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Carlson's Theorem for Different Measures

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Carlson's Theorem for Different Measures
by
Meredith Sargent

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<div align="right">Meredith Sargent</div>

Washington University in St. Louis

May 2018

Dedicated to my parents, Debora and Richard Sargent.

ABSTRACT OF THE DISSERTATION

Carlson's Theorem for Different Measures

by

Meredith Sargent

Doctor of Philosophy in Mathematics

Washington University in St. Louis, 2018

Professor John E. McCarthy, Chair

Hedenmalm, Lindqvist, and Seip in 1997 [11] revitalized the modern study of Dirichlet series by defining the space $\mathcal{H}^2$ and considering it as isometrically isomorphic to the Hardy space of the infinite polytorus $H^2(\mathbb{T}^\infty)$. This allowed a new viewpoint to be applied to classical theorems, including Carlson's theorem about the integral in the mean of a Dirichlet series. Carlson's theorem holds only for vertical lines in the right half plane, and cannot be extended to the boundary in full generality (as shown by Saksman and Seip in [14]). However, Carlson's theorem can be shown to hold on the imaginary axis for a more restrictive class of Dirichlet series, and we shall do so.

The main result contained in this dissertation is a generalized version of Carlson's theorem: given a Borel probability measure on the polytorus, a measure is constructed on the imaginary axis so that the integral in the mean is equal to the integral on the polytorus.

Chapter 1 contains background material on Dirichlet series, including questions of convergence, the Bohr lift, and spaces of Dirichlet series. Chapter 2 is the statement and proof of the main result generalizing Carlson's theorem.

Finally, Chapter 3 is a small result about weighted spaces of Dirichlet series from work done jointly with Houry Melkonian.

Chapter 1

Introduction

A well known class of theorems are so called "ergodic theorems" which, roughly speaking, say that a "time average" is equal to a "space average." That is, if one has a function over a space, and a path that covers the space (in some sense,) then the normalized integral of the function against a measure on the space should be equal to the integral in the mean of the function along the path. Formally, a path covering the space is called ergodic:

Definition (Ergodic Flow). A measurable flow S_τ on a probability space X is called an ergodic flow if all invariant sets have measure 0 or 1.

Theorem (Birkhoff-Khinchin Ergodic Theorem, (cf [9])). Let S_τ be an ergodic flow on a probability space (X, μ), and let $f : X \to \mathbb{R}$ be a μ-integrable function. Then, for μ-almost every $x_0 \in X$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(S_\tau x_0) d\tau = \int_X f(x) d\mu(x).$$

For Dirichlet series, a 1922 theorem of Carlson [8], can be viewed as a version of an ergodic theorem where the integral in the mean is along a vertical line in $\mathbb{C}_+$:

Theorem (Carlson's Theorem). If a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges in the right half plane $\mathbb{C}_+$ and is bounded in every half plane $\Re(s) \geq \delta$ for $\delta > 0$, then for each $\sigma > 0$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |f(\sigma + it)|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma}. \tag{1.0.1}$$

The ergodic interpretation requires an idea of Bohr, which allows us to consider Dirichlet series as power series on the infinite polydisk. With this context, the "path" for the ergodic

theorem viewpoint is the image of a vertical line: the left hand side of (1.0.1) is the integral along that path. The "space" is the surface of the polytorus, and the right hand side of (1.0.1) can be shown to be the integral of the lift of f on the polytorus.

It is reasonable to ask about the behavior of Carlson's theorem if $\sigma = 0$, that is, if one integrates along the imaginary axis. Unfortunately, due to matters of convergence, Carlson's theorem does not hold on this boundary, as shown by Saksman and Seip in [14].

The main goal of this work is to generalize Carlson's theorem by changing the measure on the polytorus and showing that there is a measure that can be put on the imaginary axis which gives equality. Some of the work in this dissertation is contained in [15] (under revision.) Chapter 3 contains work done jointly with Houry Melkonian in [13].

Throughout, we adopt the standard notation where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle, and $\mathbb{D}^\infty$ and $\mathbb{T}^\infty$ are their infinite Cartesian products. We also use the complex variable $s = \sigma + it$, as is common in discussion of Dirichlet series.

1.1   Dirichlet Series: Definitions and Background

A Dirichlet series is a series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \tag{1.1.1}$$

where s is a complex variable of the form $s = \sigma + it$. The classic example is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} (-1)^n n^{-s} \tag{1.1.2}$$

converging absolutely in the right half plane $\sigma > 1$. In general, if a Dirichlet series converges, it is in a right half plane which we denote $\mathbb{C}_\theta = \{s \in \mathbb{C} : \sigma > \theta\}$. The critical line of these

half planes $s = \sigma_c$, is called the abscissa of convergence and is somewhat analogous to the boundary of the disk of convergence for power series in $\mathbb{C}$. In the power series case, if the series converges, it is in an open disk of radius R, with the series converging uniformly and absolutely on every closed disk of slightly smaller radius, where the function represented by the series is bounded on that disk.

For Dirichlet series, in contrast, there is an abscissa for each type of convergence: σ_a for absolute convergence, σ_u for uniform convergence, and σ_b for convergence to a bounded function. Let us define these more formally.

Definition (Abscissae of Convergence).

- $\sigma_c = \inf \left\{ \Re(s) : \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges } \right\}$

- $\sigma_a = \inf \left\{ \Re(s) : \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges absolutely } \right\}$

- $\sigma_u = \inf \left\{ \theta : \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges uniformly in } \mathbb{C}_\theta \right\}$

- $\sigma_b = \inf \left\{ \theta : \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges to a bounded function in } \mathbb{C}_\theta \right\}$

The easy relationships between the abscissae can be summarized as $\sigma_c \leq \sigma_b \leq \sigma_u \leq \sigma_a$. It is interesting to consider the example of the alternating Riemann zeta function

$$\tilde{\zeta}(s) = \sum_{n=1}^{\infty} n^{-s}$$

which converges in the positive right half plane $\mathbb{C}_+$, and converges absolutely in $\mathbb{C}_1$, showing that unlike in the power series case, convergence does not imply absolute convergence in a slightly smaller half plane. It can be shown, however, that $\sigma_a \leq \sigma_c + 1$.

A theorem of Bohr [7] gives equality of σ_u and σ_b. The idea behind this, presented in another paper of Bohr [6] is the celebrated Bohr lift of Dirichlet series to power series on the infinite dimensional polydisk, $\mathbb{D}^\infty$, using the fundamental theorem of arithmetic and the

change of variables

$$z_1 = 2^{-s}, \; z_2 = 3^{-s}, \; \ldots, \; z_j = p_j^{-s}, \ldots$$

where p_j denotes the jth prime number. This yields the formal correspondence

$$f(s) \mapsto \mathcal{B}f(z)$$

$$\sum_{\alpha} a_{\alpha} \, [p_1^{\alpha_1} \cdots p_d^{\alpha_d}]^{-s} \mapsto \sum a_n z_1^{\alpha_1} \cdots z_d^{\alpha_d}. \tag{1.1.3}$$

1.2 Hilbert and Banach Spaces of Dirichlet Series

1.2.1 Hedenmalm, Lindqvist, and Seip

Bohr provided a foundation, and in 1997, Hedenmalm, Lindqvist, and Seip [11] began the modern study of Dirichlet series. In this paper, they introduced

$$\mathcal{H}^2 = \{f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : \sum_{n=1}^{\infty} |a_n|^2 < \infty\}, \tag{1.2.1}$$

a Dirichlet analog of the classical Hardy space:

$$H^2(\mathbb{T}) = \{f(z) = \sum_{n=1}^{\infty} a_n z^n : \sum_{n=1}^{\infty} |a_n|^2 < \infty\}.$$

This analogy to the disk can be attacked from another direction: the authors show that the Bohr lift provides an isometric isomorphism between $\mathcal{H}^2$ and $H^2(\mathbb{T}^{\infty})$, where $\mathbb{T}^{\infty}$ is the polytorus.

An easy application of the Cauchy-Schwarz inequality shows that all functions in $\mathcal{H}^2$ converge absolutely on $\mathbb{C}_{1/2}$. It can also be shown that $\mathcal{H}^2$ is a reproducing kernel Hilbert space, that is, for all $w \in \mathbb{C}_{1/2}$, that there is a kernel function K_w such that $\langle K_w, f \rangle = f(w)$,

$\forall f \in \mathcal{H}$. Interestingly, the reproducing kernel for $\mathcal{H}^2$ is a translation of the Riemann zeta function: $K_w(s) = \zeta(\bar{w} + s)$.

Viewing $\mathcal{H}^2$ this way, Hedenmalm, Lindqvist, and Seip are able to characterize the multipliers of $\mathcal{H}^2$. This is a central starting question for much of functional analysis on the disk:

Definition (Multiplier Algebra). Given a Hilbert space H, the multiplier algebra, $\text{Mult}(H)$, is the algebra of functions ϕ such that for every $f \in H$, $\phi f \in H$.

For the classical Hardy space $H^2(\mathbb{T})$, the multiplier algebra is H^∞, the space of functions that are bounded and analytic on the unit disk $\mathbb{D}$. The infinite dimensional analog of the Hardy space, $H^2(\mathbb{T}^\infty)$, has the comparable space $H^\infty(\mathbb{T}^\infty)$ as its multiplier algebra, and through the isomorphism of $H^2(\mathbb{T}^\infty)$ and $\mathcal{H}^2$, Hedenmalm, Lindqvist, and Seip showed that the multiplier algebra of $\mathcal{H}^2$ is $\mathcal{H}^\infty$: the space of Dirichlet series bounded and analytic on the open right half place $\mathbb{C}_+$. $\mathcal{H}^\infty$ is also isometrically isomorphic to $H^\infty(\mathbb{T}^\infty)$ under the Bohr lift (see [11], the discussion before Lemma 2.3 and the remark after the proof of Theorem 3.1).

1.2.2 Banach Spaces of Dirichlet Series

In [4], Bayart used the Bohr lift to extend the definition from [11] to the Banach space case:

Definition ($\mathcal{H}^p$, $1 \leq p < \infty$). On the Dirichlet polynomials, (Dirichlet series with finitely many terms) define a norm

$$\|P\|_{\mathcal{H}^p}^p = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |P(it)|^p dt.$$

Now choose $\mathcal{H}^p$ ($1 \leq p < \infty$) to be the completion under this norm.

It is important to note that the limit in the mean on the imaginary axis may not be defined for all Dirichlet series. Bayart shows that the functions in $\mathcal{H}^p$ converge absolutely in $\mathbb{C}_{1/2}$, but this is not enough to know about the convergence on the imaginary axis. However, completing the Dirichlet polynomials under this norm gives Banach spaces that are analogous to the classical H^p spaces.

Theorem 1 (Bayart [4])**.** The map $\mathcal{B} : \mathcal{P} \to H^p(\mathbb{T}^\infty)$ extends to an isometric isomorphism from $\mathcal{H}^p$ onto $H^p(\mathbb{T}^\infty)$.

In particular, for $p = 2$, Bayart's definition agrees with the definition from Hedenmalm, Lindqvist, and Seip. Bayart also finds the multiplier algebra:

Theorem 2 ([4])**.** Let $1 \leq p < \infty$. the set of multipliers of $\mathcal{H}^p$ is $\mathcal{H}^\infty$.

As an easy corollary, using that $\mathcal{H}^\infty$ is isometrically isomorphic to $H^\infty(\mathbb{T}^\infty)$, we can add that we have equality of the supremum norm and the multiplier norm, as in the $\mathcal{H}^2$ case:

Corollary 3. For $\phi \in Mult(\mathcal{H}^p)$, $\|\phi\|_{Mult} = \|\phi\|_\infty$.

Proof. Let ϕ be a multiplier on $\mathcal{H}^p$. Then, following the proof of Theorem 2, we arrive at

$$\|\mathcal{B}\phi\|_{H^\infty(\mathbb{T}^\infty)} \leq \|\phi\|_{Mult},$$

implying that $\mathcal{B}\phi \in H^\infty(\mathbb{T}^\infty)$. The isometric isomorphism between $\mathcal{H}^\infty$ and $H^\infty(\mathbb{T}^\infty)$ then gives that

$$\|\phi\|_\infty \leq \|\phi\|_{Mult}.$$

For the other inequality, consider $\phi \in \mathcal{H}^\infty$, and

$$\|\phi f\|_{\mathcal{H}^p} = \|\mathcal{B}(\phi f)\|_{H^p(\mathbb{T}^\infty)} = \|\mathcal{B}(\phi)\mathcal{B}(f)\|_{H^p(\mathbb{T}^\infty)}$$

$$= \left(\int_{\mathbb{T}^\infty} |\mathcal{B}(\phi)\mathcal{B}(f)|^p dm \right)^{1/p}$$

$$\leq \|\mathcal{B}(\phi)\|_{H^\infty(\mathbb{T}^\infty)} \|\mathcal{B}(f)\|_{H^p(\mathbb{T}^\infty)}$$

$$= \|\phi\|_\infty \|f\|_{\mathcal{H}^p}$$

This implies that $\phi f \in \mathcal{H}^p$ and so $\phi \in Mult(\mathcal{H}^p)$. Moreover, because $\|\phi\|_{Mult}$ is the least upper bound, $\|\phi\|_{Mult} \leq \|\phi\|_\infty$, as needed. $\qquad \square$

Another way to generalize the space $\mathcal{H}^2$ is to consider weighted Hilbert or Banach spaces. This will be discussed in more detail in Chapter 3.

Chapter 2

Dirichlet Series and Limits in the Mean

2.1 Background

In Section 1.1, we defined the Bohr lift, which allows us to consider Dirichlet series as power series on the infinite polydisk: for a Dirichlet series $f = \sum_{n=1}^{\infty} a_n n^{-s}$, we can use the fundamental theorem of arithmetic to factor each integer n uniquely and then represent f by a power series F in the variables $\{z_j = p_j^{-s}\}$.

As discussed in [14], the Bohr lift also allows us to consider any vertical line in $\mathbb{C}$ as an ergodic flow on the infinite-dimensional polytorus $\mathbb{T}^{\infty}$:

$$(e^{i\theta_1},\, e^{i\theta_2},\, \dots) \mapsto (p_1^{-it} e^{i\theta_1},\, p_2^{-it} e^{i\theta_2},\, \dots) \in \mathbb{T}^{\infty},$$

and in particular, the imaginary axis maps to the boundary of the infinite polydisk (of radius one). We would like to compare a "space average" of the power series F on $\mathbb{T}^{\infty}$ to a "time average" of the Dirichlet series f on the ergodic flow described above. For this question, we consider $\mathcal{H}^{\infty}$, the Banach space of Dirichlet series of the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{2.1.1}$$

that converge to bounded analytic functions on $\mathbb{C}_+$. A theorem of Carlson [8] tells us about the limit in the mean of a Dirichlet series on an ergodic flow for $\sigma > 0$.

Theorem 4 (Carlson's Theorem)**.** If a Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges in the

right half plane $\mathbb{C}_+$ and is bounded in every half plane $\Re(s) \geq \delta$ for $\delta > 0$, then for each $\sigma > 0$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |f(\sigma + it)|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 n^{-2\sigma}. \tag{2.1.2}$$

Proof. Let $\epsilon > 0$. Because the Dirichlet series of f converges uniformly on $\overline{\mathbb{C}_\delta}$, there exists N large enough that

$$\left| \sum_{n=1}^{M} a_n n^{-s} - f(s) \right| < \frac{\epsilon}{\|f\|_{\infty}}, \quad M > N$$

Now consider the left hand side of (2.1.2):

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |f(\sigma + it)|^2 dt \leq \lim_{T \to \infty} \frac{1}{T} \int_0^T \left(\left| \sum_{n=1}^{M} a_n n^{-s} \right| + \left| f(\sigma + it) - \sum_{n=1}^{M} a_n n^{-s} \right| \right)^2 dt$$

$$< \lim_{T \to \infty} \frac{1}{T} \int_0^T \left| \sum_{n=1}^{M} a_n n^{-s} \right|^2 dt + O(\epsilon)$$

$$= \sum_{n=1}^{M} |a_n|^2 n^{-2\sigma} + O(\epsilon)$$

where the last equality is because

$$\int_0^T n^{-it} m^{it} dt = \begin{cases} 0, & \text{if } n \neq m, \\ 1, & \text{if } n = m. \end{cases}$$

Letting M tend to infinity yields the right hand side of (2.1.2) plus a term depending on ϵ which can be made arbitrarily small. $\qquad \square$

Saksman and Seip showed in [14] that Carlson's theorem fails to hold on the imaginary axis when we replace $f(\sigma + it)$ with its non-tangential limit $f(it)$ (which exists for almost every t.)

Theorem 5 (Saksman-Seip)**.** The following two statements hold:

(i) There exists a function f in $\mathcal{H}^\infty$ such that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |f(it)|^2 dt$$

does not exist.

(ii) Given $\epsilon > 0$, there exists a singular inner function $g = \sum_{n=1}^\infty b_n n^{-s}$ in $\mathcal{H}^\infty$ such that $\sum_{n=1}^\infty |b_n|^2 \le \epsilon$.

What this result tells us is that there can be no direct analog of Carlson's Theorem on the boundary: the limit need not exist and equality need not hold, at least not for Lebesgue measure and for all functions in $\mathcal{H}^\infty$. However, by looking at a smaller space, we can prove an analog of Carlson's theorem.

The space we consider is $\mathcal{A}(\mathbb{C}_+)$, the set of Dirichlet series which are convergent on $\mathbb{C}_+$ and define uniformly continuous functions there. In [2], Aron, Bayart, Gauthier, Maestre, and Nestoridis show that $\mathcal{A}(\mathbb{C}_+)$ is a closed subspace of $\mathcal{H}^\infty$ and prove that it consists exactly of the uniform limits of Dirichlet polynomials:

Theorem 6. Given $f : \mathbb{C}_+ \to \mathbb{C}$ the following are equivalent.

1. f is the uniform limit on $\mathbb{C}_+$ of a sequence of Dirichlet polynomials.

2. f is represented by a Dirichlet series pointwise on $\mathbb{C}_+$ and f is uniformly continuous on $\mathbb{C}_+$.

2.2   Carlson's Theorem for Different Measures

On the more restrictive space $\mathcal{A}(\mathbb{C}_+)$, we then generalize Carlson's theorem (Theorem 4):

Theorem 7. For a Dirichlet series $f(s) = \sum_{n=1}^\infty a_n n^{-s}$, let $F(z)$ be the corresponding power series on $\mathbb{T}^\infty$ under the Bohr lift.

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(i) Let μ be a Borel probability measure on the infinite torus $\mathbb{T}^\infty$. There exists a locally finite Borel measure λ on $\mathbb{R}$, such that, for all $f \in \mathcal{A}(\mathbb{C}_+)$

$$\lim_{T \to \infty} \frac{1}{\lambda([0,T])} \int_0^T |f(it)|^2 \, d\lambda(t) = \int_{\mathbb{T}^\infty} |F(z)|^2 \, d\mu(z). \tag{2.2.1}$$

(ii) Let λ be a locally finite Borel measure on $\mathbb{R}$ such that the limit on the left hand side of (2.2.1) exists and is finite for all $f \in \mathcal{A}(\mathbb{C}_+)$. Then there exists a unique Borel probability measure μ on the infinite torus $\mathbb{T}^\infty$ such that, for all $f \in \mathcal{A}(\mathbb{C}_+)$, (2.2.1) holds.

Part (ii) follows from the Riesz representation theorem and will be shown in Section 2.4.1. To prove Theorem 7(i), it is helpful to consider the following useful lemma which allows us to first consider linear combinations of point masses and construct corresponding measures λ on $\mathbb{R}$, and then use that result to construct λ for general Borel measures.

Lemma 1 ([10])**.** Let X be a compact metric space. The set V of finite linear combinations of point masses is dense in the space of finite Borel measures, $M(X)$, with the weak-$*$ topology.

2.3   The Case of Point Masses

2.3.1   Kronecker's Theorem

Before we construct λ, it is also helpful to recall Kronecker's theorem:

Theorem (Kronecker's Theorem)**.** Let $\phi_1, \ldots, \phi_k \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$ and let $\gamma_1, \ldots, \gamma_k \in \mathbb{R}$ and $T, \epsilon > 0$ be given. Then there exists $t > T$ and $q_1, \ldots, q_k \in \mathbb{Z}$ such that

$$|t\phi_j - \gamma_j - q_j| < \epsilon, \ 1 \le j \le k$$

.

Kronecker's theorem is helpful because it tells us that in finitely many dimensions, a line with irrational slope will be dense on a torus. Because we are considering a measure on the torus, this tells us that we will be able to approximate point masses on the torus by point masses on the line.
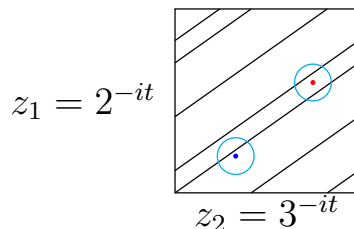
Figure 2.1: A two dimensional version where μ is a linear combination of two points, the red and blue points on the torus. The black lines represent the image of the imaginary axis under the Bohr lift, and the circles represent a "δ-ball."

To prove Kronecker's Theorem, we first need three lemmas, the proofs of which can be found in [1]. The first assures us that Fourier coefficients can be found via integration:

Lemma 2. Let $\{r_j\}$ be a sequence of distinct real numbers. For each real t and arbitrary complex numbers $c_0, \ldots, c_N$, define $f(t) = \sum_{j=0}^{N} c_j e^{itr_j}$. Then for each k we have

$$c_n = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t) e^{-itr_n} dt.$$

The second lemma rewrites our approximation question as a question about Fourier series:

Lemma 3. If t is real, let $F(t) = 1 + \sum_{j=1}^{k} e^{2\pi i(t\phi_j - \gamma_j)}$ where $\gamma_1, \ldots, \gamma_k$ and $\phi_1, \ldots, \phi_k$ are real numbers. The following are equivalent:

(i) For every $\epsilon > 0$, there exists a real t and $q_1, \ldots, q_k \in \mathbb{Z}$ such that

$$|t\phi_j - \gamma_j - q_j| < \epsilon, \ 1 \leq j \leq k$$

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.

(ii) $\limsup_{t\to\infty} |F(t)| = k + 1$.

The final lemma needed for the proof of Kronecker's theorem gives us a way to raise a sum of the coordinate functions to a power, and a bound on the number of terms.

Lemma 4. Let $g = f(x_1, \ldots, x_k) = 1 + x_1 + x_2 + \cdots + x_k$, and write

$$g^m = 1 + \sum a_j x^j \tag{2.3.1}$$

where m is a positive integer and j is a k-dimensional multi-index. The the coefficients a_j are positive integers such that

$$1 + \sum a_j = (1 + k)^m$$

and the number of terms in (2.3.1) is at most $(m + 1)^k$.

We can now present a proof also from [1]:

Proof of Kronecker's Theorem. Let $F(t) = 1 + \sum_{j=1}^{k} e^{2\pi i (t\phi_j - \gamma_j)}$. By Lemma 3 it suffices to prove that $\limsup_{t\to\infty} |F(t)| = k + 1$. Fix $m \in \mathbb{N}$ and define $\gamma = (\gamma_1, \ldots, \gamma_k)$, $\phi = (\phi_1, \ldots, \phi_k)$, $j = (j_1, \ldots, j_k)$. The mth power of F can be written as:

$$|F(t)|^m = 1 + \sum c_j e^{2\pi i t \eta_j},$$

where $\eta_j = j \cdot \phi$. By the independence of ϕ_j each η_j is distinct for different j. The c_j are the coefficients from Lemma 4 multiplied by a factor of modulus one, so

$$1 + \sum |c_j| = (k + 1)^m \tag{2.3.2}$$

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and there are at most $(m+1)^k$ terms. Now, assume for contradiction that $\limsup_{t\to\infty} F(t) < k+1$. Then there exist $M > 0$ and $\lambda < k+1$ such that $|F(t)|^m \le \lambda$, for all $t > M$.

So by Lemma 2,

$$
\begin{aligned}
|c_j| &= \left| \lim_{T\to\infty} |F(t)|^m e^{-2\pi i t \eta_j} dt \right| \\
&\le \limsup_{T\to\infty} |F(t)|^m dt \\
&\le \lambda^m
\end{aligned}
\tag{2.3.3}
$$

Thus, summing (2.3.3), recalling (2.3.2), and noting the number of terms, we have

$$
(k+1)^m = 1 + \sum |c_j| \le (m+1)^k \lambda^m,
$$

a contradiction for large m. $\qquad\square$

Kronecker's theorem will allow us to form λ by placing point masses on $\mathbb{R}$ in a strategic way.

2.3.2 Construction of λ for the Point Mass Case

Lemma 5 (Linear combination of point masses). For every measure on $\mathbb{T}^\infty$ of the form $d\mu = \sum_{j=1}^N c_j \delta_{\omega_j}$ with $\sum_{j=1}^N c_j = 1$, there exists an infinite measure on $\mathbb{R}$, λ, such that

$$
\int_{\mathbb{T}^\infty} |F(z)|^2 d\mu = \lim_{T\to\infty} \frac{1}{\lambda([0,T])} \int_0^T |f(it)|^2 d\lambda
\tag{2.3.4}
$$

for Dirichlet polynomials f.

Proof. Let $F(z) = \sum a_\alpha z^\alpha$ be a polynomial on $\mathbb{T}^\infty$ and let $f(it) = \sum a_\alpha (p_1^{\alpha_1} \cdots p_d^{\alpha_d})^{-it}$ be the corresponding Dirichlet polynomial. Note that because these have finitely many terms,

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there is some $d \in \mathbb{N}$ such that every α that appears is of the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d, 0, \ldots)$.

We will construct λ to be a sum of point masses $t \in \mathbb{R}$, using Kronecker's theorem to place them so that their images under the Bohr lift $z \in \mathbb{T}^\infty$ approximate the point masses that make up μ. In particular, we would like the images z to fall within δ-balls of $\mathbb{T}^\infty$ where given $\epsilon > 0$, δ is chosen to be small enough such that if $|\omega - z| < \delta$, then

$$\left| |F(w)|^2 - |f(it)|^2 \right| = \left| |F(w)|^2 - |F(z)|^2 \right| < \epsilon. \tag{2.3.5}$$

The first equality is because of the Bohr lift, and then we use the continuity of F.

Let us examine $|F(w)|^2$ and $|f(it)|^2$:

$$
\begin{aligned}
|F(\omega)|^2 &= \left| \sum a_\alpha \omega^\alpha \right|^2 \\
&= \sum |a_\alpha|^2 |\omega^\alpha|^2 + \sum_\alpha \sum_{\beta \neq \alpha} a_\alpha \overline{a_\beta} \omega^\alpha \overline{\omega^\beta} \\
&= \sum |a_\alpha|^2 + \sum_\alpha \sum_{\beta \neq \alpha} a_\alpha \overline{a_\beta} \omega^\alpha \overline{\omega^\beta}. \tag{2.3.6}
\end{aligned}
$$

Now, expanding $|f(it)|^2$,

$$
\begin{aligned}
|f(it)|^2 &= \left| \sum a_\alpha (p_1^{\alpha_1} \cdots p_d^{\alpha_d})^{-it} \right|^2 \\
&= \sum |a_\alpha|^2 + \sum_\alpha \sum_{\beta \neq \alpha} a_\alpha \overline{a_\beta} (p_1^{\alpha_1} \cdots p_d^{\alpha_d})^{-it} \overline{(p_1^{\beta_1} \cdots p_d^{\beta_d})^{-it}} \\
&= \sum |a_\alpha|^2 + \sum_\alpha \sum_{\beta \neq \alpha} a_\alpha \overline{a_\beta} (p_1^{\alpha_1 - \beta_1} \cdots p_d^{\alpha_d - \beta_d})^{-it}. \tag{2.3.7}
\end{aligned}
$$

So we want to place point masses t so that $(p_1^{\alpha_1 - \beta_1} \cdots p_d^{\alpha_d - \beta_d})^{-it}$ is near $\omega^\alpha \overline{\omega^\beta}$ for all α, β.

Examine both sides. Since $\omega \in \mathbb{T}^\infty$, $\omega^\alpha = \omega_1^{\alpha_1} \omega_2^{\alpha_2} \cdots \omega_d^{\alpha_d}$ and there are $\theta_1, \theta_2, \ldots, \theta_d$ so that

$$\omega^\alpha \overline{\omega^\beta} = \omega_1^{\alpha_1} \omega_2^{\alpha_2} \cdots \omega_d^{\alpha_d} \overline{\omega_1^{\beta_1} \omega_2^{\beta_2} \cdots \omega_d^{\beta_d}}$$

$$= e^{i\theta_1 \alpha_1} e^{i\theta_2 \alpha_2} \cdots e^{i\theta_d \alpha_d} e^{-i\theta_1 \beta_1} e^{-i\theta_2 \beta_2} \cdots e^{-i\theta_d \beta_d}$$

$$= e^{i\theta_1(\alpha_1 - \beta_1)} e^{i\theta_2(\alpha_2 - \beta_2)} \cdots e^{i\theta_d(\alpha_d - \beta_d)}.$$

On the other side, we have

$$\left(p_1^{\alpha_1 - \beta_1} \cdots p_d^{\alpha_d - \beta_d} \right)^{-it} = e^{-it \log\left(p_1^{\alpha_1 - \beta_1} \cdots p_d^{\alpha_d - \beta_d} \right)}$$

$$= e^{-i(\alpha_1 - \beta_1)t \log(p_1)} \cdots e^{-i(\alpha_d - \beta_d)t \log(p_d)}.$$

Note that both of these lie on the unit circle, so the problem reduces to finding t so that

$$-t \log p_r \approx \theta_r^j \bmod 2\pi.$$

We can use Kronecker's theorem to do this, however, because Kronecker's theorem only holds for a finite collection, it can only be done for finitely many primes. Because we want the measure λ to be independent of which primes appear in a polynomial, we will construct the measure in steps, so that the point masses farther from zero approximate the ω_j more accurately for more primes. This way any prime that appears in a Dirichlet polynomial will appear in our approximation at some level.

We also want the error from the poor approximations near zero to be small compared to the measure, so it will become irrelevant when we take the limit in the mean. This means that λ needs to have the property that the measures of intervals far from zero are much larger than the measures of intervals nearer to zero. We will achieve this by placing more point masses for better approximations.

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Construction of λ   First construct λ_1. By Kronecker's theorem, we can find $t_1^{1,1}, \ldots, t_1^{N,1}$ and corresponding integers q such that

$$| - t_1^{j,1} \log p_1 - \theta_1^j - 2\pi q| < 2^{-1}; \text{ for } j = 1, \ldots, N.$$

Repeat this to find $t_1^{j,2} > \max_j t_1^{j,1}$ so that there are two point masses corresponding to each component of μ. (In future steps, we will repeat this so that there are $2^k \cdot \|\lambda_{k-1}\|$ point masses for each component.) Now define

$$\lambda_1 = \sum_{j=1}^{N} c_j \left(\delta_{t_1^{j,1}} + \delta_{t_1^{j,2}} \right)$$

and note that $\|\lambda_1\| = 2$.

Inductively construct a sequence of measures $\{\lambda_k\}_{k=2}^{\infty}$:

For $k > 1$ choose $T_{k-1} > \max\{t_{k-1}^{j,m}; j = 1, \ldots, N, m = 1, \ldots, 2^{k-1}\}$. Again using Kronecker's theorem, find points $\{t_k^{j,1}\}_{j=1}^{N} > T_{k-1}$ and corresponding integers q such that

$$| - t_k^{j,1} \log p_r - \theta_r^j - 2\pi q| < 2^{-k}; \text{ for } j = 1, \ldots, N \text{ and } r = 1, \ldots, k. \tag{2.3.8}$$

This means that this inequality holds for all j and for the first k primes (or, equivalently, the first k coordinates in $\mathbb{T}^\infty$.) Repeat this to find N more points $\{t_k^{j,2}\}_{j=1}^{N}$ that satisfy (2.3.8) and such that $t_k^{j,2} > \max_j t_k^{j,1}$. Continue until there are $M_k = 2^k \cdot \|\lambda_{k-1}\|$ points for each ω_j. Define

$$\gamma_1 = \lambda_1$$

$$\gamma_k = \sum_{m=1}^{M_k} \sum_{j=1}^{N} c_j \delta_{t_k^{j,m}}, \quad \|\gamma_k\| = 2^k \|\lambda_{k-1}\|$$

and

$$\lambda_k = \lambda_{k-1} + \gamma_k = \sum_{\ell=1}^{k} \gamma_\ell \tag{2.3.9}$$

$$\|\lambda_k\| = \lambda_k\left([0, T_k]\right) = (2^k + 1)\|\lambda_{k-1}\| \tag{2.3.10}$$

and then let $\lambda = \sum_{\ell=1}^{\infty} \gamma_\ell$. Note that for any T there is some k such that $\lambda\left([0, T]\right) = \lambda_k\left([0, T]\right)$.
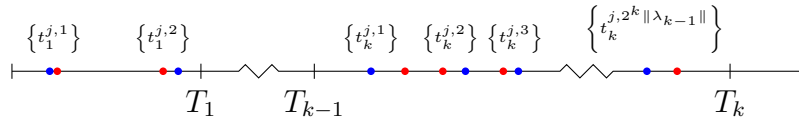
Figure 2.2: Continuing the example in Figure 2.3.1, where μ is composed of two point masses, here the point masses that make up λ are color coded corresponding to the point that they approximate. Notice that the order in which they are placed does not strictly alternate, but there are no more than two of the same color in a row. This prevents a "build up" of mass approximating a particular point mass of μ.

λ satisfies (2.3.4) Now we will verify that this measure gives the correct limit. We will use the continuity of $|F|^2$ as in (2.3.5). Given $\epsilon > 0$, there exists $\delta_j > 0$ such that $\|\omega_j - z\|_{\mathbb{T}^d} < \delta_j \Rightarrow \left||F(\omega_j)|^2 - |F(z)|^2\right| < \epsilon$. Now choose $\delta = \min_j \delta_j$ so

$$\|\omega_j - z\|_{\mathbb{T}^d} < \delta \Rightarrow \left||F(\omega_j)|^2 - |F(z)|^2\right| < \epsilon \quad \forall j. \tag{2.3.11}$$

Using (2.3.8) and the fact that the length of a chord of a circle can be bounded by the corresponding part of the circumference, for $z_{k,r}^{j,m} = e^{-it_k^{j,m} \log p_r}$, and for large k, we have

$$|\omega_{j,r} - z_{k,r}^{j,m}| <\approx 2 \cdot 2^{-k} = 2^{-k+1} \tag{2.3.12}$$

and

$$\|\omega_j - z_k^{j,m}\|_{\mathbb{T}^d}^2 = |\omega_{j,1} - z_{k,1}^{j,m}|^2 + \cdots + |\omega_{j,d} - z_{k,d}^{j,m}|^2$$

$$= (2^{-k+1})^2 \cdot d.$$

For every T there is some k such that $T \in [T_k, T_{k+1}]$, so choose T large enough that

$$(2^{-k+1})^2 \cdot d < \delta^2.$$

So for point masses $t^{j,m} \in \operatorname{supp}\lambda \cap [T_{k-1}, \infty)$, (2.3.8) holds for all j, m. (Here we omit the subscript k because the estimate works for every point mass in $[T_{k-1}, \infty)$.) Rewriting the continuity argument (2.3.11) using the Bohr lift yields

$$\left| |F(\omega_j)|^2 - |f(it^{j,m})|^2 \right| < \epsilon \quad \forall j, m. \tag{2.3.13}$$

For this T, consider

$$\left| \frac{1}{\lambda([0,T])} \int_0^T |f(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu \right|$$

$$\leq \frac{\lambda_{k-1}([0,T_{k-1}])}{\lambda_k([0,T_k])} \|f\|_\infty^2 + \left| \frac{1}{\lambda([0,T])} \int_{T_{k-1}}^T |f(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu \right|$$

$$= \frac{1}{2^k + 1} \|f\|_\infty^2 + \left| \frac{1}{\lambda([0,T])} \int_{T_{k-1}}^T |f(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu \right|.$$

The norm $\|f\|_\infty^2$ is bounded, so the first term goes to zero as T (and therefore k) goes to

infinity, so we only need to consider the second term:

$$
\left| \frac{1}{\lambda([0,T])} \int\limits_{T_{k-1}}^{T} |f(it)|^2 d\lambda - \int\limits_{\mathbb{T}^\infty} |F(z)|^2 d\mu \right|
$$

$$
= \left| \frac{1}{\lambda([0,T])} \left[\int\limits_{T_{k-1}}^{T_k} |f(it)|^2 d\lambda + \int\limits_{T_k}^{T} |f(it)|^2 d\lambda \right] - \int\limits_{\mathbb{T}^\infty} |F(z)|^2 d\mu \right|.
$$

Evaluate the integrals using the definition of λ and letting $X_j = \{t_{k+1}^{j,m}\} \in [T_k, T] \cap \operatorname{supp} \lambda$. (Note that $\lambda[T_k, T] = \sum_{j=1}^{N} c_j |X_j|$.)

$$
= \left| \frac{1}{\lambda([0,T])} \left[\sum_{m=1}^{M_k} \sum_{j=1}^{N} c_j |f(it_k^{j,m})|^2 + \sum_{j=1}^{N} c_j \sum_{t \in X_j} |f(it)|^2 \right] - \int\limits_{\mathbb{T}^\infty} |F(z)|^2 d\mu \right|.
$$

Add and subtract $|F(\omega_j)|^2$ appropriately, rearrange, and use the triangle inequality to get

$$
\leq \frac{1}{\lambda([0,T])} \left[\sum_{m=1}^{M_k} \sum_{j=1}^{N} c_j \left| |f(it_k^{j,m})|^2 - |F(\omega_j)|^2 \right| + \sum_{j=1}^{N} c_j \sum_{t \in X_j} \left| |f(it)|^2 - |F(\omega_j)|^2 \right| \right]
$$

$$
+ \left| \frac{2^k \|\lambda_{k-1}\|}{\lambda([0,T])} \int\limits_{\mathbb{T}^\infty} |F(z)|^2 d\mu - \int\limits_{\mathbb{T}^\infty} |F(z)|^2 d\mu + \frac{1}{\lambda([0,T])} \sum_{j=1}^{N} c_j \sum_{t \in X_j} |F(\omega_j)|^2 \right|.
$$

T is large enough that the continuity condition (2.3.13) holds, so

$$< \frac{1}{\lambda([0,T])} \left[\sum_{m=1}^{M_k} \sum_{j=1}^{N} c_j \epsilon + \sum_{j=1}^{N} c_j \sum_{t \in X_j} \epsilon \right]$$

$$+ \left| \frac{2^k \|\lambda_{k-1}\|}{\lambda([0,T])} \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu - \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu + \frac{1}{\lambda([0,T])} \sum_{j=1}^{N} c_j \sum_{t \in X_j} |F(\omega_j)|^2 \right|$$

$$= \frac{(2^k)\|\lambda_{k-1}\| + \lambda[T_k, T]}{(2^k + 1)\|\lambda_{k-1}\| + \lambda[T_k, T]} \epsilon$$

$$+ \left| \left(\frac{2^k \|\lambda_{k-1}\|}{\lambda([0,T])} - 1 \right) \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu + \frac{1}{\lambda([0,T])} \sum_{j=1}^{N} c_j |X_j| |F(\omega_j)|^2 \right|.$$

For large k, the first term is small, as needed. For the second term we consider three cases depending on the size of the sets X_j. When we constructed λ, we placed the point masses in sets of size N so that each mass ω_j had a representative, and then we repeated this. This means that $||X_{j_\ell}| - |X_{j_i}|| \le 1$ for $i \ne \ell$, so we can consider the cases

1. $|X_j| = 0$ for all j,

2. $|X_j| = C$ for all j, and

3. $|X_j| = C$ for $j = 1, \ldots, J$ and $|X_j| = C + 1$ for $j = J + 1, \ldots, N$.

Figure 2.3: This illustrates the reason we placed the point masses with repetitions. As discussed with Figure 2.3.2 the mass for one approximation cannot "build up", meaning that in Case 3, if $|X_j| = C$ for $j = 1, \ldots, J$, it cannot be that for $j = J + 1, \ldots, N$, $|X_j|$ is much larger than C. In this example that means that there cannot be more than one more red point than blue before T.

Case 1: $|X_j| = 0$ **for all** j In this case, $\frac{1}{\lambda([0,T])} \sum_{j=1}^{N} c_j |X_j| |F(\omega_j)|^2 = 0$ and $\lambda([0,T]) = \lambda([0,T_k]) = (2^k + 1)\|\lambda_{k-1}\|$, so we have

$$\left| \frac{2^k \|\lambda_{k-1}\|}{\lambda([0,T])} - 1 \right| \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu = \left| \frac{2^k}{2^k + 1} - 1 \right| \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu$$

which is small for large k.

Case 2: $|X_j| = C \leq M_{k+1}$ **for all** j In this case $\lambda([0,T]) = \lambda([0,T_k]) + \sum_{j=1}^{N} c_j \cdot C = (2^k + 1)\|\lambda_{k-1}\| + C$. Also, note that $\sum_{j=1}^{N} c_j |X_j| |F(\omega_j)|^2 = C \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu$. Then, substituting and simplifying gives

$$\left| \left(\frac{2^k \|\lambda_{k-1}\|}{\lambda([0,T])} - 1 \right) \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu + \frac{1}{\lambda([0,T])} \sum_{j=1}^{N} c_j |X_j| |F(\omega_j)|^2 \right|$$

$$= \left| \frac{2^k \|\lambda_{k-1}\| + C}{(2^k + 1)\|\lambda_{k-1}\| + C} - 1 \right| \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu$$

and this is small for large k.

Case 3: $|X_j| = C$ **for** $j = 1, \ldots, J$ **and** $|X_j| = C + 1$ **for** $j = J+1, \ldots, N$ Similarly to the previous case, we get

$$\left| \left(\frac{2^k \|\lambda_{k-1}\|}{\lambda([0,T])} - 1 \right) \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu + \frac{1}{\lambda([0,T])} \sum_{j=1}^{N} c_j |X_j| |F(\omega_j)|^2 \right|$$

$$= \left| \left(\frac{2^k \|\lambda_{k-1}\|}{\lambda([0,T])} - 1 \right) \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu + \frac{C}{\lambda([0,T])} \sum_{j=1}^{N} c_j |F(\omega_j)|^2 + \frac{1}{\lambda([0,T])} \sum_{j=J+1}^{N} c_j |F(\omega_j)|^2 \right|$$

$$\leq \left| \frac{2^k \|\lambda_{k-1}\| + C}{(2^k + 1)\|\lambda_{k-1}\| + C + \sum_{j=J+1}^{N} c_j} - 1 \right| \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu + \frac{1}{\lambda([0,T])} \sum_{j=J+1}^{N} c_j |F(\omega_j)|^2.$$

The first term is small as in Case 2, and in the second term $\frac{1}{\lambda([0,T])}$ is multiplied by a bounded quantity, and so this will be small for large T.

In each case,

$$\left| \frac{1}{\lambda([0,T])} \int_0^T |f(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu \right|$$

is small for large T, and so the proof is complete. $\square$

This construction also gives us the following lemma about the tail of the integral, roughly saying that the approximation error occurs near zero.

Lemma 6. If $\{\mu_n\}$ is a finite collection of measures $d\mu = \sum_{j=1}^{J_n} c_j^n \delta_{\omega_j^n}$ with $\sum_{j=1}^{J_n} c_j^n = 1$ and $\{F_m\}_{m=1}^M$ is a finite set of polynomials with corresponding Dirichlet polynomials $\{f_m\}$. If λ_n is the measure constructed as in Lemma 5 corresponding to μ_n, then given $\epsilon > 0$, there exists T_ϵ such that for sufficiently large $T \gg T_\epsilon$,

$$\left| \frac{1}{\lambda_n([T_\epsilon, T])} \int_{T_\epsilon}^T |f_m(it)|^2 d\lambda_n - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu_n \right| < \epsilon, \; \forall m, n.$$

Proof. As in the proof of Lemma 5, for each (m, n) we can find $T_{(m,n)}$ large enough that for point masses $t_\ell^j \in \operatorname{supp} \lambda_n \cap [T_{(m,n)}, \infty]$ and z_ℓ^j corresponding to t_ℓ^j,

$$|\omega_j^n - z_\ell^j| < \delta \text{ and so } \left| |F_m(\omega_j^n)|^2 - |f_m(it_\ell^j)|^2 \right| < \epsilon \, \forall j.$$

There are finitely many μ_n and F_m, so choose $T_\epsilon = \max\{T_{(m,n)}\}$. Now we have

$$\left| |F_m(\omega_j^n)|^2 - |f_m(it_\ell^j)|^2 \right| < \epsilon, \; \forall t_\ell^j \in \operatorname{supp} \lambda_n \cap [T_\epsilon, \infty] \text{ and for all } j, m, n. \qquad (2.3.14)$$

Now, for T large enough that each ω_j^n has many more than $\left\lceil \sum_{\mu_n} \sum_{j=1}^{J_n} |c_j| \right\rceil$ representatives

t_ℓ^j in $[T_\epsilon, T]$ (i.e. such that $|X_j| = |\{t_\ell^j \in [T_\epsilon, T]\}| \gg \left\lceil \sum_{\mu_n} \sum_{j=1}^{J_n} |c_j| \right\rceil$, for each j.) Consider

$$\left| \frac{1}{\lambda([T_\epsilon, T])} \int_{T_\epsilon}^{T} |f_m(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|$$

$$= \left| \frac{1}{\lambda([T_\epsilon, T])} \sum_{j=1}^{J_n} c_j^n \sum_{t_\ell^j \in X_j} \left[|f_m(it_\ell^j)|^2 - |F_m(\omega_j^n)|^2 + |F_m(\omega_j^n)|^2 \right] - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|.$$

Rearranging and using the triangle inequality yields

$$\leq \frac{1}{\lambda_n([T_\epsilon, T])} \sum_{j=1}^{J_n} c_j^n \sum_{t_\ell^j \in X_j} \left| |f_m(it_\ell^j)|^2 - |F_m(\omega_j^n)|^2 \right|$$

$$+ \left| \frac{1}{\lambda_n([T_\epsilon, T])} \sum_{j=1}^{J_n} c_j^n |X_j| |F_m(\omega_j^n)|^2 - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|$$

$$< \frac{\lambda_n([T_\epsilon, T])}{\lambda_n([T_\epsilon, T])} \epsilon + \left| \frac{1}{\lambda_n([T_\epsilon, T])} \sum_{j=1}^{J_n} c_j^n |X_j| |F_m(\omega_j^n)|^2 - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|.$$

The first term is small so we must consider the second term. As in the proof of Lemma 5 we can consider when $|X_j| = C$ for all j, and when $|X_j| = C$ for $j = 1, \ldots, J$ and $|X_j| = C + 1$ for $j = J + 1, \ldots, J_n$. Note that $C \geq \lceil \sum_{\mu_n} \sum_{j=1}^{\frac{n}{j}} |c_j| \rceil$. In the first case, we have

$$\left| \frac{1}{\lambda_n([T_\epsilon, T])} \sum_{j=1}^{J_n} c_j^n |X_j| |F_m(\omega_j^n)|^2 - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right| = \left| \frac{C}{C} \sum_{j=1}^{J_n} c_j^n |F_m(\omega_j^n)|^2 - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right| = 0.$$

In the second case

$$\left| \frac{1}{\lambda_n([T_\epsilon, T])} \sum_{j=1}^{J_n} c_j^n |X_j| |F_m(\omega_j^n)|^2 - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|$$

$$= \left| \frac{C}{C + \sum_{j=J+1}^{J_n} c_j} \sum_{j=1}^{J_n} c_j^n |F_m(\omega_j^n)|^2 - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right| + \frac{1}{\lambda_n([T_\epsilon, T])} \sum_{j=J+1}^{J_n} c_j^n |F_m(\omega_j^n)|^2$$

$$= \left| \frac{C}{C + \sum_{j=J+1}^{J_n} c_j} - 1 \right| \left| \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu + \frac{1}{\lambda_n([T_\epsilon, T])} \sum_{j=J+1}^{J_n} c_j^n |F_m(\omega_j^n)|^2 \right.$$

We chose T large enough that $C \gg \sum_{j=J+1}^{J_n} c_j$, so the first term will be small, and the second term is small because $\|f_m^2\|_\infty$ is bounded. $\square$

Remark 8. If $T' > T_\epsilon$, the upper limit T can be found so that the estimate of the lemma holds on $[T', T]$.

2.4 Proof of Theorem 7 part (i)

We may now return to the main result.

Theorem 7(i). Let μ be a Borel probability measure on the infinite torus $\mathbb{T}^\infty$. There exists a locally finite Borel measure λ on $\mathbb{R}$, such that, for all $f \in \mathcal{A}(\mathbb{C}_+)$

$$\lim_{T \to \infty} \frac{1}{\lambda([0, T])} \int_0^T |f(it)|^2 \, d\lambda(t) = \int_{\mathbb{T}^\infty} |F(z)|^2 \, d\mu(z). \tag{2.2.1}$$

Proof. Let $\{F_m\}_{m=1}^\infty$ be a countable set of polynomials which is dense in $A(\mathbb{D}^\mathbb{N})$. (By definition, the polynomials are dense in $A(\mathbb{D}^\mathbb{N})$, and there is a countable dense set of polynomials within each $A(\mathbb{D}^d)$ for finite d, so use Cantor diagonalization.) We only need to prove the theorem for F_m (corresponding to a Dirichlet polynomial f_m) in this dense set: for $F \in A(\mathbb{D}^\mathbb{N})$,

there is some F_m such that $\sup ||F_m|^2 - |F|^2| < \epsilon$ (and similarly $\sup ||f_m|^2 - |f|^2| < \epsilon$.) So

$$\left| \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu - \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu \right| < \epsilon$$

and

$$\left| \frac{1}{\lambda[0,T]} \int_0^T |f_m(it)|^2 d\lambda - \frac{1}{\lambda[0,T]} \int_0^T |f(it)|^2 d\lambda \right| < \epsilon$$

for any measure λ and for all T. So now we have

$$\left| \frac{1}{\lambda[0,T]} \int_0^T |f(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu \right|$$

$$\leq \left| \frac{1}{\lambda[0,T]} \int_0^T |f_m(it)|^2 d\lambda - \frac{1}{\lambda[0,T]} \int_0^T |f(it)|^2 d\lambda \right| + \left| \frac{1}{\lambda[0,T]} \int_0^T |f_m(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|$$

$$+ \left| \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu - \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu \right|$$

$$< \epsilon + \left| \frac{1}{\lambda[0,T]} \int_0^T |f_m(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right| + \epsilon.$$

It remains to show that

$$\left| \frac{1}{\lambda[0,T]} \int_0^T |f_m(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right| < \epsilon. \tag{2.4.1}$$

By Lemma 1 there exists a sequence of linear combinations of point masses $\{\mu_n\}$ that con-

verges weak-$*$ to μ. Now choose a subsequence $\{\mu_{n_{j,k}}\}$ such that for $m = 1, \ldots, 2^k$,

$$\left| \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu_{n_{j,k}} - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right| < 2^{-j}$$

and re-index so that $\{\mu_{j,k}\} = \{\mu_{n_{j,k}}\}$. Now, given m, for $k > \lceil \log_2 m \rceil$, we have control over the convergence:

$$\left| \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu_{j,k} - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right| < 2^{-j}. \tag{2.4.2}$$

Also, for each of these μ_j, there is a corresponding λ_j as constructed above that satisfies (2.3.4) for all Dirichlet polynomials, and in particular, works for all F_m.

Construction of λ. We construct λ using a process similar to the linear combination case: we will find an approximation and then repeat it so that better approximations appear more. However, this case, we will not be approximating with point masses, but with the measures λ_j constructed as in the previous case using the Lemma 5.

By Lemma 6, there is some T_1 such that for $j, m = 1, 2$ and for sufficiently large $T_1^{(1)}$

$$\left| \frac{1}{\lambda[T_1, T_1^{(1)}]} \int_{T_1}^{T_1^{(1)}} |f_m(it)|^2 d\lambda_j - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu_{j,k} \right| < \frac{1}{2}. \tag{2.4.3}$$

Define

$$\lambda^{(1)} = \gamma_1^{(1)} = \sum_{j=1}^{2} \frac{\lambda_j|_{[T_1, T_1^{(1)}]}}{\lambda_j([T_1, T_1^{(1)}])}$$

Note that $\operatorname{supp} \lambda^{(1)} = [T_1, T_1^{(1)}]$ and $\|\lambda^{(1)}\| = 2$.

Now find $T_2 \geq T_1^{(1)}$ such that for $j, m = 1, \ldots, 4$, and for sufficiently large T,

$$\left| \frac{1}{\lambda[T_2, T]} \int_{T_2}^{T} |f_m(it)|^2 d\lambda_j - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu_{j,k} \right| < \frac{1}{2^2}. \tag{2.4.4}$$

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Let $T_2^{(1)}$ be large enough that (2.4.4) holds. Now we repeat the approximation: by the remark after the proof of Lemma 6, there is some $T_2^{(2)}$ large enough so that for $j, m = 1, \ldots, 4$

$$\left| \frac{1}{\lambda[T_2^{(1)}, T_2^{(2)}]} \int_{T_2^{(1)}}^{T_2^{(2)}} |f_m(it)|^2 d\lambda_j - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu_{j,k} \right| < \frac{1}{2^2}.$$

Define

$$\gamma_2^{(\ell)} = \sum_{j=1}^{2^2} \frac{\lambda_j|_{[T_2^{(\ell-1)}, T_2^{(\ell)}]}}{\lambda_j([T_2^{(\ell-1)}, T_2^{(\ell)}])}, \qquad T_2^{(0)} = T_2$$

and

$$\lambda^{(2)} = \lambda^{(1)} + \sum_{\ell=1}^{\|\lambda^{(1)}\|=2} \gamma_2^{(\ell)}.$$

Note that $\|\gamma_2^{(\ell)}\| = 2^2$ and $\|\lambda^{(2)}\| = \|\lambda^{(1)}\| + 2 \cdot \|\gamma_2^{(\ell)}\|$.

Continue this process, at level k finding $T_k \geq T_{k-1}^{(\|\lambda^{(k-2)}\|)}$ such that for $j, m = 1, \ldots 2^k$ and for sufficiently large T

$$\left| \frac{1}{\lambda[T_k, T]} \int_{T_k}^{T} |f_m(it)|^2 d\lambda_j - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu_{j,k} \right| < \frac{1}{2^k}. \qquad (2.4.5)$$

Let $T_k^{(1)} = T$ be large enough that (2.4.5) holds and find $T_k^{(2)}$ such that

$$\left| \frac{1}{\lambda[T_k^{(1)}, T_k^{(2)}]} \int_{T_k^{(1)}}^{T_k^{(2)}} |f_m(it)|^2 d\lambda_j - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu_{j,k} \right| < \frac{1}{2^k}.$$

Repeat this to get an increasing sequence $\{T_k^{(\ell)}\}_{\ell=0}^{\|\lambda_{k-1}\|}$ (where $T_k^{(0)} = T_k$) such that for

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$j, m = 1, \ldots, 2^k$ and for $\ell = 1, \ldots, \|\lambda^{(k-1)}\|$,

$$\left| \frac{1}{\lambda[T_k^{(\ell-1)}, T_k^{(\ell)}]} \int_{T_k^{(\ell-1)}}^{T_k^{(\ell)}} |f_m(it)|^2 d\lambda_j - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu_{j,k} \right| < \frac{1}{2^k}. \tag{2.4.6}$$

Define for $\ell = 1, \ldots, \|\lambda^{(k-1)}\|$

$$\gamma_k^{(\ell)} = \sum_{j=1}^{2^k} \frac{\lambda_j|_{[T_k^{(\ell-1)}, T_k^{(\ell)}]}}{\lambda_j([T_k^{(\ell-1)}, T_k^{(\ell)}])}, \qquad \|\gamma_k^{(\ell)}\| = 2^k$$

and

$$\lambda^{(k)} = \lambda^{(k-1)} + \sum_{\ell=1}^{\|\lambda^{(k-1)}\|} \gamma_k^{(\ell)}.$$

Letting $\lambda = \lim_{k\to\infty} \lambda^{(k)}$, we have that $\lambda[0, T_{k+1}] = \|\lambda^{(k)}\| = (2^k + 1)\|\lambda^{(k-1)}\| = (2^k + 1)\lambda[0, T_k]$.
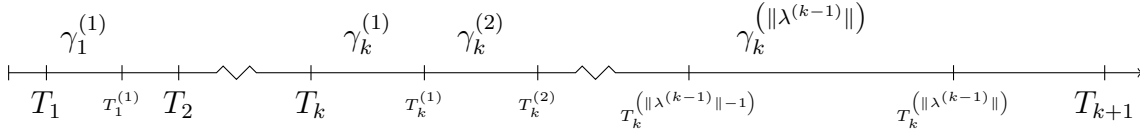

Figure 2.4: At each level k the small intervals give the same quality of estimate, as in (2.4.6). The repetition here serves the same purpose as in the point mass case: better estimates contribute more to mass of the measure over the real line.

Proof that λ satisfies (2.4.1) Now, examine

$$\left| \frac{1}{\lambda[0, T]} \int_0^T |f_m(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|.$$

For any T there is some k such that $T \in [T_{k+1}, T_{k+2}]$. Choose T large enough that $2^k \geq m$. From here, consider three cases: where $T > T_{k+1}^{(\|\lambda^{(k)}\|)}$, where $T < T_{k+1}^{(1)}$, and where $T \in [T_{k+1}^{(q)}, T_{k+1}^{(q+1)}]$ for some $1 \leq q < \|\lambda^{(k)}\|$.

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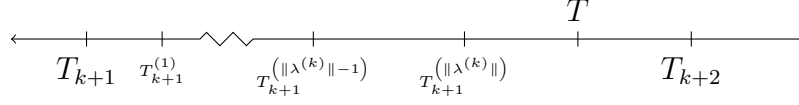**Case 1:** $T > T_{k+1}^{(\|\lambda^{(k)}\|)}$ This is where T appears after the repetitions:

Figure 2.5: T after the last $T_{k+1}^{(\ell)}$

In this case, $\lambda[0, T] = \lambda[0, T_{k+2}] = (2^{k+1} + 1)\lambda[0, T_{k+1}]$ and

$$\int\limits_{T_{k+1}}^{T} |f_m(it)|^2 d\lambda = \int\limits_{T_{k+1}}^{T_{k+2}} |f_m(it)|^2 d\lambda.$$

So

$$\left| \frac{1}{\lambda[0,T]} \int\limits_0^T |f_m(it)|^2 d\lambda - \int\limits_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|$$

$$\leq \left| \frac{1}{\lambda[0,T_{k+2}]} \int\limits_0^{T_{k+1}} |f_m(it)|^2 d\lambda \right| + \left| \frac{1}{\lambda[0,T_{k+2}]} \int\limits_{T_{k+1}}^{T_{k+2}} |f_m(it)|^2 d\lambda - \int\limits_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|$$

$$\leq \frac{\lambda[0,T_{k+1}]}{\lambda[0,T_{k+2}]} \|f_m\|_\infty^2 + \left| \frac{1}{\lambda[0,T_{k+2}]} \int\limits_{T_{k+1}}^{T_{k+2}} |f_m(it)|^2 d\lambda - \int\limits_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|$$

$$= \frac{1}{2^{k+1} + 1} \|f_m\|_\infty^2 + \left| \frac{1}{\lambda[0,T_{k+2}]} \int\limits_{T_{k+1}}^{T_{k+2}} |f_m(it)|^2 d\lambda - \int\limits_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|.$$

The first term here is small for large T, so consider the second term, using the definition of

λ and adding and subtracting $\int |F_m|^2 d\mu_{j,k}$ and $\int |F_m|^2 d\mu$ where appropriate

$$\left| \frac{1}{\lambda[0, T_{k+2}]} \int_{T_{k+1}}^{T_{k+2}} |f_m(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|$$

$$\leq \frac{1}{\lambda[0, T_{k+2}]} \sum_{\ell=1}^{\|\lambda^{(k)}\|} \sum_{j=1}^{2^{k+1}} \left| \frac{1}{\lambda_{j,k}[T_{k+1}^{\ell-1}, T_{k+1}^{\ell}]} \int_{T_{k+1}^{\ell-1}}^{T_{k+1}^{\ell}} |f_m|^2 d\lambda_{j,k} - \int_{\mathbb{T}^\infty} |F_m|^2 d\mu_{j,k} \right| \qquad (2.4.7)$$

$$+ \frac{1}{\lambda[0, T_{k+2}]} \sum_{\ell=1}^{\|\lambda^{(k)}\|} \sum_{j=1}^{2^{k+1}} \left| \int_{\mathbb{T}^\infty} |F_m|^2 d\mu_{j,k} - \int_{\mathbb{T}^\infty} |F_m|^2 d\mu \right| + \left| \frac{2^{k+1} \|\lambda^{(k)}\|}{\lambda[0, T_{k+2}]} - 1 \right| \left| \int_{\mathbb{T}^\infty} |F_m|^2 d\mu \right|.$$

Applying (2.4.6) and using that $\lambda[0, T_{k+2}] = (2^{k+1} + 1)\lambda[0, T_{k+1}] = (2^{k+1} + 1)\|\lambda^{(k)}\|$ then gives

$$< \frac{1}{2^{k+1} + 1} \sum_{j=1}^{2^{k+1}} \frac{1}{2^{k+1}} + \frac{1}{2^{k+1} + 1} \sum_{j=1}^{2^{k+1}} \left| \int_{\mathbb{T}^\infty} |F_m|^2 d\mu_{j,k} - \int_{\mathbb{T}^\infty} |F_m|^2 d\mu \right| + \frac{1}{2^{k+1} + 1} \int_{\mathbb{T}^\infty} |F_m|^2 d\mu$$

$$= \frac{1}{2^{k+1} + 1} \left[1 + \sum_{j=1}^{2^{k+1}} \left| \int_{\mathbb{T}^\infty} |F_m|^2 d\mu_{j,k} - \int_{\mathbb{T}^\infty} |F_m|^2 d\mu \right| + \int_{\mathbb{T}^\infty} |F_m|^2 d\mu \right].$$

We can get control over the second term by recalling how we chose our sequence of $\mu_{j,k}$ as in (2.4.2) because $m \leq 2^{k+1}$:

$$< \frac{1}{2^{k+1} + 1} \left[1 + \sum_{j=1}^{2^{k+1}} 2^{-j} + \int_{\mathbb{T}^\infty} |F_m|^2 d\mu \right] \qquad (2.4.8)$$

$$< \frac{1}{2^{k+1} + 1} \left[2 + \int_{\mathbb{T}^\infty} |F_m|^2 d\mu \right]$$

and this is small for large k.

Case 2: $T < T_{k+1}^{(1)}$ In this case, only $\gamma_{k+1}^{(1)}$ contributes to $\lambda[T_{k+1}, T]$, so $\lambda[T_{k+1}, T] \leq 2^{k+1}$.
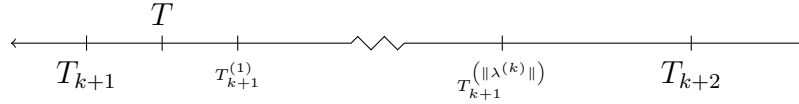

Figure 2.6: T before all the $T_{k+1}^{(\ell)}$

The first part of this case is similar to above, giving

$$
\left| \frac{1}{\lambda[0, T]} \int_0^T |f_m(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|
$$

$$
\leq \left| \frac{1}{\lambda[0, T]} \int_0^{T_k} |f_m(it)|^2 d\lambda \right| + \left| \frac{1}{\lambda[0, T]} \int_{T_k}^T |f_m(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|
$$

$$
\leq \frac{1}{2^k + 1} \|f_m\|_\infty^2 + \left| \frac{1}{\lambda[0, T]} \int_{T_k}^{T_{k+1}} |f_m(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|
$$

$$
+ \left| \frac{1}{\lambda[0, T]} \int_{T_{k+1}}^T |f_m(it)|^2 d\lambda \right|. \tag{2.4.9}
$$

Here, the first term is small as before. Similarly to Case 1, the second term simplifies down

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as

$$\left| \frac{1}{\lambda[0,T]} \int\limits_{T_k}^{T_{k+1}} |f_m(it)|^2 d\lambda - \int\limits_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|$$

$$< \frac{1}{2^k+1} \left[1 + \sum_{j=1}^{2^{k+1}} 2^{-j} \right] + \left| \frac{2^k \|\lambda^{(k-1)}\|}{\lambda[0,T]} - 1 \right| \int\limits_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu.$$

By the definition of λ, we know that $2^k \|\lambda^{(k-1)}\| = \lambda[T_k, T_{k+1}]$, so

$$< \frac{2}{2^k+1} + \left| \frac{\lambda[T_k, T_{k+1}] - \lambda[0,T]}{\lambda[0,T]} \right| \int\limits_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu$$

$$= \frac{2}{2^k+1} + \left(\frac{\lambda[0,T_k] + \lambda[T_{k+1}, T]}{\lambda[0,T]} \right) \int\limits_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu$$

$$\leq \frac{2}{2^k+1} + \left(\frac{\|\lambda^{(k-1)}\| + 2^{k+1}}{\lambda[0, T_{k+1}]} \right) \int\limits_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu.$$

Now, noting that $\lambda[0, T_{k+1}] = (2^k+1)\|\lambda^{(k-1)}\| = (2^k+1)(2^{k-1}+1)\cdots(2^2+1) \cdot 2 > 2^{\frac{k(k+1)}{2}}$, we have

$$< \frac{2}{2^k+1} + \left(\frac{1}{2^k+1} + \frac{2^{k+1}}{2^{\frac{k(k+1)}{2}}} \right) \int\limits_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu$$

and for large k, this is small.

We are now left with the last term:

$$\left| \frac{1}{\lambda[0,T]} \int\limits_{T_{k+1}}^{T} |f_m(it)|^2 d\lambda \right| \leq \frac{\lambda[T_{k+1}, T]}{\lambda[0,T]} \|f_m\|_\infty^2 \leq \frac{2^{k+1}}{2^{\frac{k(k+1)}{2}}} \|f_m\|_\infty^2.$$

Again, this is small for large k, so this case is complete.

Case 3: $T_{k+1}^{(q)} \leq T \leq T_{k+1}^{(q+1)}$ **for some** $1 \leq q < \|\lambda^{(k)}\|$
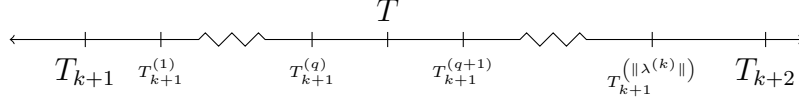

Figure 2.7: T between two of the $T_{k+1}^{(\ell)}$

In this case, $\lambda[T_k + 1, T]$ consists of the first q subintervals, and then has a contribution from $\left[T_{k+1}^{(q)}, T_{k+1}^{(q+1)}\right]$:

$$\lambda[T_{k+1}, T] = \sum_{\ell=1}^{q} \gamma_{k+1}^{(\ell)}[T_{k+1}^{\ell-1}, T_{k+1}^{\ell}] + \gamma_{k+1}^{(q+1)}[T_{k+1}^{q}, T] \tag{2.4.10}$$

$$\leq 2^{k+1}q + 2^{k+1}$$

and

$$\lambda[0, T] > \|\lambda^{(k)}\| + 2^{k+1}q > (2^{k+1} + 1)q. \tag{2.4.11}$$

The computation begins similarly to the previous cases:

$$\left| \frac{1}{\lambda[0,T]} \int_{0}^{T} |f_m(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|$$

$$\leq \frac{1}{2^k + 1} \|f_m\|_\infty^2$$

$$+ \left| \frac{1}{\lambda[0,T]} \left[\int_{T_k}^{T_{k+1}} |f_m(it)|^2 d\lambda + \int_{T_{k+1}}^{T} |f_m(it)|^2 d\lambda \right] - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|.$$

As in Case 2:

$$\leq \frac{1}{2^k+1}\|f_m\|_\infty^2 + \frac{1}{2^k+1}\left[1 + \sum_{j=1}^{2^{k+1}} 2^{-j}\right]$$

$$+ \left| \frac{1}{\lambda[0,T]} \int_{T_{k+1}}^{T} |f_m(it)|^2 d\lambda + \left(\frac{2^k\|\lambda^{(k-1)}\|}{\lambda[0,T]} - 1\right)\int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|.$$

The first two terms are small for large k, so consider the term in absolute values. This is similar to (2.4.7), where we add and subtract $\int |F_m|^2 d\mu_{j,k}$ and $\int |F_m|^2 d\mu$ appropriately, and use the triangle inequality:

$$\left| \frac{1}{\lambda[0,T]} \int_{T_{k+1}}^{T} |f_m(it)|^2 d\lambda + \left(\frac{2^k\|\lambda^{(k-1)}\|}{\lambda[0,T]} - 1\right)\int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|$$

$$\leq \frac{1}{\lambda[0,T]} \sum_{\ell=1}^{q}\sum_{j=1}^{2^{k+1}} \left| \frac{1}{\lambda_{j,k}[T_{k+1}^{(\ell-1)}, T_{k+1}^{(\ell)}]} \int_{T_{k+1}^{(\ell-1)}}^{T_{k+1}^{(\ell)}} |f_m|^2 d\lambda_{j,k} - \int_{\mathbb{T}^\infty} |F_m|^2 d\mu_{j,k} \right| \tag{I}$$

$$+ \frac{1}{\lambda[0,T]} \sum_{j=1}^{2^{k+1}} \frac{1}{\lambda_{j,k}[T_{k+1}^{(q)}, T_{k+1}^{(q+1)}]} \int_{T_{k+1}^{(q)}}^{T} |f_m(it)|^2 d\lambda_{j,k} \tag{II}$$

$$+ \frac{1}{\lambda[0,T]} \sum_{\ell=1}^{q}\sum_{j=1}^{2^{k+1}} \left| \int_{\mathbb{T}^\infty} |F_m|^2 d\mu_{j,k} - \int_{\mathbb{T}^\infty} |F_m|^2 d\mu \right| \tag{III}$$

$$+ \left| \frac{2^k\|\lambda^{(k-1)}\| + 2^{k+1}q}{\lambda[0,T]} - 1 \right| \int_{\mathbb{T}^\infty} |F_m|^2 d\mu. \tag{IV}$$

Using (2.4.11), (II) is bounded by $\frac{2^{k+1}}{\lambda[0,T_{k+1}]+2^{k+1}}\,\|f_m\|_\infty^2$. (I) can be simplified using (2.4.7) and (III) can be simplified similarly to (2.4.8) yielding:

$$\text{(I)}+\text{(II)}+\text{(III)}+\text{(IV)} < \frac{q}{\lambda[0,T]}\sum_{j=1}^{2^{k+1}}\frac{1}{2^{k+1}} + \frac{q}{\lambda[0,T]}\sum_{j=1}^{2^{k+1}}2^{-j} \qquad\qquad \text{(I+III)}$$

$$+ \frac{2^{k+1}}{\lambda[0,T_{k+1}]+2^{k+1}}\,\|f_m\|_\infty^2 \qquad\qquad\qquad\qquad \text{(II)}$$

$$+ \left|\frac{2^k\|\lambda^{(k-1)}\|+2^{k+1}q}{\lambda[0,T]}-1\right|\int_{\mathbb{T}^\infty}|F_m|^2 d\mu. \qquad\qquad \text{(IV)}$$

As before $\lambda[0,T_{k+1}] > 2^{\frac{k(k+1)}{2}}$. Also, because $q < \|\lambda^{(k)}\|$ and $\lambda[0,T] > \|\lambda^{(k)}\| + 2^{k+1}q > (2^{k+1}+1)q$,

$$\text{(I)}+\text{(II)}+\text{(III)}+\text{(IV)} < \frac{2}{2^{k+1}+1} + \frac{2^{k+1}}{2^{\frac{k(k+1)}{2}}+2^{k+1}}\,\|f_m\|_\infty^2$$

$$+ \left|\frac{2^k\|\lambda^{(k-1)}\|+2^{k+1}q}{\lambda[0,T]}-1\right|\int_{\mathbb{T}^\infty}|F_m|^2 d\mu.$$

The first two terms are clearly small for large k, so we only need to examine $\left|\frac{2^k\|\lambda^{(k-1)}\|+2^{k+1}q}{\lambda[0,T]}-1\right|$:

$$\left|\frac{2^k\|\lambda^{(k-1)}\|+2^{k+1}q}{\lambda[0,T]}-1\right| = \left|\frac{\lambda[T_k,T_{k+1}]+\lambda[T_{k+1},T_{k+1}^{(q)}]-\lambda[0,T]}{\lambda[0,T]}\right|$$

$$= \frac{\lambda[0,T_k]+\lambda[T_{k+1}^{(q)},T]}{\lambda[0,T]}.$$

Using that $\lambda[T_{k+1}^{(q)}, T] \leq 2^{k+1}$ and $\lambda[0, T] > \|\lambda^{(k)}\| + 2^{k+1} > 2^{\frac{k(k+1)}{2}} + 2^{k+1}$,

$$\frac{\lambda[0, T_k] + \lambda[T_{k+1}^{(q)}, T]}{\lambda[0, T]} \leq \frac{\lambda[0, T_k] + 2^{k+1}}{\|\lambda^{(k)}\| + 2^{k+1}}$$
$$= \frac{\|\lambda^{(k-1)}\|}{(2^k + 1)\|\lambda^{(k-1)}\| + 2^{k+1}} + \frac{2^{k+1}}{2^{\frac{k(k+1)}{2}} + 2^{k+1}}.$$

This is also small for large k, completing case three.

Because k depends on T, with T large implying that k is large, we've shown in each case that for large T, $\left| \frac{1}{\lambda[0,T]} \int_0^T |f_m(it)|^2 d\lambda - \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu \right|$ is as small as desired, so:

$$\lim_{T \to \infty} \frac{1}{\lambda[0, T]} \int_0^T |f_m(it)|^2 d\lambda = \int_{\mathbb{T}^\infty} |F_m(z)|^2 d\mu.$$

$\square$

2.4.1 Proof of Theorem 7, Part *(ii)*

Recall the statement of Theorem 7, Part *(ii)*:

Theorem 7(ii). Let λ be a locally finite Borel measure on $\mathbb{R}$ such that the limit on the left hand side of (2.2.1) exists and is finite for all $f \in \mathcal{A}(\mathbb{C}_+)$. Then there exists a unique Borel probability measure μ on the infinite torus $\mathbb{T}^\infty$ such that, for all $f \in \mathcal{A}(\mathbb{C}_+)$, (2.2.1) holds.

$$\lim_{T \to \infty} \frac{1}{\lambda([0, T])} \int_0^T |f(it)|^2 d\lambda(t) = \int_{\mathbb{T}^\infty} |F(z)|^2 d\mu(z). \tag{2.2.1}$$

Proof. We will appeal to the Riesz Representation Theorem to show that the left hand side of (2.2.1) can be used to define a positive bounded linear functional ϕ on $C(\mathbb{T}^\infty, \mathbb{R})$ which will give us our unique μ. This μ can then easily be shown to be a probability measure. We will define this functional ϕ on a subalgebra of $C(\mathbb{T}^\infty, \mathbb{R})$ and then apply Stone-Weierstrass

to show that the definition extends to the whole of $C(\mathbb{T}^\infty, \mathbb{R})$.

We choose to define ϕ on the set

$$\mathcal{U} = \left\{ \sum_{n=1}^N a_n |F_n|^2 : F_n \text{ are polynomials on } \mathbb{T}^\infty \right\}.$$

Let

$$\phi : |F|^2 \mapsto \lim_{T \to \infty} \frac{1}{\lambda([0,T])} \int_0^T |f(it)|^2 \, d\lambda(t). \tag{2.4.12}$$

Extending this linearly gives the definition of ϕ on $\mathcal{U}$, and if we can show that $\mathcal{U}$ is dense in $C(X, \mathbb{R})$, then we can extend ϕ to be a linear functional on $C(\mathbb{T}^\infty, \mathbb{R})$. To show that $\mathcal{U}$ is dense, apply Stone-Weierstrass: $\mathcal{U}$ clearly contains the constant functions and is a vector subspace of $C(X, \mathbb{R})$, and it is easy to show that it is closed under multiplication since $|F|^2|G|^2 = |FG|^2$ holds, and distribution shows that it holds for linear combinations as well. So we have that $\mathcal{U}$ is a subalgebra.

It remains to show that $\mathcal{U}$ separates points on $\mathbb{T}^\infty$: given $z \neq w \in \mathbb{T}^\infty$ we need to find a function in $\mathcal{U}$, h such that $h(z) \neq h(w)$. The points z and w must differ in at least one coordinate, so without loss of generality, assume they differ in the first one, $z_1 \neq w_1$. Consider the linear function in one variable $P(z) = az_1 + (b + ic)$, where a, b, $c \neq 0$. Given two points, the constants can be chosen such that $h(z) = |P(z)|^2$ separates those points.

So we have that $\mathcal{U}$ is dense in $C(\mathbb{T}^\infty, \mathbb{R})$, and ϕ as defined in (2.4.12) is clearly linear on $\mathcal{U}$. Extend ϕ continuously to $C(\mathbb{T}^\infty, \mathbb{R})$. Because $C(\mathbb{T}^\infty, \mathbb{R})$ is a Banach space and because of the assumption that the limit exists and is bounded for all elements in $\mathcal{A}(\mathbb{C}_+)$, ϕ is bounded on $C(\mathbb{T}^\infty, \mathbb{R})$. Then Riesz Representation gives a unique measure μ on $\mathbb{T}^\infty$ such that

$$\phi(h) = \int_{\mathbb{T}^\infty} h(z) d\mu.$$

This μ is a probability measure because $\lim_{T\to\infty} \frac{1}{\lambda([0,T])} \int_0^T |1|^2 \, d\lambda(t) = 1$, as needed. $\qquad\square$

Remark: Lebesgue measure

If μ is Lebesgue measure on $\mathbb{T}^\infty$, $\int_{\mathbb{T}^\infty} |\sum a_n z^n|^2 \, d\mu(z) = \sum_{n=1}^\infty |a_n|^2$. In this case, it is possible to choose λ to be Lebesgue measure, in addition to the measure constructed in the proof. Using the convergence in $\mathcal{A}(\mathbb{C}_+)$ then, we can see that for a sequence of Dirichlet polynomials $f_m \to f \in \mathcal{A}(\mathbb{C}_+)$ and corresponding $F_m \to F \in A(\mathbb{D}^\mathbb{N})$,

$$
\left| \frac{1}{T} \int_0^T |f(it)|^2 \, d\lambda(t) - \int_{\mathbb{T}^\infty} |F(z)|^2 \, d\mu(z) \right|
$$

$$
\leq \left| \frac{1}{T} \int_0^T |f(it)|^2 \, d\lambda(t) - \frac{1}{T} \int_0^T |f_m(it)|^2 \, d\lambda(t) \right| + \left| \frac{1}{T} \int_0^T |f(it)|^2 \, d\lambda(t) - \int_{\mathbb{T}^\infty} |F(z)|^2 \, d\mu(z) \right|
$$

$$
+ \left| \int_0^T |F_m(it)|^2 \, d\lambda(t) - \int_0^T |F(it)|^2 \, d\lambda(t) \right|
$$

$$
< \epsilon
$$

where the first and third terms are small because of the convergence, and the middle term is small for large T because Carlson's theorem holds on the boundary $\sigma = 0$ for Dirichlet polynomials. This is not hard to see by simply computing the integrals on either side. We can summarize this in the following theorem:

Theorem 9. For a Dirichlet series $f(s) = \sum_{n=1}^\infty a_n n^{-s} \in \mathcal{A}(\mathbb{C}_+)$,

$$
\lim_{T\to\infty} \frac{1}{T} \int_0^T |f(it)|^2 dt = \sum_{n=1}^\infty |a_n|^2. \tag{2.4.13}
$$

This example lets us see explicitly that given μ we do not have uniqueness for λ.

Chapter 3

Weighted Spaces of Dirichlet Series

This work is done in part with Houry Melkonian.

3.1 Background

3.1.1 McCarthy's Weighted Spaces

We discussed in Section 1.2.2 Bayart's generalization of $\mathcal{H}^2$ to the Banach spaces $\mathcal{H}^p$. In [12], McCarthy extended the definition of $\mathcal{H}^2$ in a different direction by introducing weighted Hilbert spaces of Dirichlet series. Of particular interest are the spaces

$$\mathcal{H}_\alpha^2 = \left\{ f(s) = \sum_{n=2}^\infty a_n n^{-s} : \sum_{n=2}^\infty |a_n|^2 (\log n)^\alpha < \infty \right\}$$

McCarthy termed $\alpha \leq 0$ to give "Bergman-like" spaces and showed that these have $\mathcal{H}^\infty$ as their multiplier algebra. A slight adjustment to the definition of these spaces can be made to include the constant functions without changing the multiplier algebra, and in particular, choosing $\alpha = 0$ gives the original $\mathcal{H}^2$.

McCarthy's weighted spaces can also be defined using an integral by applying Carlson's Theorem and an appropriately defined measure μ to get

$$\|f\|^2 = \sum_{n=2}^\infty |a_n|^2 (\log n)^\alpha = \lim_{c \to 0^+} \lim_{T \to \infty} \int_{-T}^{T} \int_{0}^{\infty} |f(s+c)|^2 d\mu(\sigma) dt$$

3.1.2   Weighted Banach Spaces of Dirichlet Series

This is the formulation that then gives rise to the weighted Bergman spaces of Dirichlet series defined by Bailleul and Lefèvre in [3]:

Definition ($\mathcal{A}_\mu^p$, $1 \leq p < \infty$). Let P be a Dirichlet polynomial, and define $P_\sigma(s) = P(s+\sigma)$. Let μ be a probability measure on $[0, \infty)$ such that $0 \in \mathrm{supp}(\mu)$. Define

$$\|P\|_{\mathcal{A}_\mu^p}^p = \int\limits_0^\infty \|P_\sigma\|_{\mathcal{H}^p}^p d\mu(\sigma).$$

$\mathcal{A}_\mu^p$ is the completion of the Dirichlet polynomials with respect to this norm.

Note that if μ is chosen to be the point mass at zero, the $\mathcal{A}_\mu^p$ will agree with Bayart's Banach spaces, and that if $p = 2$ only a slight adjustment is needed for these to agree with McCarthy's weighted spaces.

Bailleul and Lefèvre [3] then combined the ideas from Bayart's Banach spaces, and McCarthy's weighted Hilbert spaces to define weighted Bergman spaces of Dirichlet series:

Definition ($\mathcal{A}_\mu^p$, $1 \leq p < \infty$). Let $\sigma \geq 0$ and let P be a Dirichlet polynomial, and define $P_\sigma(s) = P(s + \sigma)$. Let μ be a probability measure on $[0, \infty)$ such that $0 \in \mathrm{supp}(\mu)$. Define

$$\|P\|_{\mathcal{A}_\mu^p}^p = \int\limits_0^\infty \|P_\sigma\|_{\mathcal{H}^p}^p d\mu(\sigma).$$

$\mathcal{A}_\mu^p$ is the completion of the Dirichlet polynomials with respect to this norm.

The measure μ can be chosen so that the Hilbert space case $p = 2$ will agree with McCarthy's weighted spaces (with a slight adjustment), and if μ is chosen to be the point mass at zero, the $\mathcal{A}_\mu^p$ will agree with Bayart's Banach spaces. In fact, more can be said about the relationship between $\mathcal{A}_\mu^p$ and $\mathcal{H}^p$:

Theorem 10 ([3]). Let $p \geq 1$ and let μ be a probability measure on $[0, \infty)$ such that $0 \in \text{supp}(\mu)$. Then

(i) $\mathcal{H}^p \subset \mathcal{A}_\mu^p$, and for every $f \in \mathcal{H}^p$, we have $\|f\|_{\mathcal{A}_\mu^p} \leq \|f\|_{\mathcal{H}^p}$

(ii) For every $f \in \mathcal{H}^p$, we have $\|f\|_{\mathcal{A}_\mu^p}^p = \int_0^\infty \|f_\sigma\|_{\mathcal{H}^p}^p d\mu(\sigma)$

(iii) For every $f \in \mathcal{A}_\mu^p$, we have $\|f\|_{\mathcal{A}_\mu^p}^p = \lim_{c \to 0^+} \|f_c\|_{\mathcal{A}_\mu^p}^p$

3.2   The Multiplier Algebra

Another way to see that these spaces are natural is to consider the multiplier algebra. Despite the weights, the multiplier algebra is the same as for the unweighted $\mathcal{H}^2$:

Theorem 11. Let μ be a probability measure on $[0, \infty)$ such that $0 \in \text{supp}(\mu)$ and let $1 \leq p < \infty$. $Mult(\mathcal{A}_\mu^p) = \mathcal{H}^\infty$ and for a multiplier, m, $\|m\|_{Mult} = \|m\|_{\mathcal{H}^\infty}$.

To prove this we will need the following definition and property of Dirichlet series.

Definition (Uniformly almost periodic). Let $f(s)$ be holomorphic in the half place $\mathbb{C}_\theta$. Let $\epsilon > 0$. A real number τ is called an ϵ translation number of f if

$$\sup_{s \in \mathbb{C}_\theta} |f(s + i\tau) - f(s)| \leq \epsilon.$$

Then $f(s)$ is called uniformly almost periodic in $\mathbb{C}_\theta$ if, for every $\epsilon > 0$, there exists a positive real number M such that every interval in $\mathbb{R}$ of length M contains at least one ϵ translation number of f.

Theorem 12 ([5], 144). Suppose that $f(s)$ is represented by a Dirichlet series that converges uniformly in the half plane $\mathbb{C}_\theta$. Then f is uniformly almost periodic in $\mathbb{C}_\theta$.

Proof of Theorem 11. To show $\mathcal{H}^\infty \subseteq Mult(\mathcal{A}_\mu^p)$, let $\phi \in \mathcal{H}^\infty$ and let P be a Dirichlet polynomial. By Theorem 2, we know that $\phi P \in \mathcal{H}^p$, so by Theorem 10 *(ii)*, and algebra on $\mathcal{H}^p$

$$\|\phi P\|_{\mathcal{A}_\mu^p}^p = \int_0^\infty \|(\phi P)_\sigma\|_{\mathcal{H}^p}^p d\mu$$

$$= \int_0^\infty \|\phi_\sigma P_\sigma\|_{\mathcal{H}^p}^p d\mu$$

Note that $\|\phi_\sigma\|_\infty \leq \|\phi\|_\infty$ and that P_σ is a Dirichlet polynomial so it is in $\mathcal{H}^p$. Now, by Corollary 3, we have

$$\|\phi P\|_{\mathcal{A}_\mu^p}^p \leq \int_0^\infty \|\phi\|_\infty^p \|P_\sigma\|_{\mathcal{H}^p}^p d\mu$$

$$= \|\phi\|_\infty^p \|P\|_{\mathcal{A}_\mu^p}^p$$

This is finite, so $M_\phi : \mathcal{P} \to \mathcal{A}_\mu^p$ is a bounded operator and since $\mathcal{P}$ is dense in its completion $\mathcal{A}_\mu^p$, M_ϕ extends to be a bounded operator on $\mathcal{A}_\mu^p$, so $\mathcal{H}^\infty \subseteq Mult(\mathcal{A}_\mu^p)$. We have also shown that $\|\phi\|_{Mult} \leq \|\phi\|_\infty$. Let us now show that $\|\phi\|_\infty \leq \|\phi\|_{Mult}$.

Now, as in the proof of Theorem 1.11 in [12], assume for contradiction that $\|\phi\|_{Mult} = 1$ and $\|\phi\|_\infty > 1$. For $\sigma > 0$, let $N_\sigma = \sup_t |\phi(\sigma + it)|$. The Phragmén-Lindelöf theorem gives that N_σ is strictly decreasing in σ.

Moreover, by Bohr's Theorem equating the abscissae of bounded and uniform convergence [6], in each half plane $\mathbb{C}_\theta$ for $\theta > 0$, the Dirichlet series of ϕ converges uniformly to ϕ. This then allows us to apply Theorem 12, which gives that ϕ is uniformly almost periodic in $\mathbb{C}_\theta$. Therefore, there exist $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 > 0$ so that for large enough T,

$$|\{t : |\phi(\sigma + it)| > 1 + \epsilon_1, -T \leq t \leq T\}| \geq \epsilon_2(2T), \ \forall \epsilon_3 \leq \sigma \leq \epsilon_3 + \epsilon_4$$

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We assumed $\|\phi\|_{Mult} = 1$, so M_ϕ is a contraction, so M_ϕ^j is a contraction for any positive integer j. Note that $1 \in \mathcal{A}_\mu^p$ for all p, and since μ is a probability measure, $\|1\|_{\mathcal{A}_\mu^p} = 1$, so we have $\|\phi^j \cdot 1\|_{\mathcal{A}_\mu^p} \leq 1$.

$$
\begin{aligned}
1 \geq \|\phi^j \cdot 1\|_{\mathcal{A}_\mu^p}^p &= \int_0^\infty \|\phi^j\|_{\mathcal{H}^p}^p d\mu(\sigma) \\
&= \int_0^\infty \left(\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T |\phi^j(\sigma + it)|^p dt \right) d\mu(\sigma) \\
&\geq \int_0^{\epsilon_4} \left(\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T |\phi^j(\sigma + it)|^p dt \right) d\mu(\sigma) \\
&> \epsilon_2 (1 + \epsilon_1)^{jp} \mu([0, \epsilon_4])
\end{aligned}
$$

Because $\mu([0, \epsilon_4]) > 0$, this goes to infinity in j, giving a contradiction, so $\|\phi\|_\infty \leq \|\phi\|_{Mult}$.

To show $Mult(\mathcal{A}_\mu^p) \subseteq \mathcal{H}^\infty$, we consider an alternate definition of $\mathcal{A}_\mu^p$. To do this, examine the norm defined on the Dirichlet polynomials $\mathcal{P}$:

$$
\|P\|_{\mathcal{A}_\mu^p}^p = \int_0^\infty \|P_\sigma\|_{\mathcal{H}^p}^p d\mu(\sigma).
$$

In particular, using the Bohr lift and Theorem 1 we can consider the $\mathcal{H}^p$ norm as the norm of a polynomial on some infinite polytorus $\mathbb{T}^\infty$:

$$
\begin{aligned}
\|P_\sigma\|_{\mathcal{H}^p}^p &= \|\mathcal{B}P_\sigma\|_{H^p(\mathbb{T}^\infty)}^p \\
&= \int_{\mathbb{T}^\infty} |\mathcal{B}P(2^{-\sigma}e^{(i\theta_1)}, 3^{-\sigma}e^{(i\theta_2)}, \dots)|^p \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \cdots
\end{aligned}
$$

Now define a measure ν on $\left(\overline{\mathbb{D}}\right)^\infty$ so that

$$\int\limits_{\left(\overline{\mathbb{D}}\right)^\infty} |\mathcal{B}P(2^{-\sigma}e^{(i\theta_1)}, 3^{-\sigma}e^{(i\theta_2)}, \dots)|^p d\nu(\sigma, \theta_1, \theta_2, \dots) = \int\limits_0^\infty \|P_\sigma\|_{\mathcal{H}^p}^p d\mu(\sigma) = \|P\|_{\mathcal{A}_\mu^p}^p \qquad (3.2.1)$$

These agree on the Dirichlet polynomials (and in fact, by Theorem 10, they agree for all elements of $\mathcal{H}^p$), so they will have the same completion, meaning we may consider $\mathcal{A}_\mu^p \subseteq L^p\left(\left(\overline{\mathbb{D}}\right)^\infty, \nu\right)$

This new definition will simplify the rest of the proof. Let $\phi \in Mult(\mathcal{A}_\mu^p)$. Then $\phi^j \in Mult(\mathcal{A}_\mu^p)$ for all $j \in \mathbb{N}$ and so $\phi^j \in \mathcal{A}_\mu^p$. Then we have

$$\|\phi\|_{Mult} = \|\phi^j\|_{Mult}^{1/j} \geq \|\phi^j\|_{\mathcal{A}_\mu^p}^{1/j} = \left(\int\limits_{\left(\overline{\mathbb{D}}\right)^\infty} |\mathcal{B}(\phi(\sigma + it)^j)|^p d\nu\right)^{1/jp} \qquad (3.2.2)$$

So $\int_{\left(\overline{\mathbb{D}}\right)^\infty} |\mathcal{B}(\phi(\sigma + it))|^{jp} d\nu \leq \|\phi\|_{Mult}$ for all $j \in \mathbb{N}$, since $\mathcal{B}(\phi^j) = \mathcal{B}(\phi)^j$ on the level of formal series. Letting j go to infinity shows that $\|\mathcal{B}\phi\|_{H^\infty\left(\left(\overline{\mathbb{D}}\right)^\infty\right)} \leq \|\phi\|_{Mult}$. In particular, the sup norm on any polytorus in $\left(\overline{\mathbb{D}}\right)^\infty$ is also less than $\|\phi\|_{Mult}$. This means that $\|\phi\|_\infty = \|\mathcal{B}\phi\|_{H^\infty(\mathbb{T}^\infty)} \leq \|\phi\|_{Mult}$ and so $\phi \in \mathcal{H}^\infty$. $\qquad\square$

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