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# Algorithmic Trading with Prior Information

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WASHINGTON UNIVERSITY IN ST. LOUIS

Department of Mathematics

# Algorithmic Trading with Prior Information

by

Xinyi Cai

A thesis presented to
The Graduate School
of Washington University in
partial fulfillment of the
requirements for the degree
of Master of Arts

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St. Louis, Missouri

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*May 2018*

Abstract

Algorithmic Trading with Prior Information

by

Xinyi Cai

A.M. in Statistics,

Washington University in St. Louis, 2018.

Professor Jos E. Figueroa-Lpez, Chair

Traders utilize strategies by using a mix of market and limit orders to generate profits. There are different types of traders in the market, some have prior information and can learn from changes in prices to tweak her trading strategy continuously(Informed Traders), some have no prior information but can learn(Uninformed Learners), and some have no prior information and cannot learn(Uninformed Traders). Alvaro C, Sebastian J and Damir K [1] proposed a model for algorithmic traders to access the impact of dynamic learning in profit and loss in 2014. The traders can employ the model to decide which strategies to use. The model considered the distribution of the prices in the future using prior information, the spread of the bid and ask prices and also the capital appreciation of inventories. I implemented the model for the case when the trader can only learn from and take positions in one asset. Compared to the uninformed traders, the informed trader using the proposed model can change the strategies along time and make higher profits.

# 1. Introduction

## 1.1 Motivation

In real world, information is important in many cases. The business man who is better informed can catch the opportunities and maximize his profits. The same thing also happens to traders. The traders who are well informed and have the ability to learn from the market dynamics can generate greater profits than the uninformed trader who have no prior information and is less qualified to learn. [1]

The algorithmic trading refers to the process of using computer programs and defined complex algorithms to place trading decisions and transactions in financial market at a speed and frequency that is impossible for a human trader. [2] The models that the algorithmic trader can use to decide how to trade in the future by using a mix of market and limit orders are discussed nowadays. A good model can generate higher and more certain profits. The model we will discuss later in this thesis is aimed to help the algorithmic traders to make good trading decisions. Market orders can guarantee execution but cost more, whereas trading with limit orders are cheaper but has uncertain time of execution. The key problem for the traders' strategy is to decide the choice of order types and the timings to submit orders. It can be affected by many factors, such as the accumulated inventory, remaining time before position has to be closed, and how good are the price predictions. As the time goes, the informed trader will have updated information about the prior and become more confident about the future price, and then she will change her trading strategies accordingly. The problem we need to solve is to determine the good combination of market and limit orders, while also considering the uncertainty of her prediction, the price changes and the inventory exposure.

### 1.2 Background Information

Before we start to talk about the model for algorithmic trading, we need to understand some basic knowledge about trading. In this section, we will learn how the electronic markets work, types of traders, limit and market orders, limit order book and some defined prices. [3]

Nowadays, a lot of financial contracts are traded in electronic markets, such as shares, bonds, preferred stock, and some derivatives. Corporations sell ordinary shares(common stock) to raise money, and the buyers own some parts of the corporations based on the amount of shares they hold. The owners have the right to receive some shares of the corporation's profit and have the voting rights in annual general meeting. Large corporations can also use bond to raise capital. The bond holders don't have the voting right but can receive guaranteed regular incomes. Preferred stock has both characteristics of bond and common stock. The preferred stock holders receive period pre-arranged incomes but have no voting rights. Compared to the bond holders, their incomes are not guaranteed. When the company is in financial distress, the preferred holders get paid after the debt holders but before the common stock holders.

The traders are classified into three classes: fundamental traders, informed traders and market makers. Fundamental (or noise or liquidity) traders are motivated by economic fundamentals outside the exchange. Informed traders make profit by using information not reflected in market prices and trading assets in anticipation of price change. Market makers are professional traders who can buy or sell securities at prices posted in an exchange's trading system on behalf of the customers. The market makers' type of trading is passive or reactive trading because they profit by using detailed and professional analysis of the trading process, and also adapt to the circumstances changes. The fundamental trader and informed traders do more active and aggressive trading.

To implement an electronic market, people signal their willingness to trade, and then a matching engine matches those wanting to buy with those wanting to sell. In basic setup, there are two types of orders: Market Orders (MOs) and Limit Orders (LOs). MOs are aggressive orders that seek to execute a trade immediately. LOs are passive orders that can wait to meet their request of certain price and quantity. Hence, MOs are more expensive than LO because it requires higher liquidity and so the traders who post MOs need to pay for it.

Orders are managed by a matching engine and a limit order book (LOB). The LOB records all incoming and outgoing orders. The matching engine uses a well-defined algorithm when a possible trade could occur, and then the criterion is used to select the orders that will be executed. Most markets prioritize MOs over LOs and then use a price-time priority. [3]

When a sell MO executes against a buy LO, it is said to hit the bid; when a buy MO executes against a sell LO, it is said to lift the offer.

There are also some defined prices we may use later in the model. [3]

The quoted spread is defined as $P_t^a$ - $P_t^b$. (best ask price - best bid price).

The mid-price is defined as $\frac{1}{2}(P_t^a + P_t^b)$.

The micro-price which includes the volumes posted is defined as

$$\frac{V_t^b}{V_t^b + V_t^a} P_t^a - \frac{V_t^b}{V_t^b + V_t^a} P_t^b \tag{1.1}$$

where $V_t^b$ and $V_t^a$ are the volumes posted at the best bid and ask.

# 2. Optimal Trading Strategy Model

In this section, the naive and optimal trading strategies used by traders are first introduced. The optimal trading strategy model is developed based on it through solving the optimization problem of the wealth of the traders' assets. The model was raised by three mathematicians in 2014, and it proves that the informed trader can perform better than the uninformed trader. [1]

## 2.1 Naive Strategy VS Advanced Strategy

Traders always have the idea of making money by buying stocks when the price is low and sell them when the price is high. The naive stratgy that traders may use is as below.

Let the asset price be $S_t$. Suppose that at time $t < T$, trader has a predcition $\hat{S}_T$ about $S_T$, $\hat{S}_T$ is a random variable. In high frequency trading, computer may use the algorithms like this:

$$\hat{S}_T - S_0 = \begin{cases} -0.2, p = 0.3 \\ 0, p = 0.2 \\ 0.2, p = 0.5 \end{cases} \tag{2.1}$$

Then, if the $E(\hat{S}_T) > S_t$, trader will make the decision to buy it at t. Since the algorithm shows that the price is expected to increase at T, traders can make money from the spread. Here, the above algorithm set for the future price $S_T$ is our prior information about $S_T$. To improve the strategy, we can incorporate the prediction $\hat{S}_T$ in the asset price process $S_t$ and also learn from the realized dynamics of the asset price. In order to realize the idea, we can have a mathematical model for it.

### 2.2 Model Setup - Stock Price Dynamics

Let $S_t^i$ be the midprice of asset i (i = 1,...,n, n $\in \mathbb{N}$) at time t $\in$ [0, T] and assume that

$$S_T^i = S_0^i + D^i \tag{2.2}$$

where $D^i$ is a random variable that represents the informed traders prior belief on the future mid-price distribution of asset i. Here, $D^i$ is only required to have finite second moment, so the IT can use any method to form the prior. By receiving new information continuously, the IT updates her prior and change strategies accordingly in the form of midprice.

In our set-up, the IT uses market information before t = 0 to form her prior belief on the joint distribution of $D^i$, and then decides how to execute a trading strategy in one or more assets between time 0 and t $\leq$ T. We assume that the midprice process is a randomized Brownian bridge (rBb) connecting the current midprice to the future midprice, i.e.

$$S_t^i - S_0^i = \sigma_i \beta_{tT}^i + \frac{t}{T} D^i \tag{2.3}$$

where $\sigma \geq 0$, $\beta_{tT}^i$ are independent standard Brownian bridges, independent of $D^i$, 1 $\leq$ i $\leq$ n, which satisfy

$$\beta_{tT}^i = W_t^i - \frac{t}{T} W_T^i \tag{2.4}$$

for t $\in$ [0, T], $W_t$ are independent standard Brownian motions, and $\beta_{0T}^i$=$\beta_{TT}^i$=0. Here, the Brownian bridge $\sigma_i \beta_{tT}^i$ models fluctuations in the asset's midprice at times t.

The informed trader can only access to the filtration $F_t$ generated by the collection of $S_t^i$, so she cannot represent $S_t^i$ in the form of $D_i$ and $\beta_{tT}^i$ only, except at T where $\beta_{TT}^i = 0$ and $S_T^i = S_0^i + D^i$.

Based on the assumed rBb process for the midprice, there is a proposition for $S_t^i$. The assets midprice process $S_t^i$ given by (2.2) satisfies the SDE

$$dS_t^i = A_i(t, S)dt + \sigma_i dW_t^i \tag{2.5}$$

where $W_t^i$ are pairwise independent $F_t$-Brownian motions.

Moreover,

$$A_i(t, S) = \frac{a_i(t, S) - (S_t^i - S_0^i)}{T - t} \tag{2.6}$$

where

$$\begin{aligned} a_i(t, S) =& E[D^i | S_t = S] \\ =& \frac{\int_{R^n} x_i \prod_{j=1}^n exp(x_j \frac{S_j - S_0^j}{\sigma_j^2 (T-t)} - \frac{1}{2} x_j^2 \frac{t}{T\sigma_j^2 (T-t)}) dF(x)}{\int_{R^n} \prod_{j=1}^n exp(x_j \frac{S_j - S_0^j}{\sigma_j^2 (T-t)} - \frac{1}{2} x_j^2 \frac{t}{T\sigma_j^2 (T-t)}) dF(x)} \end{aligned} \tag{2.7}$$

are the $F_t$-conditional expectations of $D^i$'s, and $F = F_D$ is the joint cumulative distribution function of the random variables $D^i$.

The drift part in the SDE is $A_i(t, S)$ and it shows how IT(informed trader) employs the prior information and how IT learns from the updated information. If the trader doesn't learn and believe the prices are independent arithmetic Brownian motions, the drift part $A_i(t, S) = 0$. [1] We say the trader is uninformed and denote it as UT.

## 2.3 Model Description

As we learnt in pervious section, market orders can guarantee immediate execution but are more expensive because the trader needs to pay for the liquidity taking fee. Limit orders do not have the fee and so are cheaper, but execution is not guaranteed. IT uses innovations in midprices to update her prior and then to adjust her strategy regarding the combination of market and limit orders to trade in and out of positions between now and T.

When we think about the IT's strategy, we need to consider the IT's submitted limit and market orders, wealth process, accumulated inventory, and also, the market orders sent by other participants. [1]

Let $l_t^{\pm} = \{l_t^{1\pm}, ..., l_t^{k\pm}\} \in \{0,1\}^k$ denote her decision to post a sell (+) or a buy (-) limit order for one unit of asset at time t with $l_t^{i\pm} = 0$ meaning that there is no post. $m_t^{\pm} = \{m_t^{1\pm}, ..., m_t^{k\pm}\} \in Z_+^k$ counts the total number of market orders sent by the IT up until time t.

Let $N_t^{\pm} = \{N_t^{1\pm}, ..., N_t^{k\pm}\}$ represent the total number of buy and sell market orders other participants have sent in the assets which the IT trades. The market orders which fill the IT's posted limit orders are denoted as $\bar{N}_t^{\pm} = \{\bar{N}_t^{1\pm}, \bar{N}_t^{k\pm}\}$. It is assumed to be independent Poisson processes with intensities $\lambda^{\pm}$.

The number of the IT's filled limit orders in asset-i up to time t is given by $\int_0^t l_t^{i\pm} d\bar{N}_t^{i\pm}$.

Therefore the IT's inventory in asset-i at time t is given by

$$q_t^i = -\int_0^t l_t^{i+} d\bar{N}_t^{i+} + \int_0^t l_t^{i-} d\bar{N}_t^{i-} - m_t^{i+} + m_t^{i-}, \quad q_0^i = 0 \tag{2.8}$$

We assume that the IT restricts her accumulated inventory position for all $q_t^i$ to be between $\underline{q}^i$ and $\bar{q}^i$.

Let $\epsilon^i$ represent the liquidity taking fees and $\delta^i$ be the spread, the execution prices that IT achieves for trading one unit of the asset using market orders in asset i are $S_t^i - \frac{\Delta^i}{2} - \epsilon^i$ for a sell, and $S_t^i + \frac{\Delta^i}{2} + \epsilon^i$ for a buy.

Since there is no liquidity fee for limit orders, the price that IT trades one unit of asset using limit orders is $S_t^i + \frac{\Delta^i}{2}$ for a sell and $S_t^i - \frac{\Delta^i}{2}$ for a buy.

$$\begin{aligned} dX_t = \sum_{i=1}^{k} \{ &-(S_t^i - \frac{\Delta^i}{2}) l_t^{i-} \mathbf{1}_{q_t^i \leq \bar{q}^i} d\bar{N}_t^{i-} + (S_t^i + \frac{\Delta^i}{2}) l_t^{i+} \mathbf{1}_{q_t^i \geq \bar{q}^i} d\bar{N}_t^{i+} \\ &- (S_t^i + \frac{\Delta^i}{2} + \epsilon^i) l_t^{i-} \mathbf{1}_{q_t^i \leq \bar{q}^i} d\bar{N}_t^{i-} + (S_t^i - \frac{\Delta^i}{2} - \epsilon^i) l_t^{i+} \mathbf{1}_{q_t^i \geq \bar{q}^i} d\bar{N}_t^{i+} \} \end{aligned} \tag{2.9}$$

To find the optimal strategy $v$ that maximizes the expected wealth, we need to consider the terminal wealth, the costs that the IT incurs at the terminal date $\bar{T}$ when liquidating $q_{\bar{T}}$, which is captured by parameter $\alpha^i$, and also the running penalty for the inventory risk.

The representation for the value function H admits the representation [1],

$$H(t, X, S, q) = X + \sum_{i=1}^{k} q^i S^i + g(t, S, q) \tag{2.10}$$

where g satisfies the QVI

$$
\begin{aligned}
0 = \max\{\partial_t g + \sum_{i=1}^{k}\{\frac{1}{2}\sigma_i^2\partial_{S_iS_i}g + A_i(t,S)(q^i + \partial_{S_i}g) - \phi^i(q^i)^2\} \\
+ \sum_{i=k+1}^{n}\{\frac{1}{2}\sigma_i^2\partial_{S_iS_i}g + A_i(t,S)\partial_{S^i}g\} \\
+ \sum_{i=1}^{k}\mathbf{1}_{q_t^i\le\bar{q}^i}\lambda^{i-}max_{l\in(0,1)}[\frac{\Delta^i}{2}l + g(t,S,q+\delta^i l) - g(t,S,q)] \\
+ \sum_{i=1}^{k}\mathbf{1}_{q_t^i\ge\bar{q}^i}\lambda^{i+}max_{l\in(0,1)}[\frac{\Delta^i}{2}l + g(t,S,q-\delta^i l) - g(t,S,q)]; \\
\max_{\varepsilon\in D(q)}\{-\sum_{i=1}^{k}(\frac{\Delta^i}{2}+\epsilon^i)|\varepsilon^i| + g(t,S,q+\varepsilon) - g(t,S,q)\}\},
\end{aligned}
\tag{2.11}
$$

$\delta^i$ is a k-vector with $\delta^{ij} = 0$ for $j \neq i$ and $\delta^{ii} = 1$, the set

$$D(q) = \otimes_{i=1}^{k}\{-\mathbf{1}_{q^i>\underline{q}^i}, 0, \mathbf{1}_{q^i<\bar{q}^i}\},$$

and the QVI is subject to the terminal condition

$$g(\bar{T},S,q) = -\sum_{i=1}^{k}((\frac{\Delta^i}{2}+\epsilon^i)|q^i| + \alpha^i(q^i)^2), \underline{q}^i \le q^i \le \bar{q}^i \tag{2.12}$$

In the QVI expression, the first line represents the flow of asset midprices and the updates of the priors in the assets in which the IT trades, the second line represents the flow of asset midprices and the updates of the priors in the other assets. the third and fourth lines represent the changes in the value function due to execution of the agent's posted limit orders. The last line represents the execution of market orders, and $D(q)$ is the set of allowed market order executions which respect the inventory limits imposed by the IT. [1]

# 3. Simulations - Learn from and trade in one asset

In this section, I will implement the model described above for the simplest case, where the traders learn from and trade in only one asset. I will compare the IT and UT by looking at their midprice process, stratgies used over time and their performances. The performances are evaluated by the value function introduced before. In order to simulate it, I use finite difference method to get the values of g.

## 3.1 Finite Difference Method

To solve the QVI numerically and obtain g values, we use finite difference methods to approximate the derivatives in the QVI equation. Finite difference methods are very frequently used for solving differential equations. The derivatives at a point are approximated by difference quotients over a small interval. Let us consider a function F, whose derivatives are single-values, finite and continuous functions of x, we can apply Taylor's theorem on F(x+h) and F(x-h), where h is a constant. [4]

$$F(x+h) = F(x) + hF'(x) + \frac{1}{2}h^2F''(x) + \frac{1}{6}h^3F'''(x) + ... \tag{3.1}$$

and

$$F(x-h) = F(x) - hF'(x) + \frac{1}{2}h^2F''(x) - \frac{1}{6}h^3F'''(x) + ... \tag{3.2}$$

Add (3.1) and (3.2), we can get

$$F(x+h)+F(x-h)=2F(x)+h^2F''(x)+\Theta(h^4) \tag{3.3}$$

where $\Theta$ denotes the terms containing fourth and higher powers of h. We assume that these are negligible comparing to the lower power of h. We then get

$$F''(x)\simeq\frac{1}{h^2}\{F(x+h)-2F(x)+F(x-h)\} \tag{3.4}$$

Subtract (3.2) from (3.1) and neglect the terms of order $h^3$, we get

$$F'(x)\simeq\frac{1}{2h}\{F(x+h)-F(x-h)\} \tag{3.5}$$

Here, (3.5) is called a central-difference approximation. We can also use the forward-difference formula,

$$F'(x)\simeq\frac{1}{h}\{F(x+h)-F(x)\} \tag{3.6}$$

or the backward-difference formula,

$$F'(x)\simeq\frac{1}{h}\{F(x)-F(x-h)\} \tag{3.7}$$

In our case, g(t,S,q) is the F describe above. We set the time interval of t as $\delta$, and the time interval of S as $\Delta$. Then, we can represent the following derivatives in the form of the finite difference.

$$\partial_t g(t_i,S_j,q_k)=\frac{g(t_i,S_j,q_k)-g(t_{i-1},S_j,q_k)}{\delta} \tag{3.8}$$

$$\partial_S g(t_i,S_j,q_k)=\frac{g(t_i,S_{j+1},q_k)-g(t_i,S_j,q_k)}{\delta} \tag{3.9}$$

$$\partial_{SS} g(t_i,S_j,q_k)=\frac{g(t_i,S_{j+1},q_k)-2g(t_i,S_j,q_k)+g(t_i,S_{j-1},q_k)}{\delta^2} \tag{3.10}$$

### 3.2 Midprice process

For both IT and UT traders, we assume that the midprice process follows the market dynamics we discussed in Section 2.1. Hence, the only difference between IT and UT relies on D, which is the prior distribution. We assume that for IT, D can take only two values: it can be $\delta_u$ with probability $p_u$, or $\delta_d$ with probability $p_d$, where $0 \leq S_0 + \delta_d \leq S_0 + \delta_u$ and $p_u + p_d = 1$. Hence, the stock dynamics drift is given as

$$a(t, S_t) = \delta_u \pi_u(t, S_t) + \delta_d \pi_d(t, S_t) \tag{3.11}$$

$\pi_u(t, S_t)$ and $\pi_d(t, S_t)$ are the posteriori probabilities of $S_T$ being equal to $S_0 + \delta_u$ and $S_0 + \delta_d$, conditional on the asset midprice at time t:

$$\pi_k(t, y) = P[D = \delta_k | S_t = y] = \frac{p_k \exp(\delta \frac{S_t - S_0}{\sigma^2(T-t)} - \frac{1}{2}\delta_k^2 \frac{t}{T\sigma_1^2(T-t)})}{\sum_{i=u,d} p_i \exp(\delta_i exp(\delta_i \frac{S_t - S_0}{\sigma_1^2(T-t)} - \frac{1}{2}\delta_i^2 \frac{t}{T\sigma_1^2(T-t)})} \tag{3.12}$$

For IT, we set that $\delta_u = 0.02$ and $\delta_d = -0.02$, $p_u = 0.8$ and $p_d = 0.2$. Assume that the trader will increase to $\S_0 + \delta_u$ at time T and also $\sigma = 0.02$. The $\lambda_\pm$ which is the intensities for the market orders from other participants that fill the traders posted limit orders are set to be 30 for both sell and buy. For the other parameters: T $= 1$, $\Delta = 10^{-3}$, $\epsilon = \Delta/20$, $\alpha = 5 * 20^{-3}$, $\underline{q} = -20$, $\bar{q} = 20$, and $\phi = 0$.

We generate 10 midprice process for IT and obtain the plot in Figure 3.1. We can see that they starts from $S_0$ =1 and all end at 1.02 at time T.

For UT, we simply set D $\sim N(0, \sigma\sqrt{T})$. Also, we generate 10 midprice processes and get the plot in Figure 3.2. The midprice processes start from 1 and end in different values at T, which is consistent with our assumption that UTs don't have the prior information about the value of the asset at time T, and just assume it follows a normal distribution.

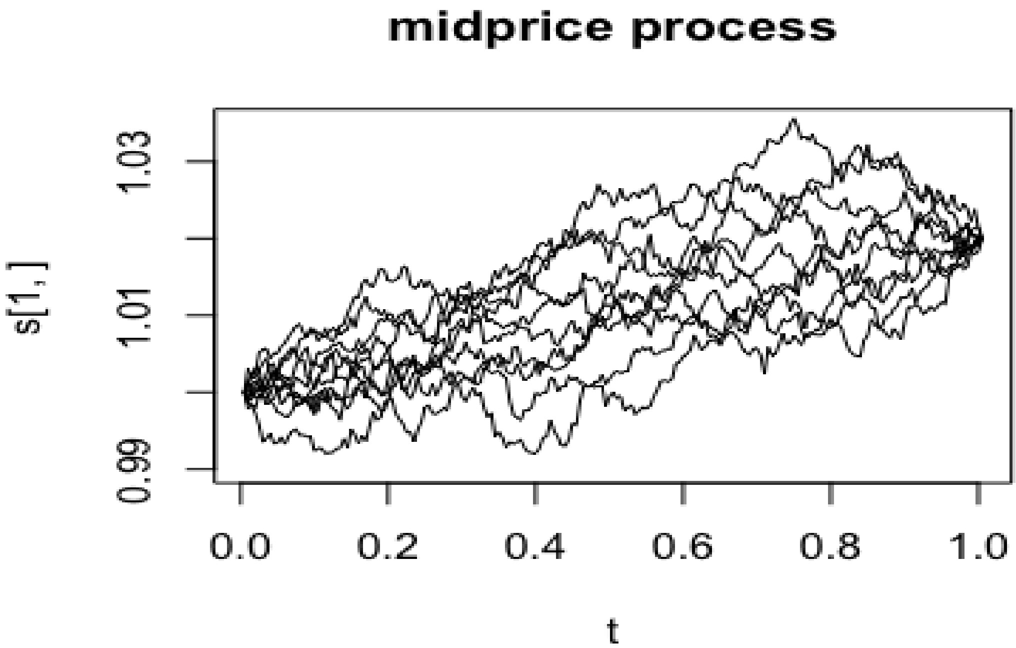



Figure 3.1. Plot of midprice process of IT

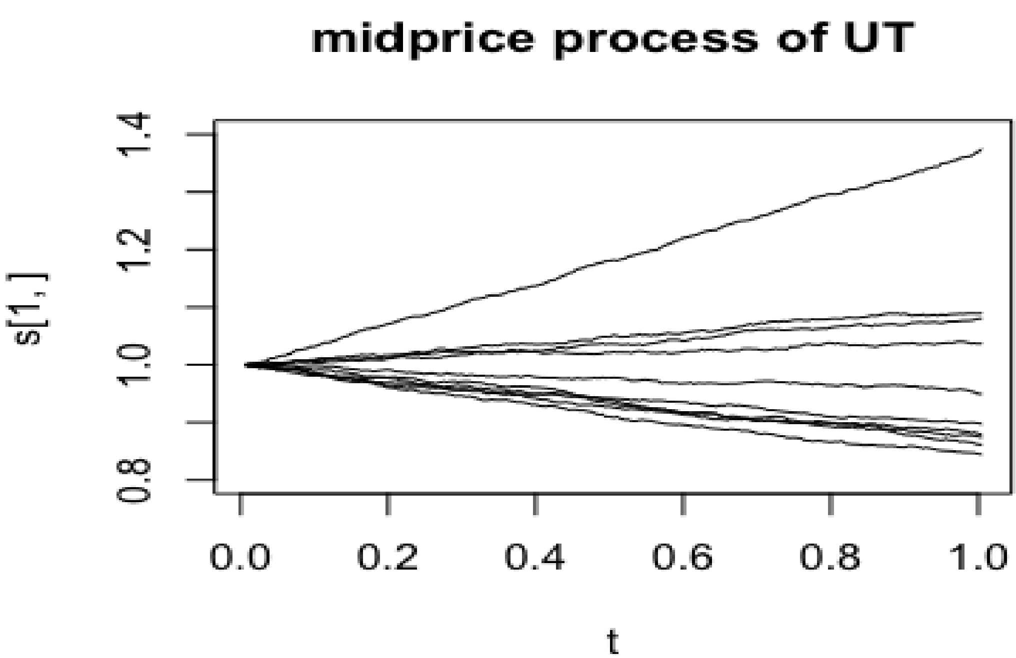
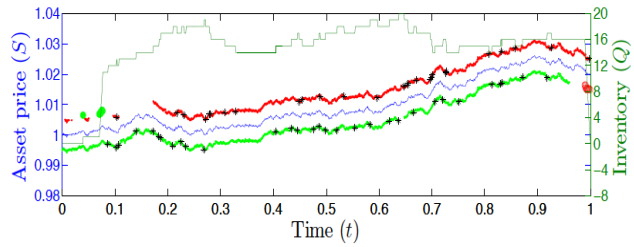
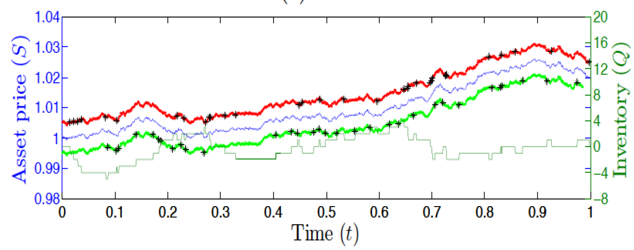



Figure 3.2. Plot of midprice process of UT




(a) IT




(b) UT

Figure 3.3. Strategies used by IT and UT

### 3.3 Strategies Used

Now, the UT and IT are assumed to follow the same midprice process as we set for IT in the previous section. Figure 3.3 shows the strategies used by UT and IT along the time. [1]

The balck line is the midprice process. The stars represent the arrival of other participants' market orders, the solid circles denote the traders' buy market orders, and the empty circles denote the traders' sell market orders. The green and red lines around the midprice path show the times when the traders post buy and sell limit orders. The green line is $\frac{1}{2}\Delta$ above the midprice path because the best ask price for the buy limit orders is $S_t + \frac{1}{2}\Delta$, and the red line is $\frac{1}{2}\Delta$ below the midprice path because the best bid price for the sell limit orders is $S_t - \frac{1}{2}\Delta$.

The green and red lines for IT are disconnected somewhere, while for UT, they are continuous all the time. This shows that the IT changes her strategies over time by submitting market orders instead of limit orders at some points, but the UT doesn't change strategies. The dark green line represents the inventory amount. In IT's plot, the inventory amount goes up at the begining and then goes back to zero as time proceeds towards T. It is consistent with our assumption about the price that will increase to 1.02 at T, because the informed trader knows the information and so will build up her inventory at low price now and then sell them at higher price in the future to make profits.

### 3.4 Value Funtion

#### 3.4.1 Uninformed Trader

For an uninformed trader, she never learns from the dynamics of the asset prices and so A(t,S)=0. Hence, the QVI can be reduced to a much simplier version as below:

$$\begin{aligned} 0 = max\{&\partial_t g + \frac{1}{2}\sigma^2 \partial_{SS} g \\ &+ \mathbf{1}_{q_t \le \bar{q}} \lambda^- [\frac{\Delta}{2} l + g(t,S,q+\delta) - g(t,S,q)]_+ \\ &+ \mathbf{1}_{q_t \ge \bar{q}} \lambda^+ [\frac{\Delta}{2} l + g(t,S,q-\delta) - g(t,S,q)]_+; \\ &\max_{\varepsilon \in \{-\mathbf{1}_{q > \underline{q}}, \mathbf{1}_{q < \bar{q}}\}} \{-(\frac{\Delta}{2} + \epsilon)|\varepsilon| + g(t,S,q+\varepsilon) - g(t,S,q)\}\}, \end{aligned} \tag{3.13}$$

subject to the terminal condition

$$g(\bar{T}, S, q) = -((\frac{\Delta}{2} + \epsilon)|q| + \alpha q^2) \tag{3.14}$$

Our goal is to find the value function of UT:

$$H(t, X, S, q) = X + qS + g(t, S, q) \tag{3.15}$$

The term $X$ and $qS$ are easy to get and don't change with time t; the tricky part is the g(t,S,q). Since we can get the values of g(t,S,q) at time T by using the terminal condition, to get the g(t,S,q) at other times, we want to represent the $g(t_{i-1}, S_j, q_k)$ in terms of $g(t_i, S_j, q_k)$.

The finite-difference method is used to solve g(t,S,q). I set $t_i = i\delta', i = 0, ..., I$, and $S_j = -\underline{S} + j\Delta', j = 0, ..., J$, $q_k \in \{\underline{q}, ..., \bar{q}\} = \underline{q} + k, k = 0, ..., \bar{q} - \underline{q}$. Note here the $\delta'$ and $\Delta'$ represent the intervals, which are different from the $\delta$ and $\Delta$ stated before. Since the inventory number is an integer, q may increase by 1 each time.

For simplicity, the first three lines of the equation(3.13) are named as part 1 and the last line is named as part 2. I discretize the differential operator defining g first and then obtain $g(t_{i-1}, S_j, q_k)$ in terms of $g(t_i, S_j, q_k)$. Since the max of part 1 and part 2 should be 0, I first set part 1 as 0, and generate the $g(t_{i-1}, S_j, q_k)$, then, I plug it into part 2 to test if part 2 is less than 0. If the part 2 is larger than 0, I set part 2 as 0. The basic idea of the simulation is as below in steps:

Step 1: Let the first part of QVI equation be 0

$$\begin{aligned}
&\frac{g(t_i, S_j, q_k) - g(t_{i-1}, S_j, q_k)}{\delta'} + \frac{1}{2}\sigma^2 \\
&\frac{g(t_i, S_{j+1}, q_k) - 2g(t_i, S_j, q_k) + g(t_i, S_{j-1}, q_k)}{\Delta'^2} \\
&+ \mathbf{1}_{q_k < \bar{q}}\lambda^-[\frac{\Delta}{2} + g(t_i, S_j, q_k + 1) - g(t_i, S_j, q_k)]_+ \\
&+ \mathbf{1}_{q_k > \underline{q}}\lambda^+[\frac{\Delta}{2} + g(t_i, S_j, q_k - 1) - g(t_i, S_j, q_k)]_+ = 0 \\
&\forall j = 0, ..., J - 1, k = 0, ..., \bar{q} - \underline{q}.
\end{aligned} \tag{3.16}$$

Then we can have $g(t_{i-1}, S_j, q_k)$ in terms of $g(t_i, S_j, q_k)$:

$$
\begin{aligned}
g(t_{i-1}, S_j, q_k) = & g(t_i, S_j, q_k) - \delta' * (-\mathbf{1}_{q_k > \underline{q}} \lambda^+ [\frac{\Delta}{2} + g(t_i, S_j, q_k - 1) - g(t_i, S_j, q_k)]_+ \\
& - \mathbf{1}_{q_k < \bar{q}} \lambda^- [\frac{\Delta}{2} + g(t_i, S_j, q_k + 1) \\
& - g(t_i, S_j, q_k)]_+ - \frac{1}{2}\sigma^2 \frac{g(t_i, S_{j+1}, q_k) - 2g(t_i, S_j, q_k) + g(t_i, S_{j-1}, q_k)}{\Delta'^2})
\end{aligned} \tag{3.17}
$$

Step 2: Check part 2

If part 2 of QVI is larger than 0, we set part 2 as 0 and get the $g(t_{i-1}, S_j, q_k + 1)$ and $g(t_{i-1}, S_j, q_k - 1)$:

If

$$
-(\frac{\Delta}{2} + \epsilon) + g(t_{i-1}, S_j, q_k + 1) - g(t_{t-1}, S_j, q_k) > 0, \tag{3.18}
$$

then

$$
g(t_{i-1}, S_j, q_k + 1) = g(t_{t-1}, S_j, q_k) + (\frac{\Delta}{2} + \epsilon). \tag{3.19}
$$

If

$$
-(\frac{\Delta}{2} + \epsilon) + g(t_{i-1}, S_j, q_k - 1) - g(t_{t-1}, S_j, q_k) > 0, \tag{3.20}
$$

then

$$
g(t_{i-1}, S_j, q_k - 1) = g(t_{t-1}, S_j, q_k) + (\frac{\Delta}{2} + \epsilon). \tag{3.21}
$$

Figure 3.4 shows the plots of the g values from time 0 to 1 when S=5, and q=-20,-15,-10,-5,0,5,10,15,20. The red line is the g when q = 0, which is the most flat one. In fact, the plots are similar for any choice of S from $\underline{S}$ and $\bar{S}$.

We can see that when the inventory q is far from 0 and near both ends of the interval of q, the g value decreases more rapidly.

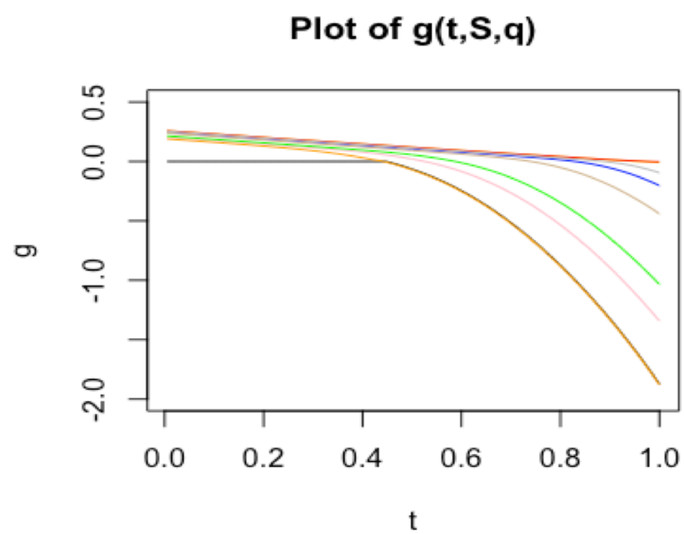



Figure 3.4. Plot of g for UT

### 3.4.2 Informed Trader

The informed trader can learn from the market dynamics, and so A(t,S) follows the (2.5) and (2.6). Hence, the QVI satisfies:

$$
\begin{aligned}
0 = & max\{\partial_t g + \frac{1}{2}\sigma^2\partial_{SS}g \\
& + A_i(t,S)(q + \partial_{S_i}g) \\
& + \mathbf{1}_{q_t \leq \bar{q}}\lambda^-[\frac{\Delta}{2}l + g(t,S,q+\delta) - g(t,S,q)]_+ \\
& + \mathbf{1}_{q_t \geq \bar{q}}\lambda^+[\frac{\Delta}{2}l + g(t,S,q-\delta) - g(t,S,q)]_+; \\
& max_{\varepsilon \in \{-\mathbf{1}_{q>\underline{q}}, \mathbf{1}_{q<\bar{q}}\}}\{-(\frac{\Delta}{2} + \epsilon)|\varepsilon| + g(t,S,q+\varepsilon) - g(t,S,q)\}\},
\end{aligned}
\tag{3.22}
$$

subject to the terminal condition

$$
g(\bar{T}, S, q) = -((\frac{\Delta}{2} + \epsilon)|q| + \alpha q^2) \tag{3.23}
$$

Same as the steps we used for UT, we generate g(t,S,q) first.

Step 1:

$$
\begin{aligned}
& \frac{g(t_i,S_j,q_k) - g(t_{i-1},S_j,q_k)}{\delta'} + \frac{1}{2}\sigma^2 \\
& \frac{g(t_i,S_{j+1},q_k) - 2g(t_i,S_j,q_k) + g(t_i,S_{j-1},q_k)}{\Delta'^2} \\
& + A_i(t,S)(q_k + \frac{g(t_i,S_j,q_k) - g(t_i,S_{j-1},q_k)}{\Delta'}) \\
& + \mathbf{1}_{q_k<\bar{q}}\lambda^-[\frac{\Delta}{2} + g(t_i,S_j,q_k+1) - g(t_i,S_j,q_k)]_+ \\
& + \mathbf{1}_{q_k>\underline{q}}\lambda^+[\frac{\Delta}{2} + g(t_i,S_j,q_k-1) - g(t_i,S_j,q_k)]_+ = 0 \\
& \forall j = 0,...,J-1, k = 0,...,\bar{q} - \underline{q}.
\end{aligned}
\tag{3.24}
$$

Then we can have $g(t_{i-1},S_j,q_k)$ in terms of $g(t_i,S_j,q_k)$:

$$
\begin{aligned}
g(t_{i-1}, S_j, q_k) =& g(t_i, S_j, q_k) - \delta' * (-\mathbf{1}_{q_k > \underline{q}} \lambda^+ [\frac{\Delta}{2} + g(t_i, S_j, q_k - 1) - g(t_i, S_j, q_k)]_+ \\
& - \mathbf{1}_{q_k < \bar{q}} \lambda^- [\frac{\Delta}{2} + g(t_i, S_j, q_k + 1) - g(t_i, S_j, q_k)]_+ \\
& - \frac{1}{2}\sigma^2 \frac{g(t_i, S_{j+1}, q_k) - 2g(t_i, S_j, q_k) + g(t_i, S_{j-1}, q_k)}{\Delta'^2} \\
& - A_i(t, S)(q_k + \frac{g(t_i, S_j, q_k) - g(t_i, S_{j-1}, q_k)}{\Delta'}))
\end{aligned}
\tag{3.25}
$$

Step 2: Check part 2

If

$$-(\frac{\Delta}{2} + \epsilon) + g(t_{i-1}, S_j, q_k + 1) - g(t_{t-1}, S_j, q_k) > 0,$$

then

$$g(t_{i-1}, S_j, q_k + 1) = g(t_{t-1}, S_j, q_k) + (\frac{\Delta}{2} + \epsilon).$$

If

$$-(\frac{\Delta}{2} + \epsilon) + g(t_{i-1}, S_j, q_k - 1) - g(t_{t-1}, S_j, q_k) > 0,$$

then

$$g(t_{i-1}, S_j, q_k - 1) = g(t_{t-1}, S_j, q_k) + (\frac{\Delta}{2} + \epsilon).$$

Figure 3.5 shows the plot of the values of g from time 0 to 1 for different q. As is the case with UT, the red line is the g when q = 0, which is the most flat one. We can find that the patterns are very similar to those for UT.

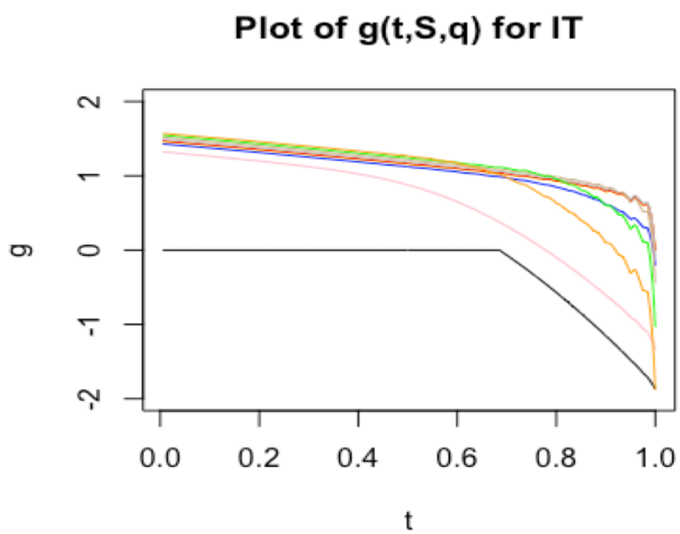



Figure 3.5. Plot of g for IT

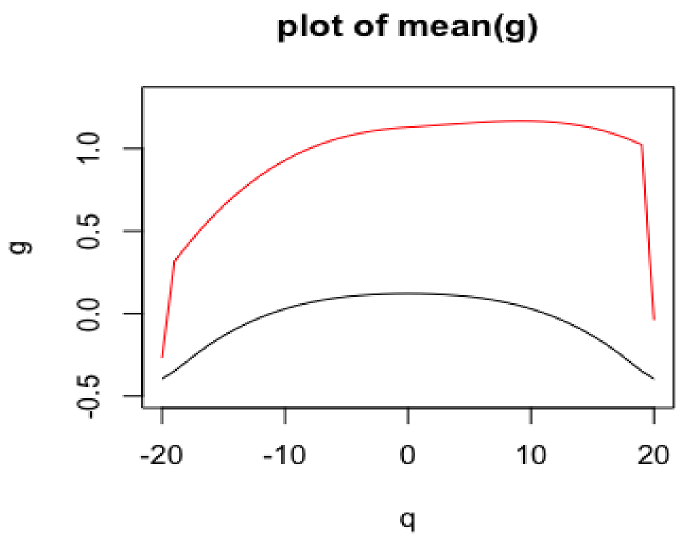



Figure 3.6. Plot of means

### 3.4.3 Comparison

Since the value function H can represent the performance of different strategies, to compare the performance of UT and IT, I compare their H(t,X,S,q) with the same X and q. For the traders who only learn from and trade one asset, the value fuction is:

$$H(t, X, S, q) = X + qS + g(t, S, q) \tag{3.26}$$

Since the X and qS are the same for IT and UT, we can compare g values instead of H to access their performance. I plot out the means of the g values for each inventory q, and then get the graph in Figure 3.6.

The red line represents the IT's means and black line represents the UT's means. We can find that the wealth of IT is overall larger than UT for every q. It proves our assumption for the model that the trader who has prior information and can learn from the market dynamics should perform better than the trader who has no prior informtion and no ability to learn.

# 4. Conclusion

In this paper, I first introduced some basic knowledge about the electronic market, e.g. the trader types, the limit and market orders, the limit order book and different defined prices. An algorithmic trader can use the model proposed by Alvaro C, Sebastian J and Damir K to decide what mix of market and limit orders will generate higher profits. The uninformed trader who has no prior information about the end price never change her stratgies along the time. However, the informed trader changes strategies over time because she has the prior distribution of $D$, incorporates it into the price process and also learns from the price changes.

The value function is used to access the performances of the informed trader and uninformed trader. Finite difference method is used to simulate the value function. The simple case when the trader learns from and takes positions in only one asset is considered. It is shown that the IT has higher wealth than UT and we can conclude that the IT can perform better than the UT.

The prior information $D$ that the informed trader knows in this paper is assumed to be a correct prior. However, if the prior is not accurate, it may hurt the profitability of the informed trader. As we can see, a good prior is essential for informed traders. In our simulation, the $D$ is simply set. The topic about how to find a proper and accurate prior can be studied further in the future.

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