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Four Generated Rank 2 Arithmetically Cohen-Macaulay Vector Bundles on General Sextic Surfaces

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by

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ABSTRACT OF THE DISSERTATION

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Vector Bundles on General Sextic Surfaces

by

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Professor N. Mohan Kumar, Chair

In this dissertation, we compute the dimension of the moduli space, of four generated indecomposable rank 2 *arithmetically Cohen-Macaulay* (ACM for short) bundles on a general sextic surface.

In Chapter One we introduce preliminaries and prove on a general sextic surface, every four generated indecomposable rank 2 ACM bundle belongs to one of fourteen cases.

In Chapter Two we prove for each of the fourteen cases, there exists an indecomposable rank 2 ACM bundle of that case on a general sextic surface.

In Chapter Three we compute for each case, the dimension of the moduli space of four generated indecomposable rank 2 ACM bundles of that case on a general sextic surface.

We do the same analysis on four generated indecomposable rank 2 ACM bundles on a general quartic surface in Chapter Four.

Chapter 1

Background

1.1 Introduction

Throughout this dissertation we work over the field of complex numbers, namely, $\mathbb{C}$.

Definition 1.1 ([1]). Let $\mathscr{F}$ be a coherent sheaf on $\mathbb{P}^n$. We say that $\mathscr{F}$ is *arithmetically Cohen-Macaulay* (ACM for short) if:

(a) $\mathscr{F}$ is Cohen-Macaulay, that is, the $\mathscr{O}_x$-module $\mathscr{F}_x$ is Cohen-Macaulay for every x in $\mathbb{P}^n$; and

(b) $H^i(\mathbb{P}^n, \mathscr{F}(j)) = 0$ for every $1 \leq i \leq \dim(\mathrm{Supp}(\mathscr{F})) - 1$ and $j \in \mathbb{Z}$.

More generally, let X be a hypersurface in $\mathbb{P}^n$ embedded by ι and $\mathscr{F}$ be a coherent sheaf on X. If $\iota_*\mathscr{F}$ is ACM on $\mathbb{P}^n$, then we say $\mathscr{F}$ is ACM on X.

A result of Horrocks ([14] page 39 Theorem 2.3.1) says that a holomorphic vector bundle $\mathscr{E}$ on $\mathbb{P}^n$ splits into a direct sum of line bundles if and only if it is ACM.

Since Horrocks' result, a lot of efforts have been directed towards classifying ACM bundles on hypersurfaces. In particular, the first area that people worked on focused on ACM bundles of low rank such as rank 2. For the moment let us write $X \subset \mathbb{P}^n$ to denote a hypersurface of degree d. Given X, the first question we ask is whether there exists an indecomposable rank 2 ACM bundle on it; if yes, then the next question is how to classify those bundles.

When $d = 2$ and X is smooth, in [8], Knörrer proved that X is of *finite Cohen-Macaulay type*, namely, up to twist by $\mathscr{O}_X(t)$, the set of isomorphism classes of indecomposable ACM bundles on X is finite.

When $d \geq 3$, $n \geq 5$ and X is *general*, in [11], Kumar, Rao, Ravindra proved that there is no indecomposable rank 2 ACM bundle on X.

Consider $n = 4$. In [3], Chiantini and Madonna proved if $d = 6$ and X is *general*, there is no indecomposable rank 2 ACM bundle on X. Later in [12], Kumar, Rao, Ravindra further proved that the same nonexistence holds if $d \geq 6$. For the case $d = 3, 4$, the geometry has been studied in great detail in [4], [6], [7]. Finally for the case $d = 5$, please refer to [13] to get a state-of-the-art summary.

Now consider $n = 3$. When $d = 3$, in [5] Faenzi classified indecomposable rank 2 ACM bundles on *any* smooth cubic surface. When $d = 4$, in [2] Chiantini and Faenzi mentioned the classification of rank 2 ACM bundles on a *general* quartic surface followed from [9]. When $d = 5$, in [2] Chiantini and Faenzi listed every possible pair of first and second Chern classes, of an initialized indecomposable rank 2 ACM bundle on a *general* quintic surface.

In this dissertation, we analyze the case when $n = 3$, $d = 6$, X is *general* and rank 2 ACM bundles are *four generated*. We say that a degree d hypersurface X in $\mathbb{P}^n$ is *general*, if it is in the complement of countably many Zariski closed proper subsets of the space parameterizing degree d hypersurfaces in $\mathbb{P}^n$. Some people call this property *very general* and use *general* for a property that is slightly different. However, the difference plays an insignificant role in this dissertation. So we use the term *general*. As a result, X is smooth and $\mathrm{Pic}(X) = \langle \mathscr{O}_X(1) \rangle = \mathbb{Z}$, according to the Noether-Lefschetz Theorem [9].

Here is our main result of this dissertation. Namely, on a general sextic surface, every four generated indecomposable rank 2 ACM bundle $\mathscr{E}$ belongs to one of fourteen cases. We present the result in the following tables. In the tables, Minimal Resolution denotes the minimal resolution of $\mathscr{E}$; c_1 denotes $c_1(\mathscr{E})$, the first Chern class of $\mathscr{E}$; Dimension of Moduli denotes the dimension of the moduli space of four generated indecomposable rank 2 ACM bundles belonging to the corresponding case on a general sextic surface.

Case	Minimal Resolution
One	$\mathscr{O}_{\mathbb{P}^3}(-5)^4 \to \mathscr{O}_{\mathbb{P}^3}(-2)^4$
Two	$\mathscr{O}_{\mathbb{P}^3}(-4)^2 \oplus \mathscr{O}_{\mathbb{P}^3}(-5)^2 \to \mathscr{O}_{\mathbb{P}^3}(-1)^2 \oplus \mathscr{O}_{\mathbb{P}^3}(-2)^2$
Three	$\mathscr{O}(-4) \oplus \mathscr{O}(-5)^2 \oplus \mathscr{O}(-6) \to \mathscr{O}(-1) \oplus \mathscr{O}(-2)^2 \oplus \mathscr{O}(-3)$
Four	$\mathscr{O}(-4)^3 \oplus \mathscr{O}(-6) \to \mathscr{O} \oplus \mathscr{O}(-2)^3$
Five	$\mathscr{O}(-3) \oplus \mathscr{O}(-5)^3 \to \mathscr{O}(-1)^3 \oplus \mathscr{O}(-3)$
Six	$\mathscr{O}(-4)^2 \oplus \mathscr{O}(-6)^2 \to \mathscr{O}(-1)^2 \oplus \mathscr{O}(-3)^2$
Seven	$\mathscr{O}(-3) \oplus \mathscr{O}(-4) \oplus \mathscr{O}(-5) \oplus \mathscr{O}(-6) \to \mathscr{O} \oplus \mathscr{O}(-1) \oplus \mathscr{O}(-2) \oplus \mathscr{O}(-3)$
Eight	$\mathscr{O}(-4)^2 \oplus \mathscr{O}(-5) \oplus \mathscr{O}(-7) \to \mathscr{O} \oplus \mathscr{O}(-2) \oplus \mathscr{O}(-3)^2$
Nine	$\mathscr{O}(-3) \oplus \mathscr{O}(-5) \oplus \mathscr{O}(-6)^2 \to \mathscr{O}(-1)^2 \oplus \mathscr{O}(-2) \oplus \mathscr{O}(-4)$
Ten	$\mathscr{O}(-3) \oplus \mathscr{O}(-5)^2 \oplus \mathscr{O}(-7) \to \mathscr{O} \oplus \mathscr{O}(-2)^2 \oplus \mathscr{O}(-4)$
Eleven	$\mathscr{O}(-3) \oplus \mathscr{O}(-4)^2 \oplus \mathscr{O}(-7) \to \mathscr{O}(1) \oplus \mathscr{O}(-2)^2 \oplus \mathscr{O}(-3)$
Twelve	$\mathscr{O}(-4)^3 \oplus \mathscr{O}(-8) \to \mathscr{O}(1) \oplus \mathscr{O}(-3)^3$
Thirteen	$\mathscr{O}(-2) \oplus \mathscr{O}(-5)^2 \oplus \mathscr{O}(-6) \to \mathscr{O} \oplus \mathscr{O}(-1)^2 \oplus \mathscr{O}(-4)$
Fourteen	$\mathscr{O}(-2) \oplus \mathscr{O}(-6)^3 \to \mathscr{O}(-1)^3 \oplus \mathscr{O}(-5)$

Table 1.1: Main Result, Part One

Case	Case Paired With	c_1	Dimension of Moduli
One		-1	21
Two		0	18
Three		-1	15
Four	Case Five	0	13
Five	Case Four	0	13
Six		-1	9
Seven		0	8
Eight	Case Nine	-1	7
Nine	Case Eight	-1	7
Ten		-1	4
Eleven	Case Thirteen	0	4
Twelve	Case Fourteen	-1	2
Thirteen	Case Eleven	0	4
Fourteen	Case Twelve	-1	2

Table 1.2: Main Result, Part Two

1.2 Preliminaries

Let X be a hypersurface in $\mathbb{P}^n$. Recall that a vector bundle $\mathscr{E}$ on X is arithmetically Cohen-Macaulay (ACM) if $H^i(X, \mathscr{E}(j)) = 0$ for $i = 1, ..., n-2$ and $j \in \mathbb{Z}$.

Theorem 1.2 ([1] Theorem A)**.** *Let $\mathscr{E}$ be an ACM bundle on a hypersurface $X \subset \mathbb{P}^n$. Then* *there is an exact sequence*

$$0 \to \oplus_{i=1}^{l} \mathscr{O}_{\mathbb{P}^n}(a_i) \to \oplus_{i=1}^{l} \mathscr{O}_{\mathbb{P}^n}(b_i) \to \mathscr{E} \to 0.$$

Conversely, if $M : \oplus_{i=1}^{l} \mathscr{O}_{\mathbb{P}^n}(a_i) \to \oplus_{i=1}^{l} \mathscr{O}_{\mathbb{P}^n}(b_i)$ *is an injective homomorphism, then the cokernel of* M *is an ACM coherent sheaf whose support is the hypersurface defined by* $\det(M) = 0$ *in* $\mathbb{P}^n$.

Remark 1.3. The homomorphism M can be naturally identified with a matrix $(m_{ij})_{l\times l}$ where m_{ij} represents the $\mathscr{O}(a_i) \to \mathscr{O}(b_j)$ component. So m_{ij} is a homogeneous polynomial of degree $b_j - a_i$.

Remark 1.4. If $\mathscr{E}$ is an ACM bundle on a hypersurface $X \subset \mathbb{P}^n$, then we will always make the short exact sequence in the theorem to be minimal, that is, $m_{ij} = 0$ whenever $a_i = b_j$. The minimal resolution of $\mathscr{E}$ is unique up to isomorphism.

The following result of Beauville allows us to relate rank 2 ACM bundles to skew-symmetric matrices.

Theorem 1.5 ([1]). *Let* $X \subset \mathbb{P}^n$ *be a smooth hypersurface of degree* d *with the property that* $\mathrm{Pic}(X) = \mathbb{Z}$. *Let* $\iota : X \hookrightarrow \mathbb{P}^n$ *be the inclusion and* f *be its defining polynomial. If* $\mathscr{E}$ *is a rank 2 ACM bundle on* X *and* $c_1(\mathscr{E}) = eH$ *where* H *is the hyperplane class, then the minimal resolution of* $\mathscr{E}$ *is of the form:*

$$0 \to F_1 := F_0^{\vee}(e-d) \xrightarrow{M} F_0 \to \mathscr{E} \to 0$$

where F_0 *is a direct sum of line bundles on* $\mathbb{P}^n$ *and* M *is a skew-symmetric matrix with its*

Pfaffian, $\mathrm{pf}(M)$ *equal to* f.

Conversely, given F_0 *that is a direct sum of line bundles on* $\mathbb{P}^n$*, and a skew-symmetric matrix* $M : F_0^\vee(\tilde{e} - d) \to F_0$ *with the property that* $\mathrm{pf}(M)$ *defines a smooth hypersurface* X *of degree* d *in* $\mathbb{P}^n$ *with* $\mathrm{Pic}(X) = \mathbb{Z}$*, then there is a rank 2 ACM bundle* $\mathscr{E}$ *on* X *such that* $\mathrm{coker}(M) = \iota_*(\mathscr{E})$*. Moreover,* $c_1(\mathscr{E}) = \tilde{e}H$*.*

Remark 1.6. Let X be a smooth hypersurface of degree d with the property that $\mathrm{Pic}(X) = \mathbb{Z}$. According to the theorem, in order to find every rank 2 ACM bundle on X, it suffices to find every skew-symmetric matrix $M : F_0^\vee(e - d) \to F_0$ such that $\mathrm{pf}(M)$ is a defining polynomial of X.

Now let us use X to denote a *general* hypersurface in $\mathbb{P}^3$ of degree d, $d \geq 4$. We want to find every indecomposable rank 2 ACM bundle on X. According to 1.6, it suffices to find every skew-symmetric matrix M whose Pfaffian is a defining polynomial of X. Let M be of size $r \times r$. First of all r is even; otherwise, as M is skew-symmetric, $\mathrm{pf}(M) \equiv 0$ and it does not define a proper hypersurface. If $r = 2$, then $F_0 \to \mathscr{E}$ becomes an isomorphism once restricted on X. So $\mathscr{E}$ splits. So $r \geq 4$. Meanwhile, $\det(M) \equiv \mathrm{pf}(M)^2$ is of degree $2d$; if $r > 2d$, then $\deg(\det(M)) > 2d$. So $r \leq 2d$. In this dissertation we analyze the cases when $r = 4$. Rank 2 ACM bundles that fall into this category are called *four generated*.

Write $F_1 := F_0^\vee(e - d) = \oplus_{i=1}^4 \mathscr{O}(a_i)$ and $F_0 = \oplus_{j=1}^4 \mathscr{O}(b_j)$. Without loss of generality, assume $a_1 \geqq a_2 \geqq a_3 \geqq a_4$ and $b_1 \geqq b_2 \geqq b_3 \geqq b_4$. Further without loss of generality for each i let $\mathscr{O}(a_i) \to \oplus_{j=1}^4 \mathscr{O}(b_j)$ be a row of M with $\mathscr{O}(a_i) \to \mathscr{O}(b_j)$ corresponding to the j^{th} column. First of all we need to find out which permutation of rows $\{\mathscr{O}(a_i) \to \oplus_{j=1}^4 \mathscr{O}(b_j) : i = 1, 2, 3, 4\}$ matches M, which is a skew symmetric matrix. For this, notice the degree decreases from left to right in each row, so for M to be skew-symmetric, rows must be arranged in such a way that in each column the degree decreases from up to down. Thus M corresponds to

$$\begin{pmatrix} \mathscr{O}(a_4) \to \mathscr{O}(b_1) & \mathscr{O}(a_4) \to \mathscr{O}(b_2) & \mathscr{O}(a_4) \to \mathscr{O}(b_3) & \mathscr{O}(a_4) \to \mathscr{O}(b_4) \\ \mathscr{O}(a_3) \to \mathscr{O}(b_1) & \mathscr{O}(a_3) \to \mathscr{O}(b_2) & \mathscr{O}(a_3) \to \mathscr{O}(b_3) & \mathscr{O}(a_3) \to \mathscr{O}(b_4) \\ \mathscr{O}(a_2) \to \mathscr{O}(b_1) & \mathscr{O}(a_2) \to \mathscr{O}(b_2) & \mathscr{O}(a_2) \to \mathscr{O}(b_3) & \mathscr{O}(a_2) \to \mathscr{O}(b_4) \\ \mathscr{O}(a_1) \to \mathscr{O}(b_1) & \mathscr{O}(a_1) \to \mathscr{O}(b_2) & \mathscr{O}(a_1) \to \mathscr{O}(b_3) & \mathscr{O}(a_1) \to \mathscr{O}(b_4) \end{pmatrix}.$$

Because M is skew-symmetric, diagonal entries must be 0. Because the resolution is minimal, off-diagonal entries must be non-units. Because X is *general*, off-diagonal entries are not 0. The last statement is implied by the following proposition.

Proposition 1.7. *If the defining equation of a degree d surface X can be written as $xy+zw = 0$ where $x, y, z, w \in \mathbb{C}[x_0, x_1, x_2, x_3]$ are of positive degrees and $d \geq 4$, then X is not general.*

Proof. Suppose to the contrary that X is general, then first of all x, y, z, w have no common zero as X would be singular at that point. Next, consider $Y = \mathscr{Z}(x, z) \subset X$. At any point on Y, either $y \neq 0$ or $w \neq 0$. Suppose $y \neq 0$, then locally at that point, X is defined by $x = -zw/y$, and $Y \subset X$ is defined by $z = 0$. So Y is locally principal in X. Similarly Y is locally principal in X if $w \neq 0$. So Y is a Cartier divisor in X. X is general, so $\mathrm{Pic}(X) = \mathbb{Z}$ and Y is a hypersurface section in X. Let $\mathscr{Z}(s)$ be the hypersurface. Then $(x, z) = (xy + zw, s)$. Because y, w are of positive degrees, $(x, z) = (s)$. So s is a common factor of x and z. This contradicts that X is smooth. □

Remark 1.8. The degree of an off-diagonal entry of M cannot be zero; otherwise, as the resolution is minimal, that entry must be zero, and $\mathrm{pf}(M)$ is of the form $xy + zw$. This contradicts our assumption that pf(M) defines a general surface.

Now let $d = 6$. Namely, consider rank 2 ACM bundles on sextic surfaces. Because $\mathrm{pf}(M) = m_{12}m_{34} - m_{13}m_{24} + m_{14}m_{23}$ is homogeneous of degree 6,

$$\deg(m_{12}) + \deg(m_{34}) = \deg(m_{13}) + \deg(m_{24}) = \deg(m_{14}) + \deg(m_{23}) = 6.$$

Because each off-diagonal m_{ij} is of positive degree, and M is arranged in such a way that the degrees decrease in each row from left to right and in each column from up to down, as a result, degreewise the upper triangular portion of M must be one of the following:

Case One:

$$\begin{pmatrix} & 3 & 3 & 3 \\ & & 3 & 3 \\ & & & 3 \\ & & & \end{pmatrix}$$

Case Two:

$$\begin{pmatrix} & 4 & 3 & 3 \\ & & 3 & 3 \\ & & & 2 \\ & & & \end{pmatrix}$$

Case Three:

$$\begin{pmatrix} 4 & 4 & 3 \\ & 3 & 2 \\ & & 2 \\ & & \end{pmatrix}$$

Case Four:

$$\begin{pmatrix} 4 & 4 & 4 \\ & 2 & 2 \\ & & 2 \\ & & \end{pmatrix}$$

Case Five:

$$\begin{pmatrix} 4 & 4 & 2 \\ & 4 & 2 \\ & & 2 \\ & & \end{pmatrix}$$

Case Six:

$$\begin{pmatrix} 5 & 3 & 3 \\ & 3 & 3 \\ & & 1 \\ & & \end{pmatrix}$$

Case Seven:

$$\begin{pmatrix} 5 & 4 & 3 \\ & 3 & 2 \\ & & 1 \\ & & \end{pmatrix}$$

Case Eight:

$$\begin{pmatrix} 5 & 4 & 4 \\ & 2 & 2 \\ & & 1 \\ & & \end{pmatrix}$$

Case Nine:

$$\begin{pmatrix} 5 & 4 & 2 \\ & 4 & 2 \\ & & 1 \\ & & \end{pmatrix}$$

Case Ten:

$$\begin{pmatrix} 5 & 5 & 3 \\ & 3 & 1 \\ & & 1 \\ & & \end{pmatrix}$$

Case Eleven:

$$\begin{pmatrix} 5 & 5 & 4 \\ & 2 & 1 \\ & & 1 \\ & & \end{pmatrix}$$

Case Twelve:

$$\begin{pmatrix} 5 & 5 & 5 \\ & 1 & 1 \\ & & 1 \\ & & \end{pmatrix}$$

Case Thirteen:

$$\begin{pmatrix} 5 & 5 & 2 \\ & 4 & 1 \\ & & 1 \\ & & \end{pmatrix}$$

Case Fourteen:

$$\begin{pmatrix} 5 & 5 & 1 \\ & 5 & 1 \\ & & 1 \\ & & \end{pmatrix}$$

1.3 Fourteen Cases

Recall that the minimal resolution of a four generated rank 2 ACM bundle $\mathscr{E}$ on a sextic surface X is of the form

$$0 \to F_1 := \oplus_{i=1}^4 \mathscr{O}_{\mathbb{P}^3}(a_i) \xrightarrow{M} F_0 := \oplus_{i=1}^4 \mathscr{O}_{\mathbb{P}^3}(b_i) \to \mathscr{E} \to 0.$$

Moreover, $F_1 = F_0^\vee(e-d)$ where $d = 6$ and $e = c_1(\mathscr{E})$. We have arranged F_1 and F_0 so that $a_1 \geq a_2 \geq a_3 \geq a_4$ and $b_1 \geq b_2 \geq b_3 \geq b_4$. For our purpose without loss of generality, we may assume $e = 0$ or $e = -1$, because $c_1(\mathscr{E}(1)) = c_1(\mathscr{E}) + 2$ and we are only interested in $\mathscr{E}$ up to twist. We have

$$\deg(m_{12}) = b_2 - a_4, \deg(m_{34}) = b_4 - a_2, \deg(m_{12}) + \deg(m_{34}) = 6.$$

So $b_2 + b_4 - a_2 - a_4 = 6$. Because $a_2 = -b_3 + e - d$ and $a_4 = -b_1 + e - d$, we get

$$\Sigma_{j=1}^4 b_j - 2e + 2d = 6 \Longrightarrow \Sigma_{j=1}^4 b_j - 2e + 6 = 0. \qquad (\star)$$

To each four generated rank 2 ACM bundle there is another rank 2 ACM bundle that is also four generated. Specifically, if $\mathscr{E}$ is a four generated rank 2 ACM bundle on a degree d hypersurface X with its minimal resolution

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathscr{E} \longrightarrow 0,$$

by restricting on X we get

$$0 \longrightarrow \mathscr{E}(-d) \longrightarrow \bar{F}_1 := F_1 \otimes \mathscr{O}_X \longrightarrow \bar{F}_0 := F_0 \otimes \mathscr{O}_X \longrightarrow \mathscr{E} \longrightarrow 0.$$

Let $\mathscr{G} = \mathrm{im}(\bar{F}_1 \longrightarrow \bar{F}_0)$ then $\mathscr{G}$ is a rank 2 vector bundle on X. It is the cokernel of the induced arrow g. In the diagram f is a defining polynomial of X.

$$\begin{tikzcd} & F_0(-d) \arrow[dl, dashed, "g"'] \arrow[d, "\cdot f"] \\ F_1 \arrow[r] & F_0 \end{tikzcd}$$

So

$$0 \longrightarrow F_0(-d) \longrightarrow F_1 \longrightarrow \mathscr{G} \longrightarrow 0.$$

According to 1.2, $\mathscr{G}$ is ACM. Do the same on $\mathscr{G}$ and we end up with

$$0 \longrightarrow F_1(-d) \longrightarrow F_0(-d) \longrightarrow \mathscr{E}(-d) \longrightarrow 0.$$

So up to twist $\mathscr{E}$ and $\mathscr{G}$ come in pairs. We say that $\mathscr{E}$ and $\mathscr{G}$ are paired to each other.

Case One: $b_2 - a_4 = b_3 - a_4 = b_4 - a_4 = b_3 - a_3 = b_4 - a_3 = b_4 - a_2 = 3$, together with the equations $a_i = -b_{5-i} + e - d$, we get $a_1 = a_2 = a_3 = a_4$ and $b_1 = b_2 = b_3 = b_4$. Now plug into $(\star)$, we get $4b_1 - 2e + 6 = 0$. So $e = -1$ and $b_1 = -2$. So in this case the minimal resolution of $\mathscr{E}$ is

$$0 \to \mathscr{O}_{\mathbb{P}^3}(-5)^4 \to \mathscr{O}_{\mathbb{P}^3}(-2)^4 \to \mathscr{E} \to 0. \tag{1.1}$$

Case Two: $b_2 - a_4 = 4, b_3 - a_4 = b_4 - a_4 = b_3 - a_3 = b_4 - a_3 = 3, b_4 - a_2 = 2$, so $a_2 - 1 = a_3 = a_4$ and $b_2 - 1 = b_3 = b_4$. With the equations $a_i = -b_{5-i} + e - d$ we get $a_1 = a_2$ and $b_1 = b_2$. Plug into $(\star)$, we get $4b_1 - 2e + 4 = 0$. So $e = 0$ and $b_1 = -1$, and the minimal resolution is

$$0 \to \mathscr{O}(-4)^2 \oplus \mathscr{O}(-5)^2 \to \mathscr{O}(-1)^2 \oplus \mathscr{O}(-2)^2 \to \mathscr{E} \to 0. \tag{1.2}$$

Case Three: $b_2 - a_4 = b_3 - a_4 = 4, b_4 - a_4 = b_3 - a_3 = 3, b_4 - a_3 = b_4 - a_2 = 2$. So $b_2 = b_3 = b_4 + 1$ and $a_2 = a_3 = a_4 + 1$. With the equations $a_i = -b_{5-i} + e - d$ we get $b_1 = b_2 + 1$ and $a_1 = a_2 + 1$. Plug into $(\star)$ and we get $4b_2 - 2e + 6 = 0$. So $e = -1$ and $b_2 = -2$, and the minimal resolution is

$$0 \to \mathscr{O}(-6) \oplus \mathscr{O}(-5)^2 \oplus \mathscr{O}(-4) \to \mathscr{O}(-1) \oplus \mathscr{O}(-2)^2 \oplus \mathscr{O}(-3) \to \mathscr{E} \to 0. \tag{1.3}$$

Case Four: $b_2 - a_4 = b_3 - a_4 = b_4 - a_4 = 4, b_3 - a_3 = b_4 - a_3 = b_4 - a_2 = 2$, so $b_2 = b_3 = b_4$ and $a_2 = a_3 = a_4 + 2$. With the equations $a_i = -b_{5-i} + e - d$ we get $a_1 = a_2$ and $b_1 = b_2 + 2$. Plug into $(\star)$ and we get $4b_2 - 2e + 8 = 0$. So $e = 0$ and $b_2 = -2$, and the minimal resolution is

$$0 \to \mathscr{O}(-6) \oplus \mathscr{O}(-4)^3 \to \mathscr{O} \oplus \mathscr{O}(-2)^3 \to \mathscr{E} \to 0. \tag{1.4}$$

Case Five: $b_2 - a_4 = b_3 - a_4 = b_3 - a_3 = 4, b_4 - a_4 = b_4 - a_3 = b_4 - a_2 = 2$, so $a_2 = a_3 = a_4$ and $b_2 = b_3 = b_4 + 2$. With the equations $a_i = -b_{5-i} + e - d$ we get $a_1 = a_2 + 2$ and $b_1 = b_2$. Plug into $(\star)$ and we get $4b_1 - 2e + 4 = 0$. So $e = 0$ and $b_1 = -1$, and the minimal resolution is

$$0 \to \mathscr{O}(-5)^3 \oplus \mathscr{O}(-3) \to \mathscr{O}(-1)^3 \oplus \mathscr{O}(-3) \to \mathscr{E} \to 0. \tag{1.5}$$

Notice its pair is

$$0 \to \mathscr{O}(-7)^3 \oplus \mathscr{O}(-9) \to \mathscr{O}(-5)^3 \oplus \mathscr{O}(-3) \to \mathscr{G} \to 0,$$

which up to twist by $\mathscr{O}(3)$ is the same short exact sequence as in Case Four. So Case Five and Case Four are paired to each other.

Case Six: $b_2 - a_4 = 5, b_3 - a_4 = b_4 - a_4 = b_3 - a_3 = b_4 - a_3 = 3, b_4 - a_2 = 1$, so $a_2 - 2 = a_3 = a_4$ and $b_2 - 2 = b_3 = b_4$. With the equations $a_i = -b_{5-i} + e - d$ we get $a_1 = a_2$ and $b_1 = b_2$. Plug into $(\star)$ and we get $4b_1 - 2e + 2 = 0$. So $e = -1$ and $b_1 = -1$, and the minimal resolution is

$$0 \to \mathscr{O}(-4)^2 \oplus \mathscr{O}(-6)^2 \to \mathscr{O}(-1)^2 \oplus \mathscr{O}(-3)^2 \to \mathscr{E} \to 0. \tag{1.6}$$

Case Seven: $b_2 - a_4 = 5, b_3 - a_4 = 4, b_4 - a_4 = b_3 - a_3 = 3, b_4 - a_3 = 2, b_4 - a_2 = 1$, so $a_2 = a_3 + 1 = a_4 + 2$ and $b_2 = b_3 + 1 = b_4 + 2$. With the equations $b_i = -a_{5-i} + e - d$ we get $a_1 = a_2 + 1$ and $b_1 = b_2 + 1$. Plug into $(\star)$ and we get $4b_1 - 2e = 0$. So $e = 0$ and $b_1 = 0$, and the minimal resolution is

$$\begin{aligned} 0 &\to \mathscr{O}(-3) \oplus \mathscr{O}(-4) \oplus \mathscr{O}(-5) \oplus \mathscr{O}(-6) \\ &\to \mathscr{O} \oplus \mathscr{O}(-1) \oplus \mathscr{O}(-2) \oplus \mathscr{O}(-3) \to \mathscr{E} \to 0. \end{aligned} \tag{1.7}$$

Case Eight: $b_2 - a_4 = 5, b_3 - a_4 = b_4 - a_4 = 4, b_3 - a_3 = b_4 - a_3 = 2, b_4 - a_2 = 1$, so $a_2 = a_3 + 1 = a_4 + 3$ and $b_2 = b_3 + 1 = b_4 + 1$. With the equations $b_i = -a_{5-i} + e - d$ we get $a_1 = a_2$ and $b_1 = b_2 + 2$. Plug into $(\star)$ and we get $4b_1 - 2e - 2 = 0$. So $e = -1$ and $b_1 = 0$ and the minimal resolution is

$$0 \to \mathscr{O}(-4)^2 \oplus \mathscr{O}(-5) \oplus \mathscr{O}(-7) \to \mathscr{O} \oplus \mathscr{O}(-2) \oplus \mathscr{O}(-3)^2 \to \mathscr{E} \to 0. \tag{1.8}$$

Case Nine: $b_2 - a_4 = 5, b_3 - a_4 = b_3 - a_3 = 4, b_4 - a_4 = b_4 - a_3 = 2, b_4 - a_2 = 1$. So $a_2 = a_3 + 1 = a_4 + 1$ and $b_2 = b_3 + 1 = b_4 + 3$. Because $b_i = -a_{5-i} + e - d$, $a_1 = a_2 + 2$ and $b_1 = b_2$. Plug into $(\star)$ and we get $4b_1 - 2e + 2 = 0$, so $e = -1$ and $b_1 = -1$, and the minimal resolution is

$$0 \to \mathscr{O}(-3) \oplus \mathscr{O}(-5) \oplus \mathscr{O}(-6)^2 \to \mathscr{O}(-1)^2 \oplus \mathscr{O}(-2) \oplus \mathscr{O}(-4) \to \mathscr{E} \to 0. \tag{1.9}$$

Notice its pair is

$$0 \to \mathscr{O}(-7)^2 \oplus \mathscr{O}(-8) \oplus \mathscr{O}(-10) \to \mathscr{O}(-3) \oplus \mathscr{O}(-5) \oplus \mathscr{O}(-6)^2 \to \mathscr{G} \to 0,$$

which up to twist by $\mathscr{O}(3)$ is the same short exact sequence as in Case Eight. So Case Nine and Case Eight are paired to each other.

Case Ten: $b_2 - a_4 = b_3 - a_4 = 5, b_4 - a_4 = b_3 - a_3 = 3, b_4 - a_3 = b_4 - a_2 = 1$, so $a_2 = a_3 = a_4 + 2$ and $b_2 = b_3 = b_4 + 2$. With the equations $a_i = -b_{5-i} + e - d$ we get $a_1 = a_2 + 2$ and $b_1 = b_2 + 2$. Plug into $(\star)$, we get $4b_1 - 2e - 2 = 0$. So $e = -1$ and $b_1 = 0$, and the minimal resolution is

$$0 \to \mathscr{O}(-3) \oplus \mathscr{O}(-5)^2 \oplus \mathscr{O}(-7) \to \mathscr{O} \oplus \mathscr{O}(-2)^2 \oplus \mathscr{O}(-4) \to \mathscr{E} \to 0. \tag{1.10}$$

Case Eleven: $b_2 - a_4 = b_3 - a_4 = 5, b_4 - a_4 = 4, b_3 - a_3 = 2, b_4 - a_3 = b_4 - a_2 = 1$, so $a_2 = a_3 = a_4 + 3$ and $b_2 = b_3 = b_4 + 1$. With the equations $a_i = -b_{5-i} + e - d$ we get $a_1 = a_2 + 1$ and $b_1 = b_2 + 3$. Plug into $(\star)$, we get $4b_1 - 2e - 4 = 0$. So $e = 0$ and $b_1 = 1$, and the minimal resolution is

$$0 \to \mathscr{O}(-3) \oplus \mathscr{O}(-4)^2 \oplus \mathscr{O}(-7) \to \mathscr{O}(1) \oplus \mathscr{O}(-2)^2 \oplus \mathscr{O}(-3) \to \mathscr{E} \to 0. \tag{1.11}$$

Case Twelve: $b_2 - a_4 = b_3 - a_4 = b_4 - a_4 = 5, b_3 - a_3 = b_4 - a_3 = b_4 - a_2 = 1$, so $a_2 = a_3 = a_4 + 4$ and $b_2 = b_3 = b_4$. With the equations $a_i = -b_{5-i} + e - d$ we get $a_1 = a_2$ and $b_1 = b_2 + 4$. Plug into $(\star)$, we get $4b_1 - 2e - 6 = 0$. So $e = -1$ and $b_1 = 1$, and the minimal resolution is

$$0 \to \mathscr{O}(-4)^3 \oplus \mathscr{O}(-8) \to \mathscr{O}(1) \oplus \mathscr{O}(-3)^3 \to \mathscr{E} \to 0. \tag{1.12}$$

Case Thirteen: $b_2 - a_4 = b_3 - a_4 = 5, b_4 - a_4 = 2, b_3 - a_3 = 4, b_4 - a_3 = b_4 - a_2 = 1$, so $a_2 = a_3 = a_4 + 1$ and $b_2 = b_3 = b_4 + 3$. With the equations $a_i = -b_{5-i} + e - d$ we get $a_1 = a_2 + 3$ and $b_1 = b_2 + 1$. Plug into $(\star)$, we get $4b_1 - 2e = 0$. So $e = 0$ and $b_1 = 0$, and the minimal resolution is

$$0 \to \mathscr{O}(-2) \oplus \mathscr{O}(-5)^2 \oplus \mathscr{O}(-6) \to \mathscr{O} \oplus \mathscr{O}(-1)^2 \oplus \mathscr{O}(-4) \to \mathscr{E} \to 0. \tag{1.13}$$

Notice its pair is

$$0 \to \mathscr{O}(-6) \oplus \mathscr{O}(-7)^2 \oplus \mathscr{O}(-10) \to \mathscr{O}(-2) \oplus \mathscr{O}(-5)^2 \oplus \mathscr{O}(-6) \to \mathscr{G} \to 0,$$

which up to twist by $\mathscr{O}(3)$ is the same short exact sequence as in Case Eleven. So Case Thirteen and Case Eleven are paired to each other.

Case Fourteen: $b_2 - a_4 = b_3 - a_4 = b_3 - a_3 = 5, b_4 - a_4 = b_4 - a_3 = b_4 - a_2 = 1$, so $a_2 = a_3 = a_4$ and $b_2 = b_3 = b_4 + 4$. With the equations $a_i = -b_{5-i} + e - d$ we get $a_1 = a_2 + 4$ and $b_1 = b_2$. Plug into $(\star)$, we get $4b_1 - 2e + 2 = 0$. So $e = -1$ and $b_1 = -1$, and the minimal resolution is

$$0 \to \mathscr{O}(-2) \oplus \mathscr{O}(-6)^3 \to \mathscr{O}(-1)^3 \oplus \mathscr{O}(-5) \to \mathscr{E} \to 0. \tag{1.14}$$

Notice its pair is

$$0 \to \mathscr{O}(-7)^3 \oplus \mathscr{O}(-11) \to \mathscr{O}(-2) \oplus \mathscr{O}(-6)^3 \to \mathscr{G} \to 0,$$

which up to twist by $\mathscr{O}(3)$ is the same short exact sequence as in Case Twelve. So Case Fourteen and Case Twelve are paired to each other.

Chapter 2

Existence in Fourteen Cases

In this chapter we prove for each of the 14 cases, there is an indecomposable rank 2 ACM bundle belonging to that case on a general sextic surface.

Let $a \geq b \geq c \geq d \geq e \geq f$ be fixed positive integers such that

$$a + f = b + e = c + d = 6.$$

Below we cite a result in [10]. Let $\mathcal{F}_{d,e,f}$ denote the Hilbert flag scheme parameterizing all inclusions $Y \subset X \subset \mathbb{P}^3$ where X is a hypersurface of degree 6 and Y is a zero dimensional complete intersection subvariety which is cut out by three hypersurfaces of degree d, e, f. Let $\mathcal{H}_6$ denote the Hilbert scheme of all degree 6 hypersurfaces in $\mathbb{P}^3$. Let $\mathcal{H}_{d,e,f}$ denote the Hilbert scheme of all zero dimensional subvariety in $\mathbb{P}^3$ with the same Hilbert polynomial as the complete intersection of three hypersurfaces of degree d, e, f. Corresponding to the projections

$$\begin{tikzcd}
\mathcal{F}_{d,e,f} \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{H}_{d,e,f} \\
\mathcal{H}_6 &
\end{tikzcd}$$

the induced morphisms between Zariski tangent spaces are described below. Namely, if T is the tangent space at the point $Y \overset{i}{\hookrightarrow} X \subset \mathbb{P}^3$ in $\mathcal{F}_{d,e,f}$, then

$$\begin{tikzcd}
T \arrow[r, "p_2"] \arrow[d, "p_1"] & H^0(Y, \mathcal{N}_{Y/\mathbb{P}}) \arrow[d, dashed, "\alpha"] \\
H^0(X, \mathcal{N}_{X/\mathbb{P}}) \arrow[r, dashed, "\beta"] & H^0(Y, i^*\mathcal{N}_{X/\mathbb{P}})
\end{tikzcd}$$

is a Cartesian diagram of vector spaces.

Each of the 14 cases is determined by a specific (a, b, c, d, e, f). To show a general sextic surface supports an indecomposable rank 2 ACM bundle belonging to a specific case, it is equivalent to showing the map

$$h : \left\{(A, B, C, D, E, F) \in \mathbb{C}[x_0, x_1, x_2, x_3]^6 : \deg(A) = a, \ldots, \deg(F) = f\right\}$$
$$\longrightarrow \mathcal{H}_6$$

sending (A, B, C, D, E, F) to the point $\mathcal{Z}(AF - BE + CD)$ is dominant. Because there is a rational dominant map from

$$\left\{(A, B, C, D, E, F) \in \mathbb{C}[x_0, x_1, x_2, x_3]^6 : \deg(A) = a, \ldots, \deg(F) = f\right\}$$

to $\mathcal{F}_{d,e,f}$ sending (A, B, C, D, E, F) to

$$\mathscr{Z}(D, E, F) \subset \mathscr{Z}(AF - BE + CD) \subset \mathbb{P}^3,$$

to show h is dominant, it suffices to show p_1 is dominant. So it suffices to find a point $Y \subset X \subset \mathbb{P}^3$ in $\mathcal{F}_{d,e,f}$ such that $p_1 : T \longrightarrow H^0(X, \mathcal{N}_{X/\mathbb{P}})$ is onto. The commutative diagram above is a Cartesian diagram of vector spaces, so p_1 is onto if $\mathrm{im}(\beta) \subset \mathrm{im}(\alpha)$.

When $Y = \mathscr{Z}(D, E, F)$ and $X = \mathscr{Z}(AF - BE + CD)$, α is described as

$$\alpha : H^0(Y, \mathscr{O}_Y(d) \oplus \mathscr{O}_Y(e) \oplus \mathscr{O}_Y(f)) \xrightarrow{[C,-B,A]} H^0(Y, \mathscr{O}_Y(6))$$

sending (x, y, z) to $xC - yB + zA$. So

$$\mathrm{im}(\alpha) \supset \overline{\{\text{degree 6 homogeneous polynomials in } (A, B, C, D, E, F)\}}.$$

Meanwhile,

$$H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6)) \longrightarrow H^0(X, \mathscr{O}_X(6)) = H^0(X, \mathcal{N}_{X/\mathbb{P}})$$

is onto, so

$$\mathrm{im}(\beta) = \mathrm{im}\left(H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(6)) \longrightarrow H^0(Y, \mathscr{O}_Y(6))\right).$$

Namely,

$$\mathrm{im}(\beta) = \overline{\{\text{degree 6 homogeneous polynomials in } \mathbb{C}[x_0, x_1, x_2, x_3]\}}.$$

If every degree 6 monomial in $\mathbb{C}[x_0, x_1, x_2, x_3]$ is in (A, B, C, D, E, F), then $\mathrm{im}(\beta) \subset \mathrm{im}(\alpha)$.

So to show p_1 is dominant, it suffices to find a specific (A, B, C, D, E, F) with the properties that $AF - BE + CD \neq 0$, D, E, F form a regular sequence and every degree 6 monomial is in (A, B, C, D, E, F).

Proposition 2.1 (Case One). *A general sextic surface X can be realized as the zero variety associated to the Pfaffian of $M : \mathscr{O}_{\mathbb{P}^3}(-5)^4 \to \mathscr{O}_{\mathbb{P}^3}(-2)^4$.*

Proof. Case One is determined by $a = b = c = d = e = f = 3$. According to the previous discussion, it suffices to find a specific (A, B, C, D, E, F) such that $AF - BE + CD \neq 0$, D, E, F form a regular sequence and every degree 6 monomial is in the ideal (A, B, C, D, E, F). Pick

$$I = (A, B, C, D, E, F) = (x_2^3, x_1x_2x_3, x_0^2x_1 + x_2^2x_3, x_1^3, x_0^3, x_3^3).$$

First of all, $AF - BE + CD = x_0^2x_1^4 + x_1^3x_2^2x_3 - x_0^3x_1x_2x_3 + x_2^3x_3^3 \neq 0$. Next, x_1^3, x_0^3, x_3^3 form a regular sequence. Finally, every degree 6 monomial can be written as $x_0^i x_1^j x_2^k x_3^l$. If $\max(i, j, k, l) \geq 3$, then $x_0^i x_1^j x_2^k x_3^l \in (x_0^3, x_1^3, x_2^3, x_3^3) \subset I$. If $\min(j, k, l) \geq 1$, then $x_0^i x_1^j x_2^k x_3^l \in (x_1x_2x_3) \subset I$. Degree 6 monomials that meet neither of these two criteria are $x_0^2x_1^2x_2^2, x_0^2x_1^2x_3^2, x_0^2x_2^2x_3^2$.

$$x_0^2x_1^2x_2^2 = \left(x_0^2x_1 + x_2^2x_3\right)x_1x_2^2 - x_1x_2^4x_3 \in I,$$

$$x_0^2x_1^2x_3^2 = \left(x_0^2x_1 + x_2^2x_3\right)x_1x_3^2 - x_1x_2^2x_3^3 \in I,$$

$$x_0^2x_2^2x_3^2 = x_0^2x_3\left(x_0^2x_1 + x_2^2x_3\right) - x_0^4x_1x_3 \in I,$$

so every degree 6 monomial is in (A, B, C, D, E, F). This proves the proposition. In 5.1 we provide another proof. $\square$

Proposition 2.2 (Case Two). *If X is a general sextic surface in $\mathbb{P}^3$, then there is a skew-*

symmetric matrix $M : \mathscr{O}_{\mathbb{P}^3}(-4)^2 \oplus \mathscr{O}_{\mathbb{P}^3}(-5)^2 \to \mathscr{O}_{\mathbb{P}^3}(-1)^2 \oplus \mathscr{O}_{\mathbb{P}^3}(-2)^2$ *such that* $\mathrm{pf}(M)$ *is a defining polynomial of* X*.*

Proof. Case Two is determined by $a = 4, b = c = d = e = 3, f = 2$. According to the previous discussion, it suffices to find a specific (A, B, C, D, E, F) such that $AF - BE + CD \neq 0$, D, E, F form a regular sequence and every degree 6 monomial is in (A, B, C, D, E, F). Pick

$$I = (A, B, C, D, E, F) = (x_0^2 x_1^2 + x_0 x_1 x_2 x_3, x_2^3, x_3^3, x_0^3, x_1^3, x_2 x_3).$$

First of all, $AF - BE + CD = x_0^2 x_1^2 x_2 x_3 + x_0 x_1 x_2^2 x_3^2 - x_1^3 x_2^3 + x_0^3 x_3^3 \neq 0$. Next, $x_0^3, x_1^3, x_2 x_3$ form a regular sequence. Finally, every degree 6 monomial can be written as $x_0^i x_1^j x_2^k x_3^l$. If $\max(i, j, k, l) \geq 3$ or $\min(k, l) \geq 1$, then $x_0^i x_1^j x_2^k x_3^l \in I$. Degree 6 monomials that meet neither of these two criteria are $x_0^2 x_1^2 x_2^2, x_0^2 x_1^2 x_3^2$.

$$x_0^2 x_1^2 x_2^2 = \left(x_0^2 x_1^2 + x_0 x_1 x_2 x_3\right) x_2^2 - x_0 x_1 x_2^3 x_3 \in I,$$

$$x_0^2 x_1^2 x_3^2 = \left(x_0^2 x_1^2 + x_0 x_1 x_2 x_3\right) x_3^2 - x_0 x_1 x_2 x_3^3 \in I,$$

so every degree 6 monomial is in (A, B, C, D, E, F). This proves the proposition. In 5.2 we provide another proof. $\square$

Proposition 2.3 (Case Three)**.** *If* X *is a general sextic surface in* $\mathbb{P}^3$*, then there is a skew-symmetric matrix*

$$M : \mathscr{O}(-6) \oplus \mathscr{O}(-5)^2 \oplus \mathscr{O}(-4) \to \mathscr{O}(-1) \oplus \mathscr{O}(-2)^2 \oplus \mathscr{O}(-3)$$

such that $\mathrm{pf}(M)$ *is a defining polynomial of* X*.*

Proof. Case Three is determined by $a = b = 4, c = d = 3, e = f = 2$. It suffices to find a specific (A, B, C, D, E, F) such that $AF - BE + CD \neq 0$, D, E, F form a regular sequence and every degree 6 monomial is in (A, B, C, D, E, F). Pick

$$I = (A, B, C, D, E, F) = (0, x_0x_1x_2x_3, x_0^3, x_1^3, x_2^2, x_3^2).$$

First of all, $AF - BE + CD = -x_0x_1x_2^3x_3 + x_0^3x_1^3 \neq 0$. Next, x_1^3, x_2^2, x_3^2 form a regular sequence. Finally, every degree 6 monomial can be written as $x_0^ix_1^jx_2^kx_3^l$. If $\max(i,j) \geq 3$ or $\max(k,l) \geq 2$, then $x_0^ix_1^jx_2^kx_3^l \in (x_0^3, x_1^3, x_2^2, x_3^2) \subset I$. The only monomial that meets neither of these criteria is $x_0^2x_1^2x_2x_3$. But $x_0^2x_1^2x_2x_3 \in (x_0x_1x_2x_3) \subset I$, so every degree 6 monomial is in (A, B, C, D, E, F). This proves the proposition. In 5.3 we provide another proof. □

Proposition 2.4 (Case Four). *If X is a general sextic surface in $\mathbb{P}^3$, then there is a skew-symmetric matrix*

$$M : \mathscr{O}(-6) \oplus \mathscr{O}(-4)^3 \to \mathscr{O} \oplus \mathscr{O}(-2)^3$$

such that $\mathrm{pf}(M)$ *is a defining polynomial of* X*.*

Proof. Case Four is determined by $a = b = c = 4, d = e = f = 2$. Step 1: If $Q = 0$ defines a general quintic curve and $S = 0$ defines a general sextic curve both in $\mathbb{P}^2$, there exist homogeneous degree 2 polynomials $P_1, P_2 \in \mathbb{C}[x_0, x_1, x_2]$ such that $Q, S \in (P_1, P_2)$. This is because Q and S intersect at 30 distinct points transversely; as Noether's theorem says that if F and G in $\mathbb{P}^2$ of degrees f and g intersect transversely at fg distinct points and if H in $\mathbb{P}^2$ passes through all fg points, then $H \in (F, G)$. Then because 5 points in general position determine a conic, taking 4 points out of the 30 intersection points by Q and S, there is an at least 1-dimensional family of conics passing through those 4 points; in particular, take two different conics P_1 and P_2. According to Noether's theorem, $Q, S \in (P_1, P_2)$.

Step 2: A defining polynomial of X can be written as

$$ax_3^6 + lx_3^5 + px_3^4 + cx_3^3 + qx_3^2 + Qx_3 + S$$

where $a, l, p, c, q, Q, S \in \mathbb{C}[x_0, x_1, x_2]$. If X is general, then both Q and S are general. According to Step 1, there are P_1, P_2 such that

$$Q = \alpha P_1 + \beta P_2, S = \gamma P_1 + \delta P_2.$$

So

$$\begin{aligned} & ax_3^6 + lx_3^5 + px_3^4 + cx_3^3 + qx_3^2 + Qx_3 + S \\ =& x_3^2(ax_3^4 + lx_3^3 + px_3^2 + cx_3 + q) + P_1(\alpha x_3 + \gamma) + P_2(\beta x_3 + \delta). \end{aligned}$$

This proves the proposition. □

Proposition 2.5 (Case Five)**.** *If X is a general sextic surface in $\mathbb{P}^3$, then there is a skew-symmetric matrix*

$$M : \mathscr{O}(-5)^3 \oplus \mathscr{O}(-3) \to \mathscr{O}(-1)^3 \oplus \mathscr{O}(-3)$$

such that $\mathrm{pf}(M)$ *is a defining polynomial of X.*

Proof. Case Five is determined by $a = b = d = 4, c = e = f = 2$. Because Case Five and Case Four are paired to each other, if there is an indecomposable rank 2 ACM bundle of Case Four on a sextic surface, there is one of Case Five on that surface. Thus this proposition is equivalent to 2.4. □

Proposition 2.6 (Case Six). *If X is a general sextic surface in $\mathbb{P}^3$, then there is a skew-symmetric matrix*

$$M : \mathscr{O}(-4)^2 \oplus \mathscr{O}(-6)^2 \to \mathscr{O}(-1)^2 \oplus \mathscr{O}(-3)^2$$

such that $\mathrm{pf}(M)$ *is a defining polynomial of X.*

Proof. Case Six is determined by $a = 5, b = c = d = e = 3, f = 1$. It suffices to find a specific (A, B, C, D, E, F) such that $AF - BE + CD \neq 0$, D, E, F form a regular sequence and every degree 6 monomial is in (A, B, C, D, E, F). Pick

$$I = (A, B, C, D, E, F) = (x_1x_2^2x_3^2, x_0^3, x_2^3, x_3^3, x_1^3, x_0).$$

First of all, $AF - BE + CD = x_0x_1x_2^2x_3^2 - x_0^3x_1^3 + x_2^3x_3^3 \neq 0$. Next, x_3^3, x_1^3, x_0 form a regular sequence. Finally, every degree 6 monomial can be written as $x_0^ix_1^jx_2^kx_3^l$. If $i \geq 1$ or $\max(j, k, l) \geq 3$, then $x_0^ix_1^jx_2^kx_3^l \in (x_0, x_1^3, x_2^3, x_3^3) \subset I$. The only monomial that meets neither of these criteria is $x_1^2x_2^2x_3^2$. But $x_1^2x_2^2x_3^2 \in (x_1x_2^2x_3^2) \subset I$, so every degree 6 monomial is in (A, B, C, D, E, F). This proves the proposition. In 5.4 we provide another proof. $\square$

Proposition 2.7 (Case Seven). *If X is a general sextic surface in $\mathbb{P}^3$, then there is a skew-symmetric matrix $M : \mathscr{O}(-3) \oplus \mathscr{O}(-4) \oplus \mathscr{O}(-5) \oplus \mathscr{O}(-6) \to \mathscr{O} \oplus \mathscr{O}(-1) \oplus \mathscr{O}(-2) \oplus \mathscr{O}(-3)$ such that* $\mathrm{pf}(M)$ *is a defining polynomial of X.*

Proof. Case Seven is determined by $a = 5, b = 4, c = d = 3, e = 2, f = 1$. It suffices to find a specific (A, B, C, D, E, F) such that $AF - BE + CD \neq 0$, D, E, F form a regular sequence and every degree 6 monomial is in (A, B, C, D, E, F). Pick

$$I = (A, B, C, D, E, F) = (0, 0, x_0^3, x_1^3, x_3^2, x_2).$$

First of all, $AF - BE + CD = x_0^3 x_1^3 \neq 0$. Next, x_1^3, x_3^2, x_2 form a regular sequence. Finally every degree 6 monomial can be written as $x_0^i x_1^j x_2^k x_3^l$. If $\max(i, j) \geq 3$ or $k \geq 1$ or $l \geq 2$, then $x_0^i x_1^j x_2^k x_3^l \in (x_0^3, x_1^3, x_2, x_3^2) \subset I$. Every degree 6 monomial satisfy the criteria above, so every degree 6 monomial is in (A, B, C, D, E, F). This proves the proposition. In 5.5 we provide another proof. $\square$

Proposition 2.8 (Case Eight). *If X is a general sextic surface in $\mathbb{P}^3$, then there is a skew-symmetric matrix $M : \mathscr{O}(-4)^2 \oplus \mathscr{O}(-5) \oplus \mathscr{O}(-7) \to \mathscr{O} \oplus \mathscr{O}(-2) \oplus \mathscr{O}(-3)^2$ such that* $\mathrm{pf}(M)$ *is a defining polynomial of X.*

Proof. Case Eight is determined by $a = 5, b = c = 4, d = e = 2, f = 1$. A defining polynomial of X can be written as

$$ax_3^6 + lx_3^5 + px_3^4 + cx_3^3 + qx_3^2 + Qx_3 + S$$

where $a, l, p, c, q, Q, S \in \mathbb{C}[x_0, x_1, x_2]$. If X is general, then both Q and S are general. According to 2.4 Step 1, there are P_1, P_2 such that

$$Q = \alpha P_1 + \beta P_2, S = \gamma P_1 + \delta P_2.$$

So

$$\begin{aligned}
&ax_3^6 + lx_3^5 + px_3^4 + cx_3^3 + qx_3^2 + Qx_3 + S \\
=&x_3x_3(ax_3^4 + lx_3^3 + px_3^2 + cx_3 + q) + P_1(\alpha x_3 + \gamma) + P_2(\beta x_3 + \delta).
\end{aligned}$$

This proves the proposition. □

Proposition 2.9 (Case Nine)**.** *If X is a general sextic surface in $\mathbb{P}^3$, then there is a skew-symmetric matrix $M : \mathscr{O}(-3) \oplus \mathscr{O}(-5) \oplus \mathscr{O}(-6)^2 \to \mathscr{O}(-1)^2 \oplus \mathscr{O}(-2) \oplus \mathscr{O}(-4)$ such that* $\mathrm{pf}(M)$ *is a defining polynomial of X.*

Proof. Case Nine is determined by $a = 5, b = d = 4, c = e = 2, f = 1$. Because Case Nine and Case Eight are paired to each other, if there is an indecomposable rank 2 ACM bundle of Case Eight on a sextic surface, there is one of Case Nine on that surface. Thus this proposition is equivalent to 2.8. □

Proposition 2.10 (Case Ten)**.** *If X is a general sextic surface in $\mathbb{P}^3$, then there is a skew-symmetric matrix $M : \mathscr{O}(-3) \oplus \mathscr{O}(-5)^2 \oplus \mathscr{O}(-7) \to \mathscr{O} \oplus \mathscr{O}(-2)^2 \oplus \mathscr{O}(-4)$ such that* $\mathrm{pf}(M)$ *is a defining polynomial of X.*

Proof. Case Ten is determined by $a = b = 5, c = d = 3, e = f = 1$. The defining polynomial of a sextic surface can be written as

$$a_0x_3^6 + a_1x_3^5 + a_2x_3^4 + a_3x_3^3 + a_4x_3^2 + a_5x_3 + a_6,$$

where $a_0, ..., a_6 \in \mathbb{C}[x_0, x_1, x_2]$. a_6 can be written as

$$a_6 = b_0x_2^6 + b_1x_2^5 + b_2x_2^4 + b_3x_2^3 + b_4x_2^2 + b_5x_2 + b_6,$$

where $b_0, ..., b_6 \in \mathbb{C}[x_0, x_1]$. b_6, being homogeneous of degree 6, splits as a product of 6 linear polynomials. So

$$\begin{aligned} &a_0x_3^6 + a_1x_3^5 + a_2x_3^4 + a_3x_3^3 + a_4x_3^2 + a_5x_3 + a_6 \\ =&x_3(a_0x_3^5 + a_1x_3^4 + a_2x_3^3 + a_3x_3^2 + a_4x_3 + a_5) \\ +&x_2(b_0x_2^5 + b_1x_2^4 + b_2x_2^3 + b_3x_2^2 + b_4x_2 + b_5) + b_6 \end{aligned}$$

and b_6 can be expressed as a product of two polynomials of degree 3 each. This proves the proposition. □

Proposition 2.11 (Case Eleven)**.** *If X is a general sextic surface in $\mathbb{P}^3$, then there is a skew-symmetric matrix $M : \mathscr{O}(-3) \oplus \mathscr{O}(-4)^2 \oplus \mathscr{O}(-7) \to \mathscr{O}(1) \oplus \mathscr{O}(-2)^2 \oplus \mathscr{O}(-3)$ such that* $\mathrm{pf}(M)$ *is a defining polynomial of X.*

Proof. Case Eleven is determined by $a = b = 5, c = 4, d = 2, e = f = 1$. The same proof as in 2.10.

$$\begin{aligned} &a_0x_3^6 + a_1x_3^5 + a_2x_3^4 + a_3x_3^3 + a_4x_3^2 + a_5x_3 + a_6 \\ =&x_3(a_0x_3^5 + a_1x_3^4 + a_2x_3^3 + a_3x_3^2 + a_4x_3 + a_5) \\ +&x_2(b_0x_2^5 + b_1x_2^4 + b_2x_2^3 + b_3x_2^2 + b_4x_2 + b_5) + b_6. \end{aligned}$$

b_6 can be expressed as a product of a polynomials of degree 2 and a polynomial of degree 4. This proves the proposition. □

Proposition 2.12 (Case Twelve)**.** *If X is a general sextic surface in $\mathbb{P}^3$, then there is a skew-symmetric matrix $M : \mathscr{O}(-4)^3 \oplus \mathscr{O}(-8) \to \mathscr{O}(1) \oplus \mathscr{O}(-3)^3$ such that* $\mathrm{pf}(M)$ *is a defining polynomial of X.*

Proof. Case Twelve is determined by $a = b = c = 5, d = e = f = 1$. The same proof as in

2.10.

$$
\begin{aligned}
&a_0x_3^6 + a_1x_3^5 + a_2x_3^4 + a_3x_3^3 + a_4x_3^2 + a_5x_3 + a_6 \\
=&x_3(a_0x_3^5 + a_1x_3^4 + a_2x_3^3 + a_3x_3^2 + a_4x_3 + a_5) \\
+&x_2(b_0x_2^5 + b_1x_2^4 + b_2x_2^3 + b_3x_2^2 + b_4x_2 + b_5) + b_6.
\end{aligned}
$$

b_6 can be expressed as a product of a polynomials of degree 1 and a polynomial of degree 5. This proves the proposition. □

Proposition 2.13 (Case Thirteen)**.** *If X is a general sextic surface in $\mathbb{P}^3$, then there is a skew-symmetric matrix $M : \mathscr{O}(-2) \oplus \mathscr{O}(-5)^2 \oplus \mathscr{O}(-6) \to \mathscr{O} \oplus \mathscr{O}(-1)^2 \oplus \mathscr{O}(-4)$ such that* $\mathrm{pf}(M)$ *is a defining polynomial of X.*

Proof. Case Thirteen is determined by $a = b = 5, c = 2, d = 4, e = f = 1$. Because Case Thirteen and Case Eleven are paired to each other, if there is an indecomposable rank 2 ACM bundle of Case Eleven on a sextic surface, there is one of Case Thirteen on that surface. Thus this proposition is equivalent to 2.11. □

Proposition 2.14 (Case Fourteen)**.** *If X is a general sextic surface in $\mathbb{P}^3$, then there is a skew-symmetric matrix $M : \mathscr{O}(-2) \oplus \mathscr{O}(-6)^3 \to \mathscr{O}(-1)^3 \oplus \mathscr{O}(-5)$ such that* $\mathrm{pf}(M)$ *is a defining polynomial of X.*

Proof. Case Fourteen is determined by $a = b = d = 5, c = e = f = 1$. Because Case Fourteen and Case Twelve are paired to each other, if there is an indecomposable rank 2 ACM bundle of Case Twelve on a sextic surface, there is one of Case Fourteen on that surface. Thus this proposition is equivalent to 2.12. □

Chapter 3

Dimension Calculation in Fourteen Cases

In this chapter we compute for each of the fourteen cases, the dimension of the moduli space of indecomposable rank 2 ACM bundles belonging to that case on a general sextic surface.

Proposition 3.1 ([13]). *Let U denote the open subset of all skew-symmetric minimal maps $F_0^\vee(e-d) \xrightarrow{M} F_0$, where each point in U determines a rank 2 ACM bundle $\mathscr{E}$ on a general hypersurface $X \subset \mathbb{P}^n$ of degree d. The group $\mathrm{Aut}(F_0)$ acts on U by $(P, M) \mapsto PMP^t$. Then the map from $U/\mathrm{Aut}(F_0)$ to the set of isomorphism classes of pairs $(X, \mathscr{E})$ is bijective.*

Proposition 3.2 ([13]). *Under the action of $\mathrm{Aut}(F_0)$ on U by $(P, M) \mapsto PMP^t$, the stabilizer of $M \in U$ is the subgroup $\mathrm{stab}(\mathscr{E}_M)$ with two connected components corresponding to $\pm\mathrm{Id}_{F_0}$. The component $\mathrm{stab}^0(\mathscr{E}_M)$ containing Id_{F_0} is described below: when $\mathscr{E}_M$ is stable, $\mathrm{stab}^0(\mathscr{E}_M)$ is*

$$\{\mathrm{Id}_{F_0} + M\tau : \tau \in \mathrm{Hom}(F_0, F_1), \tau - \tau^\vee = \tau M \tau^\vee = \tau^\vee M \tau\};$$

when $\mathscr{E}_M$ *is unstable,* $\mathrm{stab}^0(\mathscr{E}_M)$ *is*

$$\{\mathrm{Id}_{F_0} + gS + M\tau :$$
$$g \in H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(2a_1 - e)), \tau \in \mathrm{Hom}(F_0, F_1), \tau - \tau^\vee = \tau M \tau^\vee = \tau^\vee M \tau\}.$$

Here S *is predetermined,* $e = c_1(\mathscr{E}_M)$, $F_0 = \oplus_i \mathscr{O}_{\mathbb{P}^n}(a_i)$ *with* $a_1 \geq a_2 \geq \dots$.

Remark 3.3. Let $\mathscr{E}$ be an indecomposable rank 2 ACM bundle on a hypersurface $X \subset \mathbb{P}^n$. If $c_1(\mathscr{E}) = 0$ or -1, then $\mathscr{E}$ is stable if and only if $H^0(X, \mathscr{E}) = 0$.

According to 3.1 and 3.2, for each case, the set of isomorphism classes of pairs $(X, \mathscr{E})$ is in one-to-one correspondence with the set of orbits $U/\mathrm{Aut}(F_0)$. So the dimension of isomorphism classes of pairs $(X, \mathscr{E})$ is

$$dp := \dim(U) - \dim(\mathrm{Aut}(F_0)) + \dim(\mathrm{stab}(\mathscr{E}_M)).$$

Because the space of sextic surfaces is isomorphic to $\mathbb{P}^{83}$, on a general sextic surface, the dimension of indecomposable rank 2 ACM bundles belonging to that case is $dp - 83$.

Case One: The space of skew-symmetric matrices

$$F_1 = \mathscr{O}_{\mathbb{P}^3}(-5)^4 \xrightarrow{M} F_0 = \mathscr{O}_{\mathbb{P}^3}(-2)^4$$

is isomorphic to $\mathbb{C}^{120}$. U is an open subset. $\mathrm{Aut}(F_0)$ is isomorphic to $\mathrm{GL}(4, \mathbb{C})$, which is of dimension 16. In this case $H^0(X, \mathscr{E}) = 0$, so $\mathscr{E}$ is stable. Because $\mathrm{Hom}(F_0, F_1) = 0$, $\mathrm{stab}(\mathscr{E}_M)$ is zero dimensional. So $dp = 120 - 16 + 0 = 104$ and on a general sextic surface, the dimension of indecomposable rank 2 ACM bundles of Case One is $dp - 83 = 21$.

Case Two: The space of skew-symmetric matrices

$$F_1 = \mathscr{O}_{\mathbb{P}^3}(-4)^2 \oplus \mathscr{O}_{\mathbb{P}^3}(-5)^2 \xrightarrow{M} F_0 = \mathscr{O}_{\mathbb{P}^3}(-1)^2 \oplus \mathscr{O}_{\mathbb{P}^3}(-2)^2$$

is isomorphic to $\mathbb{C}^{125}$. U is an open subset. $\mathrm{Aut}(F_0)$ is of dimension 24. In this case $H^0(X, \mathscr{E}) = 0$, so $\mathscr{E}$ is stable. Because $\mathrm{Hom}(F_0, F_1) = 0$, $\mathrm{stab}(\mathscr{E}_M)$ is zero dimensional. So $dp = 125 - 24 + 0 = 101$ and on a general sextic surface, the dimension of indecomposable rank 2 ACM bundles of Case Two is $dp - 83 = 18$.

Case Three: The space of skew-symmetric matrices

$$F_1 = \mathscr{O}(-6) \oplus \mathscr{O}(-5)^2 \oplus \mathscr{O}(-4) \xrightarrow{M} \mathscr{O}(-1) \oplus \mathscr{O}(-2)^2 \oplus \mathscr{O}(-3)$$

is isomorphic to $\mathbb{C}^{130}$. U is an open subset. $\mathrm{Aut}(F_0)$ is of dimension 32. In this case $H^0(X, \mathscr{E}) = 0$, so $\mathscr{E}$ is stable. Because $\mathrm{Hom}(F_0, F_1) = 0$, $\mathrm{stab}(\mathscr{E}_M)$ is zero dimensional. So $dp = 130 - 32 + 0 = 98$ and on a general sextic surface, the dimension of indecomposable rank 2 ACM bundles of Case Three is $dp - 83 = 15$.

Case Four: The space of skew-symmetric matrices

$$F_1 = \mathscr{O}(-6) \oplus \mathscr{O}(-4)^3 \xrightarrow{M} F_0 = \mathscr{O} \oplus \mathscr{O}(-2)^3$$

is isomorphic to $\mathbb{C}^{135}$. U is an open subset. $\mathrm{Aut}(F_0)$ is of dimension 40. In this case $H^0(X, \mathscr{E}) \neq 0$, so $\mathscr{E}$ is unstable. Because $H^0(\mathbb{P}^3, \mathscr{O}(2a_1 - e)) = H^0(\mathbb{P}^3, \mathscr{O}) = 1$ and $\mathrm{Hom}(F_0, F_1) = 0$, $\mathrm{stab}(\mathscr{E}_M)$ is one dimensional. So $dp = 135 - 40 + 1 = 96$ and on a general sextic surface, the dimension of indecomposable rank 2 ACM bundles of Case Four is $dp - 83 = 13$.

Case Five: Case Five and Case Four are paired to each other, so on a general sextic surface, the dimension of indecomposable rank 2 ACM bundles of Case Five is 13.

Case Six: The space of skew-symmetric matrices

$$F_1 = \mathscr{O}(-4)^2 \oplus \mathscr{O}(-6)^2 \xrightarrow{M} F_0 = \mathscr{O}(-1)^2 \oplus \mathscr{O}(-3)^2$$

is isomorphic to $\mathbb{C}^{140}$. U is an open subset. $\mathrm{Aut}(F_0)$ is of dimension 48. In this case $H^0(X, \mathscr{E}) = 0$, so $\mathscr{E}$ is stable. Because $\mathrm{Hom}(F_0, F_1) = 0$, $\mathrm{stab}(\mathscr{E}_M)$ is zero dimensional. So $dp = 140 - 48 + 0 = 92$ and on a general sextic surface, the dimension of indecomposable rank 2 ACM bundles of Case Six is $dp - 83 = 9$.

Case Seven: The space of skew-symmetric matrices

$$F_1 = \mathscr{O}(-3) \oplus \mathscr{O}(-4) \oplus \mathscr{O}(-5) \oplus \mathscr{O}(-6)$$
$$\xrightarrow{M} F_0 = \mathscr{O} \oplus \mathscr{O}(-1) \oplus \mathscr{O}(-2) \oplus \mathscr{O}(-3)$$

is isomorphic to $\mathbb{C}^{145}$. U is an open subset. $\mathrm{Aut}(F_0)$ is of dimension 56. In this case $H^0(X, \mathscr{E}) \neq 0$, so $\mathscr{E}$ is unstable. Let us compute $\mathrm{stab}(\mathscr{E}_M)$, namely, the stabilizers of M in $G := \mathrm{Aut}(F_0)$.

P is of the form

$$P = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ k_1 & c_2 & 0 & 0 \\ q_1 & k_2 & c_3 & 0 \\ t & q_2 & k_3 & c_4 \end{pmatrix}$$

where $c_1, ..., c_4 \in \mathbb{C}$, k_1, k_2, k_3 are of degrees 1, q_1, q_2 are of degrees 2 and t is of degree 3. M is of the form

$$M = \begin{pmatrix} 0 & k & q & s_1 \\ -k & 0 & s_2 & p \\ -q & -s_2 & 0 & y \\ -s_1 & -p & -y & 0 \end{pmatrix}$$

where k is of degree 5, q is of degree 4, s_1, s_2 are of degrees 3, p is of degree 2 and y is of degree 1. And there is a short exact sequence of groups

$$1 \to N := \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ k_1 & 1 & 0 & 0 \\ q_1 & k_2 & 1 & 0 \\ t & q_2 & k_3 & 1 \end{pmatrix} \right\} \to$$

$$G \to H := \left\{ \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \end{pmatrix} \right\} \to 1.$$

First of all let us look at the stabilizers of M in N. Let

$$n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ k_1 & 1 & 0 & 0 \\ q_1 & k_2 & 1 & 0 \\ t & q_2 & k_3 & 1 \end{pmatrix},$$

then

$$n^t M n = \begin{pmatrix} 0 & z_1 & z_2 & s_1 + k_1 p + q_1 y \\ * & 0 & s_2 - q_2 y + k_3(p + k_2 y) & p + k_2 y \\ * & * & 0 & y \\ * & * & * & 0 \end{pmatrix}$$

where

$$z_1 = k - q_1 s_2 - pt + q_2(k_1 p + s_1 + q_1 y) + k_2(q + k_1 s_2 - ty),$$

$$z_2 = q + k_1 s_2 - ty + k_3(k_1 p + s_1 + q_1 y).$$

So

$$n^t M n = M \Longrightarrow p + k_2 y = p \Longrightarrow k_2 = 0.$$

$$n^t M n = M \Longrightarrow s_1 + k_1 p + q_1 y = s_1 \Longrightarrow k_1 p + q_1 y = 0.$$

Because $\gcd(p, y) = 1$, $k_1 p + q_1 y = 0$ implies that $q_1 \in (p)$. So there is some $d_1 \in \mathbb{C}$ such that $q_1 = d_1 p$, and $k_1 = -d_1 y$.

$$n^t M n = M \Longrightarrow s_2 - q_2 y + k_3(p + k_2 y) = s_2 \Longrightarrow k_3 p - q_2 y = 0.$$

Again $\gcd(p, y) = 1$, so $q_2 \in (p)$ and $q_2 = d_2 p$ for some $d_2 \in \mathbb{C}$, and $k_3 = d_2 y$.

$$\begin{aligned} n^t M n = M &\Longrightarrow z_2 = q + k_1 s_2 - ty + k_3 s_1 = q \\ &\Longrightarrow (-d_1 s_2 + d_2 s_1 - t)y = 0 \\ &\Longrightarrow t = d_2 s_1 - d_1 s_2. \end{aligned}$$

Given the previous, $z_1 \equiv k$. So the stabilizers of M in N are two dimensional and are parametrized by (d_1, d_2).

Next let us look at the stabilizers of M in G. Because G is the internal semidirect product of N by H, each $x \in G$ has a unique expression $x = nh$ where $n \in N$ and $h \in H$. $x^t M x = M$ becomes

$$h^t n^t M n h = M \Longleftrightarrow n^t M n = (h^{-1})^t M h^{-1}.$$

So we seek pairs (n, h) such that $n^t M n = h^t M h$. $n^t M n$ was calculated above. On the other hand, let

$$h = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \end{pmatrix},$$

then

$$h^t M h = \begin{pmatrix} 0 & c_1c_2k & c_1c_3q & c_1c_4s_1 \\ -c_1c_2k & 0 & c_2c_3s_2 & c_2c_4p \\ -c_1c_3q & -c_2c_3s_2 & 0 & c_3c_4y \\ -c_1c_4s_1 & -c_2c_4p & -c_3c_4y & 0 \end{pmatrix}.$$

So $n^t M n = h^t M h$ implies that

$$y = c_3c_4y \Longrightarrow c_3c_4 = 1,$$

$$p + k_2y = c_2c_4p \Longrightarrow c_2c_4 = 1,$$

$$s_1 + k_1p + q_1y = c_1c_4s_1 \Longrightarrow c_1c_4 = 1,$$

$$s_2 - q_2y + k_3p = c_2c_3s_2 \Longrightarrow c_2c_3 = 1.$$

The above imply that $c_4 = \pm 1$ and $h = \pm I$. So $n^t M n = h^t M h \Longrightarrow n^t M n = M$. Since the stabilizers of M in N have been calculated to form a 2 dimensional vector space, we know the stabilizers of M in G are also 2 dimensional.

So $dp = 145 - 56 + 2 = 91$ and on a general sextic surface, the dimension of indecomposable rank 2 ACM bundles of Case Seven is $dp - 83 = 8$.

Case Eight: The space of skew-symmetric matrices

$$F_1 = \mathscr{O}(-4)^2 \oplus \mathscr{O}(-5) \oplus \mathscr{O}(-7) \xrightarrow{M} F_0 = \mathscr{O} \oplus \mathscr{O}(-2) \oplus \mathscr{O}(-3)^2$$

is isomorphic to $\mathbb{C}^{150}$. U is an open subset. $\mathrm{Aut}(F_0)$ is of dimension 64. In this case $H^0(X, \mathscr{E}) \neq 0$, so $\mathscr{E}$ is unstable. Because $H^0(\mathbb{P}^3, \mathscr{O}(2a_1 - e)) = H^0(\mathbb{P}^3, \mathscr{O}(1)) = 4$, and

$\mathrm{Hom}(F_0, F_1) = 0$, $\mathrm{stab}(\mathscr{E}_M)$ is four dimensional. So $dp = 150 - 64 + 4 = 90$ and on a general sextic surface, the dimension of indecomposable rank 2 ACM bundles of Case Eight is $dp - 83 = 7$.

Case Nine: Case Nine and Case Eight are paired to each other, so on a general sextic surface, the dimension of indecomposable rank 2 ACM bundles of Case Nine is 7.

Case Ten: The space of skew-symmetric matrices

$$F_1 = \mathscr{O}(-3) \oplus \mathscr{O}(-5)^2 \oplus \mathscr{O}(-7) \xrightarrow{M} F_0 = \mathscr{O} \oplus \mathscr{O}(-2)^2 \oplus \mathscr{O}(-4)$$

is isomorphic to $\mathbb{C}^{160}$. U is an open subset. $\mathrm{Aut}(F_0)$ is of dimension 81. In this case $H^0(X, \mathscr{E}) \neq 0$, so $\mathscr{E}$ is unstable. Let us compute $\mathrm{stab}(\mathscr{E}_M)$, namely, the stabilizers of M in $G := \mathrm{Aut}(F_0)$.

P is of the form

$$P = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ p_1 & c_2 & c_3 & 0 \\ p_2 & c_4 & c_5 & 0 \\ q & p_3 & p_4 & c_6 \end{pmatrix}$$

where $c_1, ..., c_6 \in \mathbb{C}$, $p_1, ..., p_4$ are of degrees 2 and q is of degree 4. M is of the form

$$M = \begin{pmatrix} 0 & y_1 & y_2 & t_1 \\ -y_1 & 0 & t_2 & l_1 \\ -y_2 & -t_2 & 0 & l_2 \\ -t_1 & -l_1 & -l_2 & 0 \end{pmatrix}$$

where y_1, y_2 are of degrees 5, t_1, t_2 are of degrees 3 and l_1, l_2 are linear. And there is a short exact sequence of groups

$$1 \to N := \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ p_1 & 1 & 0 & 0 \\ p_2 & 0 & 1 & 0 \\ q & p_3 & p_4 & 1 \end{pmatrix} \right\} \to$$

$$G \to H := \left\{ \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & c_3 & 0 \\ 0 & c_4 & c_5 & 0 \\ 0 & 0 & 0 & c_6 \end{pmatrix} \right\} \to 1.$$

First of all let's look at the stabilizers of M in N. Let

$$N \ni n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ p_1 & 1 & 0 & 0 \\ p_2 & 0 & 1 & 0 \\ q & p_3 & p_4 & 1 \end{pmatrix},$$

then

$$n^t M n = \begin{pmatrix} 0 & z_1 & z_2 & l_1p_1 + l_2p_2 + t_1 \\ & 0 & -l_2p_3 + l_1p_4 + t_2 & l_1 \\ & * & 0 & l_2 \\ & * & * & 0 \end{pmatrix},$$

where

$$z_1 = y_1 - l_1 q - p_2 t_2 + p_3(l_1 p_1 + l_2 p_2 + t_1),$$

$$z_2 = y_2 - l_2 q + p_1 t_2 + p_4(l_1 p_1 + l_2 p_2 + t_1).$$

So

$$n^t M n = M \Longrightarrow l_1 p_1 + l_2 p_2 + t_1 = t_1 \Longrightarrow l_1 p_1 + l_2 p_2 = 0.$$

Because $\gcd(l_1, l_2) = 1$ and (l_1) is prime,

$$l_1 p_1 + l_2 p_2 = 0 \Longrightarrow l_2 p_2 \in (l_1) \Longrightarrow p_2 \in (l_1).$$

So there is some g_1 of degree 1 such that $p_2 = g_1 l_1$. So $p_1 = -g_1 l_2$.

$$n^t M n = M \Longrightarrow -l_2 p_3 + l_1 p_4 + t_2 = t_2 \Longrightarrow l_1 p_4 - l_2 p_3 = 0.$$

For the same reason as above, there is some g_2 of degree 1 such that $p_3 = g_2 l_1$ and $p_4 = g_2 l_2$. Next,

$$\begin{aligned} n^t M n = M &\Longrightarrow z_2 = y_2 \\ &\Longrightarrow -l_2 q + p_1 t_2 + p_4(l_1 p_1 + l_2 p_2 + t_1) = 0 \\ &\Longrightarrow -q l_2 + p_1 t_2 + p_4 t_1 = 0 \\ &\Longrightarrow q = g_2 t_1 - g_1 t_2. \end{aligned}$$

Finally, given what we have already deduced, $z_1 \equiv y_1$. So the stabilizers of M in N form an 8 dimensional vector space and is parametrized by g_1 and g_2.

Next let us look at the stabilizers of M in G. Because G is the internal semidirect product of N by H, each $x \in G$ has a unique expression $x = nh$ where $n \in N$ and $h \in H$. $x^t M x = M$ becomes

$$h^t n^t M n h = M \Longleftrightarrow n^t M n = (h^{-1})^t M h^{-1}.$$

So we seek pairs (n, h) such that $n^t M n = h^t M h$. $n^t M n$ was calculated above. On the other hand, let

$$h = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & c_3 & 0 \\ 0 & c_4 & c_5 & 0 \\ 0 & 0 & 0 & c_6 \end{pmatrix},$$

then

$$h^t M h = \begin{pmatrix} 0 & c_1(c_2 y_1 + c_4 y_2) & c_1(c_3 y_1 + c_5 y_2) & c_1 c_6 t_1 \\ * & 0 & (c_2 c_5 - c_3 c_4) t_2 & c_6(c_2 l_1 + c_4 l_2) \\ * & * & 0 & c_6(c_3 l_1 + c_5 l_2) \\ * & * & * & 0 \end{pmatrix}.$$

So

$$n^t M n = h^t M h \Longrightarrow l_2 = c_6(c_3 l_1 + c_5 l_2) \Longrightarrow c_5 c_6 = 1, c_3 c_6 = 0.$$

$$n^t M n = h^t M h \Longrightarrow l_1 = c_6(c_2 l_1 + c_4 l_2) \Longrightarrow c_2 c_6 = 1, c_4 c_6 = 0.$$

It follows that $c_3 = c_4 = 0$.

$$n^t M n = h^t M h \Longrightarrow l_1 p_1 + l_2 p_2 + t_1 = c_1 c_6 t_1 \Longrightarrow c_1 c_6 = 1.$$

$$n^t M n = h^t M h \Longrightarrow -l_2 p_3 + l_1 p_4 + t_2 = (c_2 c_5 - c_3 c_4) t_2 \Longrightarrow c_2 c_5 = 1.$$

The above equations hold because $t_1, t_2 \notin (l_1, l_2)$. $c_5 c_6 = c_2 c_6 = c_1 c_6 = c_2 c_5 = 1$ together imply that $h = \pm I$. So $n^t M n = h^t M h \Longrightarrow h = \pm I, n^t M n = M$. As the stabilizers of M in N have been calculated to form an 8 dimensional vector space, we conclude that the stabilizers of M in G are also 8 dimensional.

So $dp = 160 - 81 + 8 = 87$ and on a general sextic surface, the dimension of indecomposable rank 2 ACM bundles of Case Ten is $dp - 83 = 4$.

Case Eleven: The space of skew-symmetric matrices

$$F_1 = \mathscr{O}(-3) \oplus \mathscr{O}(-4)^2 \oplus \mathscr{O}(-7) \xrightarrow{M} F_0 = \mathscr{O}(1) \oplus \mathscr{O}(-2)^2 \oplus \mathscr{O}(-3)$$

is isomorphic to $\mathbb{C}^{165}$. U is an open subset. $\mathrm{Aut}(F_0)$ is of dimension 89. In this case $H^0(X, \mathscr{E}) \neq 0$, so $\mathscr{E}$ is unstable. Let us compute $\mathrm{stab}(\mathscr{E}_M)$, namely, the stabilizers of M in $G := \mathrm{Aut}(F_0)$.

P is of the form

$$P = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ t_1 & c_2 & c_3 & 0 \\ t_2 & c_4 & c_5 & 0 \\ q & k_1 & k_2 & c_6 \end{pmatrix}$$

where $c_1, ..., c_6 \in \mathbb{C}$, k_1, k_2 are linear, t_1, t_2 are of degrees 3 and q is of degree 4. M is of the form

$$M = \begin{pmatrix} 0 & y_1 & y_2 & u \\ -y_1 & 0 & v & l_1 \\ -y_2 & -v & 0 & l_2 \\ -u & -l_1 & -l_2 & 0 \end{pmatrix}$$

where y_1, y_2 are of degrees 5, u is of degree 4, v is of degree 2, l_1, l_2 are linear. And there is a short exact sequence of groups

$$1 \to N := \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & 1 & 0 & 0 \\ t_2 & 0 & 1 & 0 \\ q & k_1 & k_2 & 1 \end{pmatrix} \right\} \to$$

$$G \to H := \left\{ \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & c_3 & 0 \\ 0 & c_4 & c_5 & 0 \\ 0 & 0 & 0 & c_6 \end{pmatrix} \right\} \to 1.$$

First of all let us look at the stabilizers of M in N. Let

$$N \ni n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & 1 & 0 & 0 \\ t_2 & 0 & 1 & 0 \\ q & k_1 & k_2 & 1 \end{pmatrix},$$

then

$$n^t M n = \begin{pmatrix} 0 & z_1 & z_2 & l_1t_1 + l_2t_2 + u \\ * & 0 & l_1k_2 - l_2k_1 + v & l_1 \\ * & * & 0 & l_2 \\ * & * & * & 0 \end{pmatrix}$$

where

$$z_1 = -l_1q + k_1(l_1t_1 + l_2t_2 + u) - t_2v + y_1,$$

$$z_2 = -l_2q + k_2(l_1t_1 + l_2t_2 + u) + t_1v + y_2.$$

So

$$n^t M n = M \Longrightarrow l_1t_1 + l_2t_2 + u = u \Longrightarrow l_1t_1 + l_2t_2 = 0.$$

As $\gcd(l_1, l_2) = 1$ and (l_1) is prime,

$$l_1t_1 + l_2t_2 = 0 \Longrightarrow l_2t_2 \in (l_1) \Longrightarrow t_2 \in (l_1).$$

So there is some g_1 of degree 2 such that $t_2 = g_1l_1$. So $t_1 = -g_1l_2$.

$$n^t M n = M \Longrightarrow l_1k_2 - l_2k_1 + v = v \Longrightarrow l_1k_2 - l_2k_1 = 0.$$

For the same reason as above, there is some $d_1 \in \mathbb{C}$ such that $k_1 = d_1l_1$ and $k_2 = d_1l_2$.

Next,

$$\begin{aligned} n^tMn = M &\Longrightarrow z_2 = y_2 \\ &\Longrightarrow -l_2q + k_2(l_1t_1 + l_2t_2 + u) + t_1v = 0 \\ &\Longrightarrow -l_2q + k_2u + t_1v = 0 \\ &\Longrightarrow q = d_1u - g_1v. \end{aligned}$$

Finally given what we have already deduced, $z_1 \equiv y_1$. So the stabilizers of M in N form an 11 dimensional vector space and is parametrized by g_1 and d_1.

Next let us look at the stabilizers of M in G. Because G is the internal semidirect product of N by H, each $x \in G$ has a unique expression $x = nh$ where $n \in N$ and $h \in H$. $x^tMx = M$ becomes

$$h^tn^tMnh = M \Longleftrightarrow n^tMn = (h^{-1})^tMh^{-1}.$$

So we seek pairs (n, h) such that $n^tMn = h^tMh$. n^tMn was calculated above. On the other hand, let

$$h = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & c_3 & 0 \\ 0 & c_4 & c_5 & 0 \\ 0 & 0 & 0 & c_6 \end{pmatrix},$$

then

$$h^t M h = \begin{pmatrix} 0 & c_1(c_2y_1+c_4y_2) & c_1(c_3y_1+c_5y_2) & c_1c_6u \\ * & 0 & (c_2c_5-c_3c_4)v & c_6(c_2l_1+c_4l_2) \\ * & * & 0 & c_6(c_3l_1+c_5l_2) \\ * & * & * & 0 \end{pmatrix}.$$

So

$$n^tMn = h^tMh \Longrightarrow l_2 = c_6(c_3l_1+c_5l_2) \Longrightarrow c_5c_6 = 1, c_3c_6 = 0.$$

$$n^tMn = h^tMh \Longrightarrow l_1 = c_6(c_2l_1+c_4l_2) \Longrightarrow c_2c_6 = 1, c_4c_6 = 0.$$

It follows that $c_3 = c_4 = 0$.

$$n^tMn = h^tMh \Longrightarrow u = c_1c_6u \Longrightarrow c_1c_6 = 1.$$

$$n^tMn = h^tMh \Longrightarrow l_1k_2 - l_2k_1 + v = (c_2c_5 - c_3c_4)v \Longrightarrow c_2c_5 = 1.$$

The last equation holds because $v \notin (l_1, l_2)$. $c_5c_6 = c_2c_6 = c_1c_6 = c_2c_5 = 1$ together imply that $h = \pm I$. So $n^tMn = h^tMh \Longrightarrow h = \pm I, n^tMn = M$. As the stabilizers of M in N have been calculated to form an 11 dimensional vector space, we conclude that the stabilizers of M in G are also 11 dimensional.

So $dp = 165 - 89 + 11 = 87$ and on a general sextic surface, the dimension of indecomposable rank 2 ACM bundles of Case Eleven is $dp - 83 = 4$.

Case Twelve: The space of skew-symmetric matrices

$$F_1 = \mathscr{O}(-4)^3 \oplus \mathscr{O}(-8) \xrightarrow{M} F_0 = \mathscr{O}(1) \oplus \mathscr{O}(-3)^3$$

is isomorphic to $\mathbb{C}^{180}$. U is an open subset. $\mathrm{Aut}(F_0)$ is of dimension 115. In this case $H^0(X, \mathscr{E}) \neq 0$, so $\mathscr{E}$ is unstable. Because $H^0(\mathbb{P}^3, \mathscr{O}(2a_1 - e)) = H^0(\mathbb{P}^3, \mathscr{O}(3)) = 20$, and $\mathrm{Hom}(F_0, F_1) = 0$, $\mathrm{stab}(\mathscr{E}_M)$ is twenty dimensional. So $dp = 180 - 115 + 20 = 85$ and on a general sextic surface, the dimension of indecomposable rank 2 ACM bundles of Case Twelve is $dp - 83 = 2$.

Case Thirteen: Case Thirteen and Case Eleven are paired to each other, so on a general sextic surface, the dimension of indecomposable rank 2 ACM bundles of Case Thirteen is 4.

Case Fourteen: Case Fourteen and Case Twelve are paired to each other, so on a general sextic surface, the dimension of indecomposable rank 2 ACM bundles of Case Fourteen is 2.

Chapter 4

Rank 2 ACM Bundles on Quartic Surfaces

Let us apply the same machinery to study four generated indecomposable rank 2 ACM bundles on a general quartic surface. We compute in this chapter the dimension of the moduli space as we did on a general sextic surface. Here is a summary.

Case	Minimal Resolution
One	$\mathscr{O}_{\mathbb{P}^3}(-3)^4 \xrightarrow{M} \mathscr{O}_{\mathbb{P}^3}(-1)^4$
Two	$\mathscr{O}(-3)^2 \oplus \mathscr{O}(-4)^2 \xrightarrow{M} \mathscr{O}(-1)^2 \oplus \mathscr{O}(-2)^2$
Three	$\mathscr{O}(-4) \oplus \mathscr{O}(-3)^2 \oplus \mathscr{O}(-2) \xrightarrow{M} \mathscr{O} \oplus \mathscr{O}(-1)^2 \oplus \mathscr{O}(-2)$
Four	$\mathscr{O}(-5) \oplus \mathscr{O}(-3)^3 \xrightarrow{M} \mathscr{O} \oplus \mathscr{O}(-2)^3$
Five	$\mathscr{O}(-4)^3 \oplus \mathscr{O}(-2) \xrightarrow{M} \mathscr{O}(-1)^3 \oplus \mathscr{O}(-3)$

Table 4.1: ACM Bundles on Quartic Surfaces, Part One

Case	Case Paired With	c_1	Dimension of Moduli
One		0	10
Two		-1	6
Three		0	4
Four	Case Five	-1	2
Five	Case Four	-1	2

Table 4.2: ACM Bundles on Quartic Surfaces, Part Two

4.1 Five Cases

Let us use X to denote a *general* quartic surface in $\mathbb{P}^3$. So X is smooth and $\mathrm{Pic}(X) = \mathbb{Z}$. We want to find every four generated indecomposable rank 2 ACM bundle $\mathscr{E}$ on X. According to 1.5, this is equivalent to finding every 4×4 skew-symmetric minimal matrix M whose Pfaffian is a defining polynomial of X. Write $M = (m_{ij})_{4\times 4}$. Because $\mathrm{pf}(M)$ is homogeneous of degree 4,

$$\deg(m_{12}) + \deg(m_{34}) = \deg(m_{13}) + \deg(m_{24}) = \deg(m_{14}) + \deg(m_{23}) = 4.$$

According to 1.8, off-diagonal entries of M are of positive degrees. We can arranged M in such a way that the degrees decrease in each row from left to right and in each column from up to down. As a result, degreewise the upper triangular portion of M must be one of the following:

Case One:

$$\begin{pmatrix} 2 & 2 & 2 \\ & 2 & 2 \\ & & 2 \\ & & \end{pmatrix}$$

Case Two:

$$\begin{pmatrix} 3 & 2 & 2 \\ & 2 & 2 \\ & & 1 \\ & & \end{pmatrix}$$

Case Three:

$$\begin{pmatrix} 3 & 3 & 2 \\ & 2 & 1 \\ & & 1 \\ & & \end{pmatrix}$$

Case Four:

$$\begin{pmatrix} 3 & 3 & 3 \\ & 1 & 1 \\ & & 1 \\ & & \end{pmatrix}$$

Case Five:

$$\begin{pmatrix} 3 & 3 & 1 \\ & 3 & 1 \\ & & 1 \\ & & \end{pmatrix}$$

The minimal resolution of $\mathscr{E}$ is of the form

$$0 \to F_1 := \oplus_{i=1}^4 \mathscr{O}_{\mathbb{P}^n}(a_i) \xrightarrow{M} F_0 := \oplus_{i=1}^4 \mathscr{O}_{\mathbb{P}^n}(b_i) \to \mathscr{E} \to 0.$$

Moreover, $F_1 = F_0^\vee(e-d)$ where $d = 4$ and $e = c_1(\mathscr{E})$. We arrange F_1 and F_0 so that $a_1 \geq \cdots \geq a_4$ and $b_1 \geq \cdots \geq b_4$. For our purpose without loss of generality, we may assume $e = 0$ or $e = -1$. Entrywise M is of the form

$$\begin{pmatrix} \mathscr{O}(a_4) \to \mathscr{O}(b_1) & \mathscr{O}(a_4) \to \mathscr{O}(b_2) & \mathscr{O}(a_4) \to \mathscr{O}(b_3) & \mathscr{O}(a_4) \to \mathscr{O}(b_4) \\ \mathscr{O}(a_3) \to \mathscr{O}(b_1) & \mathscr{O}(a_3) \to \mathscr{O}(b_2) & \mathscr{O}(a_3) \to \mathscr{O}(b_3) & \mathscr{O}(a_3) \to \mathscr{O}(b_4) \\ \mathscr{O}(a_2) \to \mathscr{O}(b_1) & \mathscr{O}(a_2) \to \mathscr{O}(b_2) & \mathscr{O}(a_2) \to \mathscr{O}(b_3) & \mathscr{O}(a_2) \to \mathscr{O}(b_4) \\ \mathscr{O}(a_1) \to \mathscr{O}(b_1) & \mathscr{O}(a_1) \to \mathscr{O}(b_2) & \mathscr{O}(a_1) \to \mathscr{O}(b_3) & \mathscr{O}(a_1) \to \mathscr{O}(b_4) \end{pmatrix}.$$

In particular, $\deg(m_{12}) = b_2 - a_4$ and $\deg(m_{34}) = b_4 - a_2$.

$$\deg(m_{12}) + \deg(m_{34}) = 4 \Longrightarrow b_2 + b_4 - a_2 - a_4 = 4.$$

Because $a_2 = -b_3 + e - d$ and $a_4 = -b_1 + e - d$, plug into the previous equation and we get

$$\Sigma_{j=1}^{4} b_j - 2e + 2d = 4 \Longrightarrow \Sigma_{j=1}^{4} b_j - 2e + 4 = 0 \tag{$\star\star$}$$

Case One: $b_2 - a_4 = b_3 - a_4 = b_4 - a_4 = b_3 - a_3 = b_4 - a_3 = b_4 - a_2 = 2$, together with the equations $a_i = -b_{5-i} + e - d$, we get $a_1 = a_2 = a_3 = a_4$ and $b_1 = b_2 = b_3 = b_4$. Plugging into ($\star\star$), we get $4b_1 - 2e + 4 = 0$. So $e = 0$ and $b_1 = -1$. So in this case the minimal resolution of $\mathscr{E}$ is

$$0 \to \mathscr{O}_{\mathbb{P}^3}(-3)^4 \xrightarrow{M} \mathscr{O}_{\mathbb{P}^3}(-1)^4 \to \mathscr{E} \to 0. \tag{4.1}$$

Case Two: $b_2 - a_4 = 3, b_3 - a_4 = b_4 - a_4 = b_3 - a_3 = b_4 - a_3 = 2, b_4 - a_2 = 1$, so $a_2 - 1 = a_3 = a_4$ and $b_2 - 1 = b_3 = b_4$. With the equations $a_i = -b_{5-i} + e - d$, we get $a_1 = a_2$ and $b_1 = b_2$. Plugging into ($\star\star$), we get $4b_1 - 2e + 2 = 0$. So $e = -1$ and $b_1 = -1$. In this case the minimal resolution of $\mathscr{E}$ is

$$0 \to \mathscr{O}(-3)^2 \oplus \mathscr{O}(-4)^2 \xrightarrow{M} \mathscr{O}(-1)^2 \oplus \mathscr{O}(-2)^2 \to \mathscr{E} \to 0. \tag{4.2}$$

Case Three: $b_2 - a_4 = b_3 - a_4 = 3, b_4 - a_4 = b_3 - a_3 = 2, b_4 - a_3 = b_4 - a_2 = 1$, so $a_2 = a_3 = a_4 + 1$ and $b_2 = b_3 = b_4 + 1$. With the equations $a_i = -b_{5-i} + e - d$ we get $a_1 = a_2 + 1$ and $b_1 = b_2 + 1$. Plugging into ($\star\star$), we get $4b_2 - 2e + 4 = 0$. So $e = 0$ and

$b_2 = -1$. In this case the minimal resolution of $\mathscr{E}$ is

$$0 \to \mathscr{O}(-4) \oplus \mathscr{O}(-3)^2 \oplus \mathscr{O}(-2) \xrightarrow{M} \mathscr{O} \oplus \mathscr{O}(-1)^2 \oplus \mathscr{O}(-2) \to \mathscr{E} \to 0. \tag{4.3}$$

Case Four: $b_2 - a_4 = b_3 - a_4 = b_4 - a_4 = 3, b_3 - a_3 = b_4 - a_3 = b_4 - a_2 = 1$, so $a_2 = a_3 = a_4 + 2$ and $b_2 = b_3 = b_4$. With the equations $a_i = -b_{5-i} + e - d$ we get $a_1 = a_2$ and $b_1 = b_2 + 2$. Plugging into $(\star\star)$, we get $4b_2 - 2e + 6 = 0$. So $e = -1$ and $b_2 = -2$. In this case the minimal resolution of $\mathscr{E}$ is

$$0 \to \mathscr{O}(-5) \oplus \mathscr{O}(-3)^3 \xrightarrow{M} \mathscr{O} \oplus \mathscr{O}(-2)^3 \to \mathscr{E} \to 0. \tag{4.4}$$

Case Five: $b_2 - a_4 = b_3 - a_4 = b_3 - a_3 = 3, b_4 - a_4 = b_4 - a_3 = b_4 - a_2 = 1$, so $a_2 = a_3 = a_4$ and $b_2 = b_3 = b_4 + 2$. With the equations $a_i = -b_{5-i} + e - d$ we get $a_1 = a_2 + 2$ and $b_1 = b_2$. Plugging into $(\star\star)$, we get $4b_1 - 2e + 2 = 0$. So $e = -1$ and $b_1 = -1$. In this case the minimal resolution of $\mathscr{E}$ is

$$0 \to \mathscr{O}(-4)^3 \oplus \mathscr{O}(-2) \xrightarrow{M} \mathscr{O}(-1)^3 \oplus \mathscr{O}(-3) \to \mathscr{E} \to 0. \tag{4.5}$$

Notice its pair is

$$0 \to \mathscr{O}(-5)^3 \oplus \mathscr{O}(-7) \to \mathscr{O}(-4)^3 \oplus \mathscr{O}(-2) \to \mathscr{G} \to 0,$$

which up to twist by $\mathscr{O}(2)$ is the same short exact sequence as in Case Four. So Case Five and Case Four are paired to each other.

4.2 Existence in Five Cases

In this section we prove for each of the 5 cases, there is an indecomposable rank 2 ACM bundle belonging to that case on a general quartic surface.

Proposition 4.1 (Case One)**.** *A general quartic surface X can be realized as the zero variety associated to the Pfaffian of $M : \mathscr{O}_{\mathbb{P}^3}(-3)^4 \to \mathscr{O}_{\mathbb{P}^3}(-1)^4$.*

Proof. Step 1: If $Q = 0$ defines a general quartic curve and $C = 0$ defines a general cubic curve both in $\mathbb{P}^2$, then there exist homogeneous degree 2 polynomials $P_1, P_2 \in \mathbb{C}[x_0, x_1, x_2]$ such that $Q, C \in (P_1, P_2)$. This is because Q and C intersect at 12 distinct points transversely; as Noether's theorem says that if F and G in $\mathbb{P}^2$ of degrees f and g intersect transversely at fg distinct points and if H in $\mathbb{P}^2$ passes all fg points, then $H \in (F, G)$. Then because 5 points in general position determine a conic, taking 4 points out of the 12 intersection points by Q and C, there is an at least 1-dimensional family of conics passing through those 4 points; in particular, take two different conics P_1 and P_2. According to Noether's theorem, $Q, C \in (P_1, P_2)$.

Step 2: A defining polynomial of X can be written as

$$ax_3^4 + lx_3^3 + qx_3^2 + cx_3 + Q$$

where $a, l, q, c, Q \in \mathbb{C}[x_0, x_1, x_2]$. If X is general, then both c and Q are general. According to Step 1, there are P_1, P_2 such that

$$c = \alpha P_1 + \beta P_2, Q = \gamma P_1 + \delta P_2.$$

So

$$
\begin{aligned}
& ax_3^4 + lx_3^3 + qx_3^2 + cx_3 + Q \\
=& ax_3^4 + lx_3^3 + qx_3^2 + (\alpha P_1 + \beta P_2)x_3 + \gamma P_1 + \delta P_2 \\
=& x_3^2(ax_3^2 + lx_3 + q) + P_1(\alpha x_3 + \gamma) + P_2(\beta x_3 + \delta).
\end{aligned}
$$

This proves the proposition. □

Proposition 4.2 (Case Two). *If X is a general quartic surface in $\mathbb{P}^3$, then there is a skew-symmetric matrix $M : \mathscr{O}(-3)^2 \oplus \mathscr{O}(-4)^2 \to \mathscr{O}(-1)^2 \oplus \mathscr{O}(-2)^2$ such that* $\mathrm{pf}(M)$ *is a defining polynomial of X.*

Proof. As proved in 4.1, a defining polynomial of X can be written as

$$
\begin{aligned}
& ax_3^4 + lx_3^3 + qx_3^2 + cx_3 + Q \\
=& ax_3^4 + lx_3^3 + qx_3^2 + (\alpha P_1 + \beta P_2)x_3 + \gamma P_1 + \delta P_2 \\
=& x_3(ax_3^3 + lx_3^2 + qx_3) + P_1(\alpha x_3 + \gamma) + P_2(\beta x_3 + \delta).
\end{aligned}
$$

This proves the proposition. □

Proposition 4.3 (Case Three). *If X is a general quartic surface in $\mathbb{P}^3$, then there is a skew-symmetric matrix*

$$M : \mathscr{O}(-4) \oplus \mathscr{O}(-3)^2 \oplus \mathscr{O}(-2) \to \mathscr{O} \oplus \mathscr{O}(-1)^2 \oplus \mathscr{O}(-2)$$

such that $\mathrm{pf}(M)$ *is a defining polynomial of X.*

Proof. A defining polynomial of X can be written as

$$ax_3^4 + lx_3^3 + qx_3^2 + cx_3 + Q = x_3(ax_3^3 + lx_3^2 + qx_3 + c) + Q.$$

Write $Q = a_0x_2^4 + l_0x_2^3 + q_0x_2^2 + c_0x_2 + Q_0$ where $a_0, l_0, q_0, c_0, Q_0 \in \mathbb{C}[x_0, x_1]$, then $Q = x_2(a_0x_2^3 + l_0x_2^2 + q_0x_2 + c_0) + Q_0$, and Q_0 splits. So Q_0 can be written as a product of two quadratic polynomials, $Q_0 = q_1q_2$. So

$$\begin{aligned} & ax_3^4 + lx_3^3 + qx_3^2 + cx_3 + Q \\ =& x_3(ax_3^3 + lx_3^2 + qx_3 + c) + x_2(a_0x_2^3 + l_0x_2^2 + q_0x_2 + c_0) + q_1q_2. \end{aligned}$$

This proves the proposition. $\square$

Proposition 4.4 (Case Four). *If X is a general quartic surface in $\mathbb{P}^3$, then there is a skew-symmetric matrix*

$$M : \mathscr{O}(-5) \oplus \mathscr{O}(-3)^3 \to \mathscr{O} \oplus \mathscr{O}(-2)^3$$

such that $\mathrm{pf}(M)$ *is a defining polynomial of X.*

Proof. As proved in 4.3, a defining polynomial of X can be written as

$$\begin{aligned} & ax_3^4 + lx_3^3 + qx_3^2 + cx_3 + Q \\ =& x_3(ax_3^3 + lx_3^2 + qx_3 + c) + x_2(a_0x_2^3 + l_0x_2^2 + q_0x_2 + c_0) + Q_0, \end{aligned}$$

and Q_0 splits. So Q_0 can be written as a product of a cubic polynomial and a linear polynomials, $Q_0 = q_1 q_2$, and

$$\begin{aligned} & ax_3^4 + lx_3^3 + qx_3^2 + cx_3 + Q \\ =& x_3(ax_3^3 + lx_3^2 + qx_3 + c) + x_2(a_0x_2^3 + l_0x_2^2 + q_0x_2 + c_0) + q_1q_2. \end{aligned}$$

This proves the proposition. $\square$

Proposition 4.5 (Case Five)**.** *If X is a general quartic surface in $\mathbb{P}^3$, then there is a skew-symmetric matrix*

$$M : \mathscr{O}(-4)^3 \oplus \mathscr{O}(-2) \to \mathscr{O}(-1)^3 \oplus \mathscr{O}(-3)$$

such that $\mathrm{pf}(M)$ *is a defining polynomial of X.*

Proof. Because Case Five and Case Four are paired to each other, if there is an indecomposable rank 2 ACM bundle of Case Four on a quartic surface, there is one of Case Five on that surface. Thus this proposition is equivalent to 4.4. $\square$

4.3 Dimension Calculation in Five Cases

In this section we compute for each of the five cases, the dimension of the moduli space of indecomposable rank 2 ACM bundles belonging to that case on a general quartic surface.

Let U denote the open subset of all skew-symmetric minimal maps $F_0^\vee(e-d) \xrightarrow{M} F_0$, where each point in U determines a rank 2 ACM bundle $\mathscr{E}$ on a general hypersurface $X \subset \mathbb{P}^n$ of degree d. The group $\mathrm{Aut}(F_0)$ acts on U by $(P, M) \mapsto PMP^t$. According to 3.1 and 3.2, for each case, the set of isomorphism classes of pairs $(X, \mathscr{E})$ is in one-to-one correspondence with the set of orbits $U/\mathrm{Aut}(F_0)$. So the dimension of isomorphism classes of pairs $(X, \mathscr{E})$ is

$$dp := \dim(U) - \dim(\mathrm{Aut}(F_0)) + \dim(\mathrm{stab}(\mathscr{E}_M)).$$

Because the space of quartic surfaces is isomorphic to $\mathbb{P}^{34}$, on a general quartic surface, the dimension of indecomposable rank 2 ACM bundles belonging to that case is $dp - 34$.

Case One: The space of skew-symmetric matrices

$$F_1 = \mathscr{O}_{\mathbb{P}^3}(-3)^4 \xrightarrow{M} F_0 = \mathscr{O}_{\mathbb{P}^3}(-1)^4$$

is isomorphic to $\mathbb{C}^{60}$. U is an open subset. $\mathrm{Aut}(F_0)$ is isomorphic to $\mathrm{GL}(4, \mathbb{C})$, which is of dimension 16. In this case $H^0(X, \mathscr{E}) = 0$, so $\mathscr{E}$ is stable. Because $\mathrm{Hom}(F_0, F_1) = 0$, $\mathrm{stab}(\mathscr{E}_M)$ is zero dimensional. So $dp = 60-16+0 = 44$ and on a general quartic surface, the dimension of indecomposable rank 2 ACM bundles of Case One is $dp - 34 = 10$.

Case Two: The space of skew-symmetric matrices

$$F_1 = \mathscr{O}(-3)^2 \oplus \mathscr{O}(-4)^2 \xrightarrow{M} F_0 = \mathscr{O}(-1)^2 \oplus \mathscr{O}(-2)^2$$

is isomorphic to $\mathbb{C}^{64}$. U is an open subset. $\mathrm{Aut}(F_0)$ is of dimension 24. In this case $H^0(X, \mathscr{E}) = 0$, so $\mathscr{E}$ is stable. Because $\mathrm{Hom}(F_0, F_1) = 0$, $\mathrm{stab}(\mathscr{E}_M)$ is zero dimensional. So $dp = 64 - 24 + 0 = 40$ and on a general quartic surface, the dimension of indecomposable rank 2 ACM bundles of Case Two is $dp - 34 = 6$.

Case Three: The space of skew-symmetric matrices

$$F_1 = \mathscr{O}(-4) \oplus \mathscr{O}(-3)^2 \oplus \mathscr{O}(-2) \xrightarrow{M} F_0 = \mathscr{O} \oplus \mathscr{O}(-1)^2 \oplus \mathscr{O}(-2)$$

is isomorphic to $\mathbb{C}^{68}$. U is an open subset. $\mathrm{Aut}(F_0)$ is of dimension 32. In this case $H^0(X, \mathscr{E}) \neq 0$, so $\mathscr{E}$ is unstable. Let us compute $\mathrm{stab}(\mathscr{E}_M)$, namely, the stabilizers of M in $G := \mathrm{Aut}(F_0)$.

Let $P \in G$. P is of the form

$$P = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ k_1 & c_2 & c_3 & 0 \\ k_2 & c_4 & c_5 & 0 \\ q & k_3 & k_4 & c_6 \end{pmatrix}$$

where $c_1, ..., c_6 \in \mathbb{C}$, $k_1, ..., k_4$ are linear and q is of degree 2. M is of the form

$$M = \begin{pmatrix} 0 & t_1 & t_2 & q_1 \\ -t_1 & 0 & q_2 & l_1 \\ -t_2 & -q_2 & 0 & l_2 \\ -q_1 & -l_1 & -l_2 & 0 \end{pmatrix}$$

where t_1, t_2 are of degrees 3, q_1, q_2 are of degrees 2 and l_1, l_2 are linear. And there is a short exact sequence of groups

$$1 \to N := \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ k_1 & 1 & 0 & 0 \\ k_2 & 0 & 1 & 0 \\ q & k_3 & k_4 & 1 \end{pmatrix} \right\} \to$$

$$G \to H := \left\{ \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & c_3 & 0 \\ 0 & c_4 & c_5 & 0 \\ 0 & 0 & 0 & c_6 \end{pmatrix} \right\} \to 1.$$

First of all let us look at the stabilizers of M in N. As before, let

$$N \ni n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ k_1 & 1 & 0 & 0 \\ k_2 & 0 & 1 & 0 \\ q & k_3 & k_4 & 1 \end{pmatrix},$$

then

$$n^t M n = \begin{pmatrix} 0 & z_1 & z_2 & q_1 + k_1 l_1 + k_2 l_2 \\ & 0 & q_2 - k_3 l_2 + k_4 l_1 & l_1 \\ & * & 0 & l_2 \\ & * & * & 0 \end{pmatrix}$$

where

$$z_1 = t_1 - k_2 q_2 - q l_1 + q_1 k_3 + (k_1 l_1 + k_2 l_2) k_3,$$

$$z_2 = t_2 + k_1 q_2 - q l_2 + q_1 k_4 + (k_1 l_1 + k_2 l_2) k_4.$$

So

$$n^t M n = M \Longrightarrow q_1 + k_1 l_1 + k_2 l_2 = q_1 \Longrightarrow k_1 l_1 + k_2 l_2 = 0.$$

Because $\gcd(l_1, l_2) = 1$, $\deg(k_1) = \deg(k_2) = \deg(l_1) = \deg(l_2) = 1$, there is some $d_1 \in \mathbb{C}$ such that $k_1 = d_1 l_2$ and $k_2 = -d_1 l_1$.

$$n^t M n = M \Longrightarrow q_2 - k_3 l_2 + k_4 l_1 = q_2 \Longrightarrow -k_3 l_2 + k_4 l_1 = 0.$$

For the same reason, there is some $d_2 \in \mathbb{C}$ such that $k_3 = d_2 l_1$ and $k_4 = d_2 l_2$. Given what we have deduced,

$$n^t M n = M \Longleftrightarrow z_1 = t_1, z_2 = t_2 \Longleftrightarrow q = d_1 q_2 + d_2 q_1.$$

So the stabilizers of M in N are 2-dimensional and parametrized by d_1, d_2.

Next let us look at the stabilizers of M in G. Because G is the internal semidirect product of N by H, each $x \in G$ has a unique expression $x = nh$ where $n \in N$ and $h \in H$. $x^t M x = M$ becomes

$$h^t n^t M n h = M \Longleftrightarrow n^t M n = (h^{-1})^t M h^{-1}.$$

So we seek pairs (n, h) such that $n^t M n = h^t M h$. $n^t M n$ was calculated above. On the other hand, let

$$h = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & c_3 & 0 \\ 0 & c_4 & c_5 & 0 \\ 0 & 0 & 0 & c_6 \end{pmatrix},$$

then

$$\begin{aligned} h^t M h &= \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & c_4 & 0 \\ 0 & c_3 & c_5 & 0 \\ 0 & 0 & 0 & c_6 \end{pmatrix} M \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & c_3 & 0 \\ 0 & c_4 & c_5 & 0 \\ 0 & 0 & 0 & c_6 \end{pmatrix} \\ &= \begin{pmatrix} 0 & c_1t_1 & c_1t_2 & c_1q_1 \\ -c_2t_1 - c_4t_2 & -c_4q_2 & c_2q_2 & c_2l_1 + c_4l_2 \\ -c_3t_1 - c_5t_2 & -c_5q_2 & c_3q_2 & c_3l_1 + c_5l_2 \\ -c_6q_1 & -c_6l_1 & -c_6l_2 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & c_3 & 0 \\ 0 & c_4 & c_5 & 0 \\ 0 & 0 & 0 & c_6 \end{pmatrix} \\ &= \begin{pmatrix} 0 & c_1c_2t_1 + c_1c_4t_2 & c_1c_3t_1 + c_1c_5t_2 & c_1c_6q_1 \\ * & 0 & -c_4c_3q_2 + c_2c_5q_2 & c_2c_6l_1 + c_4c_6l_2 \\ * & * & 0 & c_3c_6l_1 + c_5c_6l_2 \\ * & * & * & 0 \end{pmatrix} \end{aligned}$$

So

$$n^tMn = h^tMh \Longrightarrow l_2 = c_3c_6l_1 + c_5c_6l_2 \Longrightarrow c_3c_6 = 0, c_5c_6 = 1.$$

$$n^tMn = h^tMh \Longrightarrow l_1 = c_2c_6l_1 + c_4c_6l_2 \Longrightarrow c_2c_6 = 1, c_4c_6 = 0.$$

It follows that $c_3 = c_4 = 0$.

$$n^tMn = h^tMh \Longrightarrow q_1 + k_1l_1 + k_2l_2 = c_1c_6q_1 \Longrightarrow c_1c_6 = 1.$$

$$n^tMn = h^tMh \Longrightarrow q_2 - k_3l_2 + k_4l_1 = (c_2c_5 - c_3c_4)q_2 \Longrightarrow c_2c_5 = 1.$$

$c_5c_6 = c_2c_6 = c_1c_6 = c_2c_5 = 1$ together imply that $c_6 = \pm 1$ and $h = \pm I$. So $n^tMn = h^tMh \Longrightarrow n^tMn = M$. As the stabilizers of M in N have been calculated to form a 2 dimensional vector space, we conclude that the stabilizers of M in G are also 2 dimensional.

So $dp = 68 - 32 + 2 = 38$ and on a general quartic surface, the dimension of indecomposable rank 2 ACM bundles of Case Three is $dp - 34 = 4$.

Case Four: The space of skew-symmetric matrices

$$F_1 = \mathscr{O}(-5) \oplus \mathscr{O}(-3)^3 \xrightarrow{M} F_0 = \mathscr{O} \oplus \mathscr{O}(-2)^3$$

is isomorphic to $\mathbb{C}^{72}$. U is an open subset. $\mathrm{Aut}(F_0)$ is of dimension 40. In this case $H^0(X, \mathscr{E}) \neq 0$, so $\mathscr{E}$ is unstable. Because $H^0(\mathbb{P}^3, \mathscr{O}(2a_1 - e)) = H^0(\mathbb{P}^3, \mathscr{O}(1)) = 4$ and $\mathrm{Hom}(F_0, F_1) = 0$, $\mathrm{stab}(\mathscr{E}_M)$ is four dimensional. So $dp = 72 - 40 + 4 = 36$ and on a general quartic surface, the dimension of indecomposable rank 2 ACM bundles of Case Four is $dp - 34 = 2$.

Case Five: Case Five and Case Four are paired to each other, so on a general quartic surface, the dimension of indecomposable rank 2 ACM bundles of Case Five is 2.

Chapter 5

Appendix

Proposition 5.1. *The map*

$$h : \{\textit{degree 3 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\}^6$$
$$\longrightarrow \{\textit{degree 6 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\}$$

sending (A, B, C, D, E, F) *to* $AB + CD + EF$ *is dominant.*

Proof. {degree 3 homogeneous polynomials in $\mathbb{C}[x_0, ..., x_3]\}^6$ is a vector space of dimension

120. Let

$$\{\mathbf{p^{(A)}} := (p_0^{(A)}, ..., p_3^{(A)}) : \quad p_0^{(A)}, ..., p_3^{(A)} \geq 0, \sum_{i=0}^{3} p_i^{(A)} = 3\}$$
$$\bigcup \{\mathbf{p^{(B)}} := (p_0^{(B)}, ..., p_3^{(B)}) : \quad p_0^{(B)}, ..., p_3^{(B)} \geq 0, \sum_{i=0}^{3} p_i^{(B)} = 3\}$$
$$\bigcup \{\mathbf{p^{(C)}} := (p_0^{(C)}, ..., p_3^{(C)}) : \quad p_0^{(C)}, ..., p_3^{(C)} \geq 0, \sum_{i=0}^{3} p_i^{(C)} = 3\}$$
$$\bigcup \{\mathbf{p^{(D)}} := (p_0^{(D)}, ..., p_3^{(D)}) : \quad p_0^{(D)}, ..., p_3^{(D)} \geq 0, \sum_{i=0}^{3} p_i^{(D)} = 3\}$$
$$\bigcup \{\mathbf{p^{(E)}} := (p_0^{(E)}, ..., p_3^{(E)}) : \quad p_0^{(E)}, ..., p_3^{(E)} \geq 0, \sum_{i=0}^{3} p_i^{(E)} = 3\}$$
$$\bigcup \{\mathbf{p^{(F)}} := (p_0^{(F)}, ..., p_3^{(F)}) : \quad p_0^{(F)}, ..., p_3^{(F)} \geq 0, \sum_{i=0}^{3} p_i^{(F)} = 3\}$$

be the basis. For example, $\lambda \mathbf{p^{(A)}}, \lambda \in \mathbb{C}$ denotes the monomial

$$\lambda \mathbf{x}^{\mathbf{p^{(A)}}} := \lambda x_0^{p_0^{(A)}} x_1^{p_1^{(A)}} x_2^{p_2^{(A)}} x_3^{p_3^{(A)}}.$$

{degree 6 homogeneous polynomials in $\mathbb{C}[x_0, ..., x_3]$} is a vector space of dimension 84. Let

$$\{\mathbf{q} := (q_0, ..., q_3) : q_0, ..., q_3 \geq 0, \sum_{i=0}^{3} q_i = 6\}$$

be the basis. For example, $\lambda \mathbf{q}, \lambda \in \mathbb{C}$ denotes the monomial

$$\lambda \mathbf{x}^{\mathbf{q}} := \lambda x_0^{q_0} x_1^{q_1} x_2^{q_2} x_3^{q_3}.$$

Write

$$A = \sum_{\mathbf{p^{(A)}}} a_{\mathbf{p^{(A)}}} \mathbf{x}^{\mathbf{p^{(A)}}}, B = \sum_{\mathbf{p^{(B)}}} b_{\mathbf{p^{(B)}}} \mathbf{x}^{\mathbf{p^{(B)}}}, C = \sum_{\mathbf{p^{(C)}}} c_{\mathbf{p^{(C)}}} \mathbf{x}^{\mathbf{p^{(C)}}},$$

$$D = \sum_{\mathbf{p^{(D)}}} d_{\mathbf{p^{(D)}}} \mathbf{x}^{\mathbf{p^{(D)}}}, E = \sum_{\mathbf{p^{(E)}}} e_{\mathbf{p^{(E)}}} \mathbf{x}^{\mathbf{p^{(E)}}}, F = \sum_{\mathbf{p^{(F)}}} f_{\mathbf{p^{(F)}}} \mathbf{x}^{\mathbf{p^{(F)}}}.$$

Then (A, B, C, D, E, F) in the domain has coordinates

$$(..., a_{\mathbf{p^{(A)}}}, ..., b_{\mathbf{p^{(B)}}}, ..., c_{\mathbf{p^{(C)}}}, ..., d_{\mathbf{p^{(D)}}}, ..., e_{\mathbf{p^{(E)}}}, ..., f_{\mathbf{p^{(F)}}}, ...).$$

Write G in the range as

$$G = \sum_{\mathbf{q}} g_{\mathbf{q}} \mathbf{x}^{\mathbf{q}}.$$

So G has coordinates

$$(..., g_{\mathbf{q}}, ...).$$

We will compute the Jacobian of the map h and show its rank is 84 at a certain point in the domain. Specifically, that point is

$$(A, B, C, D, E, F) = (x_1^3, x_0^3, x_3^3, x_2^3, x_1x_2x_3, x_0^2x_1 + x_2^2x_3).$$

Namely, its coordinates are

$$a_{\mathbf{p}^{(\mathbf{A})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{A})} = (0,3,0,0); \\ 0, & \text{otherwise.} \end{cases}$$

$$b_{\mathbf{p}^{(\mathbf{B})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{B})} = (3,0,0,0); \\ 0, & \text{otherwise.} \end{cases}$$

$$c_{\mathbf{p}^{(\mathbf{C})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{C})} = (0,0,0,3); \\ 0, & \text{otherwise.} \end{cases}$$

$$d_{\mathbf{p}^{(\mathbf{D})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{D})} = (0,0,3,0); \\ 0, & \text{otherwise.} \end{cases}$$

$$e_{\mathbf{p}^{(\mathbf{E})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{E})} = (0,1,1,1); \\ 0, & \text{otherwise.} \end{cases}$$

$$f_{\mathbf{p}^{(\mathbf{F})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{F})} = (2,1,0,0) \text{ or } (0,0,2,1); \\ 0, & \text{otherwise.} \end{cases}$$

First of all, the Jacobian is an 84×120 dimensional matrix. For rows, we have $\{\mathbf{q}\}$ ordered in the way that is lexicographically descending. For columns, we have each of $\{\mathbf{p}^{(\mathbf{A})}\}$,...,$\{\mathbf{p}^{(\mathbf{F})}\}$ ordered that is lexicographically descending. Moreover, the first 20 columns are with respect to $\{\mathbf{p^{(A)}}\}$,..., the last 20 columns are with respect to $\{\mathbf{p^{(F)}}\}$. Because

$$\begin{aligned} AB &= \left(\sum_{\mathbf{p^{(A)}}} a_{\mathbf{p^{(A)}}} \mathbf{x}^{\mathbf{p^{(A)}}}\right) \left(\sum_{\mathbf{p^{(B)}}} b_{\mathbf{p^{(B)}}} \mathbf{x}^{\mathbf{p^{(B)}}}\right) \\ &= \sum_{\mathbf{p^{(A)}}+\mathbf{p^{(B)}}=\mathbf{q}} a_{\mathbf{p^{(A)}}} b_{\mathbf{p^{(B)}}} \mathbf{x}^{\mathbf{q}}, \end{aligned}$$

the entry that corresponds to row $\mathbf{q}$ and column $\mathbf{p^{(A)}}$ is

$$\begin{cases} b_{\mathbf{q}-\mathbf{p^{(A)}}}, \text{ if } \mathbf{p^{(A)}} < \mathbf{q}; \\ 0, \text{ otherwise.} \end{cases}$$

Here, both $\mathbf{p^{(A)}}$ and $\mathbf{q}$ are 4-tuples, and $<$ and $-$ take the following meaning: $\mathbf{p^{(A)}} < \mathbf{q}$ if for each $i \in \{0, 1, 2, 3\}$, $p_i^{(A)} \leq q_i$; if $\mathbf{p^{(A)}} < \mathbf{q}$, then $\mathbf{q} - \mathbf{p^{(A)}} = (q_0 - p_0^{(A)}, ..., q_3 - p_3^{(A)})$. Similarly, the entry that corresponds to row $\mathbf{q}$ and column $\mathbf{p^{(B)}}$ is

$$\begin{cases} a_{\mathbf{q}-\mathbf{p^{(B)}}}, \text{ if } \mathbf{p^{(B)}} < \mathbf{q}; \\ 0, \text{ otherwise.} \end{cases}$$

The entry that corresponds to row $\mathbf{q}$ and column $\mathbf{p^{(C)}}$ is

$$\begin{cases} d_{\mathbf{q}-\mathbf{p^{(C)}}}, \text{ if } \mathbf{p^{(C)}} < \mathbf{q}; \\ 0, \text{ otherwise.} \end{cases}$$

The entry that corresponds to row $\mathbf{q}$ and column $\mathbf{p}^{(\mathbf{D})}$ is

$$\begin{cases} c_{\mathbf{q}-\mathbf{p}^{(\mathbf{D})}}, \text{ if } \mathbf{p}^{(\mathbf{D})} < \mathbf{q}; \\ 0, \text{ otherwise.} \end{cases}$$

The entry that corresponds to row $\mathbf{q}$ and column $\mathbf{p}^{(\mathbf{E})}$ is

$$\begin{cases} f_{\mathbf{q}-\mathbf{p}^{(\mathbf{E})}}, \text{ if } \mathbf{p}^{(\mathbf{E})} < \mathbf{q}; \\ 0, \text{ otherwise.} \end{cases}$$

The entry that corresponds to row $\mathbf{q}$ and column $\mathbf{p}^{(\mathbf{F})}$ is

$$\begin{cases} e_{\mathbf{q}-\mathbf{p}^{(\mathbf{F})}}, \text{ if } \mathbf{p}^{(\mathbf{F})} < \mathbf{q}; \\ 0, \text{ otherwise.} \end{cases}$$

We will show the Jacobian has rank 84 at that specific point. Namely, we will pick an 84×84 submatrix that is nonsingular. That means we will pick every row and 84 out of 120 columns of the Jacobian to form the submatrix.

For the first 20 columns that correspond to $\{\mathbf{p}^{(\mathbf{A})}\}$, we pick all of them. We claim that at that specific point, this 84×20 submatrix is

$$\begin{pmatrix} I_{20\times 20} \\ 0_{64\times 20} \end{pmatrix}.$$

The reason is that first of all in this 84×20 submatrix, an entry is 0 unless if $\mathbf{p}^{(\mathbf{A})} < \mathbf{q}$

at that entry, then the entry is $b_{\mathbf{q}-\mathbf{p}^{(\mathbf{A})}}$; next at that specific point $b_{\mathbf{p}^{(\mathbf{B})}} = 0$ unless $\mathbf{p}^{(\mathbf{B})} = (3,0,0,0)$, and $b_{(3,0,0,0)} = 1$, so only those entries whose $\mathbf{q}-\mathbf{p}^{(\mathbf{A})} = (3,0,0,0)$ are nonzero, and those entries are 1; finally we have ordered both rows and columns in the lexicographically descending order that it is precisely on the diagonal of the upper 20×20 submatrix that $\mathbf{q} - \mathbf{p}^{(\mathbf{A})} = (3,0,0,0)$. This proves the claim. The 20×20 submatrix corresponds to rows

$$\{\mathbf{q} : q_0 \geq 3\}.$$

Now come to the next 20 columns corresponding to $\{\mathbf{p}^{(\mathbf{B})}\}$ and focus on this 84×20 submatrix. Similar to the previous paragraph, at that specific point this submatrix has a 20×20 submatrix that is $I_{20\times 20}$ up to permutation and each entry outside the 20×20 submatrix is zero. This 20×20 submatrix corresponds to rows

$$\{\mathbf{q} : q_1 \geq 3\}.$$

$\{\mathbf{q} : q_0 \geq 3\}$ and $\{\mathbf{q} : q_1 \geq 3\}$ have a common row $\mathbf{q} = (3,3,0,0)$, that means in the first 40 columns there is a 39×39 submatrix that is nonsingular. Precisely, we leave out the column corresponding to $\mathbf{p}^{(\mathbf{B})} = (3,0,0,0)$; meanwhile, the 39 rows come from

$$\{\mathbf{q} : q_0 \geq 3 \text{ or } q_1 \geq 3\}.$$

Now go to the next 20 columns corresponding to $\{\mathbf{p}^{(\mathbf{C})}\}$. In a similar vein, in this 84×20 submatrix there is a 20×20 submatrix that is $I_{20\times 20}$ up to permutation and each entry outside the 20×20 submatrix is zero. This 20×20 submatrix corresponds to rows

$$\{\mathbf{q} : q_2 \geq 3\}.$$

As

$$\{\mathbf{q} : q_2 \geq 3\} \cap \{\mathbf{q} : q_0 \geq 3 \text{ or } q_1 \geq 3\} = \{(3,0,3,0),(0,3,3,0)\},$$

it means among the 20 columns corresponding to $\{\mathbf{p^{(C)}}\}$, we can pick 18 columns and with the previous 39×39 submatrix we get a 57×57 submatrix that is nonsingular. Precisely, among the first 60 columns we leave out the ones corresponding to $\mathbf{p^{(B)}} = (3,0,0,0)$, $\mathbf{p^{(C)}} = (3,0,0,0)$, $\mathbf{p^{(C)}} = (0,3,0,0)$; meanwhile, the 57 rows come from

$$\{\mathbf{q} : q_0 \geq 3 \text{ or } q_1 \geq 3 \text{ or } q_2 \geq 3\}.$$

Now we are at the next 20 columns corresponding to $\{\mathbf{p^{(D)}}\}$. Similarly, in this 84×20 submatrix there is a 20×20 submatrix that is $I_{20\times 20}$ up to permutation and each entry outside the 20×20 submatrix is zero. This 20×20 submatrix corresponds to rows

$$\{\mathbf{q} : q_3 \geq 3\}.$$

As

$$\begin{aligned}&\{\mathbf{q} : q_3 \geq 3\} \cap \{\mathbf{q} : q_0 \geq 3 \text{ or } q_1 \geq 3 \text{ or } q_2 \geq 3\}\\ =&\{(3,0,0,3),(0,3,0,3),(0,0,3,3)\},\end{aligned}$$

it means among the 20 columns corresponding to $\{\mathbf{p^{(D)}}\}$, we can pick 17 columns and with the previous 57×57 submatrix we get a 74×74 submatrix that is nonsingular. Precisely, among the first 80 columns we leave out the ones corresponding to $\mathbf{p}^{(\mathbf{B})} = (3,0,0,0)$, $\mathbf{p}^{(\mathbf{C})} = (3,0,0,0)$, $\mathbf{p}^{(\mathbf{C})} = (0,3,0,0)$, $\mathbf{p}^{(\mathbf{D})} = (3,0,0,0)$, $\mathbf{p}^{(\mathbf{D})} = (0,3,0,0)$, $\mathbf{p}^{(\mathbf{D})} = (0,0,3,0)$; meanwhile, the 74 rows come from

$$\{\mathbf{q} : q_0 \geq 3 \text{ or } q_1 \geq 3 \text{ or } q_2 \geq 3 \text{ or } q_3 \geq 3\}.$$

What remains to be done is to show from the following 10 rows

$$\{\mathbf{q} : q_i \leq 2 \text{ for all } i \in \{0,1,2,3\}\}$$

and from the last 40 columns of the Jacobian there is a 10×10 submatrix that is nonsingular at that specific point. The first 20 columns of the last 40 columns correspond to $\{\mathbf{p}^{(\mathbf{E})}\}$. We pick the following 9 columns:

$$\begin{aligned}\mathbf{p}^{(\mathbf{E})} =& (2,0,0,1),(1,2,0,0),(1,1,0,1),(0,2,0,1),\\ & (0,1,2,0),(0,1,1,1),(0,1,0,2),(0,0,2,1),(0,0,1,2).\end{aligned}$$

Meanwhile, we pick the following 9 rows:

$$\begin{aligned}\mathbf{q} =& (2,2,2,0),(2,2,1,1),(2,2,0,2),(2,1,2,1),(2,1,1,2),\\ & (2,0,2,2),(1,2,2,1),(1,1,2,2),(0,2,2,2).\end{aligned}$$

Together they form a 9×9 submatrix. We claim that this matrix is

$$\begin{pmatrix} 0_{5\times 4} & I_{5\times 5} \\ I_{4\times 4} & 0_{4\times 5} \end{pmatrix}.$$

The reason is that first of all the columns of this matrix correspond to $\mathbf{p}^{(\mathbf{E})}$, so an entry is 0 unless $\mathbf{p}^{(\mathbf{E})} < \mathbf{q}$ at that entry, then the entry is $f_{\mathbf{q}-\mathbf{p}^{(\mathbf{E})}}$; next at that specific point $f_{\mathbf{p}^{(\mathbf{F})}} = 0$ unless $\mathbf{p}^{(\mathbf{F})} = (2,1,0,0)$ or $\mathbf{p}^{(\mathbf{F})} = (0,0,2,1)$, and $f_{(2,1,0,0)} = f_{(0,0,2,1)} = 1$, so only those entries whose $\mathbf{q}-\mathbf{p}^{(\mathbf{E})} = (2,1,0,0)$ or $(0,0,2,1)$ are nonzero, and those entries are 1; finally we have ordered both rows and columns in the lexicographically descending order that it is precisely on the diagonals of the 5×5 and 4×4 submatrices that $\mathbf{q} - \mathbf{p}^{(\mathbf{E})} = (2,1,0,0)$ or $(0,0,2,1)$. This proves the claim.

Up to now we have an 83×83 submatrix that is nonsingular. There is just one row left which is $\mathbf{q} = (1,2,1,2)$. To complete we go to the last 20 columns corresponding to $\{\mathbf{p}^{(\mathbf{F})}\}$. Because each entry located at the intersection of the row $\mathbf{q} = (1,2,1,2)$ and the 9 columns in the previous paragraph is zero, to get an 84×84 submatrix that is nonsingular, it suffices to pick one column in the last 20 columns such that at its intersection with the row $\mathbf{q} = (1,2,1,2)$ the entry is nonzero. We pick the column $\mathbf{p}^{(\mathbf{F})} = (1,1,0,1)$. The entry at the intersection of the row $\mathbf{q} = (1,2,1,2)$ and the column $\mathbf{p}^{(\mathbf{F})} = (1,1,0,1)$ is $e_{\mathbf{q}-\mathbf{p}^{(\mathbf{F})}} = e_{(0,1,1,1)} = 1$.

So indeed the Jacobian has rank 84 at that specific point and h is dominant. $\square$

Proposition 5.2. *The map*

$$\begin{aligned} h :&\{\textit{degree 3 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\}^4 \\ &\oplus \{\textit{degree 2 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\} \\ &\oplus \{\textit{degree 4 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\} \\ &\longrightarrow \{\textit{degree 6 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\} \end{aligned}$$

sending (A, B, C, D, E, F) *to* $AB + CD + EF$ *is dominant.*

Proof. We follow what we did in 5.1. A difference is here the domain of h is a vector space of dimension 125. Let

$$\begin{aligned} &\{\mathbf{p^{(A)}} := (p_0^{(A)}, ..., p_3^{(A)}) : \quad p_0^{(A)}, ..., p_3^{(A)} \geq 0, \sum_{i=0}^{3} p_i^{(A)} = 3\} \\ \bigcup&\{\mathbf{p^{(B)}} := (p_0^{(B)}, ..., p_3^{(B)}) : \quad p_0^{(B)}, ..., p_3^{(B)} \geq 0, \sum_{i=0}^{3} p_i^{(B)} = 3\} \\ \bigcup&\{\mathbf{p^{(C)}} := (p_0^{(C)}, ..., p_3^{(C)}) : \quad p_0^{(C)}, ..., p_3^{(C)} \geq 0, \sum_{i=0}^{3} p_i^{(C)} = 3\} \\ \bigcup&\{\mathbf{p^{(D)}} := (p_0^{(D)}, ..., p_3^{(D)}) : \quad p_0^{(D)}, ..., p_3^{(D)} \geq 0, \sum_{i=0}^{3} p_i^{(D)} = 3\} \\ \bigcup&\{\mathbf{p^{(E)}} := (p_0^{(E)}, ..., p_3^{(E)}) : \quad p_0^{(E)}, ..., p_3^{(E)} \geq 0, \sum_{i=0}^{3} p_i^{(E)} = 2\} \\ \bigcup&\{\mathbf{p^{(F)}} := (p_0^{(F)}, ..., p_3^{(F)}) : \quad p_0^{(F)}, ..., p_3^{(F)} \geq 0, \sum_{i=0}^{3} p_i^{(F)} = 4\} \end{aligned}$$

be the basis. Let

$$\{\mathbf{q} := (q_0, ..., q_3) : q_0, ..., q_3 \geq 0, \sum_{i=0}^{3} q_i = 6\}$$

be the basis for the range. Write

$$A = \sum_{\mathbf{p}^{(\mathbf{A})}} a_{\mathbf{p}^{(\mathbf{A})}} \mathbf{x}^{\mathbf{p}^{(\mathbf{A})}}, B = \sum_{\mathbf{p}^{(\mathbf{B})}} b_{\mathbf{p}^{(\mathbf{B})}} \mathbf{x}^{\mathbf{p}^{(\mathbf{B})}}, C = \sum_{\mathbf{p}^{(\mathbf{C})}} c_{\mathbf{p}^{(\mathbf{C})}} \mathbf{x}^{\mathbf{p}^{(\mathbf{C})}},$$

$$D = \sum_{\mathbf{p}^{(\mathbf{D})}} d_{\mathbf{p}^{(\mathbf{D})}} \mathbf{x}^{\mathbf{p}^{(\mathbf{D})}}, E = \sum_{\mathbf{p}^{(\mathbf{E})}} e_{\mathbf{p}^{(\mathbf{E})}} \mathbf{x}^{\mathbf{p}^{(\mathbf{E})}}, F = \sum_{\mathbf{p}^{(\mathbf{F})}} f_{\mathbf{p}^{(\mathbf{F})}} \mathbf{x}^{\mathbf{p}^{(\mathbf{F})}}.$$

We will compute the Jacobian of the map h and show its rank is 84 at a certain point in the domain. Specifically, that point is

$$(A, B, C, D, E, F) = (x_1^3, x_0^3, x_3^3, x_2^3, x_2x_3, x_0^2x_1^2 + x_0x_1x_2x_3).$$

The point has coordinates

$$a_{\mathbf{p}^{(\mathbf{A})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{A})} = (0,3,0,0); \\ 0, & \text{otherwise.} \end{cases}$$

$$b_{\mathbf{p}^{(\mathbf{B})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{B})} = (3,0,0,0); \\ 0, & \text{otherwise.} \end{cases}$$

$$c_{\mathbf{p}^{(\mathbf{C})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{C})} = (0,0,0,3); \\ 0, & \text{otherwise.} \end{cases}$$

$$d_{\mathbf{p}^{(\mathbf{D})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{D})} = (0,0,3,0); \\ 0, & \text{otherwise.} \end{cases}$$

$$e_{\mathbf{p}^{(\mathbf{E})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{E})} = (0,0,1,1); \\ 0, & \text{otherwise.} \end{cases}$$

$$f_{\mathbf{p}^{(\mathbf{F})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{F})} = (2,2,0,0) \text{ or } (1,1,1,1); \\ 0, & \text{otherwise.} \end{cases}$$

The Jacobian is an 84×125 dimensional matrix. We will show its rank is 84 at that specific point. Namely, we will pick an 84×84 submatrix that is nonsingular. That means we will pick every row and 84 out of 125 columns of the Jacobian to form the submatrix.

The first 80 columns are exactly the same as in 5.1. As a result, what remains to be done is to show from the following 10 rows

$$\{\mathbf{q} : q_i \leq 2 \text{ for all } i \in \{0, 1, 2, 3\}\}$$

and from the last 45 columns of the Jacobian there is a 10×10 submatrix that is nonsingular at that specific point. The first 10 columns of the last 45 columns correspond to $\{\mathbf{p}^{(\mathbf{E})}\}$. We pick the following 8 columns:

$$\begin{aligned}\mathbf{p}^{(\mathbf{E})} =&(1,1,0,0),(1,0,1,0),(1,0,0,1),(0,1,1,0),\\&(0,1,0,1),(0,0,2,0),(0,0,1,1),(0,0,0,2);\end{aligned}$$

with

$$\{\mathbf{q} : q_i \leq 2 \text{ for all } i \in \{0, 1, 2, 3\}\}$$

they form a 10×8 matrix. We claim that this matrix is

$$
\begin{array}{cc}
 & \mathbf{p}^{(\mathbf{E})} \quad \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 \end{matrix} \\
\mathbf{q} & \\
\begin{matrix} 2 & 2 & 2 & 0 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 0 & 2 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 0 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 0 & 2 & 2 & 2 \end{matrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{array}
$$

The reason is that first of all an entry is 0 unless $\mathbf{p}^{(\mathbf{E})} < \mathbf{q}$ at that entry, then the entry is $f_{\mathbf{q}-\mathbf{p}^{(\mathbf{E})}}$; next at that specific point $f_{\mathbf{p}^{(\mathbf{F})}} = 0$ unless $\mathbf{p}^{(\mathbf{F})} = (2,2,0,0)$ or $\mathbf{p}^{(\mathbf{F})} = (1,1,1,1)$, and $f_{(2,2,0,0)} = f_{(1,1,1,1)} = 1$, so only those entries whose $\mathbf{q} - \mathbf{p}^{(\mathbf{E})} = (2,2,0,0)$ or $(1,1,1,1)$ are nonzero, and those entries are 1; finally we have ordered both rows and columns in the lexicographically descending order that it is precisely at the entries where we put 1 in the above matrix that $\mathbf{q}-\mathbf{p}^{(\mathbf{E})} = (2,1,0,0)$ or $(0,0,2,1)$. This proves the claim. We get an 8×8 matrix that is nonsingular by leaving out the two rows corresponding to $\mathbf{q} = (2,0,2,2)$ and $\mathbf{q} = (0,2,2,2)$ from the above 10×8 matrix. With the 74×74 submatrix from the first 80 columns of the Jacobian we get an 82×82 submatrix that is nonsingular.

Up to now there are just two rows left which are $\mathbf{q} = (2,0,2,2)$ and $\mathbf{q} = (0,2,2,2)$. To complete we go to the last 35 columns corresponding to $\mathbf{p}^{(\mathbf{F})}$. To get an 84×84 submatrix that is nonsingular, it suffices to pick two columns in the last 35 columns such that their intersections with the two rows is a 2×2 nonsingular matrix. We pick the columns $\mathbf{p}^{(\mathbf{F})} = (2,0,1,1)$ and $\mathbf{p}^{(\mathbf{F})} = (0,2,1,1)$. We get

$$
\begin{array}{cc}
 & \mathbf{p}^{(\mathbf{F})} \quad \begin{array}{cc} 2 & 0 \\ 0 & 2 \\ 1 & 1 \\ 1 & 1 \end{array} \\
\mathbf{q} & \\
\begin{array}{cccc} 2 & 0 & 2 & 2 \\ 0 & 2 & 2 & 2 \end{array} & \begin{pmatrix} e_{(0,0,1,1)} = 1 & 0 \\ 0 & e_{(0,0,1,1)} = 1 \end{pmatrix}
\end{array}
$$

which is nonsingular. So indeed the Jacobian has rank 84 at that specific point and h is dominant. $\square$

Proposition 5.3. *The map*

$$\begin{aligned} h :& \{\textit{degree 3 homogeneous polynomials in } \mathbb{C}[x_0,...,x_3]\}^2 \\ &\oplus \{\textit{degree 2 homogeneous polynomials in } \mathbb{C}[x_0,...,x_3]\} \\ &\oplus \{\textit{degree 4 homogeneous polynomials in } \mathbb{C}[x_0,...,x_3]\} \\ &\oplus \{\textit{degree 2 homogeneous polynomials in } \mathbb{C}[x_0,...,x_3]\} \\ &\oplus \{\textit{degree 4 homogeneous polynomials in } \mathbb{C}[x_0,...,x_3]\} \\ &\longrightarrow \{\textit{degree 6 homogeneous polynomials in } \mathbb{C}[x_0,...,x_3]\} \end{aligned}$$

sending (A, B, C, D, E, F) to $AB + CD + EF$ is dominant.

Proof. We follow what we did in 5.1. A difference is here the domain of h is a vector space of dimension 130. Let

$$\begin{aligned} &\{\mathbf{p^{(A)}} := (p_0^{(A)}, ..., p_3^{(A)}) : \quad p_0^{(A)}, ..., p_3^{(A)} \geq 0, \sum_{i=0}^{3} p_i^{(A)} = 3\} \\ \bigcup &\{\mathbf{p^{(B)}} := (p_0^{(B)}, ..., p_3^{(B)}) : \quad p_0^{(B)}, ..., p_3^{(B)} \geq 0, \sum_{i=0}^{3} p_i^{(B)} = 3\} \\ \bigcup &\{\mathbf{p^{(C)}} := (p_0^{(C)}, ..., p_3^{(C)}) : \quad p_0^{(C)}, ..., p_3^{(C)} \geq 0, \sum_{i=0}^{3} p_i^{(C)} = 2\} \\ \bigcup &\{\mathbf{p^{(D)}} := (p_0^{(D)}, ..., p_3^{(D)}) : \quad p_0^{(D)}, ..., p_3^{(D)} \geq 0, \sum_{i=0}^{3} p_i^{(D)} = 4\} \\ \bigcup &\{\mathbf{p^{(E)}} := (p_0^{(E)}, ..., p_3^{(E)}) : \quad p_0^{(E)}, ..., p_3^{(E)} \geq 0, \sum_{i=0}^{3} p_i^{(E)} = 2\} \\ \bigcup &\{\mathbf{p^{(F)}} := (p_0^{(F)}, ..., p_3^{(F)}) : \quad p_0^{(F)}, ..., p_3^{(F)} \geq 0, \sum_{i=0}^{3} p_i^{(F)} = 4\} \end{aligned}$$

be the basis. Let

$$\{\mathbf{q} := (q_0, ..., q_3) : q_0, ..., q_3 \geq 0, \sum_{i=0}^{3} q_i = 6\}$$

be the basis for the range. Write

$$A = \sum_{\mathbf{p^{(A)}}} a_{\mathbf{p^{(A)}}} \mathbf{x}^{\mathbf{p^{(A)}}}, B = \sum_{\mathbf{p^{(B)}}} b_{\mathbf{p^{(B)}}} \mathbf{x}^{\mathbf{p^{(B)}}}, C = \sum_{\mathbf{p^{(C)}}} c_{\mathbf{p^{(C)}}} \mathbf{x}^{\mathbf{p^{(C)}}},$$

$$D = \sum_{\mathbf{p^{(D)}}} d_{\mathbf{p^{(D)}}} \mathbf{x}^{\mathbf{p^{(D)}}}, E = \sum_{\mathbf{p^{(E)}}} e_{\mathbf{p^{(E)}}} \mathbf{x}^{\mathbf{p^{(E)}}}, F = \sum_{\mathbf{p^{(F)}}} f_{\mathbf{p^{(F)}}} \mathbf{x}^{\mathbf{p^{(F)}}}.$$

We will compute the Jacobian of the map h and show its rank is 84 at a certain point in the domain. Specifically, that point is

$$(A, B, C, D, E, F) = (x_1^3, x_0^3, x_2^2, x_0x_1x_2x_3, x_3^2, 0).$$

The point has coordinates

$$a_{\mathbf{p}^{(\mathbf{A})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{A})} = (0,3,0,0); \\ 0, & \text{otherwise.} \end{cases}$$

$$b_{\mathbf{p}^{(\mathbf{B})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{B})} = (3,0,0,0); \\ 0, & \text{otherwise.} \end{cases}$$

$$c_{\mathbf{p}^{(\mathbf{C})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{C})} = (0,0,2,0); \\ 0, & \text{otherwise.} \end{cases}$$

$$d_{\mathbf{p}^{(\mathbf{D})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{D})} = (1,1,1,1); \\ 0, & \text{otherwise.} \end{cases}$$

$$e_{\mathbf{p}^{(\mathbf{E})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{E})} = (0,0,0,2); \\ 0, & \text{otherwise.} \end{cases}$$

$$f_{\mathbf{p}^{(\mathbf{F})}} = 0, \text{ for any } \mathbf{p}^{(\mathbf{F})}.$$

The Jacobian is an 84×130 dimensional matrix. We will show its rank is 84 at that specific point. Namely, we will pick an 84×84 submatrix that is nonsingular. That means we will pick every row and 84 out of 130 columns of the Jacobian to form the submatrix.

First of all go to columns 51-85 corresponding to $\mathbf{p}^{(\mathbf{D})}$ and focus on this 84×35 submatrix. In this submatrix, an entry is zero unless $\mathbf{p}^{(\mathbf{D})} < \mathbf{q}$ at that entry, then the entry is $c_{\mathbf{q}-\mathbf{p}^{(\mathbf{D})}}$. Next at that specific point $c_{\mathbf{p}^{(\mathbf{C})}} = 0$ unless $\mathbf{p}^{(\mathbf{C})} = (0,0,2,0)$, and $c_{(0,0,2,0)} = 1$, so only those entries whose $\mathbf{q} - \mathbf{p}^{(\mathbf{D})} = (0,0,2,0)$ are nonzero, and those entries are 1. Both $\mathbf{p}^{(\mathbf{D})}$ and $\mathbf{q}$

are 4-tuples. Define a lexicographical order on 4-tuples by

$$(i_1, i_2, i_3, i_4) > (j_1, j_2, j_3, j_4) \text{ if and only if}$$

$$i_3 > j_3 \text{ or } i_3 = j_3 \text{ and } (i_1, i_2, i_4) > (j_1, j_2, j_4).$$

where $(i_1, i_2, i_4) > (j_1, j_2, j_4)$ denotes the usual lexicographical order on 3-tuples. Let us order both rows and columns of this 84×35 submatrix according to the lexicographic order just defined, then this matrix is

$$\begin{pmatrix} I_{35\times 35} \\ \\ 0_{49\times 35} \end{pmatrix}.$$

The reason is that it is precisely on the diagonal of the upper 35×35 submatrix that $\mathbf{q} - \mathbf{p^{(D)}} = (0, 0, 2, 0)$. This 35×35 submatrix corresponds to rows

$$\{\mathbf{q} : q_2 \geq 2\}.$$

Next go to columns 96-130 corresponding to $\mathbf{p^{(F)}}$. In a similar vein, in this 84×35 submatrix there is a 35×35 submatrix that is $I_{35\times 35}$ up to permutation and each entry outside the 35×35 submatrix is zero. This 35×35 submatrix corresponds to rows

$$\{\mathbf{q} : q_3 \geq 2\}.$$

The first 40 columns are exactly the same as in 5.1. As a result, there is a 39×39

submatrix that is nonsingular. This submatrix corresponds to rows

$$\{\mathbf{q} : q_0 \geq 3 \text{ or } q_1 \geq 3\}.$$

As

$$\begin{aligned}&\{\mathbf{q} : q_2 \geq 2\} \cup \{\mathbf{q} : q_3 \geq 2\} \cup \{\mathbf{q} : q_0 \geq 3 \text{ or } q_1 \geq 3\}\\ =&\{\mathbf{q} : q_0 \geq 3 \text{ or } q_1 \geq 3 \text{ or } q_2 \geq 2 \text{ or } q_3 \geq 2\}\end{aligned}$$

and

$$\{\mathbf{q}\} - \{\mathbf{q} : q_0 \geq 3 \text{ or } q_1 \geq 3 \text{ or } q_2 \geq 2 \text{ or } q_3 \geq 2\} = \{(2,2,1,1)\},$$

we conclude that from columns 1-40, 51-85 and 96-130 there is an 83×83 submatrix that is nonsingular which corresponds to rows

$$\{\mathbf{q} : q_0 \geq 3 \text{ or } q_1 \geq 3 \text{ or } q_2 \geq 2 \text{ or } q_3 \geq 2\}.$$

Moreover, there is just one row left which is $\mathbf{q} = (2,2,1,1)$. To complete we go to columns 41-50 corresponding to $\mathbf{p}^{(\mathbf{C})}$ and it suffices to pick one column there such that at its intersection with the row $\mathbf{q} = (2,2,1,1)$ the entry is nonzero. We pick the column $\mathbf{p}^{(\mathbf{C})} = (1,1,0,0)$. The entry at its intersection with the row $\mathbf{q} = (2,2,1,1)$ is $d_{\mathbf{q}-\mathbf{p}^{(\mathbf{C})}} = d_{(1,1,1,1)} = 1$, which is nonzero. So indeed the Jacobian has rank 84 at that specific point and h is dominant. $\square$

Proposition 5.4. *The map*

$$
\begin{aligned}
h :& \{\textit{degree 3 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\}^4 \\
& \oplus \{\textit{degree 1 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\} \\
& \oplus \{\textit{degree 5 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\} \\
& \longrightarrow \{\textit{degree 6 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\}
\end{aligned}
$$

sending (A, B, C, D, E, F) *to* $AB + CD + EF$ *is dominant.*

Proof. We follow what we did in 5.1. A difference is here the domain of h is a vector space of dimension 140. Let

$$
\begin{aligned}
&\{\mathbf{p}^{(\mathbf{A})} := (p_0^{(A)}, ..., p_3^{(A)}) : \quad p_0^{(A)}, ..., p_3^{(A)} \geq 0, \sum_{i=0}^{3} p_i^{(A)} = 3\} \\
\bigcup &\{\mathbf{p}^{(\mathbf{B})} := (p_0^{(B)}, ..., p_3^{(B)}) : \quad p_0^{(B)}, ..., p_3^{(B)} \geq 0, \sum_{i=0}^{3} p_i^{(B)} = 3\} \\
\bigcup &\{\mathbf{p}^{(\mathbf{C})} := (p_0^{(C)}, ..., p_3^{(C)}) : \quad p_0^{(C)}, ..., p_3^{(C)} \geq 0, \sum_{i=0}^{3} p_i^{(C)} = 3\} \\
\bigcup &\{\mathbf{p}^{(\mathbf{D})} := (p_0^{(D)}, ..., p_3^{(D)}) : \quad p_0^{(D)}, ..., p_3^{(D)} \geq 0, \sum_{i=0}^{3} p_i^{(D)} = 3\} \\
\bigcup &\{\mathbf{p}^{(\mathbf{E})} := (p_0^{(E)}, ..., p_3^{(E)}) : \quad p_0^{(E)}, ..., p_3^{(E)} \geq 0, \sum_{i=0}^{3} p_i^{(E)} = 1\} \\
\bigcup &\{\mathbf{p}^{(\mathbf{F})} := (p_0^{(F)}, ..., p_3^{(F)}) : \quad p_0^{(F)}, ..., p_3^{(F)} \geq 0, \sum_{i=0}^{3} p_i^{(F)} = 5\}
\end{aligned}
$$

be the basis. Let

$$\{\mathbf{q} := (q_0, ..., q_3) : q_0, ..., q_3 \geq 0, \sum_{i=0}^{3} q_i = 6\}$$

be the basis for the range. Write

$$A = \sum_{\mathbf{p^{(A)}}} a_{\mathbf{p^{(A)}}} \mathbf{x}^{\mathbf{p^{(A)}}}, B = \sum_{\mathbf{p^{(B)}}} b_{\mathbf{p^{(B)}}} \mathbf{x}^{\mathbf{p^{(B)}}}, C = \sum_{\mathbf{p^{(C)}}} c_{\mathbf{p^{(C)}}} \mathbf{x}^{\mathbf{p^{(C)}}},$$

$$D = \sum_{\mathbf{p^{(D)}}} d_{\mathbf{p^{(D)}}} \mathbf{x}^{\mathbf{p^{(D)}}}, E = \sum_{\mathbf{p^{(E)}}} e_{\mathbf{p^{(E)}}} \mathbf{x}^{\mathbf{p^{(E)}}}, F = \sum_{\mathbf{p^{(F)}}} f_{\mathbf{p^{(F)}}} \mathbf{x}^{\mathbf{p^{(F)}}}.$$

We will compute the Jacobian of the map h and show its rank is 84 at a certain point in the domain. Specifically, that point is

$$(A, B, C, D, E, F) = (x_1^3, x_0^3, x_3^3, x_2^3, x_0, x_1 x_2^2 x_3^2).$$

The point has coordinates

$$a_{\mathbf{p}^{(\mathbf{A})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{A})} = (0,3,0,0); \\ \\ 0, & \text{otherwise.} \end{cases}$$

$$b_{\mathbf{p}^{(\mathbf{B})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{B})} = (3,0,0,0); \\ \\ 0, & \text{otherwise.} \end{cases}$$

$$c_{\mathbf{p}^{(\mathbf{C})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{C})} = (0,0,0,3); \\ \\ 0, & \text{otherwise.} \end{cases}$$

$$d_{\mathbf{p}^{(\mathbf{D})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{D})} = (0,0,3,0); \\ \\ 0, & \text{otherwise.} \end{cases}$$

$$e_{\mathbf{p}^{(\mathbf{E})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{E})} = (1,0,0,0); \\ \\ 0, & \text{otherwise.} \end{cases}$$

$$f_{\mathbf{p}^{(\mathbf{F})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{F})} = (0,1,2,2); \\ \\ 0, & \text{otherwise.} \end{cases}$$

The Jacobian is an 84×140 dimensional matrix. We will show its rank is 84 at that specific point. Namely, we will pick an 84×84 submatrix that is nonsingular. That means we will pick every row and 84 out of 140 columns of the Jacobian to form the submatrix.

The first 80 columns are exactly the same as in 5.1. As a result, what remains to be done is to show from the following 10 rows

$$\{\mathbf{q} : q_i \leq 2 \text{ for all } i \in \{0, 1, 2, 3\}\}$$

and from the last 60 columns of the Jacobian there is a 10×10 submatrix that is nonsingular at that specific point.

The last 56 columns of the Jacobian correspond to $\{\mathbf{p}^{(\mathbf{F})}\}$. We pick the following 9 columns:

$$\begin{aligned}\mathbf{p}^{(\mathbf{F})} =&(1,2,2,0),(1,2,1,1),(1,2,0,2),\\ &(1,1,2,1),(1,1,1,2),(1,0,2,2),\\ &(0,2,2,1),(0,2,1,2),(0,1,2,2);\end{aligned}$$

with

$$\{\mathbf{q} : q_i \leq 2 \text{ for all } i \in \{0, 1, 2, 3\}\}$$

they form a 10×9 matrix. We claim that this matrix is

$$\begin{pmatrix} I_{9\times 9} \\ 0_{1\times 9} \end{pmatrix}.$$

The reason is that first of all an entry is 0 unless $\mathbf{p}^{(\mathbf{F})} < \mathbf{q}$ at that entry, then the entry is $e_{\mathbf{q}-\mathbf{p}^{(\mathbf{F})}}$; next at that specific point $e_{\mathbf{p}^{(\mathbf{E})}} = 0$ unless $\mathbf{p}^{(\mathbf{E})} = (1, 0, 0, 0)$, and $e_{(1,0,0,0)} = 1$, so only those entries whose $\mathbf{q} - \mathbf{p}^{(\mathbf{F})} = (1, 0, 0, 0)$ are nonzero, and those entries are 1; finally we have ordered both rows and columns in the lexicographically descending order that it is precisely on the diagonal of the upper 9×9 submatrix that $\mathbf{q} - \mathbf{p}^{(\mathbf{F})} = (1, 0, 0, 0)$. This proves the claim. We get a 9×9 matrix that is nonsingular by leaving out the last row which is the one corresponding to $\mathbf{q} = (0, 2, 2, 2)$ from the above 10×9 matrix. With the 74×74 submatrix from the first 80 columns of the Jacobian we get an 83×83 submatrix that is nonsingular.

To complete we go to columns 81-84 corresponding to $\mathbf{p}^{(\mathbf{E})}$. It suffices to pick one column there such that at its intersection with the row $\mathbf{q} = (0, 2, 2, 2)$ the entry is nonzero. We pick the column $\mathbf{p}^{(\mathbf{E})} = (0, 1, 0, 0)$. The entry at its intersection with the row $\mathbf{q} = (0, 2, 2, 2)$ is $f_{\mathbf{q}-\mathbf{p}^{(\mathbf{E})}} = d_{(0,1,2,2)} = 1$, which is nonzero. So indeed the Jacobian has rank 84 at that specific point and h is dominant. $\square$

Proposition 5.5. *The map*

$$\begin{aligned} h :& \{\textit{degree 3 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\}^2 \\ & \oplus \{\textit{degree 1 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\} \\ & \oplus \{\textit{degree 5 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\} \\ & \oplus \{\textit{degree 2 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\} \\ & \oplus \{\textit{degree 4 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\} \\ & \longrightarrow \{\textit{degree 6 homogeneous polynomials in } \mathbb{C}[x_0, ..., x_3]\} \end{aligned}$$

sending (A, B, C, D, E, F) to $AB + CD + EF$ is dominant.

Proof. We follow what we did in 5.1. A difference is here the domain of h is a vector space of dimension 145. Let

$$\begin{aligned}
&\{\mathbf{p^{(A)}} := (p_0^{(A)}, ..., p_3^{(A)}) : \quad p_0^{(A)}, ..., p_3^{(A)} \geq 0, \sum_{i=0}^{3} p_i^{(A)} = 3\} \\
\bigcup &\{\mathbf{p^{(B)}} := (p_0^{(B)}, ..., p_3^{(B)}) : \quad p_0^{(B)}, ..., p_3^{(B)} \geq 0, \sum_{i=0}^{3} p_i^{(B)} = 3\} \\
\bigcup &\{\mathbf{p^{(C)}} := (p_0^{(C)}, ..., p_3^{(C)}) : \quad p_0^{(C)}, ..., p_3^{(C)} \geq 0, \sum_{i=0}^{3} p_i^{(C)} = 1\} \\
\bigcup &\{\mathbf{p^{(D)}} := (p_0^{(D)}, ..., p_3^{(D)}) : \quad p_0^{(D)}, ..., p_3^{(D)} \geq 0, \sum_{i=0}^{3} p_i^{(D)} = 5\} \\
\bigcup &\{\mathbf{p^{(E)}} := (p_0^{(E)}, ..., p_3^{(E)}) : \quad p_0^{(E)}, ..., p_3^{(E)} \geq 0, \sum_{i=0}^{3} p_i^{(E)} = 2\} \\
\bigcup &\{\mathbf{p^{(F)}} := (p_0^{(F)}, ..., p_3^{(F)}) : \quad p_0^{(F)}, ..., p_3^{(F)} \geq 0, \sum_{i=0}^{3} p_i^{(F)} = 4\}
\end{aligned}$$

be the basis. Let

$$\{\mathbf{q} := (q_0, ..., q_3) : q_0, ..., q_3 \geq 0, \sum_{i=0}^{3} q_i = 6\}$$

be the basis for the range. Write

$$A = \sum_{\mathbf{p^{(A)}}} a_{\mathbf{p^{(A)}}} \mathbf{x}^{\mathbf{p^{(A)}}}, B = \sum_{\mathbf{p^{(B)}}} b_{\mathbf{p^{(B)}}} \mathbf{x}^{\mathbf{p^{(B)}}}, C = \sum_{\mathbf{p^{(C)}}} c_{\mathbf{p^{(C)}}} \mathbf{x}^{\mathbf{p^{(C)}}},$$
$$D = \sum_{\mathbf{p^{(D)}}} d_{\mathbf{p^{(D)}}} \mathbf{x}^{\mathbf{p^{(D)}}}, E = \sum_{\mathbf{p^{(E)}}} e_{\mathbf{p^{(E)}}} \mathbf{x}^{\mathbf{p^{(E)}}}, F = \sum_{\mathbf{p^{(F)}}} f_{\mathbf{p^{(F)}}} \mathbf{x}^{\mathbf{p^{(F)}}}.$$

We will compute the Jacobian of the map h and show its rank is 84 at a certain point in the domain. Specifically, that point is

$$(A, B, C, D, E, F) = (x_1^3, x_0^3, x_2, 0, x_3^2, 0).$$

The point has coordinates

$$a_{\mathbf{p}^{(\mathbf{A})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{A})} = (0,3,0,0); \\ 0, & \text{otherwise.} \end{cases}$$

$$b_{\mathbf{p}^{(\mathbf{B})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{B})} = (3,0,0,0); \\ 0, & \text{otherwise.} \end{cases}$$

$$c_{\mathbf{p}^{(\mathbf{C})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{C})} = (0,0,1,0); \\ 0, & \text{otherwise.} \end{cases}$$

$$d_{\mathbf{p}^{(\mathbf{D})}} \equiv 0$$

$$e_{\mathbf{p}^{(\mathbf{E})}} = \begin{cases} 1, & \text{if } \mathbf{p}^{(\mathbf{E})} = (0,0,0,2); \\ 0, & \text{otherwise.} \end{cases}$$

$$f_{\mathbf{p}^{(\mathbf{F})}} \equiv 0$$

The Jacobian is an 84×145 dimensional matrix. We will show its rank is 84 at that specific point. Namely, we will pick an 84×84 submatrix that is nonsingular. That means we will pick every row and 84 out of 145 columns of the Jacobian to form the submatrix.

The first 40 columns are exactly the same as in 5.1. As a result, there is a 39×39 submatrix from the first 40 columns that is nonsingular. This submatrix corresponds to rows

$$\{\mathbf{q} : q_0 \geq 3 \text{ or } q_1 \geq 3\}.$$

Next go to columns 45-100 corresponding to $\mathbf{p}^{(\mathbf{D})}$ and focus on this 84×56 submatrix.

In this submatrix, an entry is zero unless $\mathbf{p^{(D)}} < \mathbf{q}$ at that entry, then the entry is $c_{\mathbf{q}-\mathbf{p^{(D)}}}$. Next at that specific point $c_{\mathbf{p^{(C)}}} = 0$ unless $\mathbf{p^{(C)}} = (0,0,1,0)$, and $c_{(0,0,1,0)} = 1$, so only those entries whose $\mathbf{q} - \mathbf{p^{(D)}} = (0,0,1,0)$ are nonzero, and those entries are 1. Both $\mathbf{p^{(D)}}$ and $\mathbf{q}$ are 4-tuples. Define a lexicographical order on 4-tuples by

$$(i_1, i_2, i_3, i_4) > (j_1, j_2, j_3, j_4) \text{ if and only if}$$

$$i_3 > j_3 \text{ or } i_3 = j_3 \text{ and } (i_1, i_2, i_4) > (j_1, j_2, j_4).$$

where $(i_1, i_2, i_4) > (j_1, j_2, j_4)$ denotes the usual lexicographical order on 3-tuples. Let us order both rows and columns of this 84×56 submatrix according to the lexicographic order just defined, then this matrix is

$$\begin{pmatrix} I_{56\times 56} \\ 0_{28\times 56} \end{pmatrix}.$$

The reason is that it is precisely on the diagonal of the upper 56×56 submatrix that $\mathbf{q} - \mathbf{p^{(D)}} = (0,0,1,0)$. This 56×56 submatrix corresponds to rows

$$\{\mathbf{q} : q_2 \geq 1\}.$$

Finally go to columns 111-145 corresponding to $\mathbf{p^{(F)}}$ and focus on this 84×35 submatrix. In this submatrix, an entry is zero unless $\mathbf{p^{(F)}} < \mathbf{q}$ at that entry, then the entry is $e_{\mathbf{q}-\mathbf{p^{(F)}}}$. Next at that specific point $e_{\mathbf{p^{(E)}}} = 0$ unless $\mathbf{p^{(E)}} = (0,0,0,2)$, and $e_{(0,0,0,2)} = 1$, so only those entries whose $\mathbf{q} - \mathbf{p^{(F)}} = (0,0,0,2)$ are nonzero, and those entries are 1. Both $\mathbf{p^{(F)}}$ and $\mathbf{q}$

are 4-tuples. Define a lexicographical order on 4-tuples by

$$(i_1, i_2, i_3, i_4) > (j_1, j_2, j_3, j_4) \text{ if and only if}$$

$$i_4 > j_4 \text{ or } i_4 = j_4 \text{ and } (i_1, i_2, i_3) > (j_1, j_2, j_3).$$

where $(i_1, i_2, i_3) > (j_1, j_2, j_3)$ denotes the usual lexicographical order on 3-tuples. Let us order both rows and columns of this 84×35 submatrix according to the lexicographic order just defined, then this matrix is

$$\begin{pmatrix} I_{35\times 35} \\ \\ 0_{49\times 35} \end{pmatrix}.$$

The reason is that it is precisely on the diagonal of the upper 35×35 submatrix that $\mathbf{q} - \mathbf{p}^{(\mathbf{F})} = (0, 0, 0, 2)$. This 35×35 submatrix corresponds to rows

$$\{\mathbf{q} : q_3 \geq 2\}.$$

Because

$$\begin{aligned} &\{\mathbf{q} : q_0 \geq 3 \text{ or } q_1 \geq 3\} \cup \{\mathbf{q} : q_2 \geq 1\} \cup \{\mathbf{q} : q_3 \geq 2\} \\ =&\{\mathbf{q} := (q_0, ..., q_3) : q_0, ..., q_3 \geq 0, \sum_{i=0}^{3} q_i = 6\}, \end{aligned}$$

this means from columns 1-40, 45-100 and 111-145 there is an 84×84 submatrix that is nonsingular. So indeed the Jacobian has rank 84 at that specific point and h is dominant. $\square$

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