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Mass Spectrometry-based Strategies for Protein Characterization: Amyloid Formation, Protein-Ligand Interactions and Structures of Membrane Proteins in Live Cells

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Regulators on Higher Chow Groups

by

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Dedicated to my parents, Min Li and Shiping Yang.

ABSTRACT OF THE DISSERTATION

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Professor Matt Kerr, Chair

There are two natural questions one can ask about the higher Chow group of number fields: One is its torsion, the other one is its relation with the homology of GL_n. For the first question, based on some earlier work, the integral regulator on higher Chow complexes introduced here can put a lot of earlier result on a firm ground. For the second question, we give a counterexample to an earlier proof of the existence of linear representatives of higher Chow groups of number fields.

Chapter 1 gives a general picture of the two problems we are talking about. Chapter 2 contains the background material on higher Chow groups. In chapter 3, we showed the full process of proving the existence of integral regulator on higher Chow complexes, and give the explicit expression for it, and some direct application. In chapter 4, we introduced the conjecture of the (rational) surjectivity of the map from linear higher Chow group to the simplicial higher Chow group, its earlier proof and the counter example. However, it is not a global counter example, thus the original conjecture is still open.

Chapter 1

Introduction

Higher Chow groups were introduced by S. Bloch in the mid-80's as a geometric representation of algebraic K-theory [**Bl0**]. For X a smooth quasi-projective variety over an infinite field k, Bloch's Grothendieck-Riemann-Roch theorem identifies them *rationally* with certain graded pieces of K-theory:

$$CH^p(X,n) \otimes \mathbb{Q} \simeq Gr_\gamma^P K_n^{alg} X \otimes \mathbb{Q}. \tag{1.0.1}$$

There are two representation for higher Chow groups, called the cubical representation and the symplicial representation, corresponding to $\square^n := (\mathbb{P}^1 \setminus \{1\})^n$ and $\Delta^n := \mathbb{P}^n \setminus H$, where $H := \{x_0 + \cdots + x_n = 0\} \simeq \mathbb{P}^{n-1}$. For an infinite field k, we denote $CH^p(\mathrm{Spec}(k), n)$ by $CH^p(k,n)$. For $CH^n(k, 2n-1)$, there are already the Bloch-Beilinson regulator that gives us the full result of $\dim_{\mathbb{Q}}(CH^n(k, 2n-1) \otimes \mathbb{Q})$, and two natural questions beyond this will be the torsion of $CH^n(k, 2n-1)_{\mathbb{Z}}$, and the relationship between $H_{2n-1}(GL_n(k), \mathbb{Q})$ and $CH^n(k, 2n-1) \otimes \mathbb{Q}$. We'll focus on the first question by using the cubical representation in Chapter 3, and will give more details about the second question along with the simplicial representation in Chapter 4.

As Bloch showed, these groups come with natural Chern class maps

$$AJ_{\mathbb{Z}}^{p,n} : CH^p(X,n) \to H_{\mathscr{D}}^{2p-n}(X,\mathbb{Z}(p)) \tag{1.0.2}$$

to the cohomology of the underlying variety [**Bl0a**], which "interpolate" Griffiths's Abel-Jacobi maps on Chow groups (i.e. K_0) and Borel's regulators on the higher K-theory of number fields.

While abstractly defined, these maps were successfully computed in many specific cases by Bloch, Beilinson, Deninger, and others. However, an explicit general formula only emerged in the work of Kerr, Lewis and Müller-Stach [**KLM**, **KL**] in the early 00's. By introducing a subcomplex $Z_{\mathbb{R}}^p(X,\bullet) \overset{\imath}{\hookrightarrow} Z^p(X,\bullet)$ of cycles in good position with respect to the "wavefront" set of certain currents on $(\mathbb{P}^1)^n$, they are able to construct a map of complexes

$$\widetilde{AJ} : Z_{\mathbb{R}}^p(X,\bullet) \to C_{\mathscr{D}}^{2p-\bullet}(X,\mathbb{Z}(p)) \tag{1.0.3}$$

agreeing *rationally* with (1.0.2). (The explicit formula shall be recalled in §4).

At first glance, the "KLM formula" (1.0.3) looks well-adapted to detecting torsion. For example, if $X = \operatorname{Spec}(k)$, consider the portion

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & Z_{\mathbb{R}}^p(k,2p) & \xrightarrow{\partial} & Z_{\mathbb{R}}^p(k,2p-1) & \xrightarrow{\partial} & Z_{\mathbb{R}}^p(k,2p-2) & \longrightarrow \cdots \\
& {\scriptstyle W\mapsto}\big\downarrow{\scriptstyle (2\pi\mathbf{i})^p W\cap T_{2p}} & & {\scriptstyle Z\mapsto}\big\downarrow{\scriptstyle \frac{1}{(2\pi\mathbf{i})^{p-1}}\int_Z R_{2p-1}} & & \big\downarrow{\scriptstyle 0} & \\
\cdots \longrightarrow & \mathbb{Z}(p)\oplus 0\oplus 0 & \hookrightarrow & 0\oplus 0\oplus \mathbb{C} & \longrightarrow & 0\oplus 0\oplus 0 & \longrightarrow \cdots
\end{array}$$

of (1.0.3), where $T_{2p} = \mathbb{R}_{<0}^{2p} \subset (\mathbb{P}^1)^{2p}$ and R_{2p-1} is a certain $(2p-2)$-current on $(\mathbb{P}^1)^{2p-1}$. We want to detect torsion in $CH^p(k,2p-1)$ by the middle map; denote the image of $Z \in \ker(\partial)$ by $\mathscr{R}(Z) \in \mathbb{C}/\mathbb{Z}(p)$. In particular, if $Z_1 := (1-1/t, 1-t, t^{-1})_{t\in\mathbb{P}^1} \in Z^2(\mathbb{Q},3)$, we find that

$\mathscr{R}(Z_1) = \pi^2/6 \in \mathbb{C}/(2\pi\mathbf{i})^2\mathbb{Z}$, in agreement with the known result that $CH^2(\mathbb{Q},3)$ is 24-torsion (see [**Pe**]).

Unfortunately, it appears very difficult to determine whether $\imath$ is an integral quasi-isomorphism, as expected in [**KLM**]. Indeed, the proof in [**KL**] that this inclusion of complexes is a $\mathbb{Q}$-quasi-isomorphism makes essential use of Kleiman transversality in K-theory and hence of some form of (1.0.1). So the KLM formula only induces a "rational regulator"

$$AJ^{p,n}_{\mathbb{Q}} : CH^p(X,n) \to H^{2p-n}_{\mathscr{D}}(X,\mathbb{Q}(p)). \tag{1.0.4}$$

It is easy to see the problem: we could have that the class of Z in $H_{2p-1}\left(Z^p_{\mathbb{R}}(k,2p-1)\right)$ *and* its $\widetilde{AJ}$-image are m-torsion (but nonzero), whilst Z is a boundary in the larger complex (hence zero in $CH^p(k,2p-1)$). That is, there would be some $\mathcal{W} \in Z^p(k,2p) \setminus Z^p_{\mathbb{R}}(k,2p)$ with $\partial\mathcal{W} = Z$, but only $mZ \in \partial\left(Z^p_{\mathbb{R}}(k,2p)\right)$. Moreover, even if we could improve the result on $\imath$ (and eliminate this particular worry), it would remain inconvenient to find representative cycles in $Z^p_{\mathbb{R}}(X,n)$.

An alternative is to extend KLM to a formula that works on all cycles. Doing this with one map of complexes on $Z^p(X,\bullet)$ is probably too optimistic, as one can't just wish away the "wavefront sets" arising from the branch cuts in the $\{\log(z_i)\}$. Our first idea was to try an infinite family of homotopic maps on nested subcomplexes $Z^p_{\varepsilon}(X,\bullet)$ with union $Z^p(X,\bullet)$, by allowing cycles in good position with respect to "perturbations" of these branch cuts by sufficiently small nonzero "phase" $e^{\mathbf{i}\epsilon}$, $0 < \epsilon < \varepsilon$. Provided one tunes the branches of log in the regulator currents accordingly, and the same θ is used for each z_i, one gets a morphism of complexes on the ε-subcomplexes. Since the homotopy class of this morphism is independent of ϵ, this approach would define an integral refinement of $\widetilde{AJ}$ provided the $\varepsilon \to 0$ limit of the "perturbed" subcomplexes gives all of $Z^p(X,\bullet)$. Unfortunately, this is not true: there is a counterexample involving triples of functions on a curve, see §3.1. So a more subtle approach is required.

In particular, we need a way to vary phases ϵ_i independently for the branches of $\log(z_i)$, so as to place weaker demands on our cycles. But this can never lead to a morphism of complexes from $Z^p(X, \bullet)$, since this independence would conflict with the way the Bloch differential ∂ intersects cycles with all the facets. On the other hand, one has an explicit $\mathbb{Z}$-homotopy equivalence for the inclusion $\mathscr{N}^p(X, \bullet) \subset Z^p(X, \bullet)$ of the normalized cycles, on which the differential restricts to just one facet [**Bl1**]. In $\mathscr{N}^p(X, \bullet)$, we now consider the "$\varepsilon$-subcomplex" $\mathscr{N}_\varepsilon^p(X, \bullet)$, consisting of cycles which are in good position with respect to the $(e^{\mathbf{i}\epsilon_1}, \ldots, e^{\mathbf{i}\epsilon_n})$-perturbed wavefront set for any $(\epsilon_1, \ldots, \epsilon_n)$ belonging to

$$B_\varepsilon^n := \left\{ \underline{\epsilon} \in \mathbb{R}^n \middle| 0 < \epsilon_1 < \varepsilon, 0 < \epsilon_2 < e^{-1/\epsilon_1}, \ldots, 0 < \epsilon_n < e^{-1/\epsilon_{n-1}} \right\}.$$

Our main technical results are

Theorem 1. $\bigcup_{\varepsilon > 0} \mathscr{N}_\varepsilon^p(X, \bullet) = \mathscr{N}^p(X, \bullet)$.

and

Theorem 2. *Given $\underline{\epsilon}$, $\underline{\epsilon}' \in B_\epsilon^N$, the corresponding morphisms*

$$R_{\underline{\epsilon}},\, R_{\underline{\epsilon}'} \;:\; \tau_{\leq N}\mathcal{N}_\varepsilon^p(X, \bullet) \to C_{\mathscr{D}}^{2p-\bullet}(X, \mathbb{Z}(p)),$$

induced by the perturbed KLM currents, are integrally homotopic.

These results are proved in §§ 3.2 and 3.4, respectively. It is now easy to deduce that, taken over all $\underline{\epsilon}$, these morphisms induce a map of the form (1.0.2) refining (1.0.4), see §3.5. We conclude by indicating several applications of the KLM formula to torsion in §3.6 due to

[**KLM**], Petras [**Pe**], Kerr-Yang [**MY**] which are now validated by our construction, and indicate future work in this direction.

We are going to take a revisit of $\tilde{A}J : Z^p(X, \bullet) \to C_{\mathscr{D}}^{2p-\bullet}(X, \mathbb{Z}(p))$ under the simplicial representation, where now we take $Z^p(X, \bullet)$ the free abelian group on closed irreducible subvarieties of $X \times \Delta_k^m$ of codimension p. Under this construction, we can define the homology of the subcomplex $LZ^p(X, \bullet)$ given by equations *linear* in the $\{x_i\}$ defines the *linear higher Chow groups* $LCH^p(X, m)$, which map naturally to $CH^p(X, m)$. In chapter 4, we are going to give a deeper discussion about the relationship between $LCH^p(X, m)$ and $CH^p(X, m)$.

Chapter 2

Background on Higher Chow Cycles

2.1 Basic definitions

Definitions in this section follow [**KLM**].

Let X be a smooth quasiprojective algebraic variety over an infinite field k. An algebraic cycle on X is a finite linear combination $\Sigma n_V[V]$ of subvarieties $V \subset X$, where $n_V \in \mathbb{Z}$.

We define the algebraic n-cube (over k) by

$$\square^n := (\mathbb{P}^1 \backslash \{1\})^n,$$

with face inclusions $\rho_i^{\mathfrak{f}} : \square^{n-1} \to \square^n$ ($\mathfrak{f} \in \{0, \infty\}$) sending $(z_1, \ldots, z_{n-1})$ to $(z_1, \ldots, z_{i-1}, \mathfrak{f}, z_i, \ldots, z_{n-1})$, and coordinate projections $\pi_i : \square^n \to \square^{n-1}$ sending $(z_1, \ldots, z_n)$ to $(z_1, \ldots, \hat{z}_i, \ldots, z_n)$. We call

$$\partial\square^n := \bigcup_{\substack{i=1,\cdots,n \\ \mathfrak{f}=0,\infty}} (\rho_i^{\mathfrak{f}})_* \square^{n-1}$$

the facets of $\square^n$, and

$$\partial^k\square^n := \bigcup_{\substack{i_1<\cdots<i_k \\ \mathfrak{f}_1,\cdots,\mathfrak{f}_k=0,\infty}} (\rho_{i_1}^{\mathfrak{f}_1})_* \cdots (\rho_{i_k}^{\mathfrak{f}_k})_* \square^{n-k}$$

the codimension-k subfaces of $\square^n$.

Definition 3. $c^p(X,n) \subset Z^p(X \times \square^n)$ is the free abelian group on irreducible subvarieties $V \subset X \times \square^n$ of codimension p such that V meets all faces of $X \times \square^n$ properly.

Definition 4. The *degenerate cycles* $d^p(X,n) \subset c^p(X,n)$ are defined as $\sum_{i=1}^n \pi_i^*(c^p(X,n-1))$. Set $Z^p(X,n) := c^p(X,n)/d^p(X,n)$.

The Bloch differential

$$\partial_B := \sum_{i=1}^{n} (-1)^{i-1}(\rho_i^{\infty *} - \rho_i^{0*}) : Z^p(X,n) \to Z^p(X,n-1)$$

makes $Z^p(X,\bullet)$ into a complex, with the higher Chow groups $CH^p(X,n)$ given by their homology. For convenience, we shall often use cohomological indexing:

Definition 5. $CH^p(X,n) := H^{-n}\{Z^p(X,-\bullet)\}$

2.2 A moving lemma

We recall the subcomplex from [**KLM**]. Henceforth we shall take k to be a subfield of $\mathbb{C}$, so we can consider the complex analytic spaces associated to components of a cycle Z. Let $c^p_{\mathbb{R}}(X,n)$ be the set of all the cycles $Z \in c^p(X,n)$ whose components (or rather, their analytizations) intersect $X \times (T_{z_1} \cap \cdots \cap T_{z_i})$ and $X \times (T_{z_1} \cap \cdots \cap T_{z_i} \cap \partial^k\square^n)$ properly for all $1 \leq i \leq n$ and $1 \leq k < n$, and $d^p_{\mathbb{R}}(X,n) := c^p_{\mathbb{R}}(X,n) \cap d^p(X,n)$. We get a new complex $Z^p_{\mathbb{R}}(X,n) := c^p_{\mathbb{R}}(X,n)/d^p_{\mathbb{R}}(X,n)$. It is shown in [**KL**] that this subcomplex is $\mathbb{Q}$-quasi-isomorphic to the original one:

Theorem 6 (Kerr-Lewis). $Z^p_{\mathbb{R}}(X,\bullet) \xrightarrow{\simeq} Z^p(X,\bullet)$

2.3 Normalized cycles

Higher Chow groups may also be computed by complexes of cycles that have trivial boundary on all but one face.

Definition 7. $\mathscr{N}^p(X,n) := \{Z \in Z^p(X,n) | \partial_i^\infty Z = 0 \text{ for } i < n, \partial_j^0 Z = 0 \text{ for any } j\}$

In this section, we will show that there exist an explict $\mathbb{Z}$-homotopy equivalence for the inclusion $\mathscr{N}^p(X,\bullet) \subset Z^p_0(X,\bullet)$. The following proof is derived from Bloch's manuscript [**Bl1**], by replacing the notations from $(\mathbb{A}^1)^n$ which uses $\{0,1\}$ as boundary by $(\mathbb{P}^1 \setminus \{1\})^n$ which uses $\{0,\infty\}$ as boundary. (Also Bloch uses a different definition for the normalized cycle: $\mathscr{N}'^p(X,\bullet) := \{Z \in Z^p(X,n) | \partial_i^\infty Z = 0 \text{ for } i > 1, \partial_j^0 Z = 0 \text{ for any } j\}$, so we need to take a "conjugation" on Bloch's proof as well.)

Define $Z^p_{\infty,i}(X,\bullet) = \{Z \in Z^p(X,\bullet) | \partial_j^\infty Z = 0 \text{ for } j < n-i, \partial_k^0 Z = 0 \text{ for any } k\}$.We have $Z^p_{\infty,0}(X,\bullet) = \mathscr{N}^p(X,\bullet)$, and $Z^p_{\infty,i}(X,\bullet) \subset Z^P(X,\bullet)$ is a subcomplex.

Theorem 8. *The inclusion $\mathscr{N}^p(X,\bullet) \subset Z^p(X,\bullet)$ is $\mathbb{Z}$-homotopy.*

Proof: Consider $Z^p_{\infty,n-1}(X,\bullet) = \{Z \in Z^p(X,\bullet) | \partial_k^0 Z = 0 \text{ for any } k\}$. For any cycle $Z \in Z^p(X,n)$, we can always add a number of degenerate cycles (for free) to let Z lies in $Z^p_{\infty,n-1}(X,\bullet)$. So we'll consider $Z^p_{\infty,n-1}(X,\bullet)$ instead of general $Z \in Z^p(X,n)$ in the following proof.

Given integers $l \le n-1$, define $h^l : \square^{n+1} \to \square^n$ by

$$h^l(z_1,\ldots,z_{n+1}) := (z_1,\ldots,z_l, \frac{z_{l+1}z_{l+2}}{z_{l+1}+z_{l+2}-1}, z_{l+3},\ldots,z_{n+1})$$

and for $Z \in Z^p(X,n)$, define $H^l(Z) := (-1)^{n-l}(h^l)^{-1}(Z) \in Z^p(X, n+1)$. For $l \geq n$ define $H^l(Z) = 0$. Consider the following map:

$$\phi := \cdots (\mathrm{Id} - (d \circ H^l + H^l \circ d)) \circ (\mathrm{Id} - (d \circ H^{l-1} + h^{l-1} \circ d)) \circ \cdots \circ (\mathrm{Id} - (d \circ H^0 + H^0 \circ d))$$

This map stabilizes in any degree and so defines an endomorphism $\phi : Z^p(X, \bullet) \to Z^p(X, \bullet)$, and it is homotopy to the identity.

Precisely, we have

$$\begin{aligned} d \circ H^l Z = \sum_{k=1}^{n-l-1} (-1)^{n-l+k+1} \partial^{\infty}_{n-k+1} Z(z_1, \cdots, \frac{z_{l+1} z_{l+2}}{z_{l+1} + z_{l+2} - 1}, \cdots, z_n) \\ + \sum_{k=n-l+1}^{n} (-1)^{n-l+k} \partial^{\infty}_{n-k+1} Z(z_1, \cdots, \frac{z_l z_{l+1}}{z_l + z_{l+1} - 1}, \cdots, z_n) \end{aligned} \quad (2.3.1)$$

and

$$H^l \circ dZ = \sum_{k=1}^{n} (-1)^{n-l+k} \partial^{\infty}_{n-k+1} Z(z_1, \cdots, \frac{z_{l+1} z_{l+2}}{z_{l+1} + z_{l+2} - 1}, \cdots, z_n)$$

Thus we have

$$\begin{aligned} (d \circ H^l + H^l \circ d)Z = \sum_{k=n-l+1}^{n} (-1)^{n-l+k} \partial^{\infty}_{n-k+1} Z(z_1, \cdots, \frac{z_l z_{l+1}}{z_l + z_{l+1} - 1}, \cdots, z_n) \\ + \sum_{k=n-l}^{n} (-1)^{n-l+k} \partial^{\infty}_{n-k+1} Z(z_1, \cdots, \frac{z_{l+1} z_{l+2}}{z_{l+1} + z_{l+2} - 1}, \cdots, z_n) \end{aligned} \quad (2.3.2)$$

So for $Z \in Z^p_{\infty,i}(X,\bullet)$, we have $(d \circ H^l + H^l \circ d)Z = 0$ for $l \leq n-i-2$. For $l = n-i-1$, we have

$$Z' := Z - (d \circ H^l + H^l \circ d)Z = Z - \partial^\infty_{l+1} Z(z_1, \cdots, \frac{z_{l+1}z_{l+2}}{z_{l+1}+z_{l+2}-1}, \cdots, z_n)$$

and it's not hard to check that $Z' \in Z^p_{\infty,i-1}$. And then, for the next term, Z' will be mapped to some $Z'' \in Z^p_{\infty,i-2}$. Once we approaches $Z^p_{\infty,0} = \mathscr{N}'^p(X,\bullet)$, we'll have $(d \circ H^l + H^l \circ d)Z = 0$ for all the ls. Thus we have $\phi : Z^p(X,\bullet) \to \mathscr{N}^p(X,\bullet)$ and ϕ is the identity on $\mathscr{N}^p(X,\bullet)$, and ϕ composed with the inclusion is homotopy to the identity.. □

For explict expression of ϕ in low dimension, we have

$$\phi(Z(z_1,z_2)) = Z(z_1,z_2) - (\partial^\infty_1 Z)(\frac{z_1 z_2}{z_1+z_2-1})$$

$$\begin{aligned}\phi(Z(z_1,z_2,z_3)) = Z(z_1,z_2,z_3) &- (\partial^\infty_2 Z)(z_1, \frac{z_2 z_3}{z_2+z_3-1}) \\ &- (\partial^\infty_1 Z)(\frac{z_1 z_2}{z_1+z_2-1}, z_3) + (\partial^\infty_1 Z)(z_1, \frac{z_2 z_3}{z_2+z_3-1})\end{aligned} \quad (2.3.3)$$

Chapter 3

Integral regulator

3.1 Simple perturbations

The Kerr-Lewis moving lemma can only yield a rational regulator due to the passage through K-theory in the proof. Instead, one might consider maps of complexes on a nested family of subcomplexes of $Z^p_{\mathbb{R}}(X, \bullet)$, given by "perturbing" the conditions defining $Z^p_{\mathbb{R}}(X, \bullet)$. Though this turns out to be too naive, it is the first step toward a strategy that works.

Begin by defining $Z^p_{\varepsilon}(X, \bullet)$ to be the subcomplex of $Z^p(X, n)$ given by the cycles that intersect $X \times (T^{\epsilon}_{z_1} \times \cdots \times T^{\epsilon}_{z_i})$ and $X \times (T^{\epsilon}_{z_1} \times \cdots \times T^{\epsilon}_{z_i} \times \partial^k \Box^n)$ properly for all $1 \leq j \leq n$, $1 \leq k < n$ and $0 < \epsilon < \varepsilon$. Here T^{ϵ}_z is given by $\arg(z) = \pi - \epsilon$, the "perturbation" of the branch cut of $\log(z)$ in the currents defined below.

In order for this nested family of subcomplexes to be any better than $Z^p_{\mathbb{R}}(X,\bullet)$, we must have that their union gives us the original Z^p:

$$\bigcup_{\varepsilon} Z^p_{\varepsilon}(X,\bullet) = Z^p(X,\bullet). \tag{3.1.1}$$

Unfortunately, this fails in a very simple case:

Proposition 9. *For $X = Spec(\mathbb{Q}(\mathbf{i}))$, we have*

$$\bigcup_{\varepsilon} Z^2_{\varepsilon}(X,3) \subsetneq Z^2(X,3).$$

Proof. Let $F(z) = \mathbf{i}z - 1$, $G(z) = -\frac{(1+z)(1+3z)}{(1+\mathbf{i}z)(1-2z)}$, and $H(z) = \frac{\mathbf{i}z-1}{3+z}$. Then we have $Z = (F(z), G(z), H(z))_{z\in\mathbb{P}^1} \in Z^2(\text{pt},3)$; but for all $\varepsilon > 0$, $Z \notin Z^2_{\varepsilon}(\text{pt},3)$. More precisely, for any $\epsilon > 0$, we have $\dim_{\mathbb{R}}(Z \cap T^{\epsilon}_{z_1} \cap T^{\epsilon}_{z_2} \cap T^{\epsilon}_{z_3}) = 0$, not -1 (i.e. empty) as required for a proper-analytic intersection. □

Thus we need to find another way to do the "perturbation", which will be given in the next section.

3.2 Multiple perturbations

In order to have Z^{an} meet the deformations of $\{T_{z_i}\}$ for an example like that in the above proof, we clearly need to make use of the extra degrees of freedom allowed by perturbing each "branch-cut phase" independently. For convenience, we shall use the multi-index notation $\underline{\epsilon} := (\epsilon_1, \ldots, \epsilon_n)$ in what follows.

Now we are thinking of $T^{\epsilon_i}_{z_i}$ as the location of the jump in the 0-current $\log(z_i)$; these 0-currents will appear in the definition of the regulator-currents $R^{\underline{\epsilon}}_Z$ appearing in the next section. To use these currents to define Abel-Jacobi maps, we will need them to induce morphisms of complexes from a subcomplex of $Z^p(X,\bullet)$ to $C_{\mathscr{D}}^{2p-\bullet}(X,\mathbb{Z}(p))$. Unfortunately, if Z has boundaries at more than one facet of $\square^n$, say $\partial_1 Z=(\rho_1^0)*Z$ and $\partial_2 Z=(\rho_2^0)^*Z$, the residue terms in $d[R_Z^{(\epsilon_1,\ldots,\epsilon_n)}]$ will take the form $R_{\partial_1 Z}^{(\epsilon_2,\ldots,\epsilon_n)}$ resp. $R_{\partial_2 Z}^{(\epsilon_1,\epsilon_3,\ldots,\epsilon_n)}$. This clearly conflicts with having $D(T_Z^{\underline{\epsilon}},\Omega_Z,R_Z^{\underline{\epsilon}})=(T_{\partial Z}^{\underline{\epsilon}'},\Omega_{\partial Z},R_{\partial Z}^{\underline{\epsilon}'})$ for a single choice of $\underline{\epsilon}'$, so we shall need to restrict to the normalized cycles $\mathscr{N}^p(X,\bullet)$ defined in §2.3.

For $\varepsilon>0$, define B_ε as the set of infinite sequences $(\epsilon_1,\epsilon_2,\ldots)$ satisfying

$$0<\epsilon_1<\varepsilon,\ 0<\epsilon_2<\exp(-1/\epsilon_1),\ 0<\epsilon_3<\exp(-1/\epsilon_2),\ \ldots, \tag{3.2.1}$$

and define B_ε^n to comprise the n-tuples $\underline{\epsilon}$ satisfying (3.2.1).

Definition 10. $\mathscr{N}_\varepsilon^p(X,\bullet):=\{Z\in\mathscr{N}^p(X,\bullet)|Z$ intersects $X\times T_{z_1}^{\epsilon_1}\times\cdots\times T_{z_i}^{\epsilon_i}$ and $X\times T_{z_1}^{\epsilon_1}\times\cdots\times T_{z_i}^{\epsilon_i}\times\partial^k\square^n$ properly $\forall i,k,\underline{\epsilon}\in B_\varepsilon^\bullet\}$.

Theorem 11. $\bigcup_\varepsilon\mathscr{N}_\varepsilon^p(X,\bullet)=\mathscr{N}^p(X,\bullet)$

Proof. Consider the projection $(\mathbb{C}^*)^n\to(S^1)^n\cong(\mathbb{C}/\mathbb{Z}(1))^n$ defined by $(r_1e^{\mathbf{i}\epsilon_1},\cdots,r_ne^{\mathbf{i}\epsilon_n})\mapsto(\epsilon_1,\cdots,\epsilon_n)$, whose fibers are $T_{z_1}^{\epsilon_1}\times\cdots\times T_{z_n}^{\epsilon_n}$. There is also a natural $2^n:1$ map $(\mathbb{S}^1)^n\to(\mathbb{P}_\mathbb{R}^1)^n$ by taking slopes: $(\epsilon_1,\cdots,\epsilon_n)\mapsto(\tan\epsilon_1,\cdots,\tan\epsilon_n)$. The composite map $\Theta^n:(\mathbb{C}^*)^n\to(\mathbb{S}^1)^n\to(\mathbb{P}_\mathbb{R}^1)^n$ is real algebraic, sending $(x_1+\mathbf{i}y_1,\cdots,x_n+\mathbf{i}y_n)\mapsto(y_1/x_1,\cdots,y_n/x_n)$.

Now let $Z\in\mathscr{N}^p(X,n)$ be given. Set $Z^*:=\bar{Z}\cap(X\times(\mathbb{C}^*)^n)$, and let $\tilde{Z}^*$ be its resolution. The intersections of Z^* with the fibers of $\Theta_X^n:X\times(\mathbb{C}^*)^n\to X\times(\mathbb{P}_\mathbb{R}^1)^n$ are $Z^*\cap(X\times T_{z_1}^{\epsilon_1}\times\cdots\times T_{z_n}^{\epsilon_n})$. Write Θ_Z for the composition of $\tilde{Z}^*\to X\times(\mathbb{C}^*)^n$ with Θ_X^n. The set of $\underline{\epsilon}$ for which these intersections are good is the complement of the non-flat locus $\Delta\subset(\mathbb{P}_\mathbb{R}^1)^n$ of Θ_Z. Since the flat locus of an algebraic map is Zariski open, $\Delta\subset(\mathbb{P}_\mathbb{R}^1)^n$ is a real subvariety, which is proper by dimension considerations.

Therefore the preimage $\tilde{\Delta}$ of Δ in $(\mathbb{S}^1)^n$ is real analytic. By the form of the inequalities in B_ε, we know that we can choose an $\varepsilon > 0$ such that $B_\varepsilon^n \cap \tilde{\Delta} = \emptyset$. (This follows from the implicit function theorem for $\tilde{\Delta}$, and the fact that all derivatives of $e^{-1/x}$ limit to 0 at 0.) This means that Z intersects $X \times T_{z_1}^{\epsilon_1} \times \cdots \times T_{z_n}^{\epsilon_n}$ properly $\forall \underline{\epsilon} \in B_\varepsilon^\bullet$, as desired.

Repeating the argument for $X \times (\mathbb{C}^*)^i \times (\mathbb{P}^1_\mathbb{C})^{n-i}$ and $X \times (\mathbb{C}^*)^i \times (\{0, \infty\})^k \times (\mathbb{P}^1_\mathbb{C})^{n-i-k}$, we pick the minimum of the required values of ε, so that Z intersects $X \times T_{z_1}^{\epsilon_1} \times \cdots \times T_{z_i}^{\epsilon_i}$ and $X \times T_{z_1}^{\epsilon_1} \times \cdots \times T_{z_i}^{\epsilon_i} \times \partial^k \square^n$ properly $\forall i, k, \underline{\epsilon} \in B_\varepsilon^\bullet$, which means $Z \in \mathscr{N}_\varepsilon^p(X, n)$. $\square$

3.3 Abel-Jacobi maps

In this section, we'll use the strategy in [**KLM**] to define the Abel-Jacobi maps on our subcomplexes.

3.3.1 Definition of Deligne cohomology

The Deligne cohomology group $H_{\mathscr{D}}^{2p+n}(X, \mathbb{Z}(p))$ is given by the n^{th} cohomology of the complex

$$\mathcal{C}_{\mathcal{D}}^{\bullet+2p}(X, \mathbb{Z}(p)) := \{\mathcal{C}^{2p+\bullet}(X, \mathbb{Z}(p)) \oplus F^p\mathcal{D}^\bullet(X) \oplus \mathcal{D}^{\bullet-1}(X)\}$$

with differential D taking $(a, b, c) \mapsto (-\partial a, -\mathrm{d}[b], \mathrm{d}[c] - b + \delta_a)$. Here $\mathcal{D}^k(X)$ denotes currents of degree k on X^{an} and $\mathcal{C}^k(X, \mathbb{Z}(k))$ denotes C^∞ (co)chains of real codimension k and $\mathbb{Z}(k) = (2\pi\mathbf{i})^k\mathbb{Z}$ coefficients.

The cup product in Deligne cohomology is defined on the chain level by

$$(T_X, \Omega_X, R_X) \cup (T_Y, \Omega_Y, R_Y) \\ := ((2\pi\mathbf{i})^{l+n} T_X \cap T_Y, \Omega_X \wedge \Omega_Y, (-1)^l (2\pi\mathbf{i})^l \delta_{T_X} \cdot R_Y + R_X \wedge \Omega_Y).$$

It becomes commutative upon passage to cohomology. (See [**We**] for a commutative chain-level construction.)

3.3.2 KLM Currents

Firstly we'll review the currents given in [**KLM**].

The currents on $\square^n$ are given by $T_n := T_{z_1} \cap T_{z_2} \cap \ldots \cap T_{z_n}$, $\Omega_n = \frac{\mathrm{d}z_1}{z_1} \wedge \frac{\mathrm{d}z_2}{z_2} \wedge \ldots \wedge \frac{\mathrm{d}z_n}{z_n}$, and

$$R_n = \sum_{k=1}^{n} (-1)^{\binom{k}{2}} (2\pi\mathbf{i})^k \delta_{T_{z_1} \times T_{z_2} \times \ldots \times T_{z_{k-1}}} \log z_k \frac{\mathrm{d}z_{k+1}}{z_{k+1}} \wedge \ldots \wedge \frac{\mathrm{d}z_n}{z_n}.$$

For currents on X associated to a given $Z \in Z^p_{\mathbb{R}}(X, n)$, let $\pi_1 : \tilde{Z} \to \square^n$ and $\pi_2 : \tilde{Z} \to X$ be the projections (where $\tilde{Z}$ is a desingularization). Then we have:

$$\widetilde{AJ}^{p,n}_{KLM}(Z) := (2\pi\mathbf{i})^{p-n} (\pi_2)_* (\pi_1)^* ((2\pi\mathbf{i})^n T_n, \Omega_n, R_n)$$

3.3.3 Currents on $\mathscr{N}^p_\varepsilon(X, n)$

Using a similar strategy, for a normalized precycle $Z \in \mathscr{N}^p_\varepsilon(X, n)$ and $\underline{\epsilon} \in B^n_\varepsilon$, we send

$$Z \mapsto (2\pi\mathbf{i})^{p-n} (\pi_2)_* (\pi_1)^* ((2\pi\mathbf{i})^n T^{\underline{\epsilon}}_n, \Omega_n, R^{\underline{\epsilon}}_n) =: \mathcal{R}^{n,\underline{\epsilon}}_\varepsilon(Z) \tag{3.3.1}$$

where $T_n^{\underline{\epsilon}} = T_{z_1}^{\epsilon_1} \times T_{z_2}^{\epsilon_2} \times \ldots \times T_{z_n}^{\epsilon_n}$, $\Omega_Z = \frac{\mathrm{d}z_1}{z_1} \wedge \frac{\mathrm{d}z_2}{z_2} \wedge \ldots \wedge \frac{\mathrm{d}z_n}{z_n}$, and

$$R_n^{\underline{\epsilon}} = \sum_{k=1}^{n} (-1)^{\binom{k}{2}} (2\pi\mathbf{i})^k \delta_{T_{z_1}^{\epsilon_1} \times T_{z_2}^{\epsilon_2} \times \ldots \times T_{z_{k-1}}^{\epsilon_{k-1}}} \log^{\epsilon_k} z_k \frac{\mathrm{d}z_{k+1}}{z_{k+1}} \wedge \ldots \wedge \frac{\mathrm{d}z_n}{z_n}.$$

Here $\log^{\epsilon}(z)$ is the branch (0-current) with cut at T_z^{ϵ}, so that $d[\log^{\epsilon}(z)] = \frac{dz}{z} - 2\pi\mathbf{i}\delta_{T_z^{\epsilon}}$.

The formula (3.3.1) induces a map of complexes

$$\mathcal{R}_{\varepsilon}^{\bullet,\underline{\epsilon}} : \mathscr{N}_{\varepsilon}^{p}(X, -\bullet) \to \mathcal{C}_{\mathcal{D}}^{2p+\bullet}(X, \mathbb{Z}(p)). \tag{3.3.2}$$

Proposition 12. *$\mathcal{R}_{\varepsilon}^{\bullet,\underline{\epsilon}}$ is a map of complex.*

Therefore we get for each p, n, ε, and $\underline{\epsilon} \in B_{\varepsilon}$ Abel-Jacobi maps (induced by these maps of complexes)

$$AJ_{\varepsilon}^{p,n,\underline{\epsilon}} : H_n(\mathscr{N}_{\varepsilon}^{p}(X, \bullet)) \to H_{\mathscr{D}}^{2p-n}(X, \mathbb{Z}(p)).$$

3.4 Homotopies of Abel-Jacobi maps

3.4.1 Notations

Put

$$\mathcal{R}_{z_i} := (2\pi\mathbf{i}T_{z_i}, \frac{dz_i}{z_i}, \log(z_i))$$

$$\mathcal{R}_{z_i}^{\epsilon} := (2\pi\mathbf{i}T_{arg(z_i)=\pi-\epsilon}, \frac{dz_i}{z_i}, \log^{\epsilon}(z_i))$$

where $\log^{\epsilon}(z_i)$ is taking branch cut at $arg(z_i) = \pi - \epsilon$. We write $T_{arg(z_i)=\pi-\epsilon}$ as $T_{z_i}^{\epsilon}$. Define

$$\mathcal{S}_{z_i}^{\epsilon,\epsilon'} := (-\theta_{z_i}^{\epsilon,\epsilon'}, 0, 0)$$

where $\theta_{z_i}^{\epsilon,\epsilon'} := \pm\delta_{\{\arg(z_i)\in(\epsilon,\epsilon')\}}$ are 0-currents. (The sign is positive if $\epsilon > \epsilon'$, negative otherwise.) Clearly we have

$$D\mathcal{S}_{z_i}^{\epsilon,\epsilon'} = \mathcal{R}_{z_i}^{\epsilon} - \mathcal{R}_{z_i}^{\epsilon'}.$$

3.4.2 Homotopy property

In this subsection we will prove the

Theorem 13. *Given $\underline{\epsilon}$, $\underline{\epsilon}' \in B_{\varepsilon}^N$, we have $\mathcal{R}_{\varepsilon}^{\bullet,\underline{\epsilon}} \simeq_{(\mathbb{Z})} \mathcal{R}_{\varepsilon}^{\bullet,\underline{\epsilon}'}$.*

For a fixed N, consider the following double complex (truncated at N):

$$E^{a,b} := C_{\mathscr{D}}^{2a+b}((\mathbb{P}^1)^a)^{\oplus\binom{N}{a}2^{N-a}}$$

in which the components in the a^{th} column are the a-"faces" of $(\mathbb{P}^1)^N$, indexed by $\underline{I} = \{I_1, \cdots, I_{n-a}\} \subset \{1, \cdots, N\}$ (with $I_1 < \cdots < I_{n-a}$) and $f : \underline{I} \to \{0, \infty\}$. The differentials of this double complex are given by the Deligne differential D and the alternating sum of Gysin push-forwards $\delta = 2\pi\mathbf{i}\sum_{i\in\underline{I}}(-1)^{\mathrm{sgn}_{\underline{I}}(i)+\mathrm{sgn}(f(i))}(\rho_{f(i)}^i)_*$, where $\mathrm{sgn}_{\underline{I}}(i) = k$ for $i = I_k$, and $\mathrm{sgn}(0) = 0, \mathrm{sgn}(\infty) = 1$. So that on the associated simple complex we have $\mathbb{D} = d + (-1)^b\delta$.

In this double complex, we consider the set of triples $\mathcal{R}_{\square}^{\underline{\epsilon}} := \{\mathcal{R}_{\underline{n}}^{\hat{\underline{\epsilon}}} := ((2\pi\mathbf{i})^n T_{\underline{n}}^{\hat{\underline{\epsilon}}}, \Omega_n, R_{\underline{n}}^{\hat{\underline{\epsilon}}})\}_{n,\underline{I},f}$ in $E^{n,-n}$, where $0 \le n \le N$, $\{\hat{\epsilon}_1, \cdots, \hat{\epsilon}_n\} = \{\epsilon_1, \cdots, \epsilon_N\} \setminus \{\epsilon_{I_1}, \cdots, \epsilon_{I_n}\}$ a subsequence.

Proposition 14. *$\mathcal{R}_{\square}^{\underline{\epsilon}}$ is a 0-cocycle.*

Proof. According to (5.2),(5.3) and (5.4) in [**KLM**], generally we have (for $\mathcal{R}_n^{\underline{\epsilon}} \in (C^n_{\mathscr{D}}(\mathbb{P}^1)^n)_{\underline{I},f}$:

$$\begin{aligned} D\mathcal{R}_n^{\underline{\epsilon}} = (-(2\pi\mathbf{i})^n \sum_{k=1}^{n}(-1)^k((\rho_i^0)_* T_{n-1}^{\{\epsilon_1,\cdots,\hat{\epsilon_i},\cdots,\epsilon_n\}} - (\rho_i^\infty)_* T_{n-1}^{\{\epsilon_1,\cdots,\hat{\epsilon_i},\cdots,\epsilon_n\}}, \\ - 2\pi\mathbf{i}\sum_{k=1}^{n}(-1)^k\Omega(z_1,\cdots,\hat{z_i},\cdots,z_n)\delta_{(z_i)}, \\ - 2\pi\mathbf{i}\sum_{k=1}^{n}(-1)^k R^{\{\epsilon_1,\cdots,\hat{\epsilon_i},\cdots,\epsilon_n\}}(z_1,\cdots,\hat{z_i},\cdots,z_n)\delta_{(z_i)}) \\ = -(-1)^{n-1}\delta(\sum_{k=1}^{n}(\mathcal{R}_{n-1,0}^{\{\epsilon_1,\cdots,\hat{\epsilon_k},\cdots,\epsilon_n\}} + \mathcal{R}_{n-1,\infty}^{\{\epsilon_1,\cdots,\hat{\epsilon_k},\cdots,\epsilon_n\}})) \end{aligned} \quad (3.4.1)$$

where for the δ in the last line, we only consider the component mapping into $(C^n_{\mathscr{D}}(\mathbb{P}^1)^n)_{\underline{I},f}$. This tells us $D\mathcal{R}_n^{\underline{\epsilon}} + (-1)^{n-1}\delta(\sum_{k=1}^{n}(\mathcal{R}_{n-1,0}^{\{\epsilon_1,\cdots,\hat{\epsilon_k},\cdots,\epsilon_n\}} + \mathcal{R}_{n-1,\infty}^{\{\epsilon_1,\cdots,\hat{\epsilon_k},\cdots,\epsilon_n\}})) = 0$ for any n, thus each component of $\mathbb{D}\mathcal{R}_\square^{\underline{\epsilon}}$ are all 0, so $\mathcal{R}_\square^{\underline{\epsilon}} \in \mathrm{Ker}(\mathbb{D})$ is a 0-cocycle. □

For $\underline{\epsilon}$, $\underline{\epsilon}'$, consider the following (-1)-cochain:

$$\mathcal{S}_\square^{\underline{\epsilon},\underline{\epsilon}'} := \{\mathcal{S}^{\hat{\underline{\epsilon}},\hat{\underline{\epsilon}'}} := \sum_{k=1}^{n}(-1)^{k-1}\mathcal{R}_{z_1}^{\epsilon_{m_1}} \cup \cdots \cup \mathcal{R}_{z_{k-1}}^{\epsilon_{m_{k-1}}} \cup \mathcal{S}_{z_k}^{\epsilon_{m_k},\epsilon'_{m_k}} \cup \mathcal{R}_{z_{k+1}}^{\epsilon'_{m_{k+1}}} \cup \cdots \cup \mathcal{R}_{z_n}^{\epsilon'_{m_n}}\}_{n,\underline{m},\underline{i}}$$

It satisfies the following key property:

Proposition 15. $\mathbb{D}\mathcal{S}_\square^{\underline{\epsilon},\underline{\epsilon}'} = \mathcal{R}_\square^{\underline{\epsilon}} - \mathcal{R}_\square^{\underline{\epsilon}'}$

Proof.

$$\begin{aligned} D\mathcal{S}_n^{\underline{\epsilon},\underline{\epsilon}'} = \sum_{k=1}^{n}\sum_{l=1}^{k-1}(-1)^{l-1}(-1)^{k-1}\mathcal{R}_{z_1}^{\epsilon_1} \cup \cdots \cup D\mathcal{R}_{z_l}^{\epsilon_l} \cup \cdots \cup \mathcal{S}_{z_k}^{\epsilon_k,\epsilon'_k} \cup \cdots \cup \mathcal{R}_{z_n}^{\epsilon'_n} \\ + \sum_{k=1}^{n}\sum_{l=k+1}^{n}(-1)^{l}(-1)^{k-1}\mathcal{R}_{z_1}^{\epsilon_1} \cup \cdots \cup \mathcal{S}_{z_k}^{\epsilon_k,\epsilon'_k} \cup \cdots \cup D\mathcal{R}_{z_l}^{\epsilon'_l} \cup \cdots \cup \mathcal{R}_{z_n}^{\epsilon'_n} \\ + \sum_{k=1}^{n}\mathcal{R}_{z_1}^{\epsilon_1} \cup \cdots \cup D\mathcal{S}_{z_k}^{\epsilon_k,\epsilon'_k} \cup \cdots \cup \mathcal{R}_{z_n}^{\epsilon'_n} \end{aligned} \quad (3.4.2)$$

Notice that $D\mathcal{R}^{\epsilon}_{z_i} = 2\pi\mathbf{i}\delta_{(z_i)}$ and $D\mathcal{S}^{\epsilon,\epsilon'}_{z_i} = \mathcal{R}^{\epsilon}_{z_i} - \mathcal{R}^{\epsilon'}_{z_i}$, we can rewrite this expression as follow by applying telescoping method and rearranging the order of the summation:

$$
\begin{aligned}
D\mathcal{S}^{\underline{\epsilon},\underline{\epsilon}'}_{n} = 2\pi\mathbf{i}\sum_{k=1}^{n}\sum_{l=1,l\neq k}^{n}(-1)^{l-1}(-1)^{k}\delta_{(z_l)}\mathcal{R}^{\epsilon_1}_{z_1}\cup\cdots\cup\mathcal{S}^{\epsilon_k,\epsilon'_k}_{z_k}\cup\cdots\cup\mathcal{R}^{\epsilon'_n}_{z_n} \\
\text{(with the } l\text{th term omitted, either before } k \text{ or after } k) \\
+\sum_{k=1}^{n}\mathcal{R}^{\epsilon_1}_{z_1}\cup\cdots\cup(\mathcal{R}^{\epsilon_k}_{z_k}-\mathcal{R}^{\epsilon'_k}_{z_k})\cup\cdots\cup\mathcal{R}^{\epsilon'_n}_{z_n} \\
= 2\pi\mathbf{i}\sum_{l=1}^{n}(-1)^{l-1}\delta_{(z_l)}\sum_{k=1,k\neq l}^{n}(-1)^{k}\mathcal{R}^{\epsilon_1}_{z_1}\cup\cdots\cup\mathcal{S}^{\epsilon_k,\epsilon'_k}_{z_k}\cup\cdots\cup\mathcal{R}^{\epsilon'_n}_{z_n} \\
+\mathcal{R}^{\underline{\epsilon}}_{n}-\mathcal{R}^{\underline{\epsilon}'}_{n} \\
= -(-1)^{n}\delta(\sum_{k=1}^{n}(\mathcal{S}^{\{\epsilon_1,\cdots,\hat{\epsilon_k},\cdots,\epsilon_n\},\{\epsilon'_1,\cdots,\hat{\epsilon'_k},\cdots,\epsilon'_n\}}_{n-1,0}+\mathcal{S}^{\{\epsilon_1,\cdots,\hat{\epsilon_k},\cdots,\epsilon_n\},\{\epsilon'_1,\cdots,\hat{\epsilon'_k},\cdots,\epsilon'_n\}}_{n-1,\infty}) \\
+\mathcal{R}^{\underline{\epsilon}}_{n}-\mathcal{R}^{\underline{\epsilon}'}_{n} \quad (3.4.3)
\end{aligned}
$$

This tells us $D\mathcal{S}^{\underline{\epsilon},\underline{\epsilon}'}_{n} + (-1)^{n}\delta(\sum_{k=1}^{n}(\mathcal{S}^{\{\epsilon_1,\cdots,\hat{\epsilon_k},\cdots,\epsilon_n\},\{\epsilon'_1,\cdots,\hat{\epsilon'_k},\cdots,\epsilon'_n\}}_{n-1,0}+\mathcal{S}^{\{\epsilon_1,\cdots,\hat{\epsilon_k},\cdots,\epsilon_n\},\{\epsilon'_1,\cdots,\hat{\epsilon'_k},\cdots,\epsilon'_n\}}_{n-1,\infty}) = \mathcal{R}^{\underline{\epsilon}}_{n} - \mathcal{R}^{\underline{\epsilon}'}_{n}$ holds for each component of n. Thus $\mathbb{D}\mathcal{S}^{\underline{\epsilon},\underline{\epsilon}'}_{\square} = \mathcal{R}^{\underline{\epsilon}}_{\square} - \mathcal{R}^{\underline{\epsilon}'}_{\square}$ holds. □

Proof of Theorem 13. The result in Proposition 15 implies at once that $\mathcal{R}^{n,\underline{\epsilon}}_{\varepsilon}(Z) - \mathcal{R}^{n,\underline{\epsilon}'}_{\varepsilon}(Z) = D\mathcal{S}^{\underline{\epsilon},\underline{\epsilon}'}_{\varepsilon}(Z) + \mathcal{S}^{\underline{\epsilon},\underline{\epsilon}'}_{\varepsilon}(\partial Z)$, so that $\mathcal{R}^{\bullet,\underline{\epsilon}}_{\varepsilon} \simeq \mathcal{R}^{\bullet,\underline{\epsilon}'}_{\varepsilon}$ as claimed. □

3.5 The integral Abel-Jacobi map

Recall our map of complexes from (3.3.2), with n^{th} term

$$\mathcal{R}^{n,\underline{\epsilon}}_{\varepsilon} : \mathscr{N}^{p}_{\varepsilon}(X,n) \to C^{2p-n}_{\mathscr{D}}(X,\mathbb{Z}(p))$$

According to our result from the last section, we know that for $\underline{\epsilon}, \underline{\epsilon}' \in B_\varepsilon^N$, $\mathcal{R}_\varepsilon^{n,\underline{\epsilon}} \simeq \mathcal{R}_\varepsilon^{n,\underline{\epsilon}'}$; that is to say, they induce the same homomorphism after taking cohomology:

Corollary 16. *All the* $\underline{\epsilon} \in B_\varepsilon$ *induce the same map:*

$$AJ_\varepsilon^{p,n} : H_n(\mathscr{N}_\varepsilon^p(X,\bullet)) \to H_{\mathscr{D}}^{2p-n}(X,\mathbb{Z}(p)).$$

Moreover, for $\varepsilon' < \varepsilon$ and $\underline{\epsilon} \in B_{\varepsilon'} \subset B_\varepsilon$, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{N}_\varepsilon^p(X,\bullet) & \xhookrightarrow{\ \imath\ } & \mathscr{N}_{\varepsilon'}^p(X,\bullet) \\ & {\scriptstyle \mathcal{R}_\varepsilon^{\bullet,\underline{\epsilon}}} \searrow \quad \swarrow {\scriptstyle \mathcal{R}_{\varepsilon'}^{\bullet,\underline{\epsilon}}} & \\ & C_{\mathscr{D}}^{2p-\bullet}(X,\mathbb{Z}(p)) & \end{array}$$

which is straightforward from the definition. By taking homology, we have that the following diagram commutes as well:

$$\begin{array}{ccc} H_n(\mathscr{N}_\varepsilon^p(X,\bullet)) & \xrightarrow{\ [\imath]\ } & H_n(\mathscr{N}_{\varepsilon'}^p(X,\bullet)) \\ & {\scriptstyle AJ_\varepsilon^{p,n}} \searrow \quad \swarrow {\scriptstyle AJ_{\varepsilon'}^{p,n}} & \\ & H_{\mathscr{D}}^{2p-n}(X,\mathbb{Z}(p)) & \end{array}$$

In order to get the integral Abel-Jacobi map, we need the following result:

Theorem 17. $CH^p(X,n) \cong \varinjlim_{\varepsilon} H_n(\mathscr{N}_\varepsilon^p(X,\bullet))$

Proof. Since $\bigcup \mathscr{N}_\varepsilon^p(X,\bullet) = \mathscr{N}^p(X,\bullet)$, we have

$$CH^p(X,n) \subseteq \varinjlim_{\varepsilon} H_n(\mathscr{N}_\varepsilon^p(X,\bullet)).$$

For the other direction of the equation, consider $\xi \in CH^p(X,n)$, and $\tilde{\xi}$,$\tilde{\xi}'$ be two representations of ξ in the following sequence:

$$H_n(\mathscr{N}_\varepsilon^p(X,\bullet)) \to H_n(\mathscr{N}_{\varepsilon'}^p(X,\bullet)) \to \cdots \to CH^p(X,n)$$

We need to show that $\tilde{\xi}$ and $\tilde{\xi}'$ will eventually merge at some ϵ, that is to say, $\bigcup \partial \mathscr{N}_\varepsilon^p(X,n+1) = \partial \mathscr{N}^p(X,n+1)$, which directly comes from the property of normalized cycle and $\bigcup \mathscr{N}_\varepsilon^p(X,\bullet) = \mathscr{N}^p(X,\bullet)$. □

Thus we have a well-defined map

$$AJ_{\mathbb{Z}}^{p,n} : CH^p(X,n) \to H_{\mathscr{D}}^{2p-n}(X,\mathbb{Z}(p))$$

given by $AJ_{\mathbb{Z}}^{p,n} := \varinjlim_{\varepsilon} AJ_\varepsilon^{p,n}$. Precisely, for $Z \in CH^p(X,n)$ and $\tilde{Z} \in \mathrm{Ker}(\partial) \subset \mathscr{N}_\varepsilon^p(X,n)$ be any class mapping to Z and $\underline{\epsilon}$, $AJ_{\mathbb{Z}}^{p,n}(Z) = \mathcal{R}_\varepsilon^{n,\underline{\epsilon}}(\tilde{Z})$ is a well-defined map (that is to say, it lies in the same homology class for any choice of $\underline{\epsilon}$). Thus we have an explicit expression for the integral Abel-Jacobi map:

$$AJ_{\mathbb{Z}}^{p,n}(Z) = \lim_{\underline{\epsilon} \to \underline{0}} \mathcal{R}_\varepsilon^{n,\underline{\epsilon}}(\tilde{Z})$$

Moreover, for $\tilde{Z}$ a representative in $Z_{\mathbb{R}}^p(X,n) \cap \mathscr{N}^p(X,n)$, we know that $\tilde{Z}$ lies in $\mathscr{N}_\varepsilon^p$ for any $\varepsilon > 0$, and

$$\lim_{\underline{\epsilon} \to \underline{0}} \mathcal{R}_\varepsilon^{n,\underline{\epsilon}}(\tilde{Z}) = \mathcal{R}(\tilde{Z}).$$

In particular, this means that *on cycles belonging to $Z_{\mathbb{R}}^p(X,n) \cap \mathscr{N}^p(X,n)$, our integral AJ map is given by the KLM formula.*

3.6 Application to torsion cycles

Recent work of Kerr and Yang [**MY**] provides explicit representatives for generators of $CH^n(Spec(k), 2n-1)$ where k is an abelian extension of $\mathbb{Q}$, assuming the result we're giving here is correct. We'll check that when $n = 2, 3, 4$, the cycle given by [**MY**] satisfies the normal and proper intersection condition thus belongs to $Z^p_{\mathbb{R}}(X, 2p-1) \cap \mathscr{N}^p(X, 2p-1)$. For $n = 5$ and higher cases, a normalization of their given generator is needed.

Let ξ_N be an N^{th} root of 1.

Proposition 18. *The cycles given by (4.1), (4.2) and (4.3) in [*__MY__*] lie in* $Z^n_{\mathbb{R}}(\mathbb{Q}(\xi_N), 2n-1) \cap \mathscr{N}^n(\mathbb{Q}(\xi_N), 2n-1)$. *(Notice that for (4.3), we're choosing the first set of the cycles. The "Alternate" choice is not normalized.)*

The $Z^n_{\mathbb{R}}$ part is given by Remark 3.3 in [**MY**]. For the $\mathscr{N}^n$ part, it's not hard to check that the cycles of $n = 2, 3$ are normalized. For $CH^4(Spec(\mathbb{Q}(\xi)), 7)$ (and $\xi = \xi_N$), the following cycles are given by [**MY**]:

$$\mathscr{Z} = (\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{t_3}{t_3-1}, 1-\xi t_1t_2t_3, t_1^N, t_2^N, t_3^N), \mathscr{W}_1 = \frac{1}{2}(\mathscr{W}_1^{(1)} + \mathscr{W}_1^{(2)}),$$

$$\mathscr{W}_1^{(1)} = (\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1t_2}, \frac{(u-t_1^N)(u-t_2^N)}{(u-1)(u-t_1^Nt_2^N)}, \frac{u}{t_1^N}, \frac{u}{t_2^N}, \frac{1}{u}),$$

$$\mathscr{W}_1^{(2)} = (\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1t_2}, \frac{(u-t_1^N)(u-t_2^N)}{(u-1)(u-t_1^Nt_2^N)}, \frac{t_1^N}{u}, \frac{t_2^N}{u}, \frac{u}{t_1^Nt_2^N}),$$

$$\mathscr{W}_2 = -\frac{1}{2}(\frac{t_1}{t_1-1}, \frac{1}{1-\xi t_1}, \frac{(v-t_1^Nu)(v-t_1^{-N}u)}{(v-u^2)(v-1)}, \frac{(u-t_1^N)(u-vt_1^N)}{(u-v)^2}, \frac{vt_1^N}{u}, \frac{v}{t_1^Nu}, \frac{u}{v}).$$

and $\tilde{\mathscr{Z}} = \mathscr{Z} + \mathscr{W}_1 + \mathscr{W}_2$ is a generator of $CH^4(Spec(\mathbb{Q}(\xi)), 7)$. By the computation in [**MY**], we have

$$\partial\mathscr{Z} = -\partial_4^0\mathscr{Z} = -\partial_4^\infty\mathscr{W}_1^{(1)} = -\partial_4^\infty\mathscr{W}_1^{(2)},$$

$$\partial\mathscr{W}_1 = -\frac{1}{2}\partial_3^\infty\mathscr{W}_1^{(1)} + \frac{1}{2}\partial_4^\infty\mathscr{W}_1^{(1)} - \frac{1}{2}\partial_3^\infty\mathscr{W}_1^{(2)} + \frac{1}{2}\partial_4^\infty\mathscr{W}_1^{(2)},$$

$$\partial\mathscr{W}_2 = \frac{1}{2}\partial_3^\infty\mathscr{W}_1^{(1)} + \frac{1}{2}\partial_3^\infty\mathscr{W}_1^{(2)}$$

We can see that $\partial_3^\infty\tilde{\mathscr{Z}} = 0$ and $\partial_4^\infty\tilde{\mathscr{Z}} = \partial_4^0\tilde{\mathscr{Z}}$ which can be cancelled by adding (for free) a degenerate cycle, so that $\tilde{\mathscr{Z}}$ is normalized.

This puts some earlier results on firm ground as well, such as O. Petras's result in [**Pe**] that

$$Z := (1-1/t, 1-t, t^{-1}) + (1-\xi_5/t, 1-t, t^{-5}) + (1-\bar{\xi_5}/t, 1-t, t^{-5})$$

generates $CH^2(\mathbb{Q}(\sqrt{5}, 3)$ and (since we have $\mathscr{R}(Z) = \mathrm{Li}_2(1) + 5(\mathrm{Li}_2(\xi_5) + \mathrm{Li}_2(\bar{\xi_5})) = 7\pi^2/30)$ is 120-torsion.

Also according to [**MY**], for $N = 2$ ($k = \mathbb{Q}$) and $n = 4$, we have $|\frac{1}{(2\pi\mathbf{i})^4}c_{\mathscr{D}}(\tilde{\mathscr{Z}})| = \frac{7}{1440}$, which means it is 1440-torsion. The normalization of higher dimension case could be something to work out in the future.

Chapter 4

Linear higher Chow cycles

If X is smooth and k is a subfield of $\mathbb{C}$, one has Bloch's Abel-Jacobi maps

$$\mathrm{AJ}: CH^p(X,m) \to H_{\mathcal{H}}^{2p-m}\left(X_{\mathbb{C}}^{\mathrm{an}}, \mathbb{Z}(p)\right)$$

into absolute Hodge cohomology, which may be described ($\otimes\mathbb{Q}$) in terms of explicit maps of complexes $\widetilde{\mathrm{AJ}}$ [**BKLL15**]. The homology of the subcomplex $LZ^p(X,\bullet)$ given by equations *linear* in the $\{x_i\}$ defines the *linear higher Chow groups* $LCH^p(X,m)$, which map naturally to $CH^p(X,m)$.

This chapter concerns the case $CH^p(k,m)$ of a point over a number field, where $X = \mathrm{Spec}(k)$. Working $\otimes\mathbb{Q}$, this is zero unless $(p,m) = (n, 2n-1)$, in which case $CH^n(k,2n-1)_{\mathbb{Q}} \cong K_{2n-1}(k)_{\mathbb{Q}} \cong K_{2n-1}(\mathcal{O}_k)_{\mathbb{Q}}$. The linear group $LCH^n(k,2n-1)_{\mathbb{Q}}$ is (for each $n \geq 1$) the image of a canonical homomorphism

$$\psi_n: \; H_{2n-1}\left(\mathrm{GL}_n(k), \mathbb{Q}\right) \to CH^n(k, 2n-1)_{\mathbb{Q}},$$

induced by the morphism of complexes

$$\tilde{\psi}_n : C^{\mathrm{grp}}_{\bullet}(n) \to Z^n(k, \bullet)_{\mathbb{Q}}$$

given (for $\bullet = m$) by

$$(g_0, \ldots, g_m) \longmapsto \left\{ \sum_{i=0}^{m} x_i \cdot g_i \underline{v} = 0 \right\} \subset \Delta^m$$

for some choice of $\underline{v} \in k^n \setminus \{\underline{0}\}$. (Here we consider C_i^{grp} resp. $Z^n(k,i)$ to be in degree $-i$.) Now given an embedding $\sigma : k \hookrightarrow \mathbb{C}$, the Bloch-Beilinson regulator map (i.e., AJ composed with projection $\mathbb{C}/\mathbb{Q}(n) \twoheadrightarrow \mathbb{R}$) sends $CH^n(\sigma(k), 2n-1)_{\mathbb{Q}} \underset{r_{\mathrm{Be}}}{\to} \mathbb{R}$, so that composing with *all* $r = [k : \mathbb{Q}] = r_1 + 2r_2$ embeddings maps $CH^n(k, 2n-1) \to \mathbb{R}^r$. This factors through the invariants $\mathbb{R}^{d_n}$ [$d_n := r_2$ (n even) resp. $r_1 + r_2$ (n odd)] under de Rham conjugation, and is known to be equivalent to $\frac{1}{2}$ the Borel regulator $r_{\mathrm{Bo}} : K_{2n-1}(\mathcal{O}_k)_{\mathbb{Q}} \to \mathbb{R}^{d_n}$ [**Bu02**].

Given the close relation between homology of GL_n and the original context of Borel's theorem, it is natural to consider the composite morphism of complexes $\widetilde{\mathrm{AJ}} \circ \tilde{\psi}_n$. Replacing k by $\mathbb{C}$, these should yield explicit cocycles in $H^{2n-1}_{\mathrm{meas}}(\mathrm{GL}_n(\mathbb{C}), \mathbb{C}/\mathbb{Z}(n))$ lifting the Borel classes in $H^{2n-1}_{\mathrm{cont}}(\mathrm{GL}_n(\mathbb{C}), \mathbb{R})$ [**BKLL15**]. This would also deepen our understanding of the equivalence of the Beilinson and Borel regulators. The first test of this proposal is to check its simplest implication:

Conjecture 19. *For a number field k, the linear higher Chow cycles surject* (*rationally*) *onto the simplicial higher Chow groups. Equivalently, ψ_n is surjective for every $n \geq 1$.*

4.1 A strategy for surjectivity?

In fact, Conjecture 19 is claimed as Proposition 16 in R. de Jeu's paper [**dJ02**]. His approach is to fit (for each $n \geq 1$) $\tilde{\psi}_n$ into a commuting triangle

$$\begin{array}{ccc} C_\bullet^{\mathrm{grp}}(n) & \xrightarrow{\tilde{\psi}_n} & Z^n(k,\bullet) \\ & {\scriptstyle \tilde{r}_{\mathrm{Bor}}} \searrow & \downarrow {\scriptstyle \tilde{r}_{\mathrm{Be}}} \\ & & \mathbb{R}[2n-1]. \end{array} \tag{4.1.1}$$

Taking homology yields the diagram

$$\begin{array}{ccc} H_{2n-1}(GL_n(k),\mathbb{Q}) & \xrightarrow{\psi_n} & CH^n(k,2n-1)_{\mathbb{Q}} \\ & {\scriptstyle r_{\mathrm{Bor}}} \searrow & \downarrow {\scriptstyle r_{\mathrm{Be}}} \\ & & \mathbb{R}, \end{array} \tag{4.1.2}$$

in which r_{Bor} [resp. r_{Be}] is the Borel [resp. Beilinson] regulator, composed with a choice of embedding $k \hookrightarrow \mathbb{C}$. By composing with *all* embeddings (and using Borel's theorem), we get a diagram of the form

$$\begin{array}{ccc} H_{2n-1}(GL_n(k),\mathbb{R}) & \xrightarrow{\psi_n} & CH^n(k,2n-1)_{\mathbb{R}} \\ & {\scriptstyle \cong} \searrow & \downarrow {\scriptstyle \cong} \\ & & \mathbb{R}^{d_n}, \end{array} \tag{4.1.3}$$

proving Conjecture 19.

The problem here is with de Jeu's choice of Goncharov's simplicial regulator r_{Gon} [**Go95**] for $\tilde{r}_{\mathrm{Be}}$. While this appears to make (4.1.1) commute, by the calculation on pp. 228-230 of [**dJ02**], it is now known [**BKLL15**] that r_{Gon} is not a map of complexes. Specifically, in

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & Z^2(k,2n) & \xrightarrow{\partial} & Z^n(k,2n-1) & \longrightarrow & Z^n(k,2n-2) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle r_{\text{Gon}}} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{R} & \longrightarrow & 0 & \longrightarrow & \cdots \end{array} \tag{4.1.4}$$

we do not have $r_{\text{Gon}}(\partial C_{2n}) = 0$. So we must replace r_{Gon} by the "corrected" version in [**BKLL15**], which we will denote by reg_G. It is given on $Y \in Z^n(k, 2n-1)$ by

$$\text{reg}_G(Y) := \int_{Y(\mathbb{C})} r_{2n-1}\left(\tfrac{x_1+\cdots+x_{2n-1}}{-x_0}, \tfrac{x_2+\cdots+x_{2n-1}}{-x_1}, \ldots, \tfrac{x_{2n-1}}{-x_{2n-2}}\right), \tag{4.1.5}$$

which is known to induce r_{Be}.

On the group homology side, de Jeu [**dJ02**] also uses a formula of Goncharov for $\tilde{r}_{\text{Bor}}$; we denote this by reg_B. Given $(g_0, \ldots, g_{2n-1}) \in C^{\text{grp}}_{2n-1}(n)$, let $\{f_i\}_{i=1}^{2n-1}$ denote nonzero rational functions on $\mathbb{P}^{n-1}_{\mathbb{C}}$ with divisors

$$\begin{aligned} D_i = \{[\underline{X}] \in \mathbb{P}^{n-1} \mid (X_0, \ldots, X_{n-1}) \cdot g_i \underline{v} = 0\} \\ - \{[\underline{X}] \in \mathbb{P}^{n-1} \mid (X_0, \ldots, X_{n-1}) \cdot g_0 \underline{v} = 0\}. \end{aligned}$$

Then according to [**Go93**],

$$\text{reg}_B(g_0, \ldots, g_{2n-1}) := \int_{\mathbb{P}^{n-1}_{\mathbb{C}}} r_{2n-1}(f_1, \ldots, f_{2n-1}) \tag{4.1.6}$$

induces r_{Bor}. At least in the $n = 2$ case we treat below, this formula is correct. (See the calculation in §4.2 below.) Moreover, it is well-defined for any n, in the sense that the RHS of (4.1.6) is invariant when we rescale any f_i by a constant.

We tried to emulate the approach in [**dJ02**] to see if the new diagram (4.1.1) (with $\tilde{r}_{\text{Bor}} = \text{reg}_B$ unchanged and $\tilde{r}_{\text{Be}}$ corrected to reg_G) commutes, with no success. At this point, we decided to attempt the first nontrivial case by hand, and arrived at a negative result:

Proposition 20. *For $n = 2$, the amended triangle* (4.1.1) *does not commute.*

4.2 Proof of Proposition 20

In [**Go04**], Goncharov mentions the formula

$$\int_{\mathbb{P}^1} r_3(f_1, f_2, f_3) = \sum_{(x_1,x_2,x_3)\in\mathbb{C}^3} \nu_{x_1}(f_1)\nu_{x_2}(f_2)\nu_{x_3}(f_3) D_2(\mathsf{CR}(x_1, x_2, x_3, \infty)) \tag{4.2.1}$$

where $\nu_x(f)$ is the order of f at x. One easily verifies that this is correct; it will be required below.

Now take $\underline{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2$, and $(g_0, g_1, g_2, g_3) \in C_3(2)$. We can do a change of coordinate to let $g_0 = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$, $g_1 = \begin{pmatrix} 0 & * \\ 1 & * \end{pmatrix}$, $g_2 = \begin{pmatrix} a & * \\ c & * \end{pmatrix}$, $g_3 = \begin{pmatrix} b & * \\ d & * \end{pmatrix}$. For convenience, we set $\Delta := ad - bc$.

Write $z := \frac{X_1}{X_0}$ and $f_1(z) = z$, $f_2(z) = cz + a$, and $f_3(z) = dz + b$. According to (4.1.6) and (4.2.1), we have

$$\begin{aligned} \text{reg}_B(g_0, g_1, g_2, g_3) &= \int_{\mathbb{P}^1} r_3(z, cz + a, dz + b) \\ &= D_2\left(\tfrac{bc}{ad}\right). \end{aligned}$$

This is consistent with evaluating the cocycle $\varepsilon_2 \in H^3_{\text{cont}}(GL_2(\mathbb{C}), \mathbb{R})$ (cf. Intro. to [**BKLL15**]) on the "group homology chain" (g_0, g_1, g_2, g_3).

For the other side, applying $\tilde{\psi}$ to this chain produces the linear higher Chow chain $Y \subset \Delta^3$ cut out by

$$x_0 + ax_2 + bx_3 = 0 \text{ and } x_1 + cx_2 + dx_3 = 0.$$

Parametrizing $Y \cong \mathbb{P}^1$ by $t \mapsto (\Delta, \Delta t, bt - d, c - at)$, (4.1.5), (4.2.1) and the rescaling property yield $\mathrm{reg}_G(\tilde{\psi}(g_0, g_1, g_2, g_3)) =$

$$\begin{aligned}
\mathrm{reg}_G(Y) &= \int_{\mathbb{P}^1} r_3\left(\tfrac{(d-c)+(a-b-\Delta)t}{\Delta}, \tfrac{(d-c)+(a-b)t}{\Delta t}, \tfrac{at-c}{bt-d}\right) \\
&= \int_{\mathbb{P}^1} r_3\left((c-d) + (\Delta + b - a)t, \tfrac{(c-d)+(b-a)t}{t}, \tfrac{at-c}{bt-d}\right) \\
&= D_2\left(\tfrac{(d-1)\Delta}{b(c-d)}\right) - D_2\left(\tfrac{(c-1)\Delta}{a(c-d)}\right) - D_2\left(\tfrac{(b-a)(d-1)}{b(d-c)}\right) + D_2\left(\tfrac{(b-a)(c-1)}{a(d-c)}\right).
\end{aligned}$$

To check that these two results disagree, put $a = 1$, $b = -1$, $c = 1 - i$, $d = 1 + i$, so that $\Delta = 2$ and $\frac{ad}{bc} = -i$. Of course, $D_2(-i) \neq 0$. On the other hand,

$$\tfrac{(d-1)\Delta}{b(c-d)}, \quad \tfrac{(c-1)\Delta}{a(c-d)}, \quad \tfrac{(b-a)(d-1)}{b(d-c)}, \quad \tfrac{(b-a)(c-1)}{a(d-c)}$$

are all 1, D_2 of which is 0.

4.3 Concluding remarks

Naturally, it is still possible that (4.1.2) commutes, since there we restrict to *closed* chains. In fact, even if we don't accept the proof in [**dJ02**], there is the earlier result of Gerdes [**Ge91**] which gives surjectivity of ψ_n for $n = 2$. Moreover, there is the agreement between the Beilinson and Borel regulators in [**Bu02**], though this does not involve ψ_n in any way. To sum up, we conclude with the

Question 21. *Are there any techniques to prove that* (4.1.2) *commutes even though the amended diagram* (4.1.1) *does not, for* $n = 2$ *and more generally? Or is it more likely that* ψ_n *has to be somehow modified?*

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