Spring 4-8-2013

Tracial Rokhlin Property and Non-Commutative Dimensions

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by

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A dissertation presented to the
Graduate School of Arts and Sciences
of Washington University in St Louis
in partial fulfillment of the
requirements for the degree
of Doctor of Philosophy

May 2013

St. Louis, Missouri

TABLE OF CONTENTS

Page

ACKNOWLEDGMENTS

To my first advisor, Prof Weaver, thank you for your excellent support, caring and enthusiastic encouragement. Thank you for providing me with an excellent atmosphere for doing research. I always feel more confident after coming out of your office.

To my co-advisor in University of Oregon, Prof Phillips, thank you for your patient, valuable suggestions and inspiring conversations. Thank you for guiding me in many different areas, not just mathematically. It's still one of my best journey to see the lakes in Copenhagen with you.

To my second advisor, Prof McCarthy, thank you for your great support, help and encouragement. I truly appreciate your guidance on how to become a professional mathematician.

To my undergraduate mathematics advisor, Prof Wong, thank you for bring me to the math world, showing me the beauty of mathematics.

To my wife, Lu, thank you for your care and support, your unyielding devotion and love.

ABSTRACT

Tracial Rokhlin Property and Non-Commutative Dimensions

by

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Doctor of Philosophy in Mathematics,

Washington University in St. Louis, May, 2013.

Professor John E. M^{c}Carthy, Chairperson

This dissertation focuses on finite group actions with the tracial Rokhlin property and the structure of the corresponding crossed products. It consists of two major parts. For the first part, we study several different aspects of finite group actions with certain versions of the Rokhlin property. We are able to give an explicit characterization of product-type actions with the tracial Rokhlin property or strict Rokhlin property. We also show that, in good circumstances, the actions with the tracial Rokhlin property are generic.

In the second portion of this dissertation, we introduce the weak tracial Rokhlin property for actions on non-simple C*-algebras. The main results are as follows. Let A be a unital non-simple C*-algebra and α be an action of G on A with the weak tracial Rokhlin property. Assume that the crossed product $C^*(G, A, \alpha)$ is simple. Suppose A has either of the following property: tracial rank$\leq k$, stable rank one, real rank zero. Then $C^*(G, A, \alpha)$ has the same property.

1. Introduction

In this chapter we will establish the basic framework, introducing the background and some terminology. We shall assume basic knowledge of C*-algebras and finite groups. *Throughout this paper, all C*-algebras are assumed to be UNITAL, all tensor product of C*-algebras are assumed to be MINIMAL, except otherwise specified.*

Let G be a finite group and A be a C*-algebra. An *action* α of G on A is a group homomorphism from G into $\mathrm{Aut}(A)$-the group of automorphisms on A. For any element g in G, α_g stands for $\alpha(g)$, the automorphism corresponding to g.

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G. As a set, the crossed product $C^*(G, A, \alpha)$ is just the group ring $A[G]$. To make the crossed product into a C*-algebra, we will define a multiplication and involution twisted by the group action. It's not hard to recover the exact formulas from the following:

Definition 1.0.1 *(Theorem 3.18, [1]) Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C*-algebra A. Then $C^*(G, A, \alpha)$ is the universal C*-algebra generated by a unital copy of A (that is, the identity of A is supposed to be the identity of the generated C*-algebra) and unitaries u_g, for $g \in G$, subject to the relations $u_g u_h = u_{gh}$ for $g, h \in G$ and $u_g a u_g^* = \alpha_g(a)$ for $a \in A$ and $g \in G$.*

We can see from the above characterization that the group action becomes 'inner' in the crossed product. The unitaries u_g generate a copy of the group G, we shall call them the canonical unitaries for the crossed product. It's not surprising that the crossed product could carry much information of the action. On the other hand, the crossed product has been used as an important way to construct interesting new C*-algebras. There are many other good reasons to study crossed product C*-algebras. For a general reference and more motivations, see [1] and [2]. We are especially interested in the following question:

Question 1.0.2 *Since A naturally embeds in $C^*(G, A, \alpha)$, what properties of A can be inherited by the crossed product?*

The 'properties' we will investigate are the so-called noncommutative dimensions. These are noncommutative generalizations of the Lebesgue covering dimension for topological spaces. By the result of Gelfand, any commutative C*-algebra A is isomorphic to $C(X)$, for some compact topological space X (recall our convention that all C*-algebra are assumed to be unital). The covering dimension, originally defined in terms of open covers, can be reformulated in the following way:

Theorem 1.0.3 *Let X be compact, then the covering dimension of X is the smallest integer n such that for any continuous function $f\colon X \to \mathbb{R}^{n+1}$, there exists another continuous function $g\colon X \to \mathbb{R}^{n+1}\backslash\{0\}$, such that g approximates f arbitrarily well in the norm topology.*

Using complex valued functions instead of real functions, one can also define a 'complex' dimension. This immediately leads to two generalizations to 'non-commutative topological spaces':

Definition 1.0.4 *Let A be a C*-algebra. The* stable rank *(respectively,* real rank*) of A is the smallest integer, denoted by* $\mathrm{tsr}(A)$ *(respectively,* $\mathrm{RR}(A)$*), such that for each n-tuple $(x_1, x_2, \ldots, x_n)$ of elements in A (A_{sa}), with* $n \leq \mathrm{tsr}(A)$ ($\mathrm{RR}(A)$)*, and every $\varepsilon > 0$, there is an n-tuple $(y_1, y_2, \ldots, y_n)$ in A (A_{sa}), such that $\sum_k y_k^* y_k$ is invertible and*

$$\| \sum_{k=1}^{n} (x_k - y_k)(x_k^* - y_k^*) \| < \varepsilon. \tag{1.1}$$

Another version of non-commutative dimension that will also be discussed in this thesis is called the tracial rank and denoted by $\mathrm{TR}(\cdot)$. We shall give a precise definition at Chapter 4.

There is some research work on computing ranks of general C*-algebras, but the really interesting examples are C*-algebras of 'low ranks'. Most attention has been paid to the cases of *stable rank one, real rank zero* and *tracial rank zero or one.* The motivation comes from Elliott's classification program. Almost all C*-algebras being classified (before Wilhelm Winter's work) have either stable rank one or real rank zero. See for example [3], [4] and [5]. C*-algebras of tracial rank zero or one appears in Lin's classification result [6] and [7].

We should comment here that recently Wilhelm Winter and Joachim Zacharias defined a new non-commutative dimension in [8], called the nuclear dimension, which proved to be very useful in the classification of nuclear C*-algebras. Unfortunately we are not going to inverstigate it here.

Now let's go back to crossed product. Let $\alpha \colon G \to \mathrm{Aut}(A)$ be a finite group action on a C*-algebra A. The underlying set for the crossed product is the same as $C(G, A)$. But

$C(G, A)$ could also be viewed as a C*-algebra with pointwise multiplication and involution, which is isomorphic to direct sum of finite copies of A. The rank of this C*-algebra is the same as A, for any of the ranks we mentioned earlier. Therefore it's natural to guess that the dimensions of $C^*(G, A, \alpha)$ should be equal to that of A. But in general, the crossed product behaves very wildly. There are trivial examples where the stable rank of the crossed product could be less, because the matrix algebra of A has smaller stable rank than that of A (Theorem 6.1, [9]). Blackadar in [10] provided an example where the crossed product has stable rank strictly greater than the original C*-algebra, Elliott in [11] mentioned an example where the crossed product has real rank strictly greater than the original C*-algebra. Thus one need to impose some nice conditions on the group action to obtain good structure theorems. One such is called the Rokhlin property.

The Rokhlin property was originally used in ergodic theory for von Neumann algebras. The idea was used by R. Herman and V. Jones for finite group actions on C*-algebras. But they dealt only with a very small class of actions. Izumi in [12] initiated the study of finite group actions on general C*-algebras with the Rokhlin property, whose definition is given by:

Definition 1.0.5 *We say α has the **Rokhlin property** if there exist mutually orthogonal projections e_g in $A_\infty \cap A'$ which sum to 1 and have $\alpha_g(e_h) = e_{gh}$ for any g, h in G.*

Starting from Izumi, a number of authors have shown that crossed products by actions with the Rokhlin property preserve many important classes of C*-algebras, such as AF algebras, AI algebras, AT algebras, simple AH algebras with slow dimension growth and real rank

0, $\mathcal{D}$-absorbing C*-algebras for a strongly self-absorbing C*-algebra $\mathcal{D}$, and so on. In our case, if the action has the Rokhlin property, then the class of C*-algebras with stable rank one, or real rank zero, or finite tracial rank, will be preserved by crossed products. But the Rokhlin property is too strong and therefore rare. Phillips pointed out many obstructions of the Rokhlin property (and Izumi himself did too). He introduced the **tracial Rokhlin property** in [13], where he weakened the condition $\sum_{g\in G} e_g = 1$ by requiring that the defect projection $(1-\sum_{g\in G} e_g)$ be 'tracially small' (A precise definition will be given in Chapter 1). The tracial Rokhlin property is much more common, and one still gets fairly good structural theorems (of course not as good as the Rokhlin property case). In [13], Phillips proved the following:

Theorem 1.0.6 *Let A be an infinite dimensional simple separable unital C*-algebra with tracial rank zero. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A with the tracial Rokhlin property. Then the crossed product $C^*(G, A, \alpha)$ has tracial rank zero.*

There are also many other variants and generalizations of the Rokhlin property. For example, one can consider actions of more general groups, like $\mathbb{Z}$ or $\mathbb{Z}^d$. One can also try to define the tracial Rokhlin property for more general C*-algebras, non-simple or projectionless. In the first part of this dissertation we shall investigate actions with the tracial Rokhlin property in detail. For the second part we will give a generalization of Theorem 1.0.6 for non-simple C*-algebras and for other non-commutative dimensions.

2. Permanence properties of Rokhlin actions

2.1 Basic definition

In order to distinguish between two Rokhlin properties, we shall call the Rokhlin property defined by Izumi the *strict Rokhlin property.* When we say 'Rokhlin actions', we mean actions with either the strict Rokhlin property or the tracial Rokhlin property. We start with an alternative definition of the strict Rokhlin property:

Definition 2.1.1 *Let A be an infinite dimensional unital C*-algebra, let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the* strict Rokhlin property *if for every finite set $F \subset A$, every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ with $\sum_{g\in G} e_g = 1$, such that:*

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$, and

(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.

The definition of tracial Rokhlin property is then a modification of the above one:

Definition 2.1.2 *Let A be an infinite dimensional simple unital C*-algebra, let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the* tracial Rokhlin property *if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:*

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$,

(2) $\|e_g a - a e_g\| < \varepsilon$ *for all* $g \in G$ *and all* $a \in F$,

(3) Let $e = \sum_{g\in G} e_g$, *we have* $1 - e \precsim_s x$, *and*

(4) $\|exe\| > 1 - \varepsilon$.

The comparison used in condition (3) is called Blackadar's comparison, whose definition is given by:

Definition 2.1.3 *Let* B *be a* C^**-subalgebra of* A*. We say* B *is hereditary, if for any* $x \in B$ *and* $y \in A$ *with* $y \leq x$*, we have* $y \in B$*. We use* $\mathrm{Her}(x)$ *to denote the smallest hereditary subalgebra containing* x*. Let* a*,* b *be two positive elements in a* C^**-algebra* A*. We say* $a \sim_s b$ *if there exist some element* $x \in A$*, such that* $a = xx^*$ *and* $\mathrm{Her}(x^*x) = \mathrm{Her}(b)$*. We say* $a \precsim_s b$ *if there exist* $a' \in \mathrm{Her}(b)$ *such that* $a \sim_s a'$*.*

It should be pointed out here that Blackadar's comparison is a generalization of Murray-von Neumann comparison for projections. The notation is adopted from [14].

In Definition 2.1.2, we will call (3) the comparison condition and (4) the norm condition. The projections e_g for $g \in G$ will be called Rokhlin projections corresponding to F, x, and ε, or Rokhlin projections for short.

For a list of examples of actions with the tracial Rokhlin property, see Example 3.12, [15]. Also see Chapter 3 for a classification of product-type actions on UHF algebras with the Rokhlin properties.

2.2 Actions with the strict Rokhlin property

We now begin to investigate the permanence properties of Rokhlin actions. We shall consider the following ways of inducing new actions:

1. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action and p be an invariant projection. Consider the restriction $\alpha|_{pAp}$.

2. Let $(A_n)_{1<n<\infty}$ be an increasing sequence of C*-algebras and $A = \overline{\cup_{n\in\mathbb{Z}_{>0}} A_n}$. Let α_n be an action of G on A_n for each n. Suppose the actions are compatible in the sense that $\alpha_{n+1}|_{A_n} = \alpha_n$. Consider the inductive limit action $\alpha\colon G \to \mathrm{Aut}(A)$, defined by $\alpha|_{A_n} = \alpha_n$.

3. Let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be two actions. Consider the 'inner tensor' $\gamma\colon G \to \mathrm{Aut}(A \otimes B)$, defined by $\gamma_g(a \otimes b) = \alpha_g(a) \otimes \beta_g(b)$. Recall our convention that all tensor products of C*-algebras are minimal. We use $\alpha \otimes \beta$ to denote this action.

The permanence property of strict Rokhlin actions are known to specialists, we include the proof here for the reader's convenience.

We shall make use of some standard pertubation techniques, most of which could be found in Section 2.5, Chapter 2 of [16]. We single out the following:

Lemma 2.2.1 *For any $\varepsilon > 0$ and any $n \in \mathbb{Z}_{>0}$, there exists a $\delta = \delta(\varepsilon, n) > 0$ with the property that if A is a C*-algebra and if $p_1, p_2 \ldots, p_n$ are mutually orthogonal projections in A, then for any $p \in A$ with $\|pp_i - p_ip\| < \delta$ for $i \neq j$, there are mutually orthogonal projections $q_1, \ldots, q_n$ in pAp such that $\|q_i - pp_ip\| < \varepsilon$.*

Proof Combine Lemma 2.5.5 and Lemma 2.5.6 of [16]. ■

Proposition 2.2.2 *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the strict Rokhlin property, and let p be an invariant projection. Then the induced action $\alpha|_{pAp}$ has the strict Rokhlin property.*

Proof We follow essentially the same lines of Lemma 3.7 of [13]. Let $F \subset pAp$ be finite, let $\varepsilon > 0$. Set $n = \mathrm{Card}(G)$. Set

$$\varepsilon_0 = \min(\frac{1}{n},\ \frac{\varepsilon}{3}).$$

According to Lemma 2.2.1, we can choose $\delta > 0$ such that whenever B is a unital C*-algebra, $q_1, \cdots, q_n$ are mutually orthogonal projections, and $p \in B$ is a non-zero projection such that $\|pq_j - q_jp\| < \delta$ for $1 \leq j \leq n$, then there are mutually orthogonal projections $e_j \in pBp$ such that $\|e_j - pq_jp\| < \varepsilon_0$ for $1 \leq j \leq n$. We also require $\delta < \varepsilon_0$.

Since α has the strict Rokhlin property, with $F \cup \{p\}$ in place of F, with δ in place of ε, we can obtain Rokhlin projections $q_g \in A$ for $g \in G$. By the choice of δ, there are mutually orthogonal projections $e_g \in pAp$ such that $\|e_g - pq_gp\| < \varepsilon_0$ for $g \in G$. We now estimate, using $\alpha_g(p) = p$,

$$\|\alpha_g(e_h) - e_{gh}\| \leq \|e_h - pq_hp\| + \|e_{gh} - pq_{gh}p\| + \|p(\alpha_g(q_h) - q_{gh})p\| < 2\varepsilon_o + \delta \leq \varepsilon.$$

And for $a \in F$, using $pa = ap = a$,

$$\|e_ga - ae_g\| \leq 2\|e_g - pq_gp\| + \|p(q_ga - aq_g)p\| < 2\varepsilon_0 + \delta \leq \varepsilon.$$

Next, set $e = \sum_{g\in G} e_g$, then $\|e - p\| \leq \sum_{g\in G} \|e_g - pq_gp\| < n\varepsilon_o \leq 1$. But e is also a subprojection of p, this forces $e = p$, the identity of pAp. ■

Proposition 2.2.3 *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an inductive limit action, i.e. there exists an increasing sequence $(A_n)_{1<n<\infty}$ such that $A = \overline{\cup_{n\in\mathbb{Z}_{>0}} A_n}$ and each A_n is invariant under the action. Let α_n denote the induced action on A_n. If each α_n has the strict Rokhlin property, then α has the strict Rokhlin property.*

Proof Let F be a finite subset of A, and $\varepsilon > 0$. We can then find some A_n and a finite subset $\tilde{F}$ of A_n, such that for any $a \in F$, there is some b in $\tilde{F}$ with $\|a - b\| < \varepsilon/3$. Since α_n has the strict Rokhlin property, with $\tilde{F}$ and $\varepsilon/3$, we can obtain Rokhlin projections $\{e_g\}_{e\in G}$ as in Definition 2.1.1. For any $a \in F$, find b in $\tilde{F}$ such that $\|a - b\| < \varepsilon/3$, then

$$\|ae_g - e_g a\| \leq \|(a-b)e_g\| + \|be_g - e_g b\| + \|e_g(a-b)\| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

■

Proposition 2.2.4 *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the strict Rokhlin property and $\beta\colon G \to \mathrm{Aut}(B)$ be an arbitrary action. Then the tensor product $\gamma = \alpha\otimes\beta\colon G \to \mathrm{Aut}(A\otimes B)$ has the strict Rokhlin property.*

Proof Let $F \subset A$ be finite and $\varepsilon > 0$, without loss of generality we may assume that the elements of F are all elementary tensors, i.e. $F = \{a_i \otimes b_i\}$. Let $\tilde{F} = \{a_i\} \subset A$ and let $M = \max\{\|b_i\|\}$. From Definition 2.1.1, we obtain Rokhlin projections $\{e_g\}_{g\in G} \subset A$ corresponding to $\tilde{F}$ and $\varepsilon/(M+1)$. Consider projections $p_g = e_g \otimes 1 \in A \otimes B$, they are mutually orthogonal projections summing up to 1, and satisfy:

(1) $\|\alpha_g(p_h) - p_{gh}\| = \|(\alpha_g(e_h) - e_{gh}) \otimes 1\| < \varepsilon.$

(2) $\|p_g(a_i \otimes b_i) - (a_i \otimes b_i)p_g\| = \|(e_g a_i - a_i e_g) \otimes b_i\| < \varepsilon M/(M+1) < \varepsilon.$

■

2.3 C*-algebras with Property (SP)

When studying actions with the tracial Rokhlin property, we can restrict the underlying C*-algebra to a special class. A C*-algebra is said to have *Property (SP)* if every non-zero hereditary C*-subalgebra contains at least one non-zero projection. If A does not have Property (SP), then we can choose some non-zero positive element b such that $\mathrm{Her}(b)$ contains no non-zero projection. If p is a projection in A, then $p \precsim_s b$ implies $p = 0$. Hence we have the following:

Lemma 2.3.1 *(Lemma 1.13, [13]) Let A be an infinite dimensional simple separable unital C*-algebra, let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A which has the tracial Rokhlin property. Then A has Property (SP) or α has the strict Rokhlin property.*

Here we present some lemmas about C*-algebras with Property (SP) which are closely related to our construction of induced actions. The following definition of 'cutting function' will be used in later proofs and the definition of tracial rank in Chapter 5.

Definition 2.3.2 *Let $0 \leq \sigma_2 < \sigma_1 \leq 1$ be two positive numbers. The function $f_{\sigma_2}^{\sigma_1}$ is a piecewise linear function defined by:*

$$f_{\sigma_2}^{\sigma_1}(x) = \begin{cases} 0 & \text{if} \quad x < \sigma_2, \\ \frac{x-\sigma_2}{\sigma_1-\sigma_2} & \text{if} \quad \sigma_2 \leq x \leq \sigma_1, \\ 1 & \text{if} \quad x > \sigma_1. \end{cases}$$

Lemma 2.3.3 *Let A be a C*-algebra with Property (SP), let $x \in A$ be a positive element with norm 1, and let $\varepsilon > 0$. Then there exists a non-zero projection $p \in \overline{xAx}$ such that for any positive element $q \leq p$ with $\|q\| = 1$, we have:*

$$\|qx^{1/2} - q\| \leq \varepsilon, \quad \|x^{1/2}q - q\| \leq \varepsilon.$$

Compared to Lemma 1.14 of [13], here we allow q to be any positive element instead of a projection.

Proof Since $\mathrm{Her}(x) = \mathrm{Her}(x^{1/2})$, it suffices to prove the inequalities with x in place of $x^{1/2}$. Let $f = f_0^{1-\varepsilon}$, $g = f_{1-\varepsilon}^1$. (See Definition 2.3.2.) Then we have:

$$fg = g, \quad \|f(x) - x\| \leq \varepsilon, \quad \overline{g(x)Ag(x)} \subset \overline{xAx} = \overline{f(x)Af(x)}.$$

Now $fg = g$ implies that for any $c \in \overline{g(x)Ag(x)}$, $cf(x) = f(x)c = c$. Since A has Property (SP), we can choose a nonzero projection $p \in \overline{g(x)Ag(x)} \subset \overline{xAx}$. Then for any positive element $q \leq p$, $\|qx - q\| = \|qx - qf(x)\| \leq \varepsilon$ and similarly $\|xq - q\| \leq \varepsilon$. ■

A C*-algebra is elementary if it's finite dimensional or isomorphic to the C*-algebra of compact operators on some Hilbert space.

Lemma 2.3.4 *Let A be a non-elementary simple C*-algebra with Property (SP). Then for any non-zero projection p in A and any $n \in \mathbb{Z}_{>0}$, there exists n mutually orthogonal sub-projections p_i of p which are mutually Murray-von Neumann equivalent.*

Proof Lemma 3.5.7, p142 of [16]. ■

Lemma 2.3.5 *Let A be a C*-algebra. If there exists an increasing sequence of C*-algebras (A_n) such that $A = \overline{\cup_{n \in \mathbb{Z}_{>0}} A_n}$ and each A_n has Property (SP), then A itself has Property (SP).*

Proof (Private communication with C. Phillips) Let $x \in A$ be a positive element of norm 1. We need to find a non-zero projection in $\overline{xAx}$. Let $\varepsilon = 1/20$. Then we can find some $n \in \mathbb{N}^+$ and a positive element $y \in A_n$ such that $\|y - x\| < \varepsilon$. This implies $y - \varepsilon < x$. Let z be the positive part of $y - \varepsilon$. We have:

$$z \neq 0, \quad z(y-\varepsilon)z = z^3, \quad \|z - (y - \varepsilon)\| < \varepsilon \quad \text{and} \quad \|z - x\| < 2\varepsilon.$$

Let $c = zx^{1/2}$, then $1 - 2\varepsilon < \|c\| \leq 1$. Since $cc^* = zxz \geq z(y-\varepsilon)z \geq z^3$, we have:

$$\overline{cc^*Acc^*} \supset \overline{z^3Az^3} = \overline{zAz} \supset \overline{zA_nz}.$$

Let $f = f_{\varepsilon}^{1-\varepsilon}$ and $g = f_{1-\varepsilon}^{1}$. Since A_n has Property (SP) by assumption, we can a find non-zero projection $p \in \overline{g(z)A_ng(z)} \subset \overline{cc^*Acc^*}$. Now $f(z)g(z) = g(z)$ implies that $pf(z) = pg(z)f(z) = p$. Using the relation $\|z - x\| < 2\varepsilon$ and $\|pz - p\| = \|pz - pf(z)\| < \varepsilon$ repeatedly, we can show that $\|pcc^* - p\| = \|pzxz - p\| < 5\varepsilon$. Taking the adjoint we get $\|cc^*p - p\| < 5\varepsilon$. Then c^*pc is approximately a projection: $\|c^*pcc^*pc - c^*pc\| < 5\varepsilon < 1/4$. Since $c^*pc = x^{1/2}zpzx^{1/2} \in \overline{xAx}$, there exists a projection $q \in \overline{xAx}$ such that $\|q - c^*pc\| < 1/4$. The projection q is non-zero, otherwise

$$\|c^*pc\| = \|pcc^*p\| \geq \|p\| - \|(pcc^* - p)p\| = 3/4 > 1/4 > \|c^*pc\|,$$

which is a contradiction. ■

The following lemma is very useful in dealing with tensor products:

Lemma 2.3.6 *(Kirchberg's Slice Lemma). Let A and B be C^*-algebras, and let D be a non-zero hereditary sub-C^*-algebra of the minimal tensor product $A \otimes B$. Then there exists a non-zero element z in $A \otimes B$ such that z^*z is an elementary tensor $a \otimes b$, for some $a \in A_+$ and $b \in B_+$, and zz^* belongs to D.*

See lemma 4.1.9, p68 of [17] for a proof. Note that the definition of a, b in the proof shows that they are all positive. As a consequence, we can show the following:

Lemma 2.3.7 *Let A, B be two C*-algebras with Property (SP), then $A \otimes B$ has Property (SP).*

Proof Let D be a non-zero hereditary sub-C*-algebra of $A \otimes B$. By Lemma 2.3.6, we can find a non-zero element z in $A \otimes B$ such that z^*z is an elementary tensor $a \otimes b$ for some $a \in A_+$ and $b \in B_+$, and $zz^* \in D$. Since both A and B have Property (SP), there exist non-zero projections p, q in $\mathrm{Her}(a)$ and $\mathrm{Her}(b)$, respectively. Then $p \otimes q$ is a non-zero projection in $\mathrm{Her}(a \otimes b) = \mathrm{Her}(z^*z)$. But $\mathrm{Her}(z^*z)$ is isomorphic to $\mathrm{Her}(zz^*) \subset D$ (see p 218 of [18]), therefore D contains a non-zero projection. ■

2.4 Actions with the tracial Rokhlin property

Before we systematically study the inheritance of the tracial Rokhlin property, let's first present some basic properties of Blackadar's comparison first. (See Definition 2.1.3). We have the following equivalent definition:

Lemma 2.4.1 *Let a, b be two positive elements in a C*-algebra A. Let A'' be the enveloping von Neumann algebra of A. Then $a \precsim_s b$ if and only if there exist some partial isometry $v \in A''$ such that $v\mathrm{Her}(a)$ and $\mathrm{Her}(a)v$ are subsets of A, $vv^* = p_a$, where p_a is the range projection of a in A'', and $v^*\mathrm{Her}(a)v \subset \mathrm{Her}(b)$. Moreover, $a \sim_s b$ if and only if $v^*\mathrm{Her}(a)v = \mathrm{Her}(b)$.*

See Proposition 4.3 and Proposition 4.6 of [14] for a proof.

Notation 2.4.2 *By* $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ *we mean* $\mathrm{diag}\{a_1, a_2, ..., a_n\}$ *and by* $n \odot a$ *we mean* $a \oplus a \oplus \cdots \oplus a$. *We write* $a_1 \oplus a_2 \oplus \cdots \oplus a_n \precsim_s b_1 \oplus b_2 \oplus \cdots \oplus b_m$ *if and only if* $\mathrm{diag}\{a_1, a_2, \ldots, a_n, 0, \ldots, 0\} \precsim_s \mathrm{diag}\{b_1, b_2, \ldots, b_m, 0, \ldots, 0\}$ *in* $M_{m+n}(A)$.

Proposition 2.4.3 *(Proposition 3.5.3, p 141, [16]) Let A be a C*-algebra.*

(i) If $0 \le a \le b$, *then* $a \precsim_s b$.

(ii) If p and q are two projections in A, then $p \precsim_s q$ if and only if p is sub-equivalent to q in the sense of Murray and von Neumann, and $p \sim_s q$ if and only if p and q are Murray-von Neumann equivalent.

(iii) Let B be a hereditary subalgebra of A and $a, b \in B$. Then $a \precsim_s b$ in A if and only if $a \precsim_s b$ in B, and $a \sim_s b$ in A if and only if $a \sim_s b$ in B.

(iv) Let $a_1, ..., a_n$ be positive elements in A. Then $(\sum_{i=1}^n a_i) \precsim_s \oplus_{i=1}^n a_i$. If $a_i a_j = 0, \forall i \neq j$, then $(\sum_{i=1}^n a_i) \sim_s \oplus_{i=1}^n a_i$.

Now we are ready to prove analogous results for the tracial Rokhlin property:

Proposition 2.4.4 *Let A be an infinite dimensional simple C*-algebra. Let* $\alpha\colon G \to \mathrm{Aut}(A)$ *be an action with the tracial Rokhlin property, and let p be an invariant projection. Then the induced action $\alpha|_{pAp}$ has the tracial Rokhlin property.*

Proof Lemma 3.7 of [13]. ∎

Proposition 2.4.5 *Let* $\alpha\colon G \to \mathrm{Aut}(A)$ *be an inductive limit action, where* $A = \varinjlim A_n$ *and all C*-algebras are infinite dimensional and simple. Let α_n denote the induced action on A_n. If each α_n has the tracial Rokhlin property, then α has the tracial Rokhlin property.*

Proof If there are infinitely many A_n's that do not have Property (SP), then α_n has the strict Rokhlin property for infinite many n, so α also has the strict Rokhlin property. (A slight modification of Proposition 2.2.3).

Otherwise, a slight modification of Lemma 2.3.5 shows that A itself has Property (SP). Let F be a finite subset of A. Let $x \in A_+$ with $\|x\| = 1$, and $\varepsilon > 0$ be given.

Since A has Property (SP), by Lemma 2.3.3 we can find a non-zero projection p such that $\|px^{1/2} - p\| < \varepsilon$ and $\|x^{1/2}p - p\| < \varepsilon$. Set

$$\delta = \min\{\varepsilon/7, 1/2\}.$$

We can then find some $n \in \mathbb{Z}_{>0}$ and a non-zero projection $q \in A_n$ such that $\|q - p\| < \delta$. Without loss of generality, we may assume that $F \subset A_n$. Since α_n has the tracial Rokhlin property, we can then find mutually orthogonal projections $\{e_g\}_{g \in G} \subset A_n$ such that:

(1) $\|e_g a - a e_g\| < \delta < \varepsilon$,

(2) $\|\alpha_g(e_h) - e_{gh}\| < \delta < \varepsilon$,

(3) Let $e = \sum_{g \in G} e_g$, then $1 - e \precapprox_s q$, and

(4) $\|eqe\| > 1 - \delta$.

Since $\|p-q\| < \delta \leq 1/2$, p and q are Murray-von Neumann equivalent in A. But $p \in \overline{xAx}$, so $1 - e \precsim_s q \approx p \precsim_s x$. For the norm condition, we have following estimation:

$$\begin{aligned} \|exe\| &= \|ex^{1/2}x^{1/2}e\| = \|x^{1/2}ex^{1/2}\| \geq \|px^{1/2}ex^{1/2}p\| \\ &= \|(px^{1/2}p - p)(x^{1/2}p - p) + pe(x^{1/2} - p) + (px^{1/2} - p)ep + pep\| \\ &\geq \|pep\| - 3\delta \\ &\geq \|qeq\| - 3\delta - 3\delta \\ &= \|eqe\| - 6\delta > 1 - 7\delta \geq 1 - \varepsilon. \end{aligned}$$

■

Proposition 2.4.6 *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of finite group G on a simple C*-algebra A with the tracial Rokhlin property. Let $\beta\colon G \to \mathrm{Aut}(B)$ be an arbitrary action on a simple C*-algebra B. Let $\theta = \alpha \otimes \beta\colon G \to \mathrm{Aut}(A \otimes B)$ be the tensor action of α and β. If $A \otimes B$ has Property (SP), then θ has the tracial Rokhlin property.*

Proof First of all, we can assume that A has Property (SP). Otherwise the action α will have the strict Rokhlin property and therefore θ has the strict Rokhlin property by Proposition 2.2.4.

Let F be a finite subset of $A \otimes B$, $\varepsilon > 0$, $x \in A \otimes B$ be a positive element of norm 1. Without loss of generality, we may assume that F consists of elementary tensors $a_i \otimes b_i, 1 \leq i \leq n$. Set

$$\varepsilon_0 = \varepsilon/12.$$

Since $A \otimes B$ has Property (SP), by Lemma 2.3.3, we can find a non-zero projection $r \in \mathrm{Her}(x)$ such that for any positive element $s \leq r$, we have $\|sx^{1/2} - s\| \leq \varepsilon_0, \|x^{1/2}s - s\| \leq \varepsilon_0$. By

Kirchberg's Slice Lemma 2.3.6, there exist some $z \in A \otimes B$, such that $zz^* \in \mathrm{Her}(r)$, and $z^*z = a \otimes b$, for some $a \in A_+$ and $b \in B_+$. We may assume that $\|a\| = \|b\| = \|z\| = 1$. Find $z_0 = \sum_{j=1}^{k} y_j \otimes z_j$ with norm 1 such that $\|z - z_0\| < \varepsilon_0$.

Since B is simple, we can find elements $\{l_i \,|\, 1 \le i \le m\}$ such that $\sum_i l_i b l_i^* = 1$. Let $M = \max\{\|a_i\|, \|b_i\|, \|y_j\|, \|z_j\|\}$ and set

$$\delta = \frac{\varepsilon}{6 + 4(|G|)M}.$$

For $a \in A$, we first use Lemma 2.3.3 to find a non-zero projection $p \in \mathrm{Her}(a)$ such that for any projection $q \le p$, $\|qa^{1/2} - q\| \le \delta, \|a^{1/2}q - q\| \le \delta$. By Lemma 2.3.4, we can find m mutually orthogonal but equivalent subprojection $\{p_i\}_{1 \le i \le m}$ of p.

Since α has the tracial Rokhlin property, for $F' = \{a_i\} \cup \{y_k\}$, $p_1 \in A_+$ and $\delta > 0$ as chosen before, we can find projections $\{q_g\}_{g \in G}$ in A such that:

(1) $\|q_g d - d q_g\| < \delta$, $\forall d \in F'$ and $g \in G$,

(2) $\|\alpha_g(q_h) - q_{gh}\| < \delta$,

(3) With $q = \sum_{g \in G} q_g$, $1 - q \precsim_s p_1$, and

(4) $\|q p_1 q\| \ge 1 - \delta$.

Now consider the projections $e_g = q_g \otimes 1$. For the action θ, we have:

(1′) $\|e_g f - f e_g\| = \|(q_g a_i - a_i q_g) \otimes b_i\| < M\delta \le \varepsilon$, $\forall f = a_i \otimes b_i \in F$ and $g \in G$.

(2′) $\|\theta_g(e_h) - e_{gh}\| = \|(\alpha_g(q_h) - q_{gh}) \otimes 1\| < \delta \le \varepsilon$.

(3′) Let $e = \sum_{g\in G} e_g = q \otimes 1$,

$$\begin{aligned}
1 - e &= (1-q) \otimes (\sum l_i b l_i^*) \\
&= \sum_{i=1}^{m} (1 \otimes l_i)((1-q) \otimes b)(1 \otimes l_i^*) \\
&\precsim_s m \odot (1-q) \otimes b \precsim_s m \odot (p_1 \otimes b) \\
&\precsim_s a \otimes b \sim_s z^* z \precsim_s x.
\end{aligned}$$

(4′) Since $p_1 \le p$, by our choice of p, we have $\|p_1 a^{1/2} - p_1\| \le \delta$ and $\|a^{1/2} p_1 - p_1\| \le \delta$. Therefore

$$\|qaq\| = \|a^{1/2} q a^{1/2}\| \ge \|p_1 a^{1/2} q a^{1/2} p_1\| > \|p_1 q p_1\| - 2\delta = \|q p_1 q\| - 2\delta > 1 - 3\delta.$$

It follows that $\|ezz^*e\| = \|e(a \otimes b)e\| = \|(qaq) \otimes b\| > 1 - 3\delta$. Hence

$$\begin{aligned}
\|ez^*ze\| &> \|ez_0^* z_0 e\| - 2\varepsilon_0 \\
&> \|z_0^* e z_0\| - 2M|G|\delta - 2\varepsilon_0 \\
&> \|z^* e z\| - 2M|G|\delta - 4\varepsilon_0 \\
&= \|ezz^*e\| - 2M|G|\delta - 4\varepsilon_0.
\end{aligned}$$

Recall that our $r \in \mathrm{Her}(x)$ is chosen so that $\forall s \le r$, $\|sx^{1/2} - s\| \le \varepsilon_0$, $\|x^{1/2}s - s\| \le \varepsilon_0$.

Since $(zz^*)^{1/2}$ is a positive element of norm 1 in $\mathrm{Her}(r)$ and r is a projection, $(zz^*)^{1/2} \le r$.

Hence $\|(zz^*)^{1/2}x^{1/2} - (zz^*)^{1/2}\| \le \varepsilon_0$, and therefore

$$\begin{aligned}
\|exe\| &> \|(zz^*)^{1/2}x^{1/2}ex^{1/2}(zz^*)^{1/2}\| \\
&> \|(zz^*)^{1/2}e(zz^*)^{1/2}\| - 2\varepsilon_0 \\
&= \|ezz^*e\| - 2\varepsilon_0 > 1 - 3\delta - 2M|G|\delta - 6\varepsilon_0 \ge 1 - \varepsilon.
\end{aligned}$$

This completes the proof. ■

By Lemma 2.3.7, we have the following corollary:

Corollary 2.4.7 *Let* $\alpha\colon G \to \mathrm{Aut}(A)$ *be an action of finite group* G *on a simple* C^**-algebra* A*, with the tracial Rokhlin property. Let* $\beta\colon G \to \mathrm{Aut}(B)$ *be an arbitrary action on simple* C^**-algebra* B*. Let* $\theta = \alpha \otimes \beta\colon G \to \mathrm{Aut}(A \otimes B)$ *be the tensor product of* α *and* β*. If* B *has Property (SP), then* θ *has the tracial Rokhlin property.*

3. Rokhlin properties for product-type actions

3.1 Basic definition

Product-type actions are used to construct explicit examples of Rokhlin actions. We begin with the definition:

Definition 3.1.1 *Let* $A = \otimes_{i=1}^{\infty} \mathrm{B}(H_i)$, *where* H_i *is a finite dimensional Hilbert space for each* i. *Let* G *be a finite group. An action* $\alpha\colon G \mapsto \mathrm{Aut}(A)$ *is called a* product-type action *if and only if for each* i, *there exists a unitary representation* $\pi_i\colon G \to \mathrm{B}(H_i)$, *which induces an inner action* $\alpha_i\colon g \mapsto \mathrm{Ad}(\pi_i(g))$, *such that* $\alpha = \otimes_{i=1}^{\infty} \alpha_i$.

Definition 3.1.2 *Let* $\alpha\colon G \mapsto \mathrm{Aut}(A)$ *be a product-type action on a UHF-algebra* A. *A* telescope *of the action is a choice of an infinite sequence of positive integers* $1 = n_1 < n_2 < \cdots$ *and a re-expression of the action, so that* $A = \otimes_{i=1}^{\infty} B(K_i)$ *where* $K_i = \otimes_{j=n_i}^{n_{i+1}-1} H_j$, *and the action on* $B(K_i)$ *is* $\otimes_{j=n_i}^{n_{i+1}-1} \alpha_j$.

Phillips gave a characterization of product-type $\mathbb{Z}/2\mathbb{Z}$-actions with the Rokhlin properties in [19]. He then asked whether similar characterization could be found for general finite group. We answer this question affirmatively in this section.

Recall that two actions $\alpha\colon G \mapsto \mathrm{Aut}(A)$ and $\beta\colon G \mapsto \mathrm{Aut}(B)$ are said to be conjugate, if there exist an isomorphism $T\colon A \mapsto B$ such that $T \circ \alpha_g = \beta_g \circ T$, for any $g \in G$. The main result of this section is the following theorem:

Theorem 3.1.3 *Let $\alpha\colon G \mapsto \mathrm{Aut}(A)$ be a product-type action where A is UHF. Let H_i, π_i and α_i be defined as in Definition 3.1.1. Let d_i be the dimension of H_i and χ_i be the character of π_i. We will use the same notations if we do a telescope to the action. Define $\chi\colon G \mapsto \mathbb{C}$ to be the characteristic function on 1_G. Note that $\chi = \frac{1}{|G|}\chi^{reg}$, where χ^{reg} is the character of the left regular representation. Then we have:*

(i) The action α has the strict Rokhlin property if and only if there exists a telescope, such that for any $n \in \mathbb{Z}_{>0}$,

$$\frac{1}{d_n}\chi_n = \chi. \tag{3.1}$$

(ii) The action α has the tracial Rokhlin property if and only if there exists a telescope, such that for any $n \in \mathbb{Z}_{>0}$, the infinite product

$$\prod_{n \leq i < \infty} \frac{1}{d_i}\chi_i = \chi. \tag{3.2}$$

We shall deal with the finite cyclic groups first, and then extend the result to arbitrary finite groups.

3.2 Product-type action with the strict Rokhlin property

We first develop some terminology for cyclic groups as well as product-type actions.

Lemma 3.2.1 *Let G be any finite cyclic group, then there exist a commutative bihomomorphism $\zeta\colon G \times G \to S^1$, where S^1 is the unit circle of the complex plane. By a commutative bihomomorphism we mean that for any $g, h \in G$, $\zeta(\bullet, g)$ and $\zeta(h, \bullet)$ are homomorphisms and $\zeta(g, h) = \zeta(h, g)$.*

Proof Let G be a cyclic group of order n, which is generated by a primitive n-th root of unity ξ. Define $\zeta : G\times G \to S^1$ by $\zeta(\xi^i, \xi^j) = \xi^{ij}$. Computation shows that ζ is a commutative bihomomorphism. ∎

Remark 3.2.2 *From now on we will assume that a commutative bihomomorphism ζ is given whenever we have a finite cyclic group G. It's easy to see that the map $g \to \zeta(g, \bullet)$ defines an isomorphism between G and the dual group $\hat{G}$. We use $\zeta^g(h)$ to denote $\zeta(g,h)$ as an indication of this duality. It's not difficult to see that $\sum_{g\in G} \zeta^g(h) = \delta(h, 1_G)|G|$, where $\delta(\cdot,\cdot)$ is the Kronecker delta. This equality will be used in later computations.*

Product-type actions are special cases of inductive limit actions. Let $\alpha\colon G \mapsto \mathrm{Aut}(A)$ be a product-type action where A is UHF. Write $A_n = \otimes_{i=1}^n \mathrm{B}(H_i)$, and for simplicity use the same symbol α for all induced actions. By Theorem 4.5 of [1], we have

$$C^*(G, A, \alpha) = \varinjlim C^*(G, A_n, \alpha).$$

The action on each A_n is inner, so by Example 4.10 of [1], $C^*(G, A_n, \alpha) \cong C^*(G) \otimes A_n$. If G is abelian, then $C^*(G) \cong C(G)$. In particular, we have the following:

Proposition 3.2.3 *Let G be a finite cyclic group. Then $C^*(G, A_n, \alpha)$ is isomorphic to a direct sum of $|G|$ copies of A_n. We may write $C^*(G, A_n, \alpha) \cong \oplus_{g\in G} A_n^g$, where $A_n^g = A_n$ for all $g \in G$.*

Now let's find an explicit formula for the isomorphism in the above proposition as well as the corresponding direct system.

For any $g \in G$, let $V_n^g \in U(A_n)$ denote the finite tensor $\otimes_{i=1}^n \pi_i(g)$. Define $X_n^g = (\zeta^h(g) V_n^g)_{h \in G} \in \oplus_{g \in G} A_n^g$. Embed A_n into $\oplus_{g \in G} A_n^g$ by the map $a \mapsto (a^h)_{h \in G}$, where $a^h = a$ for all $h \in G$. We can check that $\{X_n^g\}_{g \in G}$ are unitary elements and in $\otimes_{g \in G} A_n^g$, the elements in $A_n \cup \{X_n^g\}_{g \in G}$ satisfy the relation:

$$X_n^g X_n^h = X_n^{gh}, \quad X_n^g a (X_n^g)^* = \alpha_g(a), \quad \forall g, h \in G, a \in A_n.$$

Let U_n^g be the canonical unitaries of $C^*(G, A_n, \alpha)$. By Lemma 3.17 of [1], there is a *-homomorphism $T_n \colon C^*(G, A_n, \alpha) \mapsto \oplus_{g \in G} A_n^g$ which sends U_g to X_g and maps A_n to $\oplus_{g \in G} A_n^g$ the same way we embedded A_n in $\oplus_{g \in G} A_n^g$. That this homomorphism is an isomorphism is equivalent to the matrix $(\zeta^h(g))_{h,g \in G}$ being invertible. But $(\zeta^h(g))_{h,g \in G}$ is actually unitary (after normalization): For any two elements $h_0, h \in G$,

$$\begin{aligned}
\sum_{g \in G} \zeta^{h_0}(g) \overline{\zeta^h(g)} &= \sum_{g \in G} \zeta^{h_0}(g) \zeta^{h^{-1}}(g) = \sum_{g \in G} \zeta^{h_0 h^{-1}}(g) \\
&= \delta(h_0 h^{-1}, 1) |G| = \delta(h_0, h) |G|.
\end{aligned}$$

For each n, we can define an isomorphism as above. This enables us to turn the direct system $\varinjlim C^*(G, A_n, \alpha)$ into a direct system $\varinjlim \oplus_{g \in G} A_n^g$. Let $\tilde{\phi}_{i,j}$ and $\phi_{i,j}$ be the connecting maps of the two direct systems, we have the following proposition:

Proposition 3.2.4 *For any positive integers $n \leq m$ and any $g \in G$, let $P_{n,m}^g$ be the element of $\otimes_{i=n}^m \mathrm{B}(H_i)$ defined by*

$$P_{n,m}^g = |G|^{-1} \sum_{h \in G} \zeta^h(g) (\otimes_{i=n}^m \pi_i(h)).$$

Then $P_{n,m}^g$ are mutually orthogonal projections thatsum up to the identity matrix I, and such that the connecting map $\phi_{n-1,m}$ of $\varinjlim \oplus_{g \in G} A_n^g$ is given by:

$$(y_{n-1}^h)_{h \in G} \mapsto (y_m^h)_{h \in G}, \quad \textit{where } y_m^h = \sum_{g \in G} y_{n-1}^g \otimes P_{n,m}^{hg^{-1}}.$$

We shall use P_n^g to denote $P_{n,n}^g$.

Proof Without loss of generality we may assume $m = n$. Then $P_n^g = |G|^{-1}\sum_{h\in G}\zeta^h(g)\pi_n(h)$.

We can compute:

$$\begin{aligned}
(P_n^g)^* &= |G|^{-1}\sum_{h\in G}\overline{\zeta^h(g)}(\pi_n(h))^* \\
&= |G|^{-1}\sum_{h\in G}\zeta^{h^{-1}}(g)\pi_n(h^{-1}) = P_n^g.
\end{aligned}$$

$$\begin{aligned}
P_n^{g_0}P_n^g &= |G|^{-2}\sum_{h,k\in G}\zeta^h(g_0)\zeta^k(g)\pi_n(h)\pi_n(k) \\
&= |G|^{-2}\sum_{h,k\in G}\zeta^h(g_0g^{-1})\zeta^{hk}(g)\pi_n(hk) \\
&= |G|^{-2}\sum_{h\in G}\zeta^h(g_0g^{-1})|G|P_n^g \\
&= |G|^{-1}\delta(g_0,g)|G|P_n^g = \delta(g_0,g)P_n^g.
\end{aligned}$$

$$\begin{aligned}
\sum_{g\in G}P_n^g &= |G|^{-1}\sum_{h,g\in G}\zeta^h(g)\pi_n(h) \\
&= |G|^{-1}\sum_{h\in G}\delta(h,1_G)|G|\pi_n(h) = \pi_n(1_G) = I.
\end{aligned}$$

The above computation shows that $\{P_n^g\}_{g\in G}$ are mutually orthogonal projections that sum up to I. Define a *-homomorphism $\phi_{n-1,n} : \oplus_{g\in G}A_{n-1}^g \to \oplus_{g\in G}A_n^g$ by:

$$(y_{n-1}^h)_{h\in G} \mapsto (y_n^h)_{h\in G}, \quad \text{where } y_n^h = \sum_{g\in G} y_{n-1}^g \otimes P_n^{hg^{-1}}.$$

The connecting map $\tilde{\phi}_{n-1,n}$ in $\varinjlim C^*(G,A_n,\alpha)$ is determined by $a \mapsto a\otimes 1$ and $U_{n-1}^g \mapsto U_n^g$. We shall verify that $\phi_{n-1,n}$ is the induced connecting map on $\varinjlim \oplus_{g\in G}A_n^g$. Adopting the notation in Proposition 3.2.3, recall that $T_n\colon C^*(G,A_n,\alpha) \mapsto \oplus_g A_n^g$ is the isomorphism such that $T_n(a) = (a, \dots, a)$ and $T_n(U^g_{n-1}) = X^g_{n-1} = (\zeta^h(g)V^g_{n-1})_{h\in G}$. For any $l \in G$, let $\phi^l_{n-1,n}$ be the composition of $\phi_{n-1,n}$ and the natural projection $\pi^l : \oplus_{g\in G} A^g_n \to A^l_n$. We have:

$$\begin{aligned}
\phi^l_{n-1,n}((a^g)_{g\in G}) &= \sum_{g\in G} a \otimes P_n^{lg^{-1}} \\
&= a \otimes \sum_{g\in G} P_n^{lg^{-1}} = a \otimes 1 \\
\phi^l_{n-1,n}(X^g_{n-1}) &= \sum_{h\in G} \zeta^h(g) V^g_{n-1} \otimes P_n^{lh^{-1}} \\
&= \sum_{h\in G} \zeta^h(g) V^g_{n-1} \otimes |G|^{-1} \sum_{k\in G} \zeta^k(lh^{-1})\pi_n(k) \\
&= |G|^{-1} \sum_{h,k\in G} \zeta^h(g)\zeta^k(lh^{-1}) V^g_{n-1} \otimes \pi_n(k) \\
&= |G|^{-1} \sum_{h,k\in G} \zeta^h(gk^{-1})\zeta^l(k) V^g_{n-1} \otimes \pi_n(k) \\
&= |G|^{-1} \sum_{k\in G} \delta(g,k)|G|\zeta^l(k) V^g_{n-1} \otimes \pi_n(k) \\
&= \zeta^l(g) V^g_{n-1} \otimes \pi_n(g) = \pi^l(X^g_n)
\end{aligned}$$

Hence $\phi_{n-1,n}(X^g_{1,n-1}) = X^g_{1,n}$. This shows that $\phi_{n-1,n} \circ T_{n-1} = T_n \circ \tilde{\phi}_{n-1,n}$, and therefore completes the proof. ∎

Remark 3.2.5 *We can get our unitaries back from the projections constructed in the above proof by the following formula:* $\pi_n(g) = \sum_{l\in G} \overline{\zeta^{g^{-1}}(l)} P^l_n$. *We could also observe that the projections* P^g_n *are invariant under the action, but some of them may be 0.*

Suppose the group G is finite cyclic, recall that $g \mapsto \zeta^g(\bullet)$ gives an isomorphism between G and its dual. The dual action on the crossed product turns out to have a very simple description under this isomorphism:

Proposition 3.2.6 *We identify the crossed product $C^*(G, A, \alpha)$ with the direct system $\varinjlim \oplus_{g \in G} A_n^g$, and identify $\hat{G}$ with G using the bihomomorphism ζ. Then the dual action $\hat{\alpha}$, when restricted to $\oplus_{g \in G} A_n^g$, is given by:*

$$\hat{\alpha}_g((y^h)_{h \in G}) = (z^h)_{h \in G}, \quad \textit{where } z^h = y^{hg}, \forall h \in G.$$

Proof The dual action $\hat{\alpha}$ is defined by $\hat{\alpha}_\phi(f)(t) = \phi(t)f(t)$, for all $\phi \in \hat{G}$ and $f \in C^*(G, A, \alpha)$. Fix a positive integer n, let $\{U_n^g\}$ be the canonical unitaries in $C^*(G, A_n, \alpha)$, then $\hat{\alpha}_h(U_n^g) = \zeta^h(g)U_n^g$. Fix an element $k \in G$, define a map $\beta_k \colon \oplus A_n^g \to \oplus A_n^g$ by $\beta((y^g)_{g \in G}) = (y^{gk})_{g \in G}$. Recall that the isomorphism T_n between $C^*(G, A_n, \alpha)$ and $\oplus A_n^g$ sends (U_n^h) to $(\zeta^g(h)V_n^h)_{g \in G}$. We have the following diagram is commutative:

$$\begin{tikzcd}
U_n^h \arrow[r, "T_n"] \arrow[d, "\hat{\alpha}_k"] & (\zeta^g(h)V_n^h)_{g \in G} \arrow[d, "\beta_k"] \\
\zeta^k(h)U_n^h \arrow[r, "T_n"] & (\zeta^{gk}(h)U_n^h)_{g \in G}
\end{tikzcd}$$

A similar commutative diagram holds for elements in A_n. Since these elements generate the corresponding C^*-algebras, we can conclude that β_k corresponds to $\hat{\alpha}_k$ under the isomorphism T. ∎

Now we are ready to state our classification result for the strict Rokhlin property:

Proposition 3.2.7 *Let $\alpha \colon G \mapsto \mathrm{Aut}(A)$ be a product-type action, where G is finite cyclic and A is UHF. Then α has the strict Rokhlin property, if and only if up to a telescope, for any $n \in \mathbb{Z}_{>0}$ the projections P_n^g for $g \in G$ constructed in Proposition 3.2.4 are mutually Murray-von Neumann equivalent.*

Proof For one direction, a telescope does not change the action, so let's assume that for any $n \in \mathbb{Z}_{>0}$, the projections P_n^g are mutually equivalent. Let F be a finite set in A. Without loss of generality we may assume that F is in A_{n-1}, for some $n \in \mathbb{Z}_{>0}$. Recall that $P_n^g = |G|^{-1} \sum_{h \in G} \zeta^h(g)\pi_n(h)$. Since these projections are equivalent, for each $g \in G$, there exists a partial isometry $W^{1,g}$ such that $W^{1,g}(W^{1,g})^* = P_n^1$ and $(W^{1,g})^* W^{1,g} = P_n^g$ (Here 1=1_G is the identity element). Let $W^{g,1}$ denote the conjugate of $W^{1,g}$ and let $W^{g,h} = W^{g,1}W^{1,h}$. Let $\delta(\cdot,\cdot)$ be the Kronecker delta, we have:

$$W^{g,g} = P_n^g, \quad W^{g,h}W^{k,l} = \delta(h,k)W^{g,l},\ \forall g,h,k,l \in G.$$

Let $Q^k = \frac{1}{|G|}\sum_{g,h \in G} \zeta^k(g^{-1}h)W^{g,h}$, for any k in G. Now we can compute that:

$$\begin{aligned}
Q^k Q^j &= \frac{1}{|G|^2}\Big(\sum_{g_1,h_1 \in G} \zeta^k(g_1^{-1}h_1)W^{g_1,h_1}\Big)\Big(\sum_{g_2,h_2 \in G} \zeta^j(g_2^{-1}h_2)W^{g_2,h_2}\Big) \\
&= \frac{1}{|G|^2} \sum_{g_1,h_1,h_2 \in G} \zeta^k(g_1^{-1}h_1)\zeta^j(h_1^{-1}h_2)W^{g_1,h_2} \\
&= \frac{1}{|G|^2} \sum_{g_1,h_1,h_2 \in G} \zeta^k(g_1^{-1})\zeta^{kj^{-1}}(h_1)\zeta^j(h_2)W^{g_1,h_2} \\
&= \delta(k,j)\frac{1}{|G|} \sum_{g_1,h_2 \in G} \zeta^k(g_1^{-1})\zeta^j(h_2)W^{g_1,h_2} \\
&= \delta(k,j)\frac{1}{|G|} \sum_{g_1,h_2 \in G} \zeta^k(g_1h_2^{-1})W^{g_1,h_2} = \delta(k,j)Q^k
\end{aligned}$$

$$\begin{aligned}
(Q^k)^* &= \frac{1}{|G|}\sum_{g,h \in G} \overline{\zeta^k(g^{-1}h)}(W^{g,h})^* \\
&= \frac{1}{|G|}\sum_{g,h \in G} \zeta^k(h^{-1}g)W^{h,g} = Q^k
\end{aligned}$$

$$
\begin{aligned}
\sum_{k\in G} Q^k &= \sum_{k\in G}\frac{1}{|G|}\sum_{g,h\in G}\zeta^k(g^{-1}h)W^{g,h}\\
&= \frac{1}{|G|}\sum_{g,h\in G}\delta(g,h)W^{g,h}\\
&= \sum_{g\in G} W^{g,g} = 1
\end{aligned}
$$

$$
\begin{aligned}
\alpha_g(Q^k) &= \pi_n(g)Q^k\pi_n(g)^*\\
&= (\sum_{l\in G}\zeta^{g^{-1}}(l)P_n^l)(\frac{1}{|G|}\sum_{r,h\in G}\zeta^k(r^{-1}h)W^{r,h})(\sum_{s\in G}\zeta^g(s)P_n^s)\\
&= \frac{1}{|G|}\sum_{l,r,h,s\in G}\zeta^{g^{-1}}(l)\zeta^k(r^{-1}h)\zeta^g(s)P_n^l W^{r,h}P_n^s\\
&= \frac{1}{|G|}\sum_{r,h}\zeta^{g^{-1}}(r)\zeta^k(r^{-1}h)\zeta^g(h)W^{r,h}\\
&= \frac{1}{|G|}\sum_{r,h}\zeta^{gk}(r^{-1}h)W^{r,h} = Q^{gk}
\end{aligned}
$$

Let's consider the projections $\{1 \otimes Q^k\}_{k\in G}$, where 1 is the identity of A_{n-1}. From the above computations, it's easy to check that they are Rokhlin projections for F, therefore α has the strict Rokhlin property.

Conversely, suppose the action has the strict Rokhlin property. Then the dual action $\hat{\alpha}$ induce a trivial action on $K_0(C^*(G,B,\alpha))$ (See Proposition 2.4 of [19] for details). Now fix $n \in \mathbb{N}$. For each $h \in G$, let $I_h = (0,\dots,1,0,\dots)$ be a projection of $\oplus_{g\in G}A_n^g$, where 1 is in the h-th position. Let $\phi_{n,\infty}$ be the connecting map from $\oplus_{g\in G}A_n^g$ to $\varinjlim \oplus_{g\in G}A_i^g$ and let $\eta_h = \phi_{n,\infty}(I_h)$. Since $\hat{\alpha}$ acts trivially on $K_0(C^*(G,A,\alpha))$, we have:

$\hat{\alpha}_h([\eta_k]) = [\eta_k]$, for any $h, k \in G$. It's easy to see that $\hat{\alpha}$ commutes with connecting maps, which implies that $[\phi_{n,\infty}(\hat{\alpha}_h(I_k))] = [\phi_{n,\infty}(I_k)]$. For the K_0 groups of the inductive limit, we have $\mathrm{Ker}(\phi_{n,\infty}) = \cup_{i=n+1}^{\infty}\mathrm{Ker}\phi_{n,i}$. So there exist some $m > n$, such that $[\phi_{n,m}(\hat{\alpha}_h(I_k))] = [\phi_{n,m}(I_k)]$. Now by Proposition 3.2.6, we have $\hat{\alpha}_h(I_k) = I_{hk}$.

Recall that $P^g_{n,m} = |G|^{-1}\sum_{h\in G}\zeta^h(g)\otimes_{i=n}^m \pi_i(h)$. By Proposition 3.2.4, $P^g_{n,m}$ are mutually orthogonal projections that sum up to 1, such that the connecting map $\phi_{n,m}$ is given by:

$$(y^h_{n-1})_{h\in G} \mapsto (y^h_n)_{h\in G}, \quad \text{where } y^h_n = \sum_{g\in G} y^g_{n-1}\otimes P^{hg^{-1}}_{n,m}.$$

Hence $\phi_{n,m}(I_k) = (1\otimes P^{gk^{-1}}_{n+1,m})_{g\in G}$. Now $[\phi_{n,m}(\hat{\alpha}_h(I_k))] = [\phi_{n,m}(I_k)]$ implies:

$$[(1\otimes P^{g(hk)^{-1}}_{n+1,m})_{g\in G}] = [(1\otimes P^{gk^{-1}}_{n+1,m})_{g\in G}], \forall h,k\in G$$

Since $K_0(\oplus A^g_n) = \oplus K_0(A^g_n)$ and $A^g_n = A_n$ is a matrix algebra, equality in the K_0 group implies that $1\otimes P^{g(hk)^{-1}}_{n+1,m}$ is Murray-von Neumann equivalent to $1\otimes P^{gk^{-1}}_{n+1,m}$, for all g,h,k in G. Therefore, for any $n \in \mathbb{Z}_{>0}$, we can find a $m = m(n)$, such that the projections $P^g_{n+1,m}$ are mutually Murray-von Neumann equivalent. Defining a telescope inductively by setting $n_1 = 1$ and $n_{i+1} = m(n_i)$ completes the proof. ■

Note that two projections are equivalent in a UHF algebra if and only if they have the same trace. This enables us to reformulate Proposition 3.2.7 in terms of the characters of the unitary representations.

Lemma 3.2.8 *Let $P^g_n = |G|^{-1}\sum_{h\in G}\zeta^h(g)\pi_n(h)$ be the projections defined as in Proposition 3.2.4. Let d_n be the dimension of H_n and let* Tr *be the unnormalized trace on* $\mathrm{B}(H_n)$.

Let $\chi_n = \mathrm{Tr} \circ \pi_n$ be the character of π_n. Then the projections P_n^g are mutually Murray-von Neumann equivalent if and only if $\chi_n(g) = \delta(g, 1_G)d_n$ for any $g \in G$, where δ is the Kronecker delta.

Proof If $\chi_n(g) = \delta(g, 1_G)|G|$ for any $g \in G$, then

$$\mathrm{Tr}(P_n^g) = |G|^{-1} \sum_{h \in G} \zeta^h(g)\chi_n(h) = |G|^{-1}\zeta^{1_G}(g)d_n = \frac{d_n}{|G|}, \quad \forall g \in G.$$

Hence these projections are mutually Murray-von Neumann equivalent. In the opposite direction, as we noted in Remark 3.2.5, we can see that $\pi_n(h) = |G| \sum_{g \in G} \zeta^{h^{-1}}(g)P_n^g$. If the projections are Murray-von Neumann equivalent, then we should have $\mathrm{Tr}(P_n^g) = \frac{d_n}{|G|}$, for all $g \in G$. Hence:

$$\chi_n(h) = |G| \sum_{g \in G} \zeta^{h^{-1}}(g) \frac{d_n}{|G|} = \delta(h^{-1}, 1_G)d_n = \delta(h, 1_G)d_n.$$

■

The above lemma, together with Proposition 3.2.7, proves the strict Rokhlin property part of Theorem 3.1.3, in the special case where G is a finite cyclic group.

3.3 Product-type action with the tracial Rokhlin property

In this section, we use the same set-up as in the previous section, and we still assume that the group G is finite cyclic and $\alpha \colon G \mapsto \mathrm{Aut}(A)$ is a product-type action on a UHF algebra A. If α has the tracial Rokhlin property, then the dual action will act trivially on the trace space of the crossed product. (See Proposition 2.5 of [19].) This suggests us to study the trace space of the crossed product.

The tracial states in $T(C^*(G, A, \alpha))$ are in one-to-one correspondence with sequences of compatible traces $\{(\tau_n)_{n=1}^{\infty} | \, \tau_n \in T(\oplus A_n^g), \tau_n = \tau_{n+1} \circ \phi_{n,n+1}\}$. Let τ_n^g denote the unique tracial state on A_n^g. When the C*-algebra is specified, we simply use τ to denote τ_n^g, as it will cause no confusion. It's easy to see that $T(\oplus A_n^g)$ can be parametrized by a standard simplex

$$\Delta^{|G|} = \{(s^g)_{g\in G} \mid \sum_{g\in G} s^g = 1, \;\; s_g \geq 0, \forall g \in G\},$$

where $(s^g)_{g\in G}$ is mapped to $\sum_{g\in G} s^g \tau_n^g$.

Let τ_{s_n} be the tracical state determined by $s_n = (s_n^g)_{g\in G}$. Computation shows that the compatibility condition is equivalent to that

$$s_n^g = \sum_{h\in G} s_{n+1}^h \tau(P_{n+1}^{hg^{-1}}),$$

where P_n^g are the projections defined as in Proposition 3.2.4.

For any n, m, define $T_{n,m}$ to be the $|G| \times |G|$ matrix whose (g,h)-entry is $\tau(P_{n,m}^{hg^{-1}})$. Write T_n to be $T_{n,n}$. The compatibility condition can be rewritten as $s_n = T_{n+1}s_{n+1}$, viewing s_n as a column vector. This lead us to study the infinite product $\prod_n T_n$.

For projections $P_{n,m}^g$, we have the following identity:

$$P_{n,n+1}^g = \sum_{l\in G} P_n^{gl^{-1}} \otimes P_{n+1}^l.$$

Let $P_{n,m}$ be the $|G| \times |G|$ matrix whose (g,h)-entry is $P_{n,m}^{hg^{-1}}$, then the above equations can be rewritten in matrix form: $P_{n,n+1} = P_n P_{n+1}$, where the multiplication of the entries is given by tensor product. Since $\tau(a \otimes b) = \tau(a)\tau(b)$, we see that $T_{n,n+1} = T_n T_{n+1}$. So telescoping corresponds to 'Adding bracket in the infinite product'. A priori, $\prod_n T_n$ may not be convergent. But we have the following:

Lemma 3.3.1 *Let T_n be the matrix $(\tau(P_n^{gh^{-1}}))_{g,h\in G}$, where P_n^g is defined as in Proposition 3.2.4. Then there exists a telescoping, such that for any $m \in \mathbb{Z}_{>0}$, the infinite product $\prod_{n=m}^{\infty} T_n$ converges. The conclusion holds no matter whether α has the tracial Rokhlin property or not.*

Proof We first observe that the matrices T_n are circular matrices, so they can be simultaneously diagonalized: Let X be the unitary matrix $\frac{1}{\sqrt{|G|}}(\zeta^h(g))_{h,g\in G}$. Then

$$XT_nX^* = \mathrm{diag}(\lambda_n^{g_1}, \lambda_n^{g_2}, \dots), \quad \text{where } \lambda_n^g = \sum_{h\in G} \zeta^g(h)\tau(P_n^h).$$

Hence convergence of the infinite product of matrices is the same thing as convergence of the infinite product of the corresponding eigenvalues $\prod_n \lambda_n^g$. Fix some g in G. We can estimate that:

$$|\lambda_n^g| \leq \sum_{h\in G} |\zeta^g(h)|\tau(P_n^h) = \tau(\sum_{h\in G} P_n^h) = \tau(I_n) = 1.$$

Hence for any $m \in \mathbb{Z}_{>0}$, the partial products $S^l = \prod_{n=m}^{l} \lambda_n^g$ are bounded by 1 for any $l > m$. We can therefore select a sub-sequence so that it converges. But choosing a sub-sequence of the partial product corresponds exactly to a telescoping. We conclude that for any $g \in G$ and any $m > 0$, the exist a telescoping, such that the infinite product $\prod_{n\geq m} \lambda_n^g$ converges. Since the composition of telescoping is again a telescoping, we can first use a Cantor's diagonal argument, then use induction on m, to find a single telescoping, such that for any $g \in G$ and any $m > 0$, the infinite product $\prod_{n=m}^{\infty} \lambda_n^g$ converges. ■

Now we are ready to state the necessary and sufficient conditions for product-type actions to have the tracial Rokhlin property:

Proposition 3.3.2 *Let $\alpha\colon G \to \otimes_{n=1}^{\infty} A_n$ be a product-type action on a UHF-algebra A, where G is finite cyclic. Adopt the notation of Lemma 3.3.1. The action α has the tracial Rokhlin property if and only if there exists a telescoping, such that for any $m \in \mathbb{Z}_{>0}$, the limit matrix $\prod_{n\geq m} T_n$ exists and has rank 1.*

Proof In one direction, suppose $\prod_{n\geq m} T_n$ has rank 1, for any $m \in \mathbb{Z}_{>0}$. Let $F \subset \otimes_n A_n$ be a finite subset. Without loss of generality we assume that $F \subset \otimes_{n=1}^{k} A_n$ for some $k \in \mathbb{Z}_{>0}$. Let $\varepsilon > 0$ be given. Let $m = k+1$.

For a circular matrix, it has rank 1 if and only if all the entries are equal. Let E be the matrix such that all the entries are equal to $\frac{1}{|G|}$. Then there exists some $l > m$, such that $\| \prod_{n=m}^{l} T_n - E \|_{\max} < \varepsilon/|G|$. The discussion before Lemma 3.3.1 shows that $\prod_{n=m}^{l} T_n = T_{m,l}$. Without loss of generality, we assume that $\tau(P^1_{m,l})$ is smallest among all the entries of $T_{m,l}$, where $1 = 1_G$ is the identity of G.

Now lets do a similar construction as in Proposition (3.2.7). Since $P^1_{m,l}$ has smallest trace among the others, we can, for each $g \in G$, find a partial isometry $W^{1,g}$ such that $W^{1,g}W^{g,1} = P^1_{m,l}$ and $W^{g,1}W^{1,g}$ is a sub-projection of $P^g_{m,l}$. Here we adopt the same notation as in Proposition 3.2.7: $W^{g,1}$ is the conjugate of $W^{1,g}$ and $W^{g,h} = W^{g,1}W^{1,h}$. Let $Q^k = \sum_{g,h\in G} \zeta^k(g^{-1}h) W^{g,h}$. By the same computation, we have:

1. $\{Q^k\}_{k\in G}$ are mutually orthogonal projections,

2. $\alpha_g(Q^k) = Q^{gk}$, and

3. Let $Q = \sum_{k\in G} Q^k$, $Q = \sum_{g\in G} W^{g,g}$.

Now

$$\tau(P^1_{m,l}) \geq \frac{1}{|G|} - \|T_{m,l} - E\|_{\max} > \frac{1-\varepsilon}{|G|}.$$

Since $W^{g,g}$ is a subprojection of $P^g_{m,l}$ and is equivalent to $P^1_{m,l}$, we have $\tau(Q) = \sum_{g\in G} \tau(W^{g,g}) > 1 - \varepsilon$, or equivalently $\tau(1-Q) < \varepsilon$. Hence $\{1 \otimes Q^k\}$ are tracial Rokhlin projections which commute with F.

For the other direction, assume that the action has the tracial Rokhlin property. Then by Proposition 2.5 of [19], the dual action induce a trivial action on the trace space $T(C^*(G,B,\alpha))$. Since $\prod_{n=m}^{\infty} T_n$ converges for all $m > 0$, we let $T_{m,\infty}$ denote the limits. Fix a vector $s \in \Delta^{|G|}$, let $s_m = T_{m,\infty}s$, for each $m > 0$. Since $T_{m,\infty} = T_m T_{m+1,\infty}$, we see that $\{s_m\}$ form a sequence of compatible traces and hence defines a trace on the crossed product $C^*(G,B,\alpha)$. Let $e = (1,0,\ldots,0) \in \oplus_{g\in G} A^g_n$, and suppose $s_m = (s^g_m)_{g\in G}$. Now the dual action acting trivially on the trace space means $s_m(\hat{\alpha}_k(e)) = s_m(e)$, for each $k \in G$. This implies $s^k_m = s^1_m$, for all k. Hence the image of $\Delta^{|G|}$ under the transformation $T_{m,\infty}$ is a single point. But the simplex $\Delta^{|G|}$ generates the whole domain of $T_{m,\infty}$, we see that $T_{m,\infty}$ has rank 1, for any $m > 0$. ■

Just as we did for strict Rokhlin property, we can reformulate Proposition 3.3.2 in terms of the characters of the unitary representations, using the following lemma:

Lemma 3.3.3 *Let $\alpha\colon G \mapsto \mathrm{Aut}(A)$ be a product-type action, where G is finite cyclic and A is UHF. Let H_n and π_n be defined as in Definition 3.1.1. Let d_n be the dimension of H_n. Let $\chi_n = \mathrm{Tr} \circ \pi_n$ be the character of π_n, and let T_n be the circular matrices defined as in the* *proof of Lemma 3.3.1. Let $\chi\colon G \mapsto \mathbb{C}$ be the characteristic function on $\{1_G\}$. Then for any $l \in \mathbb{Z}_{>0}$, $T_{l,\infty}$ has rank 1 if and only if $\prod_{l \geq m} \frac{\chi_n}{d_n} = \chi$.*

Proof Without loss of generality we may assume $l = 1$. Recall that

$$P^g_{n,m} = |G|^{-1} \sum_{h \in G} \zeta^h(g) \otimes_{i=n}^m \pi_i(h).$$

Since $\mathrm{Tr} = d_i \tau$, we have:

$$\tau(P^g_{1,m}) = |G|^{-1} \sum_{h \in G} \zeta^h(g) \prod_{i=1}^m \frac{\chi_i(h)}{d_i} \tag{3.3}$$

Since $T_{1,n}$ are circular matrices, $T_{1,\infty}$ has rank 1 if and only if $\lim_{n\to\infty} T_{1,n}$ tends to E, where E is the matrix who entries are all equal to $\frac{1}{|G|}$. This is further equivalent to $\lim_{n\to\infty} \tau(P^g_{1,n}) = \frac{1}{|G|}$, for all $g \in G$. In Equation (3.3), if we take the limit, the same computation as in the strict Rokhlin property case shows that $\lim_{n\to\infty} \tau(P^g_{1,n}) = \frac{1}{|G|}$ is equivalent to $\lim_{n\to\infty} \prod_{i=1}^n \frac{\chi_i}{d_i} = \chi$. ■

3.4 Characterization of Rokhlin actions for general finite group

So far we have proved Theorem 3.1.3 for finite cyclic groups. For general finite groups, we can actually reduce the problem to the finite cyclic case, based on the following two observations:

Lemma 3.4.1 *Let $\alpha\colon G \mapsto \mathrm{Aut}(A)$ be an action, where G is finite and A is a unital simple C^*-algebra. Let H be a subgroup of G. If α has the strict Rokhlin property, then the induced action $\alpha|_H$ also has the strict Rokhlin property. If α has the tracial Rokhlin property, then $\alpha|_H$ also has the tracial Rokhlin property.*

Proof See Lemma 5.6 of [20]. The proof of the strict Rokhlin property case is essentially the same. ■

Lemma 3.4.2 *Let $\pi\colon G \mapsto B(H)$ be a finite dimensional representation. Let H be a subgroup of G, then π will induce a representation $\pi|_H$ on H. If χ^π is the character of π and χ_H is the character of $\pi|_H$, we have $\chi_H = \chi^\pi|_H$.*

Definition 3.4.3 *(Model action) Let $r = (r_i)_{i=1}^\infty$ and $s = (s_i)_{i=1}^\infty$ be two infinite sequences of non-negative integers. Let π_i be some arbitrary representation of G on $\mathbb{C}^{s_i}$, and we write $\pi = (\pi_i)_{i=1}^\infty$. Set*

$$H_i = \underbrace{l^2(G) \oplus l^2(G) \cdots \oplus l^2(G)}_{r_i} \oplus \mathbb{C}^{s_i}.$$

Let $\tilde{\pi}_i\colon G \mapsto B(H_i)$ be the direct sum of left regular representations on each copy of $l^2(G)$ and π_i on $\mathbb{C}^{s_i}$. As in Definition 3.1.1, we get a product-type action $\alpha(r,s,\pi)$ induced by the representations $\tilde{\pi}_i$. We call $\alpha(r,s,\pi)$ the model action *for the triple (r,s,π). If $s = 0$, we write $\alpha(r) = \alpha(r,0,0)$.*

Now let's turn to the proof of Theorem 3.1.3 for general finite groups.

Proof of Theorem 3.1.3: Let $\alpha\colon G \mapsto \mathrm{Aut}(A)$ be a product-type action with the strict Rokhlin property. Write G as a finite union of cyclic subgroups $G = K_0 \cup K_1 \cup \cdots \cup K_s$. Since $\alpha|_{K_0}$ has the strict Rokhlin property, by the cyclic group version of Theorem 3.1.3, there exists a telescope such that $\frac{\chi_n|_{K_0}}{d_n} = \chi|_{K_0}$ for any $n \in \mathbb{Z}_{>0}$. We can find a successive telescope such that $\frac{\chi_n|_{K_1}}{d_n} = \chi|_{K_1}$ for any $n \in \mathbb{Z}_{>0}$. It's easy to see that we still have $\frac{\chi_n|_{K_0}}{d_n} = \chi|_{K_0}$ after the telescope. Since the composition of telescopes is again a telescope, repeating the above procedure will give us a telescope, such that $\frac{\chi_n|_{K_0}}{d_n} = \chi|_{K_i}$ for any $1 \le i \le s$. Since $G = K_0 \cup K_1 \cup \cdots \cup K_s$, we see that $\frac{\chi_n}{d_n} = \chi$, for any $n \in \mathbb{Z}_{>0}$.

Conversely, if there exists a telescope, such that $\frac{\chi_n}{d_n} = \chi$ for any $n \in \mathbb{Z}_{>0}$. Let $\iota_1, \ldots, \iota_k$ be the irreducible characters of G with dimensions $r_1, \dots, r_k$ respectively. Let $\chi_n = \sum_{i=1}^{k} a_i \iota_i$ be the irreducible decomposition, then $a_i = <\chi_n, \iota_i> = \frac{d_n r_i}{|G|}$. Since each a_i is an integer and at lease one $r_i = 1$, we see that $\frac{d_n}{|G|}$ is an integer. Hence π_n is equal to the character of the direct sum of $\frac{d_n}{|G|}$ copies of left regular representations. Therefore π_n is equivalent to a direct sum of left regular representations. Recall that two actions $\alpha\colon G \mapsto \mathrm{Aut}(A)$ and $\beta\colon G \mapsto \mathrm{Aut}(B)$ are said to be *conjugate* if and only if there exists an isomorphism $T\colon A \mapsto B$ such that $T \circ \alpha_g = \beta_g \circ T$, for any $g \in G$. Let $r = (\frac{d_n}{|G|})_{n>0}$, we see that α is conjugate to the model action $\alpha(r)$ (Definition 3.4.3). Hence α has the strict Rokhlin property.

The proof of the tracial Rokhlin property case is quite similar, we shall only prove one direction: if there exists a telescoping, such that for any $n \in \mathbb{Z}_{>0}$, $\prod_{i=n}^{\infty} \frac{\chi_n}{d_n} = \chi$, then the action has the tracial Rokhlin property. The following lemma, as a special case of Lemma 5.2 of [20], simplifies our argument for product-type actions.

Proposition 3.4.4 *Let $\alpha\colon G \mapsto \mathrm{Aut}(A)$ be a finite group action on a UHF-algebra A, let τ be the unique trace on A. Then α has the tracial Rokhlin property if and only if for any finite set $F \subset A$, any $\varepsilon > 0$, there exists mutually orthogonal projections e_g in A for $g \in G$, such that:*

(1) $\|e_g a - a e_g\| < \varepsilon$, $\forall g \in G$ and $a \in F$,

(2) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$, and

(3) $\tau(1 - e) < \varepsilon$.

Now let F be a finite subset of A, without loss of generality assume that $F \subset A_{n-1}$, for some $n > 0$. Let $\varepsilon > 0$ be given. Set $\varepsilon_0 = \frac{\varepsilon}{2|G|}$. Since $\prod_{i=n}^{\infty} \frac{\chi_i}{d_i} = \chi$, we can find some

$m > n$ such that $\| \prod_{i=n}^m \frac{\chi_i}{d_i} - \chi\|_{\max} < \varepsilon_0$. Let $\chi_{n,m} = \prod_{i=n}^m \chi_i$, we can see that $\chi_{n,m}$ is the character of the representation $\pi_{n,m} = \otimes_{i=n}^m \pi_i$, with dimension $d_{n,m} = \prod_{i=n}^m d_i$. Increasing m if necessary, we may further require that $d_{n,m} > 1/\varepsilon_0$.

In the following, we are going to show that $\pi_{n,m}$ is 'close' to a direct sum of left regular representations. Let $\iota_1, \ldots, \iota_k$ be the irreducible characters of G with dimension $r_1, \ldots, r_k$ respectively. Then the max norm of each ι_i will be less or equal to $|G|$. From now on, for characters, $\| \cdot \|$ will always denote the max norm. Let $\chi_{n,m} = \sum_{1\le i\le k} a_i \iota_i$ be the irreducible decomposition of $\chi_{n,m}$. Since $\|\frac{\chi_{n,m}}{d_{n,m}} - \chi\| < \varepsilon$, we can see that for any i,

$$\left|\frac{a_i}{d_{n,m}} - \frac{r_i}{|G|}\right| =< \frac{\chi_{n,m}}{d_{n,m}} - \chi, \iota_i > \tag{3.4}$$

$$< |G| \left\|\frac{\chi_{n,m}}{d_{n,m}} - \chi\right\| \|\iota_i\| < |G|^2 \varepsilon_0 \tag{3.5}$$

Let $d = \min_{1\le i\le k}\{\left[\frac{a_i}{r_i}\right]\}$. We can then decompose $\chi_{n,m}$ as the sum of two characters χ' and χ'', where $\chi' = \sum_i (dr_i)\iota_i$, and $\chi'' = \chi_{n,m} - \chi'$. Let π' be the direct sum d copies of left regular representations which corresponds to χ', and let π'' be a representation corresponds to χ''. Let d' and d'' be the dimensions π' and π'' respectively. Our claim is that $\frac{d''}{d'+d''} = \frac{d''}{d_{n,m}} < \varepsilon$. Note that $d' = \sum_{1\le i\le k} dr_i^2 = d|G|$. By the definition of d, there exists some i such that $\|d - a_i/r_i\| < 1$. Using equation (3.5), we can estimate:

$$\begin{aligned}
\left|\frac{d''}{d_{n,m}}\right| &= \left\|1 - \frac{d|G|}{d_{n,m}}\right\| \\
&\le \left\|1 - \frac{\frac{a_i}{r_i}|G|}{d_{n,m}}\right\| + \left\|\frac{(d - \frac{a_i}{r_i})|G|}{d_{n,m}}\right\| \\
&< \frac{r_i}{|G|}|G|^2\varepsilon_0 + |G|\varepsilon_0) \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Let's now consider the representation $\tilde{\pi} = \pi' \oplus \pi''$, let $\tilde{\alpha}$ be the inner action defined by $g \mapsto \mathrm{Ad}(\tilde{\pi}(g))$. Since $\tilde{\pi}$ contains copies of regular representation whose total dimension is d', we could find mutually orthogonal projections e_g for $g \in G$ such that $\tilde{\alpha}_h(e_g) = e_{hg}$, and $\mathrm{Tr}(e) = d'$, where $e = \sum_{g \in G} e_g$. But $\pi_{n,m}$ is equivalent to $\tilde{\pi}$ because they have the same character, therefore the induced actions $\alpha_{n,m}$ and $\tilde{\alpha}$ are conjugate. Hence we can find projections for $\alpha_{n,m}$ that satisfy the same properties. Note that $\mathrm{Tr}(e) = d'$ implies $\tau(1-e) = 1 - d'/d_{n,m} < \varepsilon$. Hence they are tracial Rokhlin projections. By Lemma 3.4.4, α has the tracial Rokhlin property.

From the above proof, we can also get the following characterization of the Rokhlin properties:

Corollary 3.4.5 *Let $\alpha\colon G \mapsto \mathrm{Aut}(A)$ be a product-type action where A is UHF. Then:*

(i) α has the strict Rokhlin property if and only if there exists some $r = (r_i)_{1 \le i < \infty}$, such that α is conjugate to the model action $\alpha(r)$ (See Definition 3.4.3).

(ii) α has the tracial Rokhlin property if and only if there exists some $r = (r_i)$, $s = (s_i)$ and $\pi = (\pi_i)$ with $\lim\limits_{i \to \infty} \frac{s_i}{r_i} = 0$, such that α is conjugate to the model action $\alpha(r, s, \pi)$.

4. Rokhlin actions are generic

4.1 Notations and terminology

In this chapter, we assume *all C*-algebras are SEPARABLE.*

The existence of Rokhlin actions depends mostly on the underlying C*-algebra. For example, if the underlying C*-algebra has no projection, then no Rokhlin action can exist for any finite group. But in other suitable circumstances, we can show that for any given finite group G, 'most' actions are Rokhlin actions. To make it precise, we let $\mathrm{Aut}_G(A)$ be the set of all G-actions on a separable C*-algebra A, and define a metrizable topology as follows: first, give $Aut(A)$ the point-norm topology, which is induced by a complete metric ρ (Lemma 3.2, [21]); then we define a metric ρ_G on $\mathrm{Aut}_G(A)$ by the formula:

$$\rho_G(\alpha, \beta) = \sup_{g \in G} \rho(\alpha_g, \beta_g). \tag{4.1}$$

The main result in this chapter is the following theorem:

Theorem 4.1.1 *Let A be a simple C*-algebra of tracial rank zero which absorbs the Jiang-Su algebra $\mathcal{Z}$ tensorially, or a simple C*-algebra which absorbs a UHF-algebra of infinite-type tensorially. Let G be any finite group. Then the set of actions with the tracial Rokhlin property forms a dense G_δ subset of* $\mathrm{Aut}_G(A)$.

Now let's explain the terms in Theorem 4.1.1 first. The Jiang-Su algebra $\mathcal{Z}$ is a stably finite simple projectionless C*-algebra which has the same K-theory as $\mathbb{C}$. (For a precise definition and it's properties, see [22].) A UHF-algebra is said to be infinite-type if the corresponding supernatural number has a prime decomposition of the form $p_1^\infty p_2^\infty p_3^\infty \ldots$. We say a C*-algebra A absorbs another C*-algebra B tensorially if and only if $A \otimes B \cong A$. The class of C*-algebras that absorb the Jiang-Su algebra or certain UHF algebra are important for classification reasons. See for example [23].

The definition of tracial rank is somewhat complicated. We define our building blocks first:

Definition 4.1.2 *We denote $\mathscr{I}^{(k)}$ to be the class of all C*-algebras which are finite direct sums of the form:*

$$P_1 M_{n_1}(C(X_1))P_1 \oplus P_1 M_{n_2}(C(X_2))P_2 \oplus \cdots \oplus P_s M_{n_s}(C(X_s))P_s,$$

where $s < \infty$ and for each i, X_i is a finite CW complex of dimension $\leq k$, P_i is a projection in $M_{n_i}(C(X_i))$.

Let B be a subset of A and a be an element in A. Let $\varepsilon > 0$ be a positive number. We use $a \in_\varepsilon B$ to mean that there exists some $b \in B$ such that $\|a - b\| < \varepsilon$. The tracial rank could be defined for any C*-algebra, but since we are interested only in simple C*-algebras here, we take the following description as our definition:

Definition 4.1.3 *(Theorem 6.13, [24]) Let A be a simple unital C*-algebra. We say that the tracial rank of A is less than or equal to k, if and only if for any finite subset $F \in A$ and any nonzero element $b \in A_+$, there exist a C*-subalgebra $B \subset A$ with $B \in \mathscr{I}^{(k)}$ and $1_B = p$ such that:*

(1) $\|[x,p]\| < \varepsilon$ *for all* $x \in F$,

(2) $pxp \in_\varepsilon B$ *for all* $x \in F$, *and*

(3) $1 - p \precsim_s b$.

The tracial rank of A, denoted by $\mathrm{TR}(A)$, is then the smallest integer k such that the tracial rank of A is less than or equal to k. When $\mathrm{TR}(A) = 0$, we also say that A is a 'tracially-AF' algebra.

The Jiang-Su algebra $\mathcal{Z}$ and UHF-algebra of infinite-type, are special examples of strongly self-absorbing C*-algebras:

Definition 4.1.4 *For* $i = 1, 2$, *let* $\phi_i\colon A \to B$ *be two homomorphisms between separable C*-algebras. We say that* ϕ_1 *and* ϕ_2 *are almost unitarily equivalent (a.u.),* $\phi_1 \approx_{a.u.} \phi_2$, *if there is a sequence* $(u_n)_{\mathrm{N}}$ *of unitaries in* B, *such that* $\|u_n(\phi_1(a))u_n^* - \phi_2(a)\| \to 0$ *as* $n \to \infty$, *for any* $a \in A$.

Definition 4.1.5 *Let* $\mathcal{D}$ *be a C*-algebra and* $\iota\colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$, $\iota(a) = 1 \otimes a$ *be the inclusion into the second tensor factor. We say* $\mathcal{D}$ *is strongly self-absorbing, if there exist an isomorphism* $\phi\colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$, *such that* $\phi \approx_{a.u.} \iota$

Strongly self-absorbing C*-algebras plays an important role in the current stage of Elliott's classification program, see [25] for an exploration.

4.2 Rokhlin actions are dense

The main ingredient to prove Theorem 4.1.1 is the following proposition:

Proposition 4.2.1 *Let $\mathcal{D}$ be a strongly self-absorbing C*-algebra and A be a separable $\mathcal{D}$-absorbing C*-algebra, i.e. $A \cong A \otimes D$. Let G be a finite group, and let $\gamma\colon G \to \mathrm{Aut}(\mathcal{D})$ be any given action. Then the set of actions*

$$\{\theta^{-1} \circ (\alpha \otimes \gamma) \circ \theta \,|\, \alpha \in Aut_G(A), \theta \text{ an isomorphism between } A \text{ and } A \otimes \mathcal{D}\} \tag{4.2}$$

is dense in $\mathrm{Aut}_G(A)$.

This is an equivariant version of Proposition 3.10 of [21] if we replace the Jiang-Su algebra $\mathcal{Z}$ there by a general strongly self-absorbing C*-algebra $\mathcal{D}$. We give a proof here for completeness.

Proof In order to prove Proposition 4.2.1, it suffices to show that for any finite set F, any positive number $\varepsilon > 0$, any action $\alpha \in \mathrm{Aut}_G(A)$, there is an action β in the set (4.2), such that $\|\beta_g(a) - \alpha_g(a)\| < \varepsilon$, for any $g \in G$ and any $a \in F$.

For any C*-algebra B, let $\iota_B\colon B \to B \otimes D$ be the inclusion defined by $\iota_B(a) = a \otimes 1$. Since $\mathcal{D}$ is strongly self-absorbing, by definition there is an isomorphism $\psi\colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ which is a.u. to $\iota_{\mathcal{D}}$. This will induce an isomorphism $\mathrm{Id} \otimes \psi\colon A \otimes \mathcal{D} \to A \otimes \mathcal{D} \otimes \mathcal{D}$, which is a.u. to $\iota_{A\otimes\mathcal{D}}$. Using the fact that $A \cong A \otimes \mathcal{D}$, we can get an isomorphism $\phi\colon A \to A \otimes \mathcal{D}$ which is a.u. to ι_A.

Let $S = \{\alpha_g(a) \,|\, g \in G, a \in T\}$. Choose a unitary u such that $\|u\phi(a)n^* - a \otimes 1\| < \varepsilon$, for any a in S. Set $\theta = Ad(u) \circ \phi$. For each $g \in G$, set

$$\beta_g = \theta^{-1} \circ (\alpha_g \otimes \gamma_g) \circ \theta.$$

It's easy to see that $g \to \beta_g$ defines an action in $\mathrm{Aut}_G(A)$. For any $g \in G$ and any $a \in T$, we have:

$$
\begin{aligned}
\|\beta_g(a) - \alpha_g(a)\| =& \|\phi^{-1}(u^*((\alpha_g \otimes \gamma_g)(u\phi(a)u^*))u) - \alpha_g(a)\| \\
\leq& \|\phi^{-1}(u^*((\alpha_g \otimes \gamma_g)(a \otimes 1))u) - \alpha_g(a)\| + \varepsilon/2 \\
=& \|\phi^{-1}(u^*(\alpha_g(a) \otimes 1)u) - \alpha_g(a)\| + \varepsilon/2 \\
=& \|\alpha_g(a) \otimes 1 - u\phi(\alpha_g(a))u^*\| + \varepsilon/2 \\
\leq& \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}
$$

This completes the proof. ■

It's easy to see that if $\phi\colon A \to B$ is an isomorphism between two C*-algebras, α is an action in $\mathrm{Aut}_G(A)$ with either the strict Rokhlin property or the tracial Rokhlin property, then $\phi \circ \alpha \circ \phi^{-1}$ is an action in $\mathrm{Aut}_G(B)$ with the same Rokhlin property. Let A be a $\mathcal{D}-$absorbing C*-algebra for a strongly self-absorbing C*-algebra $\mathcal{D}$. To show that the Rokhlin actions are generic, it suffices to find enough Rokhlin actions of the form $\alpha \otimes \gamma$, where α are actions in $\mathrm{Aut}_G(A)$ and γ is a given action in $\mathrm{Aut}_G(\mathcal{D})$. It turns out that we just need to find one action on $\mathcal{D}$ with one of the Rokhlin properties, since in most cases, an action with one of the Rokhlin properties tensored with another arbitrary action will become an action with the same Rokhlin property. For $\mathcal{D}$ to be a UHF-algebra, we have:

Proposition 4.2.2 *Let A be a simple C*-algebra which absorbs $\mathcal{D}$ tensorially, where $\mathcal{D}$ is UHF. Let G be a finite group. Suppose α is an arbitrary action in* $\mathrm{Aut}_G(A)$ *and β is an action in* $\mathrm{Aut}_G(\mathcal{D})$ *with the tracial Rokhlin property. Then the tensor product $\alpha \otimes \beta$ is an* *action in* $\mathrm{Aut}_G(A \otimes \mathcal{D})$ *with the tracial Rokhlin property. If* β *has the strict Rokhlin property, then so does* $\alpha \otimes \beta$*.*

Proof By Theorem 1.9, [26], $A \otimes \mathcal{D}$ has Property (SP). Hence the first part of the proposition follows from Proposition 2.4.6. The second part follows from Proposition 2.2.4. ■

Proposition 4.2.3 *Let* $\mathcal{D}$ *be a UHF-algebra and* G *be any finite group. Then there exists at least one action in* $\mathrm{Aut}_G(\mathcal{D})$ *with the tracial Rokhlin property. If* $\mathcal{D}$ *is the UHF-algebra of type* r^∞*, where* r *is the cardinality of* G*, then there exists an action with the strict Rokhlin property.*

Proof Write $\mathcal{D} = M_{k_1}(\mathbb{C}) \otimes M_{k_2}(\mathbb{C}) \otimes \cdots$. After suitable telescoping, we can always assume that k_i goes to ∞ as i goes to ∞. For each i, write $k_i = r_i|G| + s_i$, where r_i and s_i are non-negative integers such that $0 \leq s_i < |G|$. Therefore $\lim_{i\to\infty}(r_i/s_i) = 0$. Let π_i be trivial actions, for each i. By Corollary 3.4.5, the action $\alpha(r, s, \pi)$ has tracial Rokhlin property. Similarly, if $\mathcal{D}$ is a UHF-algebra of type r^∞, then the action $\alpha(1)$ has the strict Rokhlin property. ■

From Proposition 4.2.1, Proposition 4.2.2 and Proposition 4.2.3, we immediately get the following:

Corollary 4.2.4 *Let* $\mathcal{D}$ *be a UHF-algebra of infinite-type and let* A *be a simple separable* $\mathcal{D}$*-absorbing C*-algebra. Let* G *be a finite group, then the set of actions with the tracial Rokhlin property is dense in* $\mathrm{Aut}_G(A)$*. If* $\mathcal{D}$ *is the UHF-algebra of type* r^∞*, where* r *is the cardinality of* G*, then the set of actions with the Rokhlin property is dense in* $\mathrm{Aut}_G(A)$*.*

The Jiang-Su algebra $\mathcal{Z}$ is projectionless, hence no action on $\mathcal{Z}$ can have the tracial Rokhlin property. However, there is a generalized version of the tracial Rokhlin property that can exist on $\mathcal{Z}$. The idea is to use positive elements instead of projections:

Definition 4.2.5 *Let A be an infinite dimensional simple unital C*-algebra, let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the* projection free tracial Rokhlin property *if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are positive elements $c_g \in A$ for $g \in G$ that are mutually orthogonal, satisfy $0 < c_g \leq 1$, and such that:*

(1) $\|\alpha_g(c_h) - c_{gh}\| < \varepsilon$ for all $g, h \in G$,

(2) $\|c_g a - a c_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$,

(3) if $c = \sum_{g\in G} c_g$, then $\tau(1 - c) < \varepsilon$, for any trace τ in $T(A)$.

The same definition is called the 'weak tracial Rokhlin property' in Definition 1.2 of [21]. We will use the term 'weak tracial Rokhlin property' in the next chapter to mean something different. Compared to the original definition in [27] (Definition 2.7), we do not have the last comparison condition using Cuntz semigroup. The main reason for our new definition is to guarantee permanence property under taking tensor products. When the underlying C*-algebra is sufficiently nice, all these different generalizations of tracial Rokhlin property will coincide with the original tracial Rokhlin property:

Proposition 4.2.6 *Let A be a simple separable C*-algebra with tracial rank zero, and let G be a finite group. Let α be an action in $\mathrm{Aut}_G(A)$. Then α has the projection free tracial Rokhlin property, if and only if α has the tracial Rokhlin property (Definition 2.1.2).*

Proof A simple C*-algebra A with tracial rank zero is stably finite, has real rank zero, stable rank one, and the order of projections is determined by traces. The last property means that for any two projections p, q in A, $p \lesssim q$ if and only if $\tau(p) < \tau(q)$ for any trace τ in $T(A)$. Since A is stably finite, we can omit the norm condition in Definition 2.1.2, as in Lemma 1.16, [13]. From section 6.2 of [14], since the C*-algebra A has stable rank one, the Cuntz comparison is the same as the Blackadar comparison. Hence one can use Blackadar comparison in the last condition of Definition 2.7, [27]. Now since A has real rank zero and the order of projections are determined by traces, following the same argument as in Proposition 1.3 [21], we can show that it does not matter whether we use positive elements or projections, and the different comparison conditions are equivalent. ∎

For tensor products of actions with the projection free tracial Rokhlin property, we have the following:

Proposition 4.2.7 *Let A be a simple C*-algebra, B be a simple C*-algebra with a unique trace τ_0, and G be a finite group. Suppose α is an arbitrary action in $\mathrm{Aut}_G(A)$ and β is an action in $\mathrm{Aut}_G(B)$ with the projection free tracial Rokhlin property. Then the tensor product $\alpha \otimes \beta$ has the projection free tracial Rokhlin property.*

Proof One argues the same way as in the proof of Proposition 2.4.6. If $\{c_g\}_{g \in G}$ are the 'Rokhlin positive elements' for the action β, then we use $\{1 \otimes c_g\}_{g \in G}$ as our 'Rokhlin positive elements' for the action $\alpha \otimes \beta$. The comparison condition is much easier to deal with in this case. For if B has a unique trace τ_0, then for every trace τ in $T(A \otimes B)$, we have $\tau = \tau_1 \otimes \tau_0$ for some trace τ_1 in $T(A)$. Hence $\tau(1 - \sum 1 \otimes c_g) = \tau_0(1 - \sum c_g)$, which reduces the comparison in the tensor product to that of B. ∎

The Jiang-Su algebra $\mathcal{Z}$ is a simple C*-algebra with a unique trace. Now let's find an action on $\mathcal{Z}$ with the projection free tracial Rokhlin property. The construction is inspired by the example in [21].

Proposition 4.2.8 *Let $\mathcal{Z}^\infty$ be the infinite tensor product of $\mathcal{Z}$, which is isomorphic to $\mathcal{Z}$. Use $\mathcal{Z}_i$ to denote the i-th tensor factor, where i ranges from 0 to ∞. Set r to be the cardinality of G. Choosing any bijection between G and the set $\{1, 2, \ldots, r\}$, we can construct an action α of G on $\{1, 2, \ldots, r\}$ induced by the left regular representation. This extends to an action of G on $\mathbb{N}$, still denoted by α, such that $\alpha_g(n) = \alpha_g(n_0)$ for any $n \in \mathbb{N}$ and $g \in G$, where n_0 is the remainder of n after dividing by r. We can define an action β on $\mathcal{Z}^\infty$ by*

$$\beta_g(x_1 \otimes x_2 \otimes \ldots) = x_{\alpha_g(1)} \otimes x_{\alpha_g(2)} \otimes \ldots.$$

Then β has the projection free tracial Rokhlin property.

In view of Proposition 2.3 and Proposition 2.8 of [21], we can prove the above proposition exactly the same way, with a different index convention and a finite group version of Lemma 2.2, [21]. We shall show the following analogue of Lemma 2.2, [21] and then give a sketch of the proof. The details can be found in [21]:

Lemma 4.2.9 *Let $X = [0,1]^{\mathbb{N}}$, and let $\beta \colon G \to Homeo(X)$ be the action, given by $\beta_g(x)_k = x_{\alpha_g(k)}$, for $x = (x_k)_{k \in \mathbb{N}} \in X$ and $k \in \mathbb{N}$, where α is the action defined in Proposition 4.2.8. Let μ_0 be a Borel probability measure on $[0,1]$, and let μ be the infinite product measure on X. Suppose $\mu_0(\{t\}) = 0$ for all $t \in [0,1]$. Let r be the cardinality of G. Then for every $\varepsilon > 0$ and every $n \in \mathbb{N}$, there are $N > n \in \mathbb{N}$ and a closed set $Y_0 \in [0,1]^r$ such that the set*

$$Y = \prod_{0 \leq k < N} [0,1] \times Y_0 \times \prod_{k \geq N+r} [0,1]$$

has the properties:

1. *The sets $\beta_g(Y)$,for $g \in G$, are mutually disjoint, and*

2. $\mu(X \setminus (\cup_{g\in G} \beta_g(Y))) < \varepsilon$.

Proof Because of the periodicity in the definition of α, it's easy to see that we need only to find the desired closed set Y_0 for $n = 0$. Let $X_0 = [0,1]^r$, consider the set $Z = \{x \in X_0 \,|\, \beta_g(x) = x \text{ for some } g \in G, g \neq 1_G\}$. This is a closed $G-$invariant set. Let $x = (x_i)_{0\leq i<r}$. If $g(x) = x$, then $x_{\alpha_g(i)} = x_i$, for each i. For $g \neq 1_G$, we conclude that there exists at least two different indices i, j, such that $x_i = x_j$. Let $X_{i,j}$ be the subset of X consisting of elements $x = (x_i)_{0\leq i<r}$ such that $x_i = x_j$. Then Z is a subset of $\cup_{i\neq j} X_{i,j}$. By Fubini's theorem, each set $X_{i,j}$ has measure 0, if $i \neq j$, hence Z has measure 0.

Let X' be the set $X_0 \setminus Z$, which is an open $G-$invariant set. Consider the action

$$g \mapsto \beta'_g = \beta_g|_{X'}, \text{ for } g \in G.$$

Regard X' as a metric space with metric d inherited from the Euclidean metric on X_0. We can see that β' is a free action on X' which acts isometrically. Consider the quotient set X'/G (no topology assigned at this moment) and let $\pi\colon X' \to X'/G$ be the quotient map. We now denote $\beta'_g(x)$ simply by $g(x)$. Define a map ρ on $X'/G \times X'/G$ by the formula: $\rho([x],[y]) = \min_{g\in G} d(g(x), y)$. Using the fact that β' acts freely and isometrically, we can verify that ρ is in fact a metric, and the quotient map π becomes a metric map. In particular, π is continuous.

It's not hard to see that Z is a countable intersection of decreasing open sets in X_0. Hence $X' = X_0 \setminus Z$ is a countable union of increasing closed sets in X_0. Since X_0 is compact,

X' is a countable union of increasing compact sets. Now all the conditions in Lemma 5.1 of [28] are satisfied, therefore by the Federer-Morse lemma, we can find a Borel set E of X', such that:

1. π maps E onto X'/G,

2. for each point $y \in X'/G$, E meets $\pi^{-1}(\{y\})$ in exactly one point.

Equivalently, E is a fundamental domain of the action β', in the sense that:

1. The sets $\beta_g(E)$, for $g \in G$ are mutually disjoint,

2. $X' = \cup_{g \in G} \beta_g(E)$.

The fact that X' is an open set in X_0 and E is a Borel set in X' implies that E is a Borel set in X_0. Being a probability measure on a compact space X_0, the measure μ is inner regular. Hence we can find a compact subset Y_0 of E such that $\mu(E \backslash Y_0) < \varepsilon/r$. Use the fact that μ is invariant under the action β, we can show that:

1. The sets $\beta_g(Y_0)$, for $g \in G$ are mutually disjoint,

2. $\mu(X \backslash (\cup_{g \in G} \beta_g(Y_0))) = \mu(Z) + \sum_{g \in G} \mu(\beta_g(E \backslash Y_0)) < \varepsilon$.

This completes the proof. ■

We now sketch the proof of Proposition 4.2.8. Let $S \subset \mathcal{Z}^\infty$ be finite, let $\varepsilon > 0$, and let $n \in \mathbb{Z}_{>0}$. Without loss of generality, assume there is $M \in N$ such that $S \in \mathcal{Z}^M$, where $\mathcal{Z}^M$ is identified as $\mathcal{Z}^M \otimes 1$ in $\mathcal{Z}^\infty$. Identify $X = C([0,1]^{\mathbb{N}})$ with the infinite tensor product $\otimes_{n \in \mathbb{N}} C([0,1])$.

Let τ be the unique tracial state on $\mathcal{Z}$. For each positive element $a \in \mathcal{Z}$ with norm 1, let μ_a be the Borel probability measure on $[0,1]$ defined by $\int_0^1 f d\mu_a = \tau(f(a))$ for $f \in C([0,1])$. By Proposition 2.8, [21], we can find some $a \in \mathcal{Z}$ such that for every $t \in [0,1]$, we have $\mu_a(\{t\}) = 0$.

Let $\beta\colon G \to Homeo(X)$ be the action defined as in Lemma 4.2.9. Use the same notation to denote the induced action on $\mathrm{Aut}(C(X))$. Let $\tilde{\beta}\colon G \to \mathrm{Aut}(\mathcal{Z}^\infty)$ be the action defined as in Lemma 4.2.8. Define a map $T_0\colon C([0,1]) \to \mathcal{Z}$ by functional calculus: $T_0(f) = f(a)$ for $f \in C([0,1])$. Then there is an induced infinite tensor product homomorphism $T\colon C(X) \to \mathcal{Z}^\infty$, which satisfies $T \circ \beta = \tilde{\beta} \circ T$.

Let μ be the infinite product the μ_a. Apply Lemma 4.2.9 with μ, ε, n and M as given, obtaining integer $N > M$, a closed set $Y_0 \in [0,1]^r$, and the closed set

$$Y = \prod_{0 \le k < N} [0,1] \times Y_0 \times \prod_{k \ge N+r} [0,1]$$

with the properties:

1. The sets $\beta_g(Y)$,for $g \in G$, are mutually disjoint, and

2. $\mu(X \backslash (\cup_{g \in G} \beta_g(Y))) < \varepsilon$.

Since $[0,1]^r$ is normal, it's not hard to see that we can find an open set U_0 containing Y_0 and the open set

$$U = \prod_{0 \le k < N} [0,1] \times U_0 \times \prod_{k \ge N+r} [0,1]$$

such that the sets $\beta_g(U)$,for $g \in G$, are mutually disjoint.

By Urysohn's lemma, we can choose a continuous function $f_0\colon [0,1]^r \to [0,1]$ such that $\mathrm{supp}(f_0) \subset U_0$ and $f_0(x) = 1$ for all $x \in Y_0$. Define $f \in C([0,1])^{\mathbb{N}}$ by

$$f(\ldots, x_N, x_{N+1}, \ldots, x_{N+r}, \ldots) = f_0(x_{N+1}, \ldots, x_{N+r}).$$

For $g \in G$, define $b_g \in C([0,1]^{\mathbb{N}})$ by $b_g = \beta_g(f)$. Then set $c_g = T(b_g)$.

We verify the conditions of Definition 4.2.5. The positive elements b_g are mutually orthogonal with $0 \leq b_g \leq 1$ and $\alpha_g(b_h) = b_{gh}$. Hence the same relations hold for positive elements $c_g = T(b_g)$. For condition (2), by our assumption that $S \subset \mathcal{Z}^M$ and by our choice of N, we have $c_g a = a c_g$, for all $g \in G$ and $a \in S$. For the comparison condition, with $c = \sum_{g\in G} c_g$, we have

$$\begin{aligned}\tau(c) = \sum_{g\in G} \tau(T(b_g)) &= \sum_{g\in G} \mu(b_g)\\ &\geq \mu(\cup_{g\in G}\beta_g(Y)) > 1 - \varepsilon.\end{aligned}$$

So $\tau(1-c) < \varepsilon$.

Let $\phi\colon A \to B$ be an isomorphism between two C*-algebras, α is an action in $\mathrm{Aut}_G(A)$ with the projection free tracial Rokhlin property, then $\phi \circ \alpha \circ \phi^{-1}$ is an action in $\mathrm{Aut}_G(B)$ with the projection free tracial Rokhlin property. Combine Proposition 4.2.1, Proposition 4.2.7, Proposition 4.2.8 and Proposition 4.2.6, we get the following:

Corollary 4.2.10 *Let A be a simple separable C*-algebra which absorbs the Jiang-Su algebra $\mathcal{Z}$ tensorially. Let G be a finite group. Then the set of actions in* $\mathrm{Aut}_G(A)$ *with the projection free tracial Rokhlin property is dense in* $\mathrm{Aut}_G(A)$. *If in addition, A has tracial rank zero, then the set of actions with the tracial Rokhlin property is dense in* $\mathrm{Aut}_G(A)$.

4.3 Rokhlin actions form G_δ sets

We now turn to the proof that the actions with any of the Rokhlin properties (including the projection free tracial Rokhlin property) form a G_δ set. We establish a finite group version of Lemma 3.4 and Lemma 3.5 of [21] here.

Proposition 4.3.1 *Let A be simple separable C^*-algebra. Then the set of actions in $Aut_G(A)$ with the strict Rokhlin property is a G_δ set. The same conclusion is true if we replace strict Rokhlin property by tracial Rokhlin property, or projection free tracial Rokhlin property.*

Proof The same proof as in Lemma 3.4 and Lemma 3.5, [21] carries over without difficulty to the strict Rokhlin property case and the projection free tracial Rokhlin property case. We now work a little harder to prove the tracial Rokhlin property case. Given a finite set F in A, a positive number $\varepsilon > 0$, and a positive element $x \in A$ with norm 1, we define $W(F, \varepsilon, x)$ to be the set of actions α in $\mathrm{Aut}_G(A)$ such that there exists mutually orthogonal projections e_g, for $g \in G$ satisfying:

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$,

(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$,

(3) Let $e = \sum_{g\in G} e_g$, we have $1 - e \precsim_s x$, and

(4) $\|exe\| > 1 - \varepsilon$.

It's easy to see that an action α has the tracial Rokhlin property if and only if it's in the intersection of all sets of the form $W(F, \varepsilon, x)$. By the same argument as in Lemma 3.5 of [21], the sets $W(F, \varepsilon, x)$ are all open. Hence we are done if we can show that the sets of all actions with the tracial Rokhlin property is a countable intersection of sets of the form $W(F, \varepsilon, x)$.

Without loss of generality, we may assume that A has property (SP), otherwise it reduces to the strict Rokhlin property case. We first show that for any set $W(F, \varepsilon, x)$, there exists another finite set S, a positive number δ and a projection p, such that $W(S, \delta, p) \subset W(F, \varepsilon, x)$. Set $\delta = \varepsilon/3$. By Lemma 2.3.3, there exists a non-zero projection $p \in \overline{xAx}$ such that:

$$\|px^{1/2} - p\| \leq \delta, \quad \|x^{1/2}p - p\| \leq \delta.$$

Set $S = F \cup \{p\}$, $\delta = \varepsilon/3$. Suppose $\alpha \in W(S, \delta, p)$. By definition, there exists mutually orthogonal projections e_g, for $g \in G$ such that:

(1) $\|\alpha_g(e_h) - e_{gh}\| < \delta$ for all $g, h \in G$,

(2) $\|e_g a - a e_g\| < \delta$ for all $g \in G$ and all $a \in S$,

(3) Let $e = \sum_{g \in G} e_g$, we have $1 - e \precsim_s p$,

(4) $\|epe\| > 1 - \delta$.

By our choice of p and δ, we have:

(1') $\|\alpha_g(e_h) - e_{gh}\| < \delta < \varepsilon$ for all $g, h \in G$,

(2') $\|e_g a - a e_g\| < \delta < \varepsilon$ for all $g \in G$ and all $a \in F$,

(3') Let $e = \sum_{g \in G} e_g$, we have $1 - e \precsim_s p \precsim_s x$,

(4') $\|exe\| \geq |px^{1/2}ex^{1/2}p| \geq |pep| - 2\delta = |epe| - 2\delta > 1 - \varepsilon$.

Hence $\alpha \in W(F, \varepsilon, x)$. Now we proceed like the proof of Lemma 3.4 in [21]. Choose a countable dense subset T and let $\mathcal{F}$ consists of finite subsets of T. $\mathcal{F}$ itself is countable. Choose a countable set P of projections such that every projection in A is Murray-von Neumann equivalent to some projection in P. Then $\alpha \in \mathrm{Aut}_G(A)$ has the tracial Rokhlin property if and only if:

$$\alpha \in \cap_{F \in \mathcal{F}} \cap_{n \in N} \cap_{p \in P} W(F, \frac{1}{n}, p).$$

This completes the proof. ∎

5. Tracial Rokhlin property for non-simple C*-algebras

5.1 Weak tracial Rokhlin property

In this section, we are going to give an alternative definition of tracial Rokhlin property for non-simple C*-algebras. Although the original definition of tracial Rokhlin property makes sense for non-simple C*-algebras, it may be too strong to be distinctive from the strict Rokhlin property, as we can see from the following example:

Example 5.1.1 *Let $\alpha\colon G \mapsto \mathrm{Aut}(A)$ and $\beta\colon G \mapsto \mathrm{Aut}(B)$ be two actions of the same group G. Let $\pi\colon G \mapsto \mathrm{Aut}(A \oplus B)$ be the direct sum, i.e. $\pi_g(a,b) = (\alpha_g(a), \beta_g(b))$, for any $g \in G$, $a \in A$ and $b \in B$. Then π has the tracial Rokhlin property, if and only if both α and β have the strict Rokhlin property. In other words, π has the tracial Rokhlin property if and only if it has the strict Rokhlin property.*

Proof Suppose π has the tracial Rokhlin property. Let F be a finite subset of A and let $\varepsilon > 0$. Choose any positive element b in B with norm 1. Let $F' = \{(a,0) \mid a \in F\}$ and let $x = (0,b)$. Since π has the tracial Rokhlin property, there are mutually orthogonal projections e_g in $A \oplus B$, for $g \in G$ such that

(1) $\|\pi_g(e_h) - e_{gh}\| < \varepsilon$,

(2) $\|e_g(a,0) - (a,0)e_g\| < \varepsilon, \forall a \in F$,

(3) With $e = \sum_{g\in G} e_g$, $1 - e \precsim_s x = (0,b)$.

Let $e_g = (p_g, q_g)$. Then we can see that the projections $p_g \in A$ for g in G are mutually orthogonal projections satisfying:

(1') $\|\alpha_g(p_h) - p_{gh}\| \leq \|\alpha_g(e_h) - e_{gh}\| < \varepsilon$.

(2') $\|p_g a - a p_g\| = \|e_g(a, 0) - (a, 0)e_g\| < \varepsilon, \forall a \in F$.

(3') Let $p = \sum_{g\in G} p_g$, and $q = \sum_{g\in G} q_g$. Then $1 - e = (1 - p, 1 - q) \precapprox_s (0, b)$, hence $1 - p \precapprox_s 0$, which forces $1 - p = 0$, or $p = 1$.

Hence α has the strict Rokhlin property. The same argument shows that β also has the strict Rokhlin property. It's not hard to see that $\pi = \alpha \oplus \beta$ has the strict Rokhlin property if and only if both α and β have the strict Rokhlin property. ∎

An element a in a C*-algebra is said to be *full* if the closed ideal generated by a is the whole C*-algebra. Inspired by the above observation, we give the following alternative definition of tracial Rokhlin property:

Definition 5.1.2 *Let A be an infinite dimensional unital C*-algebra, and let $\alpha\colon G \to$ $\mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the* weak tracial Rokhlin property *if for every finite set $F \subset A$, every $\varepsilon > 0$, every positive element $b \in A$ with norm 1 and every* full *positive element $x \in A$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:*

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$,

(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$,

(3) Letting $e = \sum_{g\in G} e_g$, we have $1 - e \precapprox_s x$,

(4) $\|ebe\| > 1 - \varepsilon$.

By the same perturbation argument as in Lemma 1.17 of [13], we could have a formally stronger version of weak tracial Rokhlin property, by requiring that the defect projection be α-invariant:

Lemma 5.1.3 *Let A be an infinite dimensional unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then α has the* weak tracial Rokhlin property *if and only if for every finite set $F \subset A$, every $\varepsilon > 0$, every positive element $b \in A$ with norm 1 and every* full *positive element $x \in A$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:*

(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ *for all* $g, h \in G$,

(2) $\|e_g a - a e_g\| < \varepsilon$ *for all* $g \in G$ *and all* $a \in F$,

(3) Letting $e = \sum_{g \in G} e_g$, *then* e *is* $\alpha-$*invariant,*

(4) $1 - e \precsim_s x$, *and*

(5) $\|ebe\| > 1 - \varepsilon$.

The weak tracial Rokhlin property coincides with the original tracial Rokhlin property in the simple C*-algebra case:

Proposition 5.1.4 *Let A be an infinite dimensional simple unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be a finite group action. Then α has the tracial Rokhlin property if and only if it has the weak tracial Rokhlin property.*

Proof It's trivial that the weak tracial Rokhlin property implies the tracial Rokhlin property, since every non-zero element in a simple C*-algebra is full. So let's prove the other direction. We may assume that A has Property (SP), otherwise α has the strict Rokhlin property and therefore the tracial Rokhlin property. Let F be a finite subset of A, $\varepsilon > 0$ be a positive number, $b \in A_+$ has norm 1, and $x \in A_+$ is non-zero. Let $\delta = \frac{\varepsilon}{(2|G|+3)}$. By Lemma 2.3.3, there exists a non-zero projection $q \in \mathrm{Her}(b)$, such that for any projection $r \leq q$, we have $\|rb - r\| < \delta$. By Lemma 3.5.6 of [16], we can find a non-zero projection $p \leq q$, such that $p \precsim_s x$. Let $F' = F \cup \{p\}$, since α has tracial Rokhlin property, we can mutually orthogonal projections $e_g \in A$ such that:

(1) $\|\alpha_g(e_h) - e_{gh}\| < \delta < \varepsilon$ for all $g, h \in G$,

(2) $\|e_g a - a e_g\| < \delta < \varepsilon$ for all $g \in G$ and all $a \in F'$,

(3) For $e = \sum_{g\in G} e_g$, we have $1 - e \precsim_s p$, and

(4) $\|epe\| > 1 - \delta$.

By our choice of p, we have: $1 - e \precsim_s p \precsim_s x$. So we need only to verify that $\|ebe\| > 1 - \varepsilon$. For that, we have the following estimation:

$$
\begin{aligned}
\|ebe\| \geq \|pebep\| &> \|epbpe\| - \|(ep - pe)bpe\| - \|peb(pe - ep)\| \\
&> \|epe\| - \|e(p - pb^{1/2})pe\| - \|epb^{1/2}(b^{1/2}p - p)e\| - 2|G|\delta \\
&> 1 - \delta - \delta - \delta - 2|G|\delta = 1 - \varepsilon.
\end{aligned}
$$

■

Looking back to Example 5.1.1 which we gave at the beginning of this section, we see that the weak tracial Rokhlin property is a better definition, because we have the following:

Proposition 5.1.5 *Let $\alpha\colon G \mapsto \mathrm{Aut}(A)$ and $\beta\colon G \mapsto \mathrm{Aut}(B)$ be two actions of the same group G. Let $\pi\colon G \mapsto \mathrm{Aut}(A \oplus B)$ be the direct sum, i.e. $\pi(a,b) = (\alpha(a), \beta(b))$, for any $a \in G$ and $b \in B$. Then π has the weak tracial Rokhlin property, if and only if both α and β have the weak tracial Rokhlin property.*

For the proof, we just need to observe that for $a = (a_1, a_2)$ in $A \oplus B$, a is full if and only if a_1 and a_2 are both full in the corresponding C*-algebras.

5.2 Tracial rank of the crossed product

Recall our definition of tracial rank for simple C*-algebras in Chapter 4 (Definition 4.1.3). For non-simple C*-algebras, one could modify Definition 4.1.3 by requiring that the positive element b be full, which is called the weak tracial Rank and denoted by $TR_w(\cdot)$. This definition does not lead to good permanance properties. In [24], Lin gave a better, but more complicated definition of the tracial rank for non-simple C*-algebras. See Definition 3.5 and Definition 3.1, [24] and Theorem 2.5, [29]. We will not need the precise definition, but rather the permanence properties. Hence we collect the results in [24] here:

Proposition 5.2.1 *Let A be a C*-algebra such that $\mathrm{TR}(A) \leq k$, for some non-negative integer k. Then any unital hereditary subalgebras, or ideals, or matrix algebras over A, or sub-quotients (A/I for some ideal I), all have tracial rank $\leq k$.*

Proposition 5.2.2 *Recall that $a \in_\varepsilon B$ means there is some $b \in B$ such that $\|a - b\| < \varepsilon$. Let k be some non-negative integer. Let A be a C*-algebra with the property that, for any* *finite set F in A and any $\varepsilon > 0$, there is a C*-subalgebra B of A, such that* $\mathrm{TR}(B) \leq k$ *and* $a \in_\varepsilon B$*, for every a in B. Then* $\mathrm{TR}(A) \leq k$.

Proposition 5.2.3 *For a general C*-algebra A, $TR_w(A) \leq TR(A)$. If A is simple, then $TR_w(A) = TR(A)$.*

We have a very satisfactory result for Question 1.0.2 when the action has the strict Rokhlin property:

Theorem 5.2.4 *Let A be a separable C*-algebra with tracial rank $\leq k$. Let $\alpha \colon G \to Aut(A)$ be an action with the strict Rokhlin property. Then $C^*(A, G, \alpha)$ has tracial rank $\leq k$.*

The proof of the above theorem is based on the following lemma, which can be extracted from the proof of Theorem 2.2 in [13]:

Proposition 5.2.5 *Let A be a unital C*-algebra and $\alpha \colon G \to \mathrm{Aut}(A)$ be an action with the strict Rokhlin property. Let $n = \mathrm{card}(G) < \infty$. Then for any finite set F in the crossed product $C^*(G, A, \alpha)$ and any $\varepsilon > 0$, there exists a projection $f \in A$ and a unital homomorphism $\phi \colon M_n(fAf) \to C^*(G, A, \alpha)$ such that* $\mathrm{dist}(b, \phi(M_n(fAf)) < \varepsilon$*, for any $b \in F$.*

Now Proposition 5.2.5 together with Proposition 5.2.1 and Proposition 5.2.2 proves our Theorem 5.2.4.

The main result of this section is the following result:

Theorem 5.2.6 *Let A be an infinite dimensional separable unital C*-algebraLet $\alpha \colon G \to Aut(A)$ be an action with the weak tracial Rokhlin property. Assume that A is α-simple and*

$\mathrm{TR}(A) \leq k$. *Then* $C^*(A, G, \alpha)$ *has tracial rank* $\leq k$.

We shall break this proof into several parts, and complete it at the end of this section.

The assumption that A is α simple is used to ensure that $C^*(G, A, \alpha)$ is simple, as we have the following:

Lemma 5.2.7 *Let* A *be a unital* C^**-algebra. Let* $\alpha\colon G \to \mathrm{Aut}(A)$ *be a finite group action with the weak tracial Rokhlin property. Then* A *is* α*-simple if and only if* $C^*(G, A, \alpha)$ *is simple.*

Proof If I is a proper α-invariant ideal in A, then $C^*(G, I, \alpha)$ is a proper ideal in $C^*(G, A, \alpha)$, hence $C^*(G, A, \alpha)$ is simple implies A is α-simple. For the other direction, there are several proofs. Using the Rokhlin projections, it's not hard to show that for any $a_0, a_1, \ldots, a_n$ in A with a_0 positive, any $\varepsilon > 0$, there is a projection p such that:

$$\|pa_0p\| > \|a_0\| - \varepsilon, \quad \|pa_i - a_ip\| < \varepsilon, \quad \|p\alpha_g(p)\| < \varepsilon \text{ for } g \neq 1 \quad (*)$$

Then the same lines as Theorem 7.2 of [30] shows that $C^*(G, A, \alpha)$ is simple.

A more sophisticated way is to use Theorem 2.5 of [31]. A discrete group is exact if and only if the reduced C*-algebra is exact. In particular, finite groups are exact. When A is $\alpha-$simple, it's easy to see that the weak tracial Rokhlin property implies the residual Rokhlin* property (See Definition 2.1, [31]). By Theorem 2.5 of [31], the C*-algebra A separates the ideal in the crossed product, which is equivalent to that the crossed product is simple. ■

When A is $\alpha-$simple, we have a similar result as in Lemma 2.3.1.

Lemma 5.2.8 *Let A be a unital C^*-algebra. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a finite group action with the weak tracial Rokhlin property. If A is α-simple, then either A has Property (SP) or α has the strict Rokhlin property.*

Proof When A is α-simple, by Proposition 2.1 of [32], A is a finite direct sum of simple ideals which are permuted transitively among each other by the action of G. Write $A = I_1 \oplus \cdots \oplus I_n$. Suppose A does not have Property (SP). Then there exists some non-zero positive element $b \in A$ such that $\mathrm{Her}(b)$ contains no non-zero projection. Write $b = (b_1, \ldots, b_n)$, where b_i is a positive element in I_i. Since b is non-zero, without loss of generality assume b_1 is non-zero. Hence $\mathrm{Her}(b_1)$ contains no non-zero projection. Since the ideals are permuted transitively by the action of G, we can choose g_i such that $\alpha_{g_i}(I_1) = I_i$, for each i. Let $b'_i = \alpha_{g_i}(b_1)$ and consider the element $b' = (b'_1, \ldots, b'_n)$. Since b'_1 is a full element in I_1, we can see that b' is a full element in A. But $\mathrm{Her}(b') = \mathrm{Her}(b'_1) \oplus \cdots \oplus \mathrm{Her}(b'_n)$ contains no non-zero projections. Now if we choose Rokhlin projections corresponding to b', the defect projection will be forced to be 0, hence the Rokhlin projections must sum up to 1 and therefore α has the strict Rokhlin property. ■

The second piece we need in the proof of Theorem 3.1.3 is an analogue of Proposition 1.12 of [13].

Proposition 5.2.9 *Let A be a C^*-algebra with Property (SP) and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action with the weak tracial Rokhlin property. Then $C^*(A, G, \alpha)$ also has Property (SP). Moreover, every non-zero hereditary C^*-algebra of $C^*(A, G, \alpha)$ has a projection which is Murray-von Neumann equivalent to some projection in A.*

Proof Let B be the crossed product $C^*(A, G, \alpha)$. Let $\{u_g\}$ be the canonical unitaries. Then every element of B can be written as $\sum_{g\in G} b_g u_g$, where $b_g \in A, \forall g \in G$. Let 1 be the identity element of G, then a natural faithful expectation E from B to A can be defined by $\mathrm{E}(\sum_{g\in G} b_g u_g) = b_1$. Theorem 2.1, [33] says that if for any non-zero positive element x in B and any $\varepsilon > 0$, there is an element y in A satisfying:

$$\|y^*(x - \mathrm{E}(x))y\| < \varepsilon, \quad \|y^*\mathrm{E}(x)y\| \geq \|\mathrm{E}(x)\| - \varepsilon,$$

then B has Property (SP). We now show that the requirement above is satisfied.

Let $x = \sum_{g\in G} b_g u_g$ be a non-zero positive element of B. Then $b_1 = \mathrm{E}(x)$ must be a non-zero positive element of A: write $x = zz^*$ where $z = \sum_{g\in G} c_g u_g$, then $b_1 = \sum_{g\in G} c_g c_g^* \neq 0$. Without loss of generality, we can assume that b_1 has norm 1. Since α has the weak tracial Rokhlin property, letting $F = \{b_g \mid g \in G\}$, b_1 be the positive element of A with norm 1 and $\delta = \frac{\varepsilon}{(|G|^2 - |G| + 1)}$, we can find mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

(1) $\|\alpha_g(e_h) - e_{gh}\| < \delta$ for all $g, h \in G$,

(2) $\|e_g a - a e_g\| < \delta$ for all $g \in G$ and all $a \in F$,

(3) With $e = \sum_{g\in G} e_g$, $\|eb_1e\| > 1 - \delta$.

Since for each $g \in G$, we have $\|e_g b_1 - b_1 e_g\| < \delta$, we see that:

$$\|eb_1e - \sum_{g\in G} e_g b_1 e_g)\| = \|\sum_{g\neq h} e_g b_1 e_h\| \leq (|G|^2 - |G|)\delta + \|\sum_{g\neq h} e_g e_h b_1\| = (|G|^2 - |G|)\delta.$$

Note that since $e_g b_1 e_g$ for $g \in G$ are orthogonal to each other, we have

$$\|\sum_{g\in G} e_g b_1 e_g\| = \max\{\|e_g b_1 e_g\| \mid g \in G\}.$$

Hence there exists some $h \in G$, such that

$$\|e_h b_1 e_h\| = \|\sum_{g\in G} e_g b_1 e_g\| > \|eb_1 e\| - (|G|^2 - |G|)\delta > 1 - \varepsilon.$$

Let $y = e_h$, then $\|y\mathrm{E}(x)y\| > 1 - \varepsilon$. Also we have:

$$\begin{aligned}
\|y^*(x - \mathrm{E}(x))y\| &= \|e_h(\sum_{g\neq 1} b_g u_g)e_h\| \\
&= \|(\sum_{g\neq 1} e_h b_g \alpha_g(e_h)u_g\| \\
&\leq (\sum_{g\neq 1} \|e_h b_g e_{gh}\| + (|G| - 1)\delta \\
&\leq (\sum_{g\neq 1}(\|(e_h b_g - b_g e_h)e_{gh}\| + \|b_g e_h e_{gh}\|) + (|G| - 1)\delta \\
&\leq 2(|G| - 1)(\delta + 0) < \varepsilon.
\end{aligned}$$

This completes the proof. ■

The last piece needed for the proof of Theorem 3.1.3 is the following technical lemma, extracted from the proof of Theorem 2.2, [13]:

Lemma 5.2.10 *Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a finite group action. Let F be a finite subset of A and let u_g, $g \in G$ be the canonical unitaries implementing the action. Set $n = \mathrm{card}(G)$. Then for any $\varepsilon > 0$, there exist $\delta > 0$, such that for any family of mutually orthogonal projections $e_g \in A$, for $g \in G$ with:*

(1) $\|\alpha_g(e_h) - e_{gh}\| < \delta$,

(2) $\|e_g a - a e_g\| < \delta$ *for any* $a \in F$ *and*

(3) $e = \sum_{g\in G} e_g$ *is* $\alpha-$*invariant,*

there exists a unital homomorphism $\phi_0 \colon M_n \to eC^*(G, A, \alpha)e$ *such that* $\phi_0(v_{g,g}) = e_g$. *Here* $v_{g,h}$ *is the standard* (g, h)*- matrix units of* M_n. *Furthermore, let* $1 = 1_G$ *be the identity of* G*, if we define a unital homomorphism* $\phi \colon M_n \otimes e_1 A e_1 \to eC^*(G, A, \alpha)e$ *by:*

$$\phi(v_{g,h} \otimes a) = \phi_0(v_{g,1}) a \phi_0(v_{1,h}), \quad \textit{for } g, h \in G \quad \textit{and } a \in e_1 A e_1.$$

Then $\mathrm{dist}(eFe, \phi(M_n \otimes e_1 A e_1)) < \varepsilon$.

Proof of Theorem 5.2.6: The proof is actually a modification of Theorem 2.6 of [13]. Let $B = C^*(A, G, \alpha)$. Since the action is α-simple, B is simple by Lemma 5.2.7. By Lemma 5.2.8 and Theorem 5.2.4, we can assume that A has Property (SP).

Let S be a finite subset of B. Without loss of generality we may assume that S is of the form $F \cup \{u_g \colon g \in G\}$, where F is a finite subset of the unit ball of A and $u_g \in B$ are the canonical unitaries implementing the automorphism α_g. Let $\varepsilon > 0$ be a positive number, and let x be a nonzero positive element of B.
Set $n = |G|$, By Lemma 5.2.9, B has Property (SP). Also B is non-elementary since it's infinite dimensional unital simple. By Lemma 2.3.4, we can find $2n$ non-zero mutually orthogonal and equivalent projections $p_1, p_2, \cdots, p_{2n}$ in $\mathrm{Her}(x)$. Then by Lemma 5.2.9 again, we can find a non-zero projection $p' \in A$ such that $p' \precapprox_s p_1$.

Now we are going to find a non-zero sub-projection p of p' so that $\alpha_g(p) \precapprox_s p_1$ in B, for every $g \in G$. List the elements of G as $g_1, g_2, \cdots, g_n$. Let $f_0 \leq p_1$ such that $p' \sim f_0$. Since B is simple, by Lemma 3.5.6 of [16], there exists non-zero projections $f_1' \leq \alpha_{g_1}(p')$

and $f_1 \leq f_0 \leq p_1$ such that $f_1' \sim f_2$. Inductively, for $i = 1, 2, \ldots, n$, we can find non-zero projections f_i and f_i', such that

$$f_i' \leq \alpha_{g_i g_{i-1}^{-1}}(f_{i-1}') \leq \alpha_{g_i}(p'), \quad f_i \leq f_{i-1} \quad \text{and} \quad f_i' \sim f_i.$$

Let $p = \alpha_{g_n^{-1}}(f_n')$. Then p is a non-zero subprojection of p' such that $\alpha_{g_i}(p) \precapprox_s p_1$, for any i.

Since B is simple, there exists $s_1, s_2, \ldots, s_n$ in B such that $\sum_{i=1}^n s_i p s_i^* = 1$. Write $s_i = \sum_{g \in G} s_{i,g} u_g$. Doing a computation we can show that

$$\sum_{i=1}^{n} \sum_{g \in G} s_{i,g} \alpha_g(p) s_{i,g}^* = 1.$$

Set $\hat{p} = \sum_{g \in G} \alpha_g(p)$, then

$$\sum_{g,i} s_{i,g} \hat{p} s_{i,g}^* > \sum_{g,i} s_{i,g} \alpha_g(p) s_{i,g}^* = 1.$$

Hence $\hat{p}$ is full in A. Therefore there exists $\{z_i | 1 \leq i \leq m\} \subset A$ such that $\sum_{i=1}^m z_i p z_i^* = 1$. Let $M = \max\{\|z_i\| \mid 1 \leq i \leq m\}$. Set $\varepsilon_0 = \varepsilon/4$, then choose $\delta_0 > 0$ according to Lemma 5.2.10 for n as given and for ε_0 in place of ε. Let

$$\delta = \min\{\frac{1}{2nmM},\ \delta_0,\ \frac{\varepsilon}{8n}\}.$$

Let $F' = F \cup \{z_i | 1 \leq i \leq m\}$ be a finite subset of A and $\hat{p}$ be a full positive element of A. By Lemma 5.1.3, we can obtain mutually orthogonal projections e_g in A for $g \in G$, such that:

(1) $\|\alpha_g(e_h) - e_{gh}\| < \delta < \delta_0$ for all $g, h \in G$,

(2) $\|e_g a - a e_g\| < \delta < \delta_0$ for all $g \in G$ and all $a \in F'$,

(3) Letting $e = \sum_{g \in G} e_g$, then e is $\alpha-$invariant,

(4) $1 - e \precsim_s \hat{p}$.

Let $v_{g,h}$ be the standard matrix units for M_n. By the choice of δ, there exists a unital homomorphism $\phi_0 \colon M_n \to eBe$ such that $\phi_0(v_{g,g}) = e_g$ for all $g \in G$. Furthermore, if we define a unital homomorphism $\phi \colon M_n \otimes e_1Ae_1 \to eBe$ by $\phi(v_{g,h} \otimes a) = \phi_0(v_{g,1})a\phi_0(v_{1,h})$ for $g, h \in G$ and $a \in e_1Ae_1$, then there is a finite subset T of $M_n \otimes e_1Ae_1$ such that for every $a \in F' \cup \{u_g \mid g \in G\}$, there is $b \in T$ such that $\|\phi(b) - eae\| \leq \varepsilon_0$.

Now e_1Ae_1 is an hereditary C*-subalgebra of A and $\mathrm{TR}(A) \leq k$, we have $\mathrm{TR}(M_n \otimes e_1Ae_1) \leq k$, by Theorem 5.3 and Theorem 5.8 of [24]. In particular, $\mathrm{TR}_w(M_n \otimes e_{g_0}Ae_{g_0}) \leq k$. ($\mathrm{TR}_w(\cdot)$ is the tracial weak rank, see Definition 3.4 of [24] for the definition; Corollary 5.7 of [24] says that $\mathrm{TR}(A) \leq k$ implies $\mathrm{TR}_w(A) \leq k$)

Now consider the element $r = e_{11} \otimes e_1\hat{p}e_1$. Since $\sum_{i=1}^{n} z_i\hat{p}z_i^* = 1$, using the fact $\|e_1z_i - z_ie_1\| < \delta$, we see that:

$$
\begin{aligned}
&\|e_1 - \sum_{i=1}^{m} e_1z_ie_1\hat{p}e_1z_i^*e_1\| \\
=&\|\sum_{i=1}^{m} e_1z_i\hat{p}z_i^*e_1 - \sum_{i=1}^{m} e_1z_ie_1\hat{p}e_1z_i^*e_1\| \\
\leq&\|\sum_{i}(e_1(e_1z_i - z_ie_1)\hat{p})z_i^*e_1\| + \|\sum_{i}(e_1z_ie_1)\hat{p})(z_i^*e_1 - e_1z_i^*)e_1\| \\
\leq& mnM\delta + mnM\delta < 1
\end{aligned}
$$

This shows that the ideal generated by $e_1\hat{p}e_1$ in e_1Ae_1 contains an invertible element, hence $e_1\hat{p}e_1$ is full in e_1Ae_1. Therefore r is a full element of $M_n \otimes e_1Ae_1$. By the definition of tracial weak rank, there is a projection $p_0 \in M_n \otimes e_1Ae_1$ and an $E_0 \in \mathscr{I}^{(k)}$ with $1_{E_0} = p_0$ such that

1. $\|p_0b - bp_0\| < \varepsilon_0$ for all $b \in T$,

2. For every element $b \in T$, there is an element $b' \in E_0$ such that $\|p_0 b p_0 - b'\| < \varepsilon_0$,

3. $1 - p_0 \precsim_s r$.

Set $q = \phi(p_0)$ and $E = \phi(E_0)$. Note that the identity of E is equal to e, the sum of the Rokhlin projections. Since E is isomorphic to a sub-quotient of E_0 and $E_0 \in \mathscr{I}^k$, by the same argument as in Proposition 5.1 of [24], there exists an increasing sequence of C*-algebras C_i, such that the union $\cup_{i=1}^{\infty} C_i$ is dense in E. Therefore, we can choose some large $i = N$, such that $1_{C_N} = 1_E$ and for every $b \in T$, there is an element $b' \in E_0$ and a b'' in C_N so that $\|p_0 b b_0 - b'\| < \varepsilon_0$ and $\|\phi(b') - b''\| < \varepsilon_0$.

Let $a \in S$. Choose $b \in T$ such that $\|\phi(b) - eae\| < \varepsilon_0$. Then, using $qe = eq = q$,

$$
\begin{aligned}
\|qa - aq\| &\leq 2\|ea - ae\| + \|qeae - eaeq\| \\
&\leq 2\|ea - ae\| + 2\|eae - \phi(b)\| + \|p_0 b - b p_0\| \\
&< 2n\delta + 2\varepsilon_0 + \varepsilon_0 \leq \varepsilon
\end{aligned}
$$

Further, choosing $b' \in E_0$ such that $\|p_0 b p_0 - b'\| < \varepsilon_0$, and then choosing b'' in C_N such that $\|\phi(b') - b''\| < \varepsilon_0$. then the element $b'' \in C_N$ satisfies

$$
\begin{aligned}
\|qaq - b''\| &\leq \|qaq - q\phi(b)q\| + \|q\phi(b)q - \phi(b')\| + \|\phi(b') - b''\| \\
&\leq \|eae - \phi(b)\| + \|\phi(p_0 b p_0 - b')\| + \|\phi(b') - b''\| \\
&\leq \varepsilon_0 + \varepsilon_0 + \varepsilon_0 \leq \varepsilon.
\end{aligned}
$$

Finally, for the comparison condition, since $\phi(r) = e_1\hat{p}e_1$:

$$
\begin{aligned}
1 - q = (1-e) + (e-q) &\precapprox_s \hat{p} \oplus \phi 1 - p_0 \\
&\precapprox_s \hat{p} \oplus e_1\hat{p}e_1 \\
&\precapprox_s \oplus_{g\in G}\alpha_g(p) \oplus_{g\in G} \alpha_g(p) \\
&\precapprox_s p_1 \oplus p_2 \oplus \cdots \oplus p_{2n} \precapprox_s x.
\end{aligned}
$$

Hence $B = C^*(G, A, \alpha)$ has tracial rank less or equal to k, by Definition 4.1.3.

Remark 5.2.11 *Actually we could replace tracial rank by weak tracial rank in Theorem 5.2.6 if* $\mathrm{TR}_w(eAe) \leq \mathrm{TR}_w(A)$*, for any projection* $e \in A$*. But unfortunately this is not true in general. See Example 4.7 of [24] for a counterexample. We can also see that the norm condition is not actually used in the proof of Theorem 5.2.6. It is used in Proposition 5.2.9 which is an essential ingredient of the proof. It's possible to find a weak condition so that Proposition 5.2.9 still holds, in which case Theorem 5.2.6 is still valid.*

5.3 Crossed product with real rank zero or stable rank one

In this section, we assume that all classes of C*-algebras that we consider have the property that, if $A \cong B$ and A belongs to the class, then so is B. Following the spirit of tracially AF algebras, Elliott and Niu made the following definition of tracial approximation:

Definition 5.3.1 *(Definition 2.2, [34]) Let* $\mathscr{I}$ *be a class of C*-algebras. A unital C*-algebra* A *is said to be in the class* $\mathrm{TA}\mathscr{I}$*, if and only if for any* $\varepsilon > 0$*, any finite subset* F *of* A*, and any nonzero* $a \in A^+$*, there exist a non-zero projection* $p \in A$ *and a sub C*-algebra* $C \subset A$ *such that* $C \in \mathscr{I}$*,* $1_C = p$*, and for all* $x \in F$*,* *1.* $\|xp - px\| < \varepsilon$*,*

2. $pxp \in_\varepsilon C$*, and*

3. $1 - p \precsim_s a$*.*

By the same argument as in Example 5.1.1, we can see that if A, B are two C*-algebras such that $A \oplus B \in \mathrm{TA}\mathscr{I}$, then both A and B are in $\mathscr{I}$. Hence the comparison condition in Definition 5.3.1 may be too strong for non-simple C*-algebras. We could make the following alternative definition:

Definition 5.3.2 *(Weak tracial approximation) Let* $\mathscr{I}$ *be a class of C*-algebras. Then we define* $w\mathrm{TA}\mathscr{I}$ *to be the class of C*-algebra* A *obtained the same way as in Definition 5.3.1 with the additional requirement that the positive element* a *be full.*

The class $w\mathrm{TA}\mathscr{I}$ properly contains $\mathrm{TA}\mathscr{I}$. But in contrast with $\mathrm{TA}\mathscr{I}$, even if the class $\mathscr{I}$ is closed under taking hereditary sub-algebra, $w\mathrm{TA}\mathscr{I}$ may not have the same property. Hence we need to make this assumption in the following theorem:

Theorem 5.3.3 *Let* A *be an infinite dimensional unital C*-algebra with Property (SP). Let* G *be a finite group. Let* $\alpha\colon G \to \mathrm{Aut}(A)$ *be an action with the weak tracial Rokhlin property such that the crossed product* $C^*(G, A, \alpha)$ *is simple. Suppose* A *belongs to a sub-class* $\mathscr{I}'$ *of* $w\mathrm{TA}\mathscr{I}$*, for some class of C*-algebras* $\mathscr{I}$*. If* $\mathscr{I}'$ *is closed under taking hereditary sub-algebras and tensoring with matrix algebras, then* $C^*(G, A, \alpha)$ *belongs to* $\mathrm{TA}\mathscr{I}$*. In particular, if* A *belongs to* $\mathrm{TA}\mathscr{I}$*, and* $\mathscr{I}$ *is closed under taking hereditary subalgebras and tensoring with matrix algebras, then* $C^*(G, A, \alpha)$ *belongs to* $\mathrm{TA}\mathscr{I}$*.*

Proof Let F be a finite subset of $C^*(G, A, \alpha)$, let a be a non-zero positive element of $C^*(G, A, \alpha)$, and let $\varepsilon > 0$. Let e_g, $g \in G$ be the Rokhlin projections corresponds to F, a and ε. Let $e = \sum_{g \in G} e_g$. From the proof of Theorem 5.2.6, we can find a unital homomorphism $\phi : M_n \otimes (e_1 A e_1) \to eC^*(G, A, \alpha)e$, and a subalgebra C of $M_n \otimes (e_1 A e_1)$ with $1_C = p_0$ which is in the class $\mathscr{I}$, such that:

1. $\|x\phi(p_0) - \phi(p_0)x\| < \varepsilon$, for every $x \in F$.
2. $\phi(p_0)x\phi(p_0) \in_\varepsilon \phi(C)$, for every $x \in F$.
3. $1 - \phi(p_0) \precsim_s a$.

It's not hard to see that the homomorphism ϕ_0 defined in the proof of Theorem 5.2.6 is actually injective if δ is sufficiently small. Since $\mathscr{I}$ contains isomorphic copies of its members, we see that $C^*(G, A, \alpha)$ belongs to TA$\mathscr{I}$. By Lemma 2.3 of [34], if $\mathscr{I}$ is closed under taking unital hereditary sub-algebras and tensoring with matrix algebras, so is TA$\mathscr{I}$, hence the theorem follows. ■

As a corollary, we have the following:

Corollary 5.3.4 *Let A be an infinite dimensional unital separable C^*-algebra. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a finite group action with the weak tracial Rokhlin property such that the crossed product $C^*(G, A, \alpha)$ is simple. Suppose A has stable rank one, then $C^*(G, A, \alpha)$ also has stable rank one.*

Proof First of all, the class of stable rank one C*-algebras is preserved by strict Rokhlin actions. Hence by 5.2.8, we may assume that A has Property (SP). By Theorem 3.18 and

Theorem 3.19 of [16], the class of unital C*-algebras with stable rank one is closed under taking unital hereditary sub-algebras and tensoring with matrix algebras. It follows from Theorem 5.3.3 that the crossed product $C^*(G, A, \alpha)$ is tracially of stable rank 1. By our assumption, $C^*(G, A, \alpha)$ is simple. By Theorem 4.3 of [34], $C^*(G, A, \alpha)$ actually has stable rank one. ■

For real rank, with some modification of Theorem 4.3, we could get the following:

Lemma 5.3.5 *Let $\mathscr{I}$ be the class of unital C*-algebras with real rank 0. Let A be a simple C*-algebra in $\mathrm{TA}\mathscr{I}$, then A has real rank 0.*

Proof Any finite dimensional C*-algebra has real rank 0, hence we may assume that A is infinite dimensional. If A does not have Property (SP), we can choose a nonzero positive element such that $\mathrm{Her}(b)$ contains no nonzero projection. For any self-adjoint element $a \in A$, by Definition 5.3.1, we can find a C*-subalgebra B of real rank 0 and a projection p with the property that $pap \in_\varepsilon B$ and $1 - p \precsim_s b$. Since $\mathrm{Her}(b)$ contains no nonzero projection, $1 - p \precsim_s b$ implies $p = 1$. Hence a is norm close to a real reank 0 subalgebra, and therefore norm close to a invertible self-adjoint element. This proves that A has real rank 0.

Now we assume that A has Property (SP). Let x be a non-zero self-adjoint element in A and let $\varepsilon > 0$. Assume that x is not invertible, otherwise there's nothing to prove. We can find some $\sigma > 0$ such that $\|f_\sigma^1(x) - x\| < \varepsilon/2$. We write $f = f_\sigma^1$. Since x is not invertible, the spectrum of x contains 0. Choose a non-negative continuous function g supported in $[-\sigma, \sigma]$ such that $g(0) = 1$. Then $g(x)$ is non-zero. Since A has Property (SP), there exists a non-zero projection p in $\mathrm{Her}(g(x))$. Since A is simple, by Lemma 3.5.6 of [16], there exist non-zero projections $p_1 \leq p$ and $q_1 \leq 1 - p$, such that $p_1 \sim q_1$. A corner of real rank 0

C*-algebra is again a real rank 0 C*-algebra, hence by Lemma 2.3 of [34], $(1-p)A(1-p)$ belongs to TA$\mathscr{I}$. By the definition of TA$\mathscr{I}$, there exists a projection $q \in (1-p)A(1-p)$ and C*-subalgebra $C \subset A$ of real rank 0, such that $1_C = q$ and:

(1) $\|qf(x)q - y\| < \varepsilon/4$, for some self-adjoint element $y \in C$,

(2) $1 - p - q \precsim_s q_1$.

Since $q_1 \sim p_1 \leq p$, there exist some projection $r \leq p$ and a partial isometry v such that $vv^* = 1 - p - q$ and $v^*v = r$. Now the identity of A can be decomposed into the sum of orthogonal projections: $1 = (p-r) + r + (1-p-q) + q$. We can write $f(x)$ into a matrix form according to this decomposition. Note that $f(x) = (1-p)f(x)(1-p)$, we have:

$$f(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (1-p-q)f(x)(1-p-q) & (1-p-q)f(x)q \\ 0 & 0 & qf(x)(1-p-q) & qf(x)q \end{pmatrix}$$

Since C has real rank 0 and $\|qf(x)q - y\| < \varepsilon/4$ for some self-adjoint element $y \in C$, we could find an invertible self-adjoint element $b \in C$ such that $\|qf(x)q - b\| < \varepsilon/2$. Let $a = (1-p-q)f(x)(1-p-q)$, $c = (1-p-q)f(x)q$ and $d = qf(x)q$. Let Z be the matrix:

$$\begin{pmatrix} p-r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & 1-p-q & c^*d^{-1} \\ 0 & 0 & 0 & q \end{pmatrix}$$

Then by the same computation as in Lemma 3.1.5 of [16], we can show that:

$$f(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & c^* & d \end{pmatrix} = Z \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a - c^* d^{-1} c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} Z^*$$

Now if we consider the element:

$$x' = Z \begin{pmatrix} (\varepsilon/2)(p-r) & 0 & 0 & 0 \\ 0 & 0 & (\varepsilon/2)v* & 0 \\ 0 & (\varepsilon/2)v & a - c^* d^{-1} c & 0 \\ 0 & 0 & 0 & b \end{pmatrix} Z^*,$$

we can check that x' is an invertible self-adjoint element such that $\|f(x) - x'\| < \varepsilon/2$. Hence $\|x - x'\| \leq \|x - f(x)\| + \|f(x) - x'\| < \varepsilon$. Therefore A has real rank 0. ■

We immediately get the following corollary:

Corollary 5.3.6 *Let A be a unital C*-algebra. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be a finite group action with the weak tracial Rokhlin property, such that the crossed product $C^*(G, A, \alpha)$ is simple. Suppose A has real rank 0, then $C^*(G, A, \alpha)$ also has real rank 0.*

Proof A has real rank 0 implies that A has Property (SP). We need only to consider the case that A is infinite dimensional, because any finite dimensional C*-algebra has real rank 0. Therefore the above statement follows from Theorem 5.3.3 and Lemma 5.3.5. ■

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