Ideal Optimum Performance of Propellers, Lifting Rotors and Wind Turbines

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Ideal Optimum Performance of Propellers, Lifting Rotors
and Wind Turbines

by
Ramin Modarres

A thesis presented to the School of Engineering
of Washington University in partial fulfillment of the
requirements for the degree of
Master of Science

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Dedicated to my parents.
Chapter 1

Introduction

With the present focus on sustainable energy systems and on green aviation, it is interesting to look at the optimum efficiency of propellers, wind turbines, and helicopters. Although much work has been done on finding the optimum configurations for each of these devices, the work has left room for some very interesting and useful developments in terms both of efficient numerical solutions and compact closed-form solutions for the optimum rotor. Furthermore, since all three of these devices are governed by a unifying principle of combined potential flow and momentum theory with swirl, the formulation of all three problems can be unified in terms of application approach. This Master of Science Thesis is a compendium of three published papers—one on propellers, one on wind turbines, and one on helicopters—that apply these unifying principles to optimum rotor design.

The second chapter presents an efficient solution of Goldstein’s equations for propellers with application to rotor induced power efficiency. Betz and Prandtl (1919) presented the optimum velocity distribution for a rotor in axial flow having an infinite number of blades. Goldstein (1929) derived an expression for the circulation that would give the ideal inflow of Betz-Prandtl. Goldstein offered an elegant, numerical solution to this equation in order to find the optimum circulation to give Betz induced flow. He presented solutions for two blades at a number of inflow ratios and for four blades at one particular inflow ratio. The objective of this work is to develop a more computationally accurate and robust method of finding the optimum circulation for the ideal propeller. We look for a solution that would be taken to any desired accuracy and applied for any number of blades and any tip-speed ratio. With such a solution, one can have benchmarks against which to compare other methodologies. In addition, an accurate solution will allow computation of induced power efficiency for the Goldstein optimum such that other blade designs can be measured against it. This work was presented at the 38th European Rotorcraft Forum, Amsterdam, Netherlands, September 4-7, 2012.

The third chapter derives a compact, closed form solution for the optimum, ideal wind turbine. The classical momentum solution for the optimum induced-flow distribution of a wind turbine in the presence of wake swirl can be found in many textbooks. This standard derivation consists of two momentum balances (one for axial momentum and one for angular momentum) which are combined into a formula for power coefficient in terms of induction factors. Numerical procedures then give the proper induction factors for the optimum inflow distribution at any radial station; and this, in turn, gives the best possible power coefficient for an ideal wind turbine. The present development offers a more straightforward derivation of the optimum turbine. The final formulas give the identical conditions for the ideal wind turbine as do the classical solutions—but with several important differences in the derivation and in the form of the results. First, only one momentum balance is required (the other being redundant). Second, the solution is provided in a compact, closed form for both the induction
factors and the minimum power—rather than in terms of a numerical process. Third, the solution eliminates the singularities that are present in current published solutions. Fourth, this new approach also makes possible a closed-form solution for the optimum chord distribution in the presence of wake rotation. This work was published in the Wind Energy Journal, Feb. 6, 2013, DOI: 10.1002/we.1592.

The fourth chapter deals with the optimum performance of an actuator disk by a compact momentum theory including swirl. In this work a new compact form of momentum theory is introduced for actuator disks including swirl. The new form unifies both the axial and angular momentum balances into a single momentum equation, applicable over the entire range of thrust and power coefficients. While completely consistent with earlier momentum theories the compact form allows analytic expressions for the parameters of an optimum actuator disk and reveals additional insight into the limiting efficiency of rotors, propellers, and wind turbines. Closed-form results presented here include the optimum values of: induced flow, inflow angle, thrust, induced power, and efficiency. Closed-form expressions are also given for optimum twist, chord distribution, and solidity in the presence of profile drag (along with the resulting over-all efficiencies). For the limiting case of the optimum rotor in hover, the compact form leads to closed-form expressions for both contraction ratio and pressure distribution in the far wake. This report also gives a formal proof that the Betz inflow distribution results in the maximum figure of merit, and it further demonstrates that some approximations used in earlier actuator-disk momentum theories have been inconsistent. This work was presented at the AHS 69th Annual Forum, Phoenix, Arizona, May 21–23, 2013.

The numerical and closed-form expressions in these three papers accomplish a number of important things. First, they give insight into the nature and performance of the respective rotor systems. Second, they provide benchmark solutions against which numerical codes can be compared. Third, they provide elegant solutions that correct incorrect or inefficient derivations presently found in the literature.

Because the standard notations for propellers, wind turbines, and helicopters have each been historically distinct, each paper maintains its own list of symbols and reference lists. This should make the reading of the thesis more convenient.
Chapter 2

EFFICIENT SOLUTION OF GOLDSTEIN’S EQUATIONS FOR PROPELLERS WITH APPLICATION TO ROTOR INDUCED-POWER EFFICIENCY

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2.1. Abstract

Betz and Prandtl (1919) presented the optimum velocity distribution for a rotor in axial flow having an infinite number of blades. Goldstein (1929) derived an expression for the circulation that would give the ideal inflow of Betz-Prandtl. Goldstein offered an elegant, numerical solution to this equation in order to find the optimum circulation to give Betz induced flow. He presented solutions for two blades at a number of inflow ratios and for four blades at one particular inflow ratio. The objective of this work is to develop a more computationally accurate and robust method of finding the optimum circulation for the ideal propeller. We look for a solution that would be taken to any desired accuracy and applied for any number of blades and any tip-speed ratio. With such a solution, one can have benchmarks against which to compare other methodologies. In addition, an accurate solution will allow computation of induced power efficiency for the Goldstein optimum such that other blade designs can be measured against it.

Presented at the 38th European Rotorcraft Forum, Amsterdam, Netherlands, September 4-7, 2012.
2.2. **Nomenclature**

- $a_k$: constant part of $h'_{k}$
- $[A]$: Goldstein derivative matrix
- $b_k$: part of $h'_{k}$ due to $h_k$
- $\{B\}$: forcing factor of particular solution
- $\{c\}$: coefficients of particular solution
- $\{d\}$: coefficients of homogeneous solution
- $\{D\}$: forcing function of homogeneous solution
- $e$: Prandtl tip-correction function
- $[E]$: Galerkin stiffness matrix
- $f_k(\mu)$: velocity potential expansions
- $g_k(\mu)$: homogeneous part of $h_k$
- $[G]$: matrix of boundary conditions
- $h_k(\mu)$: correction functions
- $i$: summation index
- $I$: modified Bessel function
- $IPE$: induced power efficiency
- $j$: summation index
- $k$: harmonic number, $k = Q/2, 3Q/2, 5Q/2, \ldots$
- $K(y)$: modified Bessel function of $y$
- $K'(y)$: derivative of $K$ with respect to $y$ ($dK/dy$)
- $l$: harmonic number, $l = Q, 2Q, 3Q, \ldots$
- $\{M\}$: forcing function for boundary
- $n$: number of $k$ terms
- $N$: number of the terms in Galerkin function
- $P_j(x)$: Legendre polynomials
- $Q$: number of blades
\( r \) radial coordinate, \( m \)
\( R \) blade radius, \( m \)
\( \text{trig()} \) either \( \sin() \) or \( \cos() \)
\( v(r) \) induced velocity at disk normal to vortex sheet, \( m/sec \)
\( V \) climb rate, \( m/sec \)
\( x \) mapping coordinate, \( 2(\mu/\mu_0) - 1, \ -1 < x < 1 \)
\( z \) axial coordinate, \( m \)
\( \gamma(\mu) \) nondimensional normalized circulation, \( \Gamma \Omega/V^2 \)
\( \gamma_b \) nondimensional circulation per blade, \( \Gamma_b \Omega/V^2 \)
\( \tilde{\gamma} \) corrected circulation
\( \hat{\gamma} \) Galerkin optimum circulation
\( \Gamma(r) \) normalized nominal circulation, uncorrected, \( m^2/sec \)
\( \Gamma_b(r) \) total circulation per blade, \( m^2/sec \)
\( \kappa \) correction factor
\( \zeta \) nondimensional screw coordinate, \( \theta - \Omega z/V \)
\( \theta \) angle of screw surface, \( rad \)
\( \lambda(\mu) \) nondimensional induced velocity downstream, \( 2v(\mu) \)
\( \mu \) radial coordinate, \( \Omega r/V = \cot(\phi) \)
\( \mu_0 \) value of \( \mu \) at blade tip, \( \Omega R/V \)
\( v(\mu) \) nondimensional induced velocity at disk, \( v(r)/V \)
\( \phi \) inflow angle, \( \arctan(1/\mu) \)
\( \Phi(\mu, \zeta) \) velocity potential, normalized on \( V^2/\Omega \)
\( \Phi_j \) admissible functions of either \( \mu \) or \( x \)
\( \Omega \) rotor speed, \( rad/sec \)
2.3. Introduction

Betz and Prandtl, Ref. [1], found the optimum velocity distribution (i.e., for minimum power) for a rotor in axial flow. Although they were unable to find an exact solution for the circulation distribution that would result in such a velocity distribution, they were able to find this optimum distribution for a rotor with an infinite number of blades and offered an approximate tip correction that would account for the effect of blade number. Although the Prandtl correction factor is based on a two-dimensional inflow model, it is quite accurate and is used extensively in rotorcraft analysis to account for blade number.

It fell to Goldstein, Ref. [2], to find the exact solution for the optimal circulation on a propeller with a finite number of blades. He treated both two-bladed and four-bladed rotors at various inflow angles. The results agree nicely with computations based on Prandtl’s equation, as shown in Fig. (1)—taken from Ref. [2]—where the condition chosen is $\mu_0 = 5.0$, so the $\lambda = 0.2$ which is a fairly high climb rate.

For four blades, the two solutions are quite close, although there is a small discrepancy that occurs near the root of the blade. For the two-bladed rotor, this discrepancy is more pronounced. The behavior of the Prandtl approximation at small $\mu$ is nearly identical for all $Q$, but the Goldstein solution increasingly differs from the Prandtl solution (at small $\mu$) as $Q$ decreases.

The objective of this work is to develop a more computationally accurate and robust method of finding the optimum circulation for the ideal propeller. We look for a solution that would be taken to any desired accuracy and applied for any number of blades and any tip-speed ratio.

For formulating the problem, we will follow the general outline of Ref. [2] but proceed along what we believe is a more direct and compact approach. To begin, note that the pressure and velocity around a propeller in axial flow are governed by the following velocity potential:

$$\Phi(\mu, \zeta) = \left[\frac{\pi}{Q} - \zeta\right] \gamma(\mu) + \sum f_k(\mu) \text{trig}(k\zeta), \quad 0 < \zeta < \frac{2\pi}{Q}$$

(1)

where $\gamma(\mu)$ is some nominal circulation, $\mu$ is the nondimensional radial co-ordinate, and $Q$ is the number of blades. (Note that $\mu$ is also the cotangent of the inflow angle $\phi$.)

The first term in Eq. (1) is the nominal velocity potential that is chosen to give a nondimensional velocity distribution $\lambda(\mu) = \gamma(\mu)/\cos(\phi)$. The second part of Eq. (1), involving $f_k(\mu)$, is a correction term. (The summation is taken over appropriate $k$’s as will be defined later.)

The total velocity potential (nominal plus correction) must satisfy Laplace’s equation in helical coordinates.

$$\left(\frac{\mu \partial}{\partial \mu}\right)^2 \Phi + (1 + \mu^2) \frac{\partial^2 \Phi}{\partial \zeta^2} = 0$$

(2)

This implies that the correction functions $f_k(\mu)$ are related to a set of basic functions $h_k(\mu)$—that are governed by a differential equation that follows from Eqs. (1) and (2)—namely:
The resultant circulation per blade \( \gamma_b(\mu) \) and the resultant velocity distribution in the wake \( \lambda(\mu) \) can be given in terms of the total velocity potential:

\[
\gamma_b(\mu) = \Phi(\mu, 0) - \Phi(\mu, 2\pi/Q)
\]

\[
\lambda(\mu) = -\frac{1 + \mu^2}{\mu^2} \frac{\partial \Phi}{\partial \xi} \bigg|_{\xi=0}
\]

Therefore, once \( h_k(\mu) \) and \( f_k(\mu) \) are determined, the circulation and velocity can be found; and that is the focus of what is to follow.

Now, consider the case in which we are given the velocity distribution \( \lambda(\mu) \), and want to find the applied circulation \( \gamma_b(\mu) \) that would produce it. In order to preserve the desired velocity \( \lambda(\mu) \) from Eq. (4), we take \( \text{trig}(k\xi) = \cos(k\xi) \) in the summation of Eq. (1) with \( k = Q/2, 3Q/2, 5Q/2, \ldots \). This ensures that the derivative of \( \Phi \) will be zero at the boundary. It is then convenient to expand \( \frac{1 + \mu^2}{\mu^2} \frac{\partial \Phi}{\partial \xi} \bigg|_{\xi=0} \) in cosine terms summed over these same \( k \)'s.

\[
[\pi/Q - \zeta] = \sum_{k=\frac{Q}{2}, \frac{3Q}{2}, \frac{5Q}{2}, \ldots} \frac{2Q}{\pi k^2} \cos(k\zeta)
\]

The nominal circulation to obtain the desired velocity is:

\[
\gamma(\mu) = \frac{\mu}{\sqrt{1 + \mu^2}} \lambda(\mu)
\]

and this becomes the forcing function for the correction terms in Eq. (3). When Eq. (1) and Eq. (5) are placed into Eq. (2), it is clear that one must define the relation \( f_k(\mu) \equiv 2Qh_k(\mu)/(\pi k^2) \) in order to obtain the standard form of Eq. (3).

It follows that the total circulation distribution per blade is given from Eq. (4) as:

\[
\gamma_b(\mu) = \frac{2\pi}{Q} \left[ \gamma(\mu) + \sum_{k=\frac{Q}{2}, \frac{3Q}{2}, \frac{5Q}{2}, \ldots} \frac{2Q^2}{\pi^2 k^2} h_k(\mu) \right]
\]

For the above summation over \( k \) in Eq. (7), \( \pi k/Q = \pi/2, 3\pi/2, 5\pi/2, \ldots \). Notice that the circulation is increased due to positive \( h_k(\mu) \).
2.4. Numerical Computation

Now we need to formulate a numerical solution to the correction function. The general solution to Eq. (3) would be the sum of the particular solution and the homogenous solution.

To find the particular solution with boundary conditions \( h_k(0) = h_k(\mu_0) = 0 \), we take Eq. (3) and expand the unknown \( h_k(\mu) \) in a Galerkin series of admissible functions \( c_j \Phi_j(\mu) \).

\[
h_k(\mu) = \sum_{i=1,2,3,...}^N c_j \Phi_j(\mu) \tag{8}
\]

We then use a change of variable to map the domain onto \(-1 < x < 1\)

\[
x = 2 \frac{\mu}{\mu_0} - 1 \quad -1 < x < 1 \tag{9}
\]

This change of variable allows admissible and comparison functions to be chosen on a more convenient interval, \(-1 < x < 1\).

For test functions, we chose the combination of Legendre polynomials that have been applied to the p-version finite element method, Ref. [5],

\[
\Phi_j = \frac{P_{j+1}(x) - P_{j-1}(x)}{\sqrt{2(2j-1)}}
\]

Substituting Eq.(8) into Eq.(3), multiplying by the comparison functions \( \Phi_i(\mu)/\mu \) and integrating from zero to \( \mu_0 \), one obtains:

\[
\left[ \int_0^{\mu_0} \frac{1}{\mu} \Phi_i \left( \frac{d}{d\mu} \right) \left( \frac{d}{d\mu} \right) \Phi_j d\mu - k^2 \int_0^{\mu_0} \frac{1}{\mu} \Phi_i \Phi_j (1 + \mu^2) d\mu \right] \{ c_j \} = - \int_0^{\mu_0} \frac{1}{\mu} \Phi_i \left( \frac{d}{d\mu} \right)^2 \gamma(\mu) d\mu \tag{11}
\]

The first integral in Eq. (11) can be written in the form:

\[
\int_0^{\mu_0} \frac{1}{\mu} \Phi_i \left( \frac{d}{d\mu} \right) \left( \frac{d}{d\mu} \right) \Phi_j d\mu = \int_0^{\mu_0} \Phi_i \frac{d}{d\mu} \left( \frac{d}{d\mu} \right) \Phi_j d\mu = \int_0^{\mu_0} \Phi_i \left[ \frac{d\Phi_j}{d\mu} + \frac{\mu}{d\mu} \frac{d^2\Phi_j}{d\mu^2} \right] d\mu
\]

\[
= - \int_0^{\mu_0} \mu \frac{d}{d\mu} \frac{d\Phi_j}{d\mu} d\mu + \Phi_i \frac{d\Phi_j}{d\mu} \bigg|_0^{\mu_0} = - \int_0^{\mu_0} \mu \frac{d^2\Phi_j}{d\mu^2} d\mu \tag{12}
\]
Similarly the integral on the right hand side of Eq. (11), can be written in the form:

\[
\int_0^{\mu_0} \frac{1}{\mu} \Phi_i \left( \frac{d}{d\mu} \right)^2 \gamma(\mu) d\mu = - \int_0^{\mu_0} \mu \frac{d\Phi_i}{d\mu} \frac{dy}{d\mu} d\mu
\]  

(13)

As a result, Eq. (11) can be rewritten in the form:

\[
\int_0^{\mu_0} \mu \frac{d\Phi_i}{d\mu} \frac{d\Phi_j}{d\mu} d\mu + k^2 \int_0^{\mu_0} \frac{1 + \mu^2}{\mu} \Phi_i \Phi_j d\mu \left\{ c_j^k \right\} = - \int_0^{\mu_0} \mu \frac{d\Phi_i}{d\mu} \frac{dy}{d\mu} d\mu
\]  

(14)

Taking:

\[
[A_{ij}] = \int_0^{\mu_0} \mu \frac{d\Phi_i}{d\mu} \frac{d\Phi_j}{d\mu} d\mu
\]

\[
[E_{ij}] = \int_0^{\mu_0} \frac{1 + \mu^2}{\mu} \Phi_i \Phi_j d\mu
\]

\[
\{ B_i \} = - \int_0^{\mu_0} \mu \frac{d\Phi_i}{d\mu} \frac{dy}{d\mu} d\mu
\]

one can write Eq. (14) in the form:

\[
[A + k^2 E] \{ c_j^k \} = \{ B \}
\]  

(16)

and:

\[
\{ c_j^k \} = [A + k^2 E]^{-1} \{ B \}
\]  

(17)

The procedure for finding the homogeneous solution is similar to that used for finding the particular solution. For the homogenous case, the boundary conditions are \( h_k(0) = 0 \) and \( h_k(\mu_0) \neq 0 \), and the differential equation that follows from Eq. (3) is:

\[
\left( \frac{\mu d^2}{d\mu} \right)^2 \left[ h_k(\mu) \right] - k^2 (1 + \mu^2) h_k(\mu) = 0
\]  

(18)

For the homogenous boundary conditions we use the change of variable:
So, Eq. (18), can be written in the form:

\[
\left(\mu \frac{d}{d\mu}\right)^2 g_k(\mu) - k^2 (1 + \mu^2) g_k(\mu) = [k^2 (1 + \mu^2) - 1]\mu \tag{20}
\]

We take Eq. (20) and expand \(g_k(\mu)\) in a Galerkin series of admissible functions \(d_i \Phi_i(\mu)\).

\[
g_k(\mu) = \sum_{j=1,2,3,...}^N d_i^k \Phi_i(\mu) \tag{21}
\]

Once again, we map the domain onto \(-1 < x < 1\), and choose a combination of Legendre polynomials for our test functions:

\[
\Phi_i = \frac{P_{i+1}(x) - P_{i-1}(xf)}{\sqrt{2(2i - 1)}} \tag{22}
\]

Substituting Eq.(21) into Eq.(20), multiplying by the comparison functions \(\Phi_j(\mu)/\mu\), and integrating from zero to \(\mu_0\), yields:

\[
[A + k^2 E]\{d_i^k\} = -\int_0^{\mu_0} \Phi_j \{[k^2 (1 + \mu^2) - 1]\} d\mu \tag{23}
\]

Taking:

\[
D_j = -\int_0^{\mu_0} \Phi_j \{[k^2 (1 + \mu^2) - 1]\} d\mu \tag{24}
\]

Eq. (23) becomes:

\[
[A + k^2 E]\{d_i^k\} = \{D\} \tag{25}
\]

and:

\[
\{d_i^k\} = [A + k^2 E]^{-1}\{D\} \tag{26}
\]

The total solution for \(h\) is then the sum of the particular solution and the homogenous solution.
\[ h_k = \sum_{j=1}^{N} c_j^k \Phi_j + \left[ \sum_{i=1}^{N} d_i^k \Phi_i + \mu \right] \frac{h_k(\mu_0)}{\mu_0} \]  

(27)

With the above, we can find the solution to the potential problem. It is not difficult to see that the conditions of the problem are such that \( \Phi \) is an odd function of \( \zeta \) (or \( \frac{1}{2} \pi - \zeta \)). Furthermore, \( \Phi \) is a single-valued function of position, continuous for \( r \geq R (\mu \geq \mu_0) \). Therefore, it can be expanded, for \( r \geq R \), in a series of sines of integer multiples of \( \zeta \). Taking this expansion, differentiating term by term, and then substituting in Eq. (2), we find that the coefficients of \( \sin(l \zeta) \) must be a linear functions of \( I_l(I \mu) \) and \( K_l(I \mu) \), where \( I_l \) and \( K_l \) are the modified Bessel functions.

But \( I_l(I \mu) \) cannot occur, since \( \text{grad } \Phi \) must vanish when \( r \), or \( \mu \), is infinite. Hence we may assume:

\[ \Phi_0(\zeta, \mu) = \sum_{l=0,2Q,3Q,...} \frac{K_l(I \mu_0)}{K_l(I \mu_0)} c_l^0 \sin(l \zeta) \]  

(28)

For \( 0 \leq r \leq R (0 \leq \mu \leq \mu_0) \), the velocity potential can be obtained from Eq. (1). Since the velocity potential is a continuous function at \( r = R (\mu = \mu_0) \), it should satisfy the continuity conditions:

\[ \Phi_0(\mu_0) = \Phi(\mu_0), \quad \Phi'_0(\mu_0) = \Phi'(\mu_0) \]  

(29)

where the (') sign implies the derivative with respect to \( \mu \).

According to first continuity condition, \( \Phi \) from Eq. (28) equals \( \Phi \) from Eq. (1) at \( \mu = \mu_0 \)

\[ \sum_{l=0,2Q,3Q,...} c_l K_l(I \mu_0) \sin(l \zeta) = \left[ \frac{\pi}{Q} - \zeta \right] y(\mu_0) + \sum_{k=\frac{3Q}{2},\frac{5Q}{2},...} f_k \cos(k \zeta) \]  

(30)

Equation (5) and \( f_k(\mu) \equiv 2Q h_k(\mu)/(\pi k^2) \) can be substituted into Eq. (30). Expanding \( \cos(k \zeta) \) in sine terms:

\[ \cos(k \zeta) = \frac{4}{\pi} \sum_{l=0,2Q,3Q,...} \frac{l}{l^2 - k^2} \sin(l \zeta) \]  

(31)

One can rewrite Eq. (30) in the form:

\[ \sum_{l=0,2Q,3Q,...} c_l^0 K_l(I \mu_0) \frac{K_l(I \mu_0)}{K_l(I \mu_0)} \sin(l \zeta) \]

\[ = \sum_{k=\frac{3Q}{2},\frac{5Q}{2},...} \frac{2Q}{\pi k^2} \left[ y(\mu_0) + h_k(\mu_0) \right] \left( \frac{4}{\pi} \right) \left( \frac{l}{l^2 - k^2} \right) \sin(l \zeta) \]  

(32)
and, as a result:

\[ c^0_i = \sum_{k=\frac{3Q}{2}}^{\frac{5Q}{2}} \frac{2Q}{\pi k^2} [\gamma(\mu_0) + h_k(\mu_0)] \left( \frac{4}{\pi} \right) \left( \frac{l}{l^2 - k^2} \right) \]  

(33)

The second continuity condition implies that:

\[ l c^0_i \frac{K'(l\mu_0)}{K_0(l\mu_0)} = \sum_{k=\frac{3Q}{2}}^{\frac{5Q}{2}} \frac{2Q}{\pi k^2} [\gamma(\mu_0) + h'_k(\mu_0)] \left( \frac{4}{\pi} \right) \left( \frac{l}{l^2 - k^2} \right) \]  

(34)

Substituting Eq. (33) in Eq. (34) and simplifying, we obtain:

\[ \sum_{k=\frac{3Q}{2}}^{\frac{5Q}{2}} \frac{1}{k^2(l^2 - k^2)} \left\{ lK'_i(l\mu_0)[\gamma(\mu_0) + h_k(\mu_0)] - K_i(l\mu_0)[\gamma'(\mu_0) + h'_k(\mu_0)] \right\} = 0 \]  

(35)

On the other hand, from Eq. (27), one obtains an expression for \( h'_k \):

\[ h'_k(\mu) = \sum_{j=1}^{N} c_j^k \left\{ 2 \frac{d\Phi_j}{\mu_0 dx} \right\} + \left[ 1 + \sum_{i=1}^{N} d_i^k \left\{ 2 \frac{d\Phi_i}{\mu_0 dx} \right\} \left( \frac{h_k(\mu_0)}{\mu_0} \right) \right] \]  

(36)

and at \( \mu = \mu_0 \):

\[ h'_k(\mu_0) = \sum_{j=1}^{N} c_j^k \left\{ 2 \frac{d\Phi_j}{\mu_0 dx} \right\} \bigg|_{\mu_0} + \left[ 1 + \sum_{i=1}^{N} d_i^k \left\{ 2 \frac{d\Phi_i}{\mu_0 dx} \right\} \bigg|_{\mu_0} \left( \frac{h_k(\mu_0)}{\mu_0} \right) \right] \]  

(37)

Taking:

\[ a_k = \sum_{j=1}^{N} c_j^k \left\{ 2 \frac{d\Phi_j}{\mu_0 dx} \right\} \bigg|_{\mu_0} \]  

(38)

\[ b_k = \left[ 1 + \sum_{i=1}^{N} d_i^k \left\{ 2 \frac{d\Phi_i}{\mu_0 dx} \right\} \bigg|_{\mu_0} \left( \frac{1}{\mu_0} \right) \right] \]  

(39)
one can rewrite Eq. (37) in the form:

\[ h'(\mu_0) = a_k + b_k h_k(\mu_0) \]  

(40)

Substituting Eq. (40) in Eq. (35), and dividing the whole equation by \( K \) (to make the matrices better conditioned) one obtains:

\[
\sum_{k=\frac{Q_0^q 5Q}{2^q 2^{q-1}2^{\cdots}}} 1 \frac{1}{k^2(l^2 - k^2)} \left\{ lK'_{l}(l\mu_0) \left[ y(\mu_0)h(\mu_0) \right] \right\} = -K_{l}(l\mu_0) \left[ y'(\mu_0) + a_k + b_k h_k(\mu_0) \right] = 0 \quad (41)
\]

Taking:

\[
[G_{lk}] = lK'_{l}(l\mu_0) - b_k \quad (42)
\]

\[
\{M_l\} = \left[ \sum_{k=\frac{Q_0^q 5Q}{2^q 2^{q-1}2^{\cdots}}} 1 \frac{1}{k^2(l^2 - k^2)} \left[ y'(\mu_0) - lK'_{l}(l\mu_0) \frac{y(\mu_0)}{K(l\mu_0)} \right] \right] + \sum_{k=\frac{Q_0^q 5Q}{2^q 2^{q-1}2^{\cdots}}} \frac{a_k}{k^2(l^2 - k^2)} \quad (43)
\]

one can write Eq. (41) in the form:

\[
[G_{lk}]{h_k(\mu_0)} = \{M_l\} \quad (44)
\]

and:

\[
{h_k(\mu_0)} = [G_{lk}]^{-1} \{M_l\} \quad (45)
\]

Now, that we have \( \{c\} \) (from Eq. (17)), \( \{d\} \) (from Eq. (26)), and \( h_k(\mu_0) \) (from Eq. (45)), we can calculate the value of \( h_k(\mu) \) from Eq. (27). One may also calculate the Galerkin optimum circulation, \( \hat{\gamma} \), which is actually the expression inside the square brackets of Eq. (7).

\[
\hat{\gamma}(\mu) = \gamma(\mu) + \sum_{k=\frac{Q_0^q 5Q}{2^q 2^{q-1}2^{\cdots}}} \frac{2Q^2}{\pi^2 k^2} h_k(\mu) \quad (46)
\]

\( \gamma(\mu) \) is the non-dimensional circulation for a case with an infinite number of blades, and can be calculated from Eq. (6); but since the optimum Betz velocity distribution is \( \lambda(\mu) = \mu/\sqrt{1 + \mu^2} \), then \( \gamma(\mu) \) would be:

\[
\gamma(\mu) = \frac{\mu^2}{1 + \mu^2} \quad (47)
\]
With the Galerkin optimum circulation, one obtains the corrected circulation, \( \bar{\gamma} \), from Eq. (48):

\[
\bar{\gamma}(\mu) = \hat{\gamma}(\mu) - \hat{\gamma}(\mu_0)[1 - e]
\]

(48)

Where \( e \) is the Prandtl correction factor Ref. [1] used to make the tip correction, and \( h_k \) are the solutions from the Galerkin method. In order to maximize convergence, the Prandtl factor is used—but designed only to eliminate the residual—not correct the entire function. We add a acceleration factor, \( \kappa \), to account for the fact that the residual dies out more quickly as more terms are added. The factor is chosen so as to minimize the number of terms required for convergence in the matrix formulation. The modified Prandtl function \( e \) is therefore of the form:

\[
e = \frac{2}{\pi} \arccos[\exp(-f)]
\]

(49)

With:

\[
f = \frac{\kappa Q (1 - \frac{\mu}{\mu_0}) \sqrt{1 + \frac{\mu^2}{\mu_0^2}}}{2}
\]

(50)

the correction factor for optimized convergence has been expressed in the following form:

\[
\kappa = 1 - 5.6429 \ln\left(\frac{n}{n + 25}\right) \left[\frac{15625n}{(n + 25)^4}\right] - 3.2449 \left[\frac{25n}{(n + 25)^2}\right] + 13.3817 \left[\frac{625n}{(n + 25)^3}\right] - 3.7991 \left[\frac{15625n}{(n + 25)^4}\right]
\]

(51)

Once the solution for the corrected circulation is found, one can define the induced-power efficiency (IPE) as the ratio of the Goldstein optimum power (for a given number of blades) to the Glauert ideal power for an actuator disk:

\[
IPE = \frac{2}{\mu_0^2} \int_0^{\mu_0} \bar{\gamma} \mu d\mu
\]

(52)

## 2.5. Results

The present methodology is first used to compute cases already found in Goldstein as a verification of the convergence and accuracy of the method. Next, results, not found in earlier work are computed. Figures 2-4 compare the corrected circulation (circulation with the tip correction), and the Galerkin optimum circulation (circulation without the tip correction) for 2, 4 and 6 bladed rotors. The results are for \( n = 10 \) and \( N = 15 \) which we found was sufficient for convergence in all cases. Figure 2 shows our results in red for \( \mu_0 = 5 \) and \( Q = 2 \), for which we have a known solution from Goldstein. One can see that convergence is slow near the tip in that the zero boundary condition has not converged. Goldstein noted the same effect with his solution; and he mentions in his paper that he adds a correction to bring the tip to zero.
From the singularity at the edges the convergence may be very slow. The corresponding point in the graph may be displaced this amount if the curve can thereby be smoothed.

Goldstein

We similarly smooth the curves at the tip with our accelerated Prandtl tip-correction function, and that is shown in the blue curve which is virtually identical to the Goldstein solution. Figure 3 is for \( \mu_0 = 5 \) and \( Q = 4 \), another case for which Goldstein gives a solution. Similarly good convergence is seen. Figure 4 is for a six-bladed rotor, a result which has not heretofore been published.

We next compute the induced-power efficiency (IPE) for these cases. These are plotted versus the Glauert tip-speed ratio \( \mu_0 \) (the ratio of tip speed to free-stream velocity) and also versus its reciprocal \( \lambda = 1/\mu_0 \) in Fig. 5 for rotors with 2, 4 and 6 blades. The IPE decreases with increasing \( \lambda \) because the local blade lift is perpendicular to the vortex sheet and thus tilts—implying energy is lost in wake swirl. Figure 5 also shows how a decrease in blade number reduces efficiency because there are tip losses associated with upwash at the tip.

Figures 6, 7 and 8 compare the IPE under various induced-flow assumptions. The curve labeled "Betz Approximation" is the IPE for the infinite-blade case, which includes only the effect of lift tilt. The curve noted as "Prandtl Approximation" is the result of the Prandtl blade-number correction applied to the Glauert actuator-disk model. It includes only tip effects. The curve, "Betz-Prandtl Approximation" is the methodology suggested by Betz and Prandtl (and implemented by Goldstein in Fig. 1) in which the Prandtl correction is applied to the Betz solution. The final curve, labeled "Goldstein Exact Solution" is the result of our analysis which gives the complete solution including root losses as well as tip losses. One can see the relative effects of the various physical processes on the induced power efficiency.

Figures 6-8 reveal the magnitude of the various contributions: of lift tilt (the Glauert curves), of tip losses (the Prandtl curves), of combined tilt and tip losses (the Betz-Prandtl curves), and of root corrections (the exact curves). It is clear that the Betz-Prandtl approximation gives an almost exact result for the IPE when \( \lambda > 2.0 \), and a very good approximation even for \( \lambda < 2.0 \). This is because the Goldstein correction, clearly seen as a large effect in Fig. 1, has both positive and negative corrections to the Betz-Prandtl circulation. Thus, the net effect on efficiency is small. With the new, numerical method for finding the true Goldstein circulation, it has been possible for the first time to verify the effect of the Betz-Prandtl approximation on induced power efficiency (IPE).

2.6. Summary and Conclusions

With the use of a Galerkin procedure, we have obtained an efficient and accurate method for solving the Goldstein optimum circulation distribution for propellers with arbitrary blade number and tip-speed ratio. The numerical procedure is verified against results given by Goldstein for two specific cases, and it is then used to compute results not given by Goldstein. The results are used in order to find induced power efficiency of propellers. These results show that the effect of Goldstein’s root corrections on IPE are quite small such that the Prandtl-Betz approximation is generally adequate. However, the optimum circulation is significantly affected by Goldstein’s root effect for large wake spacing (i.e., for small blade number and small inflow ratio \( \mu \)).
2.7. References


2.8. Figures

Figure 2.1. Optimum Circulation for 2-Blades and 4-Blades Rotors

Figure 2.2. Corrected circulation vs. Galerkin optimum circulation for 2-bladed rotor
Figure 2.3. Corrected circulation vs. Galerkin optimum circulation for 4-bladed rotor

Figure 2.4. Corrected circulation vs. Galerkin optimum circulation for 6-bladed rotor
Figure 2.5. Induced power efficiency for 2, 4 and 6 blades rotor by Goldstein’s Solution
Figure 2.6. Induced power efficiency, Goldstein’s Exact Solution vs. Other approximations, 2 bladed rotor
Figure 2.7. Induced power efficiency, Goldstein’s Exact Solution vs. Other approximations, 4 bladed rotor
Figure 2.8. Induced power efficiency, Goldstein’s Exact Solution vs. Other approximations, 6 bladed rotor
Chapter 3

A COMPACT, CLOSED-FORM SOLUTION
FOR THE OPTIMUM, IDEAL WIND TURBINE

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3.1. Abstract

The classical momentum solution for the optimum induced-flow distribution of a wind turbine in the presence of wake swirl can be found in many textbooks. This standard derivation consists of two momentum balances (one for axial momentum and one for angular momentum) which are combined into a formula for power coefficient in terms of induction factors. Numerical procedures then give the proper induction factors for the optimum inflow distribution at any radial station; and this, in turn, gives the best possible power coefficient for an ideal wind turbine.

The present development offers a more straightforward derivation of the optimum turbine. The final formulas give the identical conditions for the ideal wind turbine as do the classical solutions—but with several important differences in the derivation and in the form of the results. First, only one momentum balance is required (the other being redundant). Second, the solution is provided in a compact, closed form for both the induction factors and the minimum power—rather than in terms of a numerical process. Third, the solution eliminates the singularities that are present in current published solutions. Fourth, this new approach also makes possible a closed-form solution for the optimum chord distribution in the presence of wake rotation.

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3.2. Nomenclature

- \( a \) axial induction factor, \( u/U \)
- \( a_0 \) value of \( a \) at blade tip
- \( a' \) classical swirl induction factor, \( v/\Omega r \)
- \( b \) total induction factor, \( w/U \)
- \( b_0 \) value of \( b \) at blade tip
- \( B \) number of blades
- \( c \) blade chord, \( m \)
- \( C_d \) blade drag coefficient
- \( C_L \) blade lift coefficient, \( L/\rho cV^2/2 \)
- \( C_p \) turbine power coefficient, \( P/\rho \pi R^2 U^3/2 \)
- \( C_{PN} \) net power coefficient
- \( C_Q \) turbine torque coefficient, \( Q/\rho \pi R^3 U^2/2 \)
- \( C_T \) turbine thrust coefficient, \( T/\rho \pi R^2 U^2/2 \)
- \( C_{TT} \) total thrust coefficient
- \( f \) new swirl induction factor, \( v/U \) \( (a' = f/\lambda_r) \)
- \( I_p \) integral of power due to drag
- \( I_T \) integral of thrust due to drag
- \( dL \) lift on annular ring of width \( dr \), \( N \)
- \( r \) radial position, \( m \)
- \( \bar{r} \) non-dimensional radial position \( r/R \)
- \( R \) tip radius, \( m \)
- \( P \) power delivered by wind turbine, \( N\cdot m/\text{sec} \)
- \( Q \) rotor torque, \( N\cdot m \)
- \( u \) induced velocity in axial direction, \( m/\text{sec} \)
- \( U \) wind velocity, \( m/\text{sec} \)
- \( v \) induced velocity in swirl direction, \( m/\text{sec} \)
- \( V \) total velocity relative to airfoil, \( m/\text{sec} \) \( (U_r \text{ of Ref. 1}) \)
- \( w \) total induced velocity (parallel to total thrust), \( m/\text{sec} \)
- \( x \) Ref. 1 change of variable, \( x=1-3a \)
- \( y \) present change of variable, \( y=3b-1 \)
- \( z \) intermediate variable, \( z=b^2/(1+\lambda_r^2) \)
- \( \alpha \) blade angle of attack, \( \text{rad} \)
- \( \epsilon \) small quantity, \( \epsilon \ll 1 \)
- \( \phi \) blade inflow angle, \( \text{rad} \)
- \( \eta \) nondimensional total flow, \( V/U \)
- \( \lambda_r \) local inflow ratio, \( \Omega r/U \)
- \( \lambda \) tip speed ratio, \( \Omega R/U \)
- \( \theta \) blade pitch angle, \( \text{rad} \)
- \( \rho \) air density, \( \text{kg/m}^3 \)
- \( \sigma \) local solidity, \( Bc/(2\pi r) \) \( (\sigma' \text{ of Ref. 1}) \)
- \( \Omega \) rotor angular velocity, \( \text{rad/sec} \)
3.3. Introduction

The first objective of this paper is to demonstrate that the solution for the induction factors of an ideal optimum wind turbine (and for the resultant optimum power coefficient) can be derived in a more direct manner than has been done in earlier derivations. The second objective of this paper is to show that—as the result of this more direct derivation—one can obtain a set of compact, closed-form expressions for these optimum induction factors and power coefficient. This is in contrast to past derivations, which terminate with a numerical procedure to complete the optimum rotor (rather than ending in compact formulas). Furthermore, because of the more direct approach of a unified momentum theory, singularities inherent in earlier approaches are eliminated.

The third objective of this paper is the revelation that this new, direct derivation (along with the more compact expressions) is possible because of the fact that axial momentum theory and swirl momentum theory are redundant to each other, which implies that only a single, unified momentum theory is necessary for the development. Finally, the fourth objective of this paper is to show how the optimum chord and twist distributions for a rotor (including profile drag) can also be found in closed form without the necessity of the neglect of wake swirl. It is hoped that these compact, closed-form expressions will yield additional insights into the nature of an optimum wind turbine.

3.4. Background

It is a well-known fact that the Betz optimum for an actuator disk (acting as a wind turbine rotor) is that the disk slows the wind at the disk to 2/3 of its incoming value (which implies that it is slowed to 1/3 of the incoming value far downstream). This theoretically yields 16/27 (or 59.3%) of the incoming kinetic energy converted as useful energy. However, for a disk that generates lift and torque through lifting blades, the lift is not perpendicular to the disk. The lift consequently creates a swirl velocity that imparts kinetic energy in the wake due to rotation. As a result, the distribution of induced flow for optimum power with swirl is more complicated than is the Betz solution; and the maximum possible power is smaller than 16/27.

The derivation of the optimum induced-flow distribution (including swirl)—and of the resultant optimum power—can be found in wind turbine texts, such as Refs. [1-4]. These derivations consist of several standard elements. First, a translational momentum theory is performed in the axial direction in order to relate rotor thrust to the axial induced flow, \( u \). Second, an angular momentum theory is used to find the relationship between rotor torque and swirl velocity, \( v \). These momentum results are expressed in terms of an axial induction factor and an angular induction factor (\( a \) and \( a' \), respectively) in the following form:

\[
 u = aU, \quad v = a'\Omega r
\]

\[
 \frac{a(1-a)}{a'(1+a')} = \lambda_r^2 \]

(1)

(2)
The resulting power per unit radius can be expressed in terms of tip-speed ratio $\lambda_r$ to give:

$$dC_p = 8a'(1-a) \left( \frac{\lambda_r^3}{\lambda^2} \right) d\lambda$$

Equation (2) can then be used to find $a'$ in terms of $a$, the latter of which can then be placed into Eq. (3). Noting that the derivative with respect to $a$ of Eq. (3) must equal to zero allows one (after considerable algebra) to obtain the optimality condition:

$$\lambda_r^2 = \frac{(1-a)(4a-1)^2}{(1-3a)} , \quad a' = \frac{(1-3a)}{(4a-1)}$$

The optimality condition in Eq. (4) can be written as a cubic in $a$:

$$16a^3 - 24a^2 + (9 - 3\lambda_r^2)a + \lambda_r^2 - 1 = 0$$

Although closed-form solutions exist for cubic equations, Ref. [5], the algebra is not particularly conducive to finding a closed form for the proper root of Eq. (5). Thus, the solution to Eq. (5) is traditionally done numerically. The final inflow angle then follows directly once the induction parameters ($a$ and $a'$) are determined:

$$\phi = \tan^{-1}\left( \frac{1-a}{\lambda_r(1+a')} \right) = \tan^{-1}\left( \frac{\lambda_r a'}{a} \right)$$

In order to find the total optimum induced power, Ref. [1] makes two additional changes of variable in the $C_p$ integral of Eq. (3). First, a change is made from $\lambda_r$ to $a'$; and then a change is made from $a$ to $x = 1 - 3a$. The resultant integral for $dC_p$ yields an expression that must be evaluated at the upper and lower bounds to find $C_p$:

$$C_p = \frac{8}{729\lambda^2} \left[ \frac{64}{5} x^5 + 72x^4 + 124x^3 + 38x^2 - 63x - 12 \ln(x) - \frac{4}{x} \right]_{x=1 - 3a_t}^{x=0.25}$$

where $a_t$ is the value of $a$ at the tip, (i.e., at $\lambda$). It should be noted that the above has singularities both in the limit as $\lambda$ goes to zero (the blade root) and $\lambda$ goes to infinity. Nevertheless, Eq. (7) gives a numerical method for finding the optimum turbine based on momentum theory.
3.5. Alternative Approach

What we offer here is an alternate derivation of the parameters for an optimum turbine—along with a resulting closed-form solution both for the optimum induced flow and for the total power coefficient. The new formulation differs in four ways from earlier expressions: 1.) only a single momentum balance and single induction factor is required; 2.) the optimum induction factor is found in a compact, closed form; 3.) the total optimum power is also obtained as a single, closed-form expression; and 4.) the singularities of earlier methods are removed. Although the resulting optimum turbine parameters are—of course—the same with the present method as with the previous method, the more compact results gives additional insight into the nature of the optimum wind turbine.

The geometry of the flow is crucial to the new approach for finding the optimum induced flow distribution. Figure 1 shows the various flow velocities as seen at the airfoil. The figure follows the convention of Glauert, Ref. [6]. It is important to note that both momentum considerations and vortex-tube theory show that the induced flow \( w \) and the lift vector \( L \) must be along the same line (but in opposite directions) and that this line must be perpendicular to the total flow relative to the blade. This implies that the induced flow \( w \) completely determines the local inflow angle \( \phi \). To be more specific, one can note from classical momentum developments [i.e., Ref. [1], Eqs. (3.21) – (3.28)] that the combined vector of swirl velocity and axial velocity form an induced-flow vector that is exactly parallel to the local lift vector (but in the opposite direction). Reference [4] shows that the same result follows when one considers vortex-tube theory with vorticity that is directed along the wake helix. In fact, Glauert in Ref. [6] assumes that this must be the case for the optimum rotor. The physical basis for this simple result is that of Newton’s laws of motion. For every action, there is an equal-and-opposite reaction; and the force vector must be proportional to the time rate of change of the momentum vector. Thus, it is not surprising that the induced flow and force must be opposite but parallel.

Once this factor is recognized, one can see from the geometry of Fig. 1 that—because the lift must be perpendicular to the total flow vector at the blade (the Biot-Savart Law)—it follows that the induced flow must be perpendicular to the local vortex sheet. Based on this observation, one can obtain a simple relationship between the original total flow and the total flow with induced flow based on the Pythagorean theorem.

\[
V^2 = U^2 + (\Omega r)^2 - w^2 \tag{8}
\]

Note that Fig. 1 includes several different triangles from which one can formulate \( \sin(\phi) \), \( \cos(\phi) \), and \( \tan(\phi) \). Based on this trigonometry, many useful identities can be found. For example, if one draws from the meeting point of \( \Omega r \) and \( fU \) (at the bottom of the figure) a line that ends perpendicular to the \( V \) velocity vector, then the length of that line can be expressed as \( (\Omega r) \sin(\phi) = U \cos(\phi) - w \). This equation can then be squared and solved for either \( \sin(\phi) \) or \( \cos(\phi) \) in terms of the flow variables (a very useful result).

\[
\sin(\phi) = \frac{U \sqrt{U^2 + (\Omega r)^2 - w^2 - \Omega rw}}{U^2 + (\Omega r)^2} \tag{9}
\]
The above relationships may also be written in terms of non-dimensional parameters (including the overall induction factor, $b = w/U$):

\[
\eta^2 = 1 + \lambda^2 - b^2 \\
\sin(\phi) = \frac{\sqrt{1 + \lambda^2 - b^2 - \lambda b}}{1 + \lambda^2} = \frac{v}{w} \\
\cos(\phi) = \frac{\lambda \sqrt{1 + \lambda^2 - b^2 + b}}{1 + \lambda^2} = \frac{u}{w}
\]

Therefore, the key parameter for optimum power is the total induction factor of the induced flow, $b$.

The above development offers the necessary geometric relationships to allow a derivation of the optimal wind turbine based on a single, unified momentum balance of loads versus induced flow, as shown below.

### 3.6. Momentum Theory

Reference [7] proves that momentum theory can be applied directly to a tilted lift vector to give the same induced velocity that would be obtained from vortex-tube theory. Reference [8] proves that an actuator-disk theory also gives the same answer as vortex theory (i.e., the exact answer) when applied to a tilted lift vector and to a tilted induced flow vector. Therefore, in contrast to the previous derivations which invoke both axial and angular momentum balances, it is only necessary to look at one momentum balance for the entire lift and induced flow. That momentum balance, when written for an annular ring, is:

\[
dL = 2\rho(2\pi rdr)w\left[U - w\cos(\phi)\right]
\]

This single momentum equation is all that is necessary because the swirl and axial components of $w$ are automatically included in the geometry of Fig. 1. A separate axial or swirl balance is redundant. To obtain power from Eq. (14), one can write:

\[
dP = 2\rho(2\pi rdr)w\left[U - w\cos(\phi)\right](\Omega r)\sin(\phi)
\]

The non-dimensional $C_p$ coefficient can then be written in terms of $\bar{r} = (r/R)$:

\[
dC_p = 8\lambda b[1 - b\cos(\phi)]\sin(\phi)\bar{r}d\bar{r}
\]
At this point, there are two approaches that can be taken. In the first one, trigonometric identities can be used to solve for $b \cos(\phi)$ in terms of $b \sin(\phi)$—followed by a derivative to find the maximum power. This gives an equivalent result to Eq. (5) for the optimality condition. A second approach is to substitute $\cos(\phi)$ and $\sin(\phi)$ from Eqs. (12-13) and then take a direct $b$ derivative. After considerable algebra, this results in a different cubic equation than the one in Eq. (5),

$$16z^3 - 24z^2 + 9z - \frac{1}{(1 + \lambda_r^2)} = 0$$  \hspace{1cm} (17)

where $z = b^2/(1 + \lambda_r^2)$. This cubic equation yields a closed-form result based on Ref. [5] that lends itself to a compact form for the optimum value of $b$.

$$b^2 = \left(\frac{1 + \lambda_r^2}{2}\right) \left[1 + \cos \left(\frac{2\pi}{3} + \cos^{-1} \left(\frac{1 - \lambda_r^2}{1 + \lambda_r^2}\right)\right)\right]$$  \hspace{1cm} (18)

$$\frac{1}{b} = 1 + 2 \cos \left[\frac{\pi}{3} - \frac{1}{3} \cos^{-1} \left(\frac{1 - \lambda_r^2}{1 + \lambda_r^2}\right)\right]$$  \hspace{1cm} (19)

$$\frac{1}{b} = 1 + \cos \left[\frac{1}{3} \cos^{-1} \left(\frac{1 - \lambda_r^2}{1 + \lambda_r^2}\right)\right] + \sqrt{3} \sin \left[\frac{1}{3} \cos^{-1} \left(\frac{1 - \lambda_r^2}{1 + \lambda_r^2}\right)\right]$$  \hspace{1cm} (20)

Equations (18-20) are all equivalent expressions for the optimum $b$. The axial and swirl inductions are: $a = b \cos(\phi)$ and $f = b \sin(\phi)$.

### 3.7. Complete Expressions

With the above closed-form expression for $b$, the entire optimum blade can be reduced to simple equations. At any given radial position $r$, one immediately knows the appropriate $\lambda_r = (r/R) \lambda$. From that $\lambda_r$, Eq. (20) gives the optimum $b$. One can also solve for the optimum $\lambda_r$ given $b$. In particular, from Eq. (17), one obtains:

$$\lambda_r = \frac{\sqrt{1+b(1-2b)}}{\sqrt{3b-1}}$$  \hspace{1cm} (21)

Based on the optimum $b$, one can also write relationships between $b$ and the inflow angle $\phi$.

$$b = \frac{1}{1 + 2 \cos(\phi)}$$  \hspace{1cm} (22)

$$\sin(\phi) = \frac{\sqrt{1+b} \sqrt{3b-1}}{2b}$$  \hspace{1cm} (23)
\[
\cos(\phi) = \frac{1-b}{2b}
\]  

(24)

Since one can relate \( b \) to \( \lambda_r \) and \( b \) to \( \phi \), one can also relate \( \phi \) to \( \lambda_r \).

\[
\phi = \frac{\pi}{3} - \frac{1}{3} \cos^{-1}\left(\frac{1-\lambda_r^2}{1+\lambda_r^2}\right)
\]

(25)

\[
\lambda_r = \frac{\cos^2(\phi) - \sin^2(\phi) + \cos(\phi)}{\sin(\phi)(1 + 2\cos(\phi))} = \frac{1}{\tan(3\phi/2)}
\]

(26)

Thus, Eqs. (20-26) are a complete, closed-form set of expressions of any of the optimum parameters in terms of any of the other two. Interestingly, Eq. (26) is derived in Ref. [1]—in the context of the optimum blade chord distribution—and can be found in their Eq. (3.9.105).

It is also interesting to compare these optimum parameters to the classic induction factors (\( a \) and \( a' \)) of conventional wind-turbine aerodynamics. This is easily done based on the geometry.

\[
a = b \cos(\phi) = \frac{1-b}{2} = \frac{\cos(\phi)}{1+2\cos(\phi)}
\]

(27)

\[
b = 1 - 2a
\]

(28)

\[
a' = \frac{f}{\lambda_r} = \frac{b\sin(\phi)}{\lambda_r} = \frac{\sin(\phi)}{\lambda_r(1 + 2\cos(\phi))}
\]

(29)

Due to \( \lambda_r \) in the denominator, \( a' \) is singular at \( r=0 \). The new induction parameter \( f \), however, is well-behaved. Based on the above, one can write a compact expression for the total flow at the blade in terms of \( b \).

\[
\eta^2 = 1 + \lambda_r^2 - b^2 = \frac{b^2(1+b)}{3b-1}
\]

(30)

This completes the expressions for the closed-form optimum blade parameters. Note that, at the root, \( \lambda_r = 0 \), we have \( b = 1/2 \) and \( \phi = 60^\circ \). Another interesting point to interrogate is \( \lambda_r = 1 \) (the point at which the original inflow angle = 45°). There, we have \( b = 1/(1+\sqrt{3}) = 0.366 \) and \( \phi = 30^\circ \). In the limit as \( \lambda_r \) approaches \( \infty \), we have \( b = 1/3 \) and \( \phi = 0^\circ \).
3.8. Optimum Power Coefficient

With the optimum blade parameters in compact form, it now remains to compute the optimum (i.e., maximum) power coefficient that goes with these parameters. Based on Eq. (16), the incremental optimum power is given by:

\[ dC_p = 2(1 + b)^2(1 - 2b)\bar{\varphi}d\bar{\varphi} = \frac{2(1 + b)^2(1 - 2b)\lambda_r d\lambda_r}{\lambda^2} \]

(31)

The task is to integrate this in closed form from \( \bar{\varphi} = 0 \) to \( \bar{\varphi} = 1 \) (or, alternatively, from \( \lambda_r = 0 \) to \( \lambda_r = \lambda \)). Since we have no closed-form expression for \( b \) in terms of \( \lambda_r \), our formula for \( \lambda_r \) in terms of \( b \), Eq.(21) makes a \( b \)-integration more tractable. A differential of Eq. (21) leads to:

\[ \lambda_r d\lambda_r = \frac{-6b^2(1-2b)}{(3b-1)^2} db \]

(32)

Therefore, we can write a formula for the total \( C_p \) in terms of a \( b \) integral.

\[ C_p = \frac{12}{\lambda^2} \int_{b_0}^{0.5} \left[ \frac{b(1-2b)(1+b)}{(3b-1)} \right]^2 db \]

(33)

where 0.5 is the value of \( b \) at the blade root, and \( b_0 \) is the value of \( b \) at the blade tip, \( (1/2 < b_0 < 1/3) \). Because of the singularity in the denominator, it is advisable to make a change of variable to \( y = 3b - 1 \). The term \( 1/\lambda^2 \) can also be expressed in terms of \( b_0 \) and, therefore, in terms of \( y_0 = 3b_0 - 1 \). The resultant integral for \( C_p \) is:

\[ C_p = \left( \frac{4}{27} \right) \left[ \frac{y_0}{(y_0 + 4)(1-2y_0)^2} \right] \int_{y_0}^{y_1} \left[ \frac{4}{y} - 3 - 9y - 2y^2 \right]^2 dy \]

(34)

where \( y_1 = 1/2 \) and \( 0 < y_0 < 1/2 \).

Equation (34) can be integrated in closed form. Because each integral term involves either \( \ln(y_1/y_0) \) or \( y_1^n - y_0^n \), one can factor out \( (y_1 - y_0) \) or, equivalently, factor out \( (1 - 2y_0) \). In fact, the cube can be factored out, \( (1 - 2y)^3 \). In addition, the \( y_0 \) on the outside of the integral cancels all \( y_0 \) terms in the denominator. This then creates a term of the form \( y_0 \ln(y_0) \) which removes the singularities from the final expression. The closed-form result for \( C_p \) becomes:
For convenience, we have dropped the subscript on $y_0$ such that $y = 3b_0 - 1$. Note that, for $y = (1 - \varepsilon)/2$, the term $[ln(1-\varepsilon) + \varepsilon + \varepsilon^2/2]/\varepsilon^2$ approaches -1/3, such that the formula is well-behaved. Similarly, at $y = \varepsilon$ (i.e., $\lambda_r$ approaching $\infty$), $d\eta/\varepsilon$ approaches zero such that the formula gives the Betz limit, 16/27. Since $b_0$ is known in closed form in terms of $\lambda$, Eq.(35) is the first closed-form expression for optimum $C_p$ that has been published.

### 3.9. Torque and Thrust Coefficients

The power coefficient in Eq. (35) goes to zero as $\lambda$ goes to zero, this reflects the fact that, in the limit as $\Omega$ goes to zero, there can be no power generated. However, there can be a torque in the limit as $\Omega$ approaches zero. Since $P = Q\Omega$, the torque coefficient comes immediately from the power coefficient.

$$C_v = \frac{C_p}{\lambda} = C_p \left( \frac{3\sqrt{3y}}{\sqrt{4 + y(1-2y)}} \right)$$

It follows from Eq.(35) that:

$$C_q = \frac{8}{9} \sqrt{\frac{3y}{1 + \frac{y^3}{4}}} \left[ 1 + \frac{457}{1280}y + \frac{51}{640}y^2 + \frac{y^3}{160} + \frac{3}{2} y \left\{ \ln(2y) + (1-2y) + \frac{1}{2}(1-2y)^2 \right\} \right]$$

Thus, $C_q$ approaches 0.8653 as $\lambda$ approaches zero. On the other hand, as $\lambda$ approaches infinity, $C_q$ goes to zero.

In a similar manner to the computation of power coefficient, one can also compute a closed-form expression for the thrust coefficient of the optimal rotor. From momentum theory, Eq. (14), the relationship for thrust is:

$$dT = 2\rho (2\pi r dr) w [U - w \cos(\phi)] \cos(\phi)$$

From this, the elemental thrust coefficient for the optimum rotor is:

$$dC_T = 8b [1 - b \cos(\phi)] \cos(\phi) \rho \frac{2(1-b^2)\lambda_r d\lambda_r}{\lambda^2}$$
With the change of variable into a \( b \) integral, we have:

\[
C_T = \frac{12}{\lambda^2} \int_{b_0}^{0.5} \frac{b^2 (1-2b)(1-b^2)}{(3b-1)^2} \, db
\]  

(40)

As with the power integral, a change of variable to \( y = 3b - 1 \) yields a closed-form expression for the thrust of an optimum rotor.

\[
C_T = \frac{8}{9 \left(1 + \frac{y}{4}\right)} \left\{ 1 + \frac{55}{192} y^2 - \frac{17}{64} y^3 + \frac{1}{4} y \left[ \ln(2y) + (1-2y) \right] \right\}
\]  

(41)

Note that, for small \( \lambda \), it follows that \( y = (1-\varepsilon)/2 \), where \( \varepsilon \) is a small quantity. In the limit as \( \varepsilon \) approaches zero, \( \ln(1-\varepsilon) + \varepsilon/\varepsilon^2 = -1/2 \). Also, as \( \lambda \) approaches \( \infty \), \( y = \varepsilon \), the limit of \( \varepsilon \ln(\varepsilon) = 0 \). Thus, at \( \lambda = 0 \) (y=1/2), we have \( C_T = 0.75 \); and, as \( \lambda \) approaches infinity, \( y=0 \), we have \( C_T = 8/9 \).

### 3.10. Optimal Chord and Pitch

The above derivation gives the induction factor for maximum power output. It is natural to ask as to the chord distribution and pitch angles that would give this optimum induced flow and power—under the assumption that each blade section is operating at the angle of attack \( \alpha \) that gives the maximum lift-to-drag ratio \( C_L/C_D \) for that section. Section 3.9, Eq. (3.106), of Ref. [1] gives without proof the optimum chord distribution (including swirl). In terms of our variables, the result is.

\[
c = \frac{8\pi r}{BC_i} (1 - \cos(\phi)) \quad \text{[Ref.1]}
\]  

(42)

The equivalent result can be obtained under the present approach with the use of single induction factor. In particular, because the lift from momentum theory (as well as the resultant induced flow) lies along the same axis as does the lift from blade-element theory, we may write:

\[
dL = 2\rho w [U - w \cos(\phi)] (2\pi r dr) = \left( \frac{\rho Bc}{2} \right) \left[ U^2 + (\Omega r)^2 - w^2 \right] C_i dr
\]  

(43)

The balance of momentum and blade-element lift (in terms of nondimensional variables) is:

\[
8\pi r [1 - b \cos(\phi)] = Bc C_i (1 - \lambda_i^2 - b^2)
\]  

(44)
The parameters in Eq. (44) are known from Eqs. (20–30). This allows us to solve for the optimum chord in terms of known quantities.

\[
c = \frac{4\pi r}{BC_i} \left[ \frac{(3b-1)}{b} \right] = \frac{16\pi rb^2}{BC_i} \left[ \frac{1}{(1+\lambda_r^2)} \right]
\]  

(45)

Equations (21-26) can be used to shown that Eqs. (42) and (45) are identical. A comparison of Eq. (45) with similar equations that neglect swirl (i.e., Eq. (3.79) of Ref. [1]) shows that the two expressions for chord agree for large \( \lambda_r \), which is the case for which swirl is negligible. However, results without swirl give an infinite solidity near the blade root \( \lambda_r = 0 \); whereas the formula that includes the effect of swirl is well-behaved in Eq. (45).

There are two interesting aspects of Eq. (45). First, near the blade root, the local solidity that results from the optimum chord is:

\[\sigma = \frac{Bc}{2\pi r} = \frac{8(1/2)^2}{C_i} = \frac{2}{C_i}\]

(46)

For a typical \( C_i \) of 1.0, this implies a solidity of 2 which would seem to be physically impossible (the area of blades exceeds the area of the annular ring). However, near the root, \( \phi = 60^\circ \) and \( \cos(60^\circ) = 0.5 \). Therefore, projection of the blade chord onto the rotor disk would exactly equal the available area; and the rotor would not interfere blade-to-blade. Another interesting aspect of Eq. (45) is that the ideal blade has a maximum chord that is approximately located at \( r/R = 1/\lambda \). For example, for \( \lambda = 7.0 \), the maximum would come roughly at 14\% distance from the rotor center and would give a local solidity of around 1.0—still free from blade interference. This also is the location at which \( \lambda_r = 1 \), and the local inflow angle is 30\°.

The optimum pitch angle follows directly from the relationship that the angle of attack is given by \( \alpha = \phi - \theta \). Since \( \phi \) is known in closed form from Eq. (25), and since the angle of attack for maximum \( L/D \) is known for each turbine airfoil, it follows that the optimum pitch angle is known and is given by \( \theta = \phi - \alpha \). In summary, the use of a single momentum balance—with all else following from geometry—gives a closed-form solution for the optimum rotor and allows computation of the optimum chord and pitch angle.

### 3.11. Effect of Profile Drag

It is quite straightforward to determine the effect of profile drag on the thrust, power, and efficiency of the optimum wind turbine. This is possible because the above derivation of the optimum rotor is based on a momentum theory and a blade element theory that both assume lift perpendicular to the vortex sheet. Since the profile drag is by definition along an axis parallel to the vortex sheet, it is easily included in the blade loads. What further simplifies the computation is the fact that profile drag does not affect momentum theory. This is because only the circulatory lift trails vorticity that creates induced flow. Profile drag may heat the air and produce a shear layer behind each blade; but these effects are negligible in terms of their influence on induced flow.
Thus, since it is specified that the local drag is perpendicular to the local lift (and has no effect on the momentum induced flow), we may write the net elemental thrust coefficient (due to both lift and drag) as a quantity that is proportional to:

$$C_l \cos(\phi) + C_d \sin(\phi) = C_l \cos(\phi) \left[ 1 + \left( \frac{C_d}{C_l} \right) \tan(\phi) \right] \quad (47)$$

Similarly, the elemental power coefficient is proportional to:

$$C_l \sin(\phi) - C_d \cos(\phi) = C_l \sin(\phi) \left[ 1 - \left( \frac{C_d}{C_l} \right) \cot(\phi) \right] \quad (48)$$

It follows that the existing integrals for $C_T$ and $C_P$ can be augmented with integrals that multiply $(C_d/C_l)$ in order to obtain the desired effect of drag on thrust and power.

In order to obtain insight to the effect of profile drag, we consider the case in which all airfoil sections have the same maximum lift-to-drag-ratio along the blade span. (Of course, this does not imply that each section has the same lift and drag.) Although a production wind turbine generally would not have all airfoils operating at the same lift-to-drag ratio, here we are considering only the ideal turbine. Thus, it is instructive to consider an ideal turbine with the same airfoil geometry at all sections (and thus the same optimum lift-to-drag ratio). This allows a closed-form expression for the effect of drag on the ideal optimum. It is not, strictly-speaking, the optimum for a case with drag. However, since the effect of profile drag is assumed a correction factor, one would expect the optimum induction factors not to change drastically due to the presence of drag. Thus, this approach should yield important insight.

According to Eqs. (47) and (48), the integral for thrust or power can be multiplied by either $\tan(\phi)$ or $\cot(\phi)$, respectively, in order to obtain the integrals that are to go with $C_d/C_l$ in $C_T$ and $C_P$, respectively. The integral for $C_T$ due to $C_d/C_l$ is of the form:

$$Thrust \ Integral = I_T = \frac{12}{\lambda^2} \int_{b_0}^{0.5} \frac{b^2 (1-2b)(1+b)^3}{(3b-1)^3} \, db \quad (49)$$

The integral for the effect of drag on $C_P$ is of the form:

$$Power \ Integral = I_P = \frac{12}{\lambda^2} \int_{b_0}^{0.5} \frac{b^2 (1-2b)^2 (1+b)^3 (1-b)}{(3b-1)^5} \, db \quad (50)$$

These integrals have been worked out in closed form. The thrust integral is:
\[ I_T = \frac{4y}{\sqrt{3}(y+4)(1-2y)^2} \left[ \left( \frac{1}{6}y^3 + \frac{4}{3}y^2 + \frac{8}{3}y - \frac{1}{3} + \frac{8}{3y} \right) \sqrt{y^2 + 4y} \right] \\
+ 2\ln \left( \frac{4}{(y+2) + \sqrt{y^2 + 4y}} \right) - \frac{321}{32} \] (51)

The singular term at \( y = 0.5 \) (\( \lambda = 0 \)) can be factored, giving:

\[ I_T = \frac{1}{6\sqrt{3}(y+4)} \left[ \sqrt{y(y+4)} \left( y^2 + 9y + \frac{99}{4} \right) - 48y \left( 1 - \frac{(1-2y)}{(2-y) + \sqrt{y(y+4)}} \right) \right] \\
+ \frac{107}{3} \left[ \frac{18y(y+4) + (45 - 32y - 8y^2)\sqrt{y(y+4)}}{36 + 113y - 38y^2 - 16y^3} + 9(7 - 2y)\sqrt{y(y+4)} \right] - \frac{64}{3} \left[ \frac{\sqrt{y}}{3\sqrt{y + \sqrt{y+4}}} \right] \] (52)

The power integral becomes:

\[ I_p = \frac{4y}{9\sqrt{3}(y+4)(1-2y)^2} \left[ \left( \frac{4}{15}y^4 + \frac{17}{15}y^3 - \frac{133}{45}y^2 - \frac{89}{9}y + \frac{104}{9y} + \frac{16}{9y^2} \right) \sqrt{y^2 + 4y} \right] \\
+ 8\ln \left( \frac{(y+2) + \sqrt{y^2 + 4y}}{4} \right) - \frac{1023}{80} \] (53)

It turns out that this integral is nearly linear with \( \lambda \) and can be approximated by:

\[ I_p = \frac{16}{27}\lambda \] (54)

Or to make the formula a little more accurate, one can use:

\[ I_p = \frac{27\lambda + 32\lambda^3}{54(1 + \lambda^2)} \] (55)

The maximum error of Eq. (54) is 0.024 (at \( \lambda = 0.7 \)) and that of Eq. (55) is 0.022 (at \( \lambda = 1.1 \)). These errors are, respectively, about 6% and 3% of the local integral. For all values of \( \lambda \), Eq. (55) gives lower errors than does Eq. (54). For example, the relative errors at \( \lambda = 5.0 \) are 0.011
and 0.007 respectively (0.37% and 0.24%); and the errors at $\lambda = 0.2$ are 0.016 and 0.004 (12% and 3%).

From this, the integrals for the total thrust coefficient $C_{TT}$ and the net power coefficient $C_{PN}$ follow directly:

\[
C_{TT} = C_T [\text{Eq. (41)}] + \left( \frac{C_d}{C_i} \right) I_r \quad (56)
\]

\[
C_{PN} = C_p [\text{Eq. (35)}] - \left( \frac{C_d}{C_i} \right) I_r \quad (57)
\]

### 3.12. Numerical Results

The present closed-form results are consistent with past results, but it is nonetheless informative to plot these optimum parameters based on the formulas herein. Figure 2 presents the three induction factors ($b$, $a$, $f$) versus the local speed ratio, $\lambda_r$. The total induction factor $b$ varies from 0.5 at $\lambda_r = 0$ to 0.33 for $\lambda_r >> 1$. The axial induction factor $a$ varies from 0.25 to 0.33 over the same range. The swirl induction factor $f$ shows a wider range of values, beginning at $\sqrt{3}/4$ and going to zero. Figure 3 shows the total inflow angle through this same range. The optimum angle is 60˚ for small $\lambda_r$, drops to 30˚ at $\lambda_r = 1$, and approaches 0 as $\lambda_r$ becomes large.

Figure 4 plots the same data against $1/(1 + \lambda_r^2)$ which is the square of the sine of the initial airflow angle (before the optimum induced flow is added). Thus, small inflow ratios are at the right ($\lambda_r$ tending to infinity) and large inflow ratios on the left ($\lambda_r$ tending to zero). Note that this curve is antisymmetric about $\lambda_r = 1$ and $\phi = 30^\circ$. This is a consequence of the closed-form result in Eq. (25). Figure 4, therefore, is a universal curve that shows how the optimum inflow ratio varies from 0° to 60° over the entire inflow-ratio range. A turbine with a given tip radius and root cut-out would have optimum values of the inflow angle as found from the appropriate range of $1/(1 + \lambda_r^2)$ in the figure.

Figure 5 shows a comparison between the approximate optimum chord formula (in which wake swirl is neglected) and the exact, closed-form result presented here. Results are for $\lambda = 7$, $B=3$, and $C_i = 1$. Note that the root solidity is infinite for the approximate method but is well-behaved for the true optimum.

Figure 6 presents the $C_p$ from the closed-form result in Eq. (35). We have verified that this result agrees exactly with the numerical solution in Ref. [1]. This figure gives an understanding as to why typical, production wind turbines have tip-speed ratios between 5 and 7. Values of that magnitude are needed to approach ideal efficiency, but larger values do not yield much extra power. Figure 7 shows the torque coefficient for the same case. Note that, for a stopped rotor, $\lambda=0$, there can still be a torque due to the lift on the blades in the free-stream. As rotor tip-speed becomes large, less and less torque is required to produce the same optimum power; and $C_Q$ approaches to zero. Figure 8 gives the closed-form thrust coefficient for the optimum rotor. It varies between 3/4 and 8/9 in a monotonic fashion.

Figures 9 and 10 give the optimum power and thrust including the effect of profile drag for the cases $C_d = 0.00$, 0.01, 0.02, and 0.04 and $C_i = 1.0$. Notice that, with profile drag, a given value of $C_d/C_i$ implies an optimum tip-speed ratio for maximum power. The maximum $C_p$ for $C_d/C_i =$
0.04, is 0.475 (at $\lambda = 2.85$). For $C_d/C_l = 0.02$, the maximum $C_p = 0.514$ (at $\lambda = 3.85$). For $C_d/C_l = 0.01$, the maximum $C_p = 0.541$ (at $\lambda = 5.21$). From this, one can infer an approximate formula for the $\lambda$ that gives maximum $C_p$.

$$\lambda^2 = 0.128 \left( \frac{C_l}{C_d} \right) + \sqrt{ \left[ 0.128 \left( \frac{C_l}{C_d} \right) \right]^2 + 0.559 \left( \frac{C_l}{C_d} \right)} \quad (58)$$

This sheds further insight as to why typical wind turbines with high $C_l/C_d$ have tip speeds $\lambda$ of the order of 5 to 7. Note that the effect of the profile drag becomes more pronounced at high tip-speed ratios.

### 3.13. Summary and Conclusions

An alternate derivation is provided for the parameters of an optimum, ideal wind turbine. Unlike previous derivations, only a single momentum theory is used (in the direction of the local lift) so that there are no separate accounts of axial and angular momentum. The results, also unlike previous results, are found in closed form for all variables—and the singularities of previous numerical solutions are eliminated explicitly. Although the final parameters for the optimum turbine are no different from those of conventional approaches, the closed-form nature of the results yields insight into the properties of the optimum turbine. Finally, because of the single momentum balance, it is quite straightforward also to write a closed-form expression for the optimum blade chord distribution. The true optimum does not become singular at the blade root, but rather approaches a combination of solidity and pitch angle that avoids blade-to-blade interference.

### 3.14. References


3.15. Figures

Figure 3.1. Rotor Inflow Geometry
Figure 3.2. Wake Induction Parameters as a Function of Local Speed Ratio

Figure 3.3. Optimum Inflow Angle as a Function of Local Speed Ratio
Figure 3.4. Optimized Inflow Angle as a Function of $\sin^2$ of Initial Inflow Ratio

Figure 3.5. Optimum chord, theory with wake rotation vs. theory without wake rotation
\( (\lambda = 7, C_l = 1, B = 3) \)
Figure 3.6. Power Coefficient as a Function of Tip Speed Ratio

Figure 3.7. Torque Coefficient as a Function of Tip Speed Ratio
Figure 3.8. Thrust Coefficient as a Function of Tip Speed Ratio

Figure 3.9. Power Coefficient as a Function of Tip Speed Ratio Including the Effect of Profile Drag
Figure 3.10. Thrust Coefficient as a Function of Tip Speed Ratio Including the Effect of Profile Drag

\[ \eta \]

\[ C_T \]

\[ \lambda \]

\[ C_d / C_l = 0.00 \]
\[ C_d / C_l = 0.01 \]
\[ C_d / C_l = 0.02 \]
\[ C_d / C_l = 0.04 \]
Chapter 4

OPTIMUM PERFORMANCE OF AN ACTUATOR DISK BY A COMPACT MOMENTUM THEORY INCLUDING SWIRL

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4.1. Abstract

A new compact form of momentum theory is introduced for actuator disks including swirl. The new form unifies both the axial and angular momentum balances into a single momentum equation, applicable over the entire range of thrust and power coefficients. While completely consistent with earlier momentum theories, such as that of Glauert in Ref. [1], the compact form allows analytic expressions for the parameters of an optimum actuator disk and reveals additional insight into the limiting efficiency of rotors, propellers, and wind turbines. Closed-form results presented here include the optimum values of: induced flow, inflow angle, thrust, induced power, and efficiency. Closed-form expressions are also given for optimum twist, chord distribution, and solidity in the presence of profile drag (along with the resulting over-all efficiencies). For the limiting case of the optimum rotor in hover, the compact form leads to closed-form expressions for both contraction ratio and pressure distribution in the far wake. This report also gives a formal proof that the Betz inflow distribution results in the maximum figure of merit, and it further demonstrates that some approximations used in earlier actuator-disk momentum theories have been inconsistent.

4.2. Nomenclature

\( b \) blade number
\( c \) blade chord, m
\( C_d \) local blade drag coefficient
\( C_l \) local blade lift coefficient
\( C_p \) rotor power coefficient, \( P/(\rho \pi R^2 \Omega^3 R^3) \)
\( C_{PG} \) Glauert minimum power
\( C_{PI} \) minimum induced power
\( C_{PT} \) total power, induced power plus profile power
\( C_T \) rotor thrust coefficient, \( T/(\rho \pi R^2 \Omega^2 R^2) \)
\( C_{TN} \) net thrust coefficient, lifting thrust less drag download
\( f(x) \) mapping from \( x \) to \( x_1^2 = f(x) \)
\( f(\eta, \nu_0) \) swirl-loss function, Eq. (A6)
\( F_L \) reaction force on lower momentum tube, N
\( F_U \) reaction force on upper momentum tube, N
\( g(\bar{r}) \) \( f(x)/(R^2 \nu_0^2) \)
\( g(\eta, \nu_0) \) axial drag integral, Eq. (A11)
\( h(\eta, \nu_0) \) inplane drag integral, Eq. (A12)
\( K \) contraction ratio, \( K = r_1/r = \sqrt{f(r)/r^2} \)
\( L \) local lift, N
\( P \) rotor power, watts
\( P_C \) normalized pressure required to balance centrifugal force, N/m^2
\( P_W \) normalized pressure in far wake for consistent momentum theory, N/m^2
\( r \) nondimensional radial coordinate, \( x/R \)
\( r_1 \) nondimensional radial coordinate in far wake, \( x_1/R \)
\( \bar{r} \) normalized radial coordinate, \( r/(\eta + \nu_0) \)
\( R \) rotor radius, m
\( R_1 \) radius of wake far downstream, m
\( T \) rotor thrust, N
\( u \) axial flow at airfoil, m/sec
\( \bar{u} \) normalized axial velocity, \( u/(\Omega R \nu_0) \)
\( u_1 \) axial induced flow in far wake, m/sec
\( \bar{u}_1 \) normalized axial velocity, \( u_1/(\Omega R \nu_0) \)
\( U \) rotor climb rate, m/sec
\( v \) local air speed relative to airfoil, m/sec
\( V \) dynamic-inflow mass-flow parameter, \( (\eta + 2 \nu_0) \)
\( V_T \) dynamic-inflow total-flow parameter, \( (\eta + \nu_0) \)
\( w \) local induced flow, parallel to \( L \), m/sec
\( w_0 \) nominal value of ideal inflow extrapolated to \( r = \infty \)
\( x \) radial coordinate, m
\( x_1 \) radial coordinate in far wake, m
\( \alpha \) local blade angle of attack, \( \theta - \phi \), rad
\( \Gamma \) total circulation on all blades at a radial position, m/sec^2
\( \bar{\Gamma} \) nondimensional circulation, \( \Gamma/(2\pi \Omega R^2) \)
4.3. Background

It is well-known that the optimum induced flow distribution for a lifting, actuator disk is a uniform pressure and (consequently) a uniform induced-flow distribution, Ref. [1]. This is the loading and inflow that give the minimum power for a specified thrust (or, equivalently, the maximum thrust for a given power). However, a rotor system is not a true actuator disk (i.e., does not have a loading perpendicular to the disk). Rather, the local lift of a rotor is perpendicular to the vortex sheet coming off of the blades; and the resulting tilt of the lift results in a swirl velocity with loss of energy.

Betz postulated from vortex considerations that the optimum inflow distribution is one that causes the vortex sheet to come off of the blades along a prescribed helix, Ref. [2]. He specifies the circulation distribution necessary to obtain that ideal inflow for the case of a lightly-loaded rotor with an infinite number of blades. In that same work, Prandtl gives an expression for an approximate correction for the circulation distribution that would result in this optimum induced flow for the case of a finite number of blades. Goldstein, Ref. [3], provides an exact solution for the circulation distribution that results in the optimum Betz inflow for a lightly-loaded rotor with a finite number of blades. Goldstein also demonstrates that the Prandtl approximation is quite accurate.

Reference [1] also uses the Betz optimum loading distribution—specialized to the case of hover with an infinite number of blades—to work out some numerical calculations for the optimum figure of merit for a lifting rotor. This is done for small thrust coefficients, $0 < C_T < 0.06$ for which the figure of merit lies between 1.0 and 0.89. The results offer a valid representation of how figure of merit is affected by wake swirl. However, a general formula for rotor efficiency as a function of climb rate (and valid for all thrust levels) has not been worked out. It would be beneficial to have such an expression (at least for an infinite number of blades). For such a solution to be generally useful, it should be valid throughout the entire range of rotor loadings from a lightly-loaded propeller to the case of a helicopter in hover. A result such as this
would give insight into the nature of optimum rotors and could provide a benchmark against which to compare optimization routines coupled to comprehensive codes. Presumably, the effect of a finite number of blades could be accounted for by the Prandtl approximation applied as a correction.

The purpose of this paper is to describe the optimum lifting rotor in terms of the optimum inflow distribution (as well as the optimum twist and chord that would result in such a distribution), and to find the resultant thrust, power, and efficiency of the optimum rotor as compared to the Glauert ideal for an actuator disk. Results are provided in a compact, closed form that yields insight into the nature of optimum rotors throughout the range of operating conditions. In addition, these results are compared with other references who have treated the problem; and a formula is derived for the optimum contraction ratio. A summary of the main results is given in Appendix A.

4.4. Geometry of Optimum

Figure 1, taken after Ref. [2], shows the geometry of the flow field at a typical blade section. In the figure, \( \Omega x \) is the air velocity from blade rotation, \( U \) is the air flow due to the climb rate of the rotor (positive for a rotor moving through the air in the same direction as the rotor lift), and \( w \) is the induced flow at the blade. This figure is valid for all cases—from a lightly-loaded propeller to a helicopter in hover or a lifting fan. Notice that the local lift \( L \) and the local induced flow \( w \) are parallel (but in opposite directions) to each other and that they both are perpendicular to the local flow velocity, \( u \). This is a direct result of the physics of the situation and follows either from application of vortex theory or momentum theory. As a consequence of this geometry, the local velocities and the inflow angle \( \phi \) are completely determined in terms of the rotational speed \( \Omega x \), the climb rate \( U \), and the local induced flow magnitude \( w \), as shown below.

\[
\sin(\phi) = \frac{U \sqrt{U^2 + \Omega^2 x^2 - w^2 + (\Omega x)w}}{U^2 + \Omega^2 x^2}
\]

\[
\cos(\phi) = \frac{\Omega x \sqrt{U^2 + \Omega^2 x^2 - w^2 - (U)w}}{U^2 + \Omega^2 x^2}
\]

\[
v^2 = (\Omega x)^2 + U^2 - w^2
\]

The velocity noted in Fig.1 as \( w/cos(\phi) \) is a construct in the geometry of the figure and is not a physical velocity in the system. Nonetheless, it is a useful construct in order to express the geometry. It is important to note that the induced velocity \( w \) has a vertical component \( wcos(\phi) \) that adds to \( U \) and a swirl component \( wsin(\phi) \) that subtracts from \( \Omega x \).

Betz proved that the optimum induced flow distribution \( w(x) \) is one that remains along a helix as it leaves the rotor disk. In other words, the optimum distribution has the form:

\[
w(x) = w_0 cos(\phi)
\]
where $\phi$ is the local inflow angle and $w_0$ is a nominal velocity that equals the inflow velocity extrapolated to an infinite radius, $\cos(\phi) = 1.0$. The use of $w_0$ in this parametric way makes it very convenient to express the optimum rotor for any condition in terms of this free parameter $w_0$. It is also convenient to express all quantities as normalized on the tip speed, $\Omega R$. Thus, we define the normalized velocities:

$$v = \frac{w}{\Omega R}; \quad v_0 = \frac{w_0}{\Omega R}; \quad \eta = \frac{U}{\Omega R}; \quad \mu = \frac{v}{\Omega R}; \quad r = \frac{x}{R} \quad (5)$$

With these definitions, the optimum induced flow and inflow angle become:

$$v(r) = \frac{v_0 r}{\sqrt{(\eta + v_0)^2 + r^2}}; \quad \phi = \arctan\left(\frac{\eta + v_0}{r}\right) \quad (6)$$

$$\cos(\phi) = \frac{r}{\sqrt{(\eta + v_0)^2 + r^2}} \quad (7)$$

$$\sin(\phi) = \frac{(\eta + v_0)}{\sqrt{(\eta + v_0)^2 + r^2}} \quad (8)$$

$$\mu = \frac{r^2 + \eta(\eta + v_0)}{\sqrt{(\eta + v_0)^2 + r^2}} \quad (9)$$

These are the parametric equations define the inflow for an optimum lifting rotor.

### 4.5. Momentum Theory

To find the resultant thrust and power of the Betz optimum inflow distribution requires an inflow theory that relates loading to $w$. For the lightly-loaded theory of Goldstein, Ref. [3], one can use a potential-flow, vortex model; but, for the case of finite loading, momentum theory is the model of choice. To include swirl velocity in the analysis, one would normally do both an axial and an angular momentum balance—as is done for the inflow of wind turbines, Ref. [4]. However, two balances are not necessary because angular momentum theory is redundant to axial momentum theory due to the fact that the total induced flow must be along the same axis as the local lift, Fig. 1.

Reference [5] specifically demonstrates that—when one takes the tilted lift that is on the blade but then applies that lift perpendicular to a notional, actuator disk (computing the axial, momentum-theory inflow perpendicular to that notional disk)—this computed induced flow is identical to the skewed inflow at the actual blade due to the tilted lift as determined by either vortex-tube theory or a combination of axial and angular momentum theory. Similarly, Makinen Ref. [6] shows that, when applied in this way to a lightly-loaded rotor with wake swirl and a finite number of blades, actuator-disk theory gives the same optimum induced flow distribution as found by Goldstein. Thus, it is completely rigorous to apply a single momentum balance along the lift direction in order to give both axial induced flow and the swirl flow.
Reference [7] sets out the general theory of how this might be done and works out some closed-form results in hover when the effect of drag on lift is neglected. In this work, we follow the approach of Ref. [7] and apply a single momentum balance in order to determine the optimum lifting rotor. This balance reflects that the local lift is equal to the mass-flow rate across the disk multiplied by the change in induced flow which is along the same direction as the lift (and which doubles downstream). This process results in a general momentum balance for an annular ring on the disk, as follows:

\[ dL = 4\pi \rho [U + w\cos(\phi)](w)(xdx) \]  

(10)

From this, the incremental thrust (normal to the disk) and the incremental power are:

\[ dT = 4\pi \rho [U + w\cos(\phi)][w\cos(\phi)](xdx) \]  

(11)

\[ dP = 4\pi \rho [U + w\cos(\phi)][w\sin(\phi)][\Omega x^2 dx) \]  

(12)

The nondimensional forms follow from the definitions given earlier.

\[ dC_T = 4[\eta + v\cos(\phi)][v\cos(\phi)](rdr) \]  

(13)

\[ dC_P = 4[\eta + v\cos(\phi)][v\sin(\phi)](r^2 dr) \]  

(14)

From the integrals of these expressions, one can find the thrust and power for any induced flow distribution—including the effect of swirl. Equations (11) - (14) are also applicable to wind turbines, for which \( w \) and \( v_0 \) (and, consequently, thrust and power) would be negative.

When the optimum values of \( v \) and \( \phi \) from Eqs. (6-8) are placed into Eqs. (13-14), one has the thrust and power for the particular case of the optimum lifting rotor.

\[ dC_T = \frac{(4r^3 dr)(v_0)(\eta + v_0)[r^2 + \eta(\eta + v_0)]}{[(\eta + v_0)^2 + r^2]^2} \]  

(15)

\[ dC_P = (\eta + v_0) dC_T = \frac{(4r^3 dr)(v_0)(\eta + v_0)^2[r^2 + \eta(\eta + v_0)]}{[(\eta + v_0)^2 + r^2]^2} \]  

(16)

Note that each elemental power in Eq. (15) is proportional to a corresponding elemental thrust in Eq. (16): \( dC_P = (\eta + v_0) dC_T \). Furthermore, since \( \eta C_T \) is the useful work done by the rotor, it follows that the incremental induced power is given by:

\[ dC_{\text{ind}} = dC_P - \eta dC_T = (v_0) dC_T = \frac{(4r^3 dr)(v_0)^2[\eta + v_0]^2[r^2 + \eta(\eta + v_0)]}{[(\eta + v_0)^2 + r^2]^2} \]  

(17)

Equations (15-17) can be integrated to determine the total thrust and power. The integrand of each of these integrals is identical. This is not simply a coincidence, but it is the natural consequence of having chosen the Betz optimum induced-flow distribution. The kernel integral of these coefficients is defined here as the swirl-loss function \( f(\eta, v_0) \) and is given by:

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\[
\int_{0}^{1} \frac{2[r^2 + \eta(n + v_0)]}{[(n + v_0)^2 + r^2]^2} r^3 dr
\]

(18)

Appendix B works out this integral in closed form, and it is:

\[
f(\eta, v_0) = \frac{[1 + (\eta + 2v_0)(\eta + v_0)]}{[1 + (\eta + v_0)^2]} - (\eta + 2v_0)(\eta + v_0) \ln \left[ 1 + \frac{1}{(\eta + v_0)^2} \right]
\]

(19)

This function is unity for a rotor at high tip speeds (small \( \eta \)) and decreases as tip speed becomes smaller (large \( \eta \)). It is interesting that this function depends on the same two key parameters as does dynamic inflow theory, \( V = (\eta + 2v_0) \) and \( V_T = (\eta + v_0) \).

4.6. **Induced Power and Efficiency**

With the above, one may write the thrust and power integrals in closed form as:

\[
C_T = 2v_0(\eta + v_0)f(\eta, v_0)
\]

(20)

\[
C_p = 2v_0(\eta + v_0)^2f(\eta, v_0) = (\eta + v_0)C_T
\]

(21)

\[
C_{Pl} = 2v_0^2(\eta + v_0)f(\eta, v_0) = v_0C_T
\]

(22)

The form \( C_{Pl} = v_0C_T \) is reminiscent of the relationship for the Glauert ideal case of uniform flow in which \( P = wT \). However, here \( v_0 \) is not uniform inflow but is rather the extrapolation of the optimum flow field off of the disk to where it would approach uniform flow. Thus, \( v_0 \) is greater than the induced flow anywhere on the disk; and the required induced power is greater than the Glauert minimum. Equations (19-22) give closed-form expressions for the thrust, power, and induced power of an optimum rotor that is loaded. They are written in terms of the climb rate \( \eta \) and the parameter \( v_0 \)—which is the nominal flow extrapolated to infinity. A lightly-loaded rotor is the case of \( v_0 \ll 1 \).

It is instructive to look at the induced power efficiency (IPE)—which is the minimum Glauert power for an actuator disk divided by the true induced power of the lifting rotor. The minimum Glauert power can be obtained by setting \( f(\eta, v_0) = 1 \) in Eqs. (20) and (22), and then solving for \( C_{Pl} \) in terms of \( C_T \). This \( C_{Pl} \) is, by definition, \( C_{PG} \).

\[
C_{PG} = C_T \left[ -\eta/2 + \sqrt{\eta^2/4 + C_T/2} \right]
\]

(23)

\[
C_{PG} = \frac{C_T^2/2}{\eta/2 + \sqrt{\eta^2/4 + C_T/2}}
\]

(24)

The induced power efficiency follows directly from Eqs. (22) and (24):
\[ IPE = \frac{C_T/\nu_0}{\eta + \sqrt{\eta^2 + 2C_T}} \] (25)

This can be written in terms of \( \eta \) and \( \nu_0 \) by use of the \( C_T \) formula from Eq. (20).

\[ IPE = \frac{2(\eta + \nu_0)f(\eta, \nu_0)}{\eta + \sqrt{\eta^2 + 4\nu_0(\eta + \nu_0)f(\eta, \nu_0)}} \] (26)

Equation (26) is the closed-form expression for induced power efficiency for lifting rotors at all operating conditions.

It is interesting to look at some special cases of Eq. (26). The case of a lightly-loaded rotor is given by setting \( \nu_0 = 0 \).

\[ IPE \text{ (lightly - loaded)} = f(\eta, 0) = 1 - \eta^2 \ln \left(1 + \frac{1}{\eta^2}\right) \] (27)

The case of hover is found by setting \( \eta = 0 \).

\[ IPE \text{ (hover)} = \frac{2\nu_0\sqrt{f(0, \nu_0)}}{\sqrt{4\nu_0^2}} = \frac{C_T^{3/2}}{\sqrt{2}C_p} \]

\[ = \sqrt{f(0, \nu_0)} = \sqrt{1 + \frac{\nu_0^2}{(1 + \nu_0^2)} - 2\nu_0^2 \ln \left(1 + \frac{1}{\nu_0^2}\right)} \] (28)

Equations (27-28) are identical to the formulas derived in Ref. [7]. Eq. (28) is the figure of merit (due to induced losses) for a hovering rotor.

Another interesting case is the limit as \( \eta + \nu_0 \) becomes large. In this limit, the angle of the vortex sheet approaches 90º. The resultant thrust and induced power then become:

\[ C_T \text{ (large inflow)} = \nu_0 \left[ \frac{\eta}{(\eta + \nu_0)^2} + \frac{2/3}{(\eta + \nu_0)^3} \right] \; \; ; \; \; C_{Pl} = \nu_0 C_T \] (29)

Equation (29) implies that \( C_T \) approaches zero for large \( \nu_0 \). Thus, \( C_T \) does not increase monotonically with \( \nu_0 \) but rather reaches a maximum and then tends to zero as \( \nu_0 \) increases further. Induced power \( (C_{Pl}) \), on the other hand, approaches the value of \( \eta \) for large \( \nu_0 \) and goes to zero only for the case of hover \( (\eta = 0) \). In hover, thrust reaches the maximum value \( (C_T = 0.238) \) at \( \nu_0 = 0.78 \); and, although both \( C_T \) and \( C_{Pl} \) tend to zero with large \( \nu_0 \), the induced power efficiency (i.e., figure of merit) decreases monotonically as \( \nu_0 \) increases—with a limit of zero.

The above formulas thus give a complete description of the optimum rotor in hover.
4.7. Shape of Optimum Rotor

While the above formulation gives the induced flow distribution and the resulting thrust and power coefficients for the optimum rotor, it is interesting to ask what the blade shape would be for such an optimum rotor. First, consider the form of the optimum twist distribution. It is assumed that the optimum rotor would be one for which the angle of attack \( \alpha = \theta - \phi \) is set at the fixed point for the maximum \( C_l/C_d \) at each blade section. Based on Eq. (6), it follows that the optimum blade twist would be given by:

\[
\theta = \alpha + \arctan\left(\frac{\eta + v_0}{r}\right)
\]

where \( \alpha \) is the angle of attack for minimum \( C_d/C_l \).

The optimum chord for the rotor follows directly from the fact that the \( C_l \) at every blade station is fixed at the optimum \( C_l \). From blade-element theory, this implies that:

\[
dL = \left(\frac{bpcC_l}{2}\right)[\Omega^2x^2 + U^2 - w^2]dx
\]

where \( b \) is the number of blades and \( c \) is the local chord. Blade-element lift—like momentum-theory lift—must be perpendicular to the vortex sheet. Therefore, Eq. (31) can be set equal to Eq. (10) in order to solve for the chord that will give the optimum induced flow \( w = w_0\cos(\phi) \). The resultant value for the normalized blade chord is:

\[
bc = \left(\frac{8r^2}{C_l}\right)\frac{\sqrt{\eta + v_0}}{\sqrt{(\eta + v_0)^2 + r^2}}
\]

This is the closed-form result for the optimum chord distribution.

It is also useful to present this result in terms of the local projected solidity (blade area projected on an annular ring divided by the ring area \( 2\pi r dr \)).

\[
\left(\frac{bc}{R}\right)\left(\frac{\cos(\phi)}{2\pi r}\right) = \left(\frac{4r^2}{C_l}\right)\frac{\sqrt{\eta + v_0}}{\sqrt{(\eta + v_0)^2 + r^2}}
\]

The total projected solidity \( \sigma' \) (analogous to the weighted solidity often used in helicopter design) can be found directly by integration of \( [bc/\pi R]\cos(\phi) \) from Eq. (33).

\[
\sigma' = \left(\frac{4}{C_l}\right)(\eta + v_0)^2\ln\left[1 + \frac{1}{(\eta + v_0)^2}\right] - \left(\frac{4}{C_l}\right)(\eta^2 + \eta v_0)\ln\left[1 + \frac{1}{(\eta^2 + \eta v_0)}\right]
\]

It is instructive to study the nature of the chord distribution (and projected chord distribution) in hover \( (\eta = 0) \). There, Eqs. (32–33) become, respectively:
The true solidity and the projected solidity in hover are, respectively:

\[ \frac{bc}{\pi R} = \left( \frac{8v_0^2}{C_i} \right) \left( \frac{1}{\sqrt{v_0^2 + r^2}} \right) \]  
\[ \text{(hover)} \quad (35) \]

\[ \frac{bc}{\pi R} \cos(\phi) = \left( \frac{8rv_0^2}{C_i} \right) \left( \frac{1}{v_0^2 + r^2} \right) \]  
\[ \text{(hover)} \quad (36) \]

The true solidity and the projected solidity in hover are, respectively:

\[ \sigma = \left( \frac{8v_0^2}{C_i} \right) \ln \left( \frac{1}{v_0} + \sqrt{1 + \frac{1}{v_0^2}} \right) ; \quad \sigma' = \left( \frac{4}{C_i} \right) v_0^2 \ln \left( 1 + \frac{1}{v_0^2} \right) \]  
\[ \text{(hover)} \quad (37) \]

In illustration of the above, consider \( \eta = 0, v_0 = 0.1 \) and \( C_i = 1.2 \). The various optimum values for this case are \( \sigma = 0.20 \) and \( \sigma' = 0.15 \). Despite these reasonable solidities, the chord in this blade becomes large as \( r \) approaches zero. The relationship in Eq. (35) yields a condition at \( r = 0 \), of \( \frac{bc}{\pi R} = 8v_0/C_i = 0.67 \) (or \( bc/\pi R = 2.09 \)) which would be impractical. However, one cannot truly continue any blade all the way to \( r = 0 \) (since there must be a hub); and the root section for this case \( (r < 0.1) \) has little effect on the total thrust or power. Thus, the chord very near the hub is not of practical consequence. The projected chord, on the other hand, goes to zero at the blade root; and the maximum projected chord in this example \( (r = v_0 = 0.1) \) is given by \( (bc/R) \cos(\phi) = 1.05 \). The local projected solidity at the annular ring \( (i.e., \text{the location } r = 0.1) \) is consequently \( (bc/R) \cos(\phi)/(2\pi r) = 1.67 \), implying some nominal overlap of blades—but no blade-to-blade interference.

It is also interesting to examine the blade projected chord distribution and solidity for the case of the maximum \( C_T \) possible (noted above at \( v_0^2 = 0.61, v_0 = 0.78, C_T = 0.238 \)). Although this configuration would probably not be considered for a lifting rotor (the IPE or figure of merit is only 0.45), it might be considered for the design of an air-moving fan in which the available fan area is fixed but one desires the maximum air flow with no constraint on power. (Maximum \( C_T \) implies maximum airflow.) The projected solidity for the fan with maximum air flow \( (v_0 = 0.78) \) is \( \sigma' = 1.57 \). Fans used for air handling often have blades with considerable overlap, and solidity in this range is reasonable.

The shape of the optimum rotor is related to the optimum circulation on that rotor. Appendix C derives the optimum circulation in terms is the basic inflow parameters.

4.8. Effect of Profile Drag

It is quite straightforward to determine the effect of profile drag on the thrust, power, and efficiency of the optimum rotor. This is possible because the above derivation of the optimum rotor is based on a momentum theory and blade element theory each of which assumes lift perpendicular to the vortex sheet. Since the profile drag is by definition along an axis parallel to the vortex sheet, it is easily included in the blade loads. What further simplifies the computation is the fact that profile drag does not alter momentum theory. This is because only the normal lift sheds vorticity that results in induced flow. Profile drag, on the other hand, heats the air and
produces a shear layer behind each blade; but these effects are negligible in terms of their influence on induced flow. This type of formulation gives the effect of drag on the ideal optimum. It is not, strictly-speaking, the true optimum for a case with drag. However, since the effect of profile drag is assumed a correction factor, one would expect the optimum not to change drastically due to drag. Thus, this approach should yield important insight; and we follow it.

Since it is specified that the local drag is perpendicular to the local lift (and has no effect on the momentum induced flow), we may write the net elemental thrust coefficient (due to both lift and drag) as a quantity that is proportional to:

\[ C_l \cos(\phi) - C_d \sin(\phi) = C_l \cos(\phi) \left[ 1 - \left( \frac{C_d}{C_l} \right) \tan(\phi) \right] \]  (38)

Similarly, the elemental power coefficient is proportional to:

\[ C_l \sin(\phi) + C_d \cos(\phi) = C_l \sin(\phi) \left[ 1 + \left( \frac{C_d}{C_l} \right) \cot(\phi) \right] \]  (39)

It follows that the existing integrals for \( C_T \) and \( C_P \) can be augmented with integrals that multiply \((C_d/C_l)\) in order to obtain the desired effect of drag on thrust and power.

Since both \( C_T \) and \( C_P \) involved the identical integral \( f(\eta, v_0) \) from Appendix B, the integrals for the effect of profile drag are directly derivable from that integral. According to Eqs. (38) and (39), the integral for \( f(\eta, v_0) \) can be multiplied by either the quantity \( \tan(\phi) = \left( \frac{(\eta + v_0)/r}{\eta + v_0} \right) \) or the quantity \( \cot(\phi) = \left[ r/(\eta + v_0) \right] \) in order to obtain the integrals that are to go with \((C_d/C_l)\) in \( C_T \) and \( C_P \), respectively. The integral for \( C_T \) is thus augmented by: 1.) multiplying the original integral by \( 2(\eta + v_0) \) and 2.) using \( g(\eta, v_0) \) rather than \( f(\eta, v_0) \) where \( g(\eta, v_0) \) is worked out in detail and given in Appendix B as:

\[ g(\eta, v_0) = 1 - \left( \eta + \frac{3v_0}{2} \right) \arctan \left( \frac{1}{\eta + v_0} \right) + \left( \frac{v_0}{2} \right) \left[ \frac{\eta + v_0}{1 + (\eta + v_0)^2} \right] \]  (40)

The integral for the effect of drag on \( C_P \) is similar to the original \( C_P \) integral except that: 1.) it has the added factor \((2/3)/(\eta + v_0)\) and 2.) it uses the integral \( h(\eta, v_0) \) rather than the integral \( f(\eta, v_0) \), where \( h(\eta, v_0) \) is given in Appendix B. It is:

\[ h(\eta, v_0) = 1 - 3(\eta + 2v_0)(\eta + v_0) + 3 \left( \eta + \frac{5v_0}{2} \right)(\eta + v_0)^2 \arctan \left( \frac{1}{\eta + v_0} \right) \quad \] 
\[ - \left( \frac{3v_0}{2} \right) \left[ \frac{(\eta + v_0)^3}{1 + (\eta + v_0)^2} \right] \]  (41)

From these, the formulas for the net thrust coefficient \( C_{TN} \) and the total power coefficient \( C_{PT} \) follow directly:

\[ C_{TN} = 2v_0(\eta + v_0)f(\eta, v_0) - 4 \left( \frac{C_d}{C_l} \right) v_0(\eta + v_0)^2 g(\eta, v_0) \]  (42)
\[ C_{PT} = 2v_0^2(\eta + \nu_0)f(\eta, \nu_0) + \left(\frac{4}{3}\right) \left(\frac{C_d}{C_l}\right) v_0^2 h(\eta, \nu_0) \] (43)

The total rotor efficiency (TRE) can then be found by comparison with the Glauert minimum for a specified \( C_T \).

\[
TRE = \frac{C_{TN}^2}{C_{PT} \left[ \eta + \sqrt{\eta^2 + 2C_{TN}} \right]}
\] (44)

When the expressions from Eqs. (42-43) are explicitly placed into Eq. (44), the result can be simplified to:

\[
TRE = \frac{(\eta + \nu_0)^2 \left[ f(\eta, \nu_0) - 2 \left(\frac{C_d}{C_l}\right) (\eta + \nu_0) g(\eta, \nu_0) \right]^2}{\left[ (\eta + \nu_0)f(\eta, \nu_0) + \left(\frac{2}{3}\right) \left(\frac{C_d}{C_l}\right) h(\eta, \nu_0) \right] \left[ \eta + \sqrt{\eta^2 + \frac{C_{TN}}{2}} \right]}
\] (45)

For low-inflow rotors, the effect of drag on \( C_T \) is negligible as compared to the effect of drag on \( C_p \). Thus, an accurate approximation for the efficiency of a low-inflow optimum rotor is given by:

\[
TRE = \frac{2(\eta + \nu_0)f(\eta, \nu_0)}{\left[ 1 + \left(\frac{2}{3}\right) \left(\frac{1}{\eta + \nu_0}\right) \left(\frac{C_d}{C_l}\right) \left(\frac{h(\eta, \nu_0)}{f(\eta, \nu_0)}\right) \right] \left[ \eta + \sqrt{\eta^2 + 4\nu_0(\eta + \nu_0)f(\eta, \nu_0)} \right]}
\] (46)

Equation (46) is a good approximation for low inflow rotors—with Eq. (45) being applicable to all lifting rotors.

4.9. Global Optimum for Rotor

From the previous section, one can see that, while the induced power efficiency (IPE) is highest for low inflow ratios (\( \eta + \nu_0 = 0 \)), the total rotor efficiency (TRE) goes to zero at low inflow ratios. This is because the profile drag assumes an increasingly larger proportion of total rotor power as the \( C_T \) becomes smaller. Therefore, for any given lift and drag coefficients, there is an optimum \( \nu_0 \) (i.e., an optimum \( C_T/C_l \)) that will give the greatest rotor efficiency.

As an example, consider the case of hover (\( \eta = 0 \)) for the condition that \( C_d/C_l \ll 1 \). In that case, all physically meaningful optimal solutions are inflow ratios less than unity; and the effect of drag on lift is negligible. Thus, Eq. (46) can be used to find the appropriate TRE (i.e., Figure of Merit) for a hovering rotor.

\[
TRE = Figure of Merit = \frac{\sqrt{f(0, \nu_0)}}{\left[ 1 + \left(\frac{C_d}{C_l}\right) \left(\frac{2h(0, \nu_0)}{3\nu_0 f(0, \nu_0)}\right) \right]}
\] (47)
The numerator in Eq. (47) is identical to that developed in Ref. [7]. The denominator term multiplying $C_d/C_l$ is virtually the same as in Ref. [7] except that Ref. [7] is missing a term of the order $\nu_0^2$. However, for inflows considered in Ref. [7], that term is negligible. Here, on the other hand, we consider a wider range of loading conditions and look more in detail at the shape of the optimum blade.

Setting the derivative of $TRE$ with respect to $v_0$ equal to zero gives the condition for the loading that gives the maximum figure of merit.

\[
\nu_0^2 = \left(\frac{C_d}{C_l}\right) \left[-\frac{4h}{3f'} - \frac{2v_0h}{f} + \frac{4v_0h'}{3f'}\right]
\]  

(48)

For small $C_d/C_l$, $v_0$ is also small; and one may use the Taylor series for $f(0, \nu_0)$ and $h(0, \nu_0)$:

\[
f(0, \nu_0) = 1 + 4\nu_0^2 \ln(\nu_0) \quad ; \quad h(0, \nu_0) = 1 - 6\nu_0^2
\]

(49)

The result is a simple approximation for the $\nu_0$ that results in optimum figure of merit.

\[
\nu_0^3 \ln\left(1 + \frac{1}{\nu_0^2}\right) = \left(\frac{1}{3}\right) \left(\frac{C_d}{C_l}\right)
\]

(50)

From the solution to Eq. (50) for the optimum $\nu_0^2$ with a given $C_d/C_l$, one can then find the thrust, power, $TRE$, and solidity of the global optimum rotor.

For example, consider the rotor design given in Table 1 for $C_l = 1.2$ and $C_d = 0.012$. In that Table, $\nu_0 = 0.10$; and the figure of merit ($TRE$) is 0.8937. According to Eq. (50), the optimum parametric loading for that lift and drag coefficient would be $\nu_0 = 0.0882$. Table 2 shows the parameters for this global optimum.

4.10. Numerical Results

Figure 2 shows the optimum induced-flow distribution in hover for several values of the nominal flow $\nu_0$. The flow normalized on $\nu_0$ is plotted in Fig. 2a; and the absolute inflow is plotted in Fig. 2b to give additional insight. The helical nature of the flow can be seen in the figures; and this is what makes for the Glauert optimum. In the limit as $\nu_0$ becomes large, this flow becomes linear and equal to the local blade speed. Since the inflow $w$ is defined along the optimum tilt direction, this implies that—in the limit of large $\nu_0$—the inflow cancels the local velocity, $\Omega x$. Thus, thrust goes to zero in the limit of large $\nu_0$.

Figures 3 and 4 present the optimum thrust and induced power as functions of $\nu_0$ for various inflow ratios (including the case of hover, $\eta = 0$). Note that the optimum thrust in hover peaks at a value of $C_T = 0.238$ at $\nu_0 = 0.78$. Although not practical for a lifting rotor, this would be a realistic design for a fan to provide maximum air flow (independent of the required power). Based on Eq. (56), given later in this paper, the maximum possible $C_T$ is obtained for a
constant value of $\omega = 1$ and is easily shown to be equal to 0.25. Thus, the maximum value of $C_T$ for an optimum-efficiency Betz distribution (0.238) is very close to the optimum $C_T$ for any rotor. Figure 5 gives the induced power efficiency as a function of $v_0$.

One can also plot the IPE versus $C_T$, as is done in Figure 6. However, the IPE plotted against $C_T$ doubles back on itself and heads toward the origin—since there are two values of $v_0$ and IPE for every $C_T$ below 0.238 (and none for $C_T$ larger). Figure 7 compares the induced power efficiency (IPE) of the ideal rotor with the total rotor efficiency (TRE)—which includes drag. The result given is for hover ($\eta = 0$) and $C_d/C_t = 0.010$.

Figure 8 shows the effect of $v_0$—both on the total solidity and on the projected solidity (i.e., weighted solidity) of the optimum rotor. Note that—at lower values of $v_0$—the computed solidities are typical of lifting rotors, and—at higher values of $v_0$—they are typical of lifting fans. It is interesting to consider the case of a typical lifting rotor in hover, $v_0 = 0.1$, (previously analyzed in Table 1) but now analyzed in more detail. Figures 9-10 give the twist distribution, chord distribution, and projected chord distribution for this ideal rotor. The table and figures show that this would be a realistic design with a total rotor efficiency of 0.894.

Figure 11 provides a plot of $v_0$, $C_T/\sigma$, and $C_T/\sigma'$ for the global optimum rotor as functions of the drag-to-lift ratio, $C_d/C_t$. Note that, as $v_0$ approaches zero, the thrust-to-solidity ratios approach constants. This is due to the limiting behavior of Eqs. (20), (37), and (50) and the fact that a fixed $C_t$ has been chosen for the blades. Finally, Fig. 12 provides a universal plot for the circulation $\Gamma/[2v_0(\eta + v_0)]$ as a function of normalized radial position $r/(\eta + v_0)$.

### 4.11. Comparison with Previous Hover Results

Our optimum inflow distribution in Eqs. (6) - (8) is identical to the Betz distribution used for the hover analysis of Glaeuert in Ref. [1]. As a result, the IPE calculated here in Fig. 6 for the range of $0 < C_T < 0.06$ agrees with the Figure of Merit computed in Ref. [1] as shown in Appendix D. These same results from Glaeuert are also cited in Ref. [10] which offers an alternate expression for optimum inflow that is based on a combination of Bernoulli’s principle and the concepts of angular momentum. The alternate expression gives a figure of merit lower than the Glaeuert result, and Ref. [10] claims that the new result is more accurate than the Glaeuert result. (The figure from Ref. [10] is repeated in that Appendix for comparison.)

There are reasons why the result of Ref. [10] seem suspect. First, when specialized to hover (i.e., $\eta = 0$), the formulas for thrust and power given here in Eqs. (11) and (12) are identical to the formulas presented in Eqs. (13) and (14) of Ref. [10]. Thus, one would expect that the optimum inflow distribution for either approach would be identical given that they both have the assumptions of momentum theory. Second, the “optimum” inflow distribution of Ref. [10] has a finite axial induced velocity at the rotor center in conjunction with a zero angular velocity. From vortex theory (which we have shown to be consistent with momentum considerations), a finite inflow at the blade root would imply a finite circulation at the blade root, which would imply a root vortex that would create an infinite swirl velocity at the rotor center. Thus, the optimum inflow distribution of Ref. [10] seems to be internally inconsistent.

How is it that our momentum theory and the momentum theory of Ref. [10] yield different results? Momentum theory is an approximation based on fairly simple assumptions. Although the assumptions are in some ways ad hoc, they nonetheless should be internally consistent—which then leads to consistent, useful equations. The consistency of momentum theory is one of
the reasons why it has been a staple of rotor analysis through the years. This would lead one to suspect that Eq. (51) is invalid not because momentum theory is invalid—but because the particular assumptions used in the momentum theory of Ref. [10] might be inconsistent.

The fact that the momentum theory combined with Bernoulli’s equation in Ref. [10] leads to inconsistencies was pointed out by one of the authors in a later paper, Ref. [11]. There, the inconsistency is explained by a claim that the general momentum equation (for thrust on the disk) is invalid. Based on Eq. (18) of Ref. [10], this “invalid” equation is:

\[
dT = 2\pi \rho [u_1^2 - p_0 + p_1] x_1 dx_1 = 2\pi \rho \left[ \frac{u_1^2}{2} + \left( \Omega - \frac{\omega_1}{2} \right) \omega_1 x_1^2 \right] x_1 dx_1 \tag{51}
\]

where \( u_1 \) is the axial velocity in the far wake, \( \omega_1 \) is the angular velocity of the fluid in the far wake, \( p_0 \) is the ambient air pressure, \( p_1 \) is the pressure in the far wake, and \( x_1 \) is the radial position in the far wake \( 0 < x_1 < R_1 \). The point made in Ref. [11] is that this equation from momentum theory cannot result in a far-wake pressure that balances centrifugal forces. Thus, the flow will in reality mix in the far field—which violates one of the assumptions of momentum theory. We will not treat this issue here, as it is covered in our Appendix D.

However, there is a more direct inconsistency in Ref. [10]; and it can be traced to their assumption about wake contraction. In particular, in the paragraph following Eq. (39) of Ref. [10], the authors note that—to solve for the optimum rotor—one needs to find a mapping from \( r = x/R \) at the rotor disk to \( r_1 = x_1/R \) in the far wake. The mapping takes the form of \( r_1^2 = f(r) \). The mapping then defines the contraction ratio, \( r_1/r = K(r) \). In the text after Eq. (39), they write:

The function \( r(r_1) \) can be obtained as a part of the solution of the complete flow field between the disk and the ultimate wake. . . Such a complete solution constitutes a major task demanding a large computational effort. Accordingly, in the present work, an approach which requires neither the use of Eq. (38) nor the solution for the entire flow field is developed. In this approach, the local contraction ratio \( K = r_1/r \) is taken as independent of the radial position \( r \) (or \( r_1 \)). It is recognized that in general the local contraction ratio is a function of the radial position. The results of Ref. 10, however, indicate that for heavily loaded free-running propellers, the contraction ratio is nearly independent of the radial position, except near the axis of the propeller. Ref. [10]

They then offer the approximation \( K = 1/\sqrt{2} \)—their Eq. (46)—yielding relation from Eqs. (40-41):

Ref. [10]: \( r_1 = \frac{r}{\sqrt{2}} \), \( u_1 = 2u \), \( \omega_1 = 2\omega \) \tag{52}

The problem with the above is that it is inconsistent with their own momentum theory and Bernoulli equation. In particular, in their Eq. (3), they have assumed that there is a negative pressure in the far wake that is equal to the gradient of the centrifugal force on the rotating flow and which therefore balances the centrifugal flow to hold the wake together. This implies that
the head in the far wake has not expanded to atmospheric pressure (i.e., \( p_1 \neq p_0 \)) but is instead smaller than \( p_0 \). This implies that \( u_1 \) and \( \omega_1 \) cannot have expanded to \( 2u \) and \( 2\omega \), respectively. In fact, since the pressure is negative in the far wake, \( u_1 \) and \( \omega_1 \) must be larger than \( u \) and \( \omega \), implying that the contraction ratio must be \( K < 1/\sqrt{2} \). This inconsistency makes itself known in their Eq. (16) for thrust given below:

\[
T = 2\pi \left[ \rho \int_0^{R_1} u_1^2 x_1 \, dx_1 - \int_0^{R_1} (p_0 - p_1) x_1 \, dx_1 \right]
\] (53)

The inconsistent assumption impacts this equation in that they use \( u_1 = 2u \) in the first term (which implies \( p_1 = p_0 \)), but then they apply a \( (p_1 - p_0) \) that is not equal to zero from their Eq.(3). It is this that results in an inconsistency in Eq. (18) of Ref. [10]—equivalent to Eq. (51) above. This inconsistency causes Ref. [10] to double-count the loss in thrust, thus explaining why they have a lower figure of merit than Glauert.

### 4.12. Optimum Contraction Ratio

In Appendix D, we show that—despite the pessimism of Ref. [10]—one can indeed find a consistent solution for the contraction ratio. We also show in Appendix D that, when this consistent contraction ratio is used, the proper application of Eq. (51)—when mapped to the rotor disk—results is equations that are in agreement with the equations of this present paper. With this consistent approach, one can write three equivalent formulas for thrust in hover.

\[
dT = 2\pi \rho \left[ u^2 + \left( \Omega - \frac{\Omega}{2} \right) \frac{\omega x^2}{2} \right] \, dx
\] (54)

\[
dT = 2\pi \rho (2u^2) \, dx
\] (55)

\[
dT = 2\pi \rho \left[ \left( \Omega - \frac{\Omega}{2} \right) \omega x^2 \right] \, dx
\] (56)

Equation (54) is the consistent thrust that is derived from Eq. (53) with the proper contraction ratio; Eq. (55) is the traditional equation from axial momentum theory; and Eq. (56) is identical to either Eq. (12) of Ref. [10] or Eq. (11) of our paper. These three formulas are not independent, because the first can be written as one-half the sum of the second two. Thus, any two of them comprise a consistent momentum theory.

It is also possible to formulate a consistent momentum theory with swirl under the classic contraction assumption \( K = 1/\sqrt{2} \). Under those assumptions, in order to obtain a consistent momentum theory, the unbalanced forces on the boundary of the momentum tube must be included in the thrust balance of Eq. (53).
\[ T = 2\pi \left[ \rho \int_0^{R_1} u_1^2 x_1^3 dx_1 - \int_0^{R_1} (p_0 - p_1)x_1 dx_1 \right] + F_L - F_U \quad (57) \]

where \( F_L \) and \( F_U \) are the reaction forces on the momentum tube below the disk and above the disk, respectively.

Recall that momentum theory is an artificial construct with curved momentum tubes that can have pressures on the tube walls due to lower pressure above the disk and higher pressure below the disk. Under a consistent contraction ratio, the vertical component of the inward forces on the upper contracting tube cancel the vertical forces on the tube below the disk \((F_L = F_U)\). For example, in the absence of swirl, the consistent contraction ratio is \(1/\sqrt{2}\) for all annular tubes. This is what makes the theory consistent. However, when a contraction ratio of \(1/\sqrt{2}\) is used simultaneously with the addition of wake swirl, the pressure head due to swirl, \(\rho \omega_1^2 x_1^2 / 4\), occurs as a jump across the disk so that there is no comparable upward force on the tube above the disk to balance the force below the disk.

The unbalanced pressure reactions on \(F_L - F_U\) gives an unbalanced pressure \(\rho \omega_1^2 x_1^2 / 4\). The integration of this pressure over all of the annular rings gives the unbalanced vertical force on the momentum tubes, which alters Eq. (57) to become:

\[ T = 2\pi \left[ \rho \int_0^{R_1} \left( u_1^2 x_1^3 + \frac{\omega_1^2 x_1^2}{4} \right) dx_1 - \int_0^{R_1} (p_0 - p_1)x_1 dx_1 \right] \quad (58) \]

Also from Ref. [12]:

\[ p_0 - p_1 = \frac{1}{2} \rho u_1^2 - \rho \left( \Omega - \frac{1}{2} \omega_1 \right) \omega_1 x_1^3 \quad (59) \]

Substitution of Eq. (59) back into Eq. (58) yields:

\[ T = 2\pi \rho \int_0^{R_1} \left[ u_1^2 x_1^3 + \frac{\omega_1^2 x_1^2}{4} - \frac{1}{2} u_1^2 x_1^3 + \left( \Omega - \frac{1}{2} \omega_1 \right) \omega_1 x_1^3 \right] dx_1 \quad (60) \]

Simplifying Eq. (60), one can obtain:

\[ T = 2\pi \rho \int_0^{R_1} \left[ \frac{1}{2} u_1^2 + \left( \Omega - \frac{1}{4} \omega_1 \right) \omega_1 x_1^2 \right] x_1 dx_1 \quad (61) \]

which is the consistent thrust expression. If one then substitutes \( u_1 = 2u, \omega_1 = 2\omega, \) and \( x_1 = x/\sqrt{2} \) into Eq. (61), one obtains the identical form of the thrust given in Eqs. (54) - (56). Reference [10] gives a version of Eq. (61) with a 1/2 on the second term instead of 1/4, and
this is why their figure of merit is lower than that of Glauert. It should also be noted that, although the pressures and velocities in the far wake differ depending upon the assumptions of any given momentum theory, a consistent momentum theory will always result in this same set of consistent equations for thrust and power at the rotor disk.

The above corrected thrust equation can be carried through the derivation of Ref. [10] to the optimality criterion given in their Eq. (32). Namely, that for the optimum distribution, it must hold that—in the following—the parameter $n$ remains a constant:

$$\int_{0}^{x_1} \left( \frac{\omega_1}{u_1} \right) \xi^3 d\xi + \frac{\omega_1}{u_1} x_1^4 (\Omega - \omega_1)$$  \hspace{1cm} (62)

The left-hand side of Eq. (62) arises from a variation of thrust with respect to $\omega_1$ in Eq. (51). Therefore, when corrected with the consistent formula in Eq. (61), the left-hand side of Eq. (62) becomes the consistent value, $2n x_1^2 (2\Omega - \omega_1/2)$.

We will now show that the optimality criterion under this corrected momentum assumption is indeed fulfilled by the Betz distribution. For simplicity of algebra, we use the simpler of the alternate versions ($K = 1/\sqrt{2}$) which gives $x_1 = x/\sqrt{2}$, $u_1 = 2u$, and $\omega_1 = 2\omega$. (Recall that either consistent assumption yields the same thrust and moment equations and therefore the same optimum flow.) We divide the corrected Eq. (62) by $\Omega Rx_1^2$; we change to the notation of this present paper, and we substitute the optimal Betz distribution from Eq. (6). After some algebra, this yields:

$$\left( \frac{n}{v_0 R} \right) \left[ 2 - \frac{2v_0^2}{(v_0^2 + r^2)} \right] = \left[ 2 - \frac{2v_0^2}{(v_0^2 + r^2)} \right]$$  \hspace{1cm} (63)

Since the optimality criterion is the condition that $n$ be a constant, Eq. (54) proves that the Betz distribution is indeed optimum with the appropriate constant give by $n = v_0 R$.

With this change in assumption of the analysis of Ref. [10], we find that the use of either momentum theory, vortex theory, or Bernoulli’s equation each results in the identical optimum hover inflow with the same power and figure of merit as presented here. This optimum has both components of induced velocity going to zero at the rotor center, and it has a total local lift vector always parallel and opposite to the local induced-flow vector. The derivation here, in contrast to the derivations of Glauert or Ref. [10], however, requires only a single momentum balance and does not need a separate balance of angular momentum.

### 4.13. Results in Far Wake

Figure 13 gives the universal contraction ratio $K$ versus $\bar{r}$. Both the numerical solution to the differential equation $K$ and the approximate expression $K'$ are given, and they are virtually the same. (See Appendix D.) The contraction approaches 0.707 away from the rotor centerline. Figure 14 shows the normalized axial and swirl velocities in the far wake. They both begin at a large value of 8 at the center. The axial velocity approaches 2.0 and the circumferential velocity approaches 0.0 as one moves away from the center. Figure 15 compares the normalized pressure
in the far wake $P_W$ with the pressure that would be needed to balance centrifugal forces $P_C$. As discussed in Appendix D, a momentum theory cannot give a pressure distribution in the far wake that exactly balances centrifugal forces, as proven in Ref. [11]. Nonetheless, the momentum pressure is qualitatively accurate. Figure 16 gives the normalized pressure divided by the pressure drop at the rotor tip for the case of the globally optimum rotor in Table 2, $v_0 = 0.115$. The horizontal scale is the radial position divided by $R$. Note that the residual pressure in the wake due to wake rotation is confined to a small region near the central axis, as verified by experiments.

4.14. Summary and Conclusions

A closed-form solution has been given for the optimum induced flow distribution, thrust, and induced power of a lifting rotor—valid from hover through high-speed climb (i.e., a propeller). Formulas are also provided for the induced power efficiency, which may be useful in comparison with rotor optimization from comprehensive codes. In addition, expressions are derived for the optimum pitch and chord distributions of the ideal rotor and for the effect of profile drag on the optimum performance. The optimum circulation is also found. It follows the same pattern as the Betz optimum as being proportional to the square of the cosine of the inflow angle, Appendix C.

Although results are for an infinite number of blades, it is expected that the Prandtl correction function, shown by Goldstein to be accurate for lightly-loaded rotors, would extend the results to a finite number of blades.

It was found that the claim of Ref. [10]—that the optimum hovering rotor is different from that found by Glauert—is inaccurate. The claim arises from an inconsistency in the momentum theory used in Ref. [10]. When that inconsistency is corrected, one can prove that the optimum hovering rotor is indeed the one proposed by Betz and analyzed by Glauert. With the corrections to the assumptions of Ref. [10] we also have been able to find a universal expression for the correct contraction ratio for a rotor in hover—including wake swirl in the far wake.
### 4.15. Tables

#### Table 4.1. Parameters of a Locally Optimal Lifting Rotor in Hover

<table>
<thead>
<tr>
<th>Flow</th>
<th>$\eta = 0$ (hover)</th>
<th>$\nu_0 = 0.10$</th>
<th>$\nu = 0.1r/\sqrt{r^2 + 0.01}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lift/Drag</td>
<td>$C_d = 0.012$</td>
<td>$C_l = 1.20$</td>
<td>$C_d/C_l = 0.01$</td>
</tr>
<tr>
<td>Integrals</td>
<td>$f = 0.9176$</td>
<td>$g = 0.7843$</td>
<td>$h = 0.9509$</td>
</tr>
<tr>
<td>Thrust</td>
<td>$C_T = 0.0184$</td>
<td>$C_{TN} = 0.0183^a$</td>
<td></td>
</tr>
<tr>
<td>Power</td>
<td>$C_{pl} = 0.0018$</td>
<td>$C_{pt} = 0.0020$</td>
<td></td>
</tr>
<tr>
<td>Efficiency</td>
<td>$IPE = 0.9579$</td>
<td>$TRE = 0.8937(figure of merit)$</td>
<td></td>
</tr>
<tr>
<td>Solidity</td>
<td>$\sigma = 0.1999$</td>
<td>$\sigma' = 0.1538$</td>
<td></td>
</tr>
<tr>
<td>Thrust/Solidity</td>
<td>$C_T/\sigma = 0.0918$</td>
<td>$C_T/\sigma' = 0.1193$</td>
<td></td>
</tr>
</tbody>
</table>

$^a$Effect of drag on thrust is negligible.

#### Table 4.2. Parameters of a Globally Optimal Lifting Rotor in Hover

<table>
<thead>
<tr>
<th>Flow</th>
<th>$\eta = 0$ (hover)</th>
<th>$\nu_0 = 0.0882$</th>
<th>$\nu = 0.0882r/\sqrt{r^2 + 0.0078}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lift/Drag</td>
<td>$C_d = 0.012$</td>
<td>$C_l = 1.20$</td>
<td>$C_d/C_l = 0.01$</td>
</tr>
<tr>
<td>Integrals</td>
<td>$f = 0.9321$</td>
<td>$g = 0.8078$</td>
<td>$h = 0.9609$</td>
</tr>
<tr>
<td>Thrust</td>
<td>$C_T = 0.0145$</td>
<td>$C_{TN} = 0.0145$</td>
<td></td>
</tr>
<tr>
<td>Power</td>
<td>$C_{pl} = 0.0013$</td>
<td>$C_{pt} = 0.0014$</td>
<td></td>
</tr>
<tr>
<td>Efficiency</td>
<td>$IPE = 0.9654$</td>
<td>$TRE = 0.8936(figure of merit)$</td>
<td></td>
</tr>
<tr>
<td>Solidity</td>
<td>$\sigma = 0.1618$</td>
<td>$\sigma' = 0.1260$</td>
<td></td>
</tr>
<tr>
<td>Thrust/Solidity</td>
<td>$C_T/\sigma = 0.0895$</td>
<td>$C_T/\sigma' = 0.1150$</td>
<td></td>
</tr>
</tbody>
</table>
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4.18. Appendix A: Summary of Formulas

The optimum induced-flow distribution and inflow angle at the blade are:

\[ v(r) = \frac{\nu_0 r}{\sqrt{(\eta + \nu_0)^2 + r^2}} \]  
\[ \phi = \arctan \left( \frac{\eta + \nu_0}{r} \right) \]  

The thrust, induced power, and induced power efficiency (IPE) of this optimum rotor are given by:

\[ C_T = 2\nu_0(\eta + \nu_0)f(\eta, \nu_0) \]  
\[ C_{Pl} = 2\nu_0^2(\eta + \nu_0)f(\eta, \nu_0) = \nu_0 C_T \]  
\[ IPE = \frac{2(\eta + \nu_0)f(\eta, \nu_0)}{\eta + \sqrt{\eta^2 + 4\nu_0(\eta + \nu_0)f(\eta, \nu_0)}} \]

where \( f(\eta, \nu_0) \) is the swirl-loss function:

\[ f(\eta, \nu_0) = \frac{1 + (\eta + 2\nu_0)(\eta + \nu_0)}{1 + (\eta + \nu_0)^2} - (\eta + 2\nu_0)(\eta + \nu_0)\ln \left[ 1 + \frac{1}{(\eta + \nu_0)^2} \right] \]

The optimum chord distribution for a given \( C_l \) is:

\[ \frac{bc}{\pi R} = \left( \frac{8r^2}{C_l} \right) \left[ \frac{\nu_0(\eta + \nu_0)}{[r^2 + \eta(\eta + \nu_0)]\sqrt{(\eta + \nu_0)^2 + r^2}} \right] \]

and the total projected (i.e., weighted) solidity based on \([bc/(\pi R)]\cos(\phi)\) is:

\[ \sigma' = \left( \frac{4}{C_l} \right)(\eta + \nu_0)^2 \ln \left[ 1 + \frac{1}{(\eta + \nu_0)^2} \right] - \left( \frac{4}{C_l} \right)(\eta^2 + \eta \nu_0) \ln \left[ 1 + \frac{1}{(\eta^2 + \eta \nu_0)} \right] \]

The thrust and power coefficients—including the effect of profile drag are:

\[ C_{TN} = 2\nu_0(\eta + \nu_0)f(\eta, \nu_0) - 4 \left( \frac{C_d}{C_l} \right) \nu_0(\eta + \nu_0)^2 g(\eta, \nu_0) \]  
\[ C_{PT} = 2\nu_0^2(\eta + \nu_0)f(\eta, \nu_0) + \left( \frac{4}{3} \right) \left( \frac{C_d}{C_l} \right) \nu_0^2 h(\eta, \nu_0) \]
where \( g(\eta, \nu_0) \) and \( h(\eta, \nu_0) \) are defined as:

\[
g(\eta, \nu_0) = 1 - \left( \eta + \frac{3\nu_0}{2} \right) \arctan\left( \frac{1}{\eta + \nu_0} \right) + \left( \frac{\nu_0}{2} \right) \frac{\eta + \nu_0}{[1 + (\eta + \nu_0)^2]} \tag{A11}
\]

\[
h(\eta, \nu_0) = 1 - 3(\eta + 2\nu_0)(\eta + \nu_0) + 3\left( \eta + \frac{5\nu_0}{2} \right)(\eta + \nu_0)^2 \arctan\left( \frac{1}{\eta + \nu_0} \right) - \left( \frac{3\nu_0}{2} \right) \frac{(\eta + \nu_0)^3}{[1 + (\eta + \nu_0)^2]} \tag{A12}
\]

The **total rotor efficiency** including profile drag is, therefore:

\[
TRE = \frac{C_{TN}^2}{C_{PT}(\eta + \sqrt{\eta^2 + 2C_{TN}})}
\tag{A13}
\]

### 4.19. Appendix B: Loading Integrals

The kernel of the loading integrals for thrust and power is given in Eq. (18) as:

\[
f(\eta, \nu_0) = \int_0^1 2 \left[ \frac{r^2 + \eta(\eta + \nu_0)}{[(\eta + \nu_0)^2 + r^2]^2} \right] r^3 dr
\tag{B1}
\]

This can be rewritten in a form more conducive to integration

\[
f(\eta, \nu_0) = \int_0^1 \left[ 1 - \frac{(\eta + 2\nu_0)(\eta + \nu_0)}{(\eta + \nu_0)^2 + r^2} + \frac{\nu_0(\eta + \nu_0)^3}{[(\eta + \nu_0)^2 + r^2]^2} \right] 2rdr
\tag{B2}
\]

This integrates directly into

\[
f(\eta, \nu_0) = 1 - (\eta + 2\nu_0)(\eta + \nu_0) \ln\left[ 1 + \frac{1}{(\eta + \nu_0)^2} \right] + \frac{\nu_0(\eta + \nu_0)}{1 + (\eta + \nu_0)^2}
\tag{B3}
\]

which is equivalent to Eq. (19).

For the case of the \( C_d/C_l \) integral in the thrust equation, one must multiply the integrand of Eq. (B1) by \( \tan(\phi) = (\eta + \nu_0)/r \). The resulting thrust is thus multiplied by \( 2(\eta + \nu_0) \) with a new integral defined as lacking one power of \( r \), \( g(\eta, \nu_0) \):

\[
g(\eta, \nu_0) = \int_0^1 \left[ 1 - \frac{(\eta + 2\nu_0)(\eta + \nu_0)}{(\eta + \nu_0)^2 + r^2} + \frac{\nu_0(\eta + \nu_0)^3}{[(\eta + \nu_0)^2 + r^2]^2} \right] dr
\tag{B4}
\]

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Each term in Eq. (B4) can be integrated directly giving:

\[ g(\eta, v_0) = 1 - \left( \eta + v_0 \right) \arctan \left( \frac{1}{\eta + v_0} \right) + \left( \frac{v_0}{2} \right) \left[ \arctan \left( \frac{1}{\eta + v_0} \right) + \frac{(\eta + v_0)}{1 + (\eta + v_0)^2} \right] \]

(B5)

\[ g(\eta, v_0) = 1 - \left( \eta + \frac{3v_0}{2} \right) \arctan \left( \frac{1}{\eta + v_0} \right) + \left( \frac{v_0}{2} \right) \left[ \arctan \left( \frac{1}{\eta + v_0} \right) + \frac{(\eta + v_0)}{1 + (\eta + v_0)^2} \right] \]

(B6)

For the case of the power equation, one must multiply the integrand of Eq. (B1) by \( \cot(\phi) = r/(\eta + v_0) \). The resulting integral is similar to the power integral but multiplied by \( [(2/3)/(\eta + v_0)] \) with an integral with an extra power of \( r \), \( h(\eta, v_0) \):

\[ h(\eta, v_0) = \int_0^1 \left[ 1 - \frac{(\eta + 2v_0)(\eta + v_0)}{(\eta + v_0)^2 + r^2} \right] + \frac{v_0(\eta + v_0)^3}{[(\eta + v_0)^2 + r^2]^2} 3r^2 dr \]

(B7)

The integrand of Eq. (B7) can also be rearranged for direct integration as follows:

\[ h(\eta, v_0) = \int_0^1 \left[ 3r^2 - 3(\eta + 2v_0)(\eta + v_0) + \frac{3(\eta + 2v_0)(\eta + v_0)^3}{[(\eta + v_0)^2 + r^2]^2} \right. \]

\[ + \left. \frac{3v_0(\eta + v_0)^3r^2}{[(\eta + v_0)^2 + r^2]^2} \right] dr \]

(B8)

The integral follows directly as:

\[ h(\eta, v_0) = 1 - 3(\eta + 2v_0)(\eta + v_0) + 3(\eta + 2v_0)(\eta + v_0)^2 \arctan \left( \frac{1}{\eta + v_0} \right) \\
+ \left( \frac{3v_0}{2} \right) (\eta + v_0)^2 \left[ \arctan \left( \frac{1}{\eta + v_0} \right) - \frac{\eta + v_0}{1 + (\eta + v_0)^2} \right] \]

(B9)

This can then be simplified to:

\[ h(\eta, v_0) = 1 - 3(\eta + 2v_0)(\eta + v_0) + 3 \left( \eta + \frac{5v_0}{2} \right)(\eta + v_0)^2 \arctan \left( \frac{1}{\eta + v_0} \right) \\
- \left( \frac{3v_0}{2} \right) \frac{(\eta + v_0)^3}{[1 + (\eta + v_0)^2]} \]

(B10)

This completes the momentum integrals.
4.20. Appendix C: Optimum Circulation

It is interesting to write the circulation distribution for the optimum rotor. From the relationship in Eq. (10) and the definition of circulation, we can write an expression for the total circulation of all blades at a radial location, \( x \).

\[
\frac{dL}{dx} = \rho u \Gamma = 4\pi \rho \omega x \left[ U + w \cos(\phi) \right]
\]

(C1)

If we define a nondimensional \( \bar{\Gamma} = \Gamma / (2\pi \Omega R^2) \), then Eq. (C1) becomes:

\[
\bar{\Gamma} \sqrt{r^2 + \eta^2 - \nu^2} = 2\nu r [\eta + \nu \cos(\phi)]
\]

(C2)

From Eqs. (6) - (9) for the optimum rotor, we can solve for \( \bar{\Gamma} \).

\[
\bar{\Gamma} = \frac{2\nu_0 r^2 (\eta + \nu_0)}{(\eta + \nu_0)^2 + r^2}
\]

(C3)

This can also be written in terms of the optimum inflow angle, \( \phi \).

\[
\bar{\Gamma} = 2\nu_0 (\eta + \nu_0) \cos^2(\phi) = 2\nu_0 r \cos(\phi) \sin(\phi)
\]

(C4)

To write Eq. (C3) in universal form,

\[
\frac{\bar{\Gamma}}{2\nu_0 (\eta + \nu_0)} = \frac{\tilde{r}^2}{1 + \tilde{r}^2}
\]

(C5)

where \( \tilde{r} = r / (\eta + \nu_0) \). Equation (C5) is thus a universal formula for optimum circulation.

For the special case of small \( \nu_0 r^2 \) (a lightly-loaded rotor), Eq. (C4) agrees with the optimum found by Betz, Ref. [2].

\[
\bar{\Gamma} = 2\nu_0 \eta \cos^2(\phi)
\]

(C6)

For the special case of hover, Eqs. (C3) and (C4) give:

\[
\bar{\Gamma} = \frac{2\nu_0^2 r^2}{\nu_0^2 + r^2} = 2\nu_0^2 \cos^2(\phi)
\]

(C7)

Thus, the general case has the same pattern of the optimum circulation as does the Betz lightly-loaded optimum. Namely, the circulation is a constant multiplied by \( \cos^2(\phi) \). However, the details of the variation of \( \phi \) with radial position depend on the loading and thus are not identical to the Betz form except in the limit of lightly-loaded rotors.

Because of the universal nature of Eq. (C5), it is possible to find a general formula for the sum of the integrated bound circulation over all the blades.
\[ \int_0^R \Gamma dx = 4\pi \Omega R^3 v_0 (\eta + v_0) \left[ 1 - (\eta + v_0) \arctan \left( \frac{1}{\eta + v_0} \right) \right] \quad (C8) \]

For large \((\eta + v_0)\), this approaches \((4/3)\pi \Omega R^3 [v_0/(\eta + v_0)]\).

### 4.21. Appendix D. Contraction Ratio

**Background**

In this Appendix, we derive the appropriate contraction ratio for a consistent momentum theory. In Ref. [1] it is suggested that the contraction ratio should be such that the pressures in the far wake balance the centrifugal forces on the rotating wake. However, Glauert is unable to solve the equation for contraction ratio. A solution for this contraction ratio is attempted in Ref. [10], and a differential equation for the variable contraction ratio is obtained. However, on page 22 of Ref. [10] they write:

"In fact, it is shown in Appendix C that for \(\omega_1\) bounded and \(r(r_1)\) a one to one function, no solution of Eq. (38) exists which satisfies the boundary conditions (39)." Ref. [10]

It is for this reason that they revert to the assumption of a constant contraction ratio that is \(K = 1/\sqrt{2}\). The figure of merit from both Glauert and Ref. [10] are shown in the figure at the end of this appendix, which is reproduced from Ref. [10].

On the other hand, there is a way to define the wake contraction function uniquely such that a momentum theory remains consistent. In particular, the fundamental assumption for momentum theory that leads to a consistent contraction ratio is:

**The contraction ratio in the far wake for any annular ring must be such that the induced flow at the rotor would cause a vortex sheet to come off of the rotor in a direction perpendicular to the local lift with induced flow perpendicular to the sheet.**

It is, of course, a premise of vortex theory that the sheet must be convected along the local free-stream (which is affected by the induced flow) and that the lift on the blade is perpendicular to the shed vortex sheet. For a momentum theory to be internally consistent, it must produce induced flow perpendicular to this sheet.

**Consistency Criterion**

This criterion can be written in terms of the angles formed by the local flow velocities at the rotor in hover. Namely, the angle of the lift (which is the angle of the induced flow) must be along the same line as the induced flow.

\[
\tan(\phi) = \frac{\omega x}{2u} = \frac{u}{[(\Omega - \omega/2)x]} \quad (D1)
\]
This same fundamental equation appears in Ref. [1] as Eq. (4.6) on page 197. With this assumption, one can see that either of Eqs. (54) – (56) immediately implies the other two. We have shown in the body of the paper that the simplified contraction assumption—that the wake expands to atmospheric pressure $p_a$ with a contraction ratio constant at $1/\sqrt{2}$ and that the unbalanced force from the momentum tube is included in the thrust balance—meets this criterion; and this is why the unified momentum results derived herein are accurate.

Once Eq. (D1) is accepted, it is unnecessary to find the consistent contraction ratio explicitly. Equation (D1) is sufficient in itself to derive the consistent thrust equations in Eqs. (54) - (56). Nonetheless, the optimum contraction can be found—if desired—and it can then be used to study the flow behavior in the far wake. This will be done in the paragraphs to follow.

For the more rigorous assumption that pressure remains in the wake such that it does not expand to atmospheric pressure, the consistency criterion of Eq. (D1) can be expressed by equating Eqs. (53) and (54).

\[
\left[ u^2 + \frac{1}{2} \left( \Omega - \frac{\omega}{2} \right) \omega x^2 \right] x dx = \left[ \frac{u_1^2}{2} + \left( \Omega - \frac{\omega}{2} \right) \omega_1 x_1^2 \right] x_1 dx_1 \tag{D2}
\]

The correct contraction function is the one that makes Eq. (54) balance. If we assume that there is some unknown mapping $f(x)$ that takes a fluid particle from a point $x$ on the disk to a point $x_1$ in the far wake by $x_1^2 = f(x)$, we can write from geometry, continuity, and conservation of angular momentum:

- **geometry:** $x_1^2 = f(x), \ 2x_1 dx_1 = f'(x) dx$
- **continuity:** $u_1 x_1 dx_1 = ux dx, \ u_1 = 2ux/f'$
- **angular momentum:** $\omega_1 x_1^2 = \omega x^2, \ \omega_1 = \omega x^2/f$ \tag{D3}

Placing Eq. (D3) into Eq. (D2) gives a determining equation in terms of $f(x)$ and $f'(x)$:

\[
2u^2 + \Omega \omega x^2 - \frac{\omega^2 x^2}{2} = \frac{2u^2 x}{f'} + \Omega \omega xf' - \frac{\omega^2 x^2 f'}{2f} \tag{D4}
\]

This can be solved by the quadratic formula for $f'(x)$ in terms of $f(x)$ and $x$.

\[
f' = \frac{df}{dx} = \frac{2x}{A + \sqrt{A^2 + B}}
\]

\[
A = 1 + \frac{\Omega \omega x^2}{2u^2} - \frac{\omega^2 x^2}{4u^2}
\]

\[
B = -\frac{2\Omega \omega x^2}{u^2} + \frac{\omega^2 x^4}{u^2 f} \tag{D5}
\]
Since it must be that $x = 0$ maps into $x_1 = 0$, the initial condition $f(0) = 0$ completes the system.

**Contraction for Betz Optimum**

When the Betz inflow distribution from Eq. (6) is placed into Eq. (D5), the equation for $f(x)$ simplifies greatly. For example, $A$ simplifies to $A = 2$. The result is a universal differential equation for the contraction mapping $g(\bar{r})$ where $g(\bar{r}) \equiv f(R^2 \nu_0^2)$ and $\bar{r} = x/(R \nu_0)$.

\[
\frac{dg}{d\bar{r}} = \frac{\bar{r}}{1 + \sqrt{\frac{1}{g} - \frac{1}{\bar{r}^2}}} ; \quad g(0) = 0 \quad \text{(D6)}
\]

As verification that this equation is the correct one for a consistent momentum theory, if one takes Eq. (1.13) of Ref. [1], page 193 for the case of hover and substitutes into it the general contraction mapping of Eq. (D3)—along with the Betz optimum solution—one obtains exactly Eq. (D6) above. In the text just after Eq. (1.13) of Ref. 1, Glauert writes:

*These equations, though rather complex in form, suffice to determine the relationship between the thrust and torque of the propeller and the flow in the slipstream.* H. Glauert

Thus, one can say that the fundamental premise stated above—as to what constitutes a consistent momentum theory—is consistent with the results of Glauert. Either approach results in the identical equation for the optimum contraction ratio.

One can show that the solution to Eq. (D6)—and for the contraction ratio $K$—are, in the limiting conditions:

$$
\bar{r} \ll 1, \quad g(\bar{r}) = \frac{\bar{r}^4}{16} \left(1 - \frac{7\bar{r}^2}{32}\right), \quad K = \frac{\bar{r}}{4} \left(1 - \frac{7\bar{r}^2}{64}\right)
$$

$$
\bar{r} \gg 1, \quad g(\bar{r}) = \left[\frac{\bar{r}^2}{2} - \bar{r} - \ln(\bar{r})\right], \quad K = \frac{1}{\sqrt{2}} \left[1 - \frac{1}{\bar{r}} - \frac{\ln(\bar{r})}{\bar{r}^2}\right] \quad \text{(D7)}
$$

Because the contraction ratio goes to zero near the root, it follows that the equivalent of vertical vortex lines concentrate near the root. This implies that, in the far wake, both the axial and circumferential components of normalized velocity converge to a finite value at the rotor center. In particular, the limiting case is $\bar{u}_1(0) = \bar{\omega}$, $\bar{r}_1(0) = 8$. This further implies that, near the central axis, the wake rotation varies as $\bar{\omega}_1 = 8/\bar{r}_1$. In other words, the axial vortex lines near the center are concentrated into a singularity, but it is not as strong as a pure, concentrated vortex with $\bar{\omega} \sim 1/\bar{r}_1^2$ which would give infinite circumferential velocity. Interestingly, this implies that the correct contraction ratio gives an unbounded value for $\omega_1$ which explains why Ref. [10] could prove that no solution exists for "bounded $\omega_1$." In fact, the correct solution is unbounded at the root due to the concentration of vorticity.

Equation (D6) can be solved numerically because the solution in Eq. (D7) can be used to circumvent the singularity at $\bar{r} = 0$. To be specific, one integrates from a small $\bar{r}$ (for which
$g(\tilde{r})$ is known) out to $\tilde{r} = 1/\nu_0$, the latter of which maps the location of the outside of the far wake. Figure 13 shows the resulting universal contraction ratio $K$ as a function of $\tilde{r} = r/\nu_0$. For small $\nu_0$, the contraction is slightly less than $1/\sqrt{2}$ except near the root where it goes to 0. This is in agreement with the statements made about contraction ratio in Ref. [10]. For practical purposes, a good approximation to the contraction ratio over the entire range of values is:

$$K' = \frac{\tilde{r} + 0.680\tilde{r}^2}{4 + 2.376\tilde{r} + 0.962\tilde{r}^2} \quad (D8)$$

which can be used in numerical work. $K'$ matches the first term of the small-$\tilde{r}$ expansion and the first two terms of the large-$\tilde{r}$ expansion exactly. Thus, there is only one free parameter used in obtaining this good fit. Figure 13 shows both $K$ and the approximation $K'$ plotted on the same curve, and they are virtually identical.

**Conditions in the Far Wake**

It is interesting to look at the solution for the velocities and pressures in the far wake under the assumption of a consistent contraction ratio. Figure 14 shows the axial velocity $\tilde{u}_1$ and the circumferential velocity $\tilde{\omega}_1\tilde{r}_1$ as functions of the local normalized radius $\tilde{r}_1$. This is a universal curve for any rotor radius $R$. Thus, the location of the blade tip is:

$$\tilde{r}_1(\text{blade tip}) = \sqrt{g \left( \frac{1}{\nu_0} \right)} \quad (D9)$$

Although the velocities at the central axis are zero at the rotor disk, the axial vortex lines concentrate into a singularity at the rotor center (as the flow travels downstream); and this results in finite velocities at the center in the far wake.

From these, it is possible to determine the pressure in the downstream wake from Eq. (51):

$$P_W = \frac{(p_0 - p_1)}{\rho \Omega^2 R^2 \nu_0^2} = \frac{u_1^2}{2} - \left( 1 - \frac{\tilde{\omega}_1}{2} \right) \frac{\tilde{\omega}_1}{2} \tilde{r}_1^2 \quad (D10)$$

Since the wake contraction parameter is written in terms of the normalized radius on the rotor disk, it is convenient to write both the radius in the far wake—Eq. (D9)—and the pressure in the far wake—Eq. (D10)—in terms of the generating radius at the rotor disk:

$$P_W = \left[ \frac{2\tilde{r}_1^6}{g} + \frac{2\tilde{r}_1^4}{g} - 2\tilde{r}_1^2 (1 + \tilde{r}_1^2) \right]/(1 + \tilde{r}_1^2)^2 \quad (D11)$$

where we have used the Betz optimum for the velocities. However, since $g'$ is given by the differential equation in Eq. (D6), the above can simplified to:

$$P_W = \left[ \frac{4\tilde{r}_1^5}{g'} + \frac{4\tilde{r}_1^4}{g} - 4\tilde{r}_1^2 (1 + \tilde{r}_1^2) \right]/(1 + \tilde{r}_1^2)^2 \quad (D12)$$
This is a negative pressure in the far wake and is shown in Fig. 15 as a function of $\bar{r}_1$. Note that the normalized pressure at the central axis of the far wake is 64. This is consistent with the velocities at the center, since $P_W = (8^2 + 8^2)/2 = 64$.

It is interesting to compare this pressure with the pressure that would be required to hold the centrifugal force of the Betz distribution in equilibrium. We normalize on the quantity $\rho \Omega^2 R^2 \nu_0^2$ as in Eq. (D10).

$$P_C = \int_{\bar{r}_1}^{\infty} \bar{\omega}_1^2 \bar{r}_1 d\bar{r}_1 - \int_{(1/\nu_0) \bar{K}(1/\nu_0)}^{\infty} \bar{\omega}_1^2 \bar{r}_1 d\bar{r}_1$$

(D13)

where the integrand and limits can be written in terms of $\bar{r}$ by use of Eq. (D3). This integral can be done numerically and is shown in Fig. 15 as a universal curve. To exactly match the boundary condition $P_C = 0$ at the tip, simply translate $P_C$ by the value of $P_C$ at the tip, $\bar{r}_1 = (1/\nu_0) \bar{K}(1/\nu_0)$ as indicated by Eq. (D13).

One can see that, although $P_W$ from momentum theory does not exactly match the pressure necessary to balance centrifugal forces in the far wake ($P_C$), $P_W$ is qualitatively accurate. The differences between the two are interesting. While $P_C$ goes to infinity at the central axis (a logarithmic singularity), $P_W$ remains at a finite (although very large) value. Both distributions decay with $\bar{r}$ and cross at the point $\bar{r}_1 = 0.12$ ($\bar{r} = 0.69$). They both continue to decay as radius increases. Thus, $P_C$ is higher at small radii; while $P_W$ is higher at larger radii. While $P_C$ is to be translated down by $P_C (1/\nu_0)$ in order to match the boundary condition outside of the wake, $P_W$ is independent of rotor radius such that there is always a residual pressure at the wake boundary that must be reacted by the momentum tube—in keeping with the assumptions of momentum theory.

The fact that the momentum theory result cannot exactly match the centrifugal pressure was rigorously proved in Ref. [11]. That paper shows that the equations of momentum theory imply that the flow in the annular momentum tubes cannot remain in segregated tubes and yet satisfy all of the dynamic conditions of the flow. The true solution will involve mixing among the annular tubes. This implies some further losses not predictable by momentum theory. In short, momentum theory is not an exact solution to the potential flow equations; and, therefore—while it can balance forces, moments, momentum, and angular momentum—it cannot meet all equilibrium conditions of the flow. Nonetheless, consistent momentum theory is a good approximation and represents an important upper bound on lifting rotor efficiency.
Figure 9 of Ref. [10] where $M = IPE, T_c = C_T$, and REF. 5 is Glauert.