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On the Orientation Dependence of the Casimir Force

Christopher David Markle
Washington University in St. Louis

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On the Orientation Dependence of the Casimir Force

by

Christopher D. Markle

A dissertation presented to the
Graduate School of Arts and Sciences
of Washington University in St. Louis in
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requirements for the degree
of Doctor of Philosophy

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Contents

List of Figures ............................................................. vi
List of Tables ............................................................. vii
Acknowledgements ......................................................... viii
Abstract ........................................................................ ix

1 Introduction .................................................................... 1
  1.1 Previous Calculations ....................................................... 3
     1.1.1 London-Hamaker ...................................................... 4
     1.1.2 Casimir ................................................................. 8
     1.1.3 Lifshitz ................................................................. 13
     1.1.4 Finite Temperature .................................................... 20
     1.1.5 Anisotropic Materials ................................................. 23
     1.1.6 State of Casimir Force Theory ...................................... 24
  1.2 Previous Experiments ...................................................... 24
     1.2.1 Overbeek and Sparnaay (1954) .................................... 25
     1.2.2 Derjaguin and Abrikosova (1957) ................................. 31
     1.2.3 Overbeek and Sparnaay (1960) .................................... 41
     1.2.4 Lamoreaux (1996) .................................................... 46
     1.2.5 Decca et al. (2007) ................................................... 50
     1.2.6 Munday (2009) ....................................................... 54
     1.2.7 Lamoreaux (2011) .................................................... 57
     1.2.8 State of Casimir Force Experimentation ......................... 62

2 Theory ........................................................................... 65
  2.1 Introduction ............................................................... 65
     2.1.1 Basic Approach ....................................................... 66
  2.2 Casimir Case ............................................................. 70
  2.3 Parallel-plate cavity of finite size ..................................... 73
  2.4 Single Spherical Shell .................................................... 76
  2.5 Anisotropic parallel-plate cavity ..................................... 82
     2.5.1 Isotropic Case ....................................................... 88
List of Figures

1.1 Experimental Schematic for Overbeek and Sparnaay(12) .................. 26
1.2 Schematic of Newton Rings(12) ........................................ 27
1.3 Observed Force vs Distance(12) ......................................... 28
1.4 Proximity Force Approximation(14) ..................................... 33
1.5 Experimental Results for Derjaguin and Abrikosova 1(14) ............... 35
1.6 Experimental Results for Derjaguin and Abrikosova 2(14) ............... 36
1.7 Experimental Results for Derjaguin and Abrikosova 3(14) ............... 37
1.8 Experimental Results for Derjaguin and Abrikosova 4(14) ............... 38
1.9 Experimental Results for Derjaguin and Abrikosova 5(14) ............... 39
1.10 Experimental Results for Derjaguin and Abrikosova 6(14) ............... 40
1.11 Experimental design of Overbeek and Sparnaay (1960) ................. 42
1.12 Experimental Results for flat plates(13) .................................. 43
1.13 Experimental Results for plate-sphere(13) .................................. 44
1.14 Experimental design of Lamoreaux (1996)(35) ............................ 46
1.15 Schematic of the Pendulum Design(35) .................................... 47
1.16 Single-Run Experimental Results(35) .................................... 48
1.17 Experimental Results(35) .................................................... 49
1.18 Experimental Setup for Decca et al.(20) ................................. 50
1.19 Data for Decca et al.(20) ................................................... 53
1.20 Materials for Munday et al.(16) ............................................ 54
1.21 Experimental Setup for Munday et al.(16) ............................... 55
1.22 Data for Munday et al.(16) ................................................. 56
1.23 Experimental setup for Lamoreaux et al.(34) ............................. 58
1.24 Data for Lamoreaux et al.(34) ............................................. 60
1.25 Corrected data for Lamoreaux et al.(34) ................................. 61

2.1 Two interacting surfaces $S_1$ and $S_2$ ....................................... 67
2.2 Two parallel plates separated by a distance $a$ in the z-direction ........ 74
2.3 Log-Log graph of energy vs separation .................................... 76
2.4 Schematic of the Single Sphere Problem ................................... 77
2.5 Schematic of the Anisotropic Parallel Plane Problem ..................... 83
2.6 The Normalized Casimir Potential Energy per Unit Area ................ 91

3.1 Our Experimental Setup part 1 ............................................. 98
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.2</td>
<td>Our Experimental Setup part 2</td>
<td>99</td>
</tr>
<tr>
<td>3.3</td>
<td>Our Experimental Setup part 3</td>
<td>100</td>
</tr>
<tr>
<td>3.4</td>
<td>Optical Properties of Graphite</td>
<td>104</td>
</tr>
<tr>
<td>3.5</td>
<td>Top illuminated fiber</td>
<td>105</td>
</tr>
<tr>
<td>3.6</td>
<td>Back illuminated fiber</td>
<td>106</td>
</tr>
<tr>
<td>3.7</td>
<td>Schematic of the fiber</td>
<td>111</td>
</tr>
<tr>
<td>3.8</td>
<td>Predicted Force</td>
<td>118</td>
</tr>
<tr>
<td>3.9</td>
<td>Predicted Amplitude 1</td>
<td>119</td>
</tr>
<tr>
<td>3.10</td>
<td>Predicted Amplitude 2</td>
<td>120</td>
</tr>
<tr>
<td>3.11</td>
<td>Predicted Gradient of the Casimir Force</td>
<td>122</td>
</tr>
<tr>
<td>3.12</td>
<td>Predicted Gradient Amplitude</td>
<td>123</td>
</tr>
<tr>
<td>3.13</td>
<td>Data from the digital probe</td>
<td>130</td>
</tr>
<tr>
<td>3.14</td>
<td>Data from the digital camera</td>
<td>131</td>
</tr>
<tr>
<td>3.15</td>
<td>Fourier Transform of Experimental Data (Log-Log)</td>
<td>132</td>
</tr>
<tr>
<td>3.16</td>
<td>Uncorrected Experimental Data</td>
<td>133</td>
</tr>
<tr>
<td>3.17</td>
<td>First Correction to the Experimental Data</td>
<td>134</td>
</tr>
<tr>
<td>3.18</td>
<td>First Correction to the Experimental Data with fit</td>
<td>136</td>
</tr>
<tr>
<td>3.19</td>
<td>Residuals of the experimental data</td>
<td>137</td>
</tr>
</tbody>
</table>
List of Tables

1.1 Lifshitz function for the Casimir force in the retarded regime. . . . . 20
3.1 Integrated Components . . . . . . . . . . . . . . . . . . . . . . . . . . . . 97
3.2 Sources of Error . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 129
B.1 Tabulated Data . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 144
B.2 Tabulated Data cont . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 145
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In the 83 years of its development, Casimir force theory has seen many advances. However, there still exists a need for more comprehensive approaches, particularly to complex geometries. To this end we have developed an approach to the Casimir force and demonstrated its use in predicting the Casimir force present in many cases such as the case of a single hollow sphere, finite size parallel plates, and anisotropic conductors.

Within the field of Casimir force experimentation, many predicted characteristics have not yet been demonstrated. To that end, we are performing an experiment designed to investigate the orientational dependence of the Casimir force between parallel plates with in-plane optical anisotropy. Such an experiment is the first of its kind and if an orientational dependence is observed it will unambiguously state that the Casimir force is not the result of a scalar interaction. The experiment described herein is ongoing; preliminary results our given describing the current state of the apparatus.
Chapter 1

Introduction

This dissertation describes the theoretical and experimental work performed by the candidate. Chapter 1 describes the current state of the Casimir force theory and experiment in order to motivate the need for such work. Chapter 2 details the novel approach to the Casimir force developed in house and demonstrates its use for particular cases. In Chapter 3 the experiment and its current state are described and preliminary data analyzed. We conclude the dissertation in Chapter 4 with a review and discussion of future developments.

In the following sections of this chapter we shall review the theoretical and experimental work that has shaped this field. We begin by detailing the seminal derivations in the field, beginning with the works of Fritz London(1) and H. C. Hamaker(2), proceeding with Casimir’s(4) and Lifshitz’(5) derivations, and finishing this review with derivations and comments relating to finite temperature corrections and approaches to anisotropic materials. The last half of this chapter is devoted to an experimental
review of the field, beginning with the early works of Overbeek and Sparnaay(12) and Derjaguin and Abrikosova(14) and finishing with modern works by Lamoreaux(34), Decca(20), and Munday(23).

In Chapter 2 we present novel theoretical work performed by the candidate(38) and published in Physical Review A in 2012. This work details a complete theory of the Casimir force and demonstrates its use in solving such problems as the Casimir case, the case of finite sized objects, the single hollow sphere, and anisotropic materials. At the very end of this section we detail possible areas of future theoretical work and motivate the need for further experimental work in this field.

In Chapter 3 we present the experimental work performed by the candidate. It should be noted that the experimental work described in this dissertation is preliminary; the apparatus is still being characterized and sources of systematic error are still under investigation. Chapter 3 begins with a general description of the apparatus followed by an in-depth description of the primary sub-systems, we proceed with descriptions of methods that could be used to acquire and analyze the data, we present calculations regarding the expected signal from the Casimir force, we describe in detail the known sources of error in the instrument, we present some preliminary data as means of demonstrating these sources of error, and at the very end of this section we detail several likely courses of action that we will undertake in order to complete this experiment.
1.1 Previous Calculations

The field of Casimir force research was born in 1873 with the pioneering work of Johannes Diderik van der Waals who showed that gases with pairwise attractive intermolecular forces obeyed the equation of state:

\[
\left( p - \frac{n^2a}{V^2} \right) (V - nb) = nRT \tag{1.1}
\]

This equation, known as the van der Waals equation, is a modification of the ideal gas law and relates the pressure \(p\), volume \(V\), temperature \(T\), number of moles of atoms \(n\), intermolecular attraction of atoms \(a\), and the volume excluded by a mole of atoms \(b\). The success of this equation in predicting the behavior of gases and liquids gave further credence to the idea of intermolecular forces and motivated research into the nature of these forces.

After 57 years and the birth of quantum field theory, Fritz London derived an expression for one of the intermolecular forces, now known as the London dispersion force. This force was produced by molecules with instantaneous dipole moments (no net dipole moment). The London dispersion force was soon applied to macroscopic bodies (groups of molecules), and the resulting theory became known as the London-Hamaker theory. However, experiments between macroscopic bodies were found to be in disagreement with the London-Hamaker predictions. The failure was later found to be the method of application to macroscopic bodies.

In 1948 Hendrik B. G. Casimir found a different method of extending the London force.
1.1 Previous Calculations

dispersion force to macroscopic bodies. Where London-Hamaker theory was based on
the molecular properties of the materials (polarizability of the molecules), Casimir’s
theory was based on the bulk properties of the materials (dielectric function of the
material). This shift allowed predictions based on Casimir’s work to be made and
found to be in agreement with experimental results.

In the following subsections we derive the above mentioned theories. We also go
on to extend and generalize these theories according to the seminal derivations in the
field. We conclude with a review of the state of Casimir force theory.

1.1.1 London-Hamaker

In 1930 F. London generalized the idea of intermolecular forces to bulk objects
(1). It is of some interest to derive London’s expression. To do so, we shall follow a
derivation presented in The Quantum Vacuum(6).

London found that two atoms in close proximity experience a force of attraction.
This force, known as the London dispersion force, stems from the mutual interactions
between the atoms’ instantaneous dipole moments. To derive this force, let us con-
sider two non-ionized, non-excited atoms in close proximity. In this calculation close
proximity, or the near field regime, refers to $r << c/\omega_o$ where $r$ is the separation and
$\hbar\omega_o$ is the ground state energy of the atom producing the field under consideration.

Let $p_1$ and $p_2$ be the dipole moments of atom 1 and 2, respectively. In the absence
of an external field these atoms have no permanent dipole moment, $\langle p_1 \rangle = \langle p_2 \rangle = 0$.
However, the atoms will have instantaneous dipole moments, $\bar{p}_1$ and $\bar{p}_2$. The potential
energy of dipole 1 in the electric field produced by dipole 2 is,

\[ U_1 = -\vec{p}_1 \cdot \vec{E} = -\left( \frac{3(\vec{p}_1 \cdot \hat{s})(\vec{p}_2 \cdot \hat{s}) - \vec{p}_1 \cdot \vec{p}_2}{r^3} \right) \]

\[ -\frac{qe^2}{r^3} x_1 x_2 \]  

where \( \hat{s} \) is the unit vector between dipoles 1 and 2. In the last equation we have simplified the notation using \( \vec{p}_i = e x_i \hat{\mu}_i \) where \( e \) is the charge of the electron, \( \hat{\mu}_i \) is the unit vector in the direction of the dipole moment, and \( x_i \) is the magnitude of the dipole moment per unit charge. We have also used \( q = 3(\hat{\mu}_1 \cdot \hat{s})(\hat{\mu}_2 \cdot \hat{s}) - \hat{\mu}_1 \cdot \hat{\mu}_2 \), the dimensionless orientation factor. From this expression we can derive a “force”,

\[ F_1 = -\frac{\partial}{\partial x_1} U_1 = \frac{qe^2}{r^3} x_2 \]  

This force acts on the electron in atom 1 in such a way as to create and anti-align the dipoles. As the electron in an atom acts, to a good approximation, like a harmonic oscillator of natural frequency \( \omega_0 \), we can write the equations of motion:

\[ \ddot{x}_1 + \omega_0^2 x_1 = K x_2 \]  

\[ \ddot{x}_2 + \omega_0^2 x_2 = K x_1 \]  

where \( K = \frac{qe^2}{mr^3} \). This system represents a coupled harmonic oscillator system with normal modes:

\[ \omega_\pm = \left( \omega_0^2 \pm K \right)^{1/2} \]
The Hamiltonian of a quantum harmonic oscillator(43) leads to energy eigenvalues of:

\[ E = \sum_{k\lambda} \hbar \omega_k \left( n_{k\lambda} + \frac{1}{2} \right) \]  
\[ (1.8) \]

where \( n_{k\lambda} \) is the number of photons in a state with wave vector \( k \) and polarization \( \lambda \), with \( \omega \) representing the frequency. For the purposes of this calculation we are only interested in the vacuum state, the state with \( n = 0 \). Thus the ground state energy of this system is:

\[ E = \frac{1}{2} \hbar (\omega_+ + \omega_-) \approx \hbar \omega_o - \frac{\hbar K^2}{8\omega_o^3} \]  
\[ (1.9) \]

where we expanded \( \omega_+ \) and \( \omega_- \) to third order in \( K/\omega_o^2 \). Substituting for \( K \), this implies a potential energy of:

\[ V(r) = -\frac{\hbar}{8\omega_o^3} \left( \frac{q e^2}{mr^3} \right)^2 = -\frac{3\hbar \omega_o \alpha^2}{4r^6} \]  
\[ (1.10) \]

where \( \alpha = e^2/m\omega_o^2 \) is the classical static polarizability, and we have taken the average value of \( q^2 = 2 \times 3 \), for three dimensions. Equation 1.10 represents the London dispersion force in the non-retarded regime, \( 1 << \omega_o r/c \). For the retarded regime\(^1\), \( 1 >> \omega_o r/c \), we expect a suppression of the potential energy proportional to \( \omega_o r/c \) thus,

\[ V(r) \approx -A \frac{3\hbar \omega_o \alpha^2}{4r^7} = -\frac{23\hbar \omega_o \alpha^2}{4\pi r^7} \]  
\[ (1.11) \]

where \( A = 23/3\pi \) is a proportionality constant derived by Casimir and Polder(4)

\(^1\)The regime in which the finiteness of the speed of light becomes relevant.
1.1 Previous Calculations

using far more rigorous methodology.

We would now like to calculate the net London dispersion force between two plates. Let us take the general form of the potential energy to be $V(r) = -B/r^\gamma$. Then the total potential energy between a single atom, located at $x = y = z = 0$, and the half-space, $z \geq d$, is given by,

$$V(d) = \frac{-N_1B}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{d}^{\infty} dz \left[ x^2 + y^2 + z^2 \right]^{-\gamma/2}$$  

$$= \frac{-2\pi N_1B}{(\gamma - 2)(\gamma - 3)} d^{3-\gamma}$$  

(1.13)

where $N_1$ is the number of atoms per unit volume in the half-space. Replacing the single atom with the half-space $z \leq 0$ of $N_2$ atoms per unit volume, the potential energy per unit area is

$$u(d) = -\frac{2\pi N_1 N_2 B}{(\gamma - 2)(\gamma - 3)} \int_0^\infty dR (R + d)^{3-\gamma}$$  

$$= \frac{-2\pi N_1 N_2 B}{(\gamma - 2)(\gamma - 3)(\gamma - 4)} \frac{1}{d^{\gamma-4}}$$  

(1.15)

Thus, the force per unit area is,

$$F(d) = -\frac{2\pi N_1 N_2 B}{(\gamma - 2)(\gamma - 3)} \frac{1}{d^{\gamma-3}}$$  

(1.16)

If we take the intermolecular potential energy to be the non-retarded London disper-
sion potential energy with $\gamma = 6$, $B = 3\hbar \omega_0 \alpha^2 / 4$, and let $N_1 = N_2 = N$,

$$F(d) = -\frac{\pi N^2 \hbar \omega_0 \alpha^2}{8d^3} \equiv -\frac{A}{6\pi d^3}$$

(1.17)

where $A = 3\pi^2 \hbar \omega_0 \alpha^2 / 4$ is referred to as the de Boer-Hamaker constant of the material. We can also calculate the retarded London dispersion force between two half-spaces using equation 1.16 with $\gamma = 4$ and $B = 23\hbar \omega_0 \alpha^2 / 4\pi$,

$$F(d) = -\frac{23N^2 \hbar \alpha^2}{40d^4} \equiv -\frac{A_1}{10\pi d^4}$$

(1.18)

The above two equations represent London-Hamaker theory in the retarded and non-retarded regimes. Note, the transition from the non-retarded to the retarded regime occurs for separations greater than the principal wavelength of absorption of the material, for most materials this occurs in the visible region, $\sim 0.5\mu m$. Experimental investigation of these forces begins with section 1.2.1.

### 1.1.2 Casimir

In 1948, H. B. G. Casimir constructed another means of calculating the effect of the quantum vacuum on bulk objects. We now formally derive the Casimir effect using Casimir’s original approach (described in (6)) in order to highlight the basic ideas as a starting point for the rest of this dissertation.

Rather than considering two atoms with fluctuating dipole moments as London
1.1 Previous Calculations

did, Casimir considers two parallel plates of infinite extent and the fluctuating electromagnetic field between them. In other words, let us consider the vacuum state of a photon gas contained within a rectangular cavity \((L_x \times L_y \times L_z)\) of perfectly conducting walls. This problem is equivalent to the “particle in a box”, the solution of which is well known. In terms of wave numbers, \(k_i = \pi j / L_i\) where \(j\) is some integer, the vacuum energy is represented as:

\[
E = \sum_{l,m,n}^\prime (2) \frac{1}{2} \hbar c \left[ \left( \frac{\pi l}{L_x} \right)^2 + \left( \frac{\pi m}{L_y} \right)^2 + \left( \frac{\pi n}{L_z} \right)^2 \right]^{1/2}
\]

(1.19)

where the factor of 2 in the above equation comes from the two polarization states of the electromagnetic field and the prime over the summation denotes that any term with an index of 0 is multiplied by half as that term has only 1 independent polarization. The number of linearly independent polarization states of a photon can be assessed from the transversality condition implied by Maxwell’s equation:

\[
\nabla \cdot \vec{E} = \frac{\pi l}{L_x} E_x + \frac{\pi m}{L_y} E_y + \frac{\pi n}{L_z} E_z = 0
\]

(1.20)

For fixed non-zero \((l, m, n, L_x, L_y, L_z, )\), the value of \(E_x\) is given in terms of the two independent variables \((E_y, E_z)\) indicating two independent polarizations. However, for \(m = 0\), the value of \(E_x\) is directly related to \(E_z\) indicating only 1 independent polarization.

For the case of parallel plates of infinite extent \((L_x = L_y = L = \infty)\), we can
replace the sums over $l$ and $m$ with integrals and $L_z$ with the separation, $d$,

$$\begin{align*}
E(d) &= \frac{L^2}{\pi^2} (hc) \sum_n' \int_0^\infty \int_0^\infty \left( k_x^2 + k_y^2 + \frac{n^2 \pi^2}{d^2} \right)^{1/2} dk_x dk_y \\
&= \frac{L^2}{\pi^2} (hc) \sum_n' \int_0^\infty \int_0^\infty \left( k_x^2 + k_y^2 + k_z^2 \right)^{1/2} dk_x dk_y \\
&= \frac{L^2}{\pi^2} (hc) \sum_n' \int_0^\infty \int_0^\infty \left( k_x^2 + k_y^2 + k_z^2 \right)^{1/2} dk_x dk_y dk_z
\end{align*}$$

(1.21)

This integral is clearly divergent. Let us then compare the energy of the photon gas in this situation with the energy of the photon gas inside a cavity of infinite size ($d \to \infty$) to see how the energy of the gas depends on separation.

$$\begin{align*}
E(\infty) &= \frac{L^2}{\pi^2} (hc) \frac{d}{\pi} \int_0^\infty \int_0^\infty \int_0^\infty \left( k_x^2 + k_y^2 + k_z^2 \right)^{1/2} dk_x dk_y dk_z \\
&= \frac{L^2}{\pi^2} (hc) \frac{d}{\pi} \int_0^\infty \int_0^\infty \int_0^\infty \left( k_x^2 + k_y^2 + k_z^2 \right)^{1/2} du u du dk_z f(k_z) \\
&= \frac{L^2}{\pi^2} (hc) \frac{d}{\pi} \int_0^\infty \int_0^\infty u du \left[ \sum_n' f(n) \sqrt{u^2 + \frac{n^2 \pi^2}{d^2}} - \frac{d}{\pi} \int_0^\infty dk_z f(k_z) \sqrt{u^2 + k_z^2} \right]
\end{align*}$$

(1.22)

The expression is once again divergent, however we are most interested in the difference between the two states ($U(d) = E(d) - E(\infty)$):

$$\begin{align*}
U(d) &= \frac{L^2 hc}{\pi^2} \int_0^\infty dk_x \int_0^\infty dk_y \left[ \sum_n' \sqrt{k_x^2 + k_y^2 + \frac{n^2 \pi^2}{d^2}} - \frac{d}{\pi} \int_0^\infty dk_z \sqrt{k_x^2 + k_y^2 + k_z^2} \right]
\end{align*}$$

(1.23)

The process of subtracting off the energy at infinite separation is known as renormalization. In order to evaluate the above expression some manipulation and mathematical “trickery” will have to be done. Let us start by introducing a cutoff function ($f(n)$ and $f(k_z)$) to regularize the expressions and transform to polar coordinates ($\{k_x, k_y\} \to \{u, \theta\}$):

$$\begin{align*}
U(d) &= \frac{L^2 hc}{\pi^2} \left( \frac{\pi}{2} \right) \int_0^\infty du u \left[ \sum_n' f(n) \sqrt{u^2 + \frac{n^2 \pi^2}{d^2}} - \frac{d}{\pi} \int_0^\infty dk_z f(k_z) \sqrt{u^2 + k_z^2} \right]
\end{align*}$$

(1.24)
1.1 Previous Calculations

where we have performed the integration over $\theta$, yielding the factor of $\pi/2$ seen above.

The cut-off function that we have introduced is chosen such that:

$$
f(k) = \begin{cases} 
1 & k << k_{\text{max}} \\
0 & k >> k_{\text{max}} 
\end{cases}, \quad D^n f = \begin{cases} 
0 & k << k_{\text{max}} \\
0 & k >> k_{\text{max}} 
\end{cases}
$$

(1.25)

where $k_{\text{max}}$ is some high frequency cut-off and $D^n$ is the $n$-th derivative with respect to $k$. The process of introducing a cut-off function into an expression is known as regularization. We will show later in the calculation that the value of the cut-off and the exact form of the function are irrelevant. At this point we shall make a transformation to unitless variables ($x = u^2 d^2 / \pi^2$, $\kappa = k_z d / \pi$):

$$
U(d) = \frac{L^2 hc}{4\pi} \left( \frac{\pi^3}{d^3} \right) \int_0^\infty dx \left[ \sum_{n} f(n) \sqrt{x + n^2} - \int_0^\infty d\kappa f(\kappa) \sqrt{x + \kappa^2} \right] (1.26)
$$

$$
= L^2 \left( \frac{\pi^2 hc}{4d^3} \right) \left[ \frac{1}{2} F(0) + \sum_{n=1}^\infty F(n) - \int_0^\infty d\kappa F(\kappa) \right] (1.27)
$$

where,

$$
F(\kappa) \equiv \int_0^\infty dx f \left( \frac{\pi}{d} \sqrt{x + \kappa^2} \right) \sqrt{x + \kappa^2} (1.28)
$$

In order to evaluate the difference between the sum ($S = \sum_{n=a}^b F(n)$) and the integral ($I = \int_a^b d\kappa F(\kappa)$) on the right hand side of equation 1.27 we can use the Euler-Maclaurin formula:

$$
S - I = \frac{1}{2} (F(b) + F(a)) + \sum_{i=1}^k \frac{B_{2i}}{(2i)!} \left( F^{(2i-1)'}(b) - F^{(2i-1)'}(a) \right) - R (1.29)
$$
1.1 Previous Calculations

where

$$R = \int_a^b \frac{P_{2k+1}(\kappa)}{(2k + 1)!} F^{(2k+1)'}(\kappa) d\kappa$$  \hspace{1cm} (1.30)

$B_{2i}$ and $P_{2k+1}$ are the Bernoulli numbers and periodic Bernoulli polynomials, respectively. For our case, $k = \infty$, the remainder term, $R$, certainly goes to zero, because of the $1/k!$. Using the fact that the derivatives of the cut-off function vanish at infinity we can simplify the above expression

$$S - I = \frac{1}{2} F(0) - \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!} F^{(2i-1)'}(0)$$  \hspace{1cm} (1.31)

We now need to calculate the derivatives of $F(\kappa)$ at $\kappa = 0$. We begin by noting:

$$F(\kappa) = \int_{\kappa^2}^{\infty} ds \sqrt{s} f(\frac{\kappa}{d} \sqrt{s}) , \hspace{0.5cm} F'(\kappa) = -2\kappa^2 f(\frac{\kappa}{d})$$  \hspace{1cm} (1.32)

From this it is easy to see that $F'(0) = 0$ and $F''(0) = -4$ and all higher order derivatives vanish as they all involve derivatives of the cut-off function at $\kappa = 0$. We now use equation 1.31 to solve equation 1.27 noting that the sum on the left hand side of equation 1.31 is taken from 0 to infinity instead of 1 to infinity as in equation 1.27

$$U(d) = L^2 \left( \frac{\pi^2 \hbar c}{4d^3} \right) \left[ \frac{1}{2} F(0) + \left( \frac{1}{2} F(0) - \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!} F^{(2i-1)'}(0) \right) - F(0) \right]$$  \hspace{1cm} (1.33)

$$= L^2 \left( \frac{\pi^2 \hbar c}{4d^3} \right) \left[ \frac{-1/30}{4!}(-4) \right]$$  \hspace{1cm} (1.34)

$$= - \left( \frac{\pi^2 \hbar c}{720d^3} \right) L^2$$  \hspace{1cm} (1.35)
1.1 Previous Calculations

Thus we have arrived at an expression that describes a finite energy per unit area that is independent of the cut-off function. From this we can derive the Casimir force per unit area by taking the negative of the derivative with respect to separation,

\[
\frac{F(d)}{A} = -\frac{\pi^2 \hbar c}{240d^4}
\]

where \( A = L^2 \). In the next section we generalize the ideas presented above to real materials.

1.1.3 Lifshitz

In 1956 Evgeny Mikhailovich Lifshitz generalized Casimir’s work to real materials with finite conductivity, this section derives his expression for the Casimir force between real isotropic materials in a parallel plate configuration (5).

In order to calculate the force between plates of finite conductivity, Lifshitz begins by defining the eigenmodes of the cavity and then summing over the vacuum energy of those modes. As with Casimir’s approach, the vacuum energy in the cavity needs to be renormalized and regularized. We will follow Lifshitz calculation as described in Advances in the Casimir Effect (7).

Let us start by considering a monochromatic field,

\[
E(t, \mathbf{r}) = E(\mathbf{r})e^{-i\omega t}, \quad B(t, \mathbf{r}) = B(\mathbf{r})e^{-i\omega t}
\]

13
within a parallel plate cavity. The plates themselves shall be described by their
dielectric permittivity (\( \varepsilon \)) and for simplicity we shall assume that the plates are the
same material and the gap is vacuum (\( \varepsilon_{\text{gap}} = 1 \)). Substituting equation 1.37 into
Maxwell’s equations:

\[
\begin{align*}
\nabla \cdot \mathbf{D} &= 0, \quad \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \\
\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} &= 0, \quad \nabla \cdot \mathbf{B} = 0 
\end{align*}
\]
(1.38)

leads to Helmholtz equations for the electric and magnetic fields,

\[
\begin{align*}
\nabla^2 \mathbf{E}(\mathbf{r}) + \varepsilon(\omega) \frac{\omega^2}{c^2} \mathbf{E}(\mathbf{r}) &= 0, \quad \nabla^2 \mathbf{B}(\mathbf{r}) + \varepsilon(\omega) \frac{\omega^2}{c^2} \mathbf{B}(\mathbf{r}) &= 0
\end{align*}
\]
(1.39)

The complete orthogonal set of solutions to above equations is known to be:

\[
\begin{align*}
\mathbf{E}_J(\mathbf{r}) &= e_p(z, k_\perp) e^{i k_\perp \cdot \mathbf{r}_\perp}, \quad \mathbf{B}_J(\mathbf{r}) &= g_p(z, k_\perp) e^{i k_\perp \cdot \mathbf{r}_\perp}
\end{align*}
\]
(1.40)

where \( \mathbf{r} = (x, y, z) = (\mathbf{r}_\perp, z), \quad \mathbf{k}_\perp = (k_x, k_y) \) are the position and wave vectors,
respectively. Any electromagnetic wave incident upon a plane can be expressed in
terms of a transverse electric (TE) component with \( \mathbf{E} \cdot \hat{n} = 0 \) and a transverse
magnetic (TM) component with \( \mathbf{B} \cdot \hat{n} = 0 \), where \( \hat{n} \) is the unit vector normal to the
surface. In equations 1.40 we have represented the solutions in terms of a collective
index \( \mathbf{J} = \{p, k_\perp, \omega\} \) where \( p \) denotes the polarization of the wave mode (TE, TM).
Substituting 1.40 in to 1.39 gives,

\[ e''_p(z, k_\perp) - \tilde{k}^2 e_p(z, k_\perp) = 0, \quad g''_p(z, k_\perp) - \tilde{k}^2 g_p(z, k_\perp) = 0 \]  (1.41)

where,

\[ \tilde{k}^2 \equiv k_\perp^2 - \varepsilon(\omega) \frac{\omega^2}{c^2} \]  (1.42)

is the square of the wave-number inside the plates. Inside the cavity we will define the square of the wave-number as,

\[ \tilde{q}^2 \equiv k_\perp^2 - \frac{\omega^2}{c^2} \]  (1.43)

From equations 1.41 we can write the general solution for the \( z \)-component of the electric field of a TM wave as:

\[ e_{TM,z}(z, k_\perp) = \begin{cases} 
C_1 e^{\tilde{k}z}, & z < -a/2 \\
C_2 e^{\tilde{q}z} + C_3 e^{-\tilde{q}z}, & -a/2 < z < a/2 \\
C_4 e^{-kz}, & z > a/2 
\end{cases} \]  (1.44)

From Maxwell’s equations one can derive the boundary conditions at the interface between any two materials 1 and 2:

\[ E_{1t}(t, \mathbf{r}) = E_{2t}(t, \mathbf{r}), \quad D_{1n}(t, \mathbf{r}) = D_{2n}(t, \mathbf{r}) \]  (1.45)

\[ B_{1n}(t, \mathbf{r}) = B_{2n}(t, \mathbf{r}), \quad B_{1t}(t, \mathbf{r}) = B_{2t}(t, \mathbf{r}) \]  (1.46)
1.1 Previous Calculations

The above equations and the first of Maxwell’s equations (Eq. 1.38) guarantee the continuity of \( \varepsilon e_{TM,z} \) and \( e'_{TM,z} \) at the bounding surfaces \( z = \pm a/2 \) thus implying the following system of equations:

\[
\begin{align*}
C_1 k e^{-\tilde{ka}/2} &= C_2 \tilde{q} e^{-\tilde{qa}/2} + C_3 \tilde{q} e^{\tilde{qa}/2} \\
-C_4 k e^{-\tilde{ka}/2} &= C_2 \tilde{q} e^{\tilde{qa}/2} + C_3 \tilde{q} e^{-\tilde{qa}/2} \\
C_1 \varepsilon e^{-\tilde{ka}/2} &= C_2 e^{\tilde{qa}/2} + C_3 e^{-\tilde{qa}/2} \\
C_4 \varepsilon e^{-\tilde{ka}/2} &= C_2 e^{\tilde{qa}/2} + C_3 e^{-\tilde{qa}/2}
\end{align*}
\]

(1.47) – (1.50)

This system of four equations and four unknowns \((C_1, C_2, C_3, C_4)\) has non-trivial solutions if the determinant of the coefficient matrix equals zero:

\[
\Delta_{TM}(\omega, k_\perp) \equiv e^{-\tilde{ka}} \left[ (\varepsilon \tilde{q} + \tilde{k})^2 e^{\tilde{qa}} - (\varepsilon \tilde{q} - \tilde{k})^2 e^{-\tilde{qa}} \right] = 0
\]

(1.51)

Stated another way, every combination of the parameters \((\tilde{k}, \tilde{q}, \tilde{a}, \varepsilon)\) for which \(\Delta\) equals zero represents an eigenmode of the cavity. For this reason, we consider \(\Delta\) to be a mode-generating function. Similarly, the mode-generating function for the TE modes can be found to be:

\[
\Delta_{TE}(\omega, k_\perp) \equiv e^{-ka} \left[ (\tilde{q} + \tilde{k})^2 e^{\tilde{qa}} - (\tilde{q} - \tilde{k})^2 e^{-\tilde{qa}} \right] = 0
\]

(1.52)

These two mode-generating functions describe every eigenmode of the cavity. We
now need to sum over the energies of all those eigenmodes. We can write the energy of the vacuum modes per unit area as:

$$E(a) = \frac{\hbar}{4\pi} \int_{0}^{\infty} k_{\perp}dk_{\perp} \sum_{n} \left( \omega_{k_{\perp},n}^{TM} + \omega_{k_{\perp},n}^{TE} \right)$$  \hspace{1cm} (1.53)

where $\omega_{k_{\perp},n}$ are frequencies associated with the zeros of the mode generating function.

In order to calculate the sum of these eigen-frequencies we will use the argument principle:

$$\sum_{n} \omega_{k_{\perp},n}^{TM} = \frac{1}{2\pi i} \left[ \int_{i\infty}^{-i\infty} \omega d\ln \Delta^{TM} + \int_{C^{+}} \omega d\ln \Delta^{TM} \right]$$  \hspace{1cm} (1.54)

where the contour integral above is taken to be about the right half of the complex plane. The integral along $C^{+}$ can be evaluated under the natural assumptions that the dielectric permittivity goes to 1 at infinite frequency and its derivative goes to zero. The result is divergent and independent of the separation:

$$\int_{C^{+}} \omega d\ln \Delta^{TM} = 4 \int_{C^{+}} d\omega$$  \hspace{1cm} (1.55)

Such a distance independent contribution is obviously non-physical and would produce no force. The term can be eliminated by renormalizing equation 1.53,

$$\frac{U(a)}{A} = \frac{E(a)}{A} - \lim_{a \to \infty} \frac{E(a)}{A}$$  \hspace{1cm} (1.56)

The limit of the energy per unit area is found by evaluating the limit of the mode
generating function as the separation goes to infinity:

\[
\Delta_{\infty}^{TM}(\xi_c, k_\perp) = e^{(q-k)a}(\varepsilon q + k)^2, \quad \Delta_{\infty}^{TE}(\xi_c, k_\perp) = e^{(q-k)a}(q + k)^2
\] (1.57)

where we have performed a Wick rotation, $\xi_c = -i\omega/c$ in order to regularize the integral. The variables $k$ and $q$ are given in terms of $\xi_c$:

\[
q^2 = k_\perp^2 + \xi_c^2, \quad k^2 = k_\perp^2 + \varepsilon \xi_c^2
\] (1.58)

With this we can write the potential energy per unit area as:

\[
\frac{U(a)}{A} = \frac{hc}{8\pi^2} \int_{0}^{\infty} k_\perp dk_\perp \int_{-\infty}^{\infty} \xi_c d \left[ \ln \frac{\Delta_{\infty}^{TM}}{\Delta_{\infty}^{TM}} + \ln \frac{\Delta_{\infty}^{TE}}{\Delta_{\infty}^{TE}} \right]
\] (1.59)

\[
= \frac{hc}{4\pi^2} \int_{0}^{\infty} k_\perp dk_\perp \int_{0}^{\infty} d\xi_c \left\{ \ln \left[ 1 - r_{TM}^2 e^{-2qa} \right] + \ln \left[ 1 - r_{TE}^2 e^{-2qa} \right] \right\} (1.60)
\]

where $r_{TM}$ and $r_{TE}$ are the usual Fresnel reflection coefficients for transverse magnetic and electric waves, respectively. Note, the Wick rotation has replaced oscillatory terms, $e^{-ka}$, with exponentially decaying ones, $e^{-ka}$, resulting in the regularization of equation 1.60.

Lifshitz’ equation can be solved numerically for most situations but can also be solved analytically for the Casimir case. Under the assumption of perfect conductivity
the reflection coefficients go to 1 for all frequencies.

\[
\frac{U(a)}{A} = \frac{hc}{2\pi^2} \int_0^\infty k_{\perp} dk_{\perp} \int_0^\infty d\xi_c \ln \left[ 1 - e^{-2qa} \right] \tag{1.61}
\]

\[
= \frac{hc}{2\pi^2} \int_0^\infty k_{\perp} dk_{\perp} \int_0^\infty d\xi_c \sum_{n=1}^{\infty} \frac{1}{n} e^{-2an\sqrt{\xi_c^2 + k_{\perp}^2}} \tag{1.62}
\]

\[
= -\frac{hc}{2\pi^2} \lim_{\delta \to 0} \sum_{n=1}^{\infty} \int_0^{\pi/2} \int_0^\infty e^{-2anp} \frac{1}{n} \sin \theta d\theta dp \tag{1.63}
\]

where we performed a change of variables: \( \xi_c = p \cos \theta \), \( k_{\perp} = p \sin \theta \). Integrating over \( \theta \) and integrating \( p \) by parts yields:

\[
\frac{U(a)}{A} = -\frac{hc}{8\pi^2 a^3} \sum_{n=1}^{\infty} \frac{1}{n^4} \tag{1.64}
\]

\[
= -\frac{\pi^2 hc}{720a^3} \tag{1.65}
\]

the same expression found by Casimir and derived in the previous section.

Two more cases are of particular interest, relating to separations less than or greater than the principle wavelength of absorption of the material \( \lambda_o \):

1. Non-Retarded regime, \( a << \lambda_o \),

\[
f \simeq \frac{hc}{8\pi^2 a^3} \int_0^\infty \left( \frac{\epsilon - 1}{\epsilon + 1} \right)^2 d\xi_c \tag{1.66}
\]

2. Retarded regime, \( a >> \lambda_o \),

\[
f \simeq \frac{hc}{a^4} \frac{\pi^2}{240} \left( \frac{\epsilon_s - 1}{\epsilon_s + 1} \right)^2 \phi(\epsilon_s) \tag{1.67}
\]
Table 1.1: Tabulated function $\phi$ as a function of the static dielectric constant $\epsilon_s$

<table>
<thead>
<tr>
<th>$1/\epsilon_s$</th>
<th>0</th>
<th>0.025</th>
<th>0.1</th>
<th>0.25</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi(\epsilon_s)$</td>
<td>1</td>
<td>0.53</td>
<td>0.41</td>
<td>0.35</td>
<td>0.35</td>
</tr>
</tbody>
</table>

where $\epsilon$ is the dielectric function of the material, $\epsilon_s$ is the static dielectric constant of the material, $\phi(\epsilon_s)$ is a tabulated function derived from numerical evaluation of equation 1.60, and as usual $f$ is the force per unit area. The above expressions were derived for two plates of the same material with no intervening material.

Using Lifshitz’ approach we have derived an expression for the Casimir energy per unit area between plates of real materials in a parallel-plate configuration (eq. 1.60). We then showed that Lifshitz’ expression can be solved analytically for the case of perfect reflectors to reproduce Casimir’s result. Lastly, we presented expressions appropriate to the non-retarded (eq. 1.66) and retarded (eq. 1.67) regimes.

1.1.4 Finite Temperature

In the previous section we demonstrated that the Casimir force can represented in terms of:

$$I = \frac{1}{2\pi i} \left[ \int_{i\infty}^{-i\infty} \frac{1}{2} \hbar \omega d \ln \Delta + \int_{c+} \frac{1}{2} \hbar \omega d \ln \Delta \right]$$

(1.68)

The above equation is based on the argument principle, that each possible eigenmode is represented by a zero of the renormalized mode generating function $\Delta$, and that each eigenmode contributes $\frac{1}{2}\hbar \omega$ to the total energy of the electromagnetic vacuum within the cavity. To extend this formalism to the finite temperature case we replace
the energy per mode with the free energy per mode $\eta(\omega)$:

$$
\eta(\omega) = \frac{1}{2} \hbar \omega - k_B T \ln(z) \quad (1.69)
$$

$$
\eta(\omega) = \frac{1}{2} \hbar \omega - k_B T \ln \left( \frac{1}{1 - e^{-\hbar \omega/k_B T}} \right) \quad (1.70)
$$

where $z$ is the partition function described by Bose-Einstein statistics. Therefore:

$$
I = \frac{1}{2 \pi i} \left[ \int_{i \infty}^{-i \infty} \eta(\omega) d \ln \tilde{\Delta} + \int_{C^+} \eta(\omega) d \ln \tilde{\Delta} \right] \quad (1.71)
$$

$$
= -\frac{1}{2 \pi i} \left[ \int_{i \infty}^{-i \infty} d \eta(\omega) \ln \tilde{\Delta} + \int_{C^+} d \eta(\omega) \ln \tilde{\Delta} \right] \quad (1.72)
$$

$$
= -\frac{\hbar}{4 \pi i} \left[ \int_{i \infty}^{-i \infty} \coth \left( \frac{\hbar \omega}{2 k_B T} \right) \ln \tilde{\Delta} \ d \omega + \int_{C^+} \coth \left( \frac{\hbar \omega}{2 k_B T} \right) \ln \tilde{\Delta} \ d \omega \right] \quad (1.73)
$$

In the above expression we have already written $\eta(\omega)$ explicitly as the vacuum state plus a thermal distribution. At this point we can perform the Wick rotation $\omega = i c \xi_c$. The above expression then becomes,

$$
I = -\frac{\hbar c}{4 \pi i} \int_{i \infty}^{-i \infty} \cot \left( \frac{\hbar c \xi_c}{2 k_B T} \right) \ln \tilde{\Delta} \ d \xi_c - \frac{1}{2} \frac{\hbar c}{2} \sum \text{Res} \left\{ \cot \left( \frac{\hbar c \xi_c}{2 k_B T} \right) \ln \tilde{\Delta} \right\} \quad (1.74)
$$

under the assumption that the mode-generating function is an even function of $\omega$ and $\xi_c$, equation 1.74 reduces to simply the sum over the residues in the first quadrant, hence the factor of $1/2$ in front of the sum in the above expression. For the case of
1.1 Previous Calculations

infinite parallel plates we can write:

\[
U/A = 2 \left( \frac{\hbar c}{4} \right) \left( \frac{2k_BT}{\hbar c} \right) \int_0^\infty \frac{k_\perp dk_\perp}{2\pi} \sum_{l=0}^\infty \ln \tilde{\Delta}(\xi_l) \tag{1.75}
\]

where the prime denotes that the \( l = 0 \) term should be multiplied by 1/2 and \( \xi_l = 2\pi\frac{k_BT}{\hbar c}l \) for \( l = 0, \pm 1, \pm 2, \ldots \) are the locations of the poles of \( \cot(\hbar c \xi_c/2K_BT) \).

Let us use the above expression to investigate the Casimir case. To do so we can use the mode generating functions from equations 1.51, 1.52, and 1.57 and let \( r_{TE} = r_{TM} = 1 \).

\[
U/A = 2k_BT \int_0^\infty \frac{k_\perp dk_\perp}{2\pi} \sum_{l=0}^\infty \ln(1 - e^{-2a\sqrt{\xi_l^2 + k_\perp^2}}) \tag{1.76}
\]

For \( \frac{k_BTa}{\hbar c} \gg 1 \) we can neglect all the terms with \( l \geq 1 \) and write:

\[
U/A = \frac{k_BT}{2\pi} \int_0^\infty k_\perp dk_\perp \ln(1 - e^{-2ak_\perp}) \tag{1.77}
\]

\[
= -\frac{k_BT}{2\pi} \int_0^\infty k_\perp dk_\perp \sum_{n=0}^\infty \frac{e^{-2ak_\perp n}}{n} \tag{1.78}
\]

\[
= -\frac{k_BT}{8\pi a^2} \sum_{n=0}^\infty \frac{1}{n^3} \tag{1.79}
\]

\[
= -\zeta(3)\frac{k_BT}{8\pi a^2} \tag{1.80}
\]

By using Lifshitz’ approach under the replacement \( \left( \frac{1}{2} \hbar \omega \right) \rightarrow \eta(\omega) \) we have derived the Casimir energy per unit area for plates held at a finite temperature. The transition from the zero-temperature regime to the finite-temperature regime occurs around
\[ \frac{k_B T a}{\hbar c} = 1, \] which for plates held at room temperature the transition occurs around 7\(\mu\)m. It is of some interest to note the force in the finite temperature regime goes as \(1/r^3\) just as Lifshitz' expression for the non-retarded regime, equation 1.66. However, the transitions are quite different. Equation 1.66 represents finite conductivity effects and equation 1.80 represents finite temperature effects.

The work presented in this section represents original but not novel work. I constructed this derivation from previously observed derivations.

### 1.1.5 Anisotropic Materials

Many attempts have been made to calculate the force between anisotropic materials. One such attempt was made in 1978 by Yuri Barash as detailed in (11). This approach attempts to use Lifshitz' approach by replacing the dielectric function with a dielectric tensor and then finding the solution by brute force. The difficulty with this approach is that anisotropic materials do not maintain polarization upon reflection, thus one is required to solve both the TE and TM modes simultaneously making the expressions intractably large. For comparison, the isotropic case requires one to take 2 determinants of 4x4 matrices (2 x 4! = 48 terms), the anisotropic case requires one to solve the determinant of a single 8x8 matrix (8! = 40320 terms).

Another approach to anisotropic Casimir effects was developed by Oded Kenneth and Shmuel Nussinov and detailed in (33). This last approach uses a surface current description of the Casimir force, the description of which is beyond the scope of this dissertation.
1.1.6 State of Casimir Force Theory

In this section we have demonstrated a few of the seminal derivations in the field of Casimir force theory. We started with a derivation of London-Hamaker theory, the application of London dispersion forces to groups of molecules. We then derived the Casimir force as Casimir himself derived it and found a surprisingly simple form for the solution. To generalize the Casimir force to real isotropic materials we demonstrated Lifshitz’ theory and then extended the formalism to include contributions due to the finite, non-zero temperature of the plates. We then briefly touched on extending the formalism to include anisotropic materials and found that such an approach is extremely cumbersome, thus motivating the need to find a simpler approach. While other important advancements in this field have been made, many of them are beyond the scope of this dissertation. For example, finding closed-form solutions for cylinder-plane geometry or calculating the Casimir force on a string in string theory. More relevant aspects of the Casimir force are developed in the next section, as we describe a few experiments in this field and the challenges they have had to address.

1.2 Previous Experiments

Casimir force experimentation dates back to the mid 1950’s with the works of Overbeek and Sparnaay (12; 13) and Derjaguin and Abrikosova (14), and have continued to the present with the works of Decca, Lamoreaux, and others (15; 16; 17; 19; 20; 21; 28; 34; 35; 40). Early experiments in this field were intended to dis-
cern between competing predictions of the Casimir force. However, the field quickly discovered that making an observation of the force was itself quite difficult as the measurements were plagued by systematic errors. The insights of these early experiments along with improvements in technology allowed the first precision measurement (35) of the force to be made in 1996. Experiments in the current era are focusing on measuring the Casimir force at larger separations with greater precision in order to address a long standing controversy within the field regarding the modeling of conductivity. In order to motivate the need for our experiment we will take a look at a few experiments that have shaped this field.

1.2.1 Overbeek and Sparnaay (1954)

We will begin our review of the experimental status of this field with a discussion of an experiment performed by Overbeek and Sparnaay (12) in 1954. At this point in time, two descriptions of macroscopic intermolecular forces were in competition. The first and most predominant was the application of van der Waals forces to sets of molecules, section 1.1.1. The second theory was Casimir’s theory, section 1.1.2. However, at the time Casimir’s theory had not been generalized to real materials. Thus the data in this experiment was compared to van der Waals theory by the authors. After describing the experiment and their results we shall investigate possible sources of error.

This experiment utilizes a cantilever of known spring constant in order to derive the forces from a measure of the deflection. The forces of interest act between two
1.2 Previous Experiments

glass parallel-plates one attached to the end of the cantilever the other plate rests on three air bladders that can be pressurized in order to adjust the separation between the two plates. Figure 1.1 illustrates the experimental setup.

![Schematic diagram showing the experimental setup.](image)

Figure 1.1: Schematic diagram showing the experimental setup.

In figure 1.1, A and A₂ are the glass plates. Plate A₁ was attached to the cantilever of known spring constant with aid of parts B and D (brass frame and holder respectively). Plate A₂ sits atop pins T located on three air bladders K. The pressure in all three bladders is controlled by a single pump P and measured by a manometer M. The bladders were observed to respond to an increase in pressure of 1 atm by moving plate A₂ a distance of 4μm. In this way fine positioning of plate A₂ was achieved. The deflection of A₁ was observed by measuring the change in capacitance of the condenser formed by plates C₁ and C₂, C being attached to D and C₂ being attached to frame E which is attached to frame G. The separation between the plates could

26
be directly measured by utilizing the Newton rings formed by the glass plates, figure 1.2. This entire setup was mounted in a cylindrical vacuum chamber (not shown) so that the effects of viscous air resistance could be reduced.

The apparatus was largely assembled outside the vacuum chamber after lengthy preparation of the surfaces. The apparatus was then placed inside the chamber which was subsequently pumped down to roughly 10 μm Hg. The condenser was then calibrated by tipping plate A₁ and bringing it into contact with A₂. Upon inflating the bladders A₂ and A₁ were moved by a known amount. Thus a determination of the relationship between distance and capacitance was obtained. The two plates could
be made parallel again and force-distance measurements could be obtained. Figure 1.3 displays the force vs separation for two glass plates. It was determined by optical measurements that one of these plates had a radius of curvature of 300-500 meters the radius of curvature of the other plate was too large to be measured. The best fit line corresponds to:

\[ F = \frac{A}{6\pi d^n} \]  

with obtained values of \( A = 3.8 \times 10^{-11} \) erg and \( n = 3.0 \pm 0.3 \). Note, the error in \( A \) was not reported. However, a table is given of data that had been discarded for various reasons, the values of \( A \) in that table range from \((.01 - 23) \times 10^{-11} \) erg. As the measured separations were all greater than 0.5\( \mu \)m, one would expect the results to compare well with the retarded van der Waals force; however the measured distance dependence would suggest that retardation effects are not present. Comparing the
1.2 Previous Experiments

data with equation 1.17, one would predict \( A \) to be the de Boer-Hamaker constant for glass (\( \sim 10^{-12} \) erg), making the measured value of \( A \) approximately 40 times larger than predicted.

In order to explain the discrepancy between van der Waals theory and the results of the experiment we first turn to an investigation of the experimental error. The authors reported several possible sources of error in this publication:

1. Obstacles between the plates, dust.

2. Vibrations of the plates.

3. Unexpected alterations of the condenser.

4. Electrostatic forces.

**With regard to item 1:** This source of error was minimized by thoroughly cleaning the plates before placing them in the vacuum chamber. In addition, dust particles should produce a repulsive force at short separations. Any data set exhibiting this behavior was thrown out. **With regard to item 2:** Plate vibrations only contributed to statistical readout error, which would leave the normalization of the force unaffected. **With regard to item 3:** If no sudden changes were observed in the condenser, the capacitance was assumed to be a linear function of time and the sensitivity of the condenser was measured before and after each data run. **With regard to item 4:** A radioactive preparation was placed in the vacuum chamber with the plates in order to ionize the air, eliminating any static build up on the plates.
plates were allowed to sit in the ionized air for a week and no significant decrease in the observed electrostatic effects was observed. Overbeek and Sparnaay indicated the aforementioned radioactive preparation was sufficient to discount any significant electrostatic effects.

With no clear explanation for the observed failure of the experiment to agree with London-Hamaker theory the authors looked for an error in the theory itself. They proposed the theory was inaccurate as it did not address the change in polarizability arising from the presence of other atoms.

Let us reconsider London-Hamaker theory. In section 1.1.1 we derived the London dispersion force by considering the effects of two atoms in close proximity with randomly fluctuating dipoles. We saw that each atom acts as a harmonic oscillator and that the system accordingly acts as a system of two coupled harmonic oscillators. Once we derived the London dispersion force between two atoms, we applied that force to groups of atoms in parallel plate configuration. Thus, we implicitly assumed that the London dispersion force between two groups of atoms is given as the simple sum of the two-body forces between the individual atoms, i.e., we assumed additivity. However, in general it is not accurate to represent $N$ coupled harmonic oscillators by $N/2$ sets of two coupled harmonic oscillators. As such, one would not expect the London dispersion force to be simply additive.

In this experiment, Overbeek and Sparnaay measured the force between two flat glass plates using a cantilever as a force transducer. The separation between the plates was measured using the Newton rings formed by interference between the plates. The
measured force vs separation was found to fit \( F = A/d^n \) with \( A = 3.8 \times 10^{-11} \) erg and \( n = 3 \). While the distance dependence is consistent with the London-Hamaker approach in the absence of retardation effects, the magnitude is found to be 40 times larger than expected. The authors claimed their data indicates a failure of London-Hamaker theory.

1.2.2 Derjaguin and Abrikosova (1957)

This next experiment comes from Derjaguin and Abrikosova (14) three years after the above experiment and one year after Lifshitz generalized Casimir’s theory to real materials. In this experiment the Casimir force is measured for separations 0.07 – 0.3 \( \mu \text{m} \) between three sets of plates, quartz-quartz, mixed thallium halide - mixed thallium halide, and chromium-quartz. Two particularly interesting experimental techniques are presented in this experiment. First, one of the plates has a large spherical curvature. Second, a feedback mechanism is employed to enhance the dynamic range of the balance being used. A large amount of information is gathered in this series of experiments and subsequently analyzed and presented as a comparison with Lifshitz theory.

The authors of this paper presented a very detailed description of the apparatus and its function. However, the schematics are not particularly revealing and its exact description is quite lengthy. We shall present a more general qualitative description instead.

This experiment used three pairs of plates of different materials. One of the plates
in each pair was flat while the other was spherically curved. Additionally, spherical plates of the same material but different radii were used. This procedure allowed an investigation of the effect of curvature and material properties on the observed force. All the plates were thoroughly cleaned before being placed in the vacuum chamber containing the apparatus. Additionally, a radioactive preparation was used to ionize the air within the vacuum chamber.

In this experiment one of the Casimir plates was attached to a control stage and the other to the lever arm of a torsion balance. In this configuration as the plates become close to each other the force increased causing further deflection of the balance. One of the major problems with this type of experiment is that in order to achieve sufficient precision to measure the deflection, and thus the force, at large separations a small restoring force is necessary. However, at short separations the magnitude of the force is much larger producing a large deflection that can cause the plates to stick to one another. This problem reduces the dynamic range of any such experiment. In order to address this issue the authors employed a feedback mechanism wherein the position of the balance affects the current in an electromagnetic coil that in turns acts on a permanent magnet attached to the other end of the balance. In this way the balance can be held at certain position and the Casimir force on the balance can be inferred by the compensating force produced by the electromagnet. Thus, a measure of the current going to the electromagnet is an indirect measure of the forces acting between the plates. Great care was taken in the calibration of this system in order to measure all the necessary constants of proportionality (current(force,
1.2 Previous Experiments

feedback constants, etc.). The separation between the plates was measured using Newton rings produced by light interfering between them. In this way the force was measured as a function of the separation. The process was repeated many times for various sets of plates.

In previous sections we calculated the Casimir force between real materials but only for parallel-plate geometries. This experiment highlights a technique and motivates a need for calculating the Casimir force in a broader set of geometries. Let us consider sphere-sphere geometries to demonstrate the Proximity Force Approximation.

![Diagram of sphere-sphere geometry](image)

Figure 1.4: Schematic of the variables used to calculate the force between two spheres.(14)

The Proximity Force Approximation, also known as the Derjaguin approximation or PFA, is a method of approximating an interaction between curved surfaces from knowledge of the interaction between planar ones. Figure 1.4 illustrates the variables
1.2 Previous Experiments

being used in this calculation. Let $F(H)$ be the force between two spheres separated by $H$, the distance of closest approach, then

$$F(H) \approx \int_{z=H}^{z=\infty} 2\pi r dr f(z)$$

(1.82)

where $f(z)$ is the force per unit area between parallel surfaces separated by $z$. In the above equation we have expressed the force between two spherical surfaces as the sum of the forces between concentric rings of radius $r$ on their surfaces. For spheres of radii $R_1$ and $R_2$, such an approximation is good for $H << R_1, R_2$ to the order of $H/R$.

We can simplify equation 1.82 by noting $z = H + z_1 + z_2$, and $r^2 \approx 2R_1 z_1 \approx 2R_2 z_2$ allowing us to write $rdr \approx [R_1 R_2/(R_1 + R_2)]dz$ thus

$$F(H) \approx 2\pi \left( \frac{R_1 R_2}{R_1 + R_2} \right) \int_{H}^{\infty} dz f(z)$$

(1.83)

For sphere-plane geometries we let $R_2 \to \infty$ to get the approximate Casimir force that should be observed in this experiment:

$$F(H) \approx 2\pi R \int_{H}^{\infty} dz f(z) = 2\pi R \ u(H)$$

(1.84)

where $u(H)$ is the potential energy per unit area of parallel plates separated by $H$.

The Proximity Force Approximation, although an approximation, can be made to be quite accurate. If the error is taken to be $H/R$, this approximation introduces an error of less than 0.001% at a separation of 1 μm for the spheres used in this
1.2 Previous Experiments

experiment ($R = 5 - 25$ cm).

Figure 1.5: Measured Force as a function of separation for Quartz sphere-plane geometry. Black dots $R = 11.1$ cm, Black triangles $R = 10.0$ cm, White circles $R = 25.4$ cm.

With the PFA in mind, the authors measure the force between the plates as a function of separation. They then plot the derived potential energy per unit area as a function of separation. Figure 1.5 displays the measured force as a function of the distance of closest approach $H$ on a Log-Log graph. As indicated in equation 1.84 the energy per unit area of parallel plates can be derived by dividing the force measured in this experiment by $2\pi R$. The resulting data is illustrated in figure 1.6. In both
1.2 Previous Experiments

figures 1.5 and 1.6 black dots represent the measured force between a plate and a sphere of radius $R = 11.1$ cm, black triangles $R = 10.0$ cm, white circles $R = 25.4$ cm. In figures 1.5 and 1.6 the dashed line represents the force calculated using Lifshitz

![Figure 1.6: Derived Energy as a function of separation for Quartz sphere-plane geometry. Black dots $R = 11.1$ cm, Black triangles $R = 10.0$ cm, White circles $R = 25.4$ cm.](image)

theory appropriate to the retarded regime:

\[
f(H) = \frac{\hbar c}{H^4} \frac{\pi^2}{240} \left( \frac{\epsilon_s - 1}{\epsilon_s + 1} \right)^2 \phi(\epsilon_s)
\]  

Curvature was taken into account when the dashed line was calculated.

One of the goals of this experiment was to investigate the effect of curvature on the force between the plates. With this in mind, figure 1.6 can be best interpreted as
the ratio of the force to the radius for three different radii. That is,

$$u_i(H) = \frac{F_i(H)}{2\pi R_i}$$  \hspace{1cm} (1.86)$$

where $F_i(H)$ is the measured force between a flat plate and sphere of radius $R_i$ at separation $H$, $i$ indicates the particular data set. Figure 1.6 illustrates that this ratio appears to be independent of the radius.

Figure 1.7: Measured force as a function of separation for a pair of mixed thallium halide plates in sphere-plane geometry. Dots refer to $R = 12.5$ cm and the theoretical curves I and II refer to the retarded Lifshitz theory for $\epsilon_s \cong 50$ and $\epsilon \cong n^2 = 6$, respectively. (14)
1.2 Previous Experiments

The experimental results for mixed thallium halide plates are displayed in figures 1.7 and 1.8. In figures 1.7 and 1.8 the black dots represent the measured data for $R = 12.5$ cm and the theoretical curves I and II refer to the retarded Lifshitz theory for $\epsilon_s \cong 50$ and $\epsilon \cong n^2 = 6$, respectively. In figure 1.8 crosses refer to $R = 5.2$ cm.

Figure 1.8 again demonstrates that the ratio of the force to the radius remains independent of the radius. Additionally, figures 1.7 and 1.8, by comparison with figures 1.5 and 1.6, demonstrate the effects of optical density on the force of attraction. The theoretical curves I and II indicate the sensitivity of Lifshitz’ theory on the dielectric constant of the material as well as the ambiguity in which “dielectric constant” should
be used in equation 1.67. In the current era of Casimir force research great interest and controversy surrounds the modeling of the dielectric properties of materials.

The experimental results for the chromium-quartz configuration are given in figures 1.9 and 1.10. In figure 1.9 the black dots $R = 12.5$ cm, dashed lines represent the retarded Lifshitz’ theory. In Figure 1.10 the black dots $R = 12.5$ cm, crosses represent $R = 5.4$ cm, and dashed lines represent the retarded Lifshitz’ theory. For the chromium-quartz configuration, Lifshitz’ equation in the retarded regime reduces
1.2 Previous Experiments

Figure 1.10: Derived Energy as a function of separation for Chromium-Quartz sphere-plane geometry. Black dots $R = 12.5$ cm, crosses represent $R = 5.4$ cm, and dashed lines represent the retarded Lifshitz theory.\(^{(14)}\)

where $\epsilon_s$ is the dielectric constant of the quartz at zero frequency. This data indicates that Lifshitz predictions remain accurate for materials with significant conductivity.

It is interesting to note that the predictions made by Lifshitz theory are in fairly good agreement with the data for all the sets of plates in this series of experiments. By contrast, the predictions made by the additive London-van der Waals theory are in stark contrast with the observations. For instance, in comparing the data with equation 1.18, the calculated value of $A_1$ for quartz is $1 \times 10^{-18}$ erg cm, where the value that best fits the data is $3 \times 10^{-18}$ erg cm.

\[ f = \frac{\hbar c}{H^4} \frac{\pi^2 \epsilon_s - 1}{240 \epsilon_s + 1} \phi(\epsilon_s) \]  

(1.87)
In this series of experiments the force between three sets of sphere-plate pairs has been measured and reported. From the data the authors conclude that the force of attraction between the plates is linearly dependent on the radius of the spherical plate. They also found that using London-van der Waals forces in an additive approach was inadequate. The authors concluded that Lifshitz’ theory was in agreement with the measurements. Moreover, as Lifshitz theory is a theory of electromagnetic origin, the authors concluded that intermolecular forces are entirely electromagnetic in nature. Additionally, they surmised the experiments performed by Overbeek and Sparnaay suffered contamination by background forces that were not addressed by the authors.

1.2.3 Overbeek and Sparnaay (1960)

We shall return to an experiment performed by Overbeek and Sparnaay (13) 6 years after the experiment detailed in section 1.2.1. This experiment is similar in design; however, significant improvements, leading to improved force sensitivity and repeatability, have been made. In addition to these experimental improvements, further efforts were taken to reduce systematic errors and the data was compared to a broader set of theories.

Figure 1.11 illustrates the experimental design, where A = balance-arm ; B = upper quartz plate ; C = leafsprings ; D = movable brass weights ; E = support for balance-arm ; F = metal wire, dipping into G = cylinder containing damping oil ; H = lower quartz plate ; J = brass disc carrying lower quartz plate ; K = stainless steel mounting ; L = micrometer screw ; M = vacuum tight shaft-seal ; N = arm of
One of the major improvements used in this design over the 1954 experiment was the introduction of a new damping system. Parts F and G comprise a simple but effective method of damping vibrations of the balance arm A allowing the mean deflection to be more easily and accurately measured. This system uses a brass wire, F, of diameter 0.1 cm dipped into a cylindrical container, G, filled with silicone oil to damp the vibrations of the balance arm, A, to which the wire is fixed. This damping is separation independent, thus representing an improvement over the separation
dependent air damping (41; 42).

Other changes include using water vapor to remove surface charges from the plates. This was accomplished by evaporating water in the vacuum chamber, waiting 10 minutes and then pumping it away with the vacuum pump.

Figure 1.12: Observed Force vs Distance between different flat quartz plates. (13)

Figure 1.12 displays the measured force as a function of distance between flat quartz plates with area of 1 cm$^2$ where the dots refer to the combination of plates 1 and 2, the triangles refer to the combination of plates 1 and 3, the stars refer to an
excluded data set. Figure 1.13 displays the measured force as a function of distance quartz plates in a sphere-plane geometry. In both figure 1.12 and 1.13 the dashed line refers to the line of best fit, the solid line refers to the line of best fit associated with the retarded force, the dot-dashed line corresponds to the line of best fit associated with the non-retarded force, and the dotted lines represent the assumed uncertainty corresponding to deviations of the solid line of ±0.002 dyne and ±0.02 μm (±0.01 μm in figure 1.13). From the graphs it can easily be seen that the line of best fit (solid line) is closest to the best fit line associated with Lifshitz theory in the retarded
1.2 Previous Experiments

regime (dashed line). The conclusion was that the force is best described by Lifshitz theory including retardation effects.

The authors go on to discuss possible explanations for the data obtained in their 1954 experiment. They conclude that the data in the previous experiment was taken in the presence of electrostatic forces produced by Volta potentials. These surface potentials can be 50 to 500 mV, producing a force of attraction between the plates:

\[ F = 4.45 \times 10^{-5} V^2/d^2 \]  

where \( F \) is given in dynes/cm\(^2\) for \( V \) in mV and \( d \) in microns. Interestingly, the authors recognize this background as a possible source of error for their 1954 experiment but don’t discuss this as a possible source of error for their 1960 experiment. In fact, they make no attempt at measuring or estimating the Volta potentials or the force produced by them for the 1960 experiment.

The authors conclude that their measurements are in good agreement with the Lifshitz theory in the retarded regime based on the observed distance dependence. Additionally, they conclude that the proximity force approximation is an accurate description of the force between a sphere and plate. They go on to say that their experiment demonstrates a need for more precise measurements and measurements within the non-retarded regime.
1.2.4 Lamoreaux (1996)

In 1996, more than thirty years after the experiments of Overbeek and Sparnaay, an experiment was performed by S. K. Lamoreaux. This experiment utilizes a torsion pendulum as the force transducer and uses a feedback mechanism to hold the torsion balance in place as the position of one of the Casimir plates is varied over a range of separations (0.6 - 10 μm).

The Casimir plates used in this experiment are made of gold-coated quartz in a sphere-plane geometry, the flat plate is mounted to one end of the torsion balance and the spherical lens is mounted to a positioning stage controlled by three micrometer screws and piezoelectric stack translators. The other end of the torsion balance is held in position by placing a metal plate on either side of the torsion balance, and
applying a voltage. This voltage produces a torque on the balance that is controlled using a PID circuit. This torque compensates for the Casimir force acting on the other end of the balance and thus represents an indirect measure of the force between the Casimir plates. This method of holding the balance at a set position is referred to as a capacitive feedback mechanism.

Figures 1.14 and 1.15 are schematics detailing the experimental design. Figure 1.15 shows a permanent magnet beneath the pendulum. This magnet is used to damp all vibrational modes.

The force was measured at 32 separations per sweep and 216 sweeps were performed. The absolute plate separations were not directly measured; however, the
position of the spherical plate was varied by a known amount, thus defining a set of relative separations $a_i$. The absolute separation can be expressed in terms of these relative separations plus a constant, $a = a_i + a_0$, where the constant $a_0$ is found by a fit to the data. The data for each sweep was fitted to the function:

$$F^m(i) = F_c(a_i + a_0) + \frac{\beta}{a_i + a_0} + b$$

(1.89)

where $F^m(i)$ is the measured force at the $i$-th step, $a_0$ is the fit parameter measuring the absolute separation, $b$ is a constant force offset, and the term involving $\beta$ represents a capacitive force produced by a potential between the Casimir plates.

Figure 1.16: Single-Run experimental results. (a): Force as a function of separation. (b): Data with the fitted $1/a$ term subtracted. (35)

Figure 1.16(a) shows unaltered data for a single data run and the best fit line
corresponding to equation 1.89 (the two points of smallest separation were left out of the fit). Figure 1.16(b) shows the corrected data for a single data run and a line representing the calculated Casimir force. Figure 1.17(a) shows the corrected data of all 216 data runs averaged into bins of varying width. The line corresponds to the calculated zero temperature Casimir force. Figure 1.17(b) displays the data minus the calculated zero temperature Casimir force. The line shows the estimated residuals associated with the finite temperature corrections.

The authors conclude that their data is in agreement with Lifshitz theory at the 5% level. This experiment represents the first precision measurement of the Casimir force; largely due to the authors treatment of background forces. Where previous
experiments relied solely on suppressing background forces, this experiment measured and corrected for them. They also conclude that their data is of insufficient precision to observe the finite temperature corrections or the finite conductivity corrections (the Casimir force in the non-retarded regime).

1.2.5 Decca et al. (2007)

This experiment comes from Purdue and is detailed in (20) and (21). In this experiment the authors use a micromechanical torsion oscillator to measure the Casimir force over separations 160 – 760 nm. A schematic of their experiment is shown in Figure 1.18. This experiment is designed to measure the forces acting on a gold coated micromechanical oscillator in the presence of a gold coated sapphire sphere of radius $R = 150 \mu m$. The forces are deduced through a measurement of the angu-
ilar deflection $\theta$ which is in turn deduced by capacitive measurements made through the electrodes positioned beneath the mechanical oscillator. These forces are then compared to measurements of the separation made with the optical fiber through an interferometric method.

Decca et. al. use the PFA in a manner opposite of that described in equation 1.84:

$$ f(z) \approx -\frac{1}{2\pi R} \frac{\partial F(z)}{\partial z} \quad (1.90) $$

where $f(z)$ is the Casimir force per unit area between flat plates and $F(z)$ is the force observed in the experiment. Now the equation of motion for the cantilever is given by the balance of torques:

$$ I\ddot{\theta}_T + I\omega_o^2\theta_T = F(z)b \quad (1.91) $$

where $b$ is the lever arm, $I$ is the moment of inertia of the cantilever, $\omega_o$ is the natural frequency of the balance, and $\theta_T$ is the total angular deflection of the cantilever. In equilibrium, $F(z)b = I\omega_o^2\theta_o$, where $\theta_o$ is the equilibrium deflection of the cantilever. If the separation $z$ is varied by a small amount $\delta$,

$$ I\ddot{\theta} + I\omega_o^2(\theta_o + \theta) = F(z + \delta)b \quad (1.92) $$

$$ I\ddot{\theta} + I\omega_o^2(\theta_o + \theta) \approx F(z)b + \delta \frac{\partial F(z)}{\partial z}b \quad (1.93) $$

where we have expanded $F(z + \delta)$ about $z$. We can also note $\tan \theta = \delta/b$ and for
small $\theta$ this leads to $\delta \approx b\theta$. From this we can write:

$$I\ddot{\theta} + I\omega_\theta^2(\theta) = \delta \frac{\partial F(z)}{\partial z} b$$

(1.94)

$$\ddot{\theta} + \left[ \omega_\theta^2 - \frac{b^2}{I} \frac{\partial F(z)}{\partial z} \right] \theta = 0$$

(1.95)

where the term in brackets is referred to as the resonant frequency.

$$\omega_r^2 = \omega_0^2 \left[ 1 - \frac{b^2}{I\omega_0^2} \frac{\partial F(z)}{\partial z} \right]$$

(1.96)

The gradient of the total force between the sphere and the micromechanical oscillator in this experiment is determined through a measurement of the resonant frequency. Using equations 1.96 and 1.90 the Casimir force per unit area between two parallel plates is determined

$$f(z) = \frac{I}{2\pi Rb^2} \left[ \omega_r^2 - \omega_\theta^2 \right]$$

(1.97)

and graphed below. The authors chose to represent the force per unit area as $P(z)$.

The data as displayed in Figure 1.19 has been plotted along with bands representing the theoretical predictions of two different models. The light band corresponds to the predictions of Lifshitz theory using the Leontevich impedance model (49) of conductivity to estimate the dielectric function of gold at zero frequency. The dark band represents the predictions using the Drude model of conductivity (44; 45; 46) to do the same. The data is shown to agree with the impedance model of conductivity and preclude the Drude model, the implications of which have far reaching effects in
1.2 Previous Experiments

Figure 1.19: Experimental results for the Casimir pressure as a function of separation $z$. Absolute errors are shown by black crosses in different separation regions (af). The light- and dark-gray bands represent the theoretical predictions of the impedance and Drude model approaches, respectively. The vertical width of the bands is equal to the theoretical error, and all crosses are shown in true scale.\(^{(20)}\)

the realms of thermodynamics and materials science but are beyond the scope of this dissertation. Additionally, once the contribution of the Casimir force is removed the data is used to investigate sub-millimeter forces of Yukawa-type interactions\(^{(47; 48)}\), an area of interest in gravitational and theoretical particle physics. The investigation of Yukawa-type interactions is beyond the scope of this dissertation. The reach of this experiment into so many disparate fields represents the interdisciplinary nature of this field of research.

The experimental technique used herein is quite interesting and novel. In this experiment the Casimir force per unit area between parallel plates is inferred through a measurement of the shift of the resonance frequency of a micromechanical oscillator.
in the presence of the Casimir force between a sphere and a plate. Using this technique, the Casimir force was measured over a range of separations (160 - 760 nm), the smallest separations reported in this dissertation so far.

1.2.6 Munday (2009)

This next experiment comes from Cambridge and is detailed in (16). This experiment once again utilizes sphere-plane geometry and a micromechanical oscillator, this time taking data from 20 – 150 nm. The novelty of this experiment stems from the medium in which it is performed, the interacting objects are submerged in bromobenzene. The materials used in the experiment are detailed below in Figures 1.20a and 1.20b. As the figure illustrates, this experiment is intended to measure the Casimir force between a gold coated sphere and a flat plate made of either silica or gold with bromobenzene filling the cavity. One can use Lifshitz approach to solve for the Casimir force under such conditions and find that when the intervening medium has

---

**Figure 1.20:** Repulsive quantum electrodynamical forces can exist for two materials separated by a fluid. 

- **a.** The interaction between material 1 and material 2 immersed in a fluid (material 3) is repulsive when $\varepsilon_1 > \varepsilon_3 > \varepsilon_2$, where the $\varepsilon$ terms are the dielectric functions.
- **b.** The optical properties of gold, bromobenzene and silica are such that $\varepsilon_{\text{gold}} > \varepsilon_{\text{bromobenzene}} > \varepsilon_{\text{silica}}$ and lead to a repulsive force between the gold and silica surfaces.
- **c.** A schematic of the experiment. (16)
an optical density greater than one (not both) of the materials, the resulting force is “repulsive”. The repulsive nature of the force can be understood as a “bouyant” effect. Essentially the Casimir force between the gold and bromobenzene is stronger than the force between the gold and silica, this results in more bromobenzene being pulled into the cavity and a net repulsive interaction between the gold and silica.

The force between the sphere and the plate is inferred by measuring the deflection of the cantilever that the sphere is attached to. A schematic of the experiment is detailed below in Figure 1.20c. The sphere is glued to the cantilever which in turn is attached to a piezo-electric column used to vary the separation between the sphere and plate. The forces acting between the sphere and plate cause the cantilever to bend.

Figure 1.21: a, Deflection data showing attractive interactions between a gold sphere and a gold plate. b, For the case of the same gold sphere and a silica plate, deflection data show a repulsive interaction evident during both approach and retraction. Note that the deflection voltage signal is a difference signal obtained from the detector and is proportional to the bending of the cantilever.
the deflection of which is measured by reflecting a laser beam off the cantilever onto a detector. The detector is a position sensitive device which produces a differential voltage proportional to the distance between the spot of illumination and the center of the detector. The voltage produced by the PSD thus represents the position of the incident spot which in turn reflects the angular deflection of the cantilever. This angular deflection is related to the force by the known spring constant of the cantilever. Figure 1.21 displays the some of the raw data indicating that the two cases being considered in this experiment demonstrate qualitatively different behavior. The raw data is converted into force measurements and displayed in Figure 1.22.

Figure 1.22: Attractive and repulsive Casimir force measurements. a, Measured repulsive force between a gold sphere and a silica plate in bromobenzene on a loglog scale (blue circles) and calculated force using Lifshitz theory (solid line) including corrections for the measured surface roughness of the sphere and the plate. Blue triangles are force data for another gold sphere/silica plate pair. b, Measured attractive force on a loglog scale for two gold sphere/plate pairs (circles and squares) in bromobenzene. The calculated force includes surface roughness corrections corresponding to the data represented by the circles.(16)

The data as presented in Figure 1.22 shows (a) the force between two gold sphere/-plate combinations and the calculated Casimir force and (b) the force between two gold sphere/silica plate combinations along with calculated Casimir force under those
conditions. Not only does the data show good agreement with the theory, it also demonstrates the ability of this phenomenon to be repulsive, the first experiment to do so.

It has been suggested that such a mechanism could be used to reduce friction in micromechanical machines. Consider two materials immersed in a liquid moving relative to one another. If the materials are pressed together and brought into contact, friction will be produced. However, if the materials and immersion fluid produce a repulsive Casimir force, the materials will never be put into contact as the Casimir force will dominate most other forces at some distance. This effect should substantially reduce friction and wear in any such system.

1.2.7 Lamoreaux (2011)

We shall now investigate a very recent experiment that came out of Yale (34). This experiment once again utilizes sphere-plane geometry. Unlike the previous experiments that we’ve examined, this experiment uses a macroscopic torsion pendulum and investigates the Casimir force over larger separations, 0.7–7μm. As demonstrated earlier in this dissertation, the Casimir force undergoes a fundamental shift for separations above ~5 microns. This experiment is designed to investigate the nature of that shift.

This experiment attempts to measure the Casimir force between two gold surfaces in a sphere-plane configuration at separations larger than a micron, figure 1.23 is a schematic of the instrument. The apparatus used in this experiment is nearly identical
1.2 Previous Experiments

The experiment exhibits one major improvement over their 1996 experiment. As discussed in section 1.2.4, the obtained data was fit to:

$$F_m(i) = F_c(a_i + a_0) + \frac{\beta}{a_i + a_0} + b$$  \hspace{1cm} (1.98)

where $a_0$, $\beta$, and $b$ were all parameters determined by a fit to the data at large separations. The parameter $a_0$ represents the difference between the absolute separation and the relative separation $a_i$ obtained from knowledge of the piezoelectric transducer. The terms with $\beta$ and $b$ represent background forces present in the experiment. Once these parameters are determined the separations are corrected and the background forces are subtracted. While this method allows for accurate characterization of the
1.2 Previous Experiments

experiment it suffers from diminished precision. Essentially, a certain amount of precision is used to determine the parameters, which in turn have associated uncertainties. These parameters, with uncertainty, are then used to correct the data, reintroducing error into the data. Furthermore, the fit itself is a three-parameter fit to a function of inverse powers. Such a fit is not easily and accurately obtained.

In this new experiment, the authors took steps to address the issue presented above. Once again the authors found their data indicated the existence of an electrostatic background force:

\[ F(d, V) = F_e(d) + \pi \epsilon_0 R \left[ \frac{(V - V_m)^2}{d} + \frac{V_{rms}^2}{d} \right] + b \]  \hspace{1cm} (1.99)

where \( V_{rms} \) is a potential produced by material defects, trapped charges, oxide layers, etc. A voltage, \( V \), was applied to the Casimir plates to reduce any residual voltage \( V_m \) produced by grounding differences (\( \sim 20 \) mV). Here again, \( b \) is some constant offset.

With equation 1.99 in mind, the force can be measured at a fixed separation \( d \) and varying voltage \( V \). The resulting force as a function of applied voltage \( V \) is found to be parabolic. A fit to the data will reveal \( V_m \) and \( d \). In this way the absolute separation, \( d \), can be determined and the applied voltage can be set to \( V_m \) in order to minimize background forces. This procedure was undertaken for separations of 0.7 \( \mu \text{m} \) and 7 \( \mu \text{m} \). The separations at intermediate steps were determined from the piezoelectric transducer. This process eliminates the need for the fit parameter \( a_o \) in
equation 1.98, making the fit easier and more precise, resulting in higher precision of the corrected data.

Figure 1.24: Experimental results for the total short-range force between gold plates. The data have been binned for clarity, the vertical error bars include contributions from the statistical scatter of the points as well as from uncertainties in the applied corrections. Also shown are the four theoretical models for the Casimir force: (1) the Drude model including the $T = 300K$ finite-temperature force (red), (2) the plasma model including the $T = 300K$ finite-temperature force (green), (3) the Drude model without the finite-temperature force, that is with $T = 0$ (blue), and (4) the plasma model at $T = 0$ (magenta). The data are plotted as $F \times d$ on the $y$-axis, so that the electrostatic force, proportional to $1/d$, appears as a constant offset on the plot. Inset: experimental data with each of the theoretical Casimir force models subtracted, colour as above. Electrostatic patch force $\pi \varepsilon_0 RV_{rms}^2/d$ corresponds to a constant offset. The fit to the Drude model points is shown by the black line.(34)

Figure 1.24 shows the data obtained from the 30 logarithmically spaced separations and 383 sweeps performed in this experiment along with the theoretical predictions of the Casimir force subject to various models of conductivity. This experiment had two objectives in mind, first to investigate the finite temperature regime of the Casimir force, and second to investigate the validity of various models of conductivity. From the inset in Figure 1.24 one can conclude that indeed finite temperature cor-
rections have been observed in this experiment and that the most appropriate model of conductivity appears to be the Drude model.

Figure 1.25: The short-range force data corrected for an electrostatic force with \( V_{\text{rms}} = 5.4 \text{mV} \). The error bars are the same as in Fig. 1.24. The reduced \( \chi^2 \) of 1.04 demonstrates excellent agreement with the Drude model including the thermal Casimir force at \( T = 300 \text{K} \) (red lines). The grey band represents theoretical uncertainty in the Casimir force calculation from the ellipsometry data. The data are plotted as \( F \times d^2 \) on the \( y \)-axis, so that the thermal Casimir force, corresponds to an offset of 97 \( \text{pN} \times \mu\text{m}^2 \), which dominates the force at large plate separations. In this region the Casimir force is largely independent of the material properties of the plates.\(^{34}\)

Figure 1.25 plots the data after correcting for other electrostatic interactions. This data shows excellent agreement with the predicted Casimir force under the assumptions of a Drude model of conductivity. Note, this experiment appears to directly contradict the findings of Decca\(^{(20)}\) discussed in section 1.2.5. This discrepancy remains an ongoing issue in the field. It is also important to note that the error in the data points at close separation is comparable to error in the theoretical calculation. Although this does not change the conclusions of the experimental results,
1.2 Previous Experiments

it does demonstrate that within this field precise comparison between theory and experiment is hindered by error in theoretical calculations. This issue produces complications for other experiments such as short range gravity where the Casimir force is considered a background that must be eliminated from the data in order to observe other interactions.

1.2.8 State of Casimir Force Experimentation

In this section we have described a few experiments in order to demonstrate the state of Casimir force experimentation as well as demonstrating a few common experimental techniques. Currently, the major thrust of Casimir force experimentation addresses finite temperature corrections as a means of investigating the validity of certain models of conductivity. In order to measure these corrections, experiments have focused on measuring the Casimir force at larger separations where the finite temperature corrections are most predominant. Almost all experiments in the current era measure the Casimir force between two gold surfaces utilizing a mechanical oscillator and sphere-plane geometry.

It was recognized by Sparnaay(57; 58) in 1958 after the first experiments had been completed with only marginal results that in order to measure the Casimir force accurately and precisely three requirements had to be met:

1. The plates must be free of chemical impurities and dust.

2. Precise, independent, and reproducible measurements of the separation must
be performed and effects of roughness and dust must be taken into account.

3. Low levels of electrostatic charges on the surfaces must be ensured. More generally, background signals must be suppressed and/or independently measured and addressed.

These three requirements, though simple in concept, are quite difficult to ensure individually even more so collectively. Failure to meet these requirements had resulted in the failure of the early experiments to make any conclusive statements regarding the existence of the Casimir force. Only in the last decade have experiments been performed that meet all three of these requirements simultaneously.

Other aspects of the Casimir force also need to be addressed. Although accuracy and precision are essential to any experiment, knowing what is being measured and demonstrating characteristics and properties of the object of interest are also necessary. To date, no experiment has been performed using anisotropic materials, as such no experimental evidence exists to support the idea that the measured phenomenon is in fact the result of a vector interaction, as isotropic materials suppress the vector nature of the interaction. Additionally, only a few experiments have been performed using any material other than gold. This allows one to question whether the measured phenomenon is dependent upon the optical properties of the material or some other property (density, acoustic impedance, etc.). It is the opinion of the candidate that the two fundamental characteristics of the Casimir force are:

1. The force is the result of a vector interaction.
1.2 Previous Experiments

2. The force is functionally dependent on the optical properties of the materials. The observed phenomenon has not yet been shown to exhibit these characteristics. Allowing one to question whether the observed phenomenon is indeed the Casimir force. For instance, if the observed phenomenon was found to be a function of the density of the materials, this may indicate a “fifth force”. With such considerations in mind, we have constructed an experiment utilizing anisotropic materials in order to investigate the vector nature of the interaction.
2.1 Introduction

Since Casimir’s pioneering work in 1948 many advancements have taken place in the field of Casimir force theory and its experimental confirmation. Casimir’s original work described the force that operates between uncharged parallel conductors arising from the vacuum energy of the electromagnetic field. The theory has since been extended to isotropic dielectrics (5; 8; 9) and anisotropic materials (10; 11). Experimentation in the field began in the 1950’s(12; 14) and has continued to the present day with ever-increasing precision(15; 17; 19; 20), covering a wide range of separations from 10nm – 7µm. However, many problems still remain. These problems include the modeling of material properties, the effects of temperature, and the approach to complex geometries, among others.

In this chapter we describe a novel approach(38) to the Casimir force intended
to address the issues of complex geometries. We shall present the theory first in an abstract fashion in section 2.1.1. In later sections we will use this approach to solve specific problems, and present specifics as necessary. In this way different aspects of the theory will be expressed piece by piece. We will calculate the Casimir force using a scattering approach and subsequent mode counting method. Our basic formalism is not specific to any geometry or boundary conditions; we simply assume that we have two interacting boundaries and note that the right-bound waves in the cavity are generated by reflections of all the left-bound waves and vice-versa.

2.1.1 Basic Approach

In this section we will develop the basic approach that will subsequently be used in a number of specific cases. Figure 2.1 indicates the variables being used in this derivation. This derivation proceeds as follows: (1) describe in general the problem to be solved, (2) define a complete set of functions and operations relevant to the cavity, (3) construct a closed loop path between any point on surface 1 to any point on surface 2 along any wave-vector, (4) from the consideration of the closed loop operation described above, construct a mode-generating function, (5) calculate the potential energy per path, (6) sum over all possible paths connecting the two surfaces in order to calculate the total potential energy.

The problem we wish to consider addresses the potential energy associated with a cavity of arbitrary geometry and non-trivial boundary conditions. A schematic of the problem is demonstrated in figure 2.1. While this schematic depicts a parallel-plate
2.1 Introduction

Figure 2.1: Two interacting surfaces $S_1$ and $S_2$. While the figure depicts a parallel-plate configuration, this approach makes no assumptions regarding the geometry of the interacting objects. The field between the surfaces will be denoted as the sum of a right-bound wave and a left-bound wave, $\Psi_R$ and $\Psi_L$, respectively. A reflection operation is defined at all points on surfaces 1 and 2, these reflection matrices are explicitly included when constructing the mode-generating function. The mode-generating function is constructed by considering the closed loop path from a point on $S_1$ to a point on $S_2$ and back.

cavity, we will approach the problem as if it were of arbitrary geometry. We begin by describing a complete set of waves in region 3. The complete set of waves is described in general by a set of “left-bound” waves, $\Psi_L$, and a set of “right-bound” waves, $\Psi_R$. These waves are related by a general reflection operation $R$, such that, $R\Psi_L = \Psi_R$ and $R\Psi_R = \Psi_L$. If we consider the wavelet, $\Psi_R(x_2)$, we note that the general reflection operation can be described as a function of position, $R(x_2)\Psi_R(x_2) = R_2\Psi_R(x_2) = \Psi_L(x_2)$.

The amplitude of the right-bound wave, $\Psi_R$, and left-bound wave, $\Psi_L$, at points $x_1$ and $x_2$ on surfaces $S_1$ and $S_2$ respectively are related by a generalized Huygen’s Principle:

$$\Psi_k(x_j) = \int_{S_i} \left( \frac{\partial G(x_j, x_i)}{\partial n} \right) \Psi_k(x_i) dS_i$$  \hspace{1cm} (2.1)
where \( k \) indicates whether the wave is right-bound or left-bound. Equation 2.1 is essentially Green’s theorem subject to Dirichlet boundary conditions. Let us rewrite the Green’s function in the form:

\[
\left( \frac{\partial G}{\partial n} \right) = g \Gamma
\] (2.2)

In doing so we have chosen to express the green’s function into a phase component, \( \Gamma \), and a modulus component, \( g \). With this we can write:

\[
\Psi_R(x_2) = \int_{S_1} g(x_2, x_1) \Gamma(x_2, x_1) R_1 \Psi_L(x_1) dS_1
\] (2.3)

\[
\Psi_L(x_1) = \int_{S_2} g(x_1, x_2) \Gamma(x_1, x_2) R_2 \Psi_R(x_2) dS_2
\] (2.4)

Substituting equation 2.4 into equation 2.3 we get,

\[
\Psi_R(x_2) = \int_{S_1} \int_{S'_2} g(x_2, x_1) \Gamma(x_2, x_1) R_1 g(x_1, x'_2) \Gamma(x_1, x'_2) R_2 \Psi_R(x'_2) dS_1 dS'_2
\] (2.5)

We do not expect any explicit dependence of the modulus of the wave on its phase or direction of propagation, thus, \( g \) commutes with \( \Gamma \) and \( R \). As such we can rewrite the above equation as:

\[
\Psi_R(x_2) = \int_{S_1} \int_{S'_2} g(x_2, x_1) g(x_1, x'_2) \Gamma(x_2, x_1) R_1 \Gamma(x_1, x'_2) R_2 \Psi_R(x'_2) dS_1 dS'_2
\] (2.6)

Equation 2.6 represents a closed loop operation. The right-bound wave at a point \( x_2 \)
is reflected and transported to \( x_1 \) and then reflected and transported back. Any wave that satisfy equation 2.6 is an eigen-mode of the cavity. For the above expression to be satisfied,

\[
\Gamma(x_2, x_1)R_1\Gamma(x_1, x_2)R_2 = \begin{cases} 
0 \\
I 
\end{cases}
\]  

(2.7)

If the operator on the left hand side of equation 2.7 equals 0, \( \Psi_R(x_2) = 0 \) for all \( x_2 \), this represents a trivial case. Therefore, in order for a non-trivial solution to exist,

\[
\det(I - \Gamma(x_2, x_1)R_1\Gamma(x_1, x_2)R_2) = 0
\]  

(2.8)

An eigenmode of the cavity exists for any choice of \( \omega, k, x_1, \text{or} x_2 \) which can satisfy equation 2.8. As such, \( \Delta = \det(I - \Gamma(x_2, x_1)R_1\Gamma(x_1, x_2)R_2) \) is the mode-generating function for the cavity.

In order to calculate the potential energy per path one must sum over the energies of all such eigenmodes, a task most easily performed by utilizing the argument principle:

\[
U(r, k) = \frac{1}{2} \hbar \sum_n \omega_n(r, k) = \frac{\hbar}{4\pi i} \left[ \int_{\infty}^{-i\infty} \omega d\ln \Delta(\omega, r, k) + \int_{C_+} \omega d\ln \Delta(\omega, r, k) \right]
\]  

(2.9)

where we have explicitly indicated that the mode-generating function and the potential energy are dependent on the parameters of the path, \((r, k)\).

The transport functions, \( \Gamma \), have elements proportional to \( e^{-ikr} \), making the above
integral oscillatory. To deal with this we can perform a change of variables, $\xi_c = -i\omega/c$. All the elements of $\Gamma$ will then be proportional to exponentially decaying functions and consequently the integral over the semicircle $C_+$ goes to zero. After integrating by parts the expression for total potential energy of the cavity is given by:

$$U = \frac{hc}{4\pi} \sum_{(r,k)} \int_{-\infty}^{\infty} \ln\Delta(\xi_c, r, k)d\xi_c$$

where the sum is over all unobstructed straight line paths connecting the two surfaces.

The energy represented by equation 2.10 is a binding energy associated with the energy of the vacuum state of the set of non-trivial wave modes of a certain field subject to given boundary conditions. In the following section we will use equation 2.10 and the approach that generated it to solve several specific cases.

2.2 Casimir Case

As a demonstration of how to use this approach we will solve a few specific cases and show that this approach yields results consistent with past works. While this method is general enough to be applied to other fields, we shall demonstrate its use in the context of electromagnetism.

We begin with the simplest case, the Casimir case described by a parallel-plate cavity of infinite lateral extent $L_x = L_y = \infty$ and finite spacing $L_z = a$, the walls of which are perfect conductors, $R_1 = R_2 = I$, where $I$ is the identity matrix. The cavity itself is empty with $\epsilon = 1$, a fact that allows us to say the transport functions
are diagonal (no polarization mixing upon transport). Moreover, the transport of an
eigenmode across the cavity is invariant under the change of polarization, this allows
us to write the transport functions as: $\Gamma_r = Ie^{ikr}$, where $r = (r_x, r_y, a)$ is the vector
between $x_1$ and $x_2$, and $I$ is the identity matrix. Let us express the wave vectors as
$k' = (k_\perp \cos \phi, k_\perp \sin \phi, -i\sqrt{\xi_c^2 + k_\perp^2})$ and $k = (k_\perp \cos \phi, k_\perp \sin \phi, i\sqrt{\xi_c^2 + k_\perp^2})$ are the
left and right bound wave vectors, respectively. Where, $0 \leq \phi \leq 2\pi$ and $k_\perp$ is the
magnitude of $k$ perpendicular to the surface normals. From equation 2.10 we can
express the potential energy as:

$$U = \frac{hc}{4\pi} \int d\mathbf{k} \int d\mathbf{r} \int_{-\infty}^{\infty} \ln(\det(I - e^{-ikr}e^{-ik'(-r)}))d\xi_c$$  \hspace{1cm} (2.11)$$

The integrals over $d\mathbf{k}$ and $d\mathbf{r}$ are a symbolic representations of the sum over all possible
straight line paths from any $x_1$ to any $x_2$ traveling along a wave of any wave vector
$k$. Mathematically we can express the integral over $d\mathbf{k}$ as:

$$\int d\mathbf{k} = \int_0^\infty \frac{k_\perp dk_\perp}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi}$$  \hspace{1cm} (2.12)$$

Substituting the above equation into 2.11 and using the definitions of the wave vectors
leads to:

$$U = \frac{hc}{4\pi} \int_0^\infty \frac{k_\perp dk_\perp}{2\pi} \int d\mathbf{r} \int_{-\infty}^{\infty} \ln(\det(I - Ie^{-2a\sqrt{\xi_c^2 + k_\perp^2}}))d\xi_c$$  \hspace{1cm} (2.13)$$

where the integral over $d\phi$ has been evaluated yielding a value of 1. We can express
the integrand in a different form by noting that for any 2x2 matrix, $A$, the following
2.2 Casimir Case

is true:

$$\det(\mathbf{I} - \mathbf{A}) = 1 - \text{tr}(\mathbf{A}) + \det(\mathbf{A})$$  \hspace{1cm} (2.14)

Using equation 2.14:

$$U = \frac{\hbar c}{4\pi} \int_0^\infty \frac{k_{\perp}dk_{\perp}}{2\pi} \int d\mathbf{r} \int_{-\infty}^\infty \ln(1 - 2e^{-2a\sqrt{\xi_c^2 + k_{\perp}^2}} + e^{-4a\sqrt{\xi_c^2 + k_{\perp}^2}})d\xi_c$$ \hspace{1cm} (2.15)

The integral over $d\mathbf{r}$ is less obvious than the integral over $d\mathbf{k}$. In the next section we will give expression to this integral, for now let us simply state the integral over $d\mathbf{r}$ is essentially an area integral and will yield a constant as the integrand is now independent of $\mathbf{r}$. Thus,

$$u = \frac{\hbar c}{2\pi^2} \lim_{\delta^+ \to 0} \int_0^\infty k_{\perp}dk_{\perp} \int_0^\infty d\xi_c \sum_{n=1}^{\infty} \frac{1}{n} e^{-2an\sqrt{\xi_c^2 + k_{\perp}^2}}$$ \hspace{1cm} (2.16)

Here we have written $u$ as the potential energy per unit area and eliminated the integral over $d\mathbf{r}$. We can directly evaluate the above equation by expressing the natural log as a series and performing the integration term by term:

$$u = -\frac{\hbar c}{2\pi^2} \lim_{\delta^+ \to 0} \sum_{n=1}^{\infty} \int_0^{\pi/2} d\theta \int_{-\delta}^{\delta} \frac{e^{-2anp}}{n} p^2 \sin \theta dp$$ \hspace{1cm} (2.17)

$$u = -\frac{\hbar c}{2\pi^2} \lim_{\delta^+ \to 0} \sum_{n=1}^{\infty} \int_0^{\pi/2} d\theta \int_{-\delta}^{\delta} \frac{e^{-2anp}}{n} p^2 \sin \theta dp$$ \hspace{1cm} (2.18)

here we have performed a change of variables: $\xi_c = p\cos \theta$, $k_{\perp} = p\sin \theta$. The integration over $\theta$ can be carried out readily to give a value of 1. Integrating over $p$
by parts and taking the limit as $\delta \to 0$ yields $1/(4a^3n^4)$, so that:

$$u = -\frac{hc}{8\pi^2a^3} \sum_{n=1}^{\infty} \frac{1}{n^4}$$  \hspace{1cm} (2.19)

The sum in the above expression is well known and has a definite value of $\zeta(4) = \pi^4/90$. This yields the value for the potential energy per unit area:

$$u(a) = -\frac{\pi^2hc}{720a^3}$$  \hspace{1cm} (2.20)

in conformity with the standard Casimir result(6).

Using our approach we have derived Casimir’s result without the use of renormalization. The reason for this can be traced back to the idea of triviality. Here the notion of a trivial solution must be extended to encompass any mode consistent with the non-existence of the plates. Our expression for the vacuum binding energy will always converge if the mode generating function was properly expressed and all the trivial modes were removed from it. Conversely, if the mode generating function was derived using a method that allows trivial solutions to be counted, the calculated energy will always diverge.

\section{2.3 Parallel-plate cavity of finite size}

In the last section we calculated the Casimir energy per unit area of a parallel-plate cavity of infinite extent. Although this is an interesting result it is only applicable
under the condition that the separation of the plates is much less than the size of the plates. Below we calculate the Casimir energy per unit area of a cavity of finite size.

For simplicity let us once again consider the case of a perfectly conducting parallel-plate cavity. This time, however, the side walls will only occupy the region \((-b, b)\) in the \(x\)- and \(y\)-directions. The separation of the plates will once again be \(a\) as shown in Figure 2.2.

![Figure 2.2](image.png)

Figure 2.2: Two parallel plates separated by a distance \(a\) in the \(z\)-direction. The plates are squares of side-length \(2b\). The plates are aligned such that the bottom left corners of both plates are given by the \(xy\)-coordinates \((-b, -b)\) and the top right corners are given by \((b, b)\).

To solve this problem we begin with equation 2.11 and write the integral over \(r\) as an integral over either surface weighted by the probability that the set of isotropically distributed path vectors emanating from each differential area element is intercepted by the other surface.

\[
U = \frac{hc}{2\pi^2} \int \int \frac{(dA_1 \cdot \hat{r})(dA_2 \cdot \hat{r})}{\pi r^2} \int_0^\infty k_\perp dk_\perp \int_0^\infty \ln(1 - e^{-2a\sqrt{k_\perp^2 + k_\parallel^2}}) d\xi_c \tag{2.21}
\]

The factor of \(\pi\) in the denominator of the \(r\) integration is the normalization factor
that is derived by noting that for a plate of unit area placed parallel to an infinite plane the spatial part of the integral should be unity, in order that Eq. 2.21 agree with Eq. 2.20 in the limit $b \to \infty$. We can begin solving equation 2.21 by noting that the $k$ and $\xi_c$ integrations are independent of $r$. Following the calculation in the last section yields,

\[
U = -\frac{\pi^2 hc}{720a^3} \int \int \frac{(dA_1 \cdot \hat{r})(dA_2 \cdot \hat{r})}{\pi r^2} \tag{2.22}
\]

For the plates we have chosen, $(dA \cdot \hat{r}) = dx dy \cos \theta$, where $\cos \theta = a/r$. Thus,

\[
U = -\frac{\pi^2 hc}{720a^3} \frac{a^2}{\pi} \int_{-b}^{b} \int_{-b}^{b} \int_{-b}^{b} \frac{dx_1 dx_2 dy_1 dy_2}{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + a^2]^2} \tag{2.23}
\]

We can then express the integral in unitless quantities as:

\[
U = -\frac{\pi^2 hc}{720a^3} \frac{b^2}{\pi} \frac{1}{\alpha^2} \int_{-\alpha}^{\alpha} \int_{-\alpha}^{\alpha} \int_{-\alpha}^{\alpha} \frac{dx_1 dy_1 dx_2 dy_2}{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + 1]^2} \tag{2.24}
\]

where $\alpha = b/a$. From this we can see that Eq. 2.24 differs from Eq. 2.20 solely in the manner in which the effective area is calculated.

To demonstrate the effects of finite size on the overall Casimir energy we have plotted the Casimir energy vs separation for square plates of side $2b = 10\mu m$ over separations of $0.1 \mu m$ to $100 \mu m$. The dashed line in the graph was calculated using Casimir’s expression, Eq. 2.20, simply multiplied by the area of Plate 1, and the solid line was calculated using our expression Eq. 2.24. From Figure 2.3 it is apparent that for cavities with lateral dimension much larger than their separation ($b/a >> 1$) our
2.4 Single Spherical Shell

We are now interested in calculating the Casimir force on a single spherical shell using a modified version of the approach described earlier. For simplicity we shall consider the case of perfectly reflecting boundary conditions. For this problem we will need to define two sets of waves, those propagating toward the center of the sphere expression approaches Casimir’s expression; however, for separations larger than the side-length \((b/a < 1)\) our expression begins to drop off more steeply than Casimir’s expression. This behavior is to be expected, as the plates should begin to appear as points rather than planes for large separation, leading to an additional \(1/a^2\) behavior.

![Log-Log Plot of Energy vs Separation](image)

Figure 2.3: Log-Log graph of energy vs separation. This graph was generated for square plates of side-length 10 \(\mu\text{m}\) separated by distances between .1 \(\mu\text{m}\) and 100 \(\mu\text{m}\). The dashed line represents Casimir’s expression Eq. 2.20 multiplied by the area of Plate 1. The solid line represents our expression Eq. 2.24.
2.4 Single Spherical Shell

Figure 2.4: A sphere of radius $R$. We will choose our coordinate system such that $x_1$ is at the origin. The vector from $x_1$ to $x_2$ is represented by $r$, $\Theta$ is the angle of declination and $\Phi$ is the azimuthal angle (not shown). It will prove useful to define similar coordinates centered at the center of the sphere $(R, \theta, \phi)$.

($\psi_{\text{in}}$) and those propagating away from the center ($\psi_{\text{out}}$). All the waves considered herein are within the cavity; any modes exterior to the shell are irrelevant to this calculation. In accordance with equation 2.1:

$$\psi_{\text{in}}(P) = \int_{S_1} g(P, x_1) \Gamma(P, x_1) \psi_{\text{in}}(x_1) dS_1$$  \hspace{1cm} (2.25)

where again $\Gamma$ is the transport matrix and $\Gamma^{-1} \Gamma = I$. The point $P$ is some point in the interior of the sphere such that $P \neq x_1 \neq x_2$. We have introduced this new point $P$ in order to clarify the procedure. This point has no charges associated with it and is perfectly transmitting thus anything that goes into point $P$ must come out of point...
Thus,

\[ \psi_{\text{out}}(P) = \psi_{\text{in}}(P) \]  

(2.26)

We also note:

\[ \psi_{\text{in}}(x_1) = R_1 \psi_{\text{out}}(x_1) \]  

(2.27)

\[ \psi_{\text{out}}(x_1) = g(x_1, P) \Gamma(x_1, P) \psi_{\text{out}}(P) \]  

(2.28)

where \( R \) is the reflection matrix. With these ideas in mind we wish to construct the closed loop \( \{ P \rightarrow x_2 \rightarrow P \rightarrow x_1 \rightarrow P \} \), for simplicity we will use a single "g" to represent all such terms.

\[
\int_{S_1} \int_{S_2} g \Gamma(P, x_1) R_1 \Gamma(x_1, P) \Gamma(P, x_2) R_2 \Gamma(x_2, P) \psi_{\text{in}}(P) dS_1 dS_2 = \psi_{\text{in}}(P) \]  

(2.29)

Thus,

\[ \Delta = \det \left( I - \Gamma(P, x_1) R_1 \Gamma(x_1, P) \Gamma(P, x_2) R_2 \Gamma(x_2, P) \right) = 0 \]  

(2.30)

is true for all eigenmodes of the cavity and is therefore the mode generating function that we will use with the previous approach. It is worth noting that the parallel plate problem can also be solved by introducing a "Null" point \( P \) and the result is unaffected. We did not introduce it earlier for sake of clarity.

With the new mode generating function we can start solving the problem. We will start with the general solution to any Casimir force problem indicated in equation
\[ U = \frac{\hbar c}{2\pi} \int \! \! \int \! \! \int \ln(\det(\Delta)) \, dc \] (2.31)

where \( \Delta \) is the mode generating function. This above equation is an abstract expression of the Casimir binding energy associated with any system. Form must be given to the individual elements in the expression in accordance with the specifics of that system. Let us start by giving form to the transport matrices. For this we consider the interior of the spherical shell to be empty (\( \epsilon = 1 \)), thus:

\[ \Gamma(P, x_i) = I \exp[i\vec{k}_\text{in} \cdot (P - x_i)] \] (2.32)

\[ \Gamma(x_i, P) = I \exp[i\vec{k}_\text{out} \cdot (x_i - P)] \] (2.33)

where \( \vec{k}_\text{out} \) is the wave-vector propagating toward the interior surface of the sphere and \( \vec{k}_\text{in} \) is the wave-vector propagating toward the center of the sphere. The mode generating function can thus be expressed as,

\[ \Delta = \det \left( I - I \exp \left[ i \left( (\vec{k}_\text{out} - \vec{k}_\text{in}) \cdot (x_1 - P) + (\vec{k}_\text{out} - \vec{k}_\text{in}) \cdot (x_2 - P) \right) \right] \right) \] (2.34)

where we have used the fact that for perfectly reflecting boundary conditions \( \mathbf{R} = \mathbf{R}^{-1} = \mathbf{I} \). We can also note \( \vec{k}_\text{out} = (\vec{k}_\perp, k_r) \), and \( \vec{k}_\text{in} = (\vec{k}_\perp, -k_r) \).

\[ \Delta = \det \left( I - I \exp \left[ 2i \left( k_r(x_1 - P)_r + k_r(x_2 - P)_r \right) \right] \right) \] (2.35)
2.4 Single Spherical Shell

Finally, we note that if we choose the origin to be at the center of the sphere and choose the point $P$ to be at the origin,

$$\Delta = \det \left(I - Ie^{i k \cdot R}\right)$$  \hspace{1cm} (2.36)

Returning to equation 2.31, we can write $\int \mathbf{k} = \frac{1}{(2\pi)^2} \int k_{\perp} dk_{\perp} \int d\phi$. The integral over $d\phi$ yields 1. The integral over $\mathbf{r}$ can be carried out directly as it is independent of the $k$-integration yielding $8\pi R^2$. Thus equation 2.10 can be expressed as,

$$U = \frac{\hbar c}{2\pi} (4\pi R^2) \int_0^\infty \int_0^\infty \frac{k_{\perp}}{2\pi} 2 \ln \left(1 - e^{-4\pi\sqrt{\xi^2 + k_{\perp}^2}}\right) dk_{\perp} d\xi$$ \hspace{1cm} (2.37)

which can be evaluated using techniques demonstrated earlier yielding,

$$U = -\frac{\pi^3 \hbar c}{1440 R}$$ \hspace{1cm} (2.38)

$$\mathbf{F}_{\text{cas}} = -\frac{\pi^3 \hbar c}{1440 R^2} \hat{r}$$ \hspace{1cm} (2.39)

From this equation we can see that the force is attractive (crushing) and significantly larger than the electrostatic self-repulsion of a sphere with a charge of $1e$ uniformly distributed over its surface,

$$\mathbf{F}_{\text{EM}} = \frac{\alpha \hbar c}{2R^2} \hat{r}$$ \hspace{1cm} (2.40)

In comparing equations 2.39 and 2.40, we note that they differ in sign and magnitude only. Let the ratio of the Casimir force on a sphere to the electrostatic self-repulsion
be:

\[ C = \frac{F_{\text{cas}}}{F_{\text{EM}}} \approx -\frac{137\pi^3}{720} \]  

(2.41)

Indicating that the electrostatic self-repulsion of a hollow conducting sphere with a charge of 1e could not withstand the Casimir force acting on it and would collapse.

Using the method presented in (38) we have derived the Casimir binding energy of a perfectly reflecting spherical shell. This work demonstrates that contrary to Boyer's result (39) we have found the Casimir force for this geometry to be attractive. Casimir believed this would be the case and had the idea that perhaps the attractive Casimir force might balance the self-repulsion of a charged spherical shell, and in this way on might describe the electron and derive the fine-structure constant. However, the distance dependence of the Casimir force under these conditions has been found to match that of the electrostatic self-repulsion. Additionally the magnitude of the Casimir force has been found to be roughly 12 times greater than the self-repulsion. Indicating that Casimir's model of the electron would collapse to a singularity without the introduction of another repulsive force that drops off faster than $1/R^2$.

In retrospect the failure of this model was inevitable. If the electron is an extended object the radius would have to be extremely small to have avoided experimental detection. At such small length scales the Casimir force associated with the vacuum state of other fields (weak, electron-positron, neutrino, etc.) would also have to be taken into account.
2.5 Anisotropic parallel-plate cavity

Up to this point we have concerned ourselves with the effects of geometry on the Casimir force. Let us turn our attention to the effects of reflectivity. In order to exemplify how our approach handles these effects we consider the case of a parallel-plate cavity with uniaxial boundary conditions as illustrated in Figure 2.5. Such a cavity would have side walls made of a uniaxial material, for example a wire-grid polarizer or a graphite crystal cut with in-plane optical anisotropy. When two such plates are placed close together with their optic axes rotated by an angle $\chi$ with respect to each other, both an orientationally dependent normal force and a torque tending to align the optic axes of the plates have been predicted (11)(23)(33).

In this calculation we calculate the Casimir binding energy of a cavity described by uniaxial boundary conditions (figure 2.5) using the approach described above. We begin by expressing the dielectric tensors of Plates 1 and 2 as:

$$
\epsilon_1 = \begin{pmatrix}
\epsilon_{1\parallel} & 0 & 0 \\
0 & \epsilon_{1\perp} & 0 \\
0 & 0 & \epsilon_{1\perp}
\end{pmatrix}, \quad \epsilon_2 = R_z^T[\psi] \begin{pmatrix}
\epsilon_{2\parallel} & 0 & 0 \\
0 & \epsilon_{2\perp} & 0 \\
0 & 0 & \epsilon_{2\perp}
\end{pmatrix} R_z[\psi] \quad (2.42)
$$

Where we have oriented our coordinate system such that the optic axis (the direction in which an incident wave suffers no birefringence) of Plate 1 is aligned with the x-axis ($\hat{O}_1 = \hat{x}$). Plate 2 will be oriented with its optic axis in the x-y plane, making an angle $\psi$ with the x-axis ($\hat{O}_2 = R_z^T[\psi]\hat{x}$). The subscripts $\parallel$ and $\perp$ denote the components of
2.5 Anisotropic parallel-plate cavity

the dielectric tensor parallel and perpendicular to the optic axis, respectively. Regions 1 and 2 will describe the space inside Plates 1 and 2 whose interior faces are given by the planes $z = -a/2$ and $z = a/2$, respectively. Region 3 will be filled with an isotropic medium whose dielectric function will be denoted by the scalar $\epsilon_3$.

In order to address this Casimir problem in the formalism presented in this paper, let us start with equation 2.10 relevant to a parallel-plate cavity of infinite extent:

$$u = \frac{hc}{2\pi} \int d\mathbf{k} \int_0^\infty \ln(1 - \text{tr}(R_1R_2)e^{-2a\sqrt{\xi^2 + k_z^2}} + \det(R_1R_2)e^{-4a\sqrt{\xi^2 + k_z^2}})d\xi$$

(2.43)

Where $R_1$ and $R_2$ represent the reflection matrices of Plates 1 and 2 respectively. We need only to express $R_1$ and $R_2$ and proceed as before. We choose to represent the reflection matrices as $2 \times 2$ matrices in the TE-TM basis, such that the elements

![Figure 2.5: Two parallel plates separated by a distance $a$ in the $z$-direction. The coordinate axes are oriented to coincide with the principle axes of Plate 1 (left plate) with the $x$-direction coinciding with its optic axis. Plate 2 (right plate) is rotated about the $z$-axis by an angle $\chi$ relative to the Plate 1.](image)
represent transition amplitudes between the polarization states of the incident and reflected waves:

\[
R_1 = \begin{pmatrix} r_{1EE} & r_{1ME} \\ r_{1EM} & r_{1MM} \end{pmatrix}, \quad R_2 = \begin{pmatrix} r_{2EE} & r_{2ME} \\ r_{2EM} & r_{2MM} \end{pmatrix}
\]

(2.44)

The functional definition of these matrices is to describe the reflected wave in terms of the incident wave. Thus, if we choose the incident wave, \( \psi_i \), to be a transverse electric wave with unit amplitude:

\[
\psi_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \psi_r = R\psi_i = \begin{pmatrix} r_{EE} \\ r_{EM} \end{pmatrix}
\]

(2.45)

In order to obtain an explicit expression for the matrix elements we must find the general solution for a wave in each of the three regions (the interior of the cavity and inside plates 1 and 2). We then have to match up these solutions with Maxwell’s equations providing the boundary conditions.

In the region between the plates (Region 3) the general solution of Maxwell’s equations is a linear combination of transverse plane waves described by an angular frequency \( \omega \), wave vector \( \mathbf{k} \), and polarization unit vector \( \hat{\lambda} \). For any choice of \( \omega \) we can write the solution as the sum of forward and backward traveling waves, each of which have two independent polarizations. Thus the solution for any \( \omega \) will be the
sum of four waves, written in terms of the vector potential as:

\[ A_3 = (a_3 \hat{\lambda}_E + a_4 \hat{\lambda}_{+M})e^{i(k_r - \omega t)} + (a_5 \hat{\lambda}_E + a_6 \hat{\lambda}_{-M})e^{i(k'_r - \omega t)} \]  

(2.46)

In the above equation \( a_i \) is the amplitude of the \( i \)th wave. The polarization and wave vectors can be written explicitly as:

\[
\begin{align*}
\mathbf{k} &= \begin{pmatrix} k_\perp \cos(\phi) \\ k_\perp \sin(\phi) \\ i \rho_3 \end{pmatrix}, & \mathbf{k}' &= \begin{pmatrix} k_\perp \cos(\phi) \\ k_\perp \sin(\phi) \\ -i \rho_3 \end{pmatrix} \\
\hat{\lambda}_E &= \begin{pmatrix} \sin(\phi) \\ -\cos(\phi) \\ 0 \end{pmatrix}, & \hat{\lambda}_{\pm M} &= \frac{1}{\sqrt{\rho_3^2 - k_\perp^2}} \begin{pmatrix} \cos(\phi) \rho_3 \\ \sin(\phi) \rho_3 \\ \pm i k_\perp \end{pmatrix}
\end{align*}
\]

(2.47)

In the above equations \( k_\perp \) is the magnitude of the component of \( \mathbf{k} \) perpendicular to the z-axis and \( \rho_3 = \sqrt{\epsilon_3 \epsilon_c^2 + k_\perp^2} \) where we have performed a scaled Wick rotation as before, \( \xi_c = -i \omega / c \).

The general solutions in the interior of Plates 1 and 2 (Regions 1 and 2, respectively) are similar but given in terms of a linear combination of ordinary, \( \mathbf{D} \cdot \hat{\mathbf{O}} = 0 \), and extraordinary waves, \( \mathbf{D} \cdot (\mathbf{k}_c \times \hat{\mathbf{O}}) = 0 \), where \( \mathbf{D} = \epsilon \mathbf{E} \) is the electric displacement and \( \hat{\mathbf{O}} \) is the direction of the optic axis (for a graphite, the optic axis is perpendicular to the graphene planes)(24). In contrast with Region 3, the solutions in Region 1 and 2 will have two waves each instead of four because we assume Regions 1 and 2 to
2.5 Anisotropic parallel-plate cavity

extend to infinity thereby eliminating any inbound waves. The solutions in Region 1 and 2 are then:

\[ A_1 = a_1 \hat{\lambda}_{1o} e^{i(k_{1o} \cdot r - \omega t)} + a_2 \hat{\lambda}_{1e} e^{i(k_{1e} \cdot r - \omega t)} \]

\[ A_2 = a_7 \hat{\lambda}_{2o} e^{i(k_{2o} \cdot r - \omega t)} + a_8 \hat{\lambda}_{2e} e^{i(k_{2e} \cdot r - \omega t)} \]

(2.48)

where the subscripts \( o \) and \( e \), denote ordinary and extraordinary waves. It is well known that the tangential components of the wave vector are left unchanged by refraction (Snell’s Law) and as such, only the normal component of the wave vectors and the three components of the polarization vectors in the above expression need to be worked out explicitly. This can be done by solving the set of four coupled algebraic equations; (1) the norm of the polarization vector equals one \( (\lambda^2 = 1) \), (2) the transversality of the wave \( (\mathbf{k} \cdot \mathbf{D} = 0) \), (3) the definitions of the ordinary \( (\mathbf{D} \cdot \mathbf{\hat{O}} = 0) \) and extraordinary waves \( (\mathbf{D} \cdot (\mathbf{k}_e \times \mathbf{\hat{O}}) = 0) \), (4) Snell’s law \( \left(k_z = \pm i \sqrt{\frac{c^2}{\varepsilon_1} \left(\frac{D^2}{\mathbf{E} \cdot \mathbf{D}} + k_\perp^2\right)}\right) \).

The resulting wave and polarization vectors can be written explicitly\(^1\):

\[ k_{1oz} = -i \rho_1 = -i \sqrt{\varepsilon_{1\perp} \xi_c^2 + k_{\perp}^2} \]

\[ k_{1ez} = -i \tilde{\rho}_1 = -i \sqrt{\varepsilon_{1\perp} \xi_c^2 + k_{\perp}^2 + k_{\perp} \left(\frac{\varepsilon_{1\perp}}{\varepsilon_{1\perp}} - 1\right) \cos^2[\phi]} \]

\[ k_{2oz} = i \rho_2 = i \sqrt{\varepsilon_{2\perp} \xi_c^2 + k_{\perp}^2} \]

\[ k_{2ez} = i \tilde{\rho}_2 = i \sqrt{\varepsilon_{2\perp} \xi_c^2 + k_{\perp}^2 + k_{\perp} \left(\frac{\varepsilon_{2\perp}}{\varepsilon_{2\perp}} - 1\right) \cos^2[\tilde{\phi}]} \]

\(^1\)The normalizations on the polarization vectors have been suppressed for clarity.
where \( \tilde{\phi} = \phi + \chi \). We can now use the fact that the tangential components of the electric and magnetic fields are continuous across the boundary to solve for the reflection coefficients. The matrix elements for \( R_1 \) are given explicitly in Appendix A. From the reflection coefficients it is obvious that the Casimir force depends on the angle of orientation, \( \chi \), which can not be decoupled from the azimuthal angle \( \phi \). Thus the integration over \( dk \) in equation 2.43 should be replaced with integrals explicitly
over $dk_\perp$ and $d\phi$:

$$u = \frac{\hbar c}{8\pi^3} \int_0^\infty k_\perp dk_\perp \int_0^{2\pi} d\phi \int_0^\infty \ln(1 - \text{tr}(R_1R_2)e^{-2aq} + \det(R_1R_2)e^{-4aq})d\xi_e$$ (2.51)

where $q = \sqrt{\xi_e^2 + k_\perp^2}$.

Two cases of the above expression are of particular interest, the isotropic case ($\epsilon_{i\parallel} = \epsilon_{i\perp} = \epsilon_i$) and the perfectly anisotropic case ($\epsilon_3 = \epsilon_{1\parallel} = \epsilon_{2\parallel} = 1$ and $\epsilon_{1\perp} = \epsilon_{2\perp} = \infty$). We consider these cases in the following sections.

### 2.5.1 Isotropic Case

It can be shown that in the isotropic limit the reflection matrices given in Appendix A reduce to the isotropic reflection coefficients. In this case the reflection matrices are:

$$R_1 = \begin{pmatrix} -\rho_1 + \rho_3 & 0 \\ \rho_1 + \rho_3 & 0 \\ 0 & \frac{\epsilon_3\rho_1 - \epsilon_1\rho_3}{\epsilon_3\rho_1 + \epsilon_1\rho_3} \end{pmatrix}, \quad R_2 = \begin{pmatrix} -\rho_2 + \rho_3 & 0 \\ \rho_2 + \rho_3 & 0 \\ 0 & \frac{\epsilon_3\rho_2 - \epsilon_2\rho_3}{\epsilon_3\rho_2 + \epsilon_2\rho_3} \end{pmatrix}$$

the elements of which equal the standard reflection coefficients for the isotropic case.

The integrand in equation 2.43 thus reduces to:

$$\ln[1 - (r_{1E}r_{2E} + r_{1M}r_{2M})e^{-2\rho_{3a}} + (r_{1E}r_{1M}r_{2E}r_{2M})e^{-4\rho_{3a}}]$$ (2.52)
Here we have dropped the second letter in the subscripts as the matrices are now diagonal. If we let \( \epsilon_1 = \epsilon_2 \), \((r_{1E} = r_{2E}, r_{1M} = r_{2M})\), Eq. 2.51 simplifies further to yield the standard Lifshitz expression:

\[
 u = \frac{hc}{4\pi^2} \int_0^\infty \int_0^\infty k_\perp \ln[(1 - r_{1E}e^{-2\alpha_3})(1 - r_{2M}e^{-2\alpha_1})] \, d\xi_c \, dk_\perp \tag{2.53}
\]

### 2.5.2 Perfectly Anisotropic Case

Another interesting limit of our expression is the totally anisotropic case. In this limit we let \( \epsilon_3 = \epsilon_{1\parallel} = \epsilon_{2\parallel} = 1 \) and we will let \( \epsilon_{1\perp} \) and \( \epsilon_{2\perp} \) go to infinity. The resulting reflection coefficients for Plate 1 are then:

\[
 r_{1EE} = \frac{k_\perp^2 \sin^2 \phi + \xi_c^2 (1 - 2 \cos^2 \phi) - \sqrt{k_\perp^2 + \xi_c^2} \sqrt{k_\perp^2 \sin^2 \phi + \xi_c^2}}{k_\perp^2 \sin^2 \phi + \xi_c^2 + \sqrt{k_\perp^2 + \xi_c^2} \sqrt{k_\perp^2 \sin^2 \phi + \xi_c^2}} \tag{2.54}
\]

\[
 r_{1ME} = r_{1EM} = \frac{2\xi_c \sqrt{k_\perp^2 + \xi_c^2} \cos[\phi] \sin[\phi]}{k_\perp^2 \sin^2 \phi + \xi_c^2 + \sqrt{k_\perp^2 + \xi_c^2} \sqrt{k_\perp^2 \sin^2 \phi + \xi_c^2}} \tag{2.55}
\]

\[
 r_{1MM} = \frac{-k_\perp^2 \sin^2 \phi + \xi_c^2 (1 - 2 \cos^2 \phi) - \sqrt{k_\perp^2 + \xi_c^2} \sqrt{k_\perp^2 \sin^2 \phi + \xi_c^2}}{k_\perp^2 \sin^2 \phi + \xi_c^2 + \sqrt{k_\perp^2 + \xi_c^2} \sqrt{k_\perp^2 \sin^2 \phi + \xi_c^2}} \tag{2.56}
\]

The reflection matrix for Plate 2 is identical to that of Plate 1 with the replacement of \( \phi \rightarrow \phi + \chi \). As the interaction energy depends on the product of these matrices, it is demonstrative to see the form of the product under a few simple cases, namely under
2.5 Anisotropic parallel-plate cavity

parallel ($\chi = 0$) and perpendicular ($\chi = \pi/2$) alignment. Writing $R_1R_2 = R(\chi, \phi)$:

$$ R(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & \left( \frac{\xi_c - \sqrt{k_1^2 + \xi_c^2}}{\xi_c + \sqrt{k_1^2 + \xi_c^2}} \right)^2 \end{pmatrix}, \quad R(0, \pi/2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.57) $$

$$ R(\pi/2, 0) = \begin{pmatrix} 0 & 0 \\ 0 & \left( \frac{-\xi_c + \sqrt{k_1^2 + \xi_c^2}}{\xi_c + \sqrt{k_1^2 + \xi_c^2}} \right) \end{pmatrix}, \quad R(\pi/2, \pi/2) = \begin{pmatrix} 0 & 0 \\ 0 & \left( \frac{-\xi_c + \sqrt{k_1^2 + \xi_c^2}}{\xi_c + \sqrt{k_1^2 + \xi_c^2}} \right) \end{pmatrix} \quad (2.58) $$

From the above equations, one can see that for perpendicular alignment the trace of the product of the reflection matrices is reduced with respect to parallel alignment; however, it does not go zero for all cases. Thus, one would intuitively predict that the interaction energy will not vanish for perpendicular alignment.

In order to obtain the potential energy per unit area as a function of the orientation angle ($\chi$) we have evaluated eq. 2.51 using the reflection matrices from eqs. [2.54-2.56] and plotted the values in Figure 2.6 for a separation of 1 micron. The values have been scaled with respect to the Casimir case ($u_0$) and fit to a $\cos^2[\chi]$ function.

The dependence of the energy on the angle of orientation will also produce a torque: $M = \frac{\partial u}{\partial \chi}$. This torque would exhibit the same $2\chi$ dependence as the normal force; however, it would be maximum at $\chi = \pi/4$ instead of $\chi = 0$. Such a torque would produce a displacement orthogonal to that of the normal force and could in principle be used as another method of measuring the macroscopic effects of the quantum vacuum.
2.6 Discussion of Modes

In the previous sections, we developed and described a new approach to the Casimir effect and demonstrated its use in several specific cases. In this section, we

Figure 2.6: The Normalized Casimir Potential Energy per Unit Area as a function of orientation for a cavity formed with two perfectly anisotropic plates. The values of the energy have been scaled with respect to the (isotropic) perfectly conducting case. The best fit parameters are: \( a = 22.95 \), \( b = 22.588 \), and \( c = 6.67519 \).

Of particular interest is the 180° symmetry in the orientational dependence of the force. An experiment is currently under construction whereby one of the plates will be rotated relative to the other and the variation in the force detected. This orientational dependence should allow the signal of the Casimir force to be uniquely identified. This experiment is described in the next chapter.
would like to discuss and compare this method with Casimir’s and Lifshitz’ methods.

As a case study, we shall once again consider the Casimir case.

As indicated in section 2.2, the solution to the Casimir case can be expressed as:

\[ u = \frac{hc}{2\pi^2} \int_0^\infty k\,dk \int_0^\infty \ln(1 - e^{-2ia\sqrt{\xi^2 + k^2}})d\xi \]  

(2.59)

In order to compare the modes being considered in the above equation, we would like to look at the mode-generating function for real frequencies (not Wick rotated frequencies):

\[ \Delta(\omega, k, r) = 1 - e^{-2ia\sqrt{\omega^2 - k^2}} \]  

(2.60)

This method considers those modes, such that, \( \Delta = 0 \). These modes are,

\[ \omega^2 - k^2 = \left( \frac{n\pi}{a} \right)^2 \]  

(2.61)

where, \( \omega_c \equiv \omega/c \). As the above equation indicates, this approach considers exactly the same modes as those considered by Casimir. Attempting to solve this problem in terms of real frequencies requires renormalization as Casimir demonstrated. However, upon Wick rotation this problem becomes significantly simpler, as the Wick rotation removes large amplitude oscillations of the integrand at infinity, and can be computed directly as demonstrated in section 2.2. Additionally, the solution and the modes being considered by this approach are the same as those addressed in the Lifshitz approach after renormalization.
2.7 Final Remarks

In this dissertation we have presented a general approach to Casimir problems and have used this approach to solve a number of examples. We have shown that this approach yields Casimir’s and Lifshitz’ expressions in the appropriate limits. Using this approach we have solved the single-sphere problem, demonstrating an attractive rather than repulsive force for a single spherical shell, in agreement with intuition. We have also calculated the Casimir force between anisotropic conductors and shown that such boundary conditions lead to the predicted orientational dependence.

2.8 Future Work

Investigation into the use of this approach within other contexts is currently underway. In the future we hope to demonstrate the usefulness of this approach by solving many new problems, described below.

One such problem that we hope to solve, and may already have solved, is the case of inclined planes. That is, the case of two plates whose normals are not anti-parallel. Demonstrating the solution to this problem would allow calculation of exact solutions to arbitrary geometries, as any surface can be described by an infinite number of differential inclined planes. We would then revisit the spherical shell using the inclined plane method to calculate the force. We could then cross-check the solution to the spherical shell presented in this dissertation.

Another problem we would like to investigate is the application of Casimir forces to
charged particles. The idea being, two electrons in the presence of the vacuum should experience a Casimir force. As the vacuum modes scatter off electrons, bound states should be created between two electrons producing a Casimir effect. Preliminary investigation into this idea has already yielded interesting results. The preliminary calculations are given in Appendix C
Chapter 3

Experiment

An orientational dependence of the Casimir force between anisotropic conductors has long been predicted. Such an orientational dependence is characteristic of a non-scalar interaction and its observation is fundamental to the experimental investigation of the observed phenomenon reported to be the Casimir force. This chapter details an instrument designed and constructed by the candidate with the intended use of measuring the orientational dependence of the Casimir force between Highly Ordered Pyrolitic Graphite plates with in-plane optical anisotropy over separations of 1–7 µm. The instrument is currently being characterized and methods of data acquisition and analysis are being considered. These considerations as well as preliminary data indicating the sensitivity of the instrument are presented.
3.1 Introduction

In the last 60 years many developments in the field of Casimir force research have taken place. Theoretical developments in this field led to the prediction of an orientationally dependent force in the presence of anisotropic boundary conditions (11). While numerous experiments have confirmed the existence of a force exhibiting the appropriate magnitude and distance dependence, no experiments to date have confirmed that the observed force is the result of a vector interaction, a requisite characteristic of the Casimir force. Measuring the orientational dependence of the Casimir force under anisotropic conditions is one way to confirm that the measured force is the result of a non-scalar interaction.

In this chapter we describe an apparatus designed and constructed by the candidate with the intended use of measuring the orientational dependence of the Casimir force. We begin with a general description of the apparatus, proceed to describe the individual pieces and sub-systems of the apparatus, describe possible methods of operation and data analysis, present predictions of the expected Casimir force, describe the current precision of apparatus and several sources of error, present acquired data and some preliminary analysis, and conclude with a discussion of the steps that will be taken in the future. The rest of this section is devoted to an overview of the instrument.
3.1.1 Integrated Components

This instrument utilizes several components manufactured by private companies. Below is a list of those integrated components along with citations to the products’ websites (column five). Column four indicates the letters associated with those parts in figures 3.1, 3.2, and 3.3. This table represents a select list of the integrated components and is not intended to be comprehensive.

<table>
<thead>
<tr>
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<th>Model #</th>
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<td>P3-460B-FC-1</td>
<td>(D)</td>
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<td>New Focus</td>
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<td>Rotation Motor</td>
<td>Lin Engineering</td>
<td>SilverPak 17c</td>
<td>(I)</td>
<td>(55)</td>
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<tr>
<td>10x Microscope Objective</td>
<td>Edmund Optics</td>
<td>36-132</td>
<td>(J)</td>
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</tr>
</tbody>
</table>

Table 3.1: Integrated Components. Column four is the letter associated with that component in figures 3.1, 3.2, and 3.3. Column five is a citation for the URL’s of the products; additional information can be found there.
3.1 Introduction

3.1.2 Setup

As indicated in figures 3.1, 3.2, and 3.3, the instrument that we have designed uses an inverted pendulum as the force transducer used to measure the Casimir effect. This transducer is composed of three parts, a “test” plate, (A), mounted to an optical fiber, (D), that is held from below by a fulcrum, (C). Under a force applied to the test plate, the fiber bends until the bending moment of the fiber balances the applied force. This deflection is measured using a microscope, (J), and digital camera, (G). To ensure the plates are parallel two cameras, (K), image the gap between the source plate and test plate.

Figure 3.1: Experimental Setup. The labeled parts are: (A) the test plate, (B) the source plate, (C) the fulcrum, (D) the optical fiber, (E) the pulley, (F) the translation stage. The dashed line indicates the “long axis” about which the source plate can be rotated. This image is to scale; the length of the fiber extending beyond the fulcrum is 50.5 mm.
The position of the “source” plate, (B), is varied using the translation stage, (F). The translation stage is an optically encoded translation stage New Focus Model 9067 COM-E(52) driven by a New Focus Model 8302 Picomotor actuator(51) controlled by a LabView program written in house. The combination of the picomotor and translation stage allows for incremental motion of $0.08 \pm 0.01 \mu m$ over a separation range of $1 - 7 \mu m$. The position of the stage is independently measured using an Albion Fireball digital probe, (H), with a digital resolution of $0.25 \mu m$. Additionally, the source plate can be rotated about the long axis, (dashed line), using a pulley, (E),
and rotation motor (I). The rotation motor is a SilverPak 17c DC stepper motor (55).

![Diagram of experimental setup]

Figure 3.3: Experimental Setup: This schematic is taken from the same viewpoint as Figure 3.2. The indicated components are: (K) digital cameras used for alignment purposes and (L) the vacuum chamber.

The instrument can be described in terms of three systems: the detection systems consisting of optical fiber and camera, the control systems consisting of the translation and rotation motors, and the environmental controls consisting of the vacuum chamber, (L), thermal insulation, and vibration isolation. These systems are described in depth below.
3.2 Detection Systems

The dependent variable in our experiment is the position of the mechanical oscillator. Knowledge of the position of the mechanical oscillator (the optical fiber) is used as a measure of the deflection, from which we can infer the force acting on the fiber at any particular moment. One can associate a system of interrelated parts to the measurement of the dependent variable. This system includes: the LED, the fiber, the test plate, the microscope, and the digital camera. These elements collectively form the detection system.

The detection system begins with an LED that is coupled to an optical fiber using an aspheric lens. The optical fiber is held in a fulcrum with 50.5 mm of the bare fiber extending beyond the fulcrum. Two graphite plates, of 1 cm diameter, are superglued (one on either side of the fiber, to balance the masses) to the optical fiber 27.5 mm from the fulcrum (measured from the top of the fulcrum to the bottom edge of the plates). This system is held upright with the tip pointed directly upward into the objective lens of a microscope. This lens forms an image on a ThorLabs DCC 1545M CMOS camera. The centroid of this digital image thus indicates the position of the tip of the optical fiber. The time at which the image was taken, the intensity of the image, and the centroid are all recorded, along with the positions of the translation and rotation stage. Greater detail on each of these elements will be given in the following subsections.

The precision of the measurement system depends on many things, the centroiding
precision, mechanical noise in the fiber, and precise measurements of the natural frequency and moment of inertia of the fiber. Detailed descriptions of these uncertainties are given in Section 3.7.

3.2.1 Fiber

The force transducer used in this experiment is a single mode optical fiber that also plays a primary role in the measurement system. The fiber itself is an SM460HP patch cable. The bare fiber is 245 μm in diameter consisting of a fused silica core of 125 μm diameter surrounded by an acrylic coating. One end of the patch cable is illuminated by an LED using an aspheric lens to increase optical throughput. The other end is cleaved and then checked to assure high optical output. If the fiber has been cleaved properly the light exiting the fiber should form a small gaussian spot on the camera in our system and the peak intensity should be high. If this is found to be the case, the fiber is mounted in the fulcrum and then stripped of its acrylic coating. The fiber is mounted in the fulcrum such that it extends beyond the fulcrum 50.5 mm. Once mounted, two graphite plates sandwich the fiber and are held in place using superglue. The fulcrum, fiber, and plates are then mounted beneath the microscope objective.

Mechanical oscillators are often used as force measuring devices. The novelty of this mechanical oscillator is that it is an optical fiber and allows direct determination of the deflection of the fiber under the influence of any external force. Other experiments using mechanical oscillators measure the deflection of the oscillator through
interferometry or simple reflection, that is, the balance is illuminated and the reflected light is recorded. Our apparatus combines the oscillator and light source into a single unit, allowing us to achieve higher luminosity and precision.

3.2.2 Plates

For this experiment we have chosen a parallel plate configuration using Highly Ordered Pyrolytic Graphite as the material. The plates are cut from a single piece of HOPG to be 1 cm in diameter with the optic axis in the plane of the surface. Once cut the plates are sanded to the desired mass and polished to \( \sim \lambda \), an equivalent root-mean-square roughness of \( \delta = 0.5 \mu m \). Two such plates are used to sandwich the fiber and form the “test” plate. Where, we have used two plates in order to balance the mass on the fiber. Another plate of HOPG is used as the “source” plate.

The optical properties of graphite have been measured and tabulated in Taft and Philipp(37). Unfortunately, data regarding the dielectric response function along the optic axis is less easily obtained, for this data we have used the predictions given in Johnson and Dresselhaus(36). Figure 3.4 shows the measured and predicted optical properties of graphite. Where,

\[
\epsilon = \begin{pmatrix}
\epsilon_1^\parallel + i\epsilon_2^\parallel & 0 & 0 \\
0 & \epsilon_1^\perp + i\epsilon_2^\perp & 0 \\
0 & 0 & \epsilon_1^\perp + i\epsilon_2^\perp
\end{pmatrix}
\]

(3.1)

is the dielectric tensor represented in an orthogonal coordinate system with \( x \) in
3.2 Detection Systems

Figure 3.4: Optical properties of graphite given in units of $\epsilon_0$. (a) The real part of the in-plane dielectric response function. (b) The imaginary part of the in-plane dielectric response function. (c) The real (left) and imaginary (right) parts of the cross-plane dielectric response functions. These graphs were taken from Johnson and Dresselhaus\(^{36}\). Plots (a) and (b) contain optical data from Taft and Philipp\(^{37}\) (light solid line) and all three graphs contain predictions of a Slonczewski-Weiss-McClure model (dashed line). The dark solid line was calculated using the same model with an additional Drude contribution.

the direction of the optic axis and $z$ normal to the surface. As the above data shows, graphite is optically anisotropic implying electromagnetic waves of different polarization interact with the material differently. Predictions of the Casimir force between plates of such material are given in section 3.6.

3.2.3 Optical system

The forces acting between the two plates are inferred through a measurement of the deflection of the fiber. This measurement is obtained by imaging the tip of the fiber with a digital camera. The digital camera used in this experiment is a Thorlabs DCC1545M cmos monochromatic digital camera. This camera connects to the computer through a USB2.0 port which also powers the camera. The camera settings can be controlled by the computer and typically set to a pixel clock of 40
3.2 Detection Systems

Hz, exposure time of 2 ms, and a 100×100 pixel Region of Interest centered around the tip of the fiber. Each pixel is 5.2 square μm in physical size. The signal from the pixels are converted to digital counts with an 8 bit encoder. Thus the data received by the computer, comes in the form of an integer array whose elements range from 0 – 255 digital counts.

![Top illuminated fiber.](image)

Figure 3.5: Top illuminated fiber.

The raw data from the camera is acquired in the form of an array. Every element of this array represents a single pixel on the camera. The value of the element is an integer between 0-255 indicating the optical energy absorbed by that pixel within the exposure time. This array can then be displayed as a monochromatic image as in figure 3.5. Note, figure 3.5 was not generated under operating conditions. Rather the image was generated by top illuminating the tip of the fiber. Under operating conditions the fiber is back-illuminated by an LED. The resulting image on the camera
is formed by the light coming out of the core of the fiber plus ambient light. One such image is displayed in figure 3.6. Each image is run through two digital filters, one to set the dark level to zero (dark-fielding) and another to correct for variations in pixel sensitivity (flat-fielding). Once these filters have been applied, the centroid of the image can be calculated. The centroid of the image represents the intensity weighted average of the image, specifically, it is the set of first moments of the image:

\[ M_{ij} = \sum_{xy} x^i y^j I(x, y) \]  

(3.2)
where $M_{00}$ is the total intensity. The centroid is thus,

$$(x, y) = \left(\frac{M_{10}}{M_{00}}, \frac{M_{01}}{M_{00}}\right)$$

(3.3)

This centroiding process produces a coordinate pair representing the center of the image in terms of pixel number, which can be converted to a length using the apparent pixel size. In this way the position of the fiber can be recorded as a function of time.

### 3.3 Control Systems

To control the position and motion of the source plate we have chosen to mount one of the plates on the end of a rod carried by a New Focus translation stage powered by a picomotor actuator capable of translating at a step size of $0.08 \pm 0.01 \, \mu m$. This picomotor is computer controlled using a LabView program developed in house. Furthermore, independent measurement of the position of the stage is performed by a Fireball digital probe with a precision of $0.25 \, \mu m$.

To control the orientation of the source plate we have chosen to use a SilverPak 17c DC stepper motor. This motor is also computer controlled using the same LabView program used to control the translation stage. The rod to which the source plate is attached can be rotated about its long axis. This rotation is accomplished by transferring the rotation of the stepper motor to the rod using a pulley and driving belt. Mounted onto the pulley is an encoder providing a measurement of the orientation of the rod. The encoder itself is composed of a photogate and transparent film onto
which lines have been printed at specific angular intervals. The pulley is driven by the SilverPak 17c, in turn driving the mounting rod and encoder, until the next printed line obstructs the photogate. In this way the orientation of the source plate can be moved from one angular position on the encoder to the next. While the precision of this method is low ($\sim 2^\circ$) the repeatability and reliability are high, allowing multiple measurements at the same rotational position to be made easily.

3.4 Environmental Controls

The instrument described above and used in this experiment to measure the Casimir force is based on the response of a mechanical oscillator to an applied force field. As such, this experiment is susceptible to external mechanical vibrations, barometric fluctuations, and temperature variations, which introduce additional noise in the experiment. In order to make the most precise measurement of the Casimir force in this experiment considerable effort must be made to isolate the system from external influences.

An important source of external mechanical vibrations in this experiment is air currents within the room. These air currents can be generated by a number of sources particularly air conditioning within the lab room. In order to isolate the experiment from such noise we have placed the entire experiment inside a sealed box lined with layers of thermally insulating and thermally conducting material. This arrangement protects the apparatus from large air currents within the room. The thermal insula-
tion reduces the thermal fluctuations within the box, while the thermally conducting material helps reduce thermal gradients within the box that may drive convection currents and lead to asymmetric thermal expansion.

The second largest external source of noise in this experiment is generated by seismic vibrations. At present, the experiment is performed within a sealed box sitting on the floor of laboratory. As such, vibrations passing through the floor also pass through the fiber. Investigation into the most suitable type of vibration isolation is ongoing.

The third isolation system installed in this instrument is a vacuum chamber that encloses the fiber. At present the vacuum system is capable of reaching approximately 10 \( \mu \text{m Hg} \). After being evacuated, the system is sealed and the pump detached to prevent vibrations from being introduced through the vacuum hardware. This system eliminates air currents around the mechanical oscillator. For the preliminary studies reported here the vacuum system has not been turned on. Preliminary results indicate that the system becomes highly sensitive under vacuum, requiring new controls and experimental procedures to be developed. Investigation into the best method of utilizing the vacuum chamber is ongoing.

Despite such lengthy precautions the presence of a person within the room introduces a noticeable level of noise. Data is therefore acquired at night when nobody is present on the same floor as the laboratory and the laboratory is vacant and locked. The data is taken remotely and motions of the source plate are carried out in an automated fashion using a predetermined computer program.
3.5 Methodology

The data in this experiment comes from two sources, (1) the digital camera giving information related to the position of the fiber and (2) from the digital probe giving information related to the position of the source plate. A significant degree of processing must take place in order to derive the force as a function of separation.

At every frame the centroid is calculated and recorded along with a time stamp and the total intensity of the image. The centroid values calculated using equation 3.3 are given in the camera basis (pixel column, pixel row). In order to convert to the translation stage basis, the translation stage was used to move the fiber along its direction of motion. From this data the angle between the camera basis and the translation stage basis can be determined.

The centroid indicated above describes the position of the fiber. However, we are interested in the position of the test plate attached to the fiber a distance 18 mm below the tip. Figure 3.7 indicates the dimensions involved. The position of the plate can be determined from the position of the fiber, $X_p = \alpha X_f$. Where, $\alpha$ is the conversion between “pixels of centroid motion” to “microns of test plate motion”. To determine this quantity the plates are brought into contact where data is taken from both the camera and digital probe. The source plate is then moved in the direction of the test plate. By comparing the reading of the centroid with the reading from the digital probe a direct measurement of $\alpha$ is determined. The value obtained in this manner is $\alpha = 0.2651 \pm 0.0002 \, \mu m/p$, a 0.1% relative error, where $\mu m/p$ is the unit
The centroid values from the camera along with the digital probe data, the measured conversion factor $\alpha$, the measured mass of the plates, and natural frequency of the fiber can now be brought together to calculate the force as a function of the plate separation.

3.5.1 Equation of Motion

This subsection is concerned with the general equation of motion of the fiber. From this equation two methods of data acquisition and analysis will be proposed and described in the following subsections (Method I and Method II).

The force transducer in this experiment is an inverted pendulum described by the
3.5 Methodology

equation of motion for a damped-driven harmonic oscillator:

\[ m\ddot{X}_p - \gamma \dot{X}_p + m\omega^2 X_p = F(X_S - X_p) \quad (3.4) \]

where, \( m \) is the mass on the fiber, \( \gamma \) is the damping coefficient, \( F(X_S - X_p) \) is the total external force applied to the fiber for a separation of \( X_S - X_p \). The positions of the test plate and source plate are \( X_p \) and \( X_S \), respectively, as measured from the fulcrum (see figure 3.7). For this analysis we shall assume the natural frequency of the fiber, \( \omega \), is a function of the separation:

\[ \omega^2 = (\omega(X_S - X_p))^2 \equiv \omega_o^2 + \frac{C_e(X_S - X_p)}{m} \quad (3.5) \]

where \( \omega_o \) is the natural frequency at large separations and \( C_e \) is a separation dependent elastic coefficient. This is motivated by the observations of Overbeek and Sparnaay, section 1.2.1, that indicated the presence of air damping. This air damping was found to be a function of separation and, as indicated in the literature (41; 42), the presence of air can have an elastic effect as well as a damping effect on the oscillator.

If we perturb this system by a small amount:

\[ X_p = \bar{X}_p + \delta(t) \quad (3.6) \]
This leads to:

\[ \ddot{\delta} - \gamma \dot{\delta} + [\omega^2 - 2\omega' \delta(t)] (\bar{X}_p + \delta(t)) = \frac{F(X_S - \bar{X}_p)}{m} - \frac{F'(X_S - \bar{X}_p)}{m} \delta(t) \]  

(3.7)

This equation has two components of interest, a time independent component and a time dependent component. These components are given by

\[ \omega^2 \bar{X}_p = \frac{F(X_S - \bar{X}_p)}{m} \]  

(3.8)

and,

\[ \ddot{\delta} - \gamma \dot{\delta} + [\omega^2 - 2\omega' \bar{X}_p] \delta(t) = \frac{-F'(X_S - \bar{X}_p)}{m} \delta(t) \]  

(3.9)

where we have dropped terms proportional to $\delta^2$. These equations represent a static method of analysis and a dynamic method of analysis, respectively.

### 3.5.2 Method I

The following describes a method of measuring the force between the plates as a function of separation. This method utilizes equation 3.8, which indicates that the force acting on the fiber can be calculated if the position of the test plate and source plate, the mass of the test plate, and the natural frequency of the fiber are known. Below we discuss the procedure for data acquisition associated with this method of analysis.

Every data run begins with the plates in contact. After 100 seconds the source is
3.5 Methodology

pulled back to approximately 50 \(\mu m\) separation, where data is taken for another 150 seconds. These steps are used as calibration for the rest of the data. All deflections of the fiber are calculated relative to the mean position of the fiber at the \(\sim 50 \mu m\) separation \((\tilde{X}_{f0})\). Note, in general \(\tilde{X}_{f0}\) will not equal the true position of null deflection, rather \(\tilde{X}_{f0} = F(d_0)/k\) where \(d_0\) is the separation of the plates at that source plate position. The translation stage is then stepped back toward the fiber until the plates are once again in contact. The mean position of the fiber at each step \(i\) can be used to calculate the force acting between the plates:

\[
f_i = m\omega^2\alpha(X_{fi} - \tilde{X}_{f0})
\]

where \(\alpha\) is the conversion from unit pixel to micron and \(X_{fi}\) is the \(i\)-th position of the fiber.

As every data run starts with the plates in contact \((d = 0)\), the separation can be calculated throughout the experiment by keeping track of the position of the source plate and test plate.

\[
d_i = (X_{Si} - X_{Sc}) - \alpha(X_{fi} - X_{fc})
\]

where \(X_{Si}\) is the value of the digital probe at the \(i\)-th position and \(X_{Sc}\) is the value at contact. Similarly, \(X_{fi}\) is the position of the fiber centroid at the \(i\)-th position and \(X_{fc}\) is the position of the fiber when the plates were in contact.

This method can be used to directly measure the forces acting on the plate as a function of separation. However, this method is sensitive to considerable errors. The
most difficult error to address with this method is drift of the undeflected position of the fiber \( X_{f_0} \). Drift in this quantity results in drift of the inferred forces. To address this drift, one would want to complete an entire data sweep such that the drift in the fiber is less than the precision of the experiment, \( v_d \tau \leq \Delta F \), where \( \tau \) is the duration of the data sweep, \( v_d \) is assumed drift velocity, and \( \Delta F \) is the uncertainty in the force measured at any data point.

### 3.5.3 Method II

The following describes another method of observing the Casimir effect. This method utilizes equation 3.9, which indicates that a measure of the resonant frequency can be used to infer the gradients of external forces acting on the fiber. Rearranging 3.9:

\[
\ddot{\delta} - \gamma \dot{\delta} + \left[ \omega^2 - 2\omega \omega' \dot{X}_p - \frac{F'(X_S - \dot{X}_p)}{m} \right] \delta(t) = 0 \tag{3.12}
\]

indicating the frequency of resonance is:

\[
\omega_r^2 = \omega^2 - 2\omega \omega' \dot{X}_p - \frac{F'(X_S - \dot{X}_p)}{m} \tag{3.13}
\]

Using the definition, \( \omega^2 = \omega_o^2 + C_e/m \):

\[
\omega_r^2 = \omega_o^2 + \frac{C_e}{m} - \frac{C_e'}{m} \dot{X}_p - \frac{F'(X_S - \dot{X}_p)}{m} \tag{3.14}
\]
3.5 Methodology

The above equation indicates that the gradient of external forces can be calculated from knowledge of the resonant frequency of the fiber. The procedure associated with this method of analysis is described below.

As with Method I, every data sweep begins with the plates in contact. The plates are then brought to separation of $\sim 50 \mu m$ and slowly brought back into contact. At every separation the centroid of the fiber as a function of time is recorded. This data is then fourier transformed to display the amplitude of oscillation as a function of frequency. This data can be fit to a Lorentzian:

$$ A(\nu) = a_1 + \frac{a_2}{\sqrt{\left(\nu_r^2 - \nu^2\right)^2 + (2\gamma\nu)^2}} $$  \hspace{1cm} (3.15)

where $\nu$ is the frequency in Hz, $\nu_r$ is the resonant frequency of the fiber, $\gamma$ is the damping constant, and $a_1$ and $a_2$ are fit parameters. Note, $\nu_r$, $\gamma$, $a_1$, and $a_2$ are parameters determined by the fit. In this way the resonant frequency at every separation can be measured and compared with equation 3.14 in order to calculate the gradient of external forces.

This method can be used to directly measure the gradient of the forces acting on the fiber as a function of separation. While this method is insensitive to drift in the undeflected position of the fiber, $X_{f0}$, it is sensitive to drift in the natural frequency of the fiber, $\omega_0$, and requires knowledge of the separation dependence of the natural frequency, $C_e$. The distance dependence of the natural frequency is investigated in section 3.8.
3.6 Expected signal from the Casimir Force

In order to make a well-founded estimate of the Casimir effect in a particular experiment many aspects need to be addressed including the optical properties of the materials, temperature at which the measurement is being made, the size and orientation of the plates, and the geometry of the plates including roughness corrections. This section is devoted to making theoretical estimates of the Casimir force that should be observed within this experiment.

All the necessary pieces of information such as the temperature at which the measurement is being made (300 K), the size (0.5 cm radius disk), the uncertainty in the alignment of the surface normals of the plates (< 0.1 mrad), the geometry (plate-plate) and roughness (stochastic distribution of 0.5 μm roughness) have been described in previous sections. We can therefore calculate the magnitude of the Casimir force in our instrument using the theory developed in section 2.5. We will take into account finite temperature effects (section 1.1.4), roughness effects (below), and finite conductivity effects (section 3.2.2). In order to account for a possible misalignment of no more than 0.1 mrad, we will use the the Proximity Force Approximation (equation 1.82), and to account for roughness corrections we will use an expression given in the book *Advances in the Casimir Effect* (7):

\[
P(a) = P_C(a) \left[ 1 + 10 \frac{2\delta^2}{a^2} + 105 \frac{4\delta^4}{a^4} \right]
\]

where \(\delta\) is the rms-roughness and \(P_C\) is the Casimir pressure between perfectly smooth
3.6 Expected signal from the Casimir Force

surfaces. The total force being the combination of all the above is given by the following expression:

\[
F(a) = \frac{k_B T}{4 \pi^2} \sum_{l=0}^{\infty} \int_0^R \int_0^{2\pi} \int_0^{\infty} \int_0^{2\pi} \Delta(l, \xi) \left[ 1 + 10 \frac{(a)}{a^2} + 105 \frac{(a)}{a^4} \right] k_1 dk_1 d\phi r dr d\theta
\]

(3.17)

where \( \tilde{a} = a + \delta + \varphi_0 R \cos \theta \), \( \varphi_0 \) is the misalignment angle, \( a \) is the measured separation, \( R \) is the radius of the plates, and \( \theta \) is the integration variable describing an azimuthal angle. The mode generating function, \( \Delta \), is defined in equation 2.8 using the optical data above for the values of \( \epsilon \), the dielectric tensor.

Log–Log Plot of Force vs Separation

Figure 3.8: Predicted Casimir Force in our Experiment. The dashed line was calculated assuming a 0.1 mrad misalignment between the normals of the surfaces and optic aligned optic axes. The solid line assumed no misalignment.
Numerical evaluation of equation 3.17 is displayed in Figure 3.8 under the assumption of a 0.1 mrad misalignment (dashed line) and no misalignment (solid line), both were calculated with the plates’ optic axes aligned. The most notable aspect of this graph is the “knee” around 3 μm where the slope of the graph changes. The main contributing factor giving rise to this change in slope is the effect of non-zero temperature. Below 3 μm, the Casimir force is dominated by vacuum modes. Above 3 μm, the Casimir force is dominated by thermal modes. As described in Section 1.2.7, there is a lot of interest in observing and measuring this change and only one experiment to date has been able to do so with precision and reliability.

Log–Log Plot of Amplitude vs Separation

Figure 3.9: Predicted Amplitude of the Casimir Force in our Experiment. This graph represents the difference between the Casimir force at parallel and perpendicular alignment of the optic axes. The dashed line was calculated with a misalignment of the normal of the surfaces of 0.1 mrad. The solid line was calculated with no misalignment.
3.6 Expected signal from the Casimir Force

One of the major difficulties in observing this “knee” is the sub-dominant nature of the Casimir force at such distances. In order to address the presence of background forces, one could investigate the maximal deviation in the force over one rotation. As it is unlikely that any other force will exhibit 180 deg rotational symmetry, this could be used to significantly reduce background effects. Figure 3.9 displays the predicted amplitude of variation as a function of separation. This graph also indicates the presence of a “knee” at 5 μm. This “knee” has the same origin as that in Figure 3.8; as such its observation is just as valuable to this field.

![Amplitude vs Separation](image)

**Figure 3.10:** Predicted Amplitude of the Casimir Force in our Experiment. This graph represents the difference between the Casimir force at parallel and perpendicular alignment of the optic axes as a percentage of the force at parallel alignment. The magenta line assumes the normal of the surfaces are parallel. The dashed line assumes a 0.1 mrad misalignment angle. The solid line assumes no misalignment.
3.6 Expected signal from the Casimir Force

Figure 3.10 displays the same information as a percentage variation on a linear scale. From this plot, it is easier to see that the effects of anisotropy become diminished in the finite temperature regime. More simply stated, in the finite temperature regime the plates appear nearly isotropic as far as the Casimir force is concerned. This is representative of a well known phenomenon of “bandwidth” averaging that takes place in the finite temperature regime of the Casimir force. Within this regime, substantial bandwidth averaging occurs in the calculations, resulting in a lack of uniqueness in the calculated force curve. A similar effect gives rise to the apparent lack of anisotropy within this region.

Based on preliminary data, our experiment has sufficient sensitivity to measure the Casimir force for separations ranging from $\sim 1 - 7 \mu m$. This would be accomplished by measuring the deflection of the fiber as a function of separation (Method I). However, systematic errors may become significant and make this type of analysis difficult at present.

In its current state, the best method of running the experiment involves observing the effect of the Casimir force on the resonant frequency of the fiber (Method II). As indicated in section 3.5.3, the presence of a force with non-vanishing gradient will affect the resonant frequency of the fiber:

$$\omega_r^2 = \omega_0^2 + \frac{C_e}{m} - \frac{C_e'}{m} x_p - \frac{F'(x_s - x_p)}{m}$$  \hspace{1cm} (3.18)

where $F_{ext}$ is some external force, and $C_e$ is the separation dependence of the natural
3.6 Expected signal from the Casimir Force

Figure 3.11: Predicted gradient of the Casimir Force in our Experiment. Calculated under the assumption of parallel optic axes. The solid line indicates the gradient of the force no misalignment between the surface normals. The dashed line indicates gradient of the force for a misalignment angle of 0.1 mrad.

frequency of the fiber. From this equation, one can see that a measure of the resonant frequency is a measure of the gradient of external forces. This method has certain advantages. For instance, this method is unaffected by drift of the undeflected position of the fiber, as it depends solely on the separation (easily measurable) and the resonant frequency of the fiber at that separation (easily measurable). This will allow us to take longer data sets and achieve a higher level of precision. The expected gradient of the Casimir force is given in figure 3.11. Again, we predict our instrument has sufficient sensitivity to measure the gradient in the Casimir force over a range of $\sim 1 - 7 \ \mu m$. 

122
3.6 Expected signal from the Casimir Force

![Amplitude vs Separation](image)

Figure 3.12: Predicted amplitude of the gradient of the Casimir Force in our Experiment. Calculated under the assumption of parallel optic axes. The solid line indicates the gradient of the force no misalignment between the surface normals. The dashed line indicates gradient of the force for a misalignment angle of 0.1 mrad.

It is worth noting that the effect of the Casimir force on the resonant frequency is negative. That is, the Casimir force will shift the resonant frequency to lower frequencies. For sufficiently large gradients this effect will produce a negative resonant frequency, indicating the oscillator will undergo collapse. This would be observed in our experiment by the test plate being pulled onto and sticking to the test plate. Preliminary calculations indicate this will occur around 3 $\mu$m unless other effects are present.
3.7 Uncertainty

Three types of uncertainty are seen in this experiment. The first type is nominal or scaling error. The second is statistical error and the third is systematic error. In the following subsections the statistical and systematic errors are reviewed. The values of all the sources of uncertainty are presented in table 3.2.

Let us take a brief look at the nominal errors in this experiment. As mentioned in section 3.5, the force derived in this experiment is inferred by the deflection of a cantilever. In order to convert the measured deflection to a force several constants were measured. Each one of these values has an associated uncertainty that contributes to the overall error of the measured force. We note that these uncertainties are associated with values that are constant for all data in this experiment, thus representing uncertainty in the scale of the force rather than in the measurement of the force.

The constants with associated nominal or scaling errors are: the mass of the plates \(m\), the natural frequency of the cantilever \(\omega_0\), and the conversion factor \(\alpha\). The uncertainty of the mass is the nominal error of the measuring device, thus \(m = 0.134 \pm 0.005\) g. As discussed in section 3.5, the conversion factor is \(\alpha = 0.2651 \pm 0.0002\) \(\mu m/p\).

The natural frequency of the cantilever was measured by fourier transforming the data. In the fourier domain a prominent peak is observed (figure 3.15), reflecting the
natural frequency of the fiber. This peak can be fit to a Lorentzian curve:

\[
A(\nu) = a + \frac{b}{\sqrt{(\nu_o^2 - \nu^2)^2 + (2\gamma\nu)^2}}
\]

where \(A(\nu)\) is the amplitude of oscillation at frequency \(\nu\) and \(\nu_o\) is the natural frequency in Hz. The damping constant, \(\gamma\), is a fit parameter along with \(a\), \(b\), and \(\nu_o\). The error in the natural frequency is thus given by the error in the fit parameter \(\nu_o\). Data at large separations \(d > 60 \, \mu m\) was analyzed in this fashion yielding, \(\nu_o = 1.984 \pm 0.003\) Hz.

### 3.7.1 Statistical Error

This type of error represents randomness inherent in the apparatus. Examples of this type of error include optical noise, seismic noise, and readout noise in the digital probe. One property of statistical error is that it can be reduced by multiple measurements. Averaging multiple measurements reduces the statistical error of the mean by \(\sqrt{N}\), where \(N\) is the number of measurements. In this section we will review the sources of statistical error in this experiment.

From the raw image data the centroid is calculated and recorded. This centroid represents the position of the test plate. While not typically recorded, another piece of information can be calculated from the raw image, the uncertainty in the centroid
of the image:

\[
\sigma_x^2 = \frac{1}{M_{00}} \sum_{xy} (x - \bar{x})^2 I(x, y) \quad , \quad \sigma_y^2 = \frac{1}{M_{00}} \sum_{xy} (y - \bar{y})^2 I(x, y) \quad (3.20)
\]

where \( \sigma^2 \) is the variance, \( I(x, y) \) is the intensity recorded by the pixel at \((x, y)\), and \( M_{00} \) is the total intensity of the image, the standard error is thus \( \sigma / \sqrt{M_{00}} \). This value can be used to estimate the contribution of optical uncertainty to the overall uncertainty in the experiment. This value is rather constant as it depends only on the shape of the image and its intensity neither of which change significantly. It has been observed that the errors described in equation 3.20, represent an error of approximately \( \Delta X_f = 1 \) nm in the centroid of a single image.

The effect of optical noise and seismic noise is similar, both affect the precision of the measured position of the test plate. However, one would expect a flat noise spectrum from the optical noise because the fluctuations inferred from optical noise have no preferred frequency. On the other hand, one would not expect seismic noise to be flat, indeed the fourier transform of the data is not observed to be flat. To estimate the magnitude of the error introduced by seismic noise, the standard deviation of a small string of centroid values can be calculated. This value represents the total error in the position of the plate for a single measurement, and found to be \( \Delta X_f \approx 35 \) nm. Indicating the seismic noise present in the system is roughly 35 times the optical noise. In calculating the error of the mean position of the fiber, this number will be reduced by \( \sqrt{N} \) for a data length of \( N \) images.
3.7 Uncertainty

The next source of statistical error deals with the digital probe. This probe has a digital precision of $\Delta X_s = 0.25 \ \mu m$. Once again the standard deviation of the mean can be used to determine the error in the mean position of the source plate.

3.7.2 Systematic Error

Systematic errors are biases in an experiment that lead to a non-trivial alteration of the mean of a measured value, such that the measured mean is different than the true mean for an infinite set of measurements. Common examples would include calibration errors leading to a constant value being added to all measurements. These kinds of errors are unacceptable as they can lead to seriously erroneous conclusions.

This experiment contains significant systematic errors introduced through many sources. Background forces dominate the systematic error present in this experiment. However, the magnitudes of these backgrounds are not, in general, known \textit{a priori}. Other known sources of systematic error include: misalignment of the camera to the x-y plane defined by the translation stage and drift of the fiber.

As mentioned in section 3.5, the drift of the fiber affects the accuracy of Method I. However, Method II is insensitive to this source of error. The value of the drift rate was measured several times over several days, and was found to be $\sim 0.5 \ nm/s$. It should be understood that the drift rate can be as high this value but was rarely observed to be larger. It should also be understood that the magnitude and direction of the drift within any data sweep is impossible to measure. Therefore, its effects on the data of any particular sweep are unknown. In order to address this type of error,
the data sweep would need to be completed in less than $\tau \cong \Delta F/(m\omega_0^2 v_d) \cong 17$ s, in order to ensure the effects of drift are immeasurable. At present this duration is prohibitively short, making Method II preferable.

Once the centroids of the image have been measured, they are used in some fashion (Method I or Method II) to measure of the forces acting in a particular direction. As such, it is extremely important to know what direction the plate is moving if the centroid of the image is observed to change along some direction. This misalignment angle has been measured using the following method.

The source plate was brought into contact with the test plate. The plates were left in contact in order to gather information from the camera and digital probes. The source plate was then moved further in the direction of the test plate, leaving the two in contact. From the observed change in the $x$ and $y$ centroids the misalignment angle can be calculated. This allowed a change of basis matrix to be constructed. This matrix is used to calculate the centroid values in the translation stage basis. The measured angle between these bases was found to be $\theta = 1.663$ radian, the value of which is essentially arbitrary. Of greater interest is the uncertainty in this value 0.001 radian, indicating that the mixing of observed $x$ and $y$ fluctuations is than 1 part in one thousand.

The last source of systematic error stems from background forces. These forces can effect the mean position of the fiber and the resonant frequency of the fiber. One example of an unknown background force would be an electrostatic capacitive force between the plates. This force has been described in section 1.2.4. While we
expect this force to be present in the experiment, there is no means of independently measuring or calculating this force. As such, the best method of addressing this source of systematic error is to use the data at large separations, separations where the Casimir force is much less than the precision, to measure the background forces with longer range than the Casimir force. Measuring the background forces can be accomplished by finding a fit to the data at large separations. In the measurements made and presented in this dissertation no significant background forces of this type were observed.

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<tr>
<td>$d_i$</td>
<td>1-50 $\mu$m</td>
<td>0.1 $\mu$m</td>
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Table 3.2: Sources of uncertainty and their estimated errors.

Table 3.2 summarizes the uncertainties of the quantities used in this experiment. The quantities are: the position of the fiber ($\bar{x}$), the reference position of fiber ($\bar{x}_0$), the distance from the center of the test plate to the tip of the fiber ($s$), the total length of the fiber ($L$), the conversion from “pixels of centroid motion” to “microns of test plate motion” ($\alpha$), the total mass of plates on the fiber ($m$), the radius of the test and source plates ($R$), the natural frequency of the oscillator ($\nu_0$), and the

$^1$The first value is the optical contribution, the second is the contribution from the motion of the fiber.
separation between the plates \( (d_i) \).

### 3.8 Measurements

In this section we present data acquired and analyzed in the spirit of method II, described in section 3.5.3. However, this data has been used to investigate the separation dependence of the natural frequency of the oscillator. As such, we shall consider the Casimir force to be a known quantity and attempt to ascertain a qualitative understanding the behavior of the fiber. The following data represents the average of 4 data sweeps taken in the presence of air. In accordance with Method II, the plates were brought into contact and allowed to settle for several minutes. The data

![Position Xs vs Time](image)

Figure 3.13: The output of the digital probe as a function of time.
recording begins with the plates in contact, where they are allowed to remain for 100 seconds of data recording. The source plate was then pulled back to a separation of \( \sim 30 \, \mu m \), where it was allowed to remain for 360 seconds. After which the separation was decreased by \( \sim 3 \, \mu m \), where again it remained for 360 seconds. This process was continued until the plates were brought back into contact, resulting in 11 separations per sweep. The plate separation was determined using equation 3.11. Four sweeps were performed.

![Graph](image)

**Figure 3.14:** The x-centroid (\( \mu m \)) as a function of time.

Figure 3.13 displays the output of the digital probe as a function of time. This data indicates the position of the source plate and is used to calculate the separation according to equation 3.11. It can be seen that at \( t \approx 3200 \) seconds the output of the digital probe becomes extremely noisy. This is the result of an instability in the
digital probe at that location. This instability does not affect the mean value of the readout, only the statistical error associated with it.

Figure 3.14 displays the x-centroid ($\mu$m) as a function of time. This data is used to calculate the separation according to 3.11. It is also used to measure the resonant frequency of the fiber according to equation 1.96. As figure 3.14 shows the position of the fiber spikes periodically. These spikes are the result of the motion of the source plate. As the source plate is translated toward the fiber, the resulting air pressure displaces the position of the fiber resulting in the observed spike.

![Log–Log Plot of $F(x)$](image)

Figure 3.15: Fourier transform of the x-centroid at a separation of 20 $\mu$m.

At every separation the data was fourier transformed yielding the amplitude of oscillation as a function of frequency. Figure 3.15 displays the fourier transform of the x-position of the test plate for one such separation. The most interesting feature
of this graph is the peak at 2 Hz. This peak is the resonant frequency of the fiber, all other peaks were found to be independent of the separation and most likely the result of external vibrations.

By fitting a Lorentzian (equation 3.15) to the peak at 2 Hz the resonant frequency as a function of separation can be measured. The measured resonant frequencies are displayed in figure 3.16. From figure 3.16 it can be seen that the resonant frequency at large separations, \( d > 5 \, \mu m \), remains fairly constant. This constant value represents the natural frequency of the fiber \( \sim 1.984 \) Hz. Note, the data points at the closest separation of each data sweep have mean plate separations approximately equal to the amplitude of the plates motion. As such these points will be ignored in the following analysis.
From the data one can immediately infer that the Casimir force is not the only force acting on the fiber. This inference is based on the observed increase in the resonant frequency with decreasing separation. The Casimir force, and any attractive force, will decrease the resonant frequency of the fiber. Leading us to conclude that an air cushion or some repulsive force is being expressed in this data. To investigate the nature of this interaction we shall consider the Casimir force to be known and given by the predictions described in section 3.6, and remove its expected contribution. The data, after being corrected in this fashion, is displayed in figure 3.17.

![Plot of (Resonant Frequency)^2 vs Separation](image)

Figure 3.17: Measured resonant frequency corrected for the predicted Casimir Effect.

According to equation 3.14, the data, presented in this fashion, is expected to
3.8 Measurements

Conform to:

\[ \nu_r^2 + \frac{1}{m(2\pi)^2} \frac{\partial F_{\text{cas}}(x)}{\partial x} = \nu_o^2 + \frac{C_e + C'_{\text{bg}} X_p}{m(2\pi)^2} - \frac{1}{m(2\pi)^2} \frac{\partial F_{\text{bg}}(x)}{\partial x} \] (3.21)

where we intend to determine the coefficient of the elastic term, \(c_e\), and \(F_{\text{bg}}\) are unknown background forces.

In order to determine the right-hand side of equation 3.21 and its distance dependence we have generated several fits to the data. Each of these fits includes an offset representing the natural frequency of the fiber. In addition, the fits each have one term representing an inverse power:

\[ f_n(x) = a + \frac{b}{x^n} \] (3.22)

where \(a\) is the natural frequency of the fiber, \(b\) is the magnitude of the elastic coefficient, and \(n\) is an integer between 1 and 5. The data, along with fits, is presented in figure 3.18. From figure 3.18 one can easily see that only power laws of order \(n = 4\) or \(n = 5\) fit the data sufficiently well. The best fit parameters for \(n = 4\) and \(n = 5\) are \(62 \pm 4 \text{ Hz}^2\) and \(168 \pm 14 \text{ Hz}^2\), respectively. The data, after this effect was removed, is displayed in figure 3.19.

The data in figure 3.19 displays no significant or obvious deviation from zero. Indicating that no other statistically significant background forces exist.

One aspect of the data not addressed by the associated uncertainties, is the appar-
3.8 Measurements

Figure 3.18: Measured resonant frequency corrected for the predicted Casimir Effect. Also shown are the best fits lines corresponding to equation 3.22. The colors correspond to; Red: n=1, Orange: n=2, Green: n=3, Blue: n=4, Black: n=5.

ent scatter in the frequency shift at large separations. The variance in the measured resonant frequencies at large separations is approximately $\sim 0.05$ Hz, significantly larger than the error in these measurements, $\sim 0.003$ Hz. As such, the scatter does not appear to represent a statistical error in the data. Possible explanations of this scatter include small varying background forces and drift in the natural frequency of the fiber. While drift in the natural frequency of the fiber is unavoidable, the effect of small varying background forces could potentially be reduced. If the background forces are electrostatic in nature, their magnitude and more importantly their variance might be reduced by the introduction of a UV LED into the apparatus to ionize the air. This idea is currently being investigated.
3.8 Measurements

Following a method similar to Method II, we have measured the resonant frequency of the fiber as a function of separation. This data indicates the presence of an elastic contribution to the resonant frequency or a repulsive force of unknown origin. The distance dependence of this effect appears to be $1/x^4$ or $1/x^5$ with magnitudes of $62 \text{ Hz}^2$ and $168 \text{ Hz}^2$, respectively.

This data has demonstrated a very intriguing property of this system; it did not collapse. As indicated in section 3.6, if the only forces acting on the fiber are attractive, the resonant frequency should become imaginary for separations less than 3 $\mu$m. At which point the test plate would have fallen and stuck to the source plate. This would be quite prohibitive as separations less than 3 $\mu$m would be inaccessible.

Figure 3.19: Residuals of the data after correcting for the predicted Casimir Effect and a best fit to $y = a + b/x^n$. Red and black dots are the residuals after correcting for the fits with $n = 4$ and $n = 5$, respectively.
3.9 Future Work

However, this system did not undergo collapse and data could be taken at shorter separations. If the nature of this effect can be understood, it would significantly improve the dynamic range of this system.

3.9 Future Work

Many developments regarding this experiment are currently underway. The most immediate task is to repeat and refine Method II, described in section 3.5.3 and presented in section 3.8. This method has shown significant promise and could potentially be utilized to obtain data over an extremely large range of separations. Additionally, we would like to develop Method I, described in section 3.5.2. This method has the potential to make a stronger argument than Method II, as it does not require such large corrections. In order to improve the apparatus, regardless of method used, we plan to improve on the environmental controls and vibration isolation systems. Investigation into the use of a UV LED to reduce electrostatic background forces is currently underway.
Chapter 4

Conclusions

This dissertation is concerned with the Casimir force, particularly the Casimir force between anisotropic materials. In chapter 1, the current state of Casimir force theory and experiment have been reviewed to motivate the need for additional theoretical work and further experimental investigation. In chapter 2, a theory has been presented to address the need for a simpler and more comprehensive approach to the Casimir force. This theory represents novel work developed by the candidate. In chapter 3, an experiment has been described and its data has been presented. While this experiment is still in its infancy, the preliminary data indicates this apparatus is capable of measuring the orientational dependence of the Casimir force. Data regarding the orientational dependence has not yet been taken. The preliminary data has indicated the existence of an unknown background. Further investigation into this phenomenon is currently underway.
Appendix A

Reflection Matrices

This formalism requires explicitly defined local reflection coefficients, in order to obtain them for boundaries defined by in-plane optical anisotropy we will match the general solutions given in Eqs. 2.46 and 2.48 using Maxwell’s equations as the boundary conditions, which state that the tangential components of the electric and magnetic fields must be continuous across the boundary. Explicitly, the boundary conditions for Plate 1 can be written in the form \( \psi \cdot \mathbf{v}_i = 0 \) where,

\[
\psi = (\psi_{I,E}, \psi_{I,-M}, \psi_{RE}, \psi_{R,+M}, \psi_{To}, \psi_{Te}) \tag{A.1}
\]

here \( \psi_{i,j} \) is the amplitude of the \( i \)th wave (\( i \) standing for Incident, Reflected or Transmitted) with \( j \)th polarization (\( j \) standing for transverse Electric, transverse Magnetic, ordinary, or extraordinary) and \( \lambda_{jk} \) is the \( k \)th component (\( k \) standing for...
the $x$ or $y$ directions) of the $j$th polarization unit vector.

$$
\begin{align*}
\mathbf{v}_1 &= \begin{pmatrix} 
\lambda_{Ex} \\
\lambda_{-Mx} \\
\lambda_{Ex} \\
\lambda_{+Mx} \\
-\lambda_{ox} \\
-\lambda_{ex}
\end{pmatrix}, \\
\mathbf{v}_2 &= \begin{pmatrix} 
\lambda_{Ey} \\
\lambda_{-My} \\
\lambda_{Ey} \\
\lambda_{+My} \\
-\lambda_{oy} \\
-\lambda_{ey}
\end{pmatrix} \\
\mathbf{v}_3 &= \begin{pmatrix} 
(k \times \hat{\lambda}_E)_x \\
(k \times \hat{\lambda}_{-M})_x \\
(k' \times \hat{\lambda}_E)_x \\
(k' \times \hat{\lambda}_{+M})_x \\
-(k_{1o} \times \hat{\lambda}_o)_x \\
-(k_{1e} \times \hat{\lambda}_e)_x
\end{pmatrix}, \\
\mathbf{v}_4 &= \begin{pmatrix} 
(k \times \hat{\lambda}_E)_y \\
(k \times \hat{\lambda}_{-M})_y \\
(k' \times \hat{\lambda}_E)_y \\
(k' \times \hat{\lambda}_{+M})_y \\
-(k_{1o} \times \hat{\lambda}_o)_y \\
-(k_{1e} \times \hat{\lambda}_e)_y
\end{pmatrix}
\end{align*}
$$

(A.2) (A.3)

From these four equations, the amplitudes of the reflected waves can be written in terms of the incident waves, conveniently arranged in a $2 \times 2$ matrix whose elements are given explicitly below.
\[ r_{1EE} = \frac{1}{N_1} \left[ k^2_1 \sin^2[\phi] \rho_3 (\rho_3 - \rho_1) \{ A + \epsilon_{1\perp} \tilde{\rho}_1^2 \} + \rho_1^2 \rho_3 \rho_1 \{ A + \epsilon_{1\perp} \tilde{\rho}_1^2 \} \right. \\
\left. + \rho_1 \left\{ k^2_1 \epsilon_{1\perp} \sin^2[\phi] \rho_3 (\rho_3^2 - k_1^2) - A(k^2_1 \sin^2[\phi] + \cos[2\phi] \rho_3^2) \tilde{\rho}_1 \right\} \\
+ \epsilon_{1\perp} \rho_3 (k_1^2 - \rho_3^2) \tilde{\rho}_1^2 - \epsilon_{1\perp} (k_1^2 \sin^2[\phi] + \cos[2\phi] \rho_3^2) \tilde{\rho}_1^3 \right] \\
- \rho_1^2 \left\{ k^2_1 \cos^2[\phi] \left( k^2_1 (\epsilon_{1\parallel} - \epsilon_{1\perp}) \sin^2[\phi] + \epsilon_{1\perp} \tilde{\rho}_1^2 \right) \\
+ \rho_3^2 (k_1^2 \sin^2[\phi] \left( \epsilon_{1\perp} \cos[2\phi] - 2\epsilon_{1\parallel} \cos^2[\phi] \right) + \epsilon_{1\perp} \cos[2\phi] \tilde{\rho}_1^2) \right\} \right] \tag{A.4} \]

\[ r_{1EM} = \frac{2}{N_1} \cos[\phi] \sin[\phi] \rho_3 \sqrt{\rho_3^2 - k_1^2} \left[ k^2_1 (\epsilon_{1\parallel} - \epsilon_{1\perp}) \sin^2[\phi] (k_1^2 - \rho_1^2) \right. \\
+ A \rho_1 \rho_3 + \epsilon_{1\perp} (k_1^2 - \rho_1^2) \tilde{\rho}_1^2 + \epsilon_{1\perp} \rho_1 \tilde{\rho}_1^3 \right] \tag{A.5} \]

\[ r_{1ME} = \frac{2}{N_1} \cos[\phi] \sin[\phi] \rho_1 \rho_3 \sqrt{\rho_3^2 - k_1^2} \left( \rho_1 - \rho_1 \right) \{ A + \epsilon_{1\perp} \tilde{\rho}_1^2 \} \tag{A.6} \]

\[ r_{1MM} = \frac{1}{N_1} \left[ k^2_1 \sin^2[\phi] \left\{ -A \rho_3^2 + \epsilon_{1\perp} \rho_1 \rho_3 (\rho_3^2 - k_1^2) \right\} \\
+ \rho_1^2 \left( k^2_1 (\epsilon_{1\parallel} - \epsilon_{1\perp}) \cos^2[\phi] - (2\epsilon_{1\parallel} \cos^2[\phi] - \epsilon_{1\perp} \cos[2\phi]) \rho_3^2 \right) \right] \\
+ A \left\{ k^2_1 \sin^2[\phi] \rho_1 - (k^2_1 \sin^2[\phi] - \rho_1^2) \rho_3 + \cos[2\phi] \rho_1 \rho_3^2 \right\} \tilde{\rho}_1 \\
+ \epsilon_{1\perp} \left\{ -k_1^2 \sin^2[\phi] \rho_3^2 + \rho_1 \rho_3 \left( k_1^2 - \rho_3^2 \right) + \rho_1^2 (k_1^2 \cos^2[\phi] - \cos[2\phi] \rho_3^2) \right\} \tilde{\rho}_1^2 \\
+ \epsilon_{1\perp} \left\{ k_1^2 \sin^2[\phi] \rho_1 + (\rho_1^2 - k_1^2 \sin^2[\phi]) \rho_3 + \cos[2\phi] \rho_1 \rho_3^2 \right\} \tilde{\rho}_1^3 \right] \tag{A.7} \]
where,

\[ A = -k_\perp^2 \left( \epsilon_{1\parallel} - (\epsilon_{1\parallel} - \epsilon_{1\perp}) \sin^2[\phi] \right) \]

\[ N_1 = (\rho_1 + \rho_3) \left[ k_\perp^2 \sin^2[\phi] (\rho_3 + \tilde{\rho}_1) \{A + \epsilon_{1\perp} \tilde{\rho}_1^2\} + \rho_1 \left[ \epsilon_{1\perp} \rho_3^2 \left\{ k_\perp^2 \sin^2[\phi] - \rho_1^2 \right\} \right. \right. \\
\left. \left. - \rho_3 \rho_1 \{A + \epsilon_{1\perp} \tilde{\rho}_1^2\} + k_\perp^2 \cos^2[\phi] \left\{ k_\perp^2 (\epsilon_{1\parallel} - \epsilon_{1\perp}) \sin^2[\phi] + \epsilon_{1\perp} \tilde{\rho}_1^2 \right\} \right]\right] \]

Here \( r_{1ij} \) represents the amplitude of the reflected wave with \( j \) polarization in terms of an incident wave of \( i \) polarization at the surface of Plate 1. The reflection coefficients for Plate 2 can be obtained by the simple replacement of \( \phi \rightarrow \phi + \chi, \rho_1 \rightarrow \rho_2, \tilde{\rho}_1 \rightarrow \tilde{\rho}_2, \epsilon_{1\parallel} \rightarrow \epsilon_{2\parallel}, \epsilon_{1\perp} \rightarrow \epsilon_{2\perp} \).
Appendix B

Tabulated Data

Below is table of the data used in section 3.8.

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Table B.2: Tabulated Data cont.
Appendix C

Fundamental Forces

In this appendix we present preliminary work into extending the formalism presented in this dissertation to fundamental interactions. The idea being that a pair of charged particles, having non-trivial interaction with the vacuum, experience a force deriving from the existence of a non-trivial vacuum state of an associated field. Additionally, we challenge, or investigate, the ideas behind $\vec{F} = q\vec{E}$. In this appendix we present the calculated force between a pair of electrons resulting from the electromagnetic vacuum and a pair of point masses resulting from the graviton vacuum. In doing so, we conclude that the coloumb and newton interactions can be understood as the interaction of charges, acting as boundary conditions, on their associated fields.

C.1 Coulomb

In this section we would like to generalize our approach to other interactions. We will start by considering the interaction between two charged particles separated by a
distance $a$. These charged particles will be immersed in the electromagnetic vacuum at zero temperature. We wish to know what effect the vacuum has on these particles. We shall imagine that the particles define the side-walls of a cavity and once again investigate the energy contained within the eigenmodes of the cavity. Each mode will have $\frac{1}{2}\hbar\omega$ worth of energy. Thus we can start with Eq. 10 from the previous paper (38).

\[ E = \frac{\hbar c}{4\pi} \sum_{\text{paths}} \int_{-\infty}^{\infty} \ln(\text{det}(I - \Gamma(x_2, x_1)R_1\Gamma(x_1, x_2)R_2))d\xi_c \]  

(C.1)

We will once again replace the transport functions by $Ie^{-a\sqrt{k^2+k^2_\perp}}$. We can replace the reflection matrices by $I$, since the interaction is elastic. Thus,

\[ E = \frac{\hbar c}{2\pi^2} \int d\vec{r} \int_{0}^{\infty} k_\perp dk_\perp \int_{0}^{\infty} \ln(1 - e^{-2a\sqrt{k^2+k^2_\perp}})d\xi_c \]  

(C.2)

At this point we need to state what is meant by the area integration $\int d\vec{r}$. Assuming the particles are point-like particles we can replace the area integration with the
interaction cross-section.

\[
E = \frac{\hbar c}{2\pi^2} \int_0^{\infty} k_{\perp} dk_{\perp} \int_0^{\infty} \frac{q_1 q_2}{\xi_c^2 + k_{\perp}^2} \ln(1 - e^{-2a\sqrt{\xi_c^2 + k_{\perp}^2}}) d\xi_c \tag{C.3}
\]

\[
= -\frac{\hbar c q_1 q_2}{2\pi^2} \int^{\pi/2}_{0} \int_0^{\infty} \frac{1}{p^2} \sum_{n=1}^{\infty} \frac{e^{-2anp}}{n} p^2 \sin \theta dp d\theta \tag{C.4}
\]

\[
= -\frac{\hbar c q_1 q_2}{2\pi^2} \sum_{n=1}^{\infty} \int_0^{\infty} e^{-2np} \frac{1}{n} dp \tag{C.5}
\]

\[
= -\frac{\hbar c q_1 q_2}{4\pi^2 a} \sum_{n=1}^{\infty} \frac{1}{n^2} \tag{C.6}
\]

\[
= -\frac{\hbar c q_1 q_2}{24a} \tag{C.7}
\]

where \((\tilde{q}_1, \tilde{q}_2)\) are the unitless charges associated with particles 1 and 2 respectively.

If we then identify those charges with the standard electric charges by substituting

\[
(\tilde{q}_1, \tilde{q}_2) = i\sqrt{\frac{24}{4\pi\epsilon_0\hbar c}} (q_1, q_2) = \frac{i\sqrt{24}}{q_p} (q_1, q_2)
\]

we recover potential energy of the Coulomb interaction,

\[
E = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{a} \tag{C.9}
\]

where \(q_p\) is the Planck charge. Let us now turn our attention to another interaction namely gravitation. The procedure is identical to the above with the replacement

\[
(\tilde{q}_1, \tilde{q}_2) = \sqrt{\frac{24G}{hc}} (m_1, m_2) = \frac{\sqrt{24}}{m_p} (m_1, m_2).
\]

Yielding,

\[
E = -G \frac{m_1 m_2}{a} \tag{C.10}
\]
C.2 Finite Temperature Effects Revisited

The above calculation was performed assuming the vacuum states were at zero temperature. However we know that temperature has a significant effect on these calculations. Let us recalculate the gravitational force at finite temperature. Starting with Eq. 1.76:

\[ E = \frac{k_B T}{\pi} \sum_{l=0}^{\infty} \int_0^\infty k_\perp \, dk_\perp \, \frac{\tilde{q}_1 \tilde{q}_2}{\xi^2_l + k_\perp^2} \ln(1 - e^{-2a \sqrt{\xi^2_l + k_\perp^2}}) \]  \hspace{1cm} (C.11)

\[ = -\frac{k_B T}{\pi} \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \int_0^\infty k_\perp \, dk_\perp \, \frac{\tilde{q}_1 \tilde{q}_2}{\xi^2_l + k_\perp^2} \frac{e^{-2an \sqrt{\xi^2_l + k_\perp^2}}}{n} \]  \hspace{1cm} (C.12)

The above integral is diverges as \( k_\perp \to 0 \). To remedy this we will introduce a low frequency cut-off representing a diffraction limit.

\[ E = -\frac{k_B T}{\pi} \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \int_b^\infty k_\perp \, dk_\perp \, \frac{\tilde{q}_1 \tilde{q}_2}{\xi^2_l + k_\perp^2} \frac{e^{-2an \sqrt{\xi^2_l + k_\perp^2}}}{n} \hspace{1cm} b = \frac{1}{2a} e^{-\frac{b^2}{8a^2}} \]  \hspace{1cm} (C.13)

Far into the finite temperature regime \((a >> \lambda_T)\) where \( \lambda_T \) is the thermal wavelength) we only need to take the \( l = 0 \) term in the sum and \( b \to 1/2a \).

\[ E = -\frac{k_B T}{2\pi} \sum_{n=1}^{\infty} \int_b^\infty dk_\perp \, \frac{\tilde{q}_1 \tilde{q}_2}{k_\perp} \frac{e^{-2an k_\perp}}{n} \]  \hspace{1cm} (C.14)

\[ E = -\frac{k_B T \tilde{q}_1 \tilde{q}_2}{2\pi} \sum_{n=1}^{\infty} \frac{\Gamma(0, 2abn)}{n} \]  \hspace{1cm} (C.15)
C.2 Finite Temperature Effects Revisited

The sum in the above expression is not defined to my knowledge. However we can calculate the force instead.

\[
F = -\frac{\partial}{\partial a} E = -\frac{k_B T q_1 q_2}{2a} \sum_{n=1}^{\infty} e^{-2a b n} \qquad (C.16)
\]

\[
= -\frac{k_B T q_1 q_2}{a} \frac{\ln \left(\frac{e}{e^{-1}}\right)}{2\pi} \qquad (C.17)
\]

\[
= -G \frac{m_1 m_2}{a} 12k_B T \frac{\ln \left(\frac{e}{e^{-1}}\right)}{\hbar c \pi} \qquad (C.18)
\]

It is of interest to ask what temperature we are referring to in this section. The temperature referred to in this section is the temperature of a thermal graviton field. As the interaction of matter and gravity is so weak, we conclude that the graviton field is not in thermal equilibrium with the matter. Rather, the best representation of the temperature of the graviton field is given as the Unruh temperature of the universe:

\[
T = \frac{\hbar \tilde{a}}{2\pi ck_B} \qquad (C.19)
\]

where the \( \tilde{a} \) is the acceleration. We shall take the acceleration to be the gravitational acceleration associated with the mass of the universe:

\[
a = G \frac{M_U}{R_U^2} \qquad (C.20)
\]

where \( M_U \approx 10^{60} \) kg is the mass of the universe and \( R_U \approx 10^{26} \) is the radius of the observable universe. This gives a temperature of \( T \approx 10^{-22} \) K. Therefore, the transition from the zero temperature regime to the finite temperature regime should
occur at $\lambda_T \cong 5 \text{ kpc}$. This leads naturally to the idea that this mechanism might explain, at least in part, the observed galactic rotation data. Further investigation and rigor is needed before such a conclusion is warranted.

In this appendix we have shown how this approach might be extended to fundamental interactions. At present these calculations are preliminary and further investigation into the legitimacy of the conclusions is underway.
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