a manifold is a "function" which behaves under change of coordinates in the following way:

$$\mu(y) = |\det\{\partial x^i/\partial y^j\}|^{\kappa}\mu(x). \tag{3.6}$$

Let x, y be two points of M and $\{x^k\}, \{y^k\}$ coordinate systems at x and y respectively; and let us consider the determinant $p_{y,x} = |\det \Phi_{y,x}|$ for parallel transport $\Phi_{y,x} : T_x^*M \to T_y^*M$ with respect to the connection Γ on M. Obviously $p_{y,x}$ depends on the choice of coordinates at x and y and we can check easily $p_{y,x}$ is a 1-density in y and (-1)-density in x.

Definition 3.6. We denote by $\Psi^m(\Gamma)$ the class which consists of all ψDOs A such that its Schwartz kernel $\mathcal{A}(x,y)$ is smooth outside the diagonal in $M \times M$ and within a sufficiently small neighborhood of the diagonal, $\mathcal{A}(x,y)$ has the form of an oscillatory integral

$$\mathcal{A}(x,y) = (2\pi)^{-d} p_{y,x} \int_{T_x^*M} e^{\mathbf{i}\langle \exp_x^{-1}(y),\xi\rangle}\, a(x,\xi)\, \mathrm{d}\xi, \quad a(x,\xi) \in \mathrm{S}^m(\Gamma). \tag{3.7}$$

We call $a(x,\xi) \in \mathrm{S}^m(\Gamma)$ *the symbol of the* ψDO A and denote it by $\sigma_A(x,\xi)$.

Remark 4. We notice that the integral (3.7) over T_x^*M depends on the coordinates $\{x^k\}$, or equivalently, the choice of basis of T_x^*M. However, when combined with the weight factor $p_{y,x}$, (3.7) becomes a 0-density in x and 1-density in y, and is thus a well-defined Schwartz kernel. See [MS11] for a similar but more general discussion.

Remark 5. For a Riemannian manifold, the Lebesgue measure on a cotangent plane is canonical, independent of the choice of ξ. In this case, we need to further assume the coordinates $\{x^k\}$ at x is orthonormal so that $p_{y,x}$ becomes a 0-density in x and 1-density in y.

The next Proposition states that the operator defined above is a ψDO in the sense of Definition 3.5.

Proposition 3.7. *The Schwartz kernel* (3.7) *defines a* ψ*DO.*

Proof (see [Saf97, § 4.1]). Let $U \subseteq M$ be a small coordinate chart such that $\exp_x^{-1}(y)$ is well-defined for all $x, y \in U$. Within $U \times U$, we have $\exp_x^{-1}(y) = (x,y) \cdot \Psi(x,y)\xi$ for all $(x,y) \in U \times U$, where $\Psi(x,y)$ is a smooth non-degenerate $n \times n$ matrix. By a change of variable $\zeta = \Psi\xi$ in (3.7), we conclude

$$\mathcal{A}(x,y) = (2\pi)^{-d} p_{y,x} |\det \Psi|^{-1} \int e^{\mathbf{i}(x,y)\cdot\zeta}\ a(x, \Psi^{-1}\zeta)\, \mathrm{d}\zeta, \quad \forall (x,y) \in U \times U.$$

We finish our proof by noting that $p_{y,x}|\det \Psi|^{-1} a(x, \Psi\zeta)$ is an amplitude of order m. ■

We may replace the symbol in (3.7) by an amplitude $a(y;x,\xi) \in \mathrm{S}^m(\Gamma)$ of the same order, and such defined $\mathcal{A}(x,y)$ still defines a ψDO in $\Psi^m(\Gamma)$. Precisely, we have

Lemma 3.8 ([Saf97, Proposition 4.5]). *If* $a(y;x,\xi) \in \mathrm{S}^m(\Gamma)$*, then the oscillatory integral*

$$\mathcal{A}(x,y) = (2\pi)^{-d} p_{y,x} \int_{T_x^*M} e^{\mathbf{i}\langle \exp_x^{-1}(y), \xi\rangle}\ a(y;x,\xi)\, \mathrm{d}\xi$$

coincides with the Schwartz kernel of a ψDO $A \in \Psi^m(\Gamma)$ such that

$$\sigma_A(x,\xi) \sim \sum \frac{1}{\alpha!} \mathbf{D}_\xi^\alpha \nabla_y^\alpha \, a(y;x,\xi)|_{y=x} \in \mathrm{S}^m(\Gamma) \tag{3.8}$$

as $|\xi| \to \infty$.

In fact, the classes $\Psi^m(\Gamma)$ are independent of the connection Γ and will later be denoted by Ψ^m if no specific connection is needed, so the choice of Γ affects only the full symbols. If $a \in \mathrm{S}^{-\infty}$ is a smoothing symbol (or amplitude), then the distribution in (3.7) is smooth and the corresponding ψDO is in $\Psi^{-\infty}$.

Finally at the end of this section we need to point out the symbol σ_A for a ψDO $A \in \Psi^m$ is unique modulo $\mathrm{S}^{-\infty}$:

Corollary 3.9 ([Saf97, Corollary 4.6])**.** *The map $A \mapsto \sigma_A$ is an isomorphism of the factor-classes $\Psi^m/\Psi^{-\infty}$ and $\mathrm{S}^m/\mathrm{S}^{-\infty}$.*

3.2.3 Composition of ψDOs

In this section we look at the composition of two ψDOs. The following theorem, of which the proof can be found in [Saf97, § 8.1, 8.2], is of vital importance to our calculations in this thesis.

Theorem 3.10. *Let $A \in \Psi^{m_1}$ and $B \in \Psi^{m_2}$, and at least one of the ψDOs are properly supported. Then $AB \in \Psi^{m_1+m_2}$ and*

$$\sigma_{AB}(x,\xi) \sim \sum_{\alpha,\beta,\gamma} \frac{1}{\alpha!}\frac{1}{\beta!}\frac{1}{\gamma!} P_{\beta,\gamma} \mathbf{D}_\xi^{\alpha+\beta} \sigma_A(x,\xi)\, \mathbf{D}_\xi^\gamma \nabla_x^\alpha \sigma_B(x,\xi), \tag{3.9}$$

as $|\xi| \to \infty$*, where* $P_{\beta,\gamma} \in C^\infty(T^*M)$ *are polynomials in* ξ *satisfying*

$$\deg P_{\beta,\gamma} \leqslant \min\{|\beta|, |\gamma|, \frac{1}{3}(|\beta| + |\gamma|)\}, \tag{3.10}$$

so that (3.9) *forms an asymptotic series.*

3.3 Symbols of Differential Operators

For a fixed point $x \in M$ and local coordinates $\{x^k\}$ near x, there is a unique normal coordinate system $\{y^k\}$ centered at x such that the Jacobian $(\partial y^i/\partial x^j) = I$. For such choice of coordinates $\{x^k\}$ and $\{y^k\}$, we denote $p_{y,x}$ by $\tau_x(y)$, which is considered as a function of y. Note $\tau_x(y)$ here is indeed a function, i.e., a 0-density in y, for that $\{y^k\}$ is determined by $\{x^k\}$.

Suppose A is a differential operator, one can define its full symbol $\sigma_A(x,\xi)$ as (see [Saf97, §5])

$$\sigma_A(x,\xi) = A(y, \mathbf{D}_y)(e^{i(y-x)\cdot\xi}\tau_x^{-1}(y))|_{y=x}, \tag{3.11}$$

which is a sum of endomorphisms that is positively homogeneous in ξ. For example, if

$$A = \sum_k (-\mathbf{i}) \cdot a^k \partial_{x^k} + a^0$$

is a first-order differential operator with $a^k,\ a^0 \in C^\infty(M)$, then its symbol is

$$\sigma_A(x,\xi) = \sum_k a^k(x)\xi_k + \mathbf{i}\sum_{j,k} a^k(x)\Gamma^j_{kj}(x) + a^0(x)$$

for an arbitrary coordinate system $\{x^k\}$. Note that the leading term $\sum_k a^k(x)\xi^k$ is exactly the principal symbol, which we will denote by σ'_A.

3.4 Elliptic Differential Operators

Recall that a differential operator $A : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ is elliptic if the principal symbol $\sigma'_A(x,\xi) \in \mathrm{End}(\mathcal{E}_x)$ is invertible for all $\xi \neq 0$. By formula (3.11), the full symbol of an elliptic operator A of order m has the form

$$\sigma_A(x,\xi) = a_m(x,\xi) + \cdots + a_0(x,\xi) \tag{3.12}$$

where $a_m = \sigma'_A$ is the principal symbol and each $a_k(x,\xi)$ is homogeneous in ξ of degree k. If $\xi \neq 0$, we can rewrite formula (3.12) into

$$\begin{aligned} \sigma_A(x,\xi) &= a_m(x,\xi)[I + b_{-1}(x,\xi) + \cdots + b_{-m}(x,\xi)] \\ &= a_m(x,\xi)\Big[I + \frac{b_{-1}(x,\eta)}{t} + \cdots + \frac{b_{-m}(x,\eta)}{t^m}\Big] \end{aligned} \tag{3.13}$$

where $b_{k-m}(x,\xi) = a_k(x,\xi) \cdot a_m^{-1}(x,\xi)$ is homogeneous in ξ of degree $k-m$ and $|\eta| = 1$ is a covector such that $\xi = t\eta$ for some $t > 0$.

For a compact subset $K \subseteq M$, let

$$R(K) = (m+1) \cdot \Big(\sum_k \sup_{x\in K, |\eta|=1} |b_{-k}(x,\eta)| + 1\Big) \tag{3.14}$$

so that

$$\left|\frac{b_{-k}(x,\eta)}{t^k}\right| \leqslant \left|\frac{b_{-k}(x,\eta)}{t}\right| < \frac{1}{m+1}$$

for all $|\xi| = t > R(K)$, and

$$\left|\sum_k b_{-k}(x,\xi)\right| = \left|\sum_k \frac{b_{-k}(x,\eta)}{t^k}\right| < \frac{m}{m+1}.$$

Therefore for all $(x, \xi) \in T^*M$ with $x \in K$ and $|\xi| > R(K)$, $\sum_k b_{-k}(x, \xi)$ has no eigenvalue of -1, so $[I + \sum_k b_{-k}(x, \xi)]$, and thus $\sigma_A(x, \xi)$, is invertible. Moreover, we observe the symbol $\sigma_A(x, \xi)$ of an elliptic differential operator A of order m satisfies the following condition: for any compact $K \subseteq M$, there is a positive constant c_K such that

$$(1 + |\xi|)^m \leqslant c_K |\sigma_A(x, \xi)|, \tag{3.15}$$

for all $(x, \xi) \in T^*M$ with $x \in K$ and $|\xi| \geqslant R(K)$.

Remark 6. The proof of formula (3.15) above generalizes [Shu01, Proposition 5.1].

The next lemma shows the inverse of A, if exists, is a symbol. Its proof and more general results can be found in [Saf97, § 10].

Lemma 3.11. *Suppose the symbol $\sigma_A(x, \xi)$ of a ψDO $A \in \Psi^m(\Gamma)$ satisfies* (3.15)*. If $b(x, \xi) \in$ End$(\mathcal{E})$ and for any compact $K \subseteq M$, there exists a positive constant r_K such that for all $(x, \xi) \in T^*M$ with $x \in K$, $|\xi| > r_K$,*

$$\sigma_A(x, \xi) \cdot b(x, \xi) = I,$$

then $b(x, \xi) \in \mathrm{S}^{-m}(\Gamma)$. Clearly such symbol $b(x, \xi)$ is unique modulo endomorphisms that vanish for large $|\xi|$.

Let us fix a locally finite covering $\{V_\alpha\}$ of M such that each V_α is precompact and let $\mathcal{C} := \{\overline{V_\alpha}\}$. For each $(x, \xi) \in T^*M$, let $\chi(x, \xi)$ be a smooth function on T^*M such that

$$\chi(x, \xi) = \begin{cases} 1, & \text{if } x \in K, |\xi| \geqslant R(K) + 1 \\ 0, & \text{if } x \in K, |\xi| \leqslant R(K) \end{cases} \quad \text{for all compact } K \in \mathcal{C}, \tag{3.16}$$

so $\chi(x,\xi)\sigma_A^{-1}(x,\xi)$ is well-defined on T^*M and for any (x,ξ) with $x \in K, |\xi| > R(K)+1$,

$$\sigma_A(x,\xi)\chi(x,\xi)\sigma_A^{-1}(x,\xi) = I.$$

Therefore by Lemma 3.11, $\chi(x,\xi)\sigma_A^{-1}(x,\xi) \in \mathrm{S}^{-m}$ and will be denoted by $\sigma_A^{(-1)}(x,\xi)$. To summarize our discussion in this section, we have

Corollary 3.12. *If A is an elliptic differential operator of order m, then there exists a symbol $\sigma_A^{(-1)}(x,\xi) \in \mathrm{S}^{-m}(\Gamma)$ such that*

$$\sigma_A(x,\xi) \cdot \sigma_A^{(-1)}(x,\xi) = I$$

for large $|\xi|$.

Remark 7. In general, the subsets

$$U_1 = \{(x,\xi) \in T^*M \,:\, x \in K, |\xi| \geqslant R(K)+1 \text{ for all compact } K \in \mathcal{C}\}$$

and

$$U_2 = \{(x,\xi) \in T^*M \,:\, x \in K, |\xi| \leqslant R(K) \text{ for all compact } K \in \mathcal{C}\}$$

are not closed in T^*M. But $\chi(x,\xi)$ in (3.16) is well-defined since the closures $\overline{U_1}$ and $\overline{U_2}$ are disjoint.

Chapter 4

Proof of Theorem 2.14

Since the Lie group action on M is cocompact, the injectivity radii at points of M has a lower bound L. Consider a smooth cut-off function $\chi(x, y) : M \times M \to [0, 1]$:

$$\chi(x, y) = \begin{cases} 1 & \text{if } \operatorname{dist}(x, y) \leqslant L/2; \\ 0 & \text{if } \operatorname{dist}(x, y) \geqslant L. \end{cases}$$

Now let $\boldsymbol{\nabla}$ denote the Levi-Civita connection on M, and for any symbol $a(x, \xi) \in \mathrm{S}^m(\boldsymbol{\nabla})$, let us define an integral operator

$$Au(x) = \int_M \Big(\int_{T_x^* M} e^{-\mathbf{i}\langle \exp_x^{-1}(y), \xi \rangle} \chi(x, y) a(x, \xi) \, \mathrm{d}\xi \Big) u(y) \, \mathrm{d}y, \tag{4.1}$$

of which the Schwartz kernel near the diagonal in $M \times M$ can be represented as

$$\mathcal{A}(x, y) = p_{y,x} \int_{T_x^* M} e^{-\mathbf{i}\langle \exp_x^{-1}(y), \xi \rangle} a(x, \xi) w(x, y) \, \mathrm{d}\xi, \tag{4.2}$$

where $w(x,y) = (2\pi)^d\sqrt{g(y)}/p_{y,x}$. We set the coordinate system at x to be orthonormal, so $p_{y,x}$ is a 0-density in x and 1-density in y and that the factor $w(x,y)$ is a well-defined non-vanishing function of both x and y. Clearly the amplitude $a(x,\xi)w(x,y)$ belongs to class $\mathrm{S}^m(\boldsymbol{\nabla})$, so by Lemma 3.8 and formula (3.8), A is a ψDO with symbol

$$\sigma_A(x,\xi) \sim a(x,\xi)w(x,y)|_{x=y} + \sum_{|\alpha|\geq 1} \left(\mathbf{D}_\xi^\alpha a(x,\xi)\nabla_y^\alpha w(x,y)\right)|_{y=x} \tag{4.3}$$

as $|\xi| \to \infty$. Here we use the notation $w(x,y)|_{y=x}$ instead of $w(x,x)$ to emphasize that the coordinates $\{x^k\}$ and $\{y^k\}$ at x and y are in general *different*. As a consequence, the weight factor $p_{y,x} \neq 1$ in general for $y = x$.

The Dirac operator $\not{D}_+$ is elliptic, so by Corollary 3.12 its symbol has an inverse for large $|\xi|$, which we denote by $\sigma_{\not{D}_+}^{(-1)}(x,\xi)$. Consider an operator $\tilde{A}$ similar to the one defined in (4.1):

$$\tilde{A}u(x) = \int_M \left(\int_{T_x^*M} e^{-\mathbf{i}\langle \exp_x^{-1}(y),\xi\rangle}\,\chi(x,y)\tilde{a}(x,\xi)\,\mathrm{d}\xi\right) u(y)\,\mathrm{d}y,$$

where $\tilde{a}(x,\xi) = \sigma_{\not{D}_+}^{-1}(x,\xi)\,a(x,\xi)\,w^{-1}(x,y)|_{y=x} \in \mathrm{S}^{m-1}(\boldsymbol{\nabla})$. Obviously, $\tilde{A}$ is a properly supported ψDO, so by Theorem 3.10,

$$\begin{aligned}\sigma_{\not{D}_+\tilde{A}} &\sim \sum_{\alpha,\beta,\gamma} \frac{1}{\alpha!}\frac{1}{\beta!}\frac{1}{\gamma!} P_{\beta,\gamma}(x,\xi)\,\mathbf{D}_\xi^{\alpha+\beta}\sigma_{\not{D}_+}(x,\xi)\,\mathbf{D}_\xi^\gamma\nabla_x^\alpha\sigma_{\tilde{A}}(x,\xi)\\ &= \sigma_{\not{D}_+}(x,\xi)\cdot\sigma_{\tilde{A}}(x,\xi) + R_1(x,\xi)\end{aligned}$$

as $|\xi| \to \infty$, where $R_1(x,\xi) \in \mathrm{S}^{m-r}(\boldsymbol{\nabla})$ for some positive r. (More precisely, $r \geqslant 2/3$ by (3.10).) Also by (4.3) we have

$$\begin{aligned}\sigma_{\not{D}_+}(x,\xi)\cdot\sigma_{\tilde{A}}(x,\xi) &\sim \sigma_{\not{D}_+}(x,\xi)\Big[\tilde{a}(x,\xi)w(x,y) + \sum_{|\alpha|\geq 1}\big(\mathbf{D}_\xi^\alpha \tilde{a}(x,\xi)\nabla_y^\alpha w(x,y)\big)\Big]\Big|_{y=x} \\ &= a(x,\xi) + \sigma_{\not{D}_+}(x,\xi)\cdot\sum_{|\alpha|\geq 1}\big(\mathbf{D}_\xi^\alpha \tilde{a}(x,\xi)\nabla_y^\alpha w(x,y)\big)\big|_{y=x} \\ &= a(x,\xi) + R_2(x,\xi),\end{aligned}$$

where $R_2(x,\xi) \in \mathrm{S}^{m-1}(\boldsymbol{\nabla})$ as $|\xi| \to \infty$. Therefore we conclude

$$\sigma_{\not{D}_+\tilde{A}} \sim a(x,\xi) - a'(x,\xi), \tag{4.4}$$

where $a' \in \mathrm{S}^{m-r}(\boldsymbol{\nabla})$ for some positive r. Similarly we have

$$\sigma_{\tilde{A}\not{D}_+} \sim a(x,\xi) - a''(x,\xi), \quad a''(x,\xi) \in \mathrm{S}^{m-r}(\boldsymbol{\nabla}). \tag{4.5}$$

Let $a_1 \equiv 1$ and $a_{k+1} = a'_k$ for $k \in \mathbb{N}_+$, where $a'_k \in \mathrm{S}^{-kr}(\Gamma)$ is the symbol in (4.4) when we replace a by a_k, then there exists a symbol $a \in \mathrm{S}^0(\Gamma)$ unique modulo $\mathrm{S}^{-\infty}$ such that $a \sim \sum_k a_k$ by Lemma 3.4. Therefore by setting $q(x,y) = \sigma_{\not{D}_+}^{-1}(x,\xi)\, a(x,\xi)\, w^{-1}(x,y)|_{y=x} \in \mathrm{S}^{-1}(\boldsymbol{\nabla})$, the associated integral operator

$$Qu(x) = \int_M \Big(\int_{T_x^*M} e^{-\mathbf{i}\langle \exp_x^{-1}(y),\xi\rangle}\, \chi(x,y) q(x,\xi)\,\mathrm{d}\xi\Big)\, u(y)\,\mathrm{d}y \tag{4.6}$$

satisfies $Q\not{D}_+ = I \bmod \Psi^{-\infty}$.

Analogously, we can construct a different ψDO Q' by replacing a'_k by a''_k in (4.5), so that $\not{D}_+ Q' = I \mod \Psi^{-\infty}$. Noticing $Q = Q\not{D}_+ Q' = Q' \mod \Psi^{-\infty}$, we conclude that $Q\not{D}_+ = \not{D}_+ Q = I \mod \Psi^{-\infty}$.

To finish our proof, we need to show that the Schwartz kernels of $(Q\not{D}_+ - I)$ and $(\not{D}_+ Q - I)$ are G-invariant and supported near the diagonal of $M \times M$. Let L still denote the lower bound of injectivity radii, and let ϕ and ψ be two arbitrary test functions satisfying $\operatorname{dist}(\operatorname{supp}\phi, \operatorname{supp}\psi) > L$, so that $\phi(x)\psi(y)$ is supported outside the L-neighborhood of the diagonal. One can easily see from (4.6) that $\operatorname{supp} Q\phi \subseteq N_L(\operatorname{supp}\phi)$, where $N_L(\operatorname{supp}\phi)$ is the L-neighborhood of $\operatorname{supp}\phi$. So

$$\langle (Q\not{D}_+ - I)\,\phi, \psi\rangle = \langle (\not{D}_+ Q - I)\,\phi, \psi\rangle = 0.$$

Therefore the Schwartz kernels of $(Q\not{D}_+ - I)$ and $(\not{D}_+ Q - I)$ vanish outside the L-neighborhood of the diagonal. The invariant property of the Schwartz kernel is due to the G-invariance of $q(x, \xi)$.

Chapter 5

Application

In this chapter we will generalize the Lichnerowicz vanishing theorem for spinor bundles with a proper and cocompact action. In 2013, Z. Liu has proved the Mathai-Zhang index vanishes for a spin manifold which carries a G-invariant Riemannian metric of positive scalar curvature if the group G in unimodular [Liu13]. W. Zhang in 2015 extended Liu's result to general Lie groups [Zha15]. We provide a new proof of the same result using our framework developed in this thesis.

5.1 Preliminaries

First we introduce some basic concepts and the classical Lichnerowicz theorem for compact manifolds. Let $\mathcal{S} \to M$ be a spinor bundle on an even-dimensional spin manifold M with a Levi-Civita connection $\nabla^{\mathcal{S}} : \Gamma(\mathcal{S}) \to \Gamma(T^*M \otimes \mathcal{S})$ and $\not{D}$ the Dirac operator associated to $\nabla^{\mathcal{S}}$. The second covariant derivative is defined by the composition:

$$\nabla^{T^*M\otimes\mathcal{S}}\nabla^{\mathcal{S}} : \Gamma(\mathcal{S}) \xrightarrow{\nabla^{\mathcal{S}}} \Gamma(T^*M \otimes \mathcal{S}) \xrightarrow{1\otimes\nabla^{\mathcal{S}}+\nabla\otimes 1} \Gamma(T^*M \otimes T^*M \otimes \mathcal{S}) \tag{5.1}$$

so that

$$(\nabla^{T^*M\otimes\mathcal{S}}\nabla^{\mathcal{S}} u)(X,Y) = \nabla^{\mathcal{S}}_X \nabla^{\mathcal{S}}_Y u - \nabla^{\mathcal{S}}_{(\boldsymbol{\nabla}_X Y)} u,$$

where $\boldsymbol{\nabla}$ is the Levi-Civita connection on M. The connection Laplacian $\Delta^{\mathcal{S}} : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ is defined by

$$\Delta^{\mathcal{S}} u := -\mathrm{tr}(\nabla^{T^*M\otimes\mathcal{S}}\nabla^{\mathcal{S}} u). \tag{5.2}$$

Here the trace $\mathrm{tr}(E)$ is the contraction of any $E \in \Gamma(T^*M \otimes T^*M \otimes \mathcal{S})$ with the metric tensor $g = g^{ij}\,\partial_{x^i} \otimes \partial_{x^j}$.

The Riemannian curvature R on a Riemannian manifold M is a $(1,3)$-tensor defined by

$$R(X,Y)Z := \boldsymbol{\nabla}_X\boldsymbol{\nabla}_Y Z - \boldsymbol{\nabla}_Y\boldsymbol{\nabla}_X Z - \boldsymbol{\nabla}_{[X,Y]}Z.$$

Given a local frame $\{\partial_{x^i}\}$ of the tangent bundle TM, we define a $(0,4)$-tensor R_{ijkl} using the Riemannian metric g:

$$R_{ijkl} := g(R(\partial_{x^i}, \partial_{x^j})\partial_{x^k}, \partial_{x^l}).$$

A scalar curvature **scal** is a real number defined by

$$\mathbf{scal} := \sum_{lm} R_{lmlm}.$$

Theorem 5.1 (Lichnerowicz Formula, see [LM89, Chap. II,Theorem 8.8]). *Let M be a spin manifold and suppose $\mathcal{S}$ is a spinor bundle over M endowed with a Riemannian connection $\nabla^{\mathcal{S}}$. Then*

$$\not{D}^2 = \Delta^{\mathcal{S}} + \frac{\mathbf{scal}}{4}. \tag{5.3}$$

5.2 Spinor Bundles over Compact Manifolds

Theorem 5.2 (Lichnerowicz Vanishing Theorem, see [LM89, Chap. II,Corollary 8.9])**.** *Let M be a* compact *spin manifold and $\mathcal{S}$ a spinor bundle over M. Suppose the scalar curvature of M is non-negative and strictly positive at some point. Then* $\ker \not{D} = 0$ *and* $\operatorname{ind}(\not{D}) = 0$.

Proof. By the Lichnerowicz formula (5.3), for any section $u \in \Gamma(\mathcal{S})$ we have

$$\int_M (\not{D}^2 u, u)\,\Omega = \int_M (\Delta^{\mathcal{S}} u, u)\,\Omega + \frac{1}{4}\int_M \mathbf{scal}\cdot \|u\|^2\,\Omega \tag{5.4}$$

where Ω is a volume form on M, and $(\cdot,\cdot)$ is the Hermitian product of $\mathcal{S}$.

Let $\{e_i\}$ be an local orthonormal frame on TM. Since

$$\Delta^{\mathcal{S}} u = -\sum_i (\nabla^{\mathcal{S}}_{e_i}\nabla^{\mathcal{S}}_{e_i} - \nabla^{\mathcal{S}}_{\boldsymbol{\nabla}_{e_i} e_i})u$$

we have

$$(\Delta^{\mathcal{S}} u, u) = -\sum_i (\nabla^{\mathcal{S}}_{e_i}\nabla^{\mathcal{S}}_{e_i} u - \nabla^{\mathcal{S}}_{\boldsymbol{\nabla}_{e_i} e_i} u, u),$$

and

$$(\nabla^{\mathcal{S}}_{e_i}\nabla^{\mathcal{S}}_{e_i} u, u) = -(\nabla^{\mathcal{S}}_{e_i} u, \nabla^{\mathcal{S}}_{e_i} u) + e_i(\nabla^{\mathcal{S}}_{e_i} u, u)$$

because the connection is compatible with the Hermitian product of $\mathcal{S}$. Hence we have

$$(\Delta^{\mathcal{S}} u, u) = \sum_i (\nabla^{\mathcal{S}}_{e_i} u, \nabla^{\mathcal{S}}_{e_i} u) - e_i(\nabla^{\mathcal{S}}_{e_i} u, u) + (\nabla^{\mathcal{S}}_{\boldsymbol{\nabla}_{e_i} e_i} u, u) \tag{5.5}$$

Let α be a one-form such that for any smooth vector field X, $\alpha(X) := -(\nabla^{\mathcal{S}}_X u, u)$. The divergence of α is

$$\operatorname{div}\alpha = \operatorname{tr}(\boldsymbol{\nabla}\alpha) = \sum_i e_i\alpha(e_i) - \alpha(\boldsymbol{\nabla}_{e_i}e_i) = \sum_i -e_i(\nabla^{\mathcal{S}}_{e_i}u, u) + (\nabla^{\mathcal{S}}_{\boldsymbol{\nabla}_{e_i}e_i}u, u). \tag{5.6}$$

Together with (5.5) we have

$$(\Delta^{\mathcal{S}}u, u) = (\nabla^{\mathcal{S}}u, \nabla^{\mathcal{S}}u) + \operatorname{div}\alpha. \tag{5.7}$$

Since $\int_M \operatorname{div}(\alpha)\,\Omega = 0$, by taking the integral of both sides of (5.7), we have

$$\int_M (\Delta^{\mathcal{S}}u, u)\,\Omega = \int_M \|\nabla^{\mathcal{S}}u\|^2\,\Omega.$$

If u is in the kernel of the Dirac operator $\not{D}$, then

$$0 = \int_M (\not{D}^2 u, u)\,\Omega = \int_M \|\boldsymbol{\nabla}u\|^2\,\Omega + \frac{1}{4}\int_M \mathbf{scal}\cdot\|u\|^2\,\Omega.$$

Under the hypothesis that $\mathbf{scal} > 0$ we know $\boldsymbol{\nabla}u = 0$ and $\mathrm{d}(u,u) = (\boldsymbol{\nabla}u, u) + (u, \boldsymbol{\nabla}u)$ implies $\|u\|$ is a constant. Together with $\int_M \mathbf{scal}\cdot\|u\|^2\,\Omega = 0$ we conclude $u = 0$, that is, the kernel of $\not{D}$ vanishes. Since $\not{D}$ is self-adjoint, its cokernel also vanishes. We conclude that $\operatorname{ind}(\not{D}) = 0$. ■

As a consequence of the Atiyah-Singer Index Theorem, one has from Theorem 5.2 the following theorem:

Corollary 5.3 ([LM89, Chap. II,Theorem 8.11])**.** *Let M be a compact spin manifold of dimension $4k$. If M admits a metric of positive scalar curvature, then the $\hat{A}$-genus $\hat{A}(M) = 0$.*

5.3 Spinor Bundles with Lie Group Action

In this section $\mathcal{S} \to M$ is still a spinor bundle on an even-dimensional spin manifold, and the scalar curvature of M is non-negative and positive at some point. Let G be a Lie group acting on M properly and cocompactly, and assume the G-action preserves the spin structure of M. We also assume G acts on $\mathcal{S}$ equivariantly. Then we have the following generalization of Theorem 5.2:

Theorem 5.4. *Suppose M is a even-dimensional spin manifold, $\mathcal{S}$ is a spinor bundle over M. Let G be a Lie group which acts on M properly and cocompactly. We assume that G acts on $\mathcal{S}$ equivariantly and that the Dirac operator $\not{D}$ associated to the Levi-Civita connection $\nabla^{\mathcal{S}}$ on the spinor bundle $\mathcal{S}$ is G-invariant. If g is a G-invariant Riemannian metric on M and the scalar curvature with respect to g is non-negative and strictly positive at some point on M, then the index of $\not{D}$ as a Fredholm operator on $L^2_\lambda(\mathcal{S})$ is 0.*

Proof. Let $c(x)$ be a cut-off function in (2.1) and Ω is a G-invariant volume form. For any $u \in \Gamma_\lambda(\mathcal{S})^G$, by the Lichnerowicz formula we have

$$\int_M c(x)(\not{D}^2 u, u)\,\Omega = \int_M c(x)(\Delta^{\mathcal{S}} u, u)\,\Omega + \frac{1}{4}\int_M \mathbf{scal} \cdot \|u\|^2\,\Omega. \tag{5.8}$$

Together with (5.7) and Lemma 2.9 we have

$$\int_M c(x)(\Delta^{\mathcal{S}} u, u)\,\Omega = \int_M c(x)\|\boldsymbol{\nabla} u\|^2\,\Omega.$$

Therefore if u is in the kernel of $\not{D}$, then

$$0 = \int_M c(x)(\not{D}^2 u, u)\,\Omega = \int_M c(x)\|\boldsymbol{\nabla} u\|^2\,\Omega + \frac{1}{4}\int_M c(x)\,\mathbf{scal} \cdot \|u\|^2\,\Omega.$$

So if **scal** > 0, similar to the proof of Theorem 5.2, we must have $u = 0$ and that $\ker \not{D} = \{0\}$. By Theorem 2.11, $\not{D}$ is self-adjoint in $L^2_\lambda(\mathcal{S})$, so its cokernel is also trivial. We conclude that the index of $\not{D}$ in $L^2_\lambda(\mathcal{S})$ is 0. ■

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