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Essays on Liquidity, Informational Frictions, and Monetary Policy

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by

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Dedicated to my parents and wife

ABSTRACT OF THE DISSERTATION

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The dissertation, which consists of two chapters, is devoted to exploring the role of informational friction in monetary economics and finance.

Chapter Ⅰ: COUNTERFEITING, SCREENING AND GOVERNMENT POLICY.

In this chapter, I construct a search theoretic model of money in which counterfeit money can be produced at a cost but agents can screen for fake money also at a cost. Counterfeiting can occur in equilibrium when both costs and the inflation rate are sufficiently low. Optimal monetary policy is the Friedman rule. However, the rationale for the Friedman rule in an economy with the circulation of counterfeit money differs from the conventional mechanism that holds in the model when counterfeiting does not occur. I also study optimal anti-counterfeiting policy that determines the counterfeiting cost and the screening cost.

Chapter Ⅱ: CENTRAL BANK PURCHASES OF PRIVATE ASSETS: AN EVALUATION

In this chapter, I develop a model of asset exchange and monetary policy, augmented to incorporate a housing market and a frictional financial market. Homeowners take out mortgages with banks using their residential properties as collateral to finance consumption. Banks use mortgages and government liabilities as collateral to secure deposit contracts, but they have an incentive to fake the quality of mortgages at a cost. Quantitative easing (QE) in the form of central bank purchases of mortgages from private banks has effects on the composition of assets in the economy, and on the incentive structure of the private sector. When the incentive problem is severe, the central bank can unambiguously improve welfare by purchasing mortgages. However, when it is not severe, the central bank's mortgage purchases cause a housing construction boom and sometimes can lower exchange in the economy, hence reducing welfare.

Chapter 1

Counterfeiting, Screening and Government Policy

1.1 Introduction

Counterfeiting of money is centuries old problem and has been one of major issues that monetary authority must deal with. In spite of its importance, there is relatively small literature that studies counterfeiting in a general equilibrium model. Furthermore, quite often in previous studies, counterfeiting works as a threat and does exist in equilibrium. Thus, the following questions still need to be addressed: Under what conditions does counterfeit money exist as an equilibrium outcome? How does counterfeiting, or its potential threat, distort economic agents' behavior and the allocation of resources? Is it optimal to eradicate counterfeiting? How does monetary policy affect counterfeiting activity?

We address these questions by developing a monetary search model, building on Lagos and Wright (2005) framework, in which counterfeiting can occur as an equilibrium outcome. In the model economy, an asset is necessary for an exchange to take place, and the only asset is fiat money supplied by the government. To study counterfeiting, we first introduce costly counterfeiting into the model: Agents can produce fake money at a cost that is indistinguishable from genuine money. The previous work on counterfeiting, such as Nosal and Wallace (2007) and Li, Rocheteau, and Weill (2012) among others, has typically focused on the counterfeiting incentive with this costly counterfeiting technology. In this paper, we take a step forward and incorporate costly counterfeit detection through screening as an additional ingredient of a theory of counterfeiting. By considering these conflicting incentives together, we provide new insights that counterfeiting can occur in equilibrium in contrast to Li, Rocheteau, and Weill (2012) despite using similar equilibrium concepts and economic environments.[1]

More precisely, the model shows that counterfeiting occurs if and only if there is screening activity. The intuition for this result is as follows. If an agent produces counterfeits for exchanges and money is not screened, the agent can always find a profitable deviation that prevents counterfeiting from taking place by reducing the quantity of the money transfer.[2] However, preventing counterfeiting by restricting the money transfer is costly because it limits the volume of exchanges. Screening can relax this constraint on the money transfer and the trade volume increases, but for an agent to find it optimal to screen, there must be some counterfeits in the economy.

Equilibrium can be one of three types: *no threat of counterfeiting equilibrium, threat of counterfeiting equilibrium,* and *counterfeiting equilibrium.* First, in the *no threat of counterfeiting equilibrium,* the counterfeiting cost is so high that there is no incentive to produce fake money. Thus, economic activities are the same as in an economy where counterfeiting is not a possibility. Second, in the *threat of counterfeiting equilibrium*, the counterfeiting cost is not too high but the screening cost is relatively high, so no agents screen money in a trade to check its authenticity. Therefore, counterfeits do not exist in this equilibrium. However, the counterfeiting cost matters for real allocations because it restricts the volume of exchange, as in Li and Rocheteau (2011), Li, Rocheteau, and Weill (2012), and Shao (2014). Finally, when both the counterfeiting cost and the screening cost are sufficiently low, the economy is in the *counterfeiting equilibrium* where both counterfeiting and screening occur, so genuine and counterfeit monies coexist.

In the last two equilibria, counterfeiting, or the threat of counterfeiting, generates a distortion in the allocation. First, the quantity traded is inefficiently small in the *threat of*

[1]In their extension, Li, Rocheteau, and Weill (2012) make contracts incomplete and derive counterfeiting in equilibrium. In particular, they assume that the shock on the counterfeiting cost is realized after the offers have been made, so the offers cannot be contingent on specific types of agents.

[2]This is why it was difficult to generate counterfeiting in equilibrium in previous models that studied counterfeiting without a screening decision.

counterfeiting equilibrium because of the restriction on the money transfer. Second, in the *counterfeiting equilibrium*, the counterfeiting probability decreases with the quantity traded as a result of the strategic behavior of agents, which implies that the marginal money transfer for an additional unit of goods traded decreases as the trade volume increases. Because of this pecuniary effect of increasing the trade volume, the quantity traded is larger in the *counterfeiting equilibrium* than the one in the economy without counterfeiting possibility.

One of key messages of our analysis is that the inflation rate plays a critical role in which equilibria exist, unless the counterfeiting cost is too high, and it is more likely that counterfeits circulate in the economy with low inflation. In our model, the quantity of goods that agents want to trade in a decentralized market decreases with inflation, and the incentive to make counterfeit money increases as the trade volume rises. Thus, when inflation is high enough, the quantity traded is so small that agents have no incentive to produce bogus money (i.e., *no threat of counterfeiting equilibrium*). As inflation falls, agents want to trade more goods, which raises the incentive to make counterfeits. However, if the desired trade volume is not sufficiently large, then agents restrict the trade volume such that counterfeiting is not optimal, and fake money does not exist in equilibrium (i.e., *threat of counterfeiting equilibrium*). However, when inflation is sufficiently low, it is too costly to prevent counterfeiting by restricting the trade volume because agents want to trade relatively large amount of goods, so counterfeits circulate in the economy with screening activity to achieve sufficient amount of trade (i.e., *counterfeiting equilibrium*). Surprisingly, lowering the inflation rate in the *counterfeiting equilibrium* reduces counterfeiting activities because of the strategic behavior of agents.

Finally, we extend the model to study optimal government policies: monetary and anti-counterfeiting policies. Monetary policy determines the inflation rate through changing the money supply. Anti-counterfeiting policy determines the counterfeiting environment, i.e., the counterfeiting cost and the screening cost, by investing resources in counterfeit deterrence measures, for example, embedding new security features into banknotes.

One result of the welfare analysis is that welfare decreases with inflation, regardless of counterfeit deterrence policy, while monetary policy has different effects on the economy, depending on the counterfeiting environment. This directly implies that optimal monetary policy is the Friedman rule, i.e., contracting the money supply at a rate equal to the agent's rate of time preference. In a standard money search model, the Friedman rule is optimal because it corrects the monetary distortion and thus supports an efficient amount of trade in the economy. This mechanism still works in our model as long as counterfeiting does not occur in equilibrium. A novelty here is that the Friedman rule does not maximize the trade surplus as in a standard Lagos and Wright (2005) model if counterfeits circulate in the economy. Indeed, when the inflation rate is close to the rate of time preference in the *counterfeiting equilibrium*, the trade volume is inefficiently high and lowering the inflation rate reduces the trade surplus. The rationale behind the Friedman rule here, is that it minimizes counterfeiting activity and therefore also minimizes its welfare costs.

In addition to monotonicity, welfare increases discontinuously when the economy switches from the *threat of counterfeiting equilibrium* to the *counterfeiting equilibrium*. This discontinuity implies that if equilibria with and without counterfeiting are possible under the same economic conditions, then welfare is higher in the economy with circulation of counterfeits than without. This is because when both equilibria are possible, the quantity traded without counterfeiting is too small.

For anti-counterfeiting policy, the model suggests that the government should focus on improving only one dimension: either increasing the counterfeiting cost or reducing the screening cost. This is because welfare depends on the counterfeiting cost in the *threat of counterfeiting equilibrium*, while it depends on the screening cost only in the *counterfeiting equilibrium*. Thus, any government efforts to improve one of counterfeiting environments–the counterfeiting cost and screening cost–that does not matter for welfare are just waste of resources. Further, because counterfeiting is more likely to occur when the counterfeiting cost and the screening cost are both low, this result implies that it could be optimal for the government to tolerate some amount of counterfeit money in the economy especially when the government can reduce the screening cost more effectively than increasing the counterfeiting cost.

We are certainly not the first to study counterfeiting in the context of the New Monetarist framework that specifies monetary arrangements explicitly.[3] Kultti (1996) and Green and Weber (1996) are earlier works that studied counterfeiting in the context of a monetary search model with indivisible money. Williamson (2002) and Nosal and Wallace (2007) extended previous papers and showed that counterfeiting is only a threat that does not occur in equilibrium, but such a threat could potentially lead to the collapse of a monetary equilibrium. Li and Rocheteau (2011) modify Nosal and Wallace (2007) and show that a monetary equilibrium always exists, but the threat of counterfeiting affects the real allocation.[4] Shao (2014) extends the previous literature by using divisible money, and shows that the threat of counterfeiting generates an endogenous resalability constraint under competitive price posting. Li, Rocheteau, and Weill (2012) introduce the threat of counterfeiting into a search theoretic model of asset market to study its implication on liquidity and asset prices. In contrast to other papers, Monnet (2005) and Cavalcanti and Nosal (2011) adopt a mechanism design approach to study the effects of a counterfeiting environment on allocations.

This paper contributes to the literature in three respects. First, our model goes beyond earlier models by endogenizing verification efforts to detect counterfeits with a costly screening process instead of assuming fixed signals of the authenticity of money. Fung and Shao (2016) take a costly verification technology that is similar to ours in spirit into account in their model to endogenize detection efforts.[5] However, they assume that sellers can either

[3]New Monetarist approach is surveyed in Lagos, Rocheteau, and Wright (forthcoming), Nosal and Rocheteau (2011), and Williamson and Wright (2011).

[4]Nosal and Wallace (2007) assume that the seller's belief that he faces a genuine money holder for any offer is a probability equal to the fraction of genuine money holders among all buyers. This belief system is more restrictive than is required by the Intuitive Criterion, which leads to miss some monetary equilibria, and could give inconsistent results with the assumption that genuine money holders and counterfeiters make the same offer in uninformed matches (see Li and Rocheteau, 2011, for more information).

[5]Quercioli and Smith (2015) also use a costly verification effort to study counterfeiting in a game-theoretic model.

invest in the verification technology or not, excluding stochastic investment decisions that contributed to the non-existence problem of monetary equilibrium when the inflation rate and the verification cost are sufficiently high. By allowing stochastic screening, we show that monetary equilibrium always exists.

Second, in our model, both types of equilibria *with and without* counterfeiting exist making it possible to study how economic factors, such as monetary and anti-counterfeiting policies, affect a counterfeiting state in the economy. In contrast to our result, counterfeiting does not occur in equilibrium in Williamson (2002), Nosal and Wallace (2007), Li and Rocheteau (2011), Li, Rocheteau, and Weill (2012), and Shao (2014).[6] On the other hand, some papers, such as Kultti (1996), Green and Weber (1996), and Fung and Shao (2016), focused only on equilibrium with counterfeits.[7] Therefore, these previous works cannot rationalize some of stylized facts about counterfeiting; for example, why counterfeiting is a problem in some countries but fraud is not present (or negligible at least) in other countries?

Third, we explore the effects of monetary and anti-counterfeiting policies comprehensively to characterize optimal government policy. Cavalcanti and Nosal (2011) study optimal allocation by using a mechanism design approach, and Lengwiler (1997) analyses the optimal security level of banknotes using a game theoretic model involving the central bank and a counterfeiter. However, those researchers do not study how monetary and anti-counterfeiting policies interact with each other to find optimal policy.

Our paper is also related to the literature that studies a private information problem with a costly information acquisition in a monetary search framework. Kim (1996) and Berentsen

[6]Li, Rocheteau, and Weill (2012) and Shao (2014) extend their models to make contracts incomplete, i.e., non-contingent on buyers' type, and derive monetary equilibrium with counterfeiting. In the extension, Li and Rocheteau (2011) show that their model can have monetary equilibrium with counterfeiting by relaxing the assumption of full confiscation of counterfeits or by allowing sellers to set terms of trade in some matches. Our study complements those papers in the sense that we obtain counterfeiting equilibrium by introducing costly screening to detect counterfeits and show conditions under which counterfeits exist or not.

[7]Kultti (1996) assumes exogenous stock of counterfeits in the economy and studies the conditions under which a monetary equilibrium can be sustained. Green and Weber (1996) deal more directly with counterfeiting using costly counterfeiting technology, but in their model, money holding is restricted to either zero or one and trades are deterministic, so genuine money holders cannot give signal to deviate from a candidate pooling equilibrium. Fung and Shao (2016) study counterfeiting in a pooling equilibrium, but they do not impose any restriction on sellers' out-of-equilibrium beliefs, which leads to indeterminacy of equilibrium.

and Rocheteau (2004) considered a costly inspection technology that improves the ability to recognize the quality of goods supplied in the market, and studied the role of money that is universally recognized under private information concerning the quality of goods. Lester, Postlewaite, and Wright (2012) endogenized recognizability and liquidity of assets by letting agents invest in information to distinguish the quality (genuine or fake) of certain assets that can be used as a medium of exchanges. However, to use standard bargaining theory to determine the terms of trade, they made the assumption that fake assets can be produced at zero cost, which implies unrecognized assets are not accepted in a bilateral match. On the other hand, in our paper, we explicitly deal with the bargaining problem under asymmetric information using costly counterfeiting and costly screening.

The rest of the paper is organized as follows. Section 1.2 presents the environment of the baseline model and section 1.3 contains the construction and analysis of equilibrium. In section 1.4, we extend the model to study optimal government policy. Section 1.5 is the conclusion.

1.2 Environment

The general framework is built on Lagos and Wright (2005) with heterogeneous agents similar to Lagos and Rocheteau (2005) and Rocheteau and Wright (2005) incorporating the counterfeiting technology from Li, Rocheteau, and Weill (2012) and the screening technology. Time is indexed by $t = 0, 1, 2 \ldots$, and there are two subperiods within each period; the centralized market (CM) followed by the decentralized market (DM).

There are two types of economic agents, each with unit mass: buyers and sellers. Each buyer has preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t [X_t - H_t + u(x_t)],$$

and each seller has preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t [X_t - H_t - h_t].$$

Here, $\beta \in (0,1)$ is the discount factor, X_t and H_t are consumption and labor supply, respectively, in the CM, x_t is consumption in the DM, and h_t is labor supply in the DM. We assume that $u(\cdot)$ is a strictly increasing, strictly concave, and twice continuously differentiable function with $u(0) = 0, u'(0) = \infty, u'(\infty) = 0$, $-x\frac{u''(x)}{u'(x)} < 1$ for all $x \geq 0$, and with the property that there exists some $\widehat{x}$ such that $u(\widehat{x}) = \widehat{x}$. Define x^* by $u'(x^*) = 1$.

Each agent can produce one unit of the perishable consumption good for each unit of labor supply. Notice that buyers want to consume but cannot produce in the DM while sellers can produce but do not wish to consume in the DM, which generates a double coincidence problem. The CM is a centralized Walrasian market in which agents trade numeraire CM goods and an asset. There are bilateral meetings between buyers and sellers in the DM.

In this economy there is no memory or recordkeeping, so that in any meeting, traders have no knowledge of each other's histories. Also no one can be forced to work, so lack of memory implies that there can be no unsecured credit. Hence, an asset is essential for trade to occur. The only asset in this economy is fiat money which is traded at the price ϕ_t in terms of numeraire goods in the CM in period t. Money is supplied by the government at the beginning of the CM with a lump-sum transfer $T_t = (\mu - 1)\phi_t M_{t-1}$ to each buyer. Thus, the money stock grows at the constant gross rate μ. In the following, we consider the case where $\frac{\phi_t}{\phi_{t+1}} > \beta$.[8]

A key assumption is that a buyer can produce any quantity of fake money if he or she incurs a fixed cost of k in the CM that can represent the cost of acquiring a sophisticated reprographic machine and photo editing software. Counterfeit money is indistinguishable from genuine money in the DM. However, we assume that money is perfectly recognizable

[8] In a stationary equilibrium, which we will focus on when characterizing equilibrium, $\frac{\phi_t}{\phi_{t+1}} = \mu$, and there is no equilibrium if $\mu < \beta$ as is standard in Lagos and Wright (2005) setups.

in the CM, so all counterfeits are confiscated in the CM.[9] As a result, there is no incentive for sellers to receive counterfeits in a DM meeting. On the other hand, we assume that sellers have a screening technology that can detect counterfeit money with γ units of labor in the DM. One can think of the screening cost γ as the cost of installing an authentication device or as the time and effort to scrutinize security features without error. At this moment, we treat k and γ as parameters and we endogenize them in section 4 where we discuss optimal government policy.

In this economy, the sequence of moves in each period is as follows: 1) At the beginning of the CM in period t, the government transfers or taxes money to buyers in a lump-sum way. 2) The buyer chooses a portfolio of m_t genuine money and m_t^c counterfeit money in terms of period $t+1$ CM goods. 3) In the DM, the buyer is matched with a seller and makes an offer (x_t, d_t) that specifies the quantity of DM goods produced by the seller x_t and the money transfer d_t in terms of CM goods in period $t+1$ from the buyer to the seller. 4) The seller decides whether to accept the offer or not. 5) If the seller accepts the offer, the buyer hands over $\widehat{d}_t$ genuine money and $\widehat{d}_t^c$ counterfeit money with $\widehat{d}_t + \widehat{d}_t^c = d_t$. 6) Then, the seller decides whether to screen the money. If the seller detects counterfeits through screening, then the seller makes a new offer (x_t^s, d_t^s) over DM goods production and legal money transfer, and the buyer decides whether to accept the seller's offer or not.[10] Otherwise, the trade goes through according to the proposed offer by the buyer.

[9] We make this assumption for the following reasons. First, if counterfeits are not recognizable in the CM and buyers make counterfeits in equilibrium, then buyers would produce an infinite amount of counterfeits in a given period to use them for trades in both the CM and DM (if possible) in all periods, which could threaten the existence of monetary equilibrium in which money is used as a medium of exchanges. Second, notice that it is weakly optimal for a buyer to produce an infinite quantity of counterfeits once he incurs the fixed cost k. Thus, without the 100% confiscation of counterfeits, there would be two types of buyers when counterfeits are produced: one with counterfeits from the previous period, and the other without any counterfeits from the previous period. This would generate another signaling problem in addition to the one we explore in this paper, which complicates the analysis. The limited ability of the government to take counterfeits out of circulation can provide interesting new insights, but we leave this to future research.

[10] Without this assumption, the buyer may hold a portfolio of genuine money and counterfeits together in equilibrium, which complicates the analysis than is necessary. By giving a whole bargaining power to the seller when a counterfeit is detected, we can simplify a set of buyer's actions for portfolio choice. See lemma 1 for more detail.

1.3 Equilibrium

1.3.1 Payoffs

To set the stage for equilibrium characterization, we first describe agents' payoff in the game in a given period t. Because of the quasi-linearity of preferences in Lagos and Wright models, the agent's value in the CM is linear in his wealth and the choice of asset portfolio is independent of his initial wealth when he entered the period and the government transfers. Then, the buyer's payoff in the game excluding the government transfers, given the sequence of moves described above, is:

$$S_t^b = -\left(\frac{\phi_t}{\phi_{t+1}} - \beta\right) m_t - k\mathtt{I}_{\{m_t^c>0\}} + \mathtt{I}_{\{r=A\}} \left\{ \begin{array}{c} (1-\mathtt{I}_{\{\widehat{d}_t^c>0\}})\left[u(x_t)-\beta d_t\right] \\ +\mathtt{I}_{\{\widehat{d}_t^c>0\}} \left\{ \begin{array}{c} \mathtt{I}_{\{s=Y\}}\mathtt{I}_{\{r^b=A\}}\left[u(x_t^s)-\beta d_t^s\right] \\ +(1-\mathtt{I}_{\{s=Y\}})\left[u(x_t)-\beta \widehat{d}_t\right] \end{array} \right\} \end{array} \right\},$$

where $\mathtt{I}_{\{m_t^c>0\}}$, $\mathtt{I}_{\{r=A\}}$, $\mathtt{I}_{\{\widehat{d}_t^c>0\}}$, $\mathtt{I}_{\{s=Y\}}$, and $\mathtt{I}_{\{r^b=A\}}$ are all indicator functions that equal one if $m_t^c > 0$, the seller accepts the buyer's offer ($r = A$), the buyer hands over a positive amount of counterfeits to the seller ($\widehat{d}_t^c > 0$), the seller screens the money ($s = Y$), and the buyer accepts the seller's offer once counterfeits are detected by the seller ($r^b = A$), respectively.

The term $\left(\frac{\phi_t}{\phi_{t+1}} - \beta\right) m_t$ is the cost of holding m_t legal money while in order to produce $m_t^c > 0$ counterfeits, the buyer must incur the fixed cost k. The terms in the big curly bracket are the payoff from a trade with a seller. If the buyer does not hand over any counterfeits, i.e., $d_t = \widehat{d}_t$, then the seller produces x_t units of DM goods with certainty once he accepts the offer. However, if the buyer hands over some counterfeits $\widehat{d}_t^c$, then the terms of trade depend on whether the seller screens the money or not as well as the quantity of genuine money that the buyer transferred, $\widehat{d}_t$. First, if the seller screens the money and detects counterfeits, then he makes a new offer (x_t^s, d_t^s) to the buyer where $d_t^s \leq \widehat{d}_t$. However, if the seller does not screen, then the buyer consumes x_t units of DM goods for the exchange of $\widehat{d}_t$ units of legal money.

In a bilateral meeting, the seller's money holding does not affect any outcome of the game, so, given the money holding costs, the seller would not bring any money into the DM. Then, the seller's payoff when he meets a buyer who offers (x_t, d_t) can be written as

$$S_t^s = \mathtt{I}_{\{r=A\}} \left\{ \mathtt{I}_{\{s=Y\}} \left\{ \begin{array}{c} (1 - \mathtt{I}_{\{\widehat{d}_t^c > 0\}})(-x_t + \beta d_t) \\ + \mathtt{I}_{\{\widehat{d}_t^c > 0\}} \mathtt{I}_{\{r^b = A\}} (-x_t^s + \beta d_t^s) - \gamma \end{array} \right\} + \left(1 - \mathtt{I}_{\{s=Y\}}\right) (-x_t + \beta \widehat{d}_t) \right\}.$$

Here, after accepting the buyer's offer, if the seller screens the money, he will trade DM goods only for the exchange of genuine money, but he has to pay the screening cost γ. On the other hand, if he does not screen the money, then the trade occurs according to the buyer's offer (x_t, d_t), but he receives $\widehat{d}_t$ units of legal money that can be equal to or less than d_t.

1.3.2 Equilibrium concept

Our definition of equilibrium is standard: given prices, all agents behave optimally and all markets clear in equilibrium. However, the environment generates a game between the buyer and the seller for which we need a solution concept. Here, we adopt Perfect Bayesian Equilibrium (PBE) as our equilibrium concept to the game: actions are sequentially rational given a system of beliefs, and beliefs are derived from equilibrium strategies through Bayes' rule whenever possible.

Before describing agents' strategies and beliefs, we can simplify a buyer's actions for portfolio choice using the following logic. After detecting counterfeits, the seller makes a take-it-or-leave-it offer to the buyer, and there is no asymmetric information on the authenticity of the money on the table, which implies $u(x_t^s) - \beta d_t^s = 0$. Then, given $u(x_t^s) - \beta d_t^s = 0$, mixing counterfeits and genuine money does not improve the buyer's payoff because of the fixed cost of counterfeiting and money holding costs. This argument leads to the next lemma.

Lemma 1 *In any equilibrium, either 1)* $d_t = \widehat{d}_t \leq m_t$ *and* $m_t^c = 0$ *or 2)* $d_t = \widehat{d}_t^c \leq m_t^c$ *and*

$m_t = 0$.

Proof. See Appendix ■

Lemma 1 basically means that the buyer will either produce counterfeits or hold legal money only; i.e., he will not hold both legal money and counterfeits in the DM. This implies that, if the seller detects any counterfeits in the meeting, there will be no trade between the buyer and the seller, i.e., $x_t^s = d_t^s = 0$. Thus, in the following analysis, we eliminate dominated actions and modify the set of agents' actions in the game, such that: 1) the buyer chooses his portfolio (m_t, m_t^c) from the set $\mathcal{M} = \{(m, 0)|m \in \mathbb{R}_+\} \cup \{(0, m^c)|m^c \in \mathbb{R}_+\}$ in the CM; 2) if the seller detects counterfeits through screening, the trade is not implemented.

Now, in the game, at a given period t, behavior strategies for a buyer include the probability measure F_t defined on the Borel algebra for the set $\mathcal{M}$ and the probability measure $\Omega_{t|(m,m^c)}$ over the Boreal algebra for the set $S_{\overline{m}} = \mathbb{R}_+ \times [0, \overline{m}]$, where $\overline{m} = Max\{m, m^c\}$, conditional on (m, m^c). Let supp(F_t) and supp$(\Omega_{t|(m,m^c)})$ denote supports of the measures F_t and $\Omega_{t|(m,m^c)}$, respectively. Seller's behavior strategies are a decision rule $\eta_t : \mathbb{R}_+^2 \to [0, 1]$ that assigns a probability of acceptance to any feasible offers and a decision rule $\pi_t : \mathbb{R}_+^2 \to [0, 1]$ that assigns a probability of screening the money for all feasible offers. Finally, the seller's belief regarding the buyer's action about counterfeiting, conditional on the offer (x, d) being made, is a mapping $\lambda_t : \mathbb{R}_+^2 \to [0, 1]$ that assigns a probability that the seller meets a genuine money holder.

Then, a PBE of the game at period t is a profile of behavior strategies $\left\{F_t, \Omega_{t|(m,m^c)}, \eta_t, \pi_t\right\}$ and belief function λ_t, such that 1) the agents' behavior strategies are sequentially rational at each information set, and 2) the belief function λ_t is derived from the strategy profile through Bayes' rule whenever possible.

More precisely, the probability measure $\Omega_{t|(m,m^c)}$, following portfolio choice (m, m^c), must be optimal, given the seller's strategies; thus, for any $(x, d) \in$ supp$(\Omega_{t|(m,m^c)})$, it must be

$$(x, d) \in \underset{(\widetilde{x}, \widetilde{d}) \in S_m}{\arg\max} \left\{\eta_t \left(\widetilde{x}, \widetilde{d}\right) \left[u(\widetilde{x}) - \beta\widetilde{d}\right] + \beta m\right\}, \tag{1}$$

if $m > 0$ and $m^c = 0$, or

$$(x,d) \in \underset{(\widetilde{x},\widetilde{d}) \in S_{m^c}}{\arg\max} \left\{ \eta_t(\widetilde{x},\widetilde{d}) \left[1 - \pi_t(\widetilde{x},\widetilde{d})\right] u(\widetilde{x}) \right\}, \tag{2}$$

if $m = 0$ and $m^c > 0$. Let $\mathcal{V}_t^g(m)$ and $\mathcal{V}_t^c(m^c)$ be the maximized value of the objective function in (1) and (2), respectively. Then, given the decision rule $\Omega_{t|(m,m^c)}$, $\mathcal{V}_t^g(m)$, and $\mathcal{V}_t^c(m^c)$, the probability measure F_t must be optimal, that is, for any $(m, m^c) \in \text{supp}(F_t)$,

$$(m, m^c) \in \underset{(\widetilde{m},\widetilde{m}^c) \in \mathcal{M}}{\arg\max} \left\{ \mathtt{I}_{\{\widetilde{m}>0\}} \left[-\frac{\phi_t}{\phi_{t+1}} \widetilde{m} + \mathcal{V}_t^g(\widetilde{m}) \right] + \mathtt{I}_{\{\widetilde{m}^c>0\}} \left[-k + \mathcal{V}_t^c(\widetilde{m}^c) \right] \right\},$$

where $\mathtt{I}_{\{\widetilde{m}>0\}}$ and $\mathtt{I}_{\{\widetilde{m}^c>0\}}$ are indicator functions that equal one if $\widetilde{m} > 0$ and $\widetilde{m}^c > 0$, respectively. Similarly, following an offer (x, d), the seller's decision to accept the offer and to screen the money must be optimal, given the buyer's strategies and the seller's belief, that is,

$$(\eta_t, \pi_t) \in \underset{(\widetilde{\eta},\widetilde{\pi}) \in [0,1]^2}{\arg\max} \left\{ \widetilde{\eta} \left\{ \begin{array}{c} \widetilde{\pi} \left[\lambda_t(x,d)(-x + \beta d) - \gamma \right] \\ +(1-\widetilde{\pi}) \left[\lambda_t(x,d)(-x+\beta d) - (1 - \lambda_t(x,d))x \right] \end{array} \right\} \right\}.$$

Finally, in order to find conditions for the seller's belief $\lambda_t(x, d)$, let the distribution $G_t(m, m^c, x, d)$ over $(m, m^c, x, d) \in \mathcal{M} \times \mathbb{R}_+^2$ define the buyer's mixed strategy that can be derived from the buyer's behavior strategies, $\left\{F_t, \Omega_{t|(m,m^c)}\right\}$.[11] Then, the seller's belief $\lambda_t(x, d)$, must satisfy

$$\lambda_t(x,d) = \frac{\int_{(\widetilde{m},\widetilde{m}^c,\widetilde{x},\widetilde{d}) \in [d,\infty) \times \{0\} \times \{x,d\}} dG_t}{\int_{(\widetilde{m},\widetilde{m}^c,\widetilde{x},\widetilde{d}) \in [d,\infty) \times \{0\} \times \{x,d\}} dG_t + \int_{(\widetilde{m},\widetilde{m}^c,\widetilde{x},\widetilde{d}) \in \{0\} \times [d,\infty) \times \{x,d\}} dG_t} \tag{3}$$

if the denominator is strictly positive.

Because seller's beliefs are not pinned down off the equilibrium path under the PBE in equation (3), the game would generate a plethora of equilibria.[12] However, in our economic

[11] In a game of perfect recall, mixed and behavior strategies are equivalent, according to Kuhn's theorem (see Fudenberg and Tirole 1991).

[12] For example, for any offer (x^o, d^o) that satisfies 1) $-\frac{\phi_t}{\phi_{t+1}} d^o + u(x^o) > Max\{0, -k + u(x^o)\}$, and 2)

environment, it is hard to apply standard refinement rules in signaling games such as the intuitive criterion because the buyer's type is chosen endogenously instead of being determined exogenously by nature, so there are additional ways to deviate that must be unprofitable in equilibrium. Instead, we adopt the concept of *Reordering Invariance (RI) equilibrium*, in order to refine the set of equilibria of the game, proposed by In and Wright (forthcoming) and applied by Li, Rocheteau, and Weill (2012) in an asset exchange model. This refinement selects equilibrium outcomes of the original game that are also equilibrium outcomes of a reordered game in which observed actions (terms of trade) are chosen before unobserved actions (portfolio choice), sharing the same reduced normal form as the original game. Reversing the order of the game does not affect equilibrium outcomes because the buyer does not gain any pay-off relevant information between his unobserved and observed actions, and this refinement rule has a strong game theoretic rationale and desirable normative properties (see In and Wright, forthcoming).

The timing of reordered game is as follows: 1) At the beginning of the CM in period t after the government transfers, the buyer posts his DM offer (x_t, d_t). 2) The buyer decides whether to accumulate genuine money or to produce counterfeits for the trade. 3) In the DM, the buyer is matched with a seller and the seller decides whether to accept or to reject the offer. 4) Once the offer is accepted, the remaining procedures are the same as in the original game. The game tree in Figure 1 depicts the sequence of moves in the reordered game.

Because of the positive legal money holding costs, the buyer will only acquire d units of real money in the CM whenever he decides to finance the trade (x, d) with legal money, i.e., $m_t = d$, and he will produce $m_t^c \geq d$ units of counterfeits if he decides to have the trade (x, d) with counterfeits. Thus, what matters in the buyer's portfolio choice given an offer (x, d) is whether to acquire legal money or to produce counterfeits. With this observation in mind, buyer's behavior strategies, in the reordered game, are the distribution $\Omega_t(x, d)$ over

$-x^o+\beta d^o > 0$, it can be supported as a part of equilibrium where $\text{prob}((m, m^c, x, d) = (d^o, 0, x^o, d^o)) = 1$ with a belief system λ_t such that $\lambda_t(x^o, d^o) = 1$ and $\lambda_t(x, d) = 0$, that implies $\eta_t(x, d) = 0$, for all $(x, d) \neq (x^o, d^o)$.

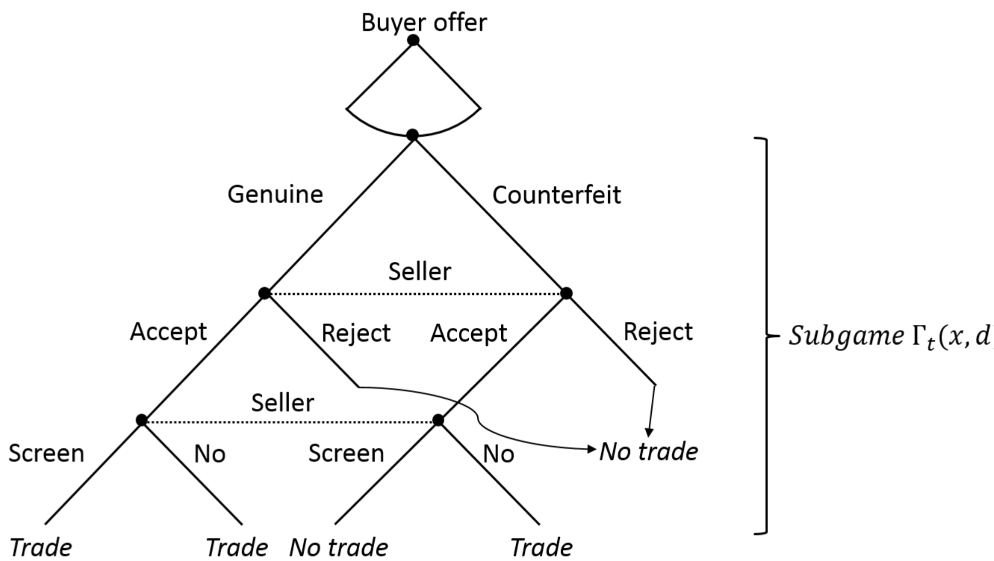

Figure 1: Game tree of reordered game

offers $(x, d) \in \mathbb{R}^2_+$ and a decision rule $\alpha_t : \mathbb{R}^2_+ \to [0, 1]$ that assigns a probability that the buyer accumulates genuine money conditional on (x, d). Seller's behavior strategies at each information set are the same as the ones in the original game.

Note, in the reordered game, that any posted offer (x, d) generates a proper subgame denoted by $\Gamma_t(x, d)$ in Figure 1. Then, by subgame perfection, agents' strategies restricted to this subgame following an offer (x, d) must form a Nash Equilibrium, so the seller can correctly infer the buyer's strategy $\alpha_t(x, d)$. This implies that we can discipline sellers' beliefs following all out-of-equilibrium offers $(x, d) \in \mathbb{R}^2_+$ in a logically consistent way as $\lambda_t(x, d) = \alpha_t(x, d)$.

We now formally state the conditions that agents' strategies must satisfy in the reordered game. If there is no risk of confusion, we drop arguments for each decision rule from now on; we use α_t instead of $\alpha_t(x, d)$, for instance. First, given the offer (x, d) and the seller's strategies regarding acceptance and screening, the buyer must minimize the cost of financing his DM trade, that is,

$$\alpha_t \in \underset{\widetilde{\alpha}\in[0,1]}{\arg\min} \left\{ \widetilde{\alpha} \left[\frac{\phi_t}{\phi_{t+1}} - (1-\eta_t)\beta \right] d + (1-\widetilde{\alpha}) \left[k + \eta_t \pi_t u(x) \right] \right\}.$$

Here, the term $\left[\frac{\phi_t}{\phi_{t+1}} - (1-\eta_t)\beta\right] d$ is the financing cost with genuine money: The holding cost, $\left(\frac{\phi_t}{\phi_{t+1}} - \beta\right) d$, plus the expected transferring cost, $\eta_t \beta d$. The term $k + \eta_t \pi_t u(x)$ relates to the financing cost with counterfeiting that consists of the fixed cost of producing counterfeits, k, and the expected cost of missing trade by screening in the DM, $\eta_t \pi_t u(x)$.

Second, the seller's decision about acceptance and screening must be optimal given the offer (x, d) and the buyer's strategy, that is,

$$(\eta_t, \pi_t) \in \underset{(\widetilde{\eta},\widetilde{\pi})\in[0,1]^2}{\arg\max} \left\{ \widetilde{\eta} \left[-x + \alpha_t \beta d + \widetilde{\pi} [(1-\alpha_t)x - \gamma] \right] \right\},$$

where we imposed the condition that $\lambda_t = \alpha_t$. The seller's payoff has two components. The first term, $-x + \alpha_t \beta d$, is the expected payoff from trade without screening, and the second term, $\widetilde{\pi}[(1-\alpha_t)x - \gamma]$, is the net payoff from screening.

Finally, given equilibrium decision rules $\{\alpha_t, \eta_t, \pi_t\}$, the buyer's optimal offer maximizes his expected payoff, that is,

$$(x_t, d_t) \in \underset{(\widetilde{x},\widetilde{d})\in\mathbb{R}_+^2}{\arg\max} \left\{ \begin{array}{c} \alpha_t \left[-\frac{\phi_t}{\phi_{t+1}} \widetilde{d} + \eta_t \left(u(\widetilde{x}) - \beta \widetilde{d} \right) + \beta \widetilde{d} \right] \\ +(1-\alpha_t) \left[-k + \eta_t (1-\pi_t) u(\widetilde{x}) \right] \end{array} \right\}.$$

Even though reordering of the game narrows down equilibria of the original game by disciplining the seller's belief λ_t in an effective way, reordering itself does not guarantee a unique equilibrium outcome in general. In particular, in our model, given (x, d), there could be multiple Nash equilibria of the subgame $\Gamma_t(x, d)$, which affects, in turn, a buyer's optimal posting decision.

It is useful to select among equilibria by restricting attention to Pareto dominant Nash equilibria of $\Gamma_t(x, d)$: there is no other Nash equilibrium that makes every player at least as well off and at least one player strictly better off.[13] Then, given the Pareto dominant Nash equilibrium conditions of the subgame $\Gamma_t(x, d)$ following all feasible offers (x, d), the buyer posts an offer (x, d) so as to maximize his payoff.

1.3.3 Characterization of Equilibrium

In this subsection, we characterize a stationary equilibrium. By stationarity, we mean that all real quantities are constant over time, and buyers and sellers play the game repeatedly in the stationary economy. This implies that the inflation rate equals the money growth rate, i.e., $\frac{\phi_t}{\phi_{t+1}} = \mu$.

We now show how to characterize an equilibrium by solving an optimization problem. Consider the following auxiliary problem,

$$(P) \qquad \overline{S}^b = \underset{x\geq 0, d\geq 0, \alpha, \eta, \pi}{Max} \left\{ \begin{array}{l} \alpha \left[-\mu d + \eta \left(u(x) - \beta d\right) + \beta d\right] \\ +(1-\alpha)\left[-k + \eta(1-\pi)u(x)\right] \end{array} \right\}$$

subject to

$$(4) \qquad \alpha \in \underset{\widetilde{\alpha}\in[0,1]}{\arg\min} \; \left\{\widetilde{\alpha}\left[\mu - (1-\eta)\beta\right] d + (1-\widetilde{\alpha})\left[k + \eta\pi u(x)\right]\right\}$$

$$(5) \qquad (\eta, \pi) \in \underset{(\widetilde{\eta},\widetilde{\pi})\in[0,1]^2}{\arg\max} \; \left\{\widetilde{\eta}\left[-x + \alpha\beta d + \widetilde{\pi}[(1-\alpha)x - \gamma]\right]\right\}.$$

The problem (P) looks similar with the actual buyer's problem in the game, but there are two major differences. First, in the maximization problem (P), we only require the Nash

[13]In Li, Rocheteau, and Weill (2012) where there is no screening technology, they did not make any assumptions about selecting equilibrium in the subgame. The difference in their model is that there exists only a pure strategy Nash equilibrium in the subgame given the optimal offer. Thus, by perturbing the offer slightly, they can make a sequence of offers such that all incentive constraints are slack, so the Nash equilibrium of the subgame following perturbed offers is unique and the sequence of offers converges to the equilibrium offer. However, in our model, there could be both a mixed and a pure strategy equilibrium following the equilibrium offer, and we cannot apply the argument of Li, Rocheteau, and Weill (2012) when a mixed strategy equilibrium exists. Thus, we resort to the Pareto dominance rule instead.

equilibrium conditions (4) and (5) of the subgame $\Gamma(x,d)$ following an offer (x,d) without the Pareto dominance condition.[14] Second, though the buyer cannot arbitrarily pick an equilibrium of the subgame $\Gamma(x,d)$ when there are multiple equilibria, $\{\alpha,\eta,\pi\}$ is a set of choice variables in the maximization problem of (P). Therefore, $\overline{S}^b$ is an upper bound of the buyer's payoff in any equilibrium of the game. Hence, if the buyer can attain $\overline{S}^b$ by posting (x,d), it must be an equilibrium offer.

It will be shown in proposition 1 below that an equilibrium can be constructed and characterized by solving (P). As a preliminary step, we provide some of properties of a solution to (P) beforehand with the interpretation of these properties as equilibrium outcomes.

To begin, observe that for any $x^o \leq \frac{\beta}{\mu}k$, $(x,d,\alpha,\eta,\pi) = \left(x^o, \frac{x^o}{\beta}, 1, 1, 0\right)$ satisfies (4) and (5). Then, the buyer's expected surplus becomes positive with sufficiently small x^o because $\lim_{x\to 0} u'(x) = \infty$, which implies that $\overline{S}^b$ must be strictly positive. Thus, any solution must feature $x > 0$ and $\eta > 0$ because the buyer's expected surplus is non-positive otherwise. Also, if $\alpha = 0$ with positive x, then $\eta = 0$ by (5). Thus, it must be $\alpha > 0$ for any solution to (P). Given these, next lemma shows a property of a solution providing a useful intermediate step to solve (P).

Lemma 2 *Any solution to (P) has following form: either 1) $\alpha = 1$ and $\pi = 0$ or 2) $\alpha \in (0,1)$ and $\pi \in (0,1)$.*

Proof. See Appendix ■

The main implication of lemma 2 is that counterfeiting occurs in equilibrium where buyers attain $\overline{S}^b$ only if screening activity exists in the economy. To get some intuition about the logic of this result, consider an offer (x,d) that induces $\alpha \in (0,1)$ and $\pi = 0$ in the subgame $\Gamma(x,d)$. The indifference condition of the buyer (4) means that $[\mu - (1-\eta)\beta]\,d = k$. Now

[14] Formally, the Pareto dominance condition of a Nash equilibrium $\{\alpha,\eta,\pi\}$ of the subgame $\Gamma(x,d)$ can be stated as there is no other $\{\alpha',\eta',\pi'\}$ that satisfies (4) and (5) such that

$$\alpha'\left[\mu - (1-\eta')\beta\right]d + (1-\alpha')\left[k + \eta'\pi' u(x)\right] \leq \alpha\left[\mu - (1-\eta)\beta\right]d + (1-\alpha)\left[k + \eta\pi u(x)\right]$$
$$\eta'\left[-x + \alpha'\beta d + \pi'[(1-\alpha')x - \gamma]\right] \geq \eta\left[-x + \alpha\beta d + \pi[(1-\alpha)x - \gamma]\right]$$

with at least one strict inequality.

consider a new offer (x, d') with $d' = \alpha d$. Then, $[\mu - (1-\eta)\beta]\, d' < k$, so it is better for the buyer to bring genuine money with certainty, i.e., $\alpha' = 1$, provided that $\eta' = \eta$ and $\pi' = 0$. Furthermore, if $\alpha' = 1$, then there is no reason for the seller to screen the money, so $\pi' = 0$. Then, the seller's payoff does not change, so $\eta' = \eta$ is still optimal. Therefore, the buyer can increase his payoff by reducing the quantity of money transfer that prevents counterfeiting from taking place while keeping the seller's expected payoff unchanged. This profitable deviation explains why counterfeiting does not emerge in equilibrium in previous studies such as Li, Rocheteau, and Weill (2012) and Shao (2014) in which there is no screening technology.

It is also clear, from (4) and (5), under what conditions, screening can exist with counterfeiting in the economy. As we argued above, for the buyer to trade with a seller in the DM with a positive probability, he must propose an offer that induces $\alpha > 0$ that requires

$$[\mu - (1-\eta)\beta]\, d \leq k + \eta\pi u(x). \tag{6}$$

Without screening, i.e., $\pi = 0$, the money transfer d and the output of DM goods x that induces $\eta > 0$ can be very low if the counterfeiting cost k is too low. By allowing a positive probability of screening by the seller, the buyer can relax this constraint and increase the trade volume in the DM. However, for the seller to find it optimal to screen the money in equilibrium, the buyer must hand over counterfeits with a positive probability to satisfy (5), so (6) must hold with equality.

Lemma 3 *For any solution to* (P), $\eta = 1$ *and* $x = \alpha\beta d$.

Proof. See Appendix ∎

Lemma 3 basically means that the buyer makes an offer that is accepted with certainty and the seller earns zero profit in expectation in equilibrium where buyers attain his maximum payoff $\overline{S}^b$. The result that $x = \alpha\beta d$ implies that the seller's expected payoff is zero, which is intuitive given that the buyer makes a take-it-or-leave-it offer to the seller. Next,

$\eta = 1$ means that the buyer makes an offer that is accepted with probability one, and Li, Rocheteau, and Weill (2012) show the same result in their model in which there is no screening technology, so counterfeiting does not occur in equilibrium. Here, we only provide the intuition for the finding of $\eta = 1$ when there are counterfeiting and screening in the economy. Suppose $(\alpha, \eta, \pi) \in (0,1)^3$. In this case, the constraint (6) binds with equality because of the buyer's indifference condition. A profitable deviation for the buyer consists of increasing the probability of acceptance by the seller, $\eta' \in (\eta, 1)$, with a change of the screening probability, $\pi' \in (0,1)$, such that the constraint (6) still holds with equality.[15] Then, $\{\eta', \alpha, \pi'\}$ satisfies (4) and (5) but the buyer's payoff increases with the higher probability of acceptance of the offer by the seller.

Given lemma 2 and 3, we can rewrite the problem (P) as

$$(P') \qquad \overline{S}^b = \underset{x \geq 0, d \geq 0, \alpha, \pi}{Max} \{-\mu d + u(x)\}$$

subject to

$$(7) \qquad x = \alpha \beta d$$

$$(8) \qquad \alpha \in \underset{\widetilde{\alpha} \in [0,1]}{\arg\max} \ \{\widetilde{\alpha}\,[-\mu d + k + \pi u(x)]\}$$

$$(9) \qquad \pi \in \underset{\widetilde{\pi} \in [0,1]}{\arg\max} \ \{\widetilde{\pi}[(1-\alpha)x - \gamma]\}\,.$$

Because $-\mu d + u(\alpha\beta d)$ is maximized with $d = \frac{1}{\beta} u'^{-1}\left(\frac{\mu}{\beta}\right)$ and $\alpha = 1$, whenever this is feasible, it must be a solution. This is possible if and only if $k \geq \frac{\mu}{\beta} u'^{-1}\left(\frac{\mu}{\beta}\right)$, and the solution to (P') is $x = u'^{-1}\left(\frac{\mu}{\beta}\right), d = \frac{1}{\beta} u'^{-1}\left(\frac{\mu}{\beta}\right), \alpha = 1$, and $\pi = 0$ in this case.

When $k < \frac{\mu}{\beta} u'^{-1}\left(\frac{\mu}{\beta}\right)$, finding a solution of (P') becomes more complicated. However, by virtue of lemma 2, we can restrict our analysis to two cases: 1) $\alpha = 1$ and $\pi = 0$, or 2) $\alpha \in (0,1)$ and $\pi \in (0,1)$. We consider these two cases separately and take the max of the

[15] Notice that by the assumption that $(\alpha, \eta, \pi) \in (0,1)^3$, $(1-\alpha)x = \gamma$. Thus, changing π does not affect the seller's expected payoff under this deviation, and $\eta' \in (0,1)$ is still optimal.

two to solve the problem (P').

First, consider the case with $(\alpha, \pi) = (1, 0)$ that requires, from (8), that

$$d \leq \frac{k}{\mu}. \tag{10}$$

This is the incentive compatibility constraint for the buyer not to commit forgery without screening. Given $k < \frac{\mu}{\beta} u'^{-1}\left(\frac{\mu}{\beta}\right)$, (10) must bind to maximize the objective function of (P'), so $x = \frac{\beta k}{\mu}$, $d = \frac{k}{\mu}$. Substituting this candidate solution into the objective function of (P'), we obtain $S^b_{tc} \equiv -k + u\left(\frac{\beta k}{\mu}\right) > 0$. Here, the trade volume in the DM is limited by the counterfeiting cost k and the buyer's surplus S^b_{tc} converges to zero as $k \to 0$.

Second, consider a candidate solution with $\alpha \in (0, 1)$ and $\pi \in (0, 1)$. The conditions (8) and (9) yield a mixed strategy equilibrium only if the following equations hold:

$$\mu d = k + \pi u(x) \tag{11}$$

$$(1 - \alpha)x = \gamma. \tag{12}$$

As one can see from (10) and (11), a positive probability of screening, $\pi > 0$, allows the buyer to make an offer with the money transfer higher than $\frac{k}{\mu}$, so the buyer could consume more DM goods. However, there is a cost for this strategy. To make the seller to screen the money with a positive probability, the buyer must produce and transfer counterfeits with a positive probability, i.e., $\alpha < 1$, so (12) holds. Then, because the seller receives legal money with a probability of α, the buyer must hand over $d = \frac{x}{\alpha\beta}$ units of money, that is higher than $\frac{x}{\beta}$, to make the seller earns non-negative profit in expectation. Thus, there is additional cost to finance DM trade which is caused by the fraud and screening. Then, the buyer optimally chooses (x, d, α, π) considering all those effects on his payoff.

Lemma 4 *Suppose* $-\frac{\mu}{\beta}\frac{x_\gamma^2}{x_\gamma-\gamma}+u(x_\gamma)\geq 0$*, where* $x_\gamma \in \left(u'^{-1}\left(\frac{\mu}{\beta}\right),\infty\right)$ *is given by*

$$\frac{\mu}{\beta}=\frac{(x_\gamma-\gamma)^2}{x_\gamma(x_\gamma-2\gamma)}u'(x_\gamma). \tag{13}$$

Then, the candidate solution of (P') *where* $(\alpha,\pi)\in(0,1)^2$ *is* $x=x_\gamma$*,* $d=\frac{x_\gamma^2}{\beta(x_\gamma-\gamma)}$*,* $\alpha=1-\frac{\gamma}{x_\gamma}$*, and* $\pi=\frac{1}{u(x_\gamma)}\left\{\frac{\mu}{\beta}\frac{x_\gamma^2}{x_\gamma-\gamma}-k\right\}$*, and the value of the objective function is* $S_c^b=-\frac{\mu}{\beta}\frac{x_\gamma^2}{x_\gamma-\gamma}+u(x_\gamma)$*.*

Proof. See Appendix ■

The final step is to compare the buyer's payoff under each candidate solution to get $\overline{S}^b=Max\{S_{tc}^b,S_c^b\}$. Subtracting S_{tc}^b from S_c^b, we obtain

$$G(\mu,k,\gamma)\equiv-\frac{\mu}{\beta}\frac{x_\gamma^2}{x_\gamma-\gamma}+u(x_\gamma)-\left\{-k+u\left(\frac{\beta k}{\mu}\right)\right\}. \tag{14}$$

Thus, if $G(\mu,k,\gamma)<0$, then $(x,d,\alpha,\pi)=(\frac{\beta k}{\mu},\frac{k}{\mu},1,0)$ is the solution to (P').[16] On the other hand, if $G(\mu,k,\gamma)>0$, then $(x,d,\alpha,\pi)=\left(x_\gamma,\frac{x_\gamma^2}{\beta(x_\gamma-\gamma)},1-\frac{\gamma}{x_\gamma},\frac{1}{u(x_\gamma)}\left\{\frac{\mu}{\beta}\frac{x_\gamma^2}{x_\gamma-\gamma}-k\right\}\right)$ solves (P'). Finally, in the knife edge case where $G(\mu,k,\gamma)=0$, $S_{tc}^b=S_c^b$, so both candidate solutions solve (P').

To gather more intuition about the influence of the counterfeiting environment, (k,γ), on the economy, take derivatives G with respect to k and γ to obtain

$$\frac{\partial G}{\partial k}=1-\frac{\beta}{\mu}u'\left(\frac{\beta k}{\mu}\right)<0,$$
$$\frac{\partial G}{\partial \gamma}=-\frac{\mu}{\beta}\left(\frac{x_\gamma}{x_\gamma-\gamma}\right)^2<0,$$

where the inequality of the first equation comes from the assumption that $k<\frac{\mu}{\beta}u'^{-1}\left(\frac{\mu}{\beta}\right)$, and we used the envelope theorem to obtain the second result. Note that $G(\mu,k,0)>0$ and

[16] In lemma 4, we derive the second candidate solution of (P') only for the case that $S_c^b\geq 0$, even though it is possible to have $S_c^b<0$ with sufficiently high γ. However, this is without loss of generality in the following sense: S_c^b is an upper bound of the maximized value of the problem (P') with constraints that $(\alpha,\pi)\in(0,1)^2$ because we do not use constraints (11) and $\pi\in(0,1)$ to get S_c^b (see the proof for details). Thus, whenever $S_c^b<0$, $(x,d,\alpha,\pi)=(\frac{\beta k}{\mu},\frac{k}{\mu},1,0)$ solves (P').

$G(\mu, k, \infty) < 0$ provided that $k < \frac{\mu}{\beta} u'^{-1} \left(\frac{\mu}{\beta} \right)$. Taken together with the result of the case where $k \geq \frac{\mu}{\beta} u'^{-1} \left(\frac{\mu}{\beta} \right)$, we define a new function $\widehat{\gamma} : [\beta, \infty) \times (0, \infty) \to \mathbb{R}$ such that

$$
\widehat{\gamma}(\mu, k) = \begin{cases} \widehat{\gamma} > 0 \text{ that satisfies } G(\mu, k, \widehat{\gamma}) = 0 & \text{if } k < \frac{\mu}{\beta} u'^{-1} \left(\frac{\mu}{\beta} \right) \\ 0 & \text{if } k \geq \frac{\mu}{\beta} u'^{-1} \left(\frac{\mu}{\beta} \right). \end{cases} \tag{15}
$$

Then, if $\gamma < \widehat{\gamma}(\mu, k)$, a solution $(x, d, \alpha, \eta, \pi)$ that solves (P) features $\alpha < 1$.

So far, we have characterized the upper bound of payoff attainable by buyers. The last step is to show that equilibrium can be characterized using (P). The basic idea, detailed in the proof, is to show that for any solution $(x, d, \alpha, \eta, \pi)$ of (P), $\{\alpha, [\eta, \pi]\}$ is a unique Pareto dominant Nash equilibrium in the following subgame $\Gamma(x, d)$. Thus, whenever the buyer posts (x, d) that solves (P), he can achieve the maximum payoff $\overline{S}^b$. This leads to the following proposition.

Proposition 1 *There exists monetary equilibrium that features:*

1. [No threat of counterfeiting] If $k \geq \frac{\mu}{\beta} u'^{-1} \left(\frac{\mu}{\beta} \right)$, *then* $x = u'^{-1} \left(\frac{\mu}{\beta} \right)$, $d = \frac{1}{\beta} u'^{-1} \left(\frac{\mu}{\beta} \right)$, $\alpha = 1$, $\eta = 1$, *and* $\pi = 0$

2. [Threat of counterfeiting] If $k < \frac{\mu}{\beta} u'^{-1} \left(\frac{\mu}{\beta} \right)$ *and* $\gamma \geq \widehat{\gamma}(\mu, k)$, *then* $x = \frac{\beta k}{\mu}$, $d = \frac{k}{\mu}$, $\alpha = 1$, $\eta = 1$, *and* $\pi = 0$

3. [Counterfeiting] If $k < \frac{\mu}{\beta} u'^{-1} \left(\frac{\mu}{\beta} \right)$ *and* $\gamma < \widehat{\gamma}(\mu, k)$, *then* $x = x_\gamma$, $d = \frac{x_\gamma^2}{\beta (x_\gamma - \gamma)}$, $\alpha = 1 - \frac{\gamma}{x_\gamma}$, $\eta = 1$, *and* $\pi = \frac{1}{u(x_\gamma)} \left\{ \frac{\mu}{\beta} \frac{x_\gamma^2}{x_\gamma - \gamma} - k \right\}$.[17]

Proof. See Appendix ■

Proposition 1 shows how the counterfeiting environment, (k, γ), and the inflation rate, μ, together determine the existence of particular equilibria. This result is illustrated with Figure 2 that depicts how the parameter space is subdivided with k on the vertical axis, and

[17] When the equilibrium type switches from the *threat of counterfeiting* to the *counterfeiting*, the set of strategies $\{x, d, \alpha, \eta, \pi\}$ changes discontinuously. Thus, in the knife edge case where $\gamma = \widehat{\gamma}(\mu, k)$, the model admits multiple equilibria. However, analysis of multiple equilibria does not give any important insight at this moment, so we assume that the economy is in the *threat of counterfeiting equilibrium* in this case. Later, we discuss multiple equilibria when we characterize welfare in section 4.

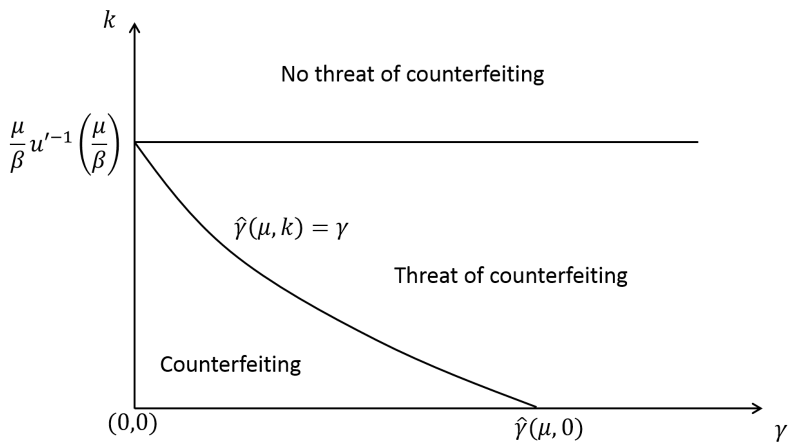

Figure 2: Equilibria with the counterfeiting environment (k, γ)

γ on the horizontal axis given the inflation rate μ.

In the *no threat of counterfeiting equilibrium*, the counterfeiting cost is too high for the buyer to produce fake money. In this case, the incentive compatibility constraint (10) does not bind. The terms of trade (x, d) are the same as the ones in an economy where there is no possibility of counterfeiting when the buyer makes a take-it-or-leave-it offer.

In the *threat of counterfeiting equilibrium*, a low counterfeiting cost is accompanied with a relatively high screening cost; $\gamma \geq \widehat{\gamma}(\mu, k)$. Because of the low counterfeiting cost, there could exist the incentive for the buyer to produce counterfeits. However, the buyer offers terms of trade that thwart entry in counterfeiting by limiting the money transfer with the binding incentive compatibility constraint (10) instead of making an offer that induces counterfeiting and screening because of the high screening cost. In this case, the usefulness of money as a medium of exchange depends on the counterfeiting cost, k, so the quantity of goods traded, x, is inefficiently low.

Finally, when both the counterfeiting cost and the screening cost are sufficiently low, the economy is in the *counterfeiting equilibrium*. If k is very low, the binding constraint (10) without screening can be too restrictive on the money transfer d. Further, if the screening cost is low such that $\gamma < \widehat{\gamma}(\mu, k)$ as in this equilibrium, the costs from having a positive probability of frauds and screening are relatively low.[18] Thus, it is better for the buyer to make an offer that induces positive probabilities of counterfeiting and screening in the following subgame, so (11) and (12) hold, instead of satisfying the constraint (10). In this case, the quantity of DM goods traded, x, the money transfer, d, and the probability that the buyer hands over genuine money, α, depend on γ but not on k. The counterfeiting cost, k, only affects the probability of screening, π. Thus, the model implies that the extent that money facilitates exchange depends on the ease of screening out fake money in the *counterfeiting equilibrium* while it depends on the counterfeiting cost in the *threat of counterfeiting equilibrium*.

One interesting feature in the *counterfeiting equilibrium* is that consumption in the DM is higher than in an economy where counterfeiting was not even a possibility: $x_\gamma > u'^{-1}\left(\frac{\mu}{\beta}\right)$ (see lemma 4). This result can be understood by looking at the indifference condition (12). The reason to offer (x, d) that induces a positive probability of screening is to consume higher quantity of DM goods x. According to (12), higher x increases α in the *counterfeiting equilibrium*. The seller can avoid producing DM goods for nothing if he detects counterfeit money. Thus, if x increases, the probability that the buyer hands over genuine money must increase to keep the seller indifferent between screening and no screening given the fixed cost of screening γ. However, non-negative profit condition of the seller, $x = \alpha\beta d$, implies that the relative money transfer to the DM goods produced, $\frac{d}{x}$, decreases with x. Because of this pecuniary effect of increasing x, the buyer offers x_γ that is strictly higher than $u'^{-1}\left(\frac{\mu}{\beta}\right)$.

As one can see from proposition 1, our model admits both equilibria *with and without* counterfeiting and a monetary equilibrium always exists. Thus, the model can explain the cross-country differences in counterfeiting experiences as an equilibrium outcome.[19] For example, two countries with the same inflation rate could have different counterfeiting experiences depending on the counterfeiting environment (k, γ). These cross-country differences

[18]Note that as $\gamma \to 0$, $\alpha \to 1$ from (12), so $d = \frac{x}{\alpha\beta} \to \frac{x}{\beta}$ and the additional cost from frauds disappears.

[19]For example, counterfeiting is a problem in some countries while it is not in other countries (see Fung and Shao 2011 for details).

could not be well explained by previous models in which counterfeiting does not occur or always exists in equilibrium.

On a related point, it seems worthwhile to discuss recent work on fraudulent practices in asset markets in the context of asset exchange models, Li, Rocheteau, and Weill (2012) and Shao (2014).[20] One of main implications in their research is that the threat of counterfeiting generates an endogenous resalability constraint similar to (10), specifying the asset's usefulness as a medium of exchange. It is this incentive compatibility constraint that prevents forgery from taking place, and it is so powerful that fraud does not occur in equilibrium even with the counterfeiting cost close to zero.

One unnoticed assumption, however, in their models is that there is no screening technology, so the only strategy for a seller in response to counterfeiting is rejecting an offer: If trade involves an asset transfer greater than the upper bound specified by the resalability constraint, then the seller would reject the offer with a positive probability. However, as argued above, this cannot be optimal for the buyer.

By incorporating a screening technology, our model shows that this resalability constraint can become ineffective thus generating fraud as an equilibrium outcome provided that the screening cost is sufficiently low. The intuition is simple: With the screening technology, the seller has an additional action to react to counterfeiting, and as long as the seller earns non-negative profit *ex-ante* with the optimal screening strategy, he would accept the buyer's offer with certainty even though the resalability constraint is not satisfied. Notice that our model encompasses those previous studies as a special case where the screening cost is sufficiently high.

Monetary policy and the equilibrium type So far, we have taken the inflation rate, μ, as given. We now study how monetary policy that determines μ affects economic agents' behaviors and hence the equilibrium type. Given the assumption that $-x\frac{u''(x)}{u'(x)} < 1$, $\frac{\mu}{\beta}u'^{-1}\left(\frac{\mu}{\beta}\right)$

[20]Observe that all agents take the inflation rate that is the inverse of the rate of return on money as given when they make their optimal strategies in the game. Thus, all of previous analysis can be applied to the economy with real assets like Li, Rocheteau, and Weill (2012).

is decreasing in μ. Thus, if $k \geq x^*$, the economy is in the *no threat of counterfeiting equilibrium* for all monetary policy $\mu \geq \beta$ (see the first part of proposition 1).

If $k < x^*$, there exists unique value $\mu_k > \beta$ that satisfies

$$k = \frac{\mu_k}{\beta} u'^{-1} \left(\frac{\mu_k}{\beta} \right). \tag{16}$$

Then, for all $\mu \geq \mu_k$, the *no threat of counterfeiting equilibrium* exists. On the other hand, when $\mu < \mu_k$, the economy is in either the *threat of counterfeiting equilibrium* or the *counterfeiting equilibrium*. Because $x_\gamma > \frac{\beta k}{\mu}$ in this case, we obtain, from (14),

$$\frac{\partial G}{\partial \mu} = \frac{1}{\mu} \left\{ -\frac{x_\gamma - \gamma}{x_\gamma - 2\gamma} x_\gamma u'(x_\gamma) + \frac{\beta k}{\mu} u' \left(\frac{\beta k}{\mu} \right) \right\} < 0,$$

so it is more likely that the *counterfeiting equilibrium* exists as μ decreases. More precisely, a decrease in μ shifts the horizontal line at point $\frac{\mu}{\beta} u'^{-1} \left(\frac{\mu}{\beta} \right)$ and the curve given by $\widehat{\gamma}(\mu, k) = \gamma$ in Figure 2 upward, so the parameter space (γ, k) for the *counterfeiting equilibrium* to exist expands.

To facilitate the presentation of the effects of μ on the equilibrium type, we derive a new critical value of μ in a following way. Note, from (15), that for all $k < x^*$, there exists unique $\widehat{\gamma}(\beta, k) > 0$ such that $G(\beta, k, \widehat{\gamma}(\beta, k)) = 0$. Then because $G(\mu, k, \gamma)$ is decreasing in each argument there exists unique $\mu_\gamma \in (\beta, \mu_k)$, for all $\gamma < \widehat{\gamma}(\beta, k)$, that satisfies

$$G(\mu_\gamma, k, \gamma) = 0. \tag{17}$$

Thus, $G(\mu, k, \gamma) > 0$ for all $\mu \in [\beta, \mu_\gamma)$, and $G(\mu, k, \gamma) \leq 0$ for all $\mu \in [\mu_\gamma, \mu_k)$. Notice that the critical value μ_γ is a decreasing function of k and γ because of the property of $G(\mu, k, \gamma)$. On the other hand, if $\gamma \geq \widehat{\gamma}(\beta, k)$, then $G(\mu, k, \gamma) \leq 0$ for all $\mu \in [\beta, \mu_k)$. In summary, we have the following proposition, whose proof is omitted, that describes how monetary policy determines the equilibrium type given k and γ.

Proposition 2 *1. Suppose $k \geq x^*$. Then, the no threat of counterfeiting equilibrium exists for all $\mu \geq \beta$.*

2. Suppose $k < x^*$ and $\gamma \geq \widehat{\gamma}(\beta, k)$. Then, i) the no threat of counterfeiting equilibrium exists for all $\mu \geq \mu_k$, and ii) the threat of counterfeiting equilibrium exists for all $\mu \in [\beta, \mu_k)$.

3. Suppose $k < x^*$ and $\gamma < \widehat{\gamma}(\beta, k)$. Then, i) the no threat of counterfeiting equilibrium exists for all $\mu \geq \mu_k$, ii) the threat of counterfeiting equilibrium exists for all $\mu \in [\mu_\gamma, \mu_k)$, and iii) the counterfeiting equilibrium exists for all $\mu \in [\beta, \mu_\gamma)$.

Figure 3 illustrates graphically proposition 2.[21] As one can see from proposition 2 and Figure 3, it is more likely that counterfeits circulate in the economy with low inflation. Put differently, for any counterfeiting environment, (k, γ), there exists a cutoff inflation rate μ_γ (or β), such that if the inflation rate is above this cutoff value, counterfeits do not exist. The intuition for this finding is as follows. When inflation is high, the cost of holding money is high so quantity of goods traded in the DM is small with genuine money. In this case, counterfeiting would be unprofitable because of its fixed cost. However, as inflation falls, the buyer can finance more DM goods with genuine money, which induces a higher incentive to produce fake money, and finally counterfeiting occurs when inflation is sufficiently low provided that $\gamma < \widehat{\gamma}(\beta, k)$.

Equilibrium Comparative Statics Propositions 1 and 2 provide comprehensive analysis of conditions under which counterfeits circulate or not in the economy. To gather more intuition for the effects of the counterfeiting environment, (k, γ), and the inflation rate, μ, on the economy, we investigate comparative statics with respect to these variables that are summarized in Table 1. Because the analysis for the other two cases are straightforward, we focus on the comparative statics in the *counterfeiting equilibrium.*

First, an increase of inflation lowers x and d because of its influence on the money holding cost (see lemma 4). Given this result, the probability of holding genuine money, α, must fall

[21]In the left panel, $\overline{k}$ satisfies $\widehat{\gamma}(\beta, \overline{k}) = \gamma$ given the assumption that $\gamma < \widehat{\gamma}(\beta, k = 0)$, and we draw the right panel with the assumption that $k < x^*$.

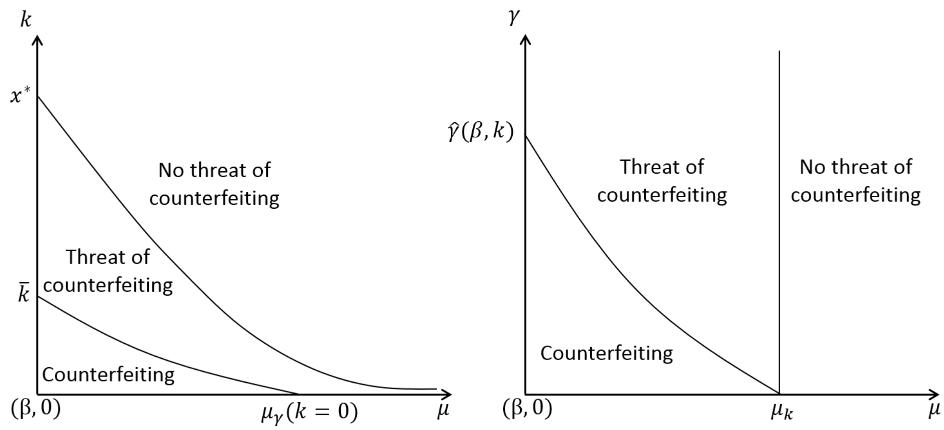

Figure 3: Equilibria with the counterfeiting environment and monetary policy

	No threat of counterfeiting				Threat of counterfeiting				Counterfeiting			
	$\frac{\partial x}{\partial \iota}$	$\frac{\partial d}{\partial \iota}$	$\frac{\partial \alpha}{\partial \iota}$	$\frac{\partial \pi}{\partial \iota}$	$\frac{\partial x}{\partial \iota}$	$\frac{\partial d}{\partial \iota}$	$\frac{\partial \alpha}{\partial \iota}$	$\frac{\partial \pi}{\partial \iota}$	$\frac{\partial x}{\partial \iota}$	$\frac{\partial d}{\partial \iota}$	$\frac{\partial \alpha}{\partial \iota}$	$\frac{\partial \pi}{\partial \iota}$
μ	−	−	·	·	−	−	·	·	−	−	−	?
k	·	·	·	·	+	+	·	·	·	·	·	−
γ	·	·	·	·	·	·	·	·	+	+	−	+

Table 1: Effects of the inflation rate and parameters in each equilibrium

with respect to μ in order to satisfy the seller's indifference condition (12). An increase in μ has an ambiguous effect on the probability of screening, π.[22]

Second, the counterfeiting cost, k, has no influence on terms of trade (x, d), and the probability of accumulating genuine money, α. It only affects the probability of screening, π, via the buyer's indifference condition (11), and π decreases with k, which is intuitive.

Finally, the effects of γ on equilibrium outcomes require more detailed analysis. From (13), we obtain

$$\frac{\partial x_\gamma}{\partial \gamma} = \frac{2\gamma x_\gamma u'(x_\gamma)}{2\gamma^2 u'(x_\gamma) - x_\gamma(x_\gamma - \gamma)(x_\gamma - 2\gamma)u''(x_\gamma)} > 0.$$

[22]We conclude this result with a numerical simulation. In general, when γ is low $\frac{\partial \pi}{\partial \mu} < 0$ whereas $\frac{\partial \pi}{\partial \mu} > 0$ when γ is relatively high. However, under some parameter values, $\pi(\mu)$ is a parabola, i.e., $\frac{\partial \pi}{\partial \mu} < 0$ for $\mu \in [\beta, \mu')$ and $\frac{\partial \pi}{\partial \mu} > 0$ for $\mu \in [\mu', \mu_\gamma)$.

Next, taking derivatives $\alpha = 1 - \frac{\gamma}{x_\gamma}$ with respect to γ, we get

$$\frac{\partial \alpha}{\partial \gamma} = \frac{(x_\gamma - \gamma)(x_\gamma - 2\gamma)u''(x_\gamma)}{2\gamma^2 u'(x_\gamma) - x_\gamma(x_\gamma - \gamma)(x_\gamma - 2\gamma)u''(x_\gamma)} < 0.$$

Thus, an increase of γ raises x but lowers α. The mechanism for these results can be found from the seller's indifference condition (12). When γ increases, the buyer must decrease α or increase x to satisfy this indifference condition. Even though higher x means more consumption in the DM, the buyer also has to consider the genuine money transfer, $d = \frac{x}{\alpha\beta}$. Because of the effects on the money transfer, buyers respond to an increase in γ with higher x but lower α in the *counterfeiting equilibrium.* The comparative statics for d come directly from the comparative statics for x and α; $\frac{\partial d}{\partial \gamma} > 0$. Last, differentiating $\pi = \frac{1}{u(x_\gamma)}\left\{\frac{\mu x_\gamma^2}{\beta(x_\gamma - \gamma)} - k\right\}$ with respect to γ, we get

$$\frac{\partial \pi}{\partial \gamma} = (1-\pi)\frac{u'(x_\gamma)}{u(x_\gamma)}\frac{\partial x_\gamma}{\partial \gamma} + \frac{1}{u(x_\gamma)}\frac{\mu}{\beta}\left(\frac{x_\gamma}{x_\gamma - \gamma}\right)^2 > 0.$$

This result appears to be counter-intuitive because one would expect that the seller will screen money less with a higher screening cost. However, as long as the seller's indifference condition (12) is satisfied, changing π does not affect the seller's expected surplus. π is adjusted to satisfy the buyer's indifference condition (11) in the *counterfeiting equilibrium.* More precisely, substituting $d = \frac{x_\gamma^2}{\beta(x_\gamma - \gamma)}$ into (11) and totally differentiating with respect to x and π, we obtain $\left\{-\frac{\mu}{\beta}\frac{x_\gamma(x_\gamma - 2\gamma)}{(x_\gamma - \gamma)^2} + \pi u'(x_\gamma)\right\}\triangle x + u(x)\triangle\pi = 0$. Then, because $-\frac{\mu}{\beta}\frac{x_\gamma(x_\gamma - 2\gamma)}{(x_\gamma - \gamma)^2} + \pi u'(x_\gamma) < 0$ by (13), π must increase with γ given $\frac{\partial x_\gamma}{\partial \gamma} > 0$.

Inflation rate and counterfeiting We close this section with a study of a relationship between the inflation rate μ and the measure of agents committing fraud, $1 - \alpha$. Proposition 2 shows that when $k < x^*$ and $\gamma < \widehat{\gamma}(\beta, k)$, the threat of counterfeiting intensifies as inflation falls, and finally counterfeiting materializes in equilibrium since μ becomes lower than μ_γ. However, $\frac{\partial(1-\alpha)}{\partial \mu} > 0$ in the *counterfeiting equilibrium.* Thus, in this environment,

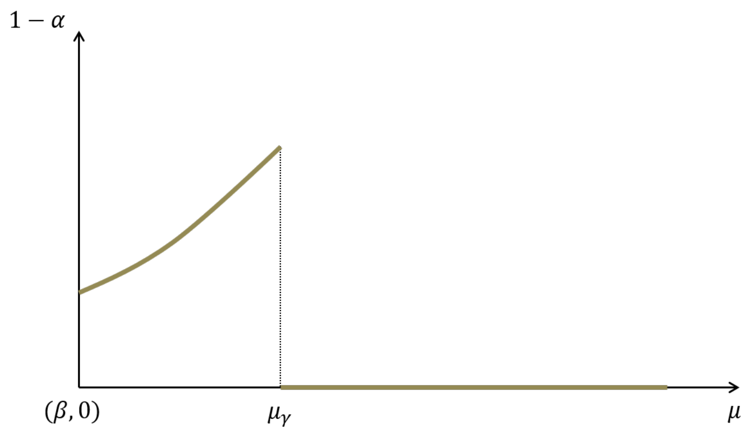

Figure 4: Measure of frauds $(1-\alpha)$ and the inflation rate μ

the measure of frauds represented by $1-\alpha$ rises as μ increases from β, but it drops suddenly to 0 once μ hits the tipping point μ_γ as described in Figure 4.[23]

This feature is useful for understanding the episodic nature of counterfeiting and the unintended results of the anti-counterfeiting policy in Canada.[24] These can be explained as an equilibrium outcome in our model using the properties that $\frac{\partial \mu_\gamma}{\partial k} < 0$ and $\frac{\partial \mu_\gamma}{\partial \gamma} < 0$. Suppose an economy is in the *threat of counterfeiting equilibrium* with the inflation rate μ close to μ_γ. If there exists a technological innovation that reduces the counterfeiting cost k such that the new critical value μ'_γ is higher than the inflation rate μ, then counterfeits rise sharply in this economy. After the government puts its effort on developing banknotes that are difficult to counterfeit or promoting screening of banknotes by retailers, then counterfeits decrease or disappear from the economy. By the same reasoning, unless the screening cost was cut down significantly such that $\gamma' \approx 0$, an inadequate reduction of γ may manifest itself as a sharp increase of counterfeits in this environment.

[23]In the discussion of Cavalcanti and Nosal (2011), Monnet (2011) shows positive relationship between inflation and counterfeiting using heterogeneous counterfeiting costs. The difference in his model is that counterfeiting increases monotonically with inflation which implies countries with hyper-inflation suffer the highest counterfeiting. However, it is quite intuitive that no one would struggle to produce fake Reichsmark, old currency of Germany, during the period from 1922 to 1923 when the country went through its worst inflation.

[24]Counterfeits of \$10 banknotes in Canada increased significantly after a launch of a new series of \$10 banknotes in 2001 (see "Bank of Canada Annual Report 2002" and "Bank of Canada Annual Report 2003" for more information).

1.4 Optimal government policy

In this section, we study optimal government policy that consists of monetary policy and anti-counterfeiting policy. Anti-counterfeiting policy determines the counterfeiting environment (k, γ). This is one of main problems that monetary authorities face whenever they develop a new series of banknotes. The government must deliberate on a reasonable trade-off between improved security and the added cost of a counterfeit deterrence measure in order to maximize welfare. In addition to anti-counterfeiting policy, counterfeiting cannot be separated from monetary policy, which affects the value of money. Therefore, these two policies must be taken into account together to find the optimal government policy.

For this purpose, we endogenize k and γ in a following way: The government taxes τ_1, τ_2, and τ_3 to buyers in a lump sum way in the CM of each period and invest in a counterfeit deterrence system to maintain $k = k(\tau_1, \tau_2)$ and $\gamma = \gamma(\tau_1, \tau_3)$. We assume that k and γ are twice continuously differentiable functions with each argument, and satisfy $k_i > 0, k_{ii} < 0, \gamma_i < 0$, and $\gamma_{ii} > 0$ where $k_i = \frac{\partial k(\tau_1,\tau_2)}{\partial \tau_i}$, for instance, with $i \in \{1, 2, 3\}$. Thus, anti-counterfeiting policy aims to improve two dimensions in the counterfeiting environment: An increase of the counterfeiting cost and a decrease of the screening cost. We further assume that $k(0, 0) = 0$, $\gamma(0, 0) = \infty$, $\lim\limits_{\tau_1 \to \infty} \lim\limits_{\tau_2 \to \infty} k(\tau_1, \tau_2) > x^*$, and $\lim\limits_{\tau_1 \to \infty} \lim\limits_{\tau_3 \to \infty} \gamma(\tau_1, \tau_2) = 0$.

For the detailed study of anti-counterfeiting policy, we introduce three types of counterfeit deterrence measures into the model to capture practices in the real world. First, observe that τ_1 affects both k and γ. We make this assumption to reflect the nature of counterfeit deterrence measures because both dimensions are inter-related. For the security features of a banknote to work as a screening device, they must be hard to counterfeit. Therefore, adding new security features to banknotes makes the screening easier while making the production of forged notes harder.

However, there are also anti-counterfeiting measures that focus only on one dimension that is captured by τ_2 and τ_3. Currently, for example, it is no longer possible to reproduce U.S. banknotes with personal computers and digital imaging tools because of a digital watermark that is embedded in banknotes.[25] This system makes counterfeiting much harder because forgers need other machines to make bogus money, but this measure does not improve the screening process. On the other hand, the main purpose of public education about the security features incorporated in banknotes and development of a portable counterfeit detector is to make the screening process easier, but these measures are not related to the production of counterfeits.

To study optimal policy, we need an aggregate welfare measure. If we measure welfare as the sum of expected utilities across agents with equal weight, we obtain

$$\mathcal{W}(\mu, \tau) = [1 - \pi(1-\alpha)][u(x) - x] - (1-\alpha)k(\tau_1, \tau_2) - \pi\gamma(\tau_1, \tau_3) - \tau_1 - \tau_2 - \tau_3, \tag{18}$$

where $\tau = (\tau_1, \tau_2, \tau_3)$. As in most Lagos and Wright (2005) setups, CM activities that involve money trades cancel out. However, the labor input to produce fake money in the CM and the labor for screening in the DM do not improve any others consumption, so they only reduce welfare. Also, when counterfeiting and screening exist, missing trade surplus, captured by $\pi(1-\alpha)[u(x) - x]$, is the deadweight loss of welfare.

As shown in propositions 1 and 2, the counterfeiting cost, k, the screening cost, γ, and monetary policy, μ, interact with each other in determining the equilibrium type, and each element has different effects on the economy depending on the other two elements. Therefore, in order to study optimality, we have to consider all policy measures together. However, understanding how optimal monetary policy depends on the counterfeiting environment is also of interest to the government. Thus, we first take τ as given, and find $\mu^*(\tau)$ that is obtained by

$$\mu^*(\tau) \in \arg\max_{\mu \geq \beta} \widetilde{\mathcal{W}}(\mu; \tau) \equiv [1 - \pi(1-\alpha)][u(x) - x] - (1-\alpha)k - \pi\gamma$$

[25] A counterfeit deterrence system (CDS) has been developed by the Central Bank Counterfeit Deterrence Group (CBCDG) to deter the use of personal computers, digital imaging equipment, and software in the counterfeiting of banknotes in 2004. For more information, visit their official website (http://www.rulesforuse.org).

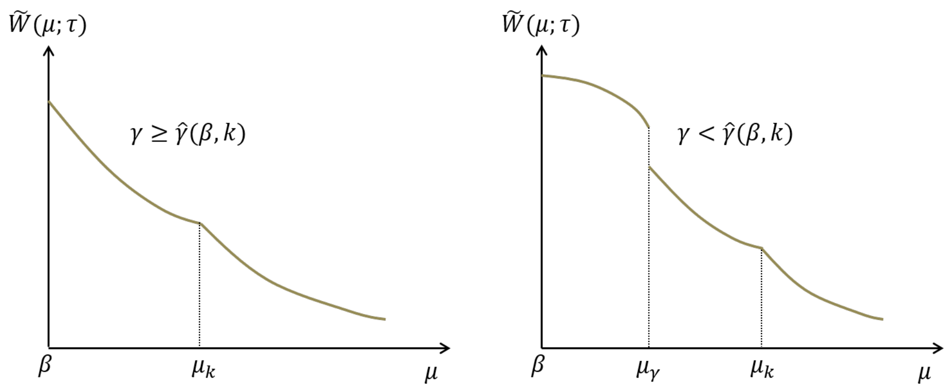

Figure 5: Welfare $\widetilde{\mathcal{W}}(\mu;\tau)$ and the inflation rate

subject to the equilibrium conditions described in proposition 1.[26] Then, the optimal government policy (μ^*, τ^*) is given by $(\mu^*(\tau^*), \tau^*)$ where τ^* maximizes $\mathcal{W}(\mu^*(\tau), \tau)$ subject to agents' optimal behaviors.

Since $k \leq x^*$ must hold with optimal government policy, we assume this condition in the following analysis without loss of generality. Suppose first that $\gamma \geq \widehat{\gamma}(\beta, k)$. Then, the economy is in either the *no threat of counterfeiting equilibrium* or the *threat of counterfeiting equilibrium* depending on μ. Substituting the allocation (x, d, α, π) under each equilibrium to $\widetilde{\mathcal{W}}(\mu;\tau)$, we obtain

$$\widetilde{\mathcal{W}}_n(\mu;\tau) = u\left(u'^{-1}\left(\frac{\mu}{\beta}\right)\right) - u'^{-1}\left(\frac{\mu}{\beta}\right) \text{ for all } \mu \geq \mu_k$$
$$\widetilde{\mathcal{W}}_t(\mu;\tau) = u\left(\frac{\beta k}{\mu}\right) - \frac{\beta k}{\mu} \text{ for all } \mu \in [\beta, \mu_k).$$

Notice that $u'^{-1}\left(\frac{\mu}{\beta}\right)$ and $\frac{\beta k}{\mu}$ are less than x^* for any $\mu \geq \beta$, so both $\widetilde{\mathcal{W}}_n(\mu;\tau)$ and $\widetilde{\mathcal{W}}_t(\mu;\tau)$ are decreasing in μ. Next, by definition of μ_k, $\widetilde{\mathcal{W}}_n(\mu_k;\tau) = \widetilde{\mathcal{W}}_t(\mu_k;\tau)$. Given these two observations, we get the left panel of Figure 5 that represents $\widetilde{\mathcal{W}}(\mu;\tau)$ as a function of μ when $\gamma \geq \widehat{\gamma}(\beta, k)$.

The story becomes richer when $\gamma < \widehat{\gamma}(\beta, k)$. In this case, the economy can be in all types

[26]Since we assume that τ is given at this moment, we use k and γ instead of $k(\tau_1, \tau_2)$ and $\gamma(\tau_1, \tau_3)$, and drop $-(\tau_1 + \tau_2 + \tau_3)$ terms in (18).

of equilibria depending on μ. By the similar arguments above, $\widetilde{\mathcal{W}}(\mu;\tau)$ is monotonically decreasing in μ for all $\mu \geq \mu_\gamma$. Substituting the expression of α and π from proposition 1 into $\widetilde{\mathcal{W}}(\mu;\tau)$, we obtain

$$\widetilde{\mathcal{W}}_c(\mu;\tau) = u(x_\gamma) - x_\gamma - \frac{\mu}{\beta}\frac{\gamma x_\gamma}{x_\gamma - \gamma} \text{ for all } \mu \in [\beta, \mu_\gamma).$$

First, taking derivatives the above expression with respect to μ and solving, we get

$$\frac{\partial \widetilde{\mathcal{W}}_c}{\partial \mu} = \frac{\mu - \beta}{\beta}\frac{\partial x_\gamma}{\partial \mu} - \frac{\gamma x_\gamma}{\beta(x_\gamma - \gamma)} < 0,$$

so $\widetilde{\mathcal{W}}(\mu;\tau)$ is decreasing in μ for all $\mu \in [\beta, \mu_\gamma)$. Second, by using the expression $-\frac{\mu_\gamma}{\beta}\frac{x_\gamma^2}{x_\gamma - \gamma} + u(x_\gamma) = -k + u\left(\frac{\beta k}{\mu_\gamma}\right)$, from (17), we obtain

$$\widetilde{\mathcal{W}}_c(\mu_\gamma;\tau) - \widetilde{\mathcal{W}}_t(\mu_\gamma;\tau) = \frac{\mu_\gamma - \beta}{\beta}\left(x_\gamma - \frac{\beta}{\mu_\gamma}k\right) > 0$$

because $x_\gamma > \frac{\beta}{\mu_\gamma}k$. Thus, there is a jump in $\widetilde{\mathcal{W}}(\mu;\tau)$ at $\mu = \mu_\gamma$ as depicted in the right panel of Figure 5 when $\gamma < \widehat{\gamma}(\beta, k)$.

It seems worthwhile to spend a little time on the multiplicity of equilibrium when $\gamma = \widehat{\gamma}(\beta, k)$. In this knife edge case, we simply assumed that the economy is in the *threat of counterfeiting equilibrium* in propositions 1 and 2. However, in principle both equilibria are possible, and more interestingly, given the same counterfeiting environment and the same inflation rate at μ_γ, the economy with counterfeiting achieves higher welfare than the one where counterfeits do not circulate. This is because the economy that admits counterfeiting supports greater trade size in the DM. However, one should not interpret this result as that counterfeiting improves welfare by working as liquidity provision. Counterfeiting occurs as a strategic outcome. Welfare is higher if counterfeiting was not possibility.

Perhaps the most interesting result in Figure 5 is that $\widetilde{\mathcal{W}}(\mu;\tau)$ is monotonically decreasing in μ for all cases: $\gamma \gtreqless \widehat{\gamma}(\beta, k)$. Thus, optimal monetary policy is the Friedman rule independent of anti-counterfeiting policy, which is re-emphasized as the next proposition.

Proposition 3 *Optimal monetary policy is the Friedman rule independent of anti-counterfeiting policy:* $\mu^*(\tau) = \beta$ *for all* $\tau \in \mathbb{R}^3_+$.

The proposition that the Friedman rule is optimal is very robust in monetary theory.[27] Conventional logic for this statement is that the Friedman rule makes an inter-temporal cost of holding money zero, so it maximizes the trade surplus, $u(x) - x$, in the DM by supporting the efficient amount of trade, x^*. In our model, when $\gamma \geq \widehat{\gamma}(\beta, k)$ and hence counterfeiting does not occur for all $\mu \geq \beta$, the Friedman rule is optimal in this conventional way.

The rationale behind the Friedman rule, however, is quite different when $\gamma < \widehat{\gamma}(\beta, k)$. In the *counterfeiting equilibrium*, a decrease in inflation increases the quantity of goods traded in the DM, x_γ, by reducing the cost of holding money. Furthermore, welfare also increases as inflation falls (see the right panel of Figure 5). Thus, is the Friedman rule optimal because it maximizes the trade surplus? The consumption in the DM when $\mu = \beta$ is strictly higher than the efficient level x^* (see lemma 4), which implies $x_\gamma > x^*$ with $\mu \approx \beta$. Thus, lowering μ near β decreases the trade surplus, $u(x_\gamma) - x_\gamma$. Then, why does the Friedman rule maximize welfare here? This is because as the inflation rate μ falls, the measure of buyers who commit forgery decreases as described in Figure 4, so does the welfare costs from counterfeiting.

We are now ready to find the optimal government policy (μ^*, τ^*) by maximizing $\mathcal{W}(\mu^*(\tau), \tau)$. By proposition 3, it suffices to let $\mu^*(\tau) = \beta$ to find τ^*. However, to obtain more general idea, we take the monetary policy μ as given and find $\tau^*(\mu)$ in a similar way to derive $\mu^*(\tau)$. To analyze the effects of τ on welfare, one has to consider the agents' optimal behaviors that determine the equilibrium type. More precisely, the maximized welfare given μ is $\mathcal{W}^*(\tau; \mu) = Max\,\{\mathcal{W}^*_t(\mu), \mathcal{W}^*_c(\mu)\}$ where

$$(Pt) \qquad \mathcal{W}^*_t(\mu) = \max_{\tau \in \mathbb{R}^3_+} \left\{ u\left(\frac{\beta k(\tau_1, \tau_2)}{\mu}\right) - \frac{\beta k(\tau_1, \tau_2)}{\mu} - \tau_1 - \tau_2 - \tau_3 \right\}$$

[27] There are some papers, however, that study conditions where deviation from the Friedman rule is optimal in monetary models (see, e.g., Sanches and Williamson 2010).

subject to

$$u\left(\frac{\beta k(\tau_1,\tau_2)}{\mu}\right) - k(\tau_1,\tau_2) \geq u(x_\gamma) - \frac{\mu}{\beta}\frac{x_\gamma^2}{x_\gamma - \gamma} \tag{19}$$

and

$$\mathcal{W}_c^*(\mu) = \max_{\tau\in\mathbb{R}_+^3}\left\{u(x_\gamma) - x_\gamma - \frac{\mu}{\beta}\frac{\gamma x_\gamma}{x_\gamma - \gamma} - \tau_1 - \tau_2 - \tau_3\right\} \tag{Pc}$$

subject to

$$u(x_\gamma) - \frac{\mu}{\beta}\frac{x_\gamma^2}{x_\gamma - \gamma} \geq u\left(\frac{\beta k(\tau_1,\tau_2)}{\mu}\right) - k(\tau_1,\tau_2). \tag{20}$$

The constraint (19) is the condition for the economy to be in the *threat of counterfeiting equilibrium* whereas the constraint (20) is the condition for the existence of the *counterfeiting equilibrium.*

Let $\tau^t(\mu)$ and $\tau^c(\mu)$ be a solution of (Pt) and (Pc) respectively. Then, optimal anti-counterfeiting policy given μ, $\tau^*(\mu)$, is

$$\tau^*(\mu) = \begin{cases} \tau^t(\mu) \text{ if } \mathcal{W}_t^*(\mu) \geq \mathcal{W}_c^*(\mu) \\ \tau^c(\mu) \text{ if } \mathcal{W}_t^*(\mu) < \mathcal{W}_c^*(\mu). \end{cases}$$

Whether $\mathcal{W}_t^*(\mu) \gtreqless \mathcal{W}_c^*(\mu)$ and the form of $\tau^*(\mu)$ depend on particular functions of $k(\tau_1,\tau_2)$, $\gamma(\tau_1,\tau_3)$, and $u(x)$. However, a certain rule of optimal anti-counterfeiting policy can be found by making two observations. First, welfare under the *threat of counterfeiting equilibrium* depends on the counterfeiting cost but not on the screening cost. Lowering the screening cost only makes the constraint (19) tighter. Thus, any solution to (Pt) must feature $\tau_3 = 0$ because the objective function can be increased otherwise. Second, welfare under the *counterfeiting equilibrium*, on the other hand, depends on the screening cost but not on the counterfeiting cost. Increasing the counterfeiting cost only tightens the constraint (20) which implies that it must be $\tau_2 = 0$ to solve (Pc). Therefore, the government should focus on either increasing the counterfeiting cost or decreasing the screening cost, so the optimal anti-counterfeiting policy is dichotomous which is formalized in the next proposition.

Proposition 4 *If $\tau^*(\mu) = \tau^t(\mu)$, then $\tau_3^*(\mu) = 0$, whereas if $\tau^*(\mu) = \tau^c(\mu)$, then $\tau_2^*(\mu) = 0$.*

After deriving $\tau^*(\mu)$, the optimal government policy is simply given by $(\beta, \tau^*(\beta))$ by virtue of proposition 3. The structure of (β, τ^*) hinges on a form of cost functions and agent's preference. For example, if $\lim_{\tau_1 \to 0} k_1 = \infty$ and $\lim_{\tau_1 \to 0} \gamma_1 = -\infty$, then the optimal government policy (β, τ^*) features $\tau_1^* > 0$ in all cases, $\mathcal{W}_t^* \gtreqless \mathcal{W}_c^*$, which means that security features must always be incorporated into banknotes. Observe that whenever $\tau^* = \tau^c(\beta)$, counterfeits circulate in the economy under the optimal government policy.

1.5 Conclusion

In this paper, we have constructed a search theoretic model of money to analyze how counterfeiting affects economic activity and to study optimal government policy. We have shown that there are three types of equilibria when an economy is susceptible to counterfeiting. When the cost of counterfeiting is high enough, the incentive to produce fake money does not exist. When the counterfeiting cost is not too high but the screening cost is relatively high, then the potential threat of counterfeiting generates a resalability constraint on money thereby affecting the allocation, but there are still no counterfeits in equilibrium. Finally, when both the counterfeiting cost and the screening cost are sufficiently low, counterfeiting occurs in equilibrium. We also show that it is more likely, in this economy, that counterfeiting occurs when inflation is low because of its effects on the value of money.

We also used the model to provide implications for government policy. First, the Friedman rule is optimal independent of anti-counterfeiting policy. When counterfeits do not exist in equilibrium, the Friedman rule is optimal because it maximizes the trade surplus. We add new insight here: When counterfeits do exist in equilibrium, the Friedman rule is optimal not because it maximizes the trade surplus but because it minimizes the welfare cost of counterfeiting. The trade surplus is not maximized with the Friedman rule in this case. Second, the model suggests that anti-counterfeiting policy must focus on improvement along one of two dimensions—increasing the counterfeiting cost or decreasing the screening cost—but not both.

A natural extension of our analysis would be to relax some of the assumptions of our model. For example, money is the only possible medium of exchanges in the model. In practice, however, other financial assets, such as government bonds and asset backed securities that are also subject to moral hazard problems, are widely used as a medium of exchanges. It would be interesting to introduce our description of screening into the counterfeiting model with multiple real assets as in Li, Rocheteau, and Weill (2012), and study how anti-counterfeiting policy for a particular asset or the supply of safe assets that are immune to faking into the economy affects counterfeiting of other types of assets. Another potentially interesting extension would be to assume that the central bank determines the rate of confiscation of counterfeits in contrast to 100% confiscation in our model, and to study how this decision affects counterfeiting and welfare.

1.6 Appendix: Proofs

Proof of Lemma 1. Suppose there exists an equilibrium in which the buyer accumulates m_t units of legal money and m_t^c units of counterfeits both in terms of CM goods of period $t+1$, makes the offer (x_t, d_t), and hands over $\widehat{d}_t > 0$ units of genuine money and $\widehat{d}_t^c > 0$ units of legal money where $\widehat{d}_t + \widehat{d}_t^c = d_t$. Then, given $u(x_t^s) - \beta d_t^s = 0$, the buyer's payoff is

$$S_t^b = -\left(\frac{\phi_t}{\phi_{t+1}} - \beta\right) m_t - k + \mathtt{I}_{\{r=A\}}(1 - \mathtt{I}_{\{s=Y\}}) \left[u(x_t) - \beta \widehat{d}_t\right].$$

Now consider another set of actions such that $m' = m_t - \widehat{d}_t$, $\widehat{d'} = 0$, and $\widehat{d^{c}}' = d_t$, and the buyer makes the same offer (x_t, d_t). The buyer's payoff with this choice of actions is

$$S_t^{b\prime} = S_t^b + \left\{ \frac{\phi_t}{\phi_{t+1}} - \beta \left[1 - \mathtt{I}_{\{r=A\}}(1 - \mathtt{I}_{\{s=Y\}}) \right] \right\} \widehat{d}_t > S_t^b,$$

so we get a contradiction. Thus, it must be $d_t = \widehat{d}_t$ or $d_t = \widehat{d}_t^c$. The proof of remaining parts is straightforward. ■

Proof of Lemma 2. We first prove that for any solution to (P), $\pi < 1$. Suppose $\pi = 1$ which requires $\alpha \in (0,1)$ by (5). Then, substituting the indifference condition (4) to the objective function, we obtain $\overline{S}^b = -k < 0$, a contradiction. Next, note that whenever $\alpha = 1$, it must be $\pi = 0$ by (5). Finally, we show that for any solution to (P), $\alpha \in (0,1)$ if and only if $\pi \in (0,1)$. The "*if*" part is trivial so we focus on proving the "*only if*" part by showing a contradiction otherwise. Suppose there exists a solution $(x, d, \alpha, \eta, \pi)$ of (P) with $\alpha < 1$ but $\pi = 0$. Then, the indifference condition to have $\alpha \in (0,1)$ is $[\mu - (1-\eta)\beta]\, d = k$. Now, consider $d' = \alpha d < d$, so $[\mu - (1-\eta)\beta]\, d' < k$. Then, $(x, d', \alpha' = 1, \eta, \pi' = 0)$ satisfies constraints (4) and (5), and attains higher value of the objective function than $(x, d, \alpha, \eta, \pi)$, a contradiction. ■

Proof of Lemma 3. 1) We first prove that $\eta = 1$. To find a contradiction, suppose there exists a solution $(x, d, \eta, \alpha, \pi)$ of (P) with $\eta < 1$. Since $\pi[(1-\alpha)x - \gamma] = 0$ with any solution by lemma 2, the indifference condition to have $\eta \in (0,1)$ is $x = \alpha\beta d$. Assume that $\alpha = 1$ so $\pi = 0$ and $x = \beta d$. Consider $\eta' = \eta + \epsilon < 1$ and $d' = d - \delta > 0$ where $\delta = \frac{\beta d \epsilon}{\mu - \beta + (\eta + \epsilon)\beta}$, so $\lim_{\epsilon \to 0} \delta = 0$. Let $x' = \beta d'$. Then, $(x', d', \alpha' = 1, \eta', \pi' = 0)$ satisfies constraints (4) and (5). However, the change in the objective function is

$$\begin{aligned} \triangle \overline{S}^b &= \eta' u(x') - \eta u(x) \\ &\approx (\eta + \epsilon) \left\{ u(x) - u'(x) \frac{\beta x \epsilon}{\mu - \beta + \beta(\eta + \epsilon)} \right\} - \eta u(x) > 0 \end{aligned}$$

with sufficiently small $\epsilon > 0$. Therefore, for the solution with $\eta < 1$ to exist, it must be that $\alpha \in (0,1)$ and $\pi \in (0,1)$. Now consider $\eta' = \eta + \epsilon$ and $\pi' = \pi - \delta$ where $\delta = \frac{\epsilon[\pi u(x) - \beta d]}{(\epsilon+\eta)u(x)}$. Since $\eta \in (0,1)$ and $\pi \in (0,1)$, there exists sufficiently small $\epsilon > 0$ such that $\eta' \in (0,1)$ and $\pi' \in (0,1)$. Then, $(x, d, \eta', \alpha, \pi')$ satisfies all constraints in (P) but delivers higher value of the objective function, a contradiction.

2) If $x > \alpha\beta d$, then $\eta = 0$ given the result of lemma 2. Thus, to prove $x = \alpha\beta d$, suppose there exists a solution $(x, d, \eta, \alpha, \pi)$ of (P) with $x < \alpha\beta d$. Assume $\alpha = 1$. Then, there exists $d' < d$ such that $x < \beta d'$. Then, $(x, d', \eta, \alpha, \pi)$ satisfies (4) and (5) but attains higher value of the objective function. Thus, the only possible case with $x < \alpha\beta d$ is that $\alpha \in (0,1)$ and $\pi \in (0,1)$. In this case, we can find $x' > x, \alpha' > \alpha$, and $\pi' < \pi$ such that 1) $x' < \alpha'\beta d$, 2) $\pi' u(x') = \pi u(x)$, and 3) $(1-\alpha')x' = \gamma$. Then, given (x', d), (η, α', π') is consistent with (4) and (5), but $(x', d, \eta, \alpha', \pi')$ gives higher value of the objective function, a contradiction. ■

Proof of Lemma 4. Rearranging (7), (11), and (12), we get $d = \frac{x^2}{\beta(x-\gamma)}, \alpha = 1 - \frac{\gamma}{x}$, and $\pi = \frac{1}{u(x)}\left\{\frac{\mu}{\beta}\frac{x^2}{x-\gamma} - k\right\}$. Substituting the expression for d into the objective function of (P'), we obtain $-\frac{\mu}{\beta}\frac{x^2}{x-\gamma} + u(x)$. This is a strictly concave function for all $x > \gamma$, which must be satisfied to have $\alpha > 0$. Then, we get the first order condition with respect to x that is (13). Because the right hand side of (13) is negative with $x < 2\gamma$, we can restrict attention to $x > 2\gamma$. Note that the right hand side of (13) is a decreasing function of x_γ, and tends to ∞ as $x_\gamma \to 2\gamma$, and to 0 as $x_\gamma \to \infty$. Thus, there exists unique $x_\gamma \in (2\gamma, \infty)$ that solves (13). Furthermore, because $\frac{(x-\gamma)^2}{x(x-2\gamma)}u'(x) > u'(x)$ for all $x > 2\gamma$, $x_\gamma > u'^{-1}\left(\frac{\mu}{\beta}\right)$. Next, for x_γ to be the candidate solution, it must be that $\alpha \in (0,1)$ and $\pi \in (0,1)$. First, because $x_\gamma > 2\gamma$, $\alpha \in (\frac{1}{2}, 1)$ for all $\gamma > 0$. Second, because $x_\gamma > u'^{-1}\left(\frac{\mu}{\beta}\right)$, $\pi > 0$ given the assumption that $k < \frac{\mu}{\beta}u'^{-1}\left(\frac{\mu}{\beta}\right)$. Finally, given $-\frac{\mu}{\beta}\frac{x_\gamma^2}{x_\gamma - \gamma} + u(x_\gamma) \geq 0$, $\pi < 1$. ■

Proof of Proposition 1. Since we already characterized a solution of (P), we only prove that equilibrium can be characterized by solving (P). Given a solution $(x, d, \alpha, \eta, \pi)$, suppose $\{\alpha, [\eta, \pi]\}$ is a unique Pareto dominant Nash equilibrium of the subgame $\Gamma(x, d)$. Then

$(x, d, \alpha, \eta, \pi)$ is an equilibrium outcome by construction of (P). Also, any equilibrium must solve (P) because the buyer can attain higher payoff by posting (x, d) that solves (P) otherwise. Thus, it suffices to show that if $(x, d, \alpha, \eta, \pi)$ is a solution to (P), then $\{\alpha, [\eta, \pi]\}$ is the unique Pareto dominant Nash equilibrium of $\Gamma(x, d)$.

Suppose $(x, d, \alpha, \eta, \pi)$ solves (P). As argued above, $x > 0$ and $\eta = 1$. Thus, any Nash equilibrium $\{\alpha', [\eta', \pi']\}$ of $\Gamma(x, d)$ with $\alpha' = 0$ is Pareto dominated by $\{\alpha, [\eta, \pi]\}$ because it must be that $\eta' = 0$. Since any solution to (P) has either $\alpha = 1$ or $\alpha \in (0, 1)$, we consider each case separately.

1) First, suppose that $\alpha = 1$ so $\pi = 0$. Thus, it must be $\mu d \leq k$ and $x = \beta d$. Consider any Nash equilibrium $\{\alpha', [\eta', \pi']\} \neq \{\alpha, [\eta, \pi]\}$ of $\Gamma(x, d)$. i) Assume $\alpha' = 1$. Then, the only Nash equilibrium, if it exists, is with $\eta' < 1$, so $\{\alpha', [\eta', \pi']\}$ is Pareto dominated by $\{\alpha, [\eta, \pi]\}$. ii) Assume $\alpha' \in (0, 1)$ which requires $\mu d - (1 - \eta')\beta d = k + \eta'\pi' u(x)$ by (4). This is possible only if $\mu d = k$ with $\eta' = 1$ and $\pi' = 0$. However, if $\pi' = 0$, then it must be $\eta' = 0$ because $-x + \alpha'\beta d < 0$, a contradiction. Therefore, if $\alpha = 1$ and $\pi = 0$, then $\{\alpha, [\eta, \pi]\}$ is the unique Pareto dominant Nash equilibrium of the subgame $\Gamma(x, d)$.

2) Second, suppose that $\alpha \in (0, 1)$ and $\pi \in (0, 1)$. Then, it must be $\mu d = k + \pi u(x)$, $(1 - \alpha)x = \gamma$, and $x = \alpha\beta d$. Now consider any Nash equilibrium $\{\alpha', [\eta', \pi']\} \neq \{\alpha, [\eta, \pi]\}$ of $\Gamma(x, d)$. i) Suppose $\alpha' \in (\alpha, 1]$. Then, the best response of the seller is $\eta' = 1$ and $\pi' = 0$. But the buyer's best response is, then, $\alpha' = 0$, because $\mu d > k$, a contradiction. ii) Consider the case that $\alpha' = \alpha$. If $\eta' < 1$, $\{\alpha', [\eta', \pi']\}$ is Pareto dominated by $\{\alpha, [\eta, \pi]\}$, and hence assume $\eta' = 1$. However, the only Nash equilibrium with $\alpha' = \alpha$ and $\eta' = 1$ is with $\pi' = \pi$, a contradiction. iii) Finally, assume $\alpha' \in (0, \alpha)$. By the same reason above, assume $\eta' = 1$. Then, it must be $\alpha'\beta d - \alpha' x - \gamma \geq 0$ with $\pi' = 1$. Otherwise $\eta' = 0$ because $\alpha' < \alpha$. However, if $(\eta', \pi') = (1, 1)$, then $\alpha' = 1$ because $\mu d < k + u(x)$, a contradiction. Therefore, if $\alpha \in (0, 1)$ and $\pi \in (0, 1)$, then $\{\alpha, [\eta, \pi]\}$ is the unique Pareto dominant Nash equilibrium of the subgame $\Gamma(x, d)$. $\blacksquare$

Chapter 2

Central Bank Purchases of Private Assets: An Evaluation

2.1 Introduction

In response to the Great Recession and its aftermath, the Fed embarked on unconventional monetary policy in the form of large-scale asset purchases, also known as quantitative easing (QE). Through successive rounds of QE, the Fed purchased, over time, long term government bonds, agency debt, and mortgage backed securities (MBS), dramatically increasing the size of the Fed's balance sheet.[28] These unconventional policy actions have generated a substantial debate in the economics profession about the effects of QE on market interest rates and the real economy. Yet the precise mechanism through which QE has affected economic activities is still not well understood.[29] The following questions still need to be addressed: How does unconventional monetary policy affect market interest rates, the incentive structure of the private sector, and real economic activities? What is the relationship between conventional and unconventional monetary policy? Can unconventional monetary policy substitute a conventional monetary policy? Is there any risk of implementing QE, and under what conditions does it improve or lower welfare?

In this paper we attempt to deal with the above questions by constructing a New Monetarist model.[30] More precisely, we study the effects of the central bank's purchases of private assets such as MBS by incorporating housing construction and asymmetric information con-

[28] During the periods from 2008 to 2014, the overall size of the Fed's balance sheet increased more than five times (Source: Board of Governors of the Federal Reserve System, Release H.4.1).

[29] Moreover, most empirical studies focused on the effects of QE on yield spreads without proper attention to its effects on real economic activities (See Williams 2011 for the survey of empirical studies).

[30] A discussion of the New Monetarist approach is in Williamson and Wright (2010, 2011) and Lagos et al. (forthcoming), and a textbook treatment is in Nosal and Rocheteau (2011).

cerning the quality of private financial assets in a model of asset exchange. In the model, the central bank's private asset purchases can relax the incentive problem faced by private banks but, at the same time, the quantity of the central bank's private asset purchases is limited by the incentive problem. Furthermore, the effects of the private asset purchase program on macroeconomic activities and welfare crucially depend on the severity of the incentive problem.

In the model, there is a fundamental role for exchange using currency and exchange with secured credit in a decentralized market, as a result of limited commitment and lack of record keeping. Then, in equilibrium financial intermediation is a type of insurance arrangement by which banks efficiently allocate liquid assets, in the form of currency and a claim on a bank, to the appropriate transactions. However, banks are inherently untrustworthy. Limited commitment implies that banks face collateral constraints, according to which banks' deposit liabilities must be secured by other financial assets.

Primitive assets in the model are government liabilities (currency, reserves, and nominal government bonds) and houses constructed by private agents. Though houses cannot be directly used as a collateral by banks, houses are useful in exchange in the decentralized market indirectly. Homeowners can take out mortgages with banks using residential properties as collateral, and then banks can pledge mortgage loans as collateral to secure their deposit claims. However, the usefulness of mortgages as collateral is limited by the threat of fraud. More precisely, banks can produce fake mortgage loans and post them as collateral at a cost. This generates an incentive problem faced by banks similar to Li, Rocheteau, and Weill (2012) and Williamson (2016), though there are key differences in the nature of the incentive problem.

We first study the effects of conventional monetary policy determining a nominal interest rate of short-maturity government bonds, on equilibrium quantities, prices, and welfare. In particular, we show that a zero nominal interest rate is optimal, at least locally, if the bank's incentive constraint binds with the low fraud cost, whereas it could be suboptimal if the incentive constraint does not bind. This is because when the incentive constraint does not bind, lowering a nominal interest rate causes a housing construction boom above the efficient level, but conventional monetary policy does not affect the housing market when the incentive constraint binds.

Next, We examine the effects of QE as unconventional monetary policy. In the model, the central bank purchases mortgage loans from banks at the market price.[31] Unlike conventional monetary policy, QE has effects on the economy only if the bank's incentive constraint binds and a rate of return differential exists between government bonds and mortgages. In this economy, government bonds and reserves act to discourage fraud by making default more costly.[32] Thus, the central bank's exchange of reserves for mortgages mitigates the incentive problem in the banking sector, and lowers the yield spread between mortgages and government bonds, consistent with the empirical findings of Hancock and Passmore (2011). In particular, the central bank can make the incentive constraint slack by purchasing a sufficient quantity of mortgages. However, the effects of the central bank's mortgage purchases on real allocations and welfare depend on whether the collateral constraint binds, which, in turn, hinges on the cost of fraud.

First, if the cost of fraud is below some threshold level, the bank's collateral constraint does not bind, and only the incentive constraint binds. In this case, the incentive problem is so severe that there are illiquid mortgages that are not used as collateral. Under QE, the central bank purchases these illiquid mortgages in exchange for reserves, and the quantity of trading in the decentralized market increases as a result. However, the central bank's mortgage purchases do not affect the mortgage price, housing price, and housing construction, because mortgages are illiquid at the margin. Therefore, in this equilibrium, the central bank's mortgage purchases unambiguously improve welfare.

Second, when the cost of fraud is above the threshold level, the bank's collateral con-

[31]Throughout this article, QE and the central bank's mortgage purchases are considered synonymous.

[32]Carapella and Williamson (2015) show a similar result that the introduction of government debt discourages default on an unsecured loan in a limited commitment model.

straint and incentive constraint both bind. In this case, the central bank's mortgage purchases have different effects on these two binding constraints. An increase of central bank's mortgage purchases tightens the binding collateral constraint because of the rate of return difference, which forces the quantity of exchange to fall. On the other hand, mortgage purchases relax the incentive constraint, which raises the quantity of exchange. In addition, the central bank's mortgage purchases boost investment in housing construction in the private sector, which increases the aggregate quantity of collateralizable assets. On net, it is not clear whether trades in the decentralized market increase or decrease when the central bank increases mortgage purchases. The effects on welfare are also ambiguous. Central bank mortgage purchases can lower welfare even when the quantity of exchange increases in the decentralized market, because mortgage purchases cause too much investment in housing construction. Under some circumstances, it is optimal for the central bank not to implement QE.

To be sure, there has been other research that studies the effects of QE theoretically. For example, Curdia and Woodford (2011), Del Negro et al. (2011), Gertler and Karadi (2011, 2013), and Gertler et al. (2012) quantitatively study the effects of QE by extending a standard New Keynesian model to include financial frictions. In their model, the central bank directly invests in private assets under "QE" that looks more like a credit market intervention by the fiscal authority. Boel and Waller (2015) and Williamson (2015, 2016, forthcoming) studied QE as asset purchases by the central bank much like the Fed's policy intervention in the United States. In their model, QE has effects on the economy by changing the composition of exchangeable assets in the private sector.

Most of those studies focused on the channel under which QE spurs economic activity without a consideration of its potential risk and welfare costs.[33] Gu and Haslag (2014) constructed an overlapping generations model and examined the effects of the central bank's private debt purchase when verifiability of private debt and a timing mismatch in debt

[33]Though New Keynesian models assumed an exogenous welfare cost of the central bank's direct lending, they are silent about where this welfare cost comes from.

settlements lead to a liquidity problem. They are interested in related issues, but approach the problem in a very different way. Williamson (2015) studied conditions under which an increase of the central bank's balance sheet with purchases of government bonds can lower welfare, but he did not examine the effects of private asset purchases.

The rest of the paper is organized as follows. Section 2 presents the environment of the model. Section 3 solves economic agents' problems, section 4 characterizes equilibrium. In section 5, we study the effects of monetary policy. Section 6 is the conclusion.

2.2 The Environment

The basic structure in the model is built on Lagos and Wright (2005) with heterogeneous agents similar to Lagos and Rocheteau (2005) and Rocheteau and Wright (2005). Time is indexed by $t = 0, 1, 2 \ldots$, and there are two subperiods within each period; the centralized market (CM) followed by the decentralized market (DM).

There exists a continuum of buyers, sellers, and bankers each with unit mass. Each buyer has preference given by

$$E_0 \sum_{t=0}^{\infty} \beta^t [X_t + \upsilon(f_t) - H_t + u(x_t)],$$

each seller has preference given by

$$E_0 \sum_{t=0}^{\infty} \beta^t [X_t - H_t - h_t],$$

and the preference of bankers who run banks is

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[X_t - H_t\right].$$

Here, $\beta \in (0, 1)$ is the discount rate, X_t and H_t are consumption and labor supply, respectively, in the CM, f_t is consumption of housing services in the CM, x_t is consumption in the DM, and h_t is labor supply in the DM. We assume that $u(\cdot)$ is a strictly increasing, strictly concave, and twice continuously differentiable function with $u(0) = 0, u'(0) = \infty, u'(\infty) = 0$, $-x\frac{u''(x)}{u'(x)} < 1$ for all $x \geq 0$, and with the property that there exists some $\widehat{x}$ such that $u(\widehat{x}) = \widehat{x}$. Let x^* denote the efficient quantity, which solves $u'(x^*) = 1$. The utility for housing services is increasing and concave with $\upsilon'(0) = \infty$, $\upsilon'(\infty) = 0$, and $-f\frac{\upsilon''(f)}{\upsilon'(f)} \leq 1$ for all $f \geq 0$.

The production technology of consumption goods available to buyers, sellers, and bankers allows the production of one unit of the perishable consumption good for each unit of labor supply. In addition to the linear production technology for consumption goods, there is a technology to construct houses. Though it does not matter for the equilibrium analysis, we assume that only sellers can access the construction technology: Sellers, in the CM, can produce I units of new houses at cost of $\chi(I)$ units of CM goods, and houses are traded at the price ψ_t in terms of the CM goods at a competitive market in the CM of period t. We assume that $\chi(\cdot)$ is a strictly increasing, strictly convex, and twice continuously differentiable function with $\chi(0) = \chi'(0) = 0$ and $\chi'(\infty) = \infty$. Each unit of housing provides y units of housing services at the beginning of the CM and housing services can be traded in a competitive rental market at the price R_t in terms of CM goods in period t.[34] We assume that houses depreciate by 100% after yielding housing services. We make this assumption to simplify the welfare analysis with a welfare measure that is widely used in a money search model, but the equilibrium characterization and main welfare implications do not hinge on this assumption.[35]

As well, homeowners can borrow in the form of a mortgage from a bank using houses as collateral in the CM. A mortgage is a promise to pay one unit of CM goods in the CM

[34] As explained in Branch, Petrosky-Nadeau, and Rocheteau (forthcoming), given the existence of the rental market for housing services, a house can be interpreted as a Lucas tree that pays a rent, which makes our analysis similar to the one in Lagos (2010), Rocheteau (2011), and Rocheteau and Wright (2013) where a Lucas tree yields the numeraire CM goods as a dividend.

[35] Following a standard approach in Lagos and Wright (2005) setups, we use the sum of expected utilities across agents in a steady state equilibrium with equal weight as our welfare measure when we conduct a welfare analysis. However, if the house does not depreciate by 100%, we need to derive all agents' value functions with dynamic programming considering the law of motion of housing stock, and impose equilibrium quantities in a steady state equilibrium into the value functions for welfare analysis. Thus, we cannot use our simple welfare measure. In order to avoid distraction associated with this problem and to focus on the central issues we wish to address here, we assume that the house depreciates by 100% every period.

of period $t+1$ and is traded at the price of q_t in units of the CM goods in period t. In the real world, mortgage loans are often packaged by financial institutions or government sponsored enterprises into MBS to improve marketability in a financial market. We could indeed introduce this step into the model, but for our purposes this is a detail. From banks' perspective, mortgage loans and MBS are the same, so we interpret mortgages as MBS whenever needed. For example, we treat mortgages as an example of ABS, and we explore the effects of the central bank's purchase of mortgages from private banks.

In addition to houses and mortgages, there are three other assets supplied by the government: currency, reserves, and nominal government bonds. Currency is a perfectly divisible and portable object that is supplied by the government at the beginning of the CM with a lump-sum transfer τ_t to each buyer. Let ϕ_t denote the price of currency in the CM of period t, in terms of the CM goods. Reserves are account balances with the central bank that can be acquired in exchange for z_t^m units of currency in the CM of period t, and each unit of reserves pays off one unit of currency in the CM of period $t+1$. A nominal government bond sells at a price z_t in the CM of period t in units of currency, and pays off one unit of currency in the CM of period $t+1$. In principle, the prices of government bonds and reserves, z_t and z_t^m respectively, could be different in the CM. However, if both assets are held in an equilibrium that we consider in this paper, their prices must be identical because both assets are perfect substitutes.[36] Thus, we impose this condition from now on: $z_t^m = z_t$.

At the beginning of the CM, all debts are paid off first. Then, the housing rental market opens and houses provide housing services to residents. In the CM, there is a centralized Walrasian market in which agents trade numeraire CM goods and assets. In the DM, there are bilateral meetings between buyers and sellers. We assume that a buyer makes a take-it-or-leave-it offer in a pairwise meeting in the DM.[37] In this economy there is no memory

[36] We include reserves though government bonds and reserves are identical because central bank private asset purchases may be infeasible otherwise.

[37] There are many ways to split the surplus from trade, including Nash bargaining, competitive search, or competitive pricing (see Rocheteau and Wright 2005). Take-it-or-leave-it offers is a special case of Nash bargaining, and is equivalent to competitive pricing given that the seller's utility is linear in labor. Allowing general bargaining rule in the DM does not appear to admit any important insight, and we use take-it-or-

or recordkeeping, so that in any meeting, traders have no knowledge of each other's history. Also, no one can be forced to work, so lack of memory implies that there can be no unsecured credit. Hence, an asset is essential for trade to occur.

In a manner of Willamson (2012), we assume limitations on the information technology in a following way: In a fraction ρ of DM meetings, a buyer will be in a *currency transaction* in which he will be matched with a seller who does not have the information technology to verify that the buyer possesses any assets other than currency. Otherwise, in a fraction $1-\rho$ of DM transactions – denoted *non-currency transactions* – the seller can verify the entire portfolio of financial assets and currency held by the buyer. At the beginning of the CM, buyers do not know what type of match (currency or non-currency transaction) they will have in the subsequent DM, but they learn this at the end of the CM, after all production and consumption decisions have taken place. We assume that the type of match in the DM is private information.

In this environment, banks can play a useful role by efficiently allocating liquid assets to appropriate types of transactions as pointed out in Williamson (2012): The bank's deposit contract will essentially allocate currency only to currency transactions, and other assets to non-currency transactions, while providing insurance to buyers, in the spirit of Diamond-Dybvig (1983). In order to prevent the banking contracts from being unwound, we assume that after a buyer learns his type at the end of the CM, he can meet at most one bank and a bank can contact buyers one-by-one.[38]

Fraud on asset backed securities (ABS) One important feature of ABS is that it is difficult to pierce the veil of ABS and learn exactly what lies behind the asset because of the complicated structure of securitization (see Gorton and Metrick 2012). This lack of recognizability problem in ABS markets caused incentive problems in the financial sector such as fraudulent asset appraisals with rating deficiencies, false documentation about the

leave-it assumption for tractability.

[38]Without this spatial separation assumption at the end of CM, the possibility of side trades can unwind bank deposit contracts. See Jacklin (1987) for more information.

underlying assets, and lax screening of borrowers without due diligence. These kinds of fraudulent practices were criticized as key factors in the financial crisis of 2008 (see Barnett 2012, Gourinchas and Jeanne 2012, Keys et al. 2010, and The Financial Crisis Inquiry Report 2011).[39]

We incorporate the incentive problem related to ABS in the model in a very simple way. First, a private bank is able to produce any quantity of fake mortgages in the CM. In order to use these fake mortgages to secure deposit liabilities, the bank must incur a fixed cost of $k > 0$ for each deposit contract. In the real world, the cost of cheating may include a proportional cost. However, adding a proportional cost does not affect the main result of analysis, so we focus on the fixed cost for simplicity.[40] Second, any securities, if they exist, that are fully or partially backed by mortgages can be faked and posted as collateral by banks without any cost, which implies that those types of ABS cannot be used as collateral to secure bank's deposit contracts.[41]

Similarly, we assume that fraud can occur in the transaction between private banks and the central bank when the central bank is willing to purchase mortgages from banks. Specifically, a private bank can produce any quantity of fake mortgages and sell them to the central bank at a fixed cost of $k_c > 0$. Note that k_c can be different from k. There are several reasons that the cost of fraud could be different when the bank deceives the central bank than when it tricks depositors. For example, the Fed only purchases MBS from its member banks that are guaranteed by Fannie Mae, Freddie Mac, and Ginnie Mae, and mortgages

[39]Robert Lucas, in his interview with the Wall Street Journal (Sep. 24, 2011) also emphasized this fraudulent practice in the financial market as the key factor of the financial crisis arguing that "Instead, the shock came because complex mortgage-related securities minted by Wall Street and "certified as safe" by rating agencies had become part of the effective liquidity supply of the system. All of a sudden, a whole bunch of this stuff turns out to be crap. It is the financial aspect that was instrumental in the meltdown of '08."

[40]Note that the fixed cost k occurs when the bank uses fake mortgages as collateral. If the bank has to incur a fixed cost to produce fake mortgages and can post fake mortgages as collateral without any cost as in Li, Rocheteau, and Weill (2012), then counterfeiting cost for each deposit contract converges to zero because the bank makes deposit contracts with continuum of buyers in equilibrium. Thus, mortgages cannot be used as collateral.

[41]We make this assumption because banks could circumvent the incentive problem on mortgages by creating a new security backed by mortgages otherwise.

underlying those MBS must undergo more strict screening and a careful approval process. Thus, it is quite conceivable that it would be more difficult for banks to cheat the Fed than depositors about the quality of MBS.

2.3 Economic Agents' problems

2.3.1 Bank's Problem

In the CM, a bank writes deposit contracts with buyers when consumption and production decisions are made, but before buyers learn what the type of their transaction (currency or non-currency) will be in the subsequent DM. A bank's deposit contract consists of three components: $[\kappa_t, c_t, d_t]$. This contract specifies that if the buyer deposits κ_t units of the CM goods with a bank in the CM of period t, the bank gives the depositor one of two options. First, the depositor can withdraw c_t units of currency in terms of the CM goods in period t, and have no other claims on the bank. Second, if the depositor does not withdraw currency, the bank gives a claim to d_t units of the CM goods in the CM of period $t+1$, and this claim is tradeable in the intervening DM.

Like all other agents, the bank is also subject to limited commitment, in that the bank borrows from depositors in the CM, and promises to give currency at the end of the current CM and the CM goods in the CM of the next period. Thus, the bank must collateralize its deposit liabilities. First, we assume that there is a strong commitment device such as ATM in which the bank can lock up currency, when it acquires its asset portfolio, to satisfy cash withdrawals. Second, the bank's deposit claims must be secured by government bonds, reserves, and mortgages.[42] Further, we assume that any agents can observe the balance sheet of any banks. Thus, if a buyer (a depositor) suspects that the bank has not acquired

[42]Here, we implicitly assume that banks cannot pledge houses as collateral. If banks can pledge houses as collateral, then banks can circumvent the incentive problem by owning the house, pledging it as collateral, and renting it. However, if banks can pledge fake houses as collateral in a same with mortgages, we get the same results.

appropriate collateral for his deposit contract in the CM, the buyer can withdraw the initial deposit κ_t and go to another bank before the realization of the buyer's transaction type in the DM. However, we assume that a buyer can leave the CM with a maximum of one deposit contract.[43]

In equilibrium, the bank maximizes the expected utility of its representative depositor, subject to the non-negative profit constraint for the bank, the collateral constraint, and the incentive constraint. In the bank's problem below, we explicitly consider the possibility that the bank might not pledge all assets in its balance sheet as collateral. Let σ_t^b, σ_t^m, and σ denote the fraction of government bonds, reserves, and mortgages, respectively, in the bank's balance sheet that are pledged as collateral by the bank. Because government bonds and reserves are perfect substitutes, we assume that $\sigma_t^b = \sigma_t^m$ without loss of generality.

Then, in equilibrium, the bank solves the following problem, by virtue of quasi-linearity of preferences, in the CM of period t:

$$\text{(P)} \qquad \Lambda_t = \underset{\kappa_t, c_t, d_t, b_t, m_t, l_t, \sigma_t^b, \sigma_t}{Max} \left\{ -\kappa_t + \rho u \left(\frac{\beta \phi_{t+1} c_t}{\phi_t} \right) + (1-\rho) u \left(\beta d_t \right) \right\}$$

[43]If there is no restriction on the number of deposits with which a buyer can enter to the DM, then the buyer can circumvent the faking incentive problem in the banking sector in a following way. First, the buyer can make a deposit contract with small d_t that is backed by sufficiently small quantity of mortgages, so a bank has no incentive to post fake mortgages at the fixed cost k. Second, the buyer can satisfy his liquidity needs by making that deposit contract with sufficiently many banks.

subject to

$$\kappa_t - \rho c_t - z_t(b_t + m_t) - q_t l_t^b - \beta(1-\rho)d_t + \beta\frac{\phi_{t+1}}{\phi_t}(b_t + m_t) + \beta l_t^b \geq 0 \tag{21}$$

$$-\beta(1-\rho)d_t + \beta\frac{\phi_{t+1}}{\phi_t}\sigma_t^b(b_t + m_t) + \beta\sigma_t l_t^b \geq 0 \tag{22}$$

$$-\beta(1-\rho)d_t + \left\{\beta\frac{\phi_{t+1}}{\phi_t} - z_t(1-\sigma_t^b)\right\}(b_t + m_t) - (q_t - \beta)\, l_t^b + k \geq 0 \tag{23}$$

$$1 - \sigma_t^b \geq 0 \tag{24}$$

$$1 - \sigma_t \geq 0 \tag{25}$$

$$\kappa_t, c_t, d_t, b_t, m_t, a_t, \sigma_t^b, \sigma_t \geq 0, \tag{26}$$

where b_t and m_t are the real quantities of government bonds and reserves in terms of CM goods in period t, respectively, acquired by the bank, and l_t^b is the demand for mortgages. The objective function in the problem (P) is the expected utility of the depositor. The buyer deposits κ_t units of CM goods with the bank in the CM of period t. Then, the buyer consumes $\frac{\beta\phi_{t+1}c_t}{\phi_t}$ units of DM goods in a currency transaction with probability ρ and consumes βd_t units of DM goods in a non-currency transaction with probability $1-\rho$, as the result of a take-it-or-leave-it offer by the buyer in a DM meeting.

The left hand side of constraint (21) is the net payoff to the bank from banking activity in the current CM and the next CM. Inequality (22) is the bank's collateral constraint, which states that the bank's deposit liabilities in the CM of period $t+1$ must be lower than the value of assets pledged as collateral. Inequality (23) is the incentive constraint for the bank not to post fake mortgages as collateral. This constraint follows from two observations. First, given the fixed cost of posting fake mortgages as collateral, the bank will pledge either the quantity of fake mortgages that is necessary to collateralize the equilibrium deposit claim or no fake mortgages at all. Second, even though the bank can fake the quality of mortgages, the bank cannot deceive depositors about its holdings of collateral assets. Therefore, the bank must acquire currency, government bonds, and reserves that are used to secure the equilibrium deposit contract to cheat depositors with fake mortgages. Otherwise, the buyer can detect that the mortgages are fake and withdraw his initial deposit, κ_t.[44] Thus, the net payoff from fraud with faking is

$$\kappa_t - \rho c_t - z_t \sigma_t^b (b_t + m_t) - k, \tag{27}$$

and the constraint (23) means that the bank's equilibrium net payoff (21) is higher than (27).[45]

In the bank's problem, it is obvious that (21) must bind, otherwise the bank could increase the value of the objective function without violating any constraints. Then, guessing that the constraint (26) does not bind for all choice variables, the first-order conditions for the bank's problem are

$$1 = \frac{\beta \phi_{t+1}}{\phi_t} u' \left(\frac{\beta \phi_{t+1}}{\phi_t} c_t \right) \tag{28}$$

$$\lambda_{1t} + \lambda_{2t} = u'(\beta d_t) - 1 \tag{29}$$

$$z_t \left[1 + \lambda_{2t} (1 - \sigma_t^b) \right] = \frac{\beta \phi_{t+1}}{\phi_t} \left[1 + \sigma_t^b \lambda_{1t} + \lambda_{2t} \right] \tag{30}$$

$$q_t (1 + \lambda_{2t}) = \beta \left[1 + \sigma_t \lambda_{1t} + \lambda_{2t} \right] \tag{31}$$

$$\lambda_{3t} = \left\{ \frac{\beta \phi_{t+1}}{\phi_t} \lambda_{1t} + z_t \lambda_{2t} \right\} (b_t + m_t) \tag{32}$$

$$\lambda_{4t} = \lambda_{1t} \beta l_t^b \tag{33}$$

where λ_{1t}, λ_{2t}, λ_{3t}, and λ_{4t} are the Lagrange multipliers for (22), (23), (24), and (25) respectively with $\lambda_{it} \geq 0$ for all $i = 1, 2, 3, 4$.

[44] Basically we are assuming that buyers believe that any deposit contract that is different from the equilibrium deposit contract with respect to terms of contract and the portfolio of collateral assets comes from a bank who intends to default on the deposit claim.

[45] In (27), the bank needs to purchase quantity of government liabilities that are posted as collateral. Alternatively, we can assume that the bank must purchase the whole quantity of government liabilities in the balance sheet in equilibrium. In this case, the net payoff from frauds is $\kappa_t - \rho c_t - z_t (b_t + m_t) - k + \frac{\beta \phi_{t+1}}{\phi_t} (1 - \sigma_t^b)(b_t + m_t)$. However, the results do not change with this alternative specification.

Quantitative Easing In our model, unconventional monetary policy — quantitative easing (QE) — takes the form that the central bank purchases l_t^g units of mortgages at the market price q_t from each private bank. In order not to make the central bank be the victim of fraud, this mortgage purchase program must satisfy the following incentive constraint

$$k_c \geq q_t l_t^g, \tag{34}$$

so that private banks have no incentive to sell fake mortgages to the central bank. Further, we assume that private banks are willing to sell l_t^g units of mortgages to the central bank given that they are indifferent. Thus, the aggregate quantity of mortgage purchases by the central bank is l_t^g given the unit measure assumption.

The constraint (34) will limit the quantity of the central bank's mortgage purchases, l_t^g, but it does not directly affect the bank's optimal deposit contract problem. In the following analysis, we assume that $k_c = \infty$ so banks cannot cheat the central bank in equilibrium. Later, we study the economy with $k_c < \infty$ when we analyze the effects of the central bank's mortgage purchases.

2.3.2 Buyer's Problem

Because of quasi-linearity and the existence of the rental market, it does not matter for the equilibrium analysis which economic agents have ownership of houses. In this paper, we assume that buyers hold all houses in the economy.

Let $W_t(a_t, l_{t-1})$ be the value function for a buyer when he enters the CM of period t holding a_t units of houses and l_{t-1} units of mortgages. Then, using quasi-linearity of preferences, the value function is

$$W_t(a_t, l_{t-1}) = \left\{ \begin{array}{c} R_t a_t - l_{t-1} + \tau_t + \Lambda_t + \underset{a_t^b \geq 0}{Max} \left\{ \upsilon(y a_t^b) - R_t a_t^b \right\} \\ + \underset{a_{t+1}, l_t^b \geq 0}{Max} \left\{ -\psi_t a_{t+1} + q_t l_t + \beta W_{t+1}(a_{t+1}, l_t) \right\} \end{array} \right\}$$

where Λ_t is the buyer's value with the equilibrium deposit contract given in the bank's problem (P), a_t^b is housing service consumption, a_{t+1} is the quantity of houses taken out of the CM in period t, and l_t is the amount of mortgages borrowed in period t. As is common in most Lagos and Wright (2005) setups, a_t^b, a_{t+1}, l_t, and Λ_t do not depend on a_t, l_{t-1} and τ_t.

Now we are ready to find the buyer's optimal choices for $\left\{a_t^b, a_{t+1}, l_t\right\}$ in the CM. First, optimal housing consumption a_t^b can be obtained by solving the first order condition;

$$R_t = yv'(ya_t^b). \tag{35}$$

However, a_{t+1} and l_t require more details because a mortgage loan must be secured with houses, otherwise the borrower would default for sure. Using linearity of $W_{t+1}(a_{t+1}, l_t)$ with respect to a_{t+1} and l_t, the buyer's problem can be written as

$$\underset{a_{t+1}, l_t \geq 0}{Max} \left\{\left(-\psi_t + \beta R_{t+1}\right) a_{t+1} + (q_t - \beta) l_t\right\}$$

subject to

$$l_t \leq R_{t+1} a_{t+1}. \tag{36}$$

Equation (36) is the buyer's collateral constraint, meaning that the payoff on loans in the CM of period $t+1$ cannot exceed the payoff on the housing collateral. Then, as long as $q_t \geq \beta$, which holds in equilibrium, we obtain the following equilibrium conditions:

$$l_t = R_{t+1} a_{t+1} \tag{37}$$

$$\psi_t = q_t R_{t+1}. \tag{38}$$

2.3.3 Seller's problem

In the CM, when a seller produces houses, he takes the housing price ψ_t as given. Thus, the seller solves

$$\underset{I_t \geq 0}{Max} \left\{ -\chi(I_t) + \psi_t I_t \right\},$$

which gives

$$\psi_t = \chi'(I_t) \tag{39}$$

as the first order condition. Then, the aggregate quantity of houses constructed in period t is I_t by unit measure assumption.

2.3.4 Government

In our model, the consolidated government consists of the fiscal authority and central bank. The fiscal authority has the power to collect a lump-sum tax from buyers in the CM. In addition, the fiscal authority issues one-period nominal government bonds in the CM of period t in nominal terms and redeems them in the next CM. Let B_t denote the quantity of newly-issued nominal government bonds held by private agents in the CM of period t. The central bank issues reserves and currency denoted by M_t and C_t in nominal terms respectively in the CM of period t, in exchange for government bonds and mortgages through open market operations. The central bank does not have the power to tax, and it transfers any income it earns through its operations to the fiscal authority.

We will describe about what the policy rules that the fiscal authority and central bank follow more explicitly later, but what matters in determining an equilibrium are the consolidated government budget constraints, which are given by

$$\phi_0 \left[C_0 + z_0(B_0 + M_0) \right] - q_0 l_0^g = \tau_0, \tag{40}$$

for period $t = 0$, and

$$\phi_t \left[C_t - C_{t-1} + z_t(B_t + M_t) - (B_{t-1} + M_{t-1}) \right] - q_t l_t^g + l_{t-1}^g = \tau_t \tag{41}$$

for all succeeding period $t \geq 1$. As one can see from equation (40), we assume that the economy starts up in period $t = 0$ with no government debts or central bank liabilities outstanding.

2.4 Equilibrium

As a preliminary step, we first describe market clearing conditions. In equilibrium, asset markets must clear in the CM. First, the representative bank's demands for currency, government bonds, and reserves are equal to the respective supplies coming from the government:

$$\rho c_t = \phi_t C_t \tag{42}$$

$$b_t = \phi_t B_t \tag{43}$$

$$m_t = \phi_t M_t. \tag{44}$$

Second, the quantity of mortgages purchased by banks equals the quantity supplied by buyers:

$$l_t = l_t^b + l_t^g. \tag{45}$$

Third, buyers' demand for houses is equal to its supply:

$$a_t^b = a_t = A_t = I_{t-1}. \tag{46}$$

We confine our attention to a stationary equilibrium. By stationarity, we mean that all real quantities are constant over time, which implies $\frac{\phi_t}{\phi_{t+1}} = \mu$ for all t where μ is the gross inflation rate. Further, from (45), we can think that the central bank is willing to purchase the fraction θ of aggregate mortgage loans, so $l^b = (1-\theta)l$ and $l^g = \theta l$. Accordingly, θ represents the intensity of the central bank's mortgage purchases, and an increase of θ can be interpreted as more aggressive quantitative easing by the central bank.

In general it will matter a great deal here how the central bank and fiscal authority interact. For instance, one authority might have a goal of its own, with the other authority optimizing against that. Here, we adopt something simple. We assume that the fiscal authority fixes exogenously the real value of the transfer in period 0 by $V > 0$, i.e., $\tau_0 = V$, so from (40), and (42)-(44), we obtain

$$\rho c + z(b+m) - q\theta l = V, \tag{47}$$

where we used $l^g = \theta l$. Then, transfers after period 0, respond passively to central bank policy, and the transfer τ that supports this fiscal policy is obtained as

$$\tau = \left(1 - \frac{1}{\mu}\right) V + (z-1)\frac{b+m}{\mu} + \theta l \left(1 - \frac{q}{\mu}\right), \tag{48}$$

from (41), (42)-(44), and (47). In this sense, the fiscal policy is fixed, and the job of the central bank is to optimize treating the fiscal policy rule as given. Thus, in determining an equilibrium, all we need to take into account is equation (47).

In the model, conventional monetary policy is the choice of a target for a nominal government bond price, z. As well, unconventional monetary policy is setting the fraction θ for mortgage purchases from private banks. Then, we define a stationary equilibrium as follows.

Definition 1 *Given fiscal policy V and monetary policies (z, θ), a stationary monetary equilibrium consists of quantities $\{C, B, M, c, d, m, b, l^b, \sigma^b, \sigma, l^g, l, a^b, a, A, I, \tau\}$, prices $\{\psi, q, R\}$, and gross inflation rate μ such that*

1. *given* $\{\psi, q, R\}$, $\{a^b, a, l\}$ *solves the buyer's problems,*
2. *given* ψ, I *solves the seller's problem,*
3. *given* $\{q, z\}$, $\{c, d, m, b, l^b, \sigma^b, \sigma\}$ *solves the bank's problem where* $l^b = (1-\theta)l$,
4. *unconventional monetary policy* θ *satisfies the incentive constraint (34) where* $l^g = \theta l$,
5. *government's budget constraints (47) and (48) hold,*
6. A *and* I *satisfy (46),*
7. *all markets clear.*

It will matter for the determination of equilibrium whether the bank's collateral constraint and incentive constraint bind. Thus, we will consider each of the four relevant cases in turn: neither the collateral constraint nor the incentive constraint binds; the collateral constraint binds and the incentive constraint does not; both constraints bind; the incentive constraint binds and the collateral constraint does not. In the following analysis, we characterize an equilibrium in terms of quantities consumed in the DM: $x_1 = \frac{\beta c}{\mu}$ and $x_2 = \beta d$ denote consumptions of each buyer in currency transactions and non-currency transactions, respectively, in the DM.

2.4.1 Non-binding collateral and incentive constraints

When neither the collateral constraint (22) nor the incentive constraint (23) binds, $\lambda_1 = \lambda_2 = 0$. Then, we obtain, from (28)-(30), that $x_1 = u'^{-1}\left(\frac{1}{z}\right)$ and $x_2 = x^*$, so consumption in non-currency transactions, x_2, in the DM is efficient and consumption in currency transactions, x_1, in the DM is pinned down by conventional monetary policy, z. Further, $\lambda_3 = \lambda_4 = 0$ by (32) and (33), so fractions of government bonds (and reserves) and mortgages that are pledged as collateral, σ^b and σ respectively, are indetermined between zero and one.

Next, from (31), we get $q = \beta$, and then rearranging (35), (38), (39), (46), we obtain

$$\psi = \chi'(A) = \beta y v'(yA), \tag{49}$$

which uniquely pins down the aggregate housing stock A, and let A^* denote this value. Given $A = A^*$, (35) gives $R = yv'(yA^*)$. Then, from (37), we obtain the aggregate quantity of mortgages, l, as

$$l = yA^*v'(yA^*) \equiv l^*. \tag{50}$$

For this to be an equilibrium requires that (22) and (23) hold with inequality. To find conditions where the collateral constraint and incentive constraint are non-binding, substitute $c = \frac{\mu}{\beta}x_1$, (30), and equilibrium conditions such as $x_1 = u'^{-1}\left(\frac{1}{z}\right)$, $x_2 = x^*$, and $l = l^*$ into (47) to get

$$\frac{\beta}{\mu}(b+m) = V - \rho u'^{-1}\left(\frac{1}{z}\right)\frac{1}{z} + \theta\beta l^*.$$

Then, substituting this expression into (22) and (23) with $\sigma^b = \sigma = 1$, we obtain the following necessary condition that neither the collateral constraint nor the incentive constraint binds:

$$V \geq \overline{V} + Max\left\{0, (1-\theta)\beta l^* - k\right\},$$

where $\overline{V}$ is given by

$$\overline{V} \equiv \rho u'^{-1}\left(\frac{1}{z}\right)\frac{1}{z} + (1-\rho)x^* - \beta l^*. \tag{51}$$

Non-currency transaction and collateralization of government liabilities As one can see from (29), consumption in non-currency transactions in the DM is efficient with $x_2 = x^*$ if and only if neither the collateral constraint nor the incentive constraint binds so $\lambda_1 + \lambda_2 = 0$. On the other hand, if one of these two constraints binds or both constraints bind, the non-currency transaction cannot attain the efficient level x^*. In addition, $\lambda_1 + \lambda_2 > 0$ implies that $\sigma^b = 1$ because $\lambda_3 > 0$ by (32). Therefore, whenever consumption in non-currency transactions in the DM is inefficient, the bank pledges all government bonds and reserves in its balance sheet as collateral. Then, given $\sigma^b = 1$, we can express x_1 as a function of x_2, from (28)-(30), such that

$$x_1 = u'^{-1}\left(\frac{u'(x_2)}{z}\right) \tag{52}$$

which is increasing in x_2 and z. In the following three subsections, we impose the equilibrium condition $\sigma^b = 1$ in the analysis.

2.4.2 Binding collateral constraint and non-binding incentive constraint

In this case, $\lambda_1 > 0$ and $\lambda_2 = 0$. Then, $\lambda_4 > 0$ by (33), so it must be $\sigma = 1$: the bank pledges all mortgages in its balance sheet as collateral. Then, from (29), (31), (35), (38), (39), and (46), we obtain

$$q = \beta u'(x_2) \tag{53}$$

$$\psi = \beta y v'(yA) u'(x_2) = \chi'(\delta A). \tag{54}$$

Then, equation (54) can be used to express the housing stock A as a function of x_2 by $\widehat{A} : \mathbb{R}_+ \to \mathbb{R}_+$ where $\widehat{A}(x_2)$ satisfies

$$\beta y v'\left(y\widehat{A}(x_2)\right) u'(x_2) = \chi'\left(\delta\widehat{A}(x_2)\right). \tag{55}$$

Similarly, from (35), (37), (46), (54), and (55), we can express mortgage outstanding l as a function of x_2, defined as

$$\widehat{l}(x_2) \equiv y\widehat{A}(x_2) v'\left(y\widehat{A}(x_2)\right). \tag{56}$$

Simple inspection gives that $\widehat{A}'(x_2) < 0$, $\widehat{l}'(x_2) < 0$ for all $x_2 > 0$, $\widehat{A}(x^*) = A^*$, and $\widehat{l}(x^*) = l^*$.

Next, substituting the binding collateral constraint (22), (28)-(30), (53), and (56) into

(47), we obtain

$$\rho x_1 u'(x_1) + (1-\rho) x_2 u'(x_2) - \beta u'(x_2)\widehat{l}(x_2) = V. \tag{57}$$

Observe that the left-hand side of (57) is strictly increasing in x_1 and x_2. Then, given fiscal policy V and conventional monetary policy z, equations (52) and (57) determine equilibrium consumptions in the DM, (x_1, x_2). Once we obtain x_2, the mortgage price, q, housing price, ψ, rental rate R, aggregate housing stock, A, and aggregate mortgage loans, l, are given by (35), and (53)-(56).

The binding collateral constraint and non-binding incentive constraint require, from (22), (23), and (29), that $x_2 < x^*$ and $k \geq (1-\theta)ql$. First, the condition $x_2 < x^*$ holds if and only if

$$V < \overline{V}$$

by (52) and (57) where $\overline{V}$ is defined in equation (51).

Finally, the condition $k \geq (1-\theta)ql$ can be rewritten, using (53) and (56), as

$$k \geq (1-\theta)\beta u'(x_2)\widehat{l}(x_2) \tag{58}$$

where the right-hand side is strictly decreasing in x_2. Because it must be $x_2 < x^*$, the necessary condition for this equilibrium to exist is that the fraud cost k is sufficiently high such that

$$k > (1-\theta)\beta l^*. \tag{59}$$

Given (59), there exists a unique threshold value of x_2, denoted by $\widehat{x} \in (0, x^*)$, such that

$$k = (1-\theta)\beta u'(\widehat{x})\widehat{l}(\widehat{x}), \tag{60}$$

and $\widehat{x}$ satisfies $\frac{\partial \widehat{x}}{\partial k} < 0$ and $\frac{\partial \widehat{x}}{\partial \theta} < 0$. Then, the condition (58) holds if and only if $x_2 \geq \widehat{x}$ that requires, from (52), (57), and (60), that

$$V \geq \rho u'^{-1}\left(\frac{u'(\widehat{x})}{z}\right)\frac{u'(\widehat{x})}{z} + (1-\rho)\widehat{x}u'(\widehat{x}) - \frac{k}{1-\theta} \equiv \widehat{V}(\theta, k). \tag{61}$$

Notice that $\widehat{V}(\theta, k)$ decreases with θ and k because $\widehat{x}$ decreases with θ and k.

2.4.3 Binding collateral and incentive constraints

First, binding collateral constraint means $\lambda_1 > 0$, so $\sigma = 1$ by (33). Then, for the collateral constraint (22) and the incentive constraint (23) both to bind, it requires $k = (1-\theta)ql$. Substituting (37), (38), (39), and (46) into this condition, we obtain

$$k = (1-\theta)\chi'(A)A. \tag{62}$$

Because the right-hand side of (62) strictly increases with A from zero to infinity, there is unique $A = \widetilde{A}$ that solves (62), and $\widetilde{A}$ increases with k and θ.

Given $A = \widetilde{A}$, (35), (39) and (46) determine the rental rate and the housing price as $R = yv'(y\widetilde{A})$ and $\psi = \chi'(\delta\widetilde{A})$, respectively. Then, from (37) and (38), we obtain the mortgage price and quantity of mortgage loans as

$$q = \frac{\chi'(\delta\widetilde{A})}{yv'(y\widetilde{A})} \equiv \widetilde{q} \tag{63}$$

$$l = yv'(y\widetilde{A})\widetilde{A} \equiv \widetilde{l}. \tag{64}$$

Next, from (22), (28)-(30), (47), and (62)-(64), we obtain

$$\rho x_1 u'(x_1) + (1-\rho)x_2 u'(x_2) - (1-\theta)\beta\widetilde{l}u'(x_2) - \frac{\theta k}{1-\theta} = V, \tag{65}$$

and then (52) and (65) solve for x_1 and x_2, given fiscal policy V and monetary policy (z, θ).

Finally, for this to be an equilibrium, lagrange multipliers of (22) and (23) must be positive. From (29), (31), and (63), we obtain

$$\lambda_1 = u'(x_2)\left\{\frac{\widetilde{q}-\beta}{\widetilde{q}}\right\}$$
$$\lambda_2 = \frac{\beta u'(x_2)-\widetilde{q}}{\widetilde{q}}.$$

First, because $\lim_{k\to(1-\theta)\beta l^*}\widetilde{A} = A^*$ and $\lim_{k\to(1-\theta)\beta l^*}\widetilde{q} = \beta$, $\lambda_1 > 0$ if and only if (59) holds. Second, it can be verified, from (55), (56), (60), (62), and (64), that $\widehat{A}(\widehat{x}) = \widetilde{A}$ and $\widehat{l}(\widehat{x}) = \widetilde{l}$. Then, we obtain, from (60), and (62)-(64), that $\lambda_2 > 0$ if and only if $x_2 < \widehat{x}$ which requires, from (52) and (65), that

$$V < \widehat{V}(\theta, k),$$

where $\widehat{V}(\theta, k)$ is defined in (61).

2.4.4 Non-binding collateral constraint and binding incentive constraint

In this case, $\lambda_1 = 0$. Then, $\sigma \leq 1$ by (33), so there are illiquid mortgages that are not used as collateral in the bank's balance sheet. Then, from (31), (35), (37)-(39), and (46), we obtain $\psi = \chi'(A^*)$, $q = \beta$, $A = A^*$, $R = yv'(yA^*)$, and $l = l^*$. Next, substituting binding incentive constraint (23), (28)-(30) into (47), we obtain

$$\rho x_1 u'(x_1) + (1-\rho)x_2 u'(x_2) - ku'(x_2) - \theta\beta l^* = V. \tag{66}$$

Then (52) and (66) solve for x_1 and x_2.

The non-binding collateral constraint (22) and the binding incentive constraint (23) require that $\sigma(1-\theta)\beta l^* \geq k$. Thus,

$$(1-\theta)\beta l^* \geq k$$

must hold for this equilibrium to exist. In addition, positive λ_2 implies $x_2 < x^*$ that requires, from (52) and (66),

$$V < \overline{V} + (1-\theta)\beta l^* - k.$$

Existence of equilibrium and the effects the fraud cost k From the previous four subsections, we can construct Figure 6 that describes how the fraud cost k and fiscal policy V together determine the existence of particular equilibria. In the figure, (CC) and (IC) represent the bank's collateral constraint and the incentive constraint, respectively, and $\overline{k}_0$ is obtained from $\widehat{V}(\theta, \overline{k}_0) = 0$.

As one can see from Figure 6, the incentive constraint tends to be non-binding with higher V. In this economy, government debts act to discourage frauds in a following sense. The bank cannot deceive depositors about its holdings of collateral government liabilities. Thus, whenever the bank defaults on a deposit claim, the pledged government bonds and reserves will also be confiscated. In this sense, government debts make default with fake mortgages more costly, and as V increases, there are more government liabilities given θ. Therefore, there is less incentive for banks to commit frauds. In particular, when both k and V are sufficiently low, the incentive constraint binds but the collateral constraint does not bind in equilibrium. In this case, there are illiquid mortgages in the bank's balance sheet that are not pledged as collateral. Thus, in principle, banks can post more mortgages to secure their deposit liabilities, and hence the collateral constraint does not bind. However, the incentive problem is so severe with low k and V that banks cannot use those illiquid mortgages as collateral to satisfy the incentive constraint.

How does the cost of fraud k affect real allocations and asset prices? In this economy, the fraud cost k affects the economy through the bank's incentive constraint. Thus, k only matters when the incentive constraint binds.

First, consider an equilibrium in which the collateral constraint and incentive constraint both bind. Because the left-hand side of (65) decreases with k, x_1 and x_2 must rise as k

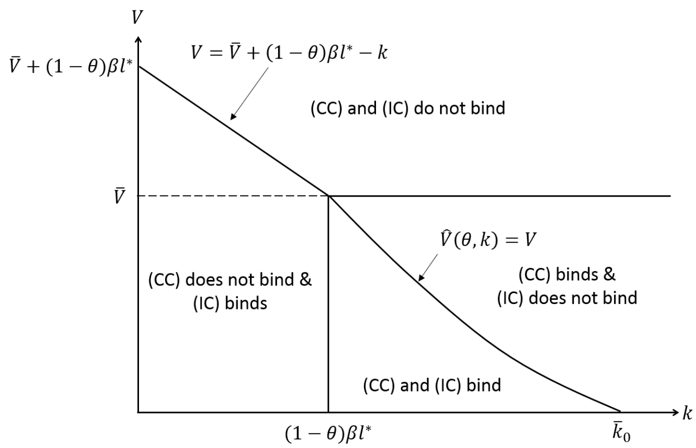

Figure 6: Equilibria with the cost of fraud k and the fiscal policy V

increases. Further, $A = \widetilde{A}$ increases with k by (62), so R decreases with k while ψ, q, and l increases with k (see subsection 4.3). The intuition is as follows. In this case, an increase in k mitigates the incentive problem in the banking sector, so the demand for mortgages rises, which lowers the mortgage interest rate. This, in turn, increases the demand for houses, the price of housing, and housing construction. Then, the rental rate falls to clear the market. Second, in an equilibrium where only the incentive constraint binds, an increase in k raises x_1 and x_2 because the left-hand side of (66) is decreasing in k. However, all macro economic variables related to houses, such as ψ, q, l, A, and R, are unaffected by the change of k because mortgages are illiquid, at the margin, as argued in subsection 4.4.

The above analysis implies that the housing price weakly increases with the fraud cost k that affects the usefulness of houses as collateral via the bank's incentive structure. This is quite different from what obtains in He et al. (2015) where they assume that agents cannot pledge more than an exogenous fraction of their housing holdings as collateral. In their model, when the borrowing constraint is tight, the housing price rises if houses become more pledgeable. However, continued increases in pledgeability eventually cause the price go back down to its fundamental value because higher pledgeability lowers the marginal value of liquidity as it further relaxes borrowing constraint and ultimately renders the constraint slack. In contrast, when we modeled the friction that limits a role for houses as collateral more explicitly, the housing price does not fall as the usefulness of houses as collateral increases.

2.5 Monetary Policy

We have so far taken the nominal government bond price, z, and the fraction of mortgages that the central bank is willing to purchase from private banks, θ, as given. Now that we have a basic working knowledge on the model, we study the effects of conventional and unconventional monetary policies on the model economy: What impact does (un)conventional monetary policy have on equilibrium quantities and prices? How does (un)conventional monetary policy affect the equilibrium type in which the economy stays? What is optimal (un)conventional monetary policy?

To set the stage for welfare analysis, we define the sum of expected utilities across agents with equal weight as our welfare measure. Then, our welfare measure can be written as

$$\mathcal{W} = \rho \left[u(x_1) - x_1\right] + (1-\rho) \left[u(x_2) - x_2\right] - \chi(A) + \beta v(yA). \tag{67}$$

Note that we discounted the utility from housing service consumption because houses constructed current period provide housing services in the next CM and depreciate by 100%. Given our welfare measure (67), the first best is obtained with $x_1 = x_2 = x^*$ and $A = A^*$.

2.5.1 Conventional Monetary Policy

We first use our model to understand the effects of the central bank's conventional monetary policy. The effects of changing z on economic variables and welfare in equilibrium where neither the collateral constraint nor the incentive constraint binds is straightforward. An increase of z raises x_1 and does not affect any other economic variables. Thus, raising z improves welfare (67).

In the following analysis, we focus on the effects of conventional monetary policy in the other three types of equilibria. For this purpose, it will be convenient to rewrite the equations (57), (65), and (66) in a general form as

$$\rho x_1 u'(x_1) + (1-\rho)x_2 u'(x_2) - \Gamma(x_2;\theta,k) = V, \tag{68}$$

where $\Gamma(x_2;\theta,k)$ has the following form depending on the equilibrium type;

$$\Gamma(x_2;\theta,k) = \begin{cases} \beta u'(x_2)\widehat{l}(x_2) & \text{with binding (CC) and non-binding (IC)} \\ (1-\theta)\beta\widetilde{l}u'(x_2) + \frac{\theta k}{1-\theta} & \text{with binding (CC) and binding (IC)} \\ ku'(x_2) + \theta\beta l^* & \text{with non-binding (CC) and binding (IC).} \end{cases} \tag{69}$$

Given fiscal policy V and unconventional monetary policy θ, equation (68) describes the menu for the central bank in terms of feasible equilibrium allocations (FEA) for consumptions in the DM, (x_1, x_2). Then, the central bank chooses equilibrium (x_1, x_2) among the feasible equilibrium allocations by determining the nominal government bond price z. More precisely, equation (68) describes a locus in (x_1, x_2) space, as depicted by the curve FEA in Figure 7. Further, the locus in (x_1, x_2) determined by (52) can be described, for example, as the line $z = z_1$ with $z_1 < 1$ in Figure 7. Then, x_1 and x_2 are determined by the intersection of FEA and $z = z_1$ in Figure 7 at point A.[46]

Now, the effects of conventional monetary policy on consumptions in the DM in equilibrium with inefficient non-currency transactions where $x_2 < x^*$ become clear from Figure 7. Suppose the central bank chooses a lower nominal interest rate, for example, pegs the price of government bonds at $z_2 > z_1$. This central bank action does not move the FEA locus in Figure 7. However, as z increases, the curve $z = z_1$ shifts down to $z = z_2$, moving equilibrium consumption (x_1, x_2) from point A to point B in Figure 7. As a result, the quantity of exchange in currency transactions x_1 in the DM rises and the quantity of ex-

[46]Note that because of the zero lower bound constraint on the nominal interest rate, $z \leq 1$, the central bank cannot choose an allocation below the curve ZLB depicted in Figure 7.

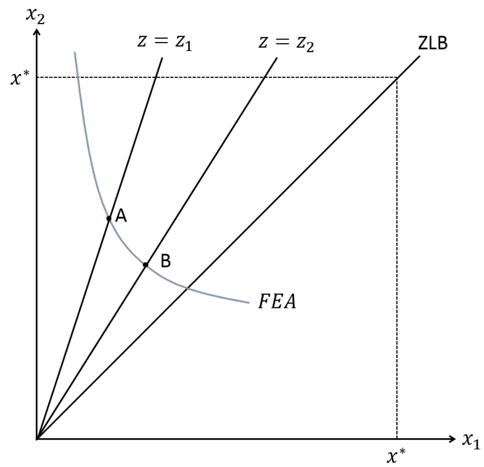

Figure 7: Conventional monetary policy when the aggregate collateral is scarce and $x_2 < x^*$

changes in non-currency transactions x_2 in the DM falls. This is because the central bank raises z by purchasing government bonds for the exchange of currency through open market operation, so there are less collateralizable assets for non-currency transactions and more currency in the economy. Then, because there are less collateralizable government liabilities for non-currency transaction with a higher z, the parameter space (k, V) for equilibrium in which the incentive constraint binds expands. More precisely, an increase of z shifts the line $V = \overline{V}$ and the curve given by $\widehat{V}(\theta, k) = V$ in Figure 6 upward.

The effects of conventional monetary policy on housing and mortgage markets depend on whether the incentive constraint binds or not. First, if the incentive constraint does not bind and only the collateral constraint binds, the mortgage price q, housing price ψ, housing stock A, and mortgage outstanding l rise and rental rate R falls as the central bank raises z (see subsection 4.2): Given that x_2 falls as z increases, there is higher demand for mortgages as collateral, which in turn increases the mortgage price, housing demand, housing price, so sellers construct more houses in response. Then, the rental rate decreases via market clearing.

However, conventional monetary policy does not affect ψ, q, l, A, and R when the bank's incentive constraint binds (see subsections 4.3 and 4.4). The intuition for this finding is as follows. When the incentive constraint binds, the fraud cost, k, limits the quantity of mortgages that can be used as collateral similar to Li, Rocheteau, and Weill (2012). If the bank were to acquire an additional unit of mortgages and pledge them as collateral, then the depositor would not make a deposit contract (or withdraws initial deposit κ) because of the counterfeiting possibility. In this sense, the mortgage is not perfectly liquid. Therefore, even though the exchange in non-currency transactions x_2 falls, it does not lead to higher demand for mortgages by the bank. Thus, the channel through which conventional monetary policy affects the housing market does not work well when the efficient banking intermediation is disrupted by the financial friction in the form of moral hazard problem.

We close this subsection with the analysis of the optimality of conventional monetary policy in equilibrium with inefficient non-currency transactions. In particular, we restrict our attention to the optimality of the zero lower bound, i.e., $z = 1$, when it is feasible under each type of equilibria.

Proposition 5 *If the bank's incentive constraint binds or if*

$$y\widehat{A}(x)\widehat{A}'(x)u'(x)\upsilon''\left(y\widehat{A}(x)\right) \leq -u''(x)\upsilon'\left(\widehat{A}(x)\right)\left[\widehat{A}(x) - x\widehat{A}'(x)\right] \tag{70}$$

where x solves $xu'(x) - \beta\widehat{l}(x)u'(x) = V$ in equilibrium where the collateral constraint binds and the incentive constraint does not bind, then $z = 1$ is optimal at least locally.

Proof. The proof is done by comparing the derivative of a level surface of the welfare function defined by (67), $\left.\frac{\partial x_2}{\partial x_1}\right|_{\mathcal{W}}$, and the derivative of the locus given by (68), $\left.\frac{\partial x_2}{\partial x_1}\right|_{FEA}$, both evaluated at the zero lower bound. From (68), $\left.\frac{\partial x_2}{\partial x_1}\right|_{FEA,z=1} = -\frac{\rho}{1-\rho-\frac{1}{u'(x)+xu''(x)}\frac{\partial\Gamma}{\partial x_2}}$ where $\frac{\partial\Gamma}{\partial x_2}$ can be obtained from (69) for each type of equilibrium. First, in equilibria where the incentive constraint binds, $\left.\frac{\partial x_2}{\partial x_1}\right|_{\mathcal{W}} = -\frac{\rho}{1-\rho}$ and $\frac{\partial\Gamma}{\partial x_2} < 0$. Thus, $\left.\frac{\partial x_2}{\partial x_1}\right|_{\mathcal{W}} < \left.\frac{\partial x_2}{\partial x_1}\right|_{FEA}$ and $z = 1$ is locally optimal. Second, in an equilibrium where the collateral constraint binds and the incentive constraint does not, $\left.\frac{\partial x_2}{\partial x_1}\right|_{\mathcal{W}} = -\frac{\rho}{1-\rho-\frac{[\chi'(\widehat{A}(x))-\beta y\upsilon'(y\widehat{A}(x))]\widehat{A}'(x)}{u'(x)-1}}$ and $\left.\frac{\partial x_2}{\partial x_1}\right|_{\mathcal{W}} \leq \left.\frac{\partial x_2}{\partial x_1}\right|_{FEA}$ if and only if (70) holds. ■

The main implication of proposition 3 is that when the incentive constraint binds, it is optimal for the central bank to set the nominal interest rate to zero at least locally.[47] On the other hand, when the incentive problem in the banking sector does not matter, the central bank should be more cautious about implementing zero nominal interest rate policy because a low nominal interest rate causes a housing construction boom. In an equilibrium where the collateral constraint binds and the incentive constraint does not bind, $A = \widehat{A}(x_2) > A^*$, and an increase of z raises A, aggravating the welfare loss from over-construction of houses. Here, the optimality of the zero lower bound depends on the curvature of $\upsilon(\cdot)$. In particular, if $\upsilon''(\upsilon) = 0$, then (70) is always satisfied so $z = 1$ is optimal at least locally.

2.5.2 Unconventional Monetary Policy

We now turn our focus to unconventional monetary policy in the form of mortgage purchases keeping V and z constant, which is main goal of this paper. Under unconventional monetary policy, the central bank decides the quantity of mortgage purchases by gearing θ.

Equilibria with unconventional monetary policy Before studying the effects of unconventional monetary policy on real allocations and welfare in our model economy, we first show how the existence of each type of equilibrium depends on θ, which is also of interest to policy makers.

Consider the case that $k > \beta l^*$. Then, it is clear, from Figure 6, that an equilibrium with efficient non-currency transactions where $x_2 = x^*$ exists for any θ if $\overline{V} \leq V$. Next, because $\widehat{V}(\theta, k)$ decreases with θ, if $\widehat{V}(\theta = 0, k) \leq V < \overline{V}$, then the collateral constraint binds and the incentive constraint does not bind in equilibrium for any θ. Finally, assume that $V < \widehat{V}(\theta = 0, k)$. Then, there exists $\widehat{\theta} \in (0, 1)$ such that $\widehat{V}(\widehat{\theta}, k) = V$, and there is an equilibrium where: (i) both constraints bind for $\theta \in [0, \widehat{\theta})$, and (ii) the collateral constraint

[47] Though we could not prove global optimality of zero lower bound analytically, numerical simulations show that whenever the conditions for $z = 1$ to be locally optimal are satisfied, zero lower bound is globally optimal.

binds and the incentive constraint does not bind for $\theta \in [\widehat{\theta}, 1]$.

Next, suppose $k \leq \beta l^*$, so there exists $\overline{\theta} \in [0,1)$ such that $k = (1-\overline{\theta})\beta l^*$. First, if $\overline{V} + \beta l^* - k \leq V$, an equilibrium with efficient non-currency transactions exists for any θ. Second, if $\overline{V} \leq V < \overline{V} + \beta l^* - k$, there exists $\widetilde{\theta} \in (0, \overline{\theta}]$ such that $\overline{V} + (1-\widetilde{\theta})\beta l^* - k = V$. Then, there exists an equilibrium where: (i) the collateral constraint does not bind and the incentive constraint binds for $\theta \in [0, \widetilde{\theta})$, and (ii) neither the collateral constraint nor the incentive constraint binds for $\theta \in [\widetilde{\theta}, 1]$. Finally, assume that $V < \overline{V}$. Note, from (55), (56), and (60), that $\lim\limits_{\theta \to \overline{\theta}} \widehat{x} = x^*$ and $\lim\limits_{\theta \to \overline{\theta}} \widehat{l} = l^*$. Thus, $\widehat{V}(\overline{\theta}, k) = \overline{V}$, and there exists $\widehat{\theta} \in (\overline{\theta}, 1)$ such that $\widehat{V}(\widehat{\theta}, k) = V$. Then, there is an equilibrium where; (i) the collateral constraint does not bind and the incentive constraint binds for $\theta \in [0, \overline{\theta}]$; (ii) both constraints bind for $\theta \in (\overline{\theta}, \widehat{\theta})$; (iii) the collateral constraint binds and the incentive constraint does not bind for $\theta \in [\widehat{\theta}, 1]$.

We can use the above analysis to construct Figure 8 that depicts how the parameter space is subdivided with θ on the horizontal axis and V on the vertical axis. One implication is that when the efficient non-currency transaction in the DM is not attainable with low V, then as the central bank increases mortgage purchases, the mortgage becomes more liquid in the following sense: Illiquid ($\sigma \leq 1$) $\rightarrow$ liquid but with restriction on the quantity that can be pledged as collateral ($\sigma = 1$ with the binding incentive constraint) $\rightarrow$ liquid without any restriction (non-binding incentive constraint). In particular, there exists a threshold level of θ, denoted by $\widehat{\theta} < 1$, such that if the central bank purchases more than $\widehat{\theta}$ fraction of mortgages in the economy, the bank's faking incentive does not matter so the mortgage can be pledged as collateral without any restriction. This is because as θ increases, banks post more government bonds and reserves and less mortgages as collateral, which mitigates the incentive for banks to commit frauds as argued in the previous section.

Effects of unconventional monetary policy in each equilibrium We now study how unconventional monetary policy θ affects equilibrium quantities, prices, and welfare in our model economy. Unlike conventional monetary policy, QE has no efficacy on macro-

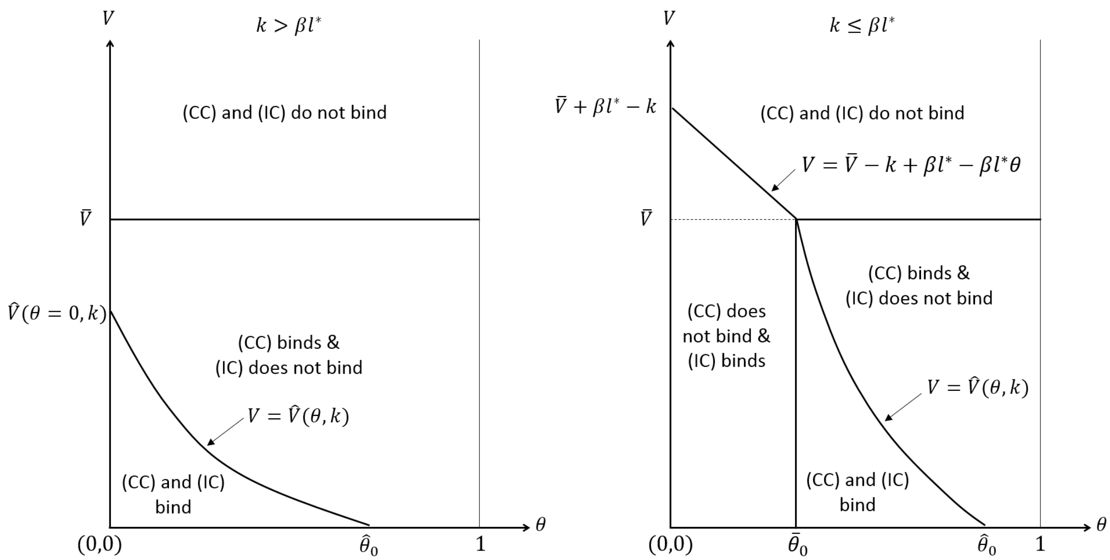

Figure 8: Equilibria with unconventional monetary policy θ and fiscal policy V

economic variables and hence welfare when the bank's incentive constraint does not bind. In the following, we focus on the effects of increasing θ in equilibrium where the bank's incentive constraint binds.

Consider the easy case, first, where only the incentive constraint binds and the collateral constraint does not. From (69), we obtain that

$$\frac{\partial \Gamma(x_2; \theta, k)}{\partial \theta} = \beta l^* > 0.$$

Thus, as θ increases, the curve FEA in Figure 7 that describes equation (68) shifts to the right while the curve $z = z_1$ stays fixed. As a result, x_1 and x_2 rise in equilibrium. The intuition behind this result is as follows. In this case, QE affects the economy only through the binding incentive constraint because the collateral constraint is slack. Then, as argued above, the central bank's mortgage purchases for the exchange of reserves always relax the binding incentive constraint which increases exchanges in the DM. Further, given that the mortgage is still illiquid at the margin in this equilibrium, its price should be fixed at its fundamental value and hence the housing price, housing stock, and mortgage outstanding are unaffected by unconventional monetary policy.

Next, consider an equilibrium where the collateral constraint and the incentive constraint both bind. By taking derivative $\Gamma(x_2;\theta,k)$ in (69) with respect to θ, we get

$$\frac{\partial\Gamma(x_2;\theta,k)}{\partial\theta} = -\beta\widetilde{l}u'(x_2) + \frac{k}{(1-\theta)^2} + (1-\theta)\beta u'(x_2)\frac{\partial\widetilde{l}}{\partial\theta}. \tag{71}$$

Here, the sign of $\frac{\partial\Gamma(x_2;\theta,k)}{\partial\theta}$ is not clear. In particular, $\frac{\partial\Gamma(x_2;\theta,k)}{\partial\theta}$ could be negative, which implies that an increase of θ moves down FEA curve and exchanges in the DM, (x_1, x_2), fall.

The intuition is in line with our earlier observations. In an equilibrium where the collateral constraint and the incentive constraint both bind, θ affects exchanges in the DM via these two binding constraints. Now suppose that the central bank purchases additional one unit of mortgages for the exchange of reserves. On the one hand, given rate of return difference between mortgages and reserves, this balanced budget mortgage purchase tightens the collateral constraint (22), which forces x_2 to fall.[48] On the other hand, this policy always relaxes the bank's incentive constraint (23), which forces x_2 to rise. The first two terms in (71) are related to these two effects of an increase of θ. Finally, central bank's mortgage purchases induce more production of houses in private sector, which will be explained below. This, in turn, provides more collateralizable assets to the economy, and the last term in (71) captures this effect.

Unlike from the case where only the incentive constraint binds, changing θ has impact on the housing market in this case. In particular, ψ, q, A, and l increase while R decreases with θ (see subsection 4.3). The intuition is as follows. In this equilibrium, the liquidity premium on the mortgage price depends on the intensity of the incentive problem in the banking sector, and an increase of θ mitigates the incentive problem as argued above. Thus, as θ

[48] More precisely, the central bank can purchase one unit of mortgage for the exchange of $\frac{q}{z\phi_t}$ units of reserves. Here, one unit of mortgage can secure one unit of deposit claims while $\frac{q}{z\phi_t}$ units of reserves can secure $\frac{q}{z\mu} = \frac{1}{\lambda_2+1}$ units of deposit claims that are less than one because $\lambda_2 > 0$.

increases, the demand for mortgages increases, which raises the mortgage price. Then, the housing price and housing construction increase and the rental rate falls by market clearing conditions.

How does an increase of θ affect the yield spread between government bonds and mortgages, $\frac{\mu}{q} - \frac{1}{z}$? First, when the incentive constraint does not bind, $q = z\mu$ (see subsections 4.1 and 4.2). Therefore, there is no yield spread, and changing θ has no effect on the yield spread. Second, if both the collateral and incentive constraints bind, then from (29)-(31) with $\sigma^b = \sigma = 1$, the yield spread is $\frac{\mu}{q} - \frac{1}{z} = \frac{1}{z}\left\{\frac{\beta u'(x_2)}{\tilde{q}} - 1\right\}$. Because $\frac{\partial \tilde{q}}{\partial \theta} > 0$, if x_2 rises as θ increases, then the spread must fall as the central bank purchases more mortgages. Now suppose x_2 falls. Substituting the equilibrium condition $k = (1-\theta)ql$ into (65), we obtain

$$\rho x_1 u'(x_1) + (1-\rho)x_2 u'(x_2) - \frac{\theta k}{1-\theta} = k\frac{\beta u'(x_2)}{q} - V. \tag{72}$$

Because the left hand side of (72) decreases with θ given the assumption that $\frac{\partial x_2}{\partial \theta} < 0$, $\frac{\beta u'(x_2)}{q}$ decreases, so that the spread, $\frac{\mu}{q} - \frac{1}{z}$, must fall as θ increases. Finally, when the collateral constraint does not bind and the incentive constraint binds, the yield spread is $\frac{\mu}{q} - \frac{1}{z} = \frac{\mu}{\beta} - \frac{1}{z}$. In this case, an increase of θ lowers μ, so it reduces the yield spread. Therefore, our model suggests that the central bank can lower the yield spread between government bonds and mortgages with the mortgage purchase program when the incentive problem in the financial sector generates the positive yield spread. This result is consistent with empirical evidences on QE of Hancock and Passmore (2011).

We now study welfare implication of central bank's mortgage purchases in equilibrium with the binding incentive constraint. First, when only the incentive constraint binds, the effects of increasing θ on welfare is straightforward: It raises (x_1, x_2) and A is fixed at A^*. Therefore, welfare given by (67) increases unambiguously. Second, in equilibrium where the

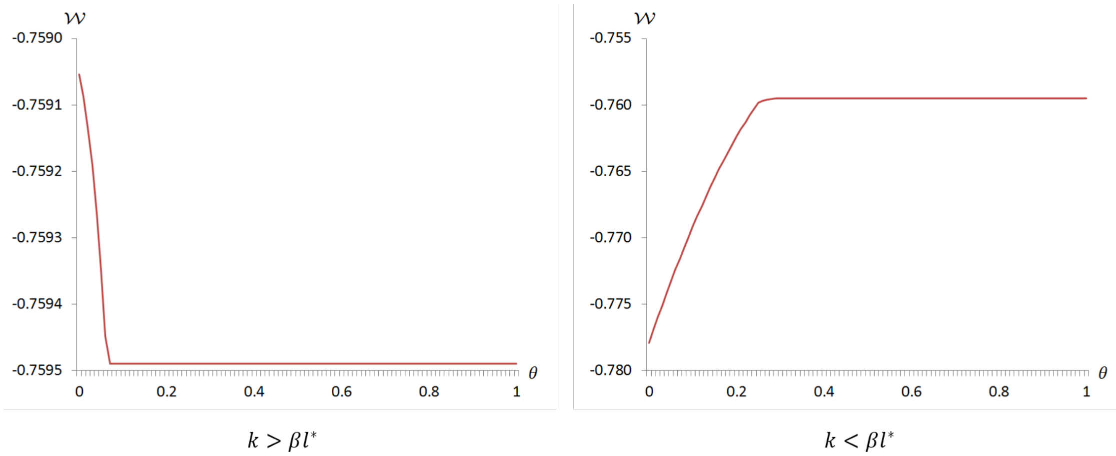

Figure 9: Welfare and Central bank asset purchases

collateral constraint and the incentive constraint both bind, the effects of θ on welfare is

$$\frac{\partial \mathcal{W}}{\partial \theta} = \underbrace{\left\{ \begin{array}{c} \rho \left[u'(x_1) - 1\right] X_1'(x_2) \\ +(1-\rho)\left[u'(x_2) - 1\right] \end{array} \right\} \frac{dx_2}{d\theta}}_{\text{Part 1 }(+,-)} - \underbrace{\left\{ \chi'\left(\widetilde{A}\right) - \beta y \upsilon'\left(y\widetilde{A}\right) \right\} \frac{\partial \widetilde{A}}{\partial \theta}}_{\text{Part 2 }(+)}$$

where $X_1(x_2) = u'^{-1}\left(\frac{u'(x_2)}{z}\right)$.[49] Thus, if $\frac{dx_2}{d\theta} \leq 0$, then an increase of θ lowers welfare for sure. If $\frac{dx_2}{d\theta} > 0$, on the other hand, the sign of $\frac{\partial \mathcal{W}}{\partial \theta}$ is ambiguous and depends on the relative magnitudes of two counteracting effects: trade surplus from the increased exchanges in the DM (part 1) and the welfare loss from the expanded residential investment above the efficient level (part 2).

Figure 9 plots two examples illustrating the effects of the central bank's mortgage purchases on welfare depending on the fraud cost k.[50] As one can see, welfare stays constant at $\mathcal{W}$ =2.9989 for sufficiently high θ in both cases because the economy stays in equilibrium with the non-binding incentive constraint once θ becomes higher than $\widehat{\theta}$. However, the effects of θ on welfare are remarkably different between two examples depending on k. On the one

[49]Given $\widetilde{A} > A^*$ and $\frac{\partial \widetilde{A}}{\partial \theta} > 0$, part 2 is positive.

[50]Functional forms to get the results are $u(x) = \frac{u_0 x^{1-\gamma}}{1-\gamma}$, $\upsilon(f) = \upsilon_0 \log(f)$, and $\chi(A) = \chi_0 A^{\omega}$. Parameter values used are $\beta = 0.95, \gamma = 0.6, u_0 = \chi_0 = 1, \upsilon_0 = 0.4, \omega = 2, y = 0.01, z = 1$, and $V = 0.55$, and we used $k = 0.3804$ for left panel and $k = 0.2850$ for right panel.

hand, when $k > \beta l^*$, central bank's mortgage purchases only lower welfare and $\theta = 0$ (no QE) is optimal.[51] On the other hand, when $k < \beta l^*$, welfare increases with θ. Thus, it is optimal for the central bank to purchase mortgages from private banks more than $\widehat{\theta}$ fraction of mortgages in the economy.[52]

QE with finite fraud cost $k_c < \infty$ We close this section with the study of the model economy where banks can make fake mortgages and sell them to the central bank at the fixed cost $k_c \in (0, \infty)$. Thus, the constraint (34), that can be rewritten as

$$k_c \geq q\theta l, \tag{73}$$

must hold in equilibrium. However, this new incentive constraint does not affect our previous results. Instead, it generates a upper bound for θ.

Consider equilibrium where $q = \beta$ and $l = l^*$. Here, (73) requires $\theta \leq \frac{k_c}{\beta l^*} \equiv \overline{\theta}_1$. Next, in equilibrium where only the collateral constraint binds, (73) can be rewritten as $\theta \leq \frac{k_c}{\beta u'(x_2)\widehat{l}(x_2)} \equiv \overline{\theta}_2$. Because x_2 increases with V in this equilibrium, $\overline{\theta}_2$ rises with V. Finally, when both the collateral and incentive constraints bind, (73) becomes $\widetilde{q}\theta\widetilde{l} \leq k_c$ that can be written as $\theta \leq \frac{k_c}{k+k_c} \equiv \overline{\theta}_3$. Figure 10 illustrates above analysis.[53] As one can see from Figure 10, the possibility that private banks can make fake mortgages and sell them to the central bank limits the quantity of mortgage purchases by the central bank.

[51] Of course, one can get different results with other parameter values. For example, it is possible that the optimal mortgage purchases is $\theta \geq \widehat{\theta}$ even though $k > \beta l^*$.

[52] In particular, numerical exercises show that optimal θ decreases with the fraud cost $k \in (0, \overline{k})$ where $\overline{k}$ satisfies $\widehat{V}(\theta, \overline{k}) = 0$, which implies that the central bank needs to purchase more mortgages as financial frictions become more severe.

[53] The left panel draw the case where $\overline{\theta}_3 < \widehat{\theta}_0 < \overline{\theta}_1 < 1$ and the right panel is the case where $\overline{\theta}_0 < \overline{\theta}_3 < \widehat{\theta}_0 < \overline{\theta}_1 < 1$. Depending on parameter values, in particular k and k_c, there are many other cases. For example, it is possible that $\overline{\theta}_1 > 1$ or $\overline{\theta}_1 < \overline{\theta}_0$, etc.

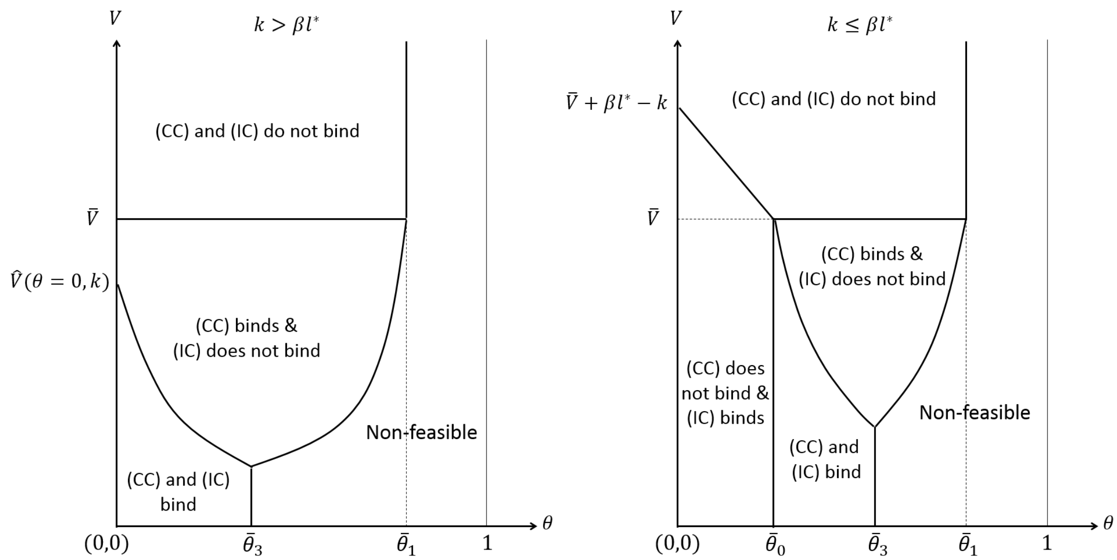

Figure 10: Equilibria with finite $k_c < \infty$

2.6 Conclusion

In this paper, we construct a New Monetarist model characterized by an endogenous private asset supply and an incentive to misrepresent the quality of private assets in the banking sector. When the incentive problem matters, it limits the extent in which private assets facilitate the exchange process. In this circumstance, QE in the form of central bank's private asset purchases from banks matters, in that it replaces private assets that are less useful as a medium of exchange with government assets. However, the effects of the central bank's private asset purchase program depend on how severe the incentive problem is. When the incentive to fake the quality of private assets is so high that some private assets cannot be used as a medium of exchange, an increase of the size of central bank's asset purchases improves welfare unambiguously. However, when the incentive problem is not so severe but still matters, it is possible that the central bank's private asset purchases only lower welfare. Thus, in some cases with weak financial frictions, it would be suboptimal for the central bank to engage in a private asset purchase program.

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