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Statistical Analysis of Short-time Option Prices Based on a Levy Model

Weiliang Wang
Washington University in St. Louis

WASHINGTON UNIVERSITY IN ST. LOUIS

Department of Mathematics

Statistical Analysis of Short-time Option Prices Based on a Lévy Model

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Weiliang Wang

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Dedicated to My Family.

ABSTRACT OF THE THESIS

Statistical Analysis of Short-time Option Prices Based on a Lévy Model

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The Black-Scholes model has been widely used to find the prices of option, while several generalizations have been made due to its limitation. In this thesis, we consider one of the generalizations—the exponential Lévy model with a mixture of CGMY process and Brownian motion. We state the main results of the first-, second- and third-order expansions for close-to-the-money call option prices under this model. Using importance sampling based on Monte Carlo method, a dataset of call option prices can be simulated. Comparing the simulated true prices with the three different order approximations, we find that the higher-order approximation is more accurate than the lower-order in most cases, which can be used for calibrating the parameters in the model. In order to verify these results, we use call option prices obtained from the Standard & Poor's 500 index options. The third-order approximation of this real dataset is not as accurate as before.

1. Introduction and Background

In this chapter, we present some basic background for this thesis. The first section begins by making an elementary introduction to financial derivatives and options. Some useful information about the Standard & Poor's 500 index options will be considered in detail. The second section briefly reviews some mathematics background. We introduce the definition and simulation of the Brownian motion. Several important concepts and propositions related to Lévy process will be shown.

1.1 Financial Background

1.1.1 Derivatives and Options

In finance, a derivative is an asset whose value is completely determined by some other underlying assets. For example, the value of a stock option (the derivative) is determined by stock (the underlying asset), the value of a crude oil futures contract (the derivative) is determined by crude oil (the underlying asset).

Options can be divided into two types—call options and put options. A call option is the right (but not the obligation) to buy an asset at a specified price on or before a specified date. Accordingly, a put option is the right (but not the obligation) to sell an asset at a specified price on or before a specified date [1]. The specified price in options is called strike price, always denoted by K. The specified date in options is known as maturity or expiration, always denoted by T.

Options can be American style or European style. An American option may be exercised any time up until expiration (include expiration date), while European options can only be exercised at the expiration date.

In this thesis, we work with European call options (put options can be reproduced from call options by the call/put parity). Suppose S_t denotes the price of the underlying asset at time t. Then S_0 is the price at $t = 0$, which is called the spot price. A call option is called at the money call, or shortly ATM call, if the strike price is equal to the spot price ($K = S_0$). Similarly, it is called in the money call if $K < S_0$ and out of the money call if $K > S_0$. Here, we assume no dividends and zero interest rates.

Based on the definition, the payoff of a call option at time t is

$$(S_t - K)^+ = \begin{cases} S_t - K, & \text{for } S_t \geq K, \\ 0, & \text{for } S_t < K. \end{cases}$$

Then, the risk-neutral price of a call option is its expectation, which is $\mathbb{E}\left[(S_t - K)^+\right]$.

1.1.2 S&P 500 Index Options

In 1973, the Chicago Board Options Exchange(CBOE) introduced the standard call options firstly. Four years later, the put options were introduced. In 2005, a new type of option, called weekly options, was introduced by CBOE. This option begins trading on each Thursday and expires on the following Friday. The popularity of weekly options cannot be ignored due to its fast growing.

The Standard & Poor's 500 index, often abbreviated as the S&P 500 index, is one of the most common benchmarks of the overall U.S. stock market. SPX Option contracts, introduced by CBOE as products of S&P 500, are the most actively traded index options in the United States.

In October 2011, SPXPM, a kind of p.m.-settled traded SPX options, was introduced by CBOE. Soon after, SPX weekly options, abbreviated as SPXW, were listed under the SPXPM in 2013. Both standard SPX and SPXW are European options. However, there are some differences between these two types. The standard SPX is expiring on the third Friday of each month. It is a.m.-settled and ceases trading on the Thursday afternoon (one day before the expiration). The SPXW, otherwise, is p.m.-settled and trading through Friday. In other words, the standard SPX settled based on the opening price of the market on the third Friday of each month while the SPXW settled based on the closing price on Friday—its expiration day [2]. Another category of SPXPM options is called Quarterly SPX (abbreviated as SPXQ), which is expiring at the end of each quarter. It is noteworthy that if the quarter ends on a Friday, the SPXW will not be listed for that day [2].

Based on the data from CBOE, at the beginning of 2011, the SPXW only accounted for roughly 10% of the trading of S&P 500 options. However, by the end of 2014, this number grew up to 40%, which is an evidence of the popularity of weekly options.

1.2 Mathematical Background

1.2.1 Brownian Motion

A Brownian motion (or a Wiener process) is a continuous-time stochastic process named after Norbert Wiener. It plays a key role in both pure and applied mathematics. It has a profound impact on finance, in particular on the Black-Scholes model. The Brownian motion is defined as follows [3]:

Definition 1.2.1 *A stochastic process $\{W_t\}_{t\geq 0}$ is called a Brownian motion or a Wiener process with variance parameter σ^2 if the following properties are satisfied :*

- $W_0 = 0$.
- *Independent increments: For any $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_n \leq t_n$, the random variables $W_{t_1} - W_{s_1}$, $W_{t_2} - W_{s_2}$, $\ldots$, $W_{t_n} - W_{s_n}$ are independent.*
- *For any $s < t$, $W_t - W_s \sim N(0, (t-s)\sigma^2)$.*
- *The paths are continuous, i.e., the function $t \longmapsto W_t$ is a continuous function of t.*

Note that the standard Brownian motion is a Brownian motion with $\sigma^2 = 1$. Besides, a Brownian motion is a Markov process (the process that the change at time t is not determined by the values before time t, but only the value at time t (details see [3])).

Now, we consider the simulation of one dimensional Brownian motion. Based on the definition, the Brownian motion can be seen as the cumulation of a sequence of variables following a normal distribution with same mean and variance. In other words, it can be simulated by the following procedures:

$$\begin{aligned}
&X_0 = 0,\\
&X_{\Delta_t} \sim X_0 + N(0, \Delta_t\sigma^2),\\
&X_{2\Delta_t} \sim X_{\Delta_t} + N(0, \Delta_t\sigma^2),\\
&\vdots\\
&X_{i\Delta_t} \sim X_{(i-1)\Delta_t} + N(0, \Delta_t\sigma^2).
\end{aligned}$$

We can generate a path of Brownian motion based on these procedures. A realization of a standard Brownian motion in one dimension with $\Delta_t = 1$ and $t = 500$ is displayed in Figure 1.1.

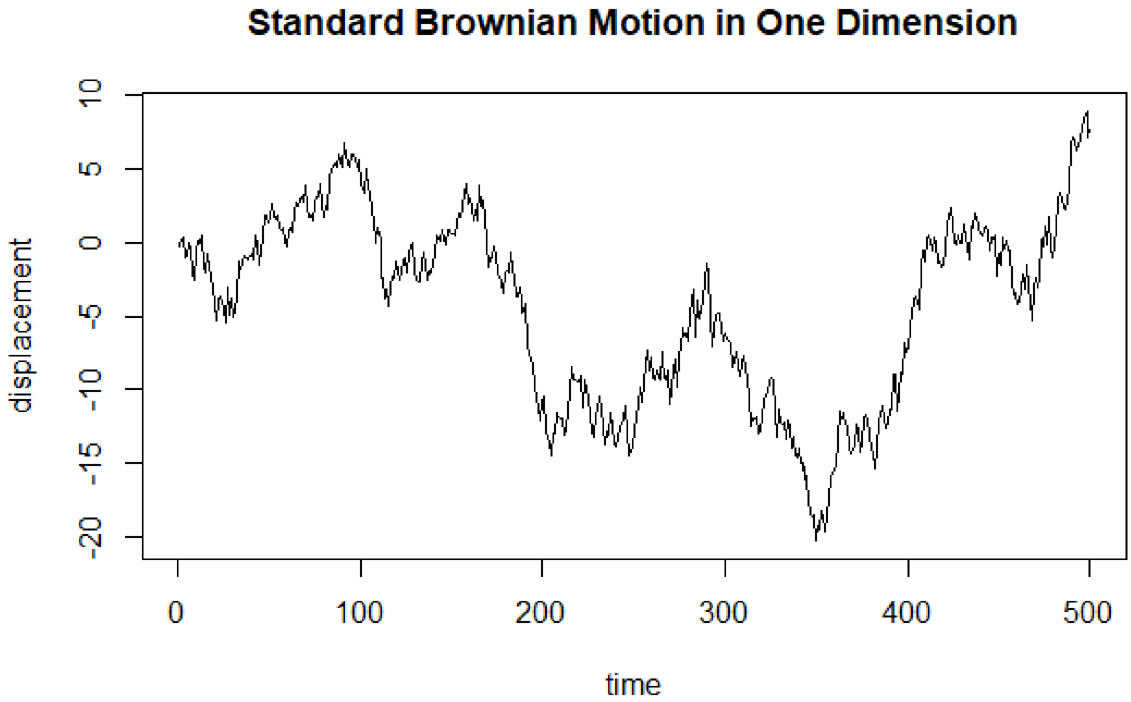

Figure 1.1. A path of a standard Brownian motion in one dimension.

1.2.2 Itô Diffusion and Geometric Brownian Motion

A diffusion process is a Markov process with almost surely continuous sample path (details see [4]). It is a solution to a stochastic differential equation. A specific type of diffusion is Itô diffusion, which is denoted by

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW_t,$$

where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion.

An important example of Itô diffusion is geometric Brownian motion, which is a common model for pricing an asset. Suppose μ is the constant excepted return and σ is the constant volatility, the stochastic differential equation of a geometric Brownian motion $\{S_t\}_{t\geq 0}$ is given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Note that $\ln S_t$ follows a Brownian motion. After some simple calculations using Itô formula (details see [3]), we can get

$$S_t = S_0 \cdot \exp\{(\mu - \sigma^2/2)t + \sigma W_t\}.$$

Such result is the basis of the Black-Scholes model.

1.2.3 Lévy Process

Lévy process, named after French mathematician Paul Lévy, plays a significant role in several fields, especially in mathematical finance. As a stochastic process in continuous time, it is the basis of many continuous-time financial models.

Definition 1.2.2 *A cadlag stochastic process $\{X_t\}_{t\geq 0}$ on R^d is called a Lévy process if the following properties are satisfied [5]:*

- $X_0 = 0$, *almost surely.*

- *Independent increments: For any* $0 \leq t_1 < t_2 < \cdots < t_n$, *the random variables* X_{t_1}, $X_{t_2} - X_{t_1}$, $\ldots$, $X_{t_n} - X_{t_{n-1}}$ *are independent.*

- *Stationary increments: The law of* $X_{t+h} - X_t$ *does not depend on* t.

- *Stochastic continuity: For any* $\epsilon > 0$ *and* $t \geq 0$, $\lim_{h \to 0} P(|X_{t+h} - X_t| > \epsilon) = 0$.

Note that it is easy to verify that a Brownian motion is an example of Lévy process. Another important concept is infinitely divisible distribution, which is shown as follows:

Definition 1.2.3 *A probability distribution* F *is said to be infinitely divisible if for all* $n \in \mathbb{N}$, *there exists independent and identically distributed (i.i.d.) random variables* X_1, $\ldots$, X_n *such that* $X_1 + \cdots + X_n$ *has distribution* F.

For example, the normal distribution is infinitely divisible. Consider $Y \sim N(\mu, \sigma^2)$, we can write $Y = X_1 + \cdots + X_n$ where X_1, $\ldots$, X_n are i.i.d. random variables with $X_k \sim N(\mu/n, \sigma^2/n)$, $k = 1, 2, \ldots, n$. When the distribution of a random variable X is infinitely divisible, there exists a triplet (μ, σ^2, ν) such that

$$\phi_X(\omega) = \mathbf{E}[e^{i\omega X}] = \exp[i\mu\omega - \frac{1}{2}\sigma^2\omega^2 + \int_R (e^{i\omega x} - 1 - i\omega x 1_{|x|<1})\nu(\mathrm{d}x)]^1,$$

where $\mu \in \mathbb{R}$, $\sigma^2 \in \mathbb{R}_{\geq 0}$ and a Lévy measure ν which satisfies the following two assumptions:

- $\nu(\{0\}) = 0$,

- $\int_R (1 \wedge |x|^2)\nu(\mathrm{d}x) < \infty$ [2].

[1] 1 is called an indicator function which is $1_{x \in A} = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$

[2] i.e. $\int_R \min\{1, |x|^2\}\nu(\mathrm{d}x) < \infty$.

The representation before is called the Lévy-Khintchine formula, which provides a complete characterization of a random variable with infinitely divisible distribution through its characteristic function. Here, μ is the drift term, σ^2 is called the diffusion or Gaussian coefficient, ν is Lévy measure. However, for a Lévy process $\{X_t\}_{t\geq 0}$, X_t is infinitely divisible(details see [6]). Therefore, for every Lévy process $\{X_t\}_{t\geq 0}$, we have

$$\mathbf{E}[e^{i\omega X_t}] = \exp[i\mu\omega - \frac{1}{2}\sigma^2\omega^2 + \int_R (e^{i\omega x} - 1 - i\omega x 1_{|x|<1})\nu(\mathrm{d}x)].$$

On the other hand, when we have a random variable Y which is infinitely divisible, we can construct a Lévy process $\{X_t\}_{t\geq 0}$ such that Y and X_t follow the same law. This conclusion can be obtained by Lévy-Itô decomposition (details see [5]). The Lévy-Itô decomposition and Lévy-Khintchine formula make a connection between Lévy processes and infinitely divisible distribution [7].

The Lévy measure ν of a Lévy processes $\{X_t\}_{t\geq 0}$ is defined by:

$$\nu(A) = \mathbf{E}[\#\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}].$$

We have seen that the Lévy measure ν is a measure satisfying:

$$\nu(\{0\}) = 0 \qquad \text{and} \qquad \int_R (1 \wedge |x|^2)\nu(\mathrm{d}x) < \infty.$$

Based on the definition, it can be seen as the excepted number of jumps per unit time of a certain size belonged to A. The Lévy measure ensures that only a finite number of large jumps can occur, while small jumps may occur finitely or infinitely many times [6]. Besides, a Lévy measure should be quadratic integrable around the origin. The Lévy measure determine the properties of Lévy process, several propositions (proof see [5]) are listed as follows:

Proposition 1.2.1 *Let $\{X_t\}_{t\geq 0}$ be a Lévy processes with triplet (μ, σ^2, ν).*

- *If* $0 < \nu(\mathrm{R}) < \infty$*, then, almost surely, all paths of* $\{X_t\}$ *have a finite excepted number of small jumps and a finite excepted number of large jumps per unit time, which is called a finite activity Lévy processes.*

- *If* $\nu(\mathrm{R}) = \infty$*, then, almost surely, all paths of* $\{X_t\}$ *have an infinite excepted number of small jumps and a finite excepted number of large jumps per unit time, which is called an infinite activity Lévy processes.*

- *If* $\sigma^2 = 0$ *and* $\int_{|x|\leq 1} |x|\,\nu(\mathrm{d}x) < \infty$*, then, almost surely, all paths of* $\{X_t\}$ *have a finite variation.*

- *If* $\sigma^2 \neq 0$ *and* $\int_{|x|\leq 1} |x|\,\nu(\mathrm{d}x) = \infty$*, then, almost surely, all paths of* $\{X_t\}$ *have an infinite variation.*

An important subclass of Lévy process is a subordinator. A process $\{X_t\}_{t\geq 0}$ is called a subordinator if it is a nondecreasing Lévy process almost surely. Since $X_0 = 0$, the value of a subordinator is non-negative. A subordinated Brownian motion W_{X_t} can be obtained through replacing the time t in Brownian motion W_t by an independent subordinator X_t.

The Gamma process is a common choice for a subordinating. The probability density at time t is given by

$$p_t(x) = \frac{\beta^{\alpha t}}{\Gamma(\alpha t)} x^{\alpha t - 1} e^{-\beta x},$$

where $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \mathrm{d}x$ is the Gamma function.

Based on the density above, the characters of Gamma process is determined by the parameters α and β. To be more specific, the mean and variance are α/β and α/β^2, respectively. Besides, the Lévy density of Gamma process is $\nu(x) = \frac{\alpha e^{-\beta x}}{x} 1_{x>0} \mathrm{d}x$. The subordinated Brownian motion of a Gamma process is called a variance gamma prcoess (details see Section 2.3).

2. The Option Pricing Model

The well-known Black-Scholes (B-S) model for a risky asset has been widely used to find a price of financial derivatives, such as options [8]. However, some assumptions in this model are not suitable in practice. In this chapter, we will consider the limitation of B-S model and present a generalization which is called exponential Lévy model. In addition, several concepts and properties of the CGMY process will be shown in this chapter. As a special case of CGMY model, some results of the variance gamma model will also be stated. In the last section, we will show the main results of expansions for close-to-the-money option prices based on exponential Lévy model with a mixture of CGMY process and Brownian motion.

2.1 Black-Scholes Model and Its Limitation

In general, the B-S model can be written as the following local form:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t,$$

or equivalently represented as the exponential form:

$$S_t = S_0 \cdot \exp\{(\mu - \sigma^2/2)t + \sigma W_t\},$$

where $\{W_t\}_{t\geq 0}$ is a standard Brownian motion. As a model built on Brownian motion whose sample path is continuous, the B-S model is one of the diffusion models. However, in practice, several jumps can be found in price behavior, which seems not meet the

Figure 2.1. Euro/Dollar exchange rate from Jan. 2014 to Dec. 2015.

property of continuity. For example, Figure 2.1 [1] shows the Euro/Dollar exchange rate from January 2014 to December 2015. We can observe some significant price changes in the graph. Several models made promotion of the B-S model by allowing the prices to jump. Two reasons for introducing jumps in financial model will be presented as follows(details see [7] and [9]):

First, in the "real" world, based on some empirical facts, the jumps exist in price and the continuous-path models may not deal with this phenomenon appropriately. Let us consider the returns (increments of the log-price) of an asset in practice (example see [7]). Several large peaks can be observed which is corresponding to jumps in the returns. This high variability is a common feature of asset returns, which leads to heavy

[1]Source of Data: http://www.macrotrends.net

tails in the distribution of returns. The diffusion models explain the heavy tails by introducing highly varying coefficients or choosing extreme values of parameters, which is unreasonable from a statistical aspect. More importantly, even though heavy tails can be generated by diffusion models, the sudden and discontinuous moves in prices cannot. However, for a model with jump, such as a Lévy model, we do not need to choose the extreme values for parameters since it is one of the generic properties of this model. Such jumps in model are helpful to capture the risk of asset.

Second, the evidence from option market show some weaknesses of diffusion models. In the "risk neutral world", consider a European call option with maturity T, strike price K and no dividends. For a fixed time t, the implied volatility depends on T and K. Figure 2.2 [2] shows a specific volatility surface. The B-S model predicts a constant value of the implied volatility which is contradicted to the empirical fact that the implied volatility is not a constant as a function of T or K. The feature of the implied volatility is known as implied volatility smile. Even though the B-S model can fit the shape of the implied volatility, it does not give an interpretation of the smile/skew feature. However, the models with jump can generate several smile/skew patterns with explanation of phenomenon in market. Moreover, when we consider the options with short maturities, the implied volatilities show a considerable skew. In continuous model, it can only be achieved by extreme values which will lead difficulty to explain or use, but jump models can handle this problem naturally. Besides, the law of returns becomes more Gaussian in diffusion models for short maturities. However, in reality, it is less Gaussian, which is the nature of jump models.

[2] Source of Data: https://www.timera-energy.com

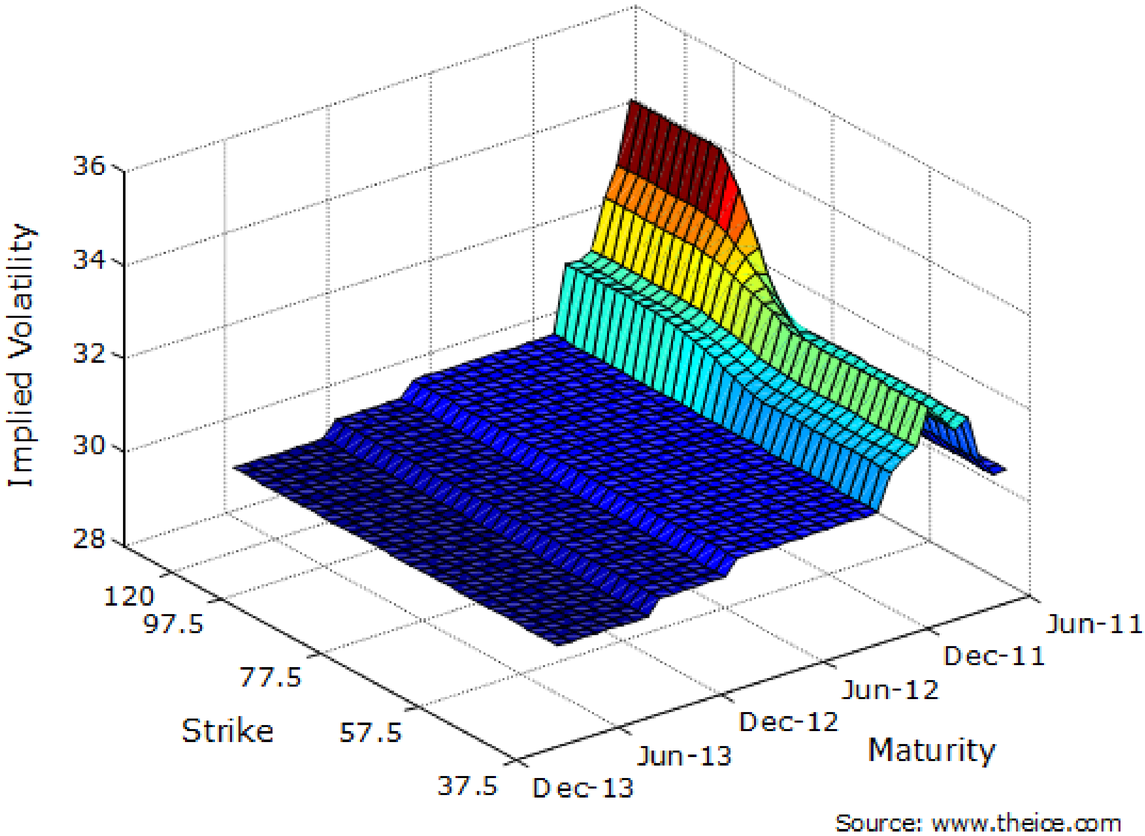

Figure 2.2. Implied volatility surface of UK gas market.

2.2 Exponential Lévy Model

To deal with the problems stated before, a common generalization of the B-S model, called the exponential Lévy model, had been considered. Suppose X represents a Lévy process, then we can construct an exponential Lévy model as follows [9]:

$$S_t = S_0 \cdot \exp\{rt + X_t\},$$

where r stands for interest rate. In our thesis, we assume zero interest rate. Then, our model becomes

$$S_t = S_0 \cdot e^{X_t}.$$

Under this model, we can derive that

$$\ln(\frac{S_t}{S_0}) = X_t.$$

Therefore, in finance, a process S_t is modeled as an exponential of a Lévy process means that the log-return follows a Lévy process.

2.3 Variance Gamma Model

So far, we have shown that a Lévy process can be finite activity or infinite activity. Jump processes with finite activity are studied by [10]. We will focus on pure jump processes with infinite activity. A common choice is the variance gamma(VG) process, which is studied by [11].

The VG process is a Brownian motion with drift at a random time which is changed by a gamma process. It can be described as a Brownian motion subject to a gamma subordinator. Suppose $b(t; \theta, \sigma)$ is a process which is a Brownian motion with drift θ and volatility σ, then

$$b(t; \theta, \sigma) = \theta t + \sigma W_t.$$

Suppose $\gamma(t;\mu,\nu)$ is a gamma process with mean rate μ and variance rate ν. Based on the density given in Section 1.2.3, we can derive the density of a gamma process at time t in this case, which is

$$p_t(x) = \frac{1}{\nu^{t/\nu}\Gamma(t/\nu)} x^{t/\nu-1} e^{-x/\nu}.$$

The VG process is defined as

$$X_{VG}(t;\sigma,\nu,\theta) = b(\gamma(t;1,\nu),\theta,\sigma) = \theta\gamma(t;1,\nu) + \sigma W_{\gamma(t;1,\nu)},$$

where θ, σ and ν are three constants. It can also be represented as the difference of two independent gamma processes, which is

$$X_{VG}(t;\sigma,\nu,\theta) = \gamma_p(t;\mu_p,\nu_p) - \gamma_n(t;\mu_n,\nu_n)$$

with the parameters given by (see [11] for details)

$$\begin{aligned}
\mu_p &= \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{v}} + \frac{\theta}{2}, \\
\mu_n &= \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{v}} - \frac{\theta}{2}, \\
\nu_p &= (\frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{v}} + \frac{\theta}{2})^2\nu, \\
\nu_n &= (\frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{v}} - \frac{\theta}{2})^2\nu.
\end{aligned}$$

Besides, the Lévy density for a VG process is also shown in [11], which is

$$K_{VG}(x) = \begin{cases} \frac{\mu_n^2}{\nu_n}\frac{exp\{-\frac{\mu_n}{\nu_n}|x|\}}{|x|}, & \text{for } x < 0, \\ \frac{\mu_p^2}{\nu_p}\frac{exp\{-\frac{\mu_p}{\nu_p}|x|\}}{|x|}, & \text{for } x > 0. \end{cases}$$

2.4 CGMY Model

As a generalization of a VG process, a CGMY process [12] has the Lévy density with four parameters, C, G, M and Y, which is

$$K_{CGMY}(x) = \begin{cases} C\frac{exp\{-G|x|\}}{|x|^{1+Y}}, & \text{for } x < 0, \\ C\frac{exp\{-M|x|\}}{|x|^{1+Y}}, & \text{for } x > 0, \end{cases}$$

where $C > 0$, $G \geq 0$, $M \geq 0$ and $Y < 2$. As we can see, when $Y = 0$, the CGMY process is exactly the VG process with parameters

$$C = \frac{1}{\nu}, \qquad G = \frac{1}{\mu_n}, \qquad M = \frac{1}{\mu_p}.$$

The four parameters in CGMY process play distinct roles to capture the characters of the model. The parameter C, which is a multiplier factor appeared both sides, can be seen as a measure of the overall level of activity. The parameter G and M, which appears in the case of $x < 0$ and $x > 0$ separately, control the rate of exponential decay on both sides of the Lévy density. When $G = M$, it is easy to verify that the density of CGMY process is symmetric. However, when $G < M$, the left tail of the distribution is heavier than the right tail, which is consistent to our observation from option market (details see [12]). The Y in different range will lead the process have different properties. The condition that $1 < Y < 2$ will be consider in our thesis. In this range, the process is completely monotone with infinite activity and infinite variation. For properties in other ranges, see [12] for details.

In fact, CGMY process is one of a tempered stable process. The tempered stable process has the Lévy density of the following form:

$$\nu(x) = \frac{c_-}{|x|^{1+\alpha}} e^{-\lambda_-|x|} 1_{x<0} + \frac{c_+}{|x|^{1+\alpha}} e^{-\lambda_+|x|} 1_{x>0},$$

where $c_- > 0$, $c_+ > 0$, $\lambda_- > 0$, $\lambda_+ > 0$ and $\alpha < 2$. It is actually the CGMY process if $c_+ = c_- = C$, $\lambda_- = G$, $\lambda_+ = M$, $\alpha = Y$.

Besides, if we allow different α for positive and negative half-lines, we call it generalized tempered stable model with Lévy measure

$$\nu(x) = \frac{c_-}{|x|^{1+\alpha_-}} e^{-\lambda_-|x|} 1_{x<0} + \frac{c_+}{|x|^{1+\alpha_+}} e^{-\lambda_+|x|} 1_{x>0}$$

with $\alpha_- < 2$ and $\alpha_+ < 2$.

2.5 Expansions for Close-to-the-Money Option Prices

In this thesis, we study the behavior of close-to-the-money European call option prices. Consider the Exponential Lévy model with the form

$$S_t = S_0 \cdot e^{X_t} \qquad \text{with} \qquad X_t = L_t + \sigma W_t.$$

Here $L = \{L_t\}_{t\geq 0}$ is a CGMY Lévy Process and $W = \{W_t\}_{t\geq 0}$ is an independent standard Brownian motion. This model can be classified into two groups—mixed model ($\sigma \neq 0$) and pure-jump model ($\sigma = 0$). We focus on the former case since several empirical evidence support that the mixed model performs better than the pure-jump one (details see [13]). For simplicity, we call our model mixed CGMY model.

Suppose $t \longmapsto \kappa_t$ is a deterministic function such that $\kappa_t \longmapsto 0$ as $t \longmapsto 0$. Here, κ_t is called log-moneyness. The strike price can be represented as $K = S_0 e^{\kappa_t}$. If $K = S_0$, we have $\kappa_t = 0$. Based on the model mentioned above, the option prices can be represented as

$$\mathbb{E}\left[(S_t - K)^+\right] = \mathbb{E}\left[(S_t - S_0 e^{\kappa_t})^+\right] = S_0 \mathbb{E}\left[(e^{X_t} - e^{\kappa_t})^+\right].$$

Consider the asymptotic behavior for close-to-the-money option prices under the mixed CGMY model. The second-order expansions for ATM call option prices have been shown as the following form[3] (details see [14]):

$$\frac{1}{S_0}\mathbb{E}[(S_t - S_0)^+] = d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3-Y}{2}} + o(t^{\frac{3-Y}{2}}), \qquad t \to 0,$$

where

$$d_1 = \frac{\sigma}{\sqrt{2\pi}} \qquad \text{and} \qquad d_2 = \frac{C 2^{\frac{1-Y}{2}} \sigma^{1-Y}}{\sqrt{\pi} Y(Y-1)} \Gamma(1 - \frac{Y}{2}).$$

[3] A function $f(t)$ is said to be "little O" of $g(t)$ at t_0, denoted as $o(g(t))$, if $f(t)/g(t) \to 0$ as $t \to t_0$. Accordingly, a function $f(t)$ is said to be "big O" of $g(t)$ at t_0, denoted as $O(g(t))$, if there exists $M > 0$ such that $|f(t)| \leq M\,|g(t)|$ as $t \to t_0$.

Based on the expressions of d_1 and d_2, only limit information contained in the second-order expansions. Specifically, d_1 only involves continuous volatility σ, while the overall level of activity C, the property parameter Y and σ are contained in d_2. However, as mentioned in Section 2.4, the parameter G and M also play important roles in the model, respectively. Therefore, in order to get more accurate results, a higher order expansion is considered, which is shown in following theorem [15]:

Theorem 2.5.1 *Suppose $t \longmapsto \kappa_t$ is a deterministic function such that $\kappa_t = o(1)$ as $t \to 0$. Let*

$$\begin{aligned}
d_{31} &:= \frac{C\Gamma(-Y)}{2}[(M-1)^Y - M^Y - (G+1)^Y + G^Y],\\
d_{32} &:= -\frac{1}{\pi}\sigma^{1-2Y}C^2\cos^2(\frac{\pi Y}{2})\Gamma^2(-Y)2^{Y-\frac{1}{2}}\Gamma(Y-\frac{1}{2}),\\
c_{\kappa,\sigma}(t) &:= \kappa_t\int_0^1 P(\sigma W_t \geq k_t w)\,dw.
\end{aligned}$$

Then for a mixed CGMY model with an independent Brownian component,

$$\frac{e^{-\kappa_t}}{S_0}\mathbb{E}[(S_t - S_0e^{\kappa_t})^+] + c_{\kappa,\sigma}(t) = d_1t^{\frac{1}{2}} + d_2t^{\frac{3-Y}{2}} + d_{31}t + d_{32}t^{\frac{5}{2}-Y} + o(\kappa_t) + o(t) + o(t^{\frac{5}{2}-Y}), t \to 0,$$

where d_1 and d_2 are same as before and both $o(t)$ and $o(t^{\frac{5}{2}-Y})$ are independent from κ_t.

Moreover, if we assume $\kappa_t = o(\sqrt{t})$ when $t \to 0$, the expansion becomes

$$\frac{e^{-\kappa_t}}{S_0}\mathbb{E}[(S_t - S_0e^{\kappa_t})^+] + \frac{\kappa_t}{2} = d_1t^{\frac{1}{2}} + d_2t^{\frac{3-Y}{2}} + d_{31}t + d_{32}t^{\frac{5}{2}-Y} + o(t) + o(t^{\frac{5}{2}-Y}) + o(\kappa_t), t \to 0.$$

The left side of the equation under this assumption is called the log-moneyness adjusted price. We can derive that if $Y \in (1, 1.5)$ and $\kappa_t = O(t)$, the third-order term of the log-moneyness adjusted price is $d_{31}t$; if $Y = 1.5$ and $\kappa_t = O(t)$, the third-order term is $(d_{31} + d_{32})t$; if $Y \in (1.5, 2)$ and $\kappa_t = O(t^{\frac{5}{2}-Y})$, the third-order term is $d_{32}t$. In this thesis, we choose the options whose strike price is very close to the spot price. Therefore, for convenience, it is reasonable to treat κ_t as zero. However, the statements before still hold since $\kappa_t = 0$ is just a the special case of the assumption $\kappa_t = O(t)$ or $\kappa_t = O(t^{\frac{5}{2}-Y})$.

3. Numerical Results

In this chapter, we firstly make a briefly introduction about our raw dataset. In addition, we present the motivation of importance sampling, which is a well-known strategy based on Monte Carlo Method. We use this strategy to simulate option prices under the mixed CGMY model. After that, the calibration of the model's parameters will be considered. We repeat the same procedures on call option prices obtained from our dataset. The main results and its comparisons will be shown at the end of this chapter. The approach presented here follows along the lines of [15].

3.1 Dataset

Our raw dataset includes the variables of standard SPX, SPXW and SPXQ in January 2014. The number of trading days in 2014 is counted as 252. The main variables are spot price S_0, option root symbol, strike price K, maturities T, bid/ask price, volume and open interest. A small part of dataset is shown in Table 3.1.

Option root symbol is a code for a option which contained several basic information. Consider an example of our dataset which symbol is SPXW140103C01300000. "SPXW" is the root symbol of the stock which stands for weekly SPX. "140103" stands for the expiration date whose 6-digit arranges in a "yymmd" form. In this example, the date is January 3rd, 2014. "C" stands for call ("P" stands for put). "01300000" is the strike price, which is 1300 in this example. The zero before 1300 allows higher strike price to

spot price	optionroot	maturities	strike	bid	ask	volume	openinterest
1831.98	SPXW140103C01300000	01/03/2014	1300	526.5	538.3	0	0
1831.98	SPXW140103C01325000	01/03/2014	1325	501.5	513.3	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1831.98	SPXW140103C01825000	01/03/2014	1825	9	10.3	6194	11180
1831.98	SPXW140103C01830000	01/03/2014	1830	5.8	6.6	2885	11445
1831.98	SPXW140103C01835000	01/03/2014	1835	3.3	3.6	4425	3610
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 3.1
Parts of raw dataset.

be shown, and the zero after, which occupied last three digits, allows the strike price accurate to three decimal places.

The bid/ask price is the latest price to buy/sell a particular option. The difference between the bid and ask is called the spread. In general, the tighter the bid/ask spread, the better the liquidity of the options. In this thesis, we take the mid values of bid and ask prices as the option prices.

The volume represents the number of contracts traded during a given period of a particular option. The open interest is the number of contracts of a particular option that have been opened but havent been closed out, exercised or expired. These two variables describe the activity of options. Generally, large volume and open interest indicate high liquidity.

Another important variable is VIX (the Volatility Index), which is calculated based on options of the S&P 500 Index and published by the CBOE. It is the first benchmark index to measure the stock markets expectation of volatility by calculating the weighted summation of the prices of S&P 500 Index calls and puts over a wide range of strike prices. To be more specific, the prices used to compute the VIX are the mid values of bid and ask prices, which coincide with our setting of option prices. The σ in this thesis can be measured by VIX divided by 100.

3.2 Call Option Prices Based on Monte Carlo Method

The Monte Carlo method was designed for approximating the integral effectively. Obviously, this method can be applied to estimate the expectation since integral always appears. Suppose $X_1, \ldots, X_n$ are i.i.d. random variables with density p. Based on the law of large numbers, the sample mean converges to the expectation, which is

$$\frac{1}{n}\sum_{i=1}^{n} f(X_i) \to \mathbb{E}_p[f(X)] = \int f(x)p(x)\mathrm{dx}.$$

The approximation follows a normal distribution around the true expectation based on the central limit theorem when n is large, with a shrinkage of variance like $1/n$. Therefore, we can estimate the expectation efficiently by drawing a large number of samples from p and calculating the average. However, sometimes drawing samples from p is not easy. One of strategies to deal with this problem is called importance sampling.

Suppose we can find another density r which is easy to simulate. This r satisfies $r(x) > 0$ whenever $p(x) > 0$. Such assumption allows us to write the equation as

$$\mathbb{E}_p[f(X)] = \int f(x)p(x)\mathrm{dx} = \int f(x)\frac{p(x)}{r(x)}r(x)\mathrm{dx} = \mathbb{E}_r[f(X)\frac{p(X)}{r(X)}].$$

Since we can draw samples $X_1, \ldots, X_n$ from r easily, we estimate the expectation $\mathbb{E}_p[f(X)]$ by

$$\frac{1}{n}\sum_{i=1}^{n} f(X_i)\frac{p(X_i)}{r(X_i)} \to \mathbb{E}_p[f(X)].$$

These procedures that we approximate the expectation under one distribution by sampling from another distribution are called importance sampling. The ratios $p(X_i)/r(X_i)$ are called importcance weight, which are random for different samples.

We use importance sampling to compute the options prices under the mixed CGMY model. For simplicity, we assume $S_0 = 1$. Consider the option prices under another probability measure $\widetilde{P}$, rather than the original measure P since it is hard for us to draw samples. After several calculations, we get the following (details see [15]):

$$\mathbb{E}\left[(e^{X_t} - e^{\kappa_t})^+\right] = \widetilde{\mathbb{E}}\left[e^{-(M-1)\bar{U_t}^{(p)}+(G+1)\bar{U_t}^{(n)}-\eta t}\left(1 - e^{\kappa_t - \bar{U_t}^{(p)} - \bar{U_t}^{(n)} - \tilde{\gamma}t - \sigma W_t^*}\right)^+\right],$$

where $\bar{U_t}^{(p)}$ and $-\bar{U_t}^{(n)}$ are independent Y-stable random variables[1] with skewness 1, location 0 and scale $(tC\,|\cos(\pi Y/2)|\,\Gamma(-Y))^{1/Y}$, which can be simulated directly. $\{W_t^*\}$ is a standard Brownian motion, $\tilde{\gamma} = -C\Gamma(-Y)[(M-1)^Y + (G+1)^Y - M^Y - G^Y] + \frac{\sigma^2}{2}$, $\eta = C\Gamma(-Y)[(M-1)^Y + (G+1)^Y]$.

Our setting of parameters is motivated by the study in [16] where the tempered stable process was considered. In that paper, based on several analyses of the real data, the following calibrated parameters were given:

$$c_+ = 0.0028, \quad c_- = 0.0025, \quad \lambda_- = 0.4087, \quad \lambda_+ = 1.9320, \quad \alpha = 1.5, \quad \sigma = 0.1.$$

For our model, we just take the midvalue of c_+ and c_- as the value of C. Then the parameters setting is

$$C = 0.00265, \quad G = 0.4087, \quad M = 1.9320, \quad Y = 1.5, \quad \sigma = 0.1.$$

Based on these setting with sample size 200000, using Monte Carlo method with importance sampling, we simulate the call option prices for one year (252 business days). Besides, we calculate the first-, second-, third-order approximations. To be more specific, since $Y = 1.5$, we get the following result:

$$d_1 t^{\frac{1}{2}}, \qquad \text{the first-order approximation,}$$

$$d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3-1.5}{2}}, \qquad \text{the second-order approximation,}$$

$$d_1 t^{\frac{1}{2}} + d_2 t^{\frac{3-1.5}{2}} + (d_{31} + d_{32})t, \qquad \text{the third-order approximation.}$$

Figure 3.1 shows the comparisons of the call option prices using Monte Carlo method with the first-, second-, third-order approximations. The first-order approximation underestimates the true values while the second-order one makes an overestimation. However,

[1] A distribution is said to be stable if a linear combination of two i.i.d random variables with this distribution has the same distribution, up to location and scale parameters. A random variable is called stable random variable if its distribution is stable. A stable distribution is also called the Lévy α-stable distribution, named after Paul Lévy.

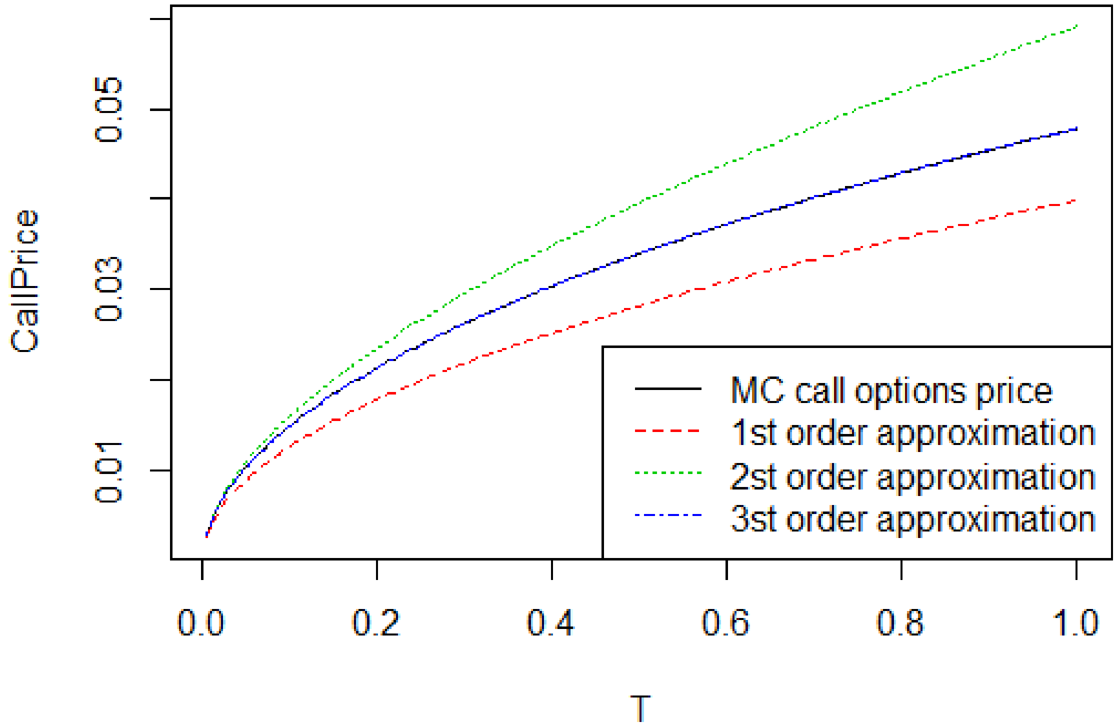

Figure 3.1. One-year call option prices under the mixed CGMY model using Monte Carlo method and the first-, second-, third-order approximations.

the third-order approximation is much precise than the others, which almost coincides with the true value.

3.3 Calibration of the Model's Parameters

In reality, errors exist in option prices, which will lead some difficulties of the calibration of model's parameters. In this section, we will show the approximations before can be used to make calibration of parameters even though the errors exist. To be more specific, we focus on the calibration of the parameter C and the volatility σ.

In fact, only the maturities t and the log-moneyness κ are available. However, we assume $\kappa = 0$ since we only consider ATM option prices or close-to-the-money option prices. Therefore, let $\Pi_i^\star := \Pi^\star(t_i)$ denote the observed option prices at maturities t_i and ϵ_i is the corresponding random errors. Let also $\hat{Y}$ be an estimate of Y. An obviously strategy to make an estimation of σ is fitting linear models to the data as follows:

$$\Pi_i^\star = d_1 t_i^{\frac{1}{2}} + \epsilon_i, \qquad \text{the first-order approximation,}$$

$$\Pi_i^\star = d_1 t_i^{\frac{1}{2}} + d_2 t_i^{\frac{3-\hat{Y}}{2}} + \epsilon_i, \qquad \text{the second-order approximation,}$$

$$\Pi_i^\star = d_1 t_i^{\frac{1}{2}} + d_2 t_i^{\frac{3-\hat{Y}}{2}} + d_{31} t_i + d_{32} t_i^{\frac{5}{2}-\hat{Y}} + \epsilon_i, \qquad \text{the third-order approximation.}$$

Using the least square estimate, we get the estimation of d_1 in these three models, which can be represented as $\hat{d}_1^{(1)}$, $\hat{d}_1^{(2)}$, $\hat{d}_1^{(3)}$, respectively. Since $d_1 = \sigma/\sqrt{2\pi}$, the estimates for σ are then given by

$$\hat{\sigma}^{(m)} = \sqrt{2\pi}\hat{d}_1^{(m)}, \qquad m = 1, 2, 3.$$

Similarly, we can apply the same strategy to estimate the parameter C. Suppose the estimations of d_2 based on the second- and third-order models are represented as $\hat{d}_2^{(2)}$ and $\hat{d}_2^{(3)}$, respectively. Since we have

$$d_2 = \frac{C 2^{\frac{1-Y}{2}} \sigma^{1-Y}}{\sqrt{\pi} Y (Y-1)} \Gamma(1 - \frac{Y}{2}) =: C\sigma^{1-Y} m_Y,$$

the estimations of C can be written as

$$\hat{C}^{(m)} = \hat{d}_2^{(m)} (\hat{\sigma}^{(m)})^{\hat{Y}-1} m_{\hat{Y}}^{-1}, \qquad m = 2, 3.$$

Moreover, the numerical results should be generated based on reasonable maturities. Since the estimates we got are based on short-time asymptotic for option prices, it is reasonable to only consider relatively small maturities, which will cause the reduction of sample size. Therefore, we should handle this tradeoff by choosing proper maturities.

Consider the S&P 500 index options which quote date is January 2nd, 2014. Based on the discussion above, we select the maturities which is less than one year. Then, we have the following maturities[2] :

$$t_i \in \{0.003, 0.022, 0.044, 0.060, 0.008, 0.140, 0.217, 0.242, 0.294, 0.467, 0.492, 0.717, 0.967\}$$

Using the same method and same parameters setting in Section 3.2, we simulate the ATM option prices at t_i. Figure 3.2 shows the comparisons of the options prices with the first-, second-, third-order approximations. Here, $\hat{Y} = 1.5$. The conclusion is similar to Section 3.2 that the third-order approximation is the most precise one which is almost overlapping with the true values.

Now we consider the sensitivity of the estimate of σ to the value of Y. Since we cannot get the exactly value of Y most of the times, we calculate the estimations of $\hat{\sigma}^{(m)}$ in the range $\hat{Y} \in [1.1, 1.8]$. Figure 3.3 shows the results of three approximations with different orders. Even though the second-order approximation is a little better than the first-order, both of them are far from the exact values. However, the third-order approximation is accurate to the range of $\hat{Y}$ with estimations of σ between 0.098 and 0.105. Therefore, based on third-order approximation, we can estimate σ well even if $\hat{Y}$ is not accurate.

On the other hand, Figure 3.4 considers the sensitivity of the estimate of C to the value of Y. The estimations based on the second-order approximation are always far from the true values, but not sensitive to the value of $\hat{Y}$, while that based on the third-order approximation are sensitive, unlike the estimate of $\sigma^{(3)}$. However, it is precise if $\hat{Y}$ around the true value.

The conclusions we reached before are based on thirteen available maturities. However, as we stated before, the approximations are under the assumption of short-time

[2] The maturities are shown in year, which can be calculated directly by the software. Here, only three decimal places are displayed.

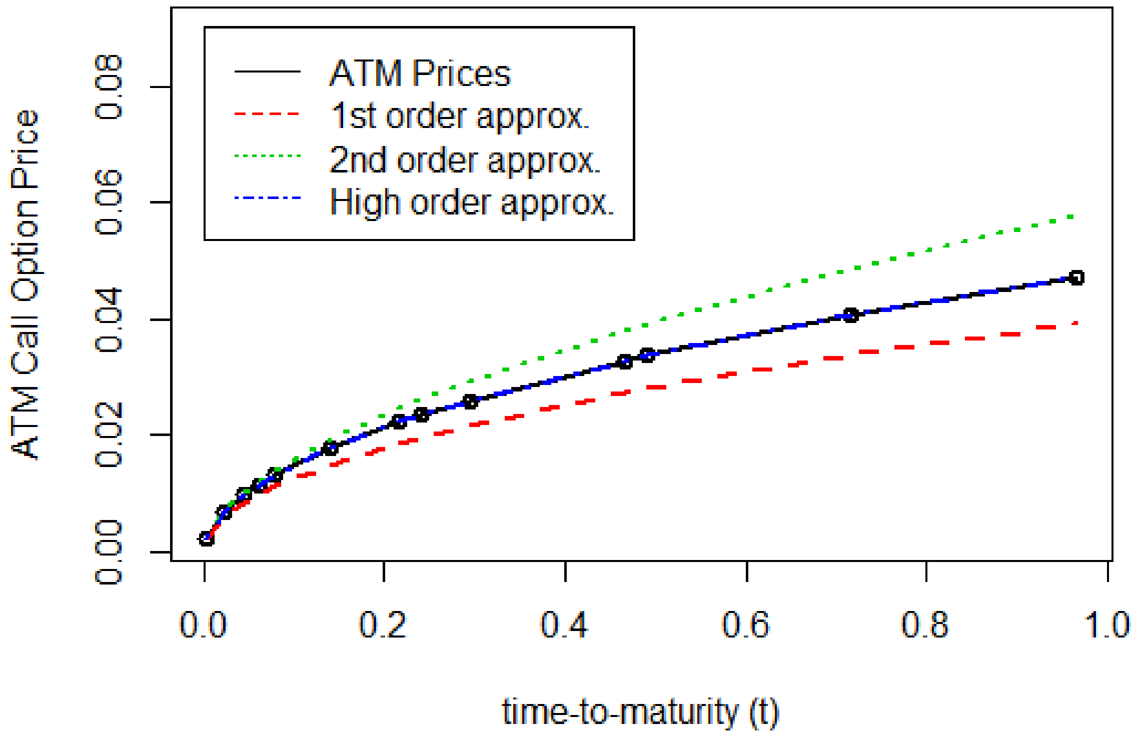

Figure 3.2. Call option prices under the maturities t_i based on the mixed CGMY model using Monte Carlo method and the first-, second-, third-order approximations.

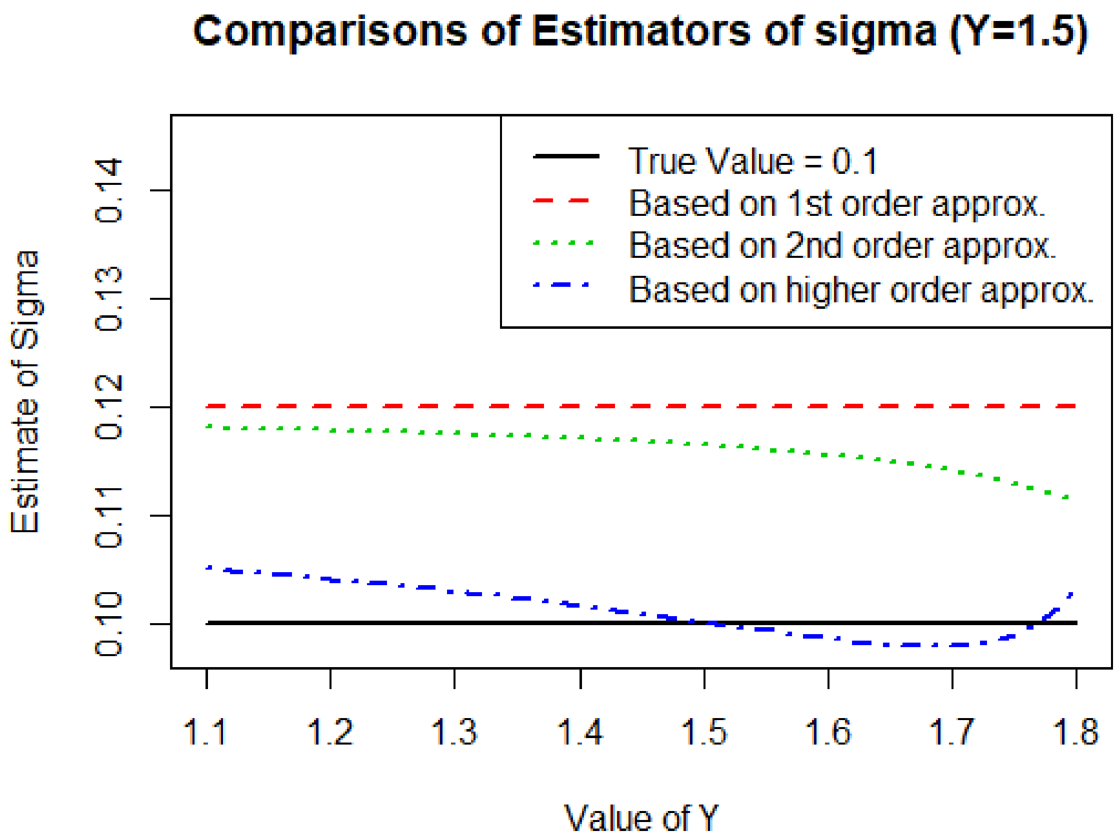

Figure 3.3. Estimates of σ based on the call option prices under the mixed CGMY model and the maturities t_i.

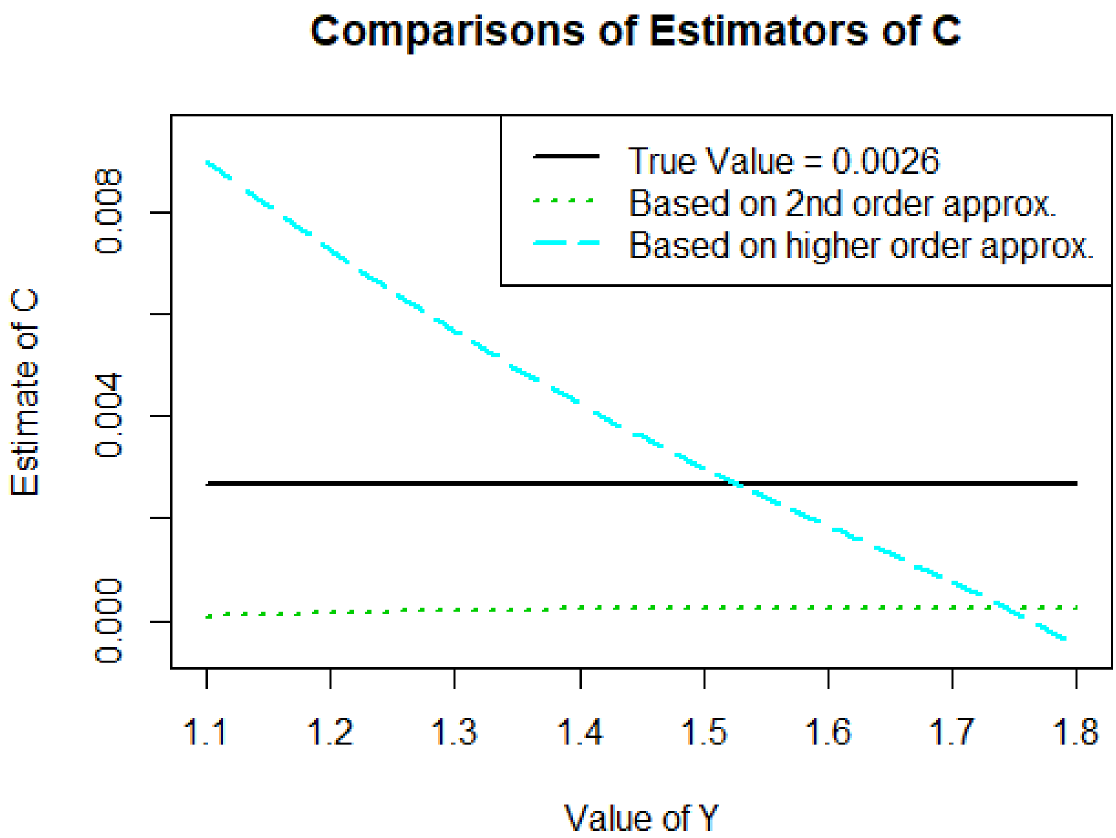

Figure 3.4. Estimates of C based on the call option prices under the mixed CGMY model and the maturities t_i.

maturity. A natural question is: If we ask for much closer maturities, i.e., using less maturities, what results we will get. We then try maturities in three conditions— within one month (the first five maturities), one quarter (the first eight maturities) and half of a year (the first eleven maturities). After repeat the procedures before, we get the results shown in Figure 3.5–3.7.

For the call option prices and the three order approximations, we can find that the higher-order approximation is always better than the lower. However, when sample size become smaller, the differences between higher- and lower- order become smaller.

For the estimates of σ, the first-order approximation is always far from the true value since it only depends on d_1, which will give us a constant estimation. The second-order approximation become better when the number of maturities is small, and it becomes more accurate when Y is greater than its true value. However, the higher-order approximation is better than the second-order most of times. The sensitivity is reflected by the figures that we can give a good estimation even if $\hat{Y}$ is not accurate based on the higher-order approximation.

The estimates of C is not very similar. We can see that even though the second-order approximation always give us an underestimation, it is less sensitive to the value of Y, The estimates become closer to the true value when the sample size is smaller. On the other hand, the higher-order approximation can give us an accurate estimate when Y is around the true value, while it is much more sensitive than the second-order, especially when sample size is small. Therefore, we may consider the second-order approximation to estimate σ if we cannot give an accurate estimate of Y or the sample size is very small.

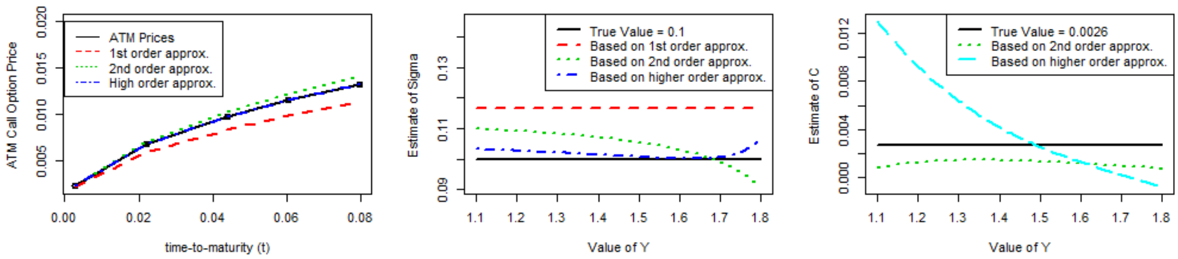

Figure 3.5. Results under the mixed CGMY model and the first five maturities using Monte Carlo method. The left panel is the call option prices and the first-, second-, third-order approximations. The middle and right panels are the estimates of σ and C, respectively.

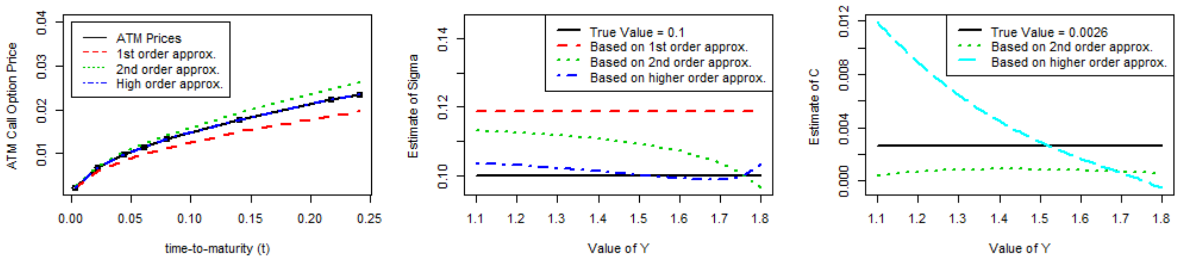

Figure 3.6. Results under the mixed CGMY model and the first eight maturities using Monte Carlo method. The left panel is the call option prices and the first-, second-, third-order approximations. The middle and right panels are the estimates of σ and C, respectively.

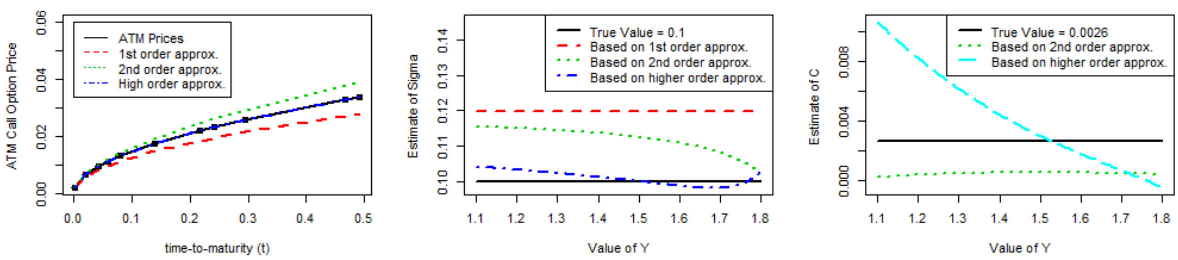

Figure 3.7. Results under the mixed CGMY model and the first eleven maturities using Monte Carlo method. The left panel is the call option prices and the first-, second-, third-order approximations. The middle and right panels are the estimates of σ and C, respectively.

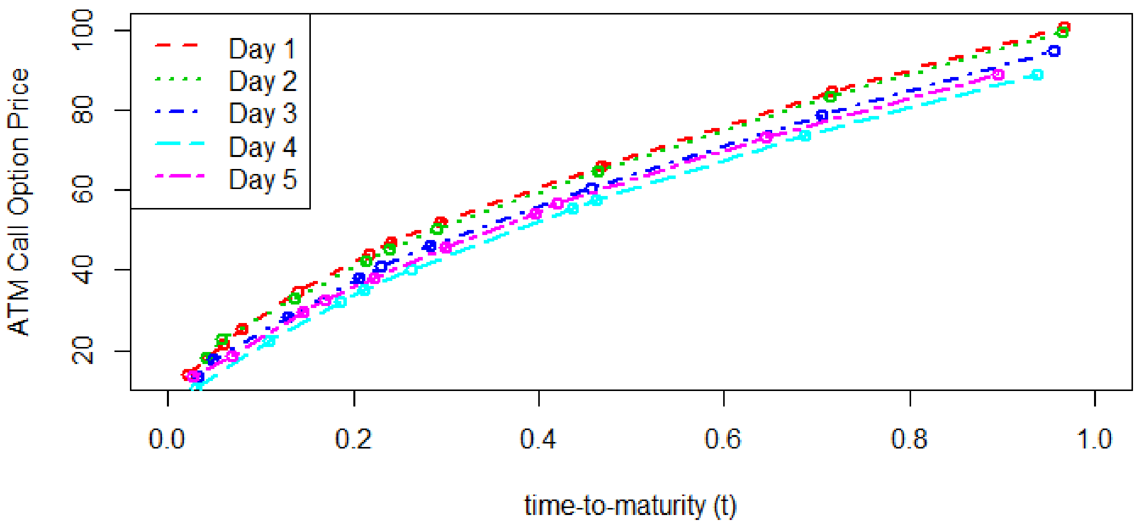

Figure 3.8. Five-day close-to-the-money S&P 500 index call option prices in January 2014.

3.4 Results Based on Real Dataset

In Section 3.3, we show some numerical results based on the call prices generated by Monte Carlo method. In this section, we apply the strategy to the call option prices obtained from S&P 500 index options. As we stated before, the option prices (calculated by mid values of bid and ask) we selected should be very close to the spot price and the maturities should not be very large. Besides, since the volume and option interest are major criterion to judge the liquidity of the market, we will drop some data which both volume and open interest are very low. For illustration, we choose our call option prices from five different quote dates in January 2014. The graph of the option prices is shown in Figure 3.8.

After selecting the call prices, we repeat the procedures in the Section 3.3. Figure 3.9 shows the estimates of σ and C for five different days.

For the estimate of σ, the second-order approximation becomes less sensitive than the third-order in most cases, which is different from the results we obtained from call prices generated by Monte Carlo method. Since VIX can reflect the level of volatility, we compare it with the mean of the volatility. The results are shown in Figure 3.10. We can see that the results generated by the second-order approximation are consistent. The trend of σ is same as the VIX and still being underestimated. However, the third-order approximation give us totally different results. Since it is sensitivity to the value of Y, especially when Y is greater than 1.5, it is hard for us to obtain a precise estimate of σ based on the third-order approximation.

For the estimate of C, the sensitivity can be seen as consistent with Section 3.3. The second-order approximation is much less sensitive than the third-order. Since we do not have the exact value of C, it may not easy for us to judge whether these approximations give us correct results. However, one assumption of our model is $C > 0$. In most cases, the higher-order approximation gives us a negative estimate, while the estimation based on second-order approximation is positive, Therefore, the second-order approximation seems more reasonable if we consider about the real data.

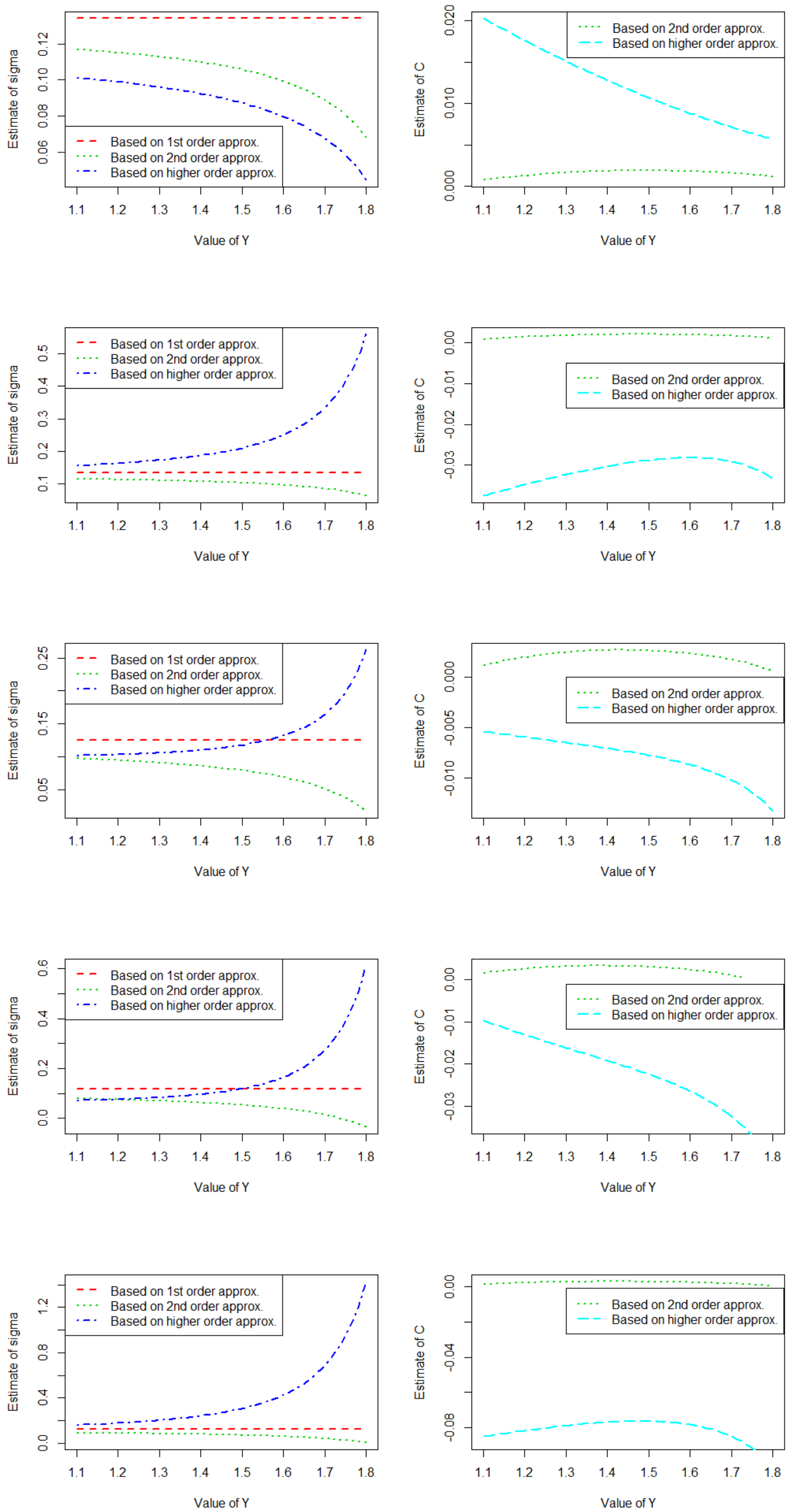

Figure 3.9. Estimates of σ and C based on the five-day option prices. Left five panels: the estimates of σ. Right five panels: the estimates of C.

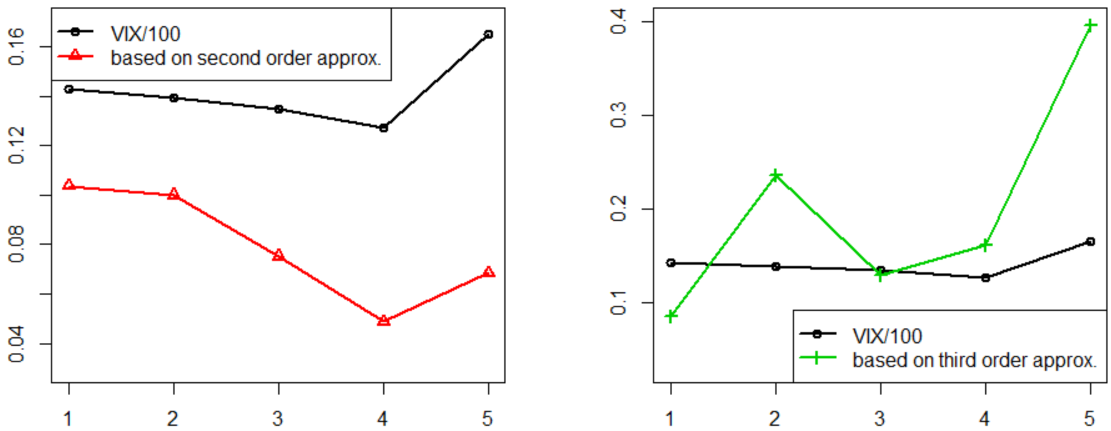

Figure 3.10. Comparisons of the VIX divided by 100 to the estimates of σ. The left and right panels are based on the second- and third-order approximation, respectively.

4. Conclusion

In this thesis, we use the exponential Lévy model with a mixture of CGMY process and an independent Brownian motion. We consider the expansions for close-to-the-money call option prices under this model. The results obtained from the call prices generated by Monte Carlo method show that the third-order approximation is usually better than the first- and second-order for the estimate of σ, no matter how many maturities we use. For the estimate of C, the higher-order is more sensitive than the second-order, which can only give us the accurate results when we know the exact value of Y. However, the results obtained from the real call prices give us a different conclusion when we consider the estimate based on the third-order approximation. I think several potential factors may explain these results.

Firstly, our raw dataset contains three types of call options— SPX, SPXW and SPWQ. Since we only consider short-time maturities, we have to mix them together to get enough sample size. It is possible that fitting the model with three types of options lead us get the different results.

Secondly, the method that we choose call prices is subjective to some degree. In Monte Carlo method, to generate the call prices precisely, we use the sample size which is large enough. Therefore, the call prices we obtained are exactly fitted the model, then the exactness of higher-order approximation is reasonable, even under small numbers of maturities. However, the prices in real dataset were chosen by hand. Even though we have our rules such as the call prices should be close to spot price with large volume and open interest, we may have lots of choices and all of them seem imperfect but with some reasonableness. It is hard to expect that our call prices obtained are as ideal as the simulation results, which makes higher-order approximation not precisely.

In addition, it is probable that our model may not fit the data perfectly. Even though it makes a generalization of the B-S model, the four parameters, C, G, M and Y, may not cover all features of the S&P 500 index options. The similar type models such as tempered stable model or generalized tempered stable model may improve our result here.

In the future work, we may try to find a reasonable way to select ideal data. Also, we treat the log-moneyness κ as zero in this thesis, we may think about the value of κ since it is not exactly equal to zero for close-to-the-money option prices. Besides, we may consider other generalizations of B-S model and then make a comparison of them, to find whether we can improve our model.

REFERENCES

[1] John Van der Hoek and Robert J. Elliott. *Binomial Models in Finance*. Springer Science & Business Media, 2006.

[2] Russell Rhoads. *Trading Weekly Options: Pricing Characteristics and Short-term Trading Strategies*. John Wiley & Sons, 2014.

[3] Gregory F. Lawler. *Introduction to Stochastic Processes*. CRC Press, 2006.

[4] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic Differential Equations and Diffusion Processes*, volume 24. Elsevier, 2014.

[5] Ken-iti Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 1999.

[6] Antonis Papapantoleon. An Introduction to Lévy Processes with Applications in Finance. *arXiv preprint arXiv:0804.0482*, 2008.

[7] Rama Cont. *Financial Modelling with Jump Processes*. Chapman and Hall, 2004.

[8] Fischer Black and Myron Scholes. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[9] Peter Tankov. Pricing and Hedging in Exponential Lévy Models: Review of Recent Results. In *Paris-Princeton Lectures on Mathematical Finance 2010*, pages 319–359. Springer, 2011.

[10] John C. Cox and Stephen A. Ross. The Valuation of Options for Alternative Stochastic Processes. *Journal of Financial Economics*, 3(1-2):145–166, 1976.

[11] Dilip B. Madan, Peter P. Carr, and Eric C. Chang. The Variance Gamma Process and Option Pricing. *Review of Finance*, 2(1):79–105, 1998.

[12] Peter Carr, Hélyette Geman, Dilip B. Madan, and Marc Yor. The Fine Structure of Asset Returns: An Empirical Investigation. *The Journal of Business*, 75(2):305–332, 2002.

[13] Yacine Aït-Sahalia and Jean Jacod. Is Brownian Motion Necessary to Model High-frequency data? *The Annals of Statistics*, pages 3093–3128, 2010.

[14] José E. Figueroa-López, Ruoting Gong, and Christian Houdré. High-order Short-time Expansions for ATM Option Prices of Exponential Lévy Models. *Mathematical Finance*, 26(3):516–557, 2016.

[15] José E. Figueroa-López, Ruoting Gong, and Christian Houdré. Third-order Short-time Expansions for Close-to-the-money Option Prices under the CGMY Model. *Applied Mathematical Finance*, 24(6):547–574, 2017.

[16] Leif Andersen and Alexander Lipton. Asymptotics for Exponential Lévy Processes and Their Volatility Smile: Survey and New Results. *International Journal of Theoretical and Applied Finance*, 16(01):1350001, 2013.