Spring 4-24-2013

On the (Co)Homology of Non-Commutative Toroidal Orbifold

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A dissertation presented to the
Graduate School of Arts and Sciences
of Washington University in
partial fulfillment of the
requirements for the degree
of Doctor of Philosophy

May 2013
St.Louis, Missouri

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ACKNOWLEDGEMENTS

ALLAH has been kind to me by providing me support through various people during these 6 years of graduate study. I am fortunate to have supportive parents and sisters at home and, also at school I have many Professors and fellow graduate students who spent their time and attention. The support extended at my home during the last 17 months is extraordinarily instrumental towards my thesis. My parents and my sisters, Saba and Sadqua worked day and night to gather my diminished courage to work hard. May they be rewarded.

My department on the other hand consists of people who are polite and patient in their approach. The most patient of all in my view is my advisor, Prof. Xiang Tang. I have benefited a lot through his dedicated weekly discussions with me, which were uninterrupted by his holidays and poor internet connection from my end. He is the best person that I could have got from the department. I thank ALLAH for making him my advisor.

I will not end this page before I acknowledge the help coming through Prof. David Wright, Prof. Mohan Kumar, members of my dissertation committee, Ms. Mary Ann Stenner, Liwei Chen, Bingyuan Liu, and the Graduate School.

Dedicated to my friend.

ABSTRACT OF THE DISSERTATION

On the (Co)Homology of Non-commutative Toroidal Orbifolds.

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Safdar Quddus

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Doctor of Philosophy in Mathematics

Washington University in St. Louis, 2013

Professor Xiang Tang, Chair

Noncommutative torus algebra was studied in the early 80's as a fundamental example of noncommutative geometry. Connes calculated its cyclic and Hochschild cohomology. In this thesis, we study noncommutative toroidal orbifolds generated by actions of finite subgroups of $SL(2,\mathbb{Z})$ on a noncommutative torus algebra.

In the first part, we calculate the Hochschild and cyclic homology of $\mathcal{A}_\theta^{alg} \rtimes \Gamma$ for all finite subgroups $\Gamma \subset SL(2,\mathbb{Z})$. In the second part we analyse the cohomology of these algebras and compute the Chern-Connes pairing between the K_0 elements of $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2$ and explicit cocycles discovered in our calculations. In the third part we discuss some partial results and conjectures about the corresponding smooth orbifolds.

CHAPTER I

Thesis Summary

1.1 Brief introduction

My thesis examines the homological properties of the algebraic and smooth non-commutative toroidal orbifolds. Let A be a commutative C*-algebra and let X be the space of multiplicative linear functionals of A. Let $\gamma : A \to C_0(X)$ be the Gelfand representation. The Gelfand-Naimark theorem states that γ is an isometric *-isomorphism from A onto $C_0(X)$. In this analogy, one can think of a non-commutative C*-algebra A as the algebra of continuous functions on a "non-commutative space"(which does not exist) M, and one studies the geometry of M by looking at the algebra A.

Let us take the non-commutative two-torus as an example. For any $\theta \in \mathbb{R}$, a non-commutative torus algebra can be defined as follows:

$$\mathcal{A}_\theta := \left\{ \sum_{(n,m)\in\mathbb{Z}^2} a_{m,n} U_1^m U_2^n | U_1^* = U_1^{-1}, U_2^* = U_2^{-1}, U_2 U_1 = e^{2\pi i\theta} U_1 U_2, (a_{m,n}) \in \mathcal{S}(\mathbb{Z}^2) \right\}$$

where, $\{a_{m,n}\} \in \mathcal{S}(\mathbb{Z}^2)$ means that each $a_{m,n}$ is a complex valued Schwartz function on $\mathbb{Z}^2$; i.e. the sequence of complex numbers $\{a_{m,n} \in \mathbb{C} | (m,n) \in \mathbb{Z}^2\}$ decreases rapidly, that is, for any $k \in \mathbb{N}_0$:

$$(1.1) \qquad ||a||_k = sup_{(m,r)\in\mathbb{Z}^2}|a_{m,r}|(1+|m|+|r|)^k < \infty.$$

We point out here that when $\theta \in \mathbb{Q}$, $\mathcal{A}_\theta$ is Morita equivalent to the algebra of smooth functions on the classical 2-torus $\mathbb{T}^2$ via the Fourier transformation. When $\theta \notin \mathbb{Q}$, then we have $\mathcal{A}_\theta$ as a non-commutative pre-C*-algebra. We can also consider the algebraic version of this non-commutative algebra as:

$$\mathcal{A}_\theta^{alg} := \left\{ \sum_{(n,m)\in\mathbb{Z}^2} a_{m,n} U_1^m U_2^n \in \mathcal{A}_\theta | (a_{m,n}) \text{ is finitely supported} \right\}.$$

In non-commutative geometry, for more than past 30 years, many properties of these (pre-)C*-algebras have been studied. An invariant widely studied for classification of algebras is the abelian group $K_0(A)$ associated to the algebra A. Given an algebra A, $K_0(A)$ is a group generated by equivalent classes of finitely generated projective modules over A, satisfying the Grothendieck group relation $[P] + [P'] = [P \oplus P']$.

In the early 80's D.Voiculescu and M. Pimsner [VP] computed the K_0 group of the non-commutative torus algebra.

Theorem 1.1. *[VP]* $K_0(\mathcal{A}_\theta) \cong \mathbb{Z}^2$.

In the later years, researches were carried out on the homological properties of these algebras. Hochschild cohomology and cyclic cohomology were calculated in a long fundamental paper [C] of A. Connes. While Hochschild homology, $H_\bullet(A, M)$, for an algebra A over its bi-module M captures the non-commutative analogue of the differential forms; cyclic homology, $HC_\bullet(A)$, captures the de-Rahm cohomology of the algebra A.

A. Connes [C] calculated the Hochschild and cyclic cohomology of $\mathcal{A}_\theta$ using a projective resolution. His result separates θ into two types, one that satisfies the Diophantine condition i.e. $|1-e^{2\pi i\theta n}|^{-1}$ is $O(n^k)$ for some k, and other which does not.

Theorem 1.2. *Let $\mathcal{A}_\theta^*$ be the dual of the algebra $\mathcal{A}_\theta$ with natural the $\mathcal{A}_\theta$ bi-module structure.*

(a). Let $\theta \notin \mathbb{Q}$. One has $H^0(\mathcal{A}_\theta, \mathcal{A}_\theta^*) = \mathbb{C}$.

(b). If $\theta \notin \mathbb{Q}$ satisfies the Diophantine condition, then $H^j(\mathcal{A}_\theta, \mathcal{A}_\theta^*)$ is of dimension 2 for j=1, and of dimension 1 for j=2.

(c). If $\theta \notin \mathbb{Q}$ does not satisfy the Diophantine condition, then H^1, H^2 are infinite dimensional non-Hausdorff spaces.

A. Connes also calculated the cyclic cohomology of $\mathcal{A}_\theta$.

Theorem 1.3. *For all values of $\theta, HC^{even}(\mathcal{A}_\theta) \cong \mathbb{C}^2$ and $HC^{odd}(\mathcal{A}_\theta) \cong \mathbb{C}^2$.*

A. Connes also computed the pairing between $K_0(\mathcal{A}_\theta)$ and $HC^{even}(\mathcal{A}_\theta)$ as a generalization of the index theory.

Theorem 1.4. *The pairing of $K_0(\mathcal{A}_\theta)$ and $H^{even}(\mathcal{A}_\theta)$ is given by:*

a) $<[1], S\tau>= 1, <[\mathfrak{S}], S\tau>= \theta \in]0,1]$

b) $<[1], \phi>= 1, <[\mathfrak{S}], \phi>= 1$

where, S is the periodicity map on cyclic cohomology, τ is the canonical trace on $\mathcal{A}_\theta$, and $[\mathfrak{S}]$ and $[1]$ are the generators of $K_0(\mathcal{A}_\theta)$

We refer to [R1]-[R3] for more results about $\mathcal{A}_\theta$ and refer to [BRT] for the comparison of our knowledge on the smooth noncommutative torus $\mathcal{A}_\theta$ and the algebraic noncommutative torus $\mathcal{A}_\theta^{alg}$.

1.2 My Results

My research studies the Hoschschild and cyclic homology of some algebras that are non-commutative analogues of the quotients of $\mathbb{T}^2$ by finite group actions. We analyse the pairing between K-theory and cyclic cohomology for the algebraic $\mathbb{Z}_2$ orbifold. We hope that with this we will have a better understanding of the K-theory of these algebras which at the time of writing is not well understood. We end the thesis with some partial results on smooth noncommutative toroidal orbifold.

1.2.1 Homology of algebraic non-commutative torus orbifold

Let $\Gamma \subset SL(2,\mathbb{Z})$ be a finite subgroup acting on $\mathcal{A}_\theta^{alg}$ by automorphism. Define the algebraic non-commutative torus Γ orbifold to be $\mathcal{A}_\theta^{alg} \rtimes \Gamma$.

A part of the thesis deals with the calculation of hochschild and cyclic homology groups of the algebra $\mathcal{A}_\theta^{alg} \rtimes \Gamma$. As it is known that up to isomorphism all the finite subgroups of $SL(2,\mathbb{Z})$ are isomorphic to one of these four groups : $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4,$ or $\mathbb{Z}_6$. The cyclic generator g, $g = \begin{bmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{bmatrix}$ acts on $\mathcal{A}_\theta^{alg}$ by:

$$gU_1 = e^{(\pi i g_{1,1} g_{2,1})\theta} U_1^{g_{1,1}} U_2^{g_{2,1}} \text{ and } gU_2 = e^{(\pi i g_{1,2} g_{2,2})\theta} U_1^{g_{1,2}} U_2^{g_{2,2}}.$$

For example, $g = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is the generator of $\mathbb{Z}_2$ as a sub-group of $SL(2,\mathbb{Z})$. $\mathbb{Z}_2$ acts on $\mathcal{A}_\theta^{alg}$ as follows:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} U_1 = U_1^{-1} \text{ and } \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} U_2 = U_2^{-1}.$$

With this action we have a non-commutative orbifold $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2$. Similarly we have other non-commutative orbifolds $\mathcal{A}_\theta^{alg} \rtimes \Gamma$ for $\Gamma = \mathbb{Z}_3, \mathbb{Z}_4$ and $\mathbb{Z}_6$.

We calculate in chapter 3 the Hochschild homology and cyclic homology of the orbifolds $\mathcal{A}_\theta^{alg} \rtimes \Gamma$ for $\Gamma = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ and $\mathbb{Z}_6$.

Theorem 1.5. *If $\theta \notin \mathbb{Q}$, the Hochschild homology groups are as follows:*

$$H_0(\mathcal{A}_\theta^{alg} \rtimes \Gamma, \mathcal{A}_\theta^{alg} \rtimes \Gamma) \cong \begin{cases} \mathbb{C}^5 & \text{for } \Gamma = \mathbb{Z}_2 \\ \mathbb{C}^7 & \text{for } \Gamma = \mathbb{Z}_3 \\ \mathbb{C}^8 & \text{for } \Gamma = \mathbb{Z}_4 \\ \mathbb{C}^9 & \text{for } \Gamma = \mathbb{Z}_6. \end{cases}$$

$$H_1(\mathcal{A}_\theta^{alg} \rtimes \Gamma, \mathcal{A}_\theta^{alg} \rtimes \Gamma) \cong 0 \text{ for all finite subgroups } \Gamma \subset SL(2,\mathbb{Z}).$$

$$H_2(\mathcal{A}_\theta^{alg} \rtimes \Gamma, \mathcal{A}_\theta^{alg} \rtimes \Gamma) \cong \mathbb{C} \text{ for all finite subgroups } \Gamma \subset SL(2,\mathbb{Z}).$$

And $H_k(\mathcal{A}_\theta^{alg} \rtimes \Gamma, \mathcal{A}_\theta^{alg} \rtimes \Gamma) \cong 0$ *for all* $k > 3$ *and all finite subgroups* $\Gamma \subset SL(2,\mathbb{Z})$.

We remark that the result of Theorem 1.5 for $\Gamma = \mathbb{Z}_2$ is obtained in [O] and the group $HH_0(\mathcal{A}_\theta^{alg} \rtimes \Gamma)$ in Theorem 1.5 is proved in [B]. Our Theorem 1.5 completes the computation of all Hochschild homology groups for $\mathcal{A}_\theta^{alg} \rtimes \Gamma$ for all $\Gamma \subset SL(2,\mathbb{Z})$. The periodic cyclic homology groups of $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2$ are as follows:

Theorem 1.6. *If $\theta \notin \mathbb{Q}$,* $HC_{even}(\mathcal{A}_\theta^{alg} \rtimes \Gamma) \cong \begin{cases} \mathbb{C}^6 & \text{for } \Gamma = \mathbb{Z}_2 \\ \mathbb{C}^8 & \text{for } \Gamma = \mathbb{Z}_3 \\ \mathbb{C}^9 & \text{for } \Gamma = \mathbb{Z}_4 \\ \mathbb{C}^{10} & \text{for } \Gamma = \mathbb{Z}_6. \end{cases}$

The odd homology $HC_{odd}(\mathcal{A}_\theta^{alg} \rtimes \Gamma) \cong 0$ *for all finite subgroups* $\Gamma \subset SL(2,\mathbb{Z})$.

1.2.2 Cohomology groups of the algebraic non-commutative torus orbifold

Theorem 1.7. *If $\theta \notin \mathbb{Q}$ the Hochschild cohomology of $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2$ can be described as follows,*

$$H^0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*) \cong \mathbb{C}^5,\ H^1(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*) \cong 0 \ \text{and}$$
$$H^2(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*) \cong \mathbb{C}.$$

Theorem 1.8 (Periodic Cyclic Cohomology). $HC^{even}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*) \cong \mathbb{C}^6$ *and*

$$HC^{odd}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*) = 0.$$

1.2.3 Homology of the smooth $\mathbb{Z}_2$ smooth torus orbifold, $\mathcal{A}_\theta \rtimes \mathbb{Z}_2$

Theorem 1.9. *For $\theta \notin \mathbb{Q}, H_2(\mathcal{A}_\theta \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta \rtimes \mathbb{Z}_2)^*) \cong \mathbb{C}$.*

We also examine the $\mathbb{Z}_2$ invariant cycles in $H_\bullet(\mathcal{A}_\theta, \mathcal{A}_\theta)$ for $\bullet = 0, 1$.

Theorem 1.10. *For $\theta \notin \mathbb{Q}$ satisfying Diophantine condition, we have*

$$H_1(\mathcal{A}_\theta, \mathcal{A}_\theta)^{\mathbb{Z}_2} = 0,\ \text{and}\ H_0(\mathcal{A}_\theta, \mathcal{A}_\theta)^{\mathbb{Z}_2} \cong \mathbb{C}.$$

We enlist some of the ingredients of our computations to provide an overview of the thesis work. These ingredients come from different fields of mathematics.

1. The Projective Resolution: We use the adjusted Connes' projective resolution for $\mathcal{A}_\theta$ to obtain a projective resolution for $\mathcal{A}_\theta^{alg}$.

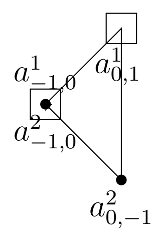
2. Paracyclic Modules: We apply the theory of paracyclic modules in our computations. This theory of paracyclic modules and spectral sequence for crossed product algebras is mainly based on the results of Ezra Geztler and J.S.D. Jones [GJ]. With such usage, we have reduced the homology computation of a particular crossed product algebra into the homology of several other simpler algebras.

3. Lattice Plots: For studying the homology groups of the individual cases we obtain from the spectral splitting of the homology of the crossed product algebra, we introduce some combinatorial tools to study the space $ker\partial/im\partial$. These tools determine the dimension of the group with a pictorial understanding of the (co)homology (co)cycles. More concretely, we plot the elements appearing in the kernel equation on the lattice plane $\mathbb{Z}^2$ and then use the image equations to study the homology. We use some induction arguments on plotted elements along with some combinatorial techniques as a key step to our proofs.

   To illustrate this, consider the following kernel equation:

$$\lambda^{-0}a^1_{-1,0} - \lambda^0 a^1_{0,1} = a^2_{-1,0} - a^2_{0,-1}.$$

   Here $a^1_{-1,0}, a^1_{0,1}, a^2_{-1,0}$ and $a^2_{0,-1}$ are elements in $ker\partial$. This can be drawn as follows:



By plotting all the non-zero lattice elements of a given kernel solution in the way described above, we have a pictorial representation of the relations that these lattice points have between themselves.

1.2.4 Chern-Connes character

Recently [ELPH], explicit projections of the smooth orbifold $\mathcal{A}_\theta \rtimes \Gamma$ generating the $K_0(\mathcal{A}_\theta \rtimes \Gamma)$ were calculated. However we do not know the description of the group $K_0(\mathcal{A}_\theta^{alg} \rtimes$

Γ), for the algebraic orbifold. A close observation reveals that except for the projection having irrational trace, all others are indeed projections in their corresponding algebraic orbifolds $\mathcal{A}_\theta^{alg} \rtimes \Gamma$. We study the pairing of these finitely generated projective modules with cyclic cocyles of $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2$. Such a computation can be viewed as the generalization of the Atiyah-Singer index theorem [AS] to the noncommutative algebraic torus. And furthermore, our computations shed some light towards the full description of $K_0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)$.

1.3 Geometric interpretation of Γ action

Given a matrix $t \in SL(2, \mathbb{Z})$ acting on $\mathbb{T}^2$, we look for the points fixed by this action. After we have identified $\mathbb{T}^2$ with $\mathbb{R}^2/\mathbb{Z}^2$, a fixed point on T^2 under this action corresponds to $(x, y) \in \mathbb{R}^2$ such that $t\cdot(x, y)-(x, y) \in \mathbb{Z}^2$. We observe that the number of points fixed in $\mathbb{R}^2/\mathbb{Z}^2$ by the action of t is same as the dimension of the Hochschild homology group $H_0(A_\theta^{alg}, {}_tA_\theta^{alg})$. This phenomenon occurs similarly in the computation of the Hochschild (co)homology of the Weyl algebra with a finite group action [AFLS] and in the paper [TPPN]. This is interesting as we now have more similarities in the homological properties of classical torus with the algebraic non-commutative torus algebra. Also, we have several other indications that this similarity would prevail for the smooth non-commutative torus. With this we have a geometrical understanding to the non-commutative torus and its associated orbifolds.

1.4 Conjectures

To our amazement we see that the groups $HC^{even}(\mathcal{A}_\theta^{alg} \rtimes \Gamma)$ and the group $K_0(\mathcal{A}_\theta \rtimes \Gamma)$ calculated in [ELPH] have same dimension. With a close reference to recent researches in related fields, we conjecture the following.

Conjecture 1.11. $K_0(\mathcal{A}_\theta^{alg} \rtimes \Gamma) \cong \begin{cases} \mathbb{Z}^5 & \text{for } \Gamma = \mathbb{Z}_2 \\ \mathbb{Z}^7 & \text{for } \Gamma = \mathbb{Z}_3 \\ \mathbb{Z}^8 & \text{for } \Gamma = \mathbb{Z}_4 \\ \mathbb{Z}^9 & \text{for } \Gamma = \mathbb{Z}_6. \end{cases}$

Conjecture 1.12. *For $\theta \notin \mathbb{Q}$, we conjecture that*

$$HC_{even}(\mathcal{A}_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{C}^6, HC_{odd}(\mathcal{A}_\theta \rtimes \mathbb{Z}_2) = 0.$$

1.5 Organization of the thesis

The thesis is presented in five chapters. In chapter one we give an overview of the motivation and background of our work. We present a summary of our results and the methods used in proving them. We end this chapter with a geometrical interpretation of the obtained results.

In chapter two we present some preliminary materials to our results. We recall the basic definitions of C*-algebras followed by an introduction to Hochschild and cyclic (co)homology. We also give the readers a panorama view to the theory of paracyclic categories and its application towards the computation of the cyclic (co)homology of the crossed product algebras.

In chapter three we prove all the results pertaining to the Hochschild and cyclic homology of the algebraic non-commutative toroidal orbifold $\mathcal{A}_\theta^{alg} \rtimes \Gamma$ for all finite subgroups $\Gamma \subset SL(2, \mathbb{Z})$.

In chapter four we calculate the Hochschild and cyclic cohomology of the algebraic non-commutative toroidal orbifold $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2$. We also compute the Chern-Connes pairing of the even cycli cocycles with the known $K_0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)$ elements.

Finally, in chapter five, we present the readers partial results of the smooth non-commutative toroidal orbifold $\mathcal{A}_\theta \rtimes \mathbb{Z}_2$. We end this thesis with a conjecture on the full description of $HC_{even}(\mathcal{A}_\theta \rtimes \mathbb{Z}_2)$ and $HC_{odd}(\mathcal{A}_\theta \rtimes \mathbb{Z}_2)$.

CHAPTER II

Background

2.1 C*-algebras

A Banach algebra $\mathfrak{A}$ is a complex normed algebra which is complete as a topological space and satisfies

$$||AB|| \leq ||A||||B|| \text{ for all } A, B \in \mathfrak{A}.$$

A Banach *-algebra is a complex Banach algebra $\mathfrak{A}$ with a conjugate linear involution *(called the adjoint) which is an anti-isomorphism. That is, for all A, B in $\mathfrak{A}$ and λ in $\mathbb{C}$,

$$(A+B)^* = A^* + B^*$$

$$(\lambda A)^* = \bar{\lambda} A^*$$

$$A^{**} = A$$

$$(AB)^* = B^* A^*.$$

A C*-algebra is a Banach algebra with the additional norm condition

$$||A^* A|| = ||A||^2.$$

Example 2.1. The algebra of all bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is a C*-algebra with the usual adjoint(*) operation.

Remark 2.2. A norm-closed subalgebra of a C*-algebra, closed under adjoints is a C*-algebra.

Example 2.3. $C^*(T)$, the norm closed self-adjoint unital subalgebra of $\mathcal{B}(\mathcal{H})$ generated by an operator $T \in \mathcal{B}(\mathcal{H})$, containing T, T^* and I, is a C*-algebra.

Example 2.4. Let X be a locally compact Hausdorff space. Then $C_0(X)$, the space of all continuous functions on X vanishing at infinity, forms a C*-algebra with complex conjugation as the adjoint operation.

Definition 2.5. The *spectrum* of an element A of a unital Banach algebra $\mathfrak{A}$ is the set

$$\sigma(A) := \{\lambda \in \mathbb{C} | \lambda I - A \text{ is not invertible } \}$$

Definition 2.6. A *multiplicative linear functional* on a commutative Banach algebra $\mathfrak{A}$ is a non-zero homomorphism of $\mathfrak{A}$ into $\mathbb{C}$. This set of all multiplicative linear functionals on $\mathfrak{A}$, $\mathcal{M}_{\mathfrak{A}}$, is called as *maximal ideal space.*

The maximal ideal space is a locally compact Hausdorff space endowed with the weak*-topology on the Banach space dual of $\mathfrak{A}$.

Hence, we can define the *Gelfand transform* Θ of a commutative Banach algebra $\mathfrak{A}$ into $C_0(\mathcal{M}_{\mathfrak{A}})$ by $\Theta(A) = \widehat{A}$, where

$$\widehat{A}(\varphi) = \varphi(A) \text{ for } \varphi \in \mathcal{M}_{\mathfrak{A}}.$$

Theorem 2.7. *The Gelfand transform is a contractive algebra homomorphism of an unital abelian Banach algebra $\mathfrak{A}$ into $C_0(\mathcal{M}_{\mathfrak{A}})$. The image algebra separates the points of $\mathcal{M}_{\mathfrak{A}}$.*

From the above theorem, one has the following.

Corollary 2.8. *In a unital commutative Banach algebra $\mathfrak{A}$, A is invertible iff $\widehat{A}$ is invertible, which is precisely when $\widehat{A}$ does not vanish on $\mathcal{M}_{\mathfrak{A}}$. Thus,*

$$\sigma(A) = \sigma(\widehat{A}) = \{\varphi(A) | \varphi \in \mathcal{M}_{\mathfrak{A}}\}.$$

Theorem 2.9 (Gelfand-Naimark)**.** *Let $\mathfrak{A}$ be an abelian C^*-algebra. Then the Gelfand transform Θ is an isometric *-isomorphism of $\mathfrak{A}$ onto $C_0(\mathcal{M}_{\mathfrak{A}})$.*

2.2 Non-commutative smooth torus

Topologically we can define the classical two-torus, $\mathbb{T}^2$ by the quotient space:

$$\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2.$$

Using the Fourier transformation one concludes that the algebra $C^\infty(\mathbb{T}^2)$ is a commutative Fourier algebra generated by two generators.

$$C^\infty(\mathbb{T}^2) = \{ \sum_{(n,m)\in\mathbb{Z}^2} a_{n,m} U_1^n U_2^m | U_2 U_1 = U_1 U_2, (a_{n,m}) \in \mathcal{S}(\mathbb{Z}^2)\}$$
$$||a||_k = sup_{(m,n)\in\mathbb{Z}^2} |a_{n,m}|(1 + |m| + |n|)^k < \infty, \forall k.$$

The algebra $C^\infty(\mathbb{T}^2)$ is a dense sub-algebra of $C(\mathbb{T}^2)$. In the light of the Gelfand Naimark duality one can define for each $\theta \in \mathbb{R}$ a non-commutative (continuous) torus algebra $\bar{A}_\theta$ and its dense *- subalgebra $\mathcal{A}_\theta$ as non-commutative analogues to $C(\mathbb{T}^2)$ and its dense subalgebra $C^\infty(\mathbb{T}^2)$.

Consider the following two unitary operators on $\mathcal{H} = L^2(S^1)$, U_1 and U_2

$$U_1 f(t) = f(t - \theta) \text{ and } U_2 f(t) = e^{2\pi i t} f(t).$$

It can be seen that

$$U_1 U_2 f(t) = (U_2 f)(t - \theta) = e^{2\pi i (t-\theta)} f(t - \theta) = e^{2\pi i (t-\theta)} (U_1 f)(t) = e^{-2\pi i \theta} (U_2 U_1 f)(t).$$

Hence,

$$U_2 U_1 = e^{2\pi i \theta} U_1 U_2 \tag{2.1}$$

and $\bar{A}_\theta$ is the universal C*-algebra generated by unitaries U_1 and U_2. $\mathcal{A}_\theta$ is the pre-C*-algebra dense in $\bar{A}_\theta$.

$$\mathcal{A}_\theta = \{ \sum_{(n,m)\in\mathbb{Z}^2} a_{n,m} U_1^n U_2^m | U_2 U_1 = e^{2\pi i\theta} U_1 U_2, (a_{n,m}) \in \mathcal{S}(\mathbb{Z}^2)\}. \tag{2.2}$$

By considering the subalgebra of $\mathcal{A}_\theta$ consisting of the elements in $\mathcal{A}_\theta$ which are finitely supported over the lattice $\mathbb{Z}^2$, we get the algebraic non-commutative torus, $\mathcal{A}_\theta^{alg}$.

$$\mathcal{A}_\theta^{alg} = \{ \sum_{(n,m)\in\mathbb{Z}^2} a_{n,m} U_1^n U_2^m | U_2 U_1 = e^{2\pi i\theta} U_1 U_2; a_{n,m} = 0 \text{ for all but finitely many}\}.$$

2.3 Crossed product of C*-algebras

Crossed product C*-algebras were introduced as a tool for making a systematic study of groups action on C*-algebras as automorpshims. They provide a larger algebra which encodes the original C*-algebra and the group action. The consequence has also been to produce new classes of interesting algebras, and new ways of looking at old ones.

A *C*-dynamical system* $(\mathfrak{A}, G, \alpha)$ consists of a C*-algebra $\mathfrak{A}$ together with a homomorphism α of a locally compact group G into $Aut(\mathfrak{A})$. We will denote by α_s the automorphism $\alpha(s)$ for s in G. Given a C*-dynamical system, a *covariant representation* is a pair (π, U) where π is a *-representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ and $s \to U_s$ is a unitary representation of G on the same space such that for all $A \in \mathfrak{A}$ and $s \in G$

$$U_s \pi(A) U_s^* = \pi(\alpha_s(A)). \tag{2.3}$$

We shall restrict our attention to the discrete group crossed product. The Haar measure on these discrete groups are just the counting measure. Let G be a countable discrete group, and let $(\mathfrak{A}, G, \alpha)$ be a C*-dynamical system. The space of continuous compactly supported $\mathfrak{A}$-valued functions on G is just the algebra $\mathfrak{A}G$ of all finite sums $f = \sum_{t\in G} A_t t$

with coefficients in $\mathfrak{A}$. Multiplication is determined by the formal rule $tAt^{-1} = \alpha_t(A)$. If $g = \sum_{t\in G} B_t t$ is another finite sum, then

$$
\begin{aligned}
fg = \sum_{t\in G}\sum_{u\in G} A_t t B_u u &= \sum_{t\in G}\sum_{u\in G} A_t (tB_u t^{-1}) tu \\
&= \sum_{t\in G}\sum_{u\in G} A_t \alpha_t(B_u) tu = \sum_{s\in G}(\sum_{t\in G} A_t\alpha_t(B_{t^{-1}s})s
\end{aligned}
$$

which is just a twisted convolution product. The adjoint is determined by the rule $s^* = s^{-1}$, so that

$$(As)^* = s^*A^* = s^{-1}A^*ss^{-1} = \alpha_s^{-1}(A^*)s^{-1}.$$

Hence,

$$f^* = \sum_{t\in G} \alpha_t(A^*_{t^{-1}})t.$$

It is easy to check that a covariant representation (π, U) of $(\mathfrak{A}, G, \alpha)$ yields a *-representation of $\mathfrak{A}G$ through the following formula

$$\sigma(f) = \sum_{t\in G} \pi(A_t)U_t.$$

One can prove that there is a 1-1 correspondence between *-representations of $\mathfrak{A}G$ and covariant representations of $(\mathfrak{A}, G, \alpha)$.

Let $\pi : \mathfrak{A} \to L(\mathcal{H}_0)$ be a faithful representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}_0$. Let $\mathcal{H} = l^2(G, \mathcal{H}_0)$ be the space of elements $\zeta = (\zeta_g)_{g\in G} \in \bigoplus_{g\in G} \mathcal{H}_0$ with the following norm

$$\langle (\zeta_g)_{g\in G}, (\eta_g)_{g\in G}\rangle = \sum_{g\in G} \langle \zeta_g, \eta_g\rangle.$$

We then define the map $\sigma : \mathfrak{A}G \to L(\mathcal{H})$ as follows. For $f = \sum A_t \cdot t \in \mathfrak{A}G$, we define

$$(\sigma(f)\zeta)_h = \sum_{t\in G} \pi(\alpha_h^{-1}(A_t))(\zeta_{t^{-1}h}) \ .$$

For a finite group G, the *maximal crossed product* $\mathfrak{A} \times_\alpha G$ is the C*-algebra $\mathfrak{A}G$ with the C*-algebra norm

$$||f|| = sup_\sigma ||\sigma(f)||.$$

It is easy to check that $\mathfrak{A}G$ is complete with the above norm. By the standard theory, the norm $||f|| = sup_\sigma ||\sigma(f)||$ is therefore the *only* norm in which $\mathfrak{A}G$ is a C*-algebra. In particular this means that the norm $|| \cdot ||$ does not depend on π.

2.4 K-theory

K_0 is a functor that assigns an ordered abelian group to each ring based on the structure of idempotents in the matrix algebra over the ring. It turns out to be a useful and frequently computable algebraic invariant for C*-algebras.

In a ring $\mathcal{R}$, say that two idempotents P and Q are (von Neumann) *equivalent* if there are elements X, Y in $\mathcal{R}$ such that

$$P = XY \text{ and } Q = YX$$

In a C*-algebra $\mathfrak{A}$, two projections P and Q are **-equivalent* if there is a partial isometry X in $\mathfrak{A}$ such that

$$P = X^*X \text{ and } Q = XX^*.$$

It is evident that this an equivalence relation. In a C*-algebra, it is more natural and convenient to consider only projections rather than arbitrary idempotents. The first easy proposition shows that this choice does not affect things. Moreover, since equivalence and *-equivalence are the same relation on projections in a C*-algebra, the following theorem establishes an equivalence between these equivalences.

Proposition 2.10. *In a C^*-algebra, every idempotent is equivalent to a projection; and equivalent projections are *-equivalent.*

Let $\mathcal{P}(\mathfrak{A})$ denote the collection of all projections in $\cup_{n\geq 1}\mathcal{M}_n(\mathfrak{A})$. This is a semigroup under the operation of direct sum. Say that two projections in $\mathcal{P}(\mathfrak{A})$, say P in $\mathcal{M}_m(\mathfrak{A})$ and Q in $\mathcal{M}_n(\mathfrak{A})$ with $m \leq n$, are *equivalent*(write $P \sim Q$) if $P \oplus 0_{n-m}$ is equivalent to Q in $\mathcal{M}_n(\mathfrak{A})$. Say that P and Q are *stably equivalent* (write $P \approx Q$) if there is a projection R in $\mathcal{P}(\mathfrak{A})$ so that $P \oplus R \sim Q \oplus R$. Stable equivalence is evidently an equivalence relation. Let $K_0^+(\mathfrak{A})$ denote the collection of stable equivalence classes, denoted by $[P]$ for P in $\mathcal{P}(\mathfrak{A})$.

Lemma 2.11. *Let P, P', Q, Q' and R be projections in $\mathcal{P}(\mathfrak{A})$.*

(a) If $P \approx P'$ and $Q \approx Q'$, then $P \oplus Q \approx P' \oplus Q'$.

(b) $P \oplus Q \sim Q \oplus P$.

(c) If $PQ = 0$ in $\mathcal{M}_n(\mathfrak{A})$, then $P + Q \sim Q + P$.

(d) If $P \oplus Q \approx Q \oplus P$, then $P \approx Q$.

Hence $K_0^+(\mathfrak{A})$ is an abelian cancellation semigroup with the operation $[P]+[Q] := [P\oplus Q]$ and zero element $[0]$.

An abelian cancellation semigroup S generates an abelian group G, called the *Grothendieck group* of S, consisting of all 'differences' of elements of S modulo the natural equivalence. That is, consider the collection of all formal differences $s - t$ for s, t in S and identify $s_1 - t_1$ with $s_2 - t_2$ iff $s_1 + t_2 = s_2 + t_1$ in S.

Define addition by the rule $(s_1 - t_1) + (s_2 - t_2) = (s_1 + s_2) - (t_1 + t_2)$. It is routine to verify that addition is well defined, commutative and associative. The inverse of $s - t$ is $t - s$, and the zero is $0 - 0$. So G is an abelian group.

If $\mathfrak{A}$ is a C*-algebra, we define $K_0(\mathfrak{A})$ to be the Grothendieck group of $K_0^+(\mathfrak{A})$.

2.5 Hochschild homology

2.5.1 Hochschild complex and Hochschild homology groups

Let k be a field and A be a k-algebra. A *bimodule* over A is a symmetric k-module M on which A operates linearly on the left and on the right in such a way that $(am)a' = a(ma')$ for $a, a' \in A$ and $m \in M$. The actions of A and k on M are always supposed to be compatible, for instance: $(\lambda a)m = \lambda(am) = a(\lambda m)$, $\lambda \in k$, $a \in A$, $m \in M$. When A has a unit element 1, we always assume that $1 \cdot m = m \cdot 1 = m$ for all $m \in M$. Under this unital hypothesis, the bimodule M is equivalent to a right $A \otimes A^{op}$-module via $m(a' \otimes a) = ama'$.

The product map of A is usually denoted as $\mu : A \otimes A \to A, \mu(a, b) = ab$.

Consider the vector space $C_n(A, M) := M \otimes_k A^{\otimes n}$. The *Hochschild boundary* is th k-linear map $b : M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$ given by the formula

$$b(m, a_1, ..., a_n) := (ma_1, a_2, ..., a_n) + \sum_{i=1}^{n-1} (-1)^i (m, a_1, ..., a_i a_{i+1}, ..., a_n) \\ + (-1)^n (a_n m, a_1, ..., a_{n-1}).$$

This formula makes sense because A is an algebra and M is an A-bimodule. The main example we are going to look at is when $M = A$. It is useful to introduce the operators $d_i : M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$ given by

$$d_0(m, a_1, ..., a_n) := (ma_1, a_2, ..., a_n),$$

$$d_i(m, a_1, ..., a_n) := (m, a_1, ..., a_i a_{i+1}, ..., a_n) \text{ for } 1 \leq i < n,$$

$$d_n(m, a_1, ..., a_n) := (a_n m, a_1, ..., a_{n-1}).$$

With this notation, one has:

$$b = \sum_{i=0}^{n} (-1)^i d_i.$$

It is direct to check that $b \circ b = 0$ in the above complex.

Therefore, we get a complex (C_*, b), which is called as the *Hochschild complex.*

$$C(A, M) = C_*(A, M) = ... \xrightarrow{b} M \otimes_k A^{\otimes n} \xrightarrow{b} M \otimes_k A^{\otimes n-1} \xrightarrow{b} ...... \xrightarrow{b} M \otimes_k A \xrightarrow{b} M,$$

where, $C_n(A, M) := M \otimes_k A^{\otimes n}$.

In the case where $M = A$, we get the following Hochschild complex.

$$C(A) = C_*(A) = ... \xrightarrow{b} A^{\otimes n+1} \xrightarrow{b} A^{\otimes n} \xrightarrow{b} ...... \xrightarrow{b} A^{\otimes 2} \xrightarrow{b} A.$$

By definition the *n-th Hochschild homology group* $H_n(A, M)$ of the unital k-algebra A with coefficient in the A-bimodule M is the n-th homology group of the Hochschild complex $(C_*(A, M), b)$. When $M = A$, we also use $HH_n(A)$ to denote the group $H_n(A, A)$.

We compute some lower degree Hochschild homology groups.

The group

$$H_0(A, M) = M_A = M/\{am - ma | a \in A, m \in M\}$$

is also called the *module of covariants* of M by A. Let $[A, A]$ denote the additive commutator sub-k-module generated by $[a, a'] = aa' - a'a$, for $a, a' \in A$. Then $HH_0(A) = A/[A, A]$. If A is commutative, then $HH_0(A) = A$. When $A = k$ the Hochschild complex for $M = k$ is

$$... \to k \xrightarrow{1} k \xrightarrow{0} ... \xrightarrow{1} k \xrightarrow{0} k$$

therefore $HH_0(k) = k$ and $HH_n(k) = 0$ for $n > 0$.

Below we compute $HH_1(A)$ when A is unital and commutative.

Definition 2.12. For A unital and commutative, let $\Omega^1_{A/k}$ be the A-module of differentials generated by the k-linear symbols da for $a \in A$ satisfying $d(\mu a + \gamma b) = \mu da + \gamma db$ for $\mu, \gamma \in k$ and $a, b \in A$. It also satisfies the relation

$$d(ab) = a(db) + (da)b; \text{ for } a, b \in A.$$

Proposition 2.13. *If A is a unital and commutative, then, there is a canonical isomorphism between $HH_1(A)$ and the algebra of differentials on A, $\Omega^1_{A/k}$. If M is a symmetric bimodule (i.e. $am = ma$ for all $a \in A$ and $m \in M$), then $H_1(A, M) \cong M \otimes_A \Omega^1_{A/k}$.*

2.5.2 Bar complex

Let $A^e = A \otimes A^{op}$ be the enveloping algebra of the associative and unital algebra A. The left A^e-module structure of A is given by $(a \otimes a')c = aca'$. Consider the following complex, called the *bar complex*

$$C^{bar}_*(A) = ... \xrightarrow{b'} A^{\otimes n+1} \xrightarrow{b'} A^{\otimes n} \xrightarrow{b'} ...... \xrightarrow{b'} A^{\otimes 2}$$

where $A^{\otimes 2}$ is in degree 0 and where $b' = \sum_{i=0}^{n-1}(-1)^i d_i$. With the augmentation map $b' = \mu : A \otimes A \to A$, sending $(a \otimes b)$ to ab.

Proposition 2.14. *Let A be a unital k-algebra. The complex $C^{bar}_*(A)$ is a resolution of the A^e-module A.*

Proof. It is immediate to see that the cokernel of the last map in C_* is the augmentation map μ. The operator

$$s : A^{\otimes n} \to A^{\otimes n+1}, (a_1, a_2, ..., a_n) \mapsto (1, a_1, ..., a_n)$$

satisfies the formula $d_i s = s d_{i-1}$ for $i = 1, 2, ..., n-1$ and $d_0 s = id$. Therefore $b's + sb' = id$ and s is a contracting homotopy, showing that the b'-complex is acyclic. □

It can be seen from the above proposition that $H_\bullet(A, M) = Tor_\bullet(A, M)$, where Tor is the functor on the category of $A \otimes A^{op}$-modules.

2.6 Hochschild cohomology

With the notations above, we can define Hochschild homology to be the homology of the complex $M \otimes_{A^e} C_*^{bar}(A)$, where $C_*^{bar}(A)$ is the bar resolution of A. So we define *Hochschild cohomology* of A with coefficients in M as

$$H^n(A, M) = H_n(Hom_{A^e}(C_*^{bar}(A), M)).$$

The coboundary map β' in the Hom-complex is given by

$$\beta'(\varphi) = -(-1)^n \varphi \circ b'$$

for any cochain φ in $Hom_{A^e}(C_n^{bar}(A), M)$. Explicitly, such a cochain φ is completely determined by a k-linear map $f : A^{\otimes n} \to M$.

The formula for the coboundary map is

$$\begin{aligned} \beta'(f)(a_1, ...a_{n+1}) = a_1 f(a_2, ..., a_{n+1}) + \sum_{0<i<\leq n} (-1)^i f(a_1, ..., a_i a_{i+1}, ..., a_{n+1}) \\ + (-1)^{n+1} f(a_1, ..., a_n) a_{n+1}. \end{aligned}$$

Hence, $H^n(A, M)$ is the homology of the complex $(C^n(A, M), \beta)$, where

$$C^n(A, M) = Hom_k(A^{\otimes n}, M).$$

The cohomological group $H^n(A, M)$ is a $Z(A)$-module where $Z(A)$ is the center of A. For a fixed A, $H^n(A, M)$ is a functor from the category of A-bimodules to the category of $Z(A)$-modules. When commutative, $H^n(A, M)$ is also an A-module.

We compute some lower degree Hochschild cohomology groups.

For $n = 0$, we have:

$$H^0(A, M) = M^A = \{m \in M | am = ma \text{ for any } a \in A\}.$$

For $n = 1$, we have a 1-cocylce as a k-module homomorphism $D : A \to M$ satisfying the identity:

$$D(aa') = aD(a') + D(a)a' \text{ for } a, a' \in A.$$

The k-module of all maps satisfying the above relation is denoted by $Der(A, M)$. It is a coboundary if it has the form $ad_m(a) = [m, a] = ma - am$ for some fixed $m \in M; ad_m$ is called an *inner derivation*. Therefore

$$H^1(A, M) = Der(A, M)/\{ \text{ inner derivations } \}.$$

2.7 Cyclic homology

2.7.1 Cyclic homology groups

Definition 2.15. The cyclic group $\mathbb{Z}_{n+1}$ action on the module $A^{\otimes n+1}$ is given by letting its generator $t = t_n$ act by

$$t_n(a_0, a_1, ..., a_n) = (-1)^n(a_n, a_0, ..., a_{n-1})$$

on the generators of $A^{\otimes n+1}$. It is then extended by linearity; this is called as the *cyclic operator*. The corresponding *norm operator* is $N = 1 + t + t^2 + ... + t^n$.

2.7.2 Connes complex

The cokernel $A^{\otimes n+1}/(1-t)$ of the endomorphism $(1-t)$ of $A^{\otimes n+1}$ is the coinvariant space of $A^{\otimes n+1}$ for the action of the cyclic group $\mathbb{Z}_{n+1}$. We denote this complex by $C_n^\lambda(A) := A^{\otimes n+1}/(1-t)$. We define $HC_n(A) := H_n(C_*^\lambda(A))$, where

$$C_*^\lambda(A) := ... \xrightarrow{b} C_n^\lambda(A) \xrightarrow{b} C_{n-1}^\lambda \xrightarrow{b} ... \xrightarrow{b} C_0^\lambda(A)$$

is called the *Connes complex*.

We can see that $HC_0(A) = HH_0(A) = A/[A, A]$. So if A is commutative, then $HC_0(A) = HH_0(A) = A$.

Remark 2.16. For any unital commutative k-algebra A one has

$$HC_1(A) = \Omega^1_{A/k}/dA.$$

2.7.3 Connes' periodicity exact sequence

Theorem 2.17 (Connes' Periodicity Exact Sequence)**.** *For any associative and not necessarily unital k-algebra A such that k contains $\mathbb{Q}$, there is a natural long exact sequence.*

$$... \to HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} ....$$

2.8 Cyclic cohomology

Let A be an associative algebra, and denote the dual $A^* = Hom(A, k)$ of A by $C^0(A)$. More generally we put $C^n(A) = Hom(A^{\otimes n+1}, k)$.

A cochain f in $C^n(A)$ is said to be *cyclic* if it satisfies the relation

$$f(a_0, a_1, ..., a_n) = (-1)^n f(a_n, a_0, ..., a_{n-1}), \text{ for } a_i \in A.$$

These *cyclic cochains* form a sub-k-module of $C^n(A)$, $C^n_\lambda(A)$. Further observation reveals that b^* is a well defined chain map on $C^*_\lambda(A)$. We then define the *cyclic cohomology* of A to be the homology of the cochain complex $C^*_\lambda(A)$:

$$HC^n(A) := H^n(C^*_\lambda(A)).$$

2.8.1 Connes periodicity exact sequence for cyclic cohomology

Connes [C] discovered a long exact sequence relating $HC^{n+1}(A)$ and $HH^{n+1}(A)$.

Theorem 2.18 (Connes Periodicity Exact Sequence)**.** *Let A be any associative k-algebra. We have the following long exact sequence*

$$... \xrightarrow{I} HH^n(A) \xrightarrow{B} HC^n(A) \xrightarrow{S} HC^{n+1}(A) \xrightarrow{I} HH^{n+1}(A) \xrightarrow{B} ...,$$

2.9 Hochschild and cyclic cohomology of $C^\infty(V)$

Hochschild cohomology for a commutative C*-algebra $C^\infty(V)$ for a smooth manifold V, can be seen as the space of *de-Rahm currents* on V.

Definition 2.19. For a compact manifold V, a k de-Rahm current C on it is a functional on the space of k-differential forms on V. We denote by $\mathcal{D}_k$ the space all k de-Rahm currents on V.

Proposition 2.20. *Let V ba a compact smooth manifold, and consider $\mathcal{A} = C^\infty(V)$ as a locally convex topological algebra, then $H^k(\mathcal{A}, \mathcal{A}^*)$ is canonically isomorphic with $\mathcal{D}_k$, the space of de-Rahm currents of dimension k on V. To the (k+1)-linear functional φ is associated the current C such that*

$$\langle C, f^0 df^1 \wedge ... \wedge df^k \rangle = \sum_{\sigma \in S_k} \epsilon(\sigma)\varphi(f^0, f^{\sigma(1)}, f^{\sigma(2)}, ..., f^{\sigma(k)}).$$

Similarly we have an analogous result for cyclic cohomology.

Proposition 2.21. *For each k, $H^k_\lambda(\mathcal{A})$ is canonically isomorphic to the direct sum*

$$Kerb(\subset \mathcal{D}_k) \oplus H_{k-2}(V, \mathbb{C}) \oplus H_{k-4}(V, \mathbb{C}) \oplus ...$$

where $H_*(V, \mathbb{C})$ is the usual de-Rahm homology of the manifold V and b is the de-Rahm homology differential.

2.10 Chern character

Given a commutative and unital k-algebra A(where $\mathbb{Q} \subset k$), there exists a homomorphism

$$ch_0 : K_0(A) \to H^{ev}_{dR}(A).$$

The homomorphism above is called as the Chern map in classical differential geometry. We have similar homomorphism for case when A is noncommutative k-algebra.

Theorem 2.22. *If 2 is regular in k, then the map* $ch^{\lambda}_{0,n} : K_0(A) \to HC_{2n}(A)$*, given by* $ch^{\lambda}_{0,n}([e]) = tr((-1)^n e^{\otimes 2n+1})$*, is well defined and functorial in A.*

2.10.1 Chern-Connes pairing

In the cohomological framework the Chern map can be replaced by a pairing

$$\langle -, - \rangle : K_0(A) \times HC^{2n}(A) \to k.$$

which is constructed as follows. Let $f \in C^{2n}(A)$ be a cyclic cocycle, that is a linear functional $f : \mathcal{M}(A)^{\otimes 2n+1} \to k$, which satisfies the equation $b \circ f = 0$ and which is cyclic. Then the Chern-Connes pairing is given by

$$\langle [e], [f] \rangle = f(e, e, ..., e).$$

It is easy to check that this pairing can be lifted to a pairing

$$\langle -, - \rangle : K_0(A) \times HC^{2n}(A) \to k.$$

This pairing can be viewed as the generalization of the Atiyah-Singer index theorem [AS] for noncommutative geometry [C].

2.11 Paracyclic categories

We describe in this section the theory of paracyclic categories, which is instrumental in calculating the cohomology of the crossed product algebras. We refer the reader to the paper of Ezra Getzler and J.S.D.Jones [GJ] for details therein.

For a unital algebra A over a commutative ring k with a group G acting on it, we have the associated crossed product algebra $A \rtimes G$.

We shall try to understand the cyclic module $(A \rtimes G)^{\natural}(n) := k[G^{n+1}] \otimes A^{n+1}$. Let us denote the elementary tensor

$$(a_0 \otimes g_0) \otimes ... \otimes (a_n \otimes g_n) \in (A \rtimes G)^\natural(n)$$

by $(g_0, ..., g_n | h_0^{-1} a_0, ..., h_n^{-1} a_n)$. where $h_i = g_i \cdots g_n$.

We have the following formulas for d, s and $t = d \cdot s$ acting on $(A \rtimes G)^\natural(n)$:

$$d(g_0, ..., g_n | a_0, ..., a_n) = (g_n g_0, g_1, ..., g_{n-1} | g_n((g^{-1} a_n) a_0), g_n a_1, ..., g_n a_{n-1}),$$

$$s(g_0, ..., g_n | a_0, ..., a_n) = (1, g_0, ..., g_n | 1, a_0, ..., a_n),$$

$$t(g_0, ..., g_n | a_0, ..., a_n) = (g_n, g_0, ..., g_{n-1} | g_n g^{-1} a_n, g_n a_0, ..., g_n a_{n-1}).$$

where, $g = g_0 \cdots g_n$.

Also, we define here a cyclic bimodule, $A\natural G$ whose diagonal is the cyclic module $(A \rtimes G)^\natural$. We define $A\natural G(p,q) := k[G^{p+1}] \otimes A^{q+1}$, spanned by the elementary tensors products which we denote by $(g_0, ..., g_p | a_0, ..., a_q)$. For this bimodule we define the actions of $(\bar{d}, \bar{s}, \bar{t})$ and (d, s, t) as follows:

$$\bar{d}(g_0, ..., g_p | a_0, ..., a_q) = (g_p g_0, g_1, .., g_{p-1} | g_p a_0, ..., g_p a_q),$$

$$\bar{s}(g_0, ..., g_p | a_0, ..., a_q) = (g_0, ..., g_p | 1, a_0, ..., a_q),$$

$$\bar{t}(g_0, ..., g_p | a_0, ..., a_q) = (g_p, g_0, ..., g_{p-1} | g_p a_0, ..., g_p a_q),$$

$$d(g_0, ..., g_p | a_0, ..., a_q) = (g_0, ..., g_p | (g^{-1} a_q) a_0, a_1, ..., a_{q-1}),$$

$$s(g_0, ..., g_p | a_0, ..., a_q) = (g_0, ..., g_p | 1, a_0, ..., a_q),$$

$$t(g_0, ..., g_p | a_0, ..., a_q) = (g_0, ..., g_p | g^{-1} a_q, a_0, .., a_{q-1}),$$

where $g = g_0 \cdots g_p$. Since, $\bar{t}^{p+1} \neq 1 \neq t^{q+1}$, this is not a bi-cyclic structure on $A\natural G(p,q)$, though $T := \bar{t}^{p+1} = t^{q+1}$ satisfies:

$$T(g_0, ..., g_p | a_0, ..., a_q) = (g_0, ..., g_p | g a_0, ..., g a_q).$$

Definition 2.23. For a given algebra A, define the graded module $A_G^\natural(n) := (A\natural G)(0,n) \cong k[G] \otimes A^{n+1}$.

Theorem 2.24 (E. Getzler and J Jones)**.** *If G is finite and $|G|$ is invertible in k, then there is a natural isomorphism of cyclic homology and*

$$HC_\bullet(A \rtimes G; W) = HC_\bullet(H_0(G, A_G^\sharp); W),$$

where $(H_0(G, A_G^\sharp)$ is the cyclic module

$$H_0(G, A_G^\sharp)(n) = H_0(G, k[G] \otimes A^{n+1}).$$

CHAPTER III

Hochschild and cyclic homology

3.1 $SL(2,\mathbb{Z})$ action on $\mathcal{A}_\theta^{alg}$

The special linear group $SL(2,\mathbb{Z})$ acts on $\mathcal{A}_\theta^{alg}$ in the following way. An element

$$g = \begin{bmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{bmatrix} \in SL(2,\mathbb{Z})$$

acts on the generators U_1 and U_2 by:

$$gU_1 = e^{(\pi i g_{1,1} g_{2,1})\theta} U_1^{g_{1,1}} U_2^{g_{2,1}} \text{ and } gU_2 = e^{(\pi i g_{1,2} g_{2,2})\theta} U_1^{g_{1,2}} U_2^{g_{2,2}}$$

To illustrate this we consider the case of $\mathbb{Z}_2$ action on $\mathcal{A}_\theta^{alg}$. With the $SL(2,\mathbb{Z})$ action on $\mathcal{A}_\theta^{alg}$ described above, we see that the generator of $\mathbb{Z}_2$ in $SL(2,\mathbb{Z})$, $g = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ acts on $\mathcal{A}_\theta^{alg}$ as follows:

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} U_1 = U_1^{-1} \text{ and } \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} U_2 = U_2^{-1}.$$

With these actions we have the corresponding crossed product algebra, and we state the main result of the homology groups of algebraic non-commutative torus orbifolds in this section.

Theorem 3.1 (Hochschild homology)**.** *Let Γ be any finite subgroup $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ of $SL(2,\mathbb{Z})$. Then for $\theta \notin \mathbb{Q}$, we have*

$$HH_0(\mathcal{A}_\theta^{alg} \rtimes \Gamma) = \begin{cases} \mathbb{C}^5 & for\ \Gamma = \mathbb{Z}_2 \\ \mathbb{C}^7 & for\ \Gamma = \mathbb{Z}_3 \\ \mathbb{C}^8 & for \Gamma = \mathbb{Z}_4 \\ \mathbb{C}^9 & for\ \Gamma = \mathbb{Z}_6. \end{cases}$$

$$HH_1(\mathcal{A}_\theta^{alg} \rtimes \Gamma) = 0 \ for\ \Gamma \subset SL(2,\mathbb{Z}).$$

$$HH_2(\mathcal{A}_\theta^{alg} \rtimes \Gamma) \cong \mathbb{C} \ for\ \Gamma \subset SL(2,\mathbb{Z}).$$

$$HH_k(\mathcal{A}_\theta^{alg} \rtimes \Gamma) = 0 \ for\ \Gamma \subset SL(2,\mathbb{Z}) \ and\ k \geq 3.$$

According to the results from the paracyclic techniques it is clear that to calculate the homology, we need to calculate $H_0(\Gamma, \mathcal{A}_{\theta,\Gamma}^{alg\natural})$ first. As Γ is abelian, we can conclude that the group homology $H_0(\Gamma, \mathcal{A}_{\theta,\Gamma}^{alg\natural})$ splits the complex into $|\Gamma|$ disjoint parts.

$$H_0(\Gamma, \mathcal{A}_{\theta,\Gamma}^{alg\natural})(n) = H_0(\Gamma, k[\Gamma] \otimes (\mathcal{A}_\theta^{alg})^{\otimes n+1}) = \bigoplus_{t\in\Gamma}((\mathcal{A}_{\theta,t}^{alg})^{\otimes n+1})^\Gamma$$

For each $t \in \Gamma$, the algebra $\mathcal{A}_{\theta,t}^{alg}$ is setwise $\mathcal{A}_\theta^{alg}$ with the Hochschild differential ${}_tb$ in the Hochschild complex $C_\bullet(\mathcal{A}_{\theta,t}^{alg}, \mathcal{A}_{\theta,t}^{alg})$. Using the result in [2] for the algebra $\mathcal{A}_\theta^{alg} \rtimes \Gamma$, we can write its Hochschild homology, $HH_\bullet(\mathcal{A}_\theta^{alg} \rtimes \Gamma)$ as follows

$$HH_\bullet(\mathcal{A}_\theta^{alg} \rtimes \Gamma) = HH_\bullet(H_0(\Gamma, \mathcal{A}_{\theta,\Gamma}^{alg\natural})) = \bigoplus_{t\in\Gamma} HH_\bullet((\mathcal{A}_{\theta,t}^{alg\bullet})^\Gamma).$$

It is enough to calculate $HH_\bullet((\mathcal{A}_{\theta,t}^{alg\bullet})^\Gamma)$ for each $t \in \Gamma$. To calculate these individual homology groups, we describe some lemmas below.

Lemma 3.2. *Let*

$$J_* := 0 \xleftarrow{d_v} A \xleftarrow{d_v} (A^{\otimes 2}) \xleftarrow{d_v} (A^{\otimes 3}) \xleftarrow{d_v} (A^{\otimes 4}) \xleftarrow{d_v} (A^{\otimes 5}) \xleftarrow{d_v} ...$$

be a chain complex. For a given Γ *action on* A*, consider the following chain complex, with chain map* $d_v^\Gamma : (A^{\otimes n})^\Gamma \to (A^{\otimes n-1})^\Gamma$ *induced from the map* $d_v : A^{\otimes n} \to A^{\otimes n-1}$.

$$J_*^\Gamma := 0 \xleftarrow{d_v^\Gamma} A^\Gamma \xleftarrow{d_v} (A^{\otimes 2})^\Gamma \xleftarrow{d_v^\Gamma} (A^{\otimes 3})^\Gamma \xleftarrow{d_v^\Gamma} (A^{\otimes 4})^\Gamma \xleftarrow{d_v^\Gamma} (A^{\otimes 5})^\Gamma \xleftarrow{d_v^\Gamma} \ldots$$

With the Γ *action commuting with the differential* d_v*. We have the following group equality* $H_q(J_*^\Gamma, d_v^\Gamma) = H_q(J_*, d_v)^\Gamma$.

Proof. Consider the following bicomplex E.

$$\begin{array}{ccccccccc}
 & & \downarrow d_v & & \downarrow d_v & & \downarrow d_v & & \\
\longrightarrow & & k[\Gamma^2] \otimes A^3 & \xrightarrow{d_h} & k[\Gamma] \otimes A^3 & \xrightarrow{d_h} & A^3 & \xrightarrow{d_h} & 0 \\
 & & \downarrow d_v & & \downarrow d_v & & \downarrow d_v & & \\
\longrightarrow & & k[\Gamma^2] \otimes A^2 & \xrightarrow{d_h} & k[\Gamma] \otimes A^2 & \xrightarrow{d_h} & A^2 & \xrightarrow{d_h} & 0 \\
 & & \downarrow d_v & & \downarrow d_v & & \downarrow d_v & & \\
\longrightarrow & & k[\Gamma^2] \otimes A & \xrightarrow{d_h} & k[\Gamma] \otimes A & \xrightarrow{d_h} & A & \xrightarrow{d_h} & 0 \\
 & & \downarrow d_v & & \downarrow d_v & & \downarrow d_v & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We know from theory of spectral sequence that $E_0^{p,q}$ can be defined as $H_p(E^{\bullet,q}, d_h)$, which turns out to be 0 for $p > 0$ and $E^{0,q} = (A^q)^\Gamma$. Further calculation reveals that $E_1^{p,q} := H_p(E_0^{p,\bullet}, d_v) = 0$ for $p > 0$ and $E_1^{0,q} = H_q(J_*^\Gamma, d_v)$.

On the other hand if we define $E'^{p,q}_0 := H_q(E^{p,\bullet}, d_v) = k[\Gamma^p] \otimes H_q(J_*, d_v)$, we have $E'^{p,q}_1 := H(E'^{\bullet,q}_0, d_h) = 0$ for $p > 0$ and it equals $(H_q(J_*, d_v))^\Gamma$ at $p = 0$.

Hence, $H_q(J_*^\Gamma, d_v^\Gamma) = H_q(J_*, d_v)^\Gamma$. □

3.2 Revisiting Connes' resolution

Firstly we have to ensure if the projective resolution introduced in [1] can be adjusted to the algebraic noncommutative torus algebra:

Lemma 3.3. *The following is a projective resolution of* $\mathcal{A}_\theta^{alg}$

$$\mathcal{A}_\theta^{alg} \xleftarrow{\epsilon} \mathfrak{B}_\theta^{alg} \xleftarrow{b_1} \mathfrak{B}_\theta^{alg} \bigoplus \mathfrak{B}_\theta^{alg} \xleftarrow{b_2} \mathfrak{B}_\theta^{alg}$$

where, $\mathfrak{B}_\theta^{alg} = \mathcal{A}_\theta^{alg} \otimes (\mathcal{A}_\theta^{alg})^{op}$

$\epsilon(a \otimes b) = ab,$

$b_1(1 \otimes e_j) = 1 \otimes U_j - U_j \otimes 1$ *and*

$b_2(1 \otimes (e_1 \wedge e_2)) = (U_2 \otimes 1 - \lambda \otimes U_2) \otimes e_1 - (\lambda U_1 \otimes 1 - 1 \otimes U_1) \otimes e_2.$

Proof. Since $\epsilon b_1(1 \otimes e_j) = U_j - U_j = 0$, hence we have $im(b_1) \subset ker(\epsilon)$. Let an element $x \in ker(\epsilon)$ be

$$x = \sum a_{\nu,\nu'} X^\nu Y^{\nu'}, \text{ where } X^\nu = U_1^{n_1} U_2^{n_2} \otimes 1, Y^{\nu'} = 1 \otimes U_1^{n_1'} U_2^{n_2'}.$$

Hence, $x = \sum a_{\nu,\nu'} X^\nu (Y^{\nu'} - X^{\nu'})$. Consider the following equality

$$(1 \otimes {U_2}^{n_2})(1 \otimes {U_1}^{n_1}) - ({U_1}^{n_1} \otimes 1)({U_2}^{n_2} \otimes 1) = (1 \otimes {U_2}^{n_2})(\sum_{j=0}^{n_1-1} U_1^j \otimes U_1^{n_1-1-j})(1 \otimes U_1 - U_1 \otimes 1)$$
$$+ ({U_1}^{n_1} \otimes 1)(\sum_{j=0}^{n_2-1} U_2^j \otimes U_2^{n_2-1-j})(1 \otimes U_2 - U_2 \otimes 1).$$

Since $x = \sum a_{\nu,\nu'} X^\nu (Y^{\nu'} - X^{\nu'})$ can be written in terms of the image of the right hand side coefficients in above equation. Hence $ker(\epsilon)$ is generated by $(1 \otimes U_1 - U_1 \otimes 1)$ and $(1 \otimes U_2 - U_2 \otimes 1)$.

Given an element $x = x_1 \otimes e_1 - x_2 \otimes e_2$ where, $x_1, x_2 \in \mathcal{A}_\theta^{alg} \in ker(b_1)$, we have,

$$x_1(1 \otimes U_1 - U_1 \otimes 1) = x_2(1 \otimes U_2 - U_2 \otimes 1).$$

To prove $x \in Imb_2$, it is enough to find $y \in \mathcal{A}_\theta^{alg} \otimes^{op} \mathcal{A}_\theta^{alg}$ such that $x_1 = y(U_2 \otimes 1 - \lambda \otimes U_2)$. Let $Z = (\lambda U_2^{-1} \otimes U_2)$ then we have the following

$$x_1(\sum_{-\infty}^{\infty} Z^k) = 0.$$

Also one can calculate that

$$x_1(\sum_{-\infty}^{\infty} Z^k)(1 \otimes U_1 - U_1 \otimes 1) = x_1(1 \otimes U_1 - U_1 \otimes 1) \sum_{-\infty}^{\infty} (U_2^{-1} \otimes U_2)^k =$$
$$x_2(1 \otimes U_2 - U_2 \otimes 1) \sum_{-\infty}^{\infty} (U_2^{-1} \otimes U_2)^k = 0.$$

After writing $x_1 = \sum a_k Z^k$, where (a_k) is a finitely supported over the elements of the closed subalgebra generated by $U_1 \otimes 1, U_2 \otimes 1, 1 \otimes U_1$. We have

$$x_1 = \sum a_k (Z^k - 1) = \sum a_k (\sum_0^{k-1} Z^k)(Z - 1).$$

It can be observed that for $y := \sum b_k Z^k$ with $b_k = \sum_{j-1=k}^{\infty} a_j$ satisfies, $x_1 = y(U_2 \otimes 1 - \lambda \otimes U_2)$.

Hence $y \in \mathcal{A}_\theta^{alg}$. □

3.3 $HH_\bullet((\mathcal{A}_{\theta,1}^{alg})^\Gamma)$

In this section we compute the Hochschild homology for the $g = 1$ part appearing as one of the decomposed cases while we compute the Hochschild homology for the crossed product algebras $\mathcal{A}_\theta^{alg} \rtimes \Gamma$. We calculate these groups for each of the four cases, corresponding to the groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$.

We notice that from Lemma 3.2 we can simplify the group as follows

$$HH_\bullet((\mathcal{A}_{\theta,1}^{alg})^\Gamma) = (HH_\bullet(\mathcal{A}_{\theta,1}^{alg}))^\Gamma$$

BY the Kozul complex in Lemma 3.3, the Hochschild homology of $\mathcal{A}_\theta^{alg}$ is computed by the following complex

$$0 \leftarrow \mathcal{A}_\theta^{alg} \xleftarrow{1\otimes_{\mathfrak{B}_\theta^{alg}} b_1} \mathcal{A}_\theta^{alg} \bigoplus \mathcal{A}_\theta^{alg} \xleftarrow{1\otimes_{\mathfrak{B}_\theta^{alg}} b_2} \mathcal{A}_\theta^{alg}$$

We have

$$H_2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg}) = ker(1 \otimes b_2),$$

where,

$$(1 \otimes b_2)(a \otimes I) = a \otimes_{\mathfrak{B}_\theta^{alg}} (U_2 \otimes I - \lambda \otimes U_2) \otimes e_1 - a \otimes_{\mathfrak{B}_\theta^{alg}} (\lambda U_1 \otimes I - I \otimes U_1) \otimes e_2 =$$
$$(aU_2 - \lambda U_2 a, U_1 a - \lambda a U_1) \ .$$

It is clear that $H_2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg}) = \langle \varphi \in \mathcal{A}_\theta^{alg} | \varphi_{n,m-1} = \lambda^{n+1}\varphi_{n,m-1} = \lambda^m \varphi_{n,m-1} \rangle$. To solve the above condition on a cycle in $H_2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg})$, we observe that we need to have $m = 0$ for the second equality to hold and also we need that $n = -1$ for the first equality. Hence we see that $\varphi_{-1,-1}U_1^{-1}U_2^{-1}$, the elements of $\mathcal{A}_\theta^{alg}$ supported at $U_1^{-1}U_2^{-1}$ is the required generator for $H_2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg})$.

To calculate the Γ invariant algebra we need to push this cycle $\varphi_{-1,-1}U_1^{-1}U_2^{-1}$ into the bar complex C_* using the map $h_2 : J_2(\mathcal{A}_\theta^{alg}) \to C_2(\mathcal{A}_\theta^{alg})$, where

$$h_2(1 \otimes (e_1 \wedge e_2)) = I \otimes U_2 \otimes U_1 - \lambda \otimes U_1 \otimes U_2.$$

After considering the action of $t \in \Gamma$ on the element $(1 \otimes h_2)(\varphi_{-1,-1}U_1^{-1}U_2^{-1})$. Then we pull back the element $t \cdot ((1 \otimes h_2)(\varphi_{-1,-1}U_1^{-1}U_2^{-1}))$ to the Kozul complex to check its invariance. Using this technique we have the following computations.

3.3.1 $H_2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg})^\Gamma$

$(1 \otimes h_2)(\varphi_{-1,-1}U_1^{-1}U_2^{-1}) = \varphi_{-1,-1}(1 \otimes h_2)(U_1^{-1}U_2^{-1})$

$= a_{-1,-1}(U_1^{-1}U_2^{-1} \otimes U_2 \otimes U_1 - \lambda U_1^{-1}U_2^{-1} \otimes U_1 \otimes U_2)$

After $t \in \mathbb{Z}_2$ acts on the element $a_{-1,-1}U_1^{-1}U_2^{-1} \otimes U_2 \otimes U_1 - \lambda U_1^{-1}U_2^{-1} \otimes U_1 \otimes U_2$ we get

$$a_{-1,-1}(U_1U_2 \otimes U_2^{-1} \otimes U_1^{-1} - \lambda U_1U_2 \otimes U_1^{-1} \otimes U_2^{-1})$$

Now, we consider the element $(1 \otimes k_2)(a_{-1,-1}(U_1U_2 \otimes U_2^{-1} \otimes U_1^{-1} - \lambda U_1U_2 \otimes U_1^{-1} \otimes U_2^{-1}))$, where for $\nu = (n_1, n_2)$ and $\mu = (m_1, m_2)$ the chain map $k_2 : C_2(\mathcal{A}_\theta^{alg}) \to J_2(\mathcal{A}_\theta^{alg})$ is given by the following formula

$$k_2(I \otimes U^\nu \otimes U^\mu) =$$
$$(U_1 \otimes I)^{n_1} \frac{\lambda^{n_2 m_1}(U_1 \otimes I)^{m_1} - \lambda^{m_1 m_2}(I \otimes U_1)^{m_1}}{\lambda^{n_2}(U_1 \otimes I) - \lambda^{-m_2}(I \otimes U_1)} \frac{(U_2 \otimes I)^{n_2} - \lambda^{n_2}(I \otimes U_2)^{n_2}}{(U_2 \otimes I) - \lambda(I \otimes U_2)} (I \otimes U_2)^{m_2} \otimes e_1 \wedge e_2.$$

After $t \in \mathbb{Z}_2$ acts on $(1 \otimes h_2)(\varphi_{-1,-1}U_1^{-1}U_2^{-1})$, we consider the pull back of $-1 \cdot (1 \otimes h_2)(\varphi_{-1,-1}U_1^{-1}U_2^{-1})$ on the Kozul complex.

$$\begin{aligned}&(1 \otimes k_2)a_{-1,-1}(U_1U_2 \otimes U_2^{-1} \otimes U_1^{-1} - \lambda U_2U_1 \otimes U_1^{-1} \otimes U_2^{-1})\\ &= a_{-1,-1}U_1U_2 \frac{\lambda(U_1 \otimes 1)^{-1} - (I \otimes U_1)^{-1}}{\lambda^{-1}(U_1 \otimes I) - (I \otimes U_1)} \frac{(U_2 \otimes I)^{-1} - \lambda^{-1}(I \otimes U_2)^{-1}}{(U_2 \otimes I) - \lambda(I \otimes U_2)} - 0.\end{aligned}$$

To simply the relations above we notice that

$$\frac{\lambda(U_1 \otimes 1)^{-1} - (I \otimes U_1)^{-1}}{\lambda^{-1}(U_1 \otimes I) - (I \otimes U_1)} = -\lambda(U_1^{-1} \otimes U_1^{-1}) \text{ and}$$
$$\frac{(U_2 \otimes I)^{-1} - \lambda^{-1}(I \otimes U_2)^{-1}}{(U_2 \otimes I) - \lambda(I \otimes U_2)} = -\lambda^{-1}(U_2^{-1} \otimes U_2^{-1}).$$

Hence we have

$(1 \otimes k_2)a_{-1,-1}(U_1U_2 \otimes U_2^{-1} \otimes U_1^{-1} - \lambda U_2U_1 \otimes U_1^{-1} \otimes U_2^{-1})$

$= a_{-1,-1}U_1U_2 \cdot -\lambda(U_1^{-1} \otimes U_1^{-1}) \cdot -\lambda^{-1}(U_2^{-1} \otimes U_2^{-1})$

$= a_{-1,-1}U_2U_1^{-1} \cdot (U_2^{-1} \otimes U_2^{-1}) = a_{-1,-1}U_2^{-1}U_2U_1^{-1}U_2^{-1} = a_{-1,-1}U_1^{-1}U_2^{-1}.$

Hence we have, $(H_2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg}))^{\mathbb{Z}_2} \cong \mathbb{C}$.

Similarly, we can compute the groups $(H_2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg}))^\Gamma$ for $\Gamma = \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ and find them to be isomorphic to $\mathbb{C}$.

3.3.2 $H_1(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg})^\Gamma$

We compute this group in a similar way to our calculation of the group $(H_2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg}))^\Gamma$. We notice that we have the group $H_1(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg})$ equals

$$ker(1 \otimes b_1)/im(1 \otimes b_2).$$

$(1 \otimes b_1)(a \otimes I \otimes e_j) = a \otimes_{\mathfrak{B}_\theta^{alg}} (I \otimes U_j - U_j \otimes I) = U_j a - aU_j$, Hence,

$ker(1 \otimes b_1) = \langle(\varphi^1, \varphi^2) \in \mathcal{A}_\theta^{alg} \oplus \mathcal{A}_\theta^{alg} | U_1\varphi^1 - \varphi^1 U_1 = \varphi^2 U_2 - U_2\varphi^2\rangle$.

While we know that

$$im(1 \otimes b_2) = \langle(\varphi^1, \varphi^2) \in \mathcal{A}_\theta^{alg} \oplus \mathcal{A}_\theta^{alg} | \exists\varphi \in \mathcal{A}_\theta^{alg}; \varphi^1 = \varphi U_2 - \lambda U_2 \varphi \text{ and } \varphi^2 = U_1\varphi - \lambda\varphi U_1\rangle$$

Hence we see that $H_1(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg}) \cong \mathbb{C}^2$, and is generated by the equivalence classes of $\varphi^{\bar{1}}_{-1,0}$ and $\varphi^{\bar{2}}_{0,-1}$.

We need to check the invariance of $H_1(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg})$ under each of the finite subgroups to get the desired group $(H_1(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg}))^\Gamma$.

Theorem 3.4. *$H_1(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg})^\Gamma = 0$.*

Proof. For $\Gamma = \mathbb{Z}_2$, to check the invariance we need to push the cycle to the bar complex, $C_*(\mathcal{A}_\theta^{alg})$ and then consider the natural action that exists on the bar complex.

Using the map $h_1 : J_1(\mathcal{A}_\theta^{alg}) \to C_1(\mathcal{A}_\theta^{alg})$,

$$h_1(I \otimes e_i) = I \otimes U_j$$

we obtain $(1 \otimes h_1)(\varphi^1_{-1,0}U_1^{-1}, 0) = \varphi^1_{-1,0}U_1^{-1} \otimes U_1$, thereafter we consider the action of $-1 \in \mathbb{Z}_2$. Hence we obtain the element

$$\varphi^1_{-1,0}U_1 \otimes U_1^{-1}.$$

We now pull this transformed element on back to the Kozul complex for its comparison with the original cycle using the map

$$k_1(I \otimes U^{\nu}) = A_{\nu} \otimes e_1 + B_{\nu} \otimes e_2, \text{ where } \nu = (n_1, n_2).$$

Coefficient $A_{\nu} = (I \otimes U_2)^{n_2}((U_1 \otimes I)^{n_1} - (I \otimes U_1)^{n_1})((U_1 \otimes I) - (I \otimes U_1))^{-1}$ while the coefficient $B_{\nu} = (I \otimes U_1)^{n_1}((U_2 \otimes I)^{n_2} - (I \otimes U_2)^{n_2})((U_2 \otimes I) - (I \otimes U_2))^{-1}$. These maps induce quasi-isomorphism, whence we have the push forwards and pull backs of cycles as cycles in their corresponding complexes.

Proceeding further with a general cycle $\varphi \in H_1(\mathcal{A}_{\theta}^{alg}, {}_{-1}\mathcal{A}_{\theta}^{alg})$, we can express it in terms of generators $\varphi^{\bar{1}}_{-1,0}$ and $\varphi^{\bar{2}}_{0,-1}$ as follows,

$$\varphi = aU_1^{-1} \otimes e_1 + bU_2^{-1} \otimes e_2.$$

$$(1 \otimes k_1)(-1 \cdot (1 \otimes h_1)(\varphi)) = (1 \otimes k_1)(-1 \cdot (aU_1^{-1} \otimes U_1 + bU_2 \otimes U_2^{-1})) =$$

$$(1 \otimes k_1)(aU_1 \otimes U_1^{-1} + bU_2 \otimes U_2^{-1}) = aU_1A \otimes e_1 + aU_1B \otimes e_2 + bU_2A' \otimes e_1 + bU_2B' \otimes e_2 =$$

$$(aU_1A + bU_2A') \otimes e_1 + (aU_1B + bU_2B') \otimes e_2.$$

Let us calculate $U_1A \in \mathcal{A}_{\theta}^{alg}$. We know from the formula described above that

$$A = \frac{(U_1 \otimes I)^{-1} - (I \otimes U_1)^{-1}}{(U_1 \otimes I) - (I \otimes U_1)}.$$

We can simplify the formula for A,

$$A = \frac{(U_1 \otimes I)^{-1} - (I \otimes U_1)^{-1}}{(U_1 \otimes I) - (I \otimes U_1)} = -U_1^{-1} \otimes U_1^{-1}.$$

Hence using $\mathcal{A}_{\theta}^{alg}$ as a $\mathfrak{B}_{\theta}^{alg}$ module, we have $U_1A = -U_1^{-1}$. Similarly we compute that $U_2B' = -U_2^{-1}$. As for A', B; we see that from the formula above that $A' = B = 0$, hence we have

$$(1 \otimes k_1)(-1 \cdot (1 \otimes h_1)(aU_1^{-1} \otimes e_1 + bU_2^{-1} \otimes e_2)) = -(aU_1^{-1} \otimes e_1 + bU_2^{-1} \otimes e_2).$$

Whence,

$$(H_1(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg}))^{\mathbb{Z}_2} = 0.$$

□

Similarly we get that for $\Gamma = \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$

$$(H_1(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg}))^{\Gamma} = 0.$$

3.3.3 $H_0(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg})^{\Gamma}$

For the zeroth homology we have a simple calculation, through the fact that $k_0 = h_0 = id$. Hence the natural action on the bar complex translates into the Kozul complex without any changes to it.

We know that

$$H_0(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg}) = \mathcal{A}_\theta^{alg} / \langle im(1 \otimes b_1) \rangle$$

$$im(1 \otimes b_1) = \langle \varphi \in \mathcal{A}_\theta^{alg} | \varphi = U_1\varphi^1 - \varphi^1 U_1 + U_2\varphi^2 - \varphi^2 U_2 \rangle$$

Since $U_1U_2 \in im(1 \otimes b_1)$ we have the group $H_0(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg}) = \langle \bar{\varphi_{0,0}} \rangle$. It is clearly invariant under the $\mathbb{Z}_2$ action. Hence we have the following result

$$H_0(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg})^{\mathbb{Z}_2} \cong \mathbb{C}.$$

In general we have that

$$H_0(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg})^{\Gamma} \cong \mathbb{C}.$$

for $\Gamma = \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$.

3.4 $\mathbb{Z}_2$ action on $\mathcal{A}_\theta^{alg}$

3.4.1 Hochschild homology

In this sub-section, we prove the following theorem.

Theorem 3.5. *$HH_0((\mathcal{A}_{\theta,-1}^{alg\bullet})^{\mathbb{Z}_2}) \cong \mathbb{C}^4$, while $HH_k((\mathcal{A}_{\theta,-1}^{alg\bullet})^{\mathbb{Z}_2})$ is a trivial group for all $k \geq 1$*

Lemma 3.6. *Consider the following chain complex $J_{*,-1}^{\mathbb{Z}_2}$*

$$J_{*,-1}^{\mathbb{Z}_2} := 0 \xleftarrow{-1b} (\mathcal{A}_{\theta,-1}^{alg})^{\mathbb{Z}_2} \xleftarrow{-1b} ((\mathcal{A}_{\theta,-1}^{alg})^{\otimes 2})^{\mathbb{Z}_2} \xleftarrow{-1b} ((\mathcal{A}_{\theta,-1}^{alg})^{\otimes 3})^{\mathbb{Z}_2} ...$$

where,

$${}_{-1}b(a_0 \otimes a_1 \otimes .... \otimes a_n) = b'(a_0 \otimes a_1 \otimes .... \otimes a_n) + (-1)^n((-1 \cdot a_n)a_0 \otimes a_1 \otimes .... \otimes a_{n-1}.)$$

Then,

$$H_\bullet(J_{*,-1}^{\mathbb{Z}_2}, {}_{-1}b) = (H_\bullet(J_*(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg})), b)^{\mathbb{Z}_2}.$$

Proof. By considering ${}_{-1}\mathcal{A}_\theta^{alg}$ as a twisted $\mathcal{A}_\theta^{alg}$ module, we can integrate the twisted part $((-1 \cdot a_n)a_0 \otimes a_1 \otimes .... \otimes a_{n-1})$ in the module structure of ${}_{-1}\mathcal{A}_\theta^{alg}$. With this we have the following equality

$$H_\bullet(J_{*,-1}^{\mathbb{Z}_2}, {}_{-1}b) = H_\bullet((J_*(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg})^{\mathbb{Z}_2}, b).$$

The map ${}_{-1}b$ has been modified into the regular Hochschild map, b of a Hochschild complex $C(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg})$, with twisted bimodule structure of ${}_{-1}\mathcal{A}_\theta$ that is as below

$$a\alpha = (-1 \cdot a)\alpha \text{ and } \alpha a = \alpha a; a \in \mathcal{A}_\theta, \alpha \in {}_{-1}\mathcal{A}_\theta.$$

Using Lemma 3.2 we can simplify $H_\bullet((J_*(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg})^{\mathbb{Z}_2}, b)$.

$$H_\bullet((J_*(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg})^{\mathbb{Z}_2}, b) = (H_\bullet(J_*(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg}), b))^{\mathbb{Z}_2}$$

□

Hence using the adjusted Connes' complex for the algebraic case we can now calculate the homology groups.

- $H_0(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg}) = {}_{-1}\mathcal{A}_\theta^{alg} \otimes_{\mathfrak{B}_\theta^{alg}} \mathfrak{B}_\theta^{alg} / Image(1 \otimes b_1)$,
- $H_1(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg}) = Ker(1 \otimes b_1)/Image(1 \otimes b_2)$,
- $H_2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg}) = Ker(1 \otimes b_2)$.

Lemma 3.7. $H_2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg}) \cong 0$.

Proof. We consider the map $(1 \otimes b_2)$ in the tensor complex. To calculate the kernel of this map we have a closer look at the following map,

$$(1 \otimes b_2)(a \otimes I) = a \otimes_{\mathfrak{B}_\theta^{alg}} (U_2 \otimes I - \lambda \otimes U_2) \otimes e_1 - a \otimes_{\mathfrak{B}_\theta^{alg}} (\lambda U_1 \otimes I - I \otimes U_1) \otimes e_2.$$

Using the twisted bimodule structure of ${}_{-1}\mathcal{A}_\theta^{alg}$ over $\mathcal{A}_\theta^{alg}$, the equation can be simplified to the following.

$$(1 \otimes b_2)(a \otimes I) = (aU - \lambda U_2^{-1} a, U_1^{-1} a - \lambda a U_1).$$

Hence we obtain the following relation over an element $(a \otimes 1)$ to reside in $ker(1 \otimes b_2)$.

$$H_2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg}) = \left\{ a \in {}_{-1}\mathcal{A}_\theta^{alg} | a_{n,m} = \lambda^{m-1} a_{n-2,m}; a_{n-1,m} = \lambda^n a_{n-1,m-2} \right\}.$$

Since, no such nontrivial element exists in ${}_{-1}\mathcal{A}_\theta^{alg}$ because if it did then all $a_{m,n}$ have to be same up to multiple of λ but since the algebra consists of finitely supported elements so they are reduced to zero. Therefore, we have the desired result,

$$H_2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg}) = 0.$$

□

Lemma 3.8. $H_0(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg})^{\mathbb{Z}_2} \cong \mathbb{C}^4$.

Proof. We have the map $(1 \otimes b_1)$ in the tensor complex defined below.

$$(1 \otimes b_1)(a \otimes I \otimes e_j) = a \otimes_{\mathfrak{B}_\theta^{alg}} (I \otimes U_j - U_j \otimes I) = U_j^{-1} a - a U_j.$$

As before if we use the twisted bicomplex structure of ${}_{-1}\mathcal{A}_\theta^{alg}$ over the algebra $\mathcal{A}_\theta^{alg}$, the map$(1 \otimes b_1)$ can be simplified as follows,

$$b_1(a_1, 0) = a_1 U_1 - U_1^{-1} a_1 \text{ and } b_1(0, a_2) = a_2 U_2 - U_2^{-1} a_2.$$

Observe that $b_1(a_1, a_2) = b_1(a_1, 0) + b_1(0, a_2)$. Further we can simplify the calculation by considering only elements of the type $b_1(U_1^n U_2^m, 0)$, and similarly for $b_1(0, a_2)$ we can consider the elements of the type $b_1(0, U_1^n U_2^m)$. We observe that

$$b_1(U_1^n U_2^m, 0) = U_1^{-1}({U_1}^n {U_2}^m) - ({U_1}^n {U_2}^m) U_1 = U_1^{n-1} U_2^m (1 - \lambda^{-m} U_1^2),$$

and

$$b_1(0, U_1^n U_2^{m-1}) = U_2^{-1}({U_1}^n {U_2}^{m-1}) - ({U_1}^n {U_2}^m) = \lambda^{-1} U_1^n U_2^{m-2} - U_1^n U_2^m = (\lambda^{-n} - U_2^2) U_1^n U_2^{m-2}.$$

Let us consider the case $m = n = 1$. In this case we can see that the elements U_j and U_j^{-1} are linearly related in $H_0(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg})$.

$$b_1(U_1 U_2, 0) = U_2(1 - \lambda^{-1} U_1^2) \text{ and } b_1(0, U_1 U_2) = (\lambda^{-1} - U_2^2) U_1.$$

Hence we can conclude that

$$H_0(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg}) = {}_{-1}\mathcal{A}_\theta^{alg} / \langle (U_1^{n-1} U_2^m (1 - \lambda^{-m} U_1^2), (\lambda^{-n} - U_2^2) U_1^n U_2^{m-2} \rangle.$$

Now to obtain $(H_0(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg}))^{\mathbb{Z}_2}$ we consider complex map $h : J_* \to C_*$, since the map $h_0 : J_0(\mathcal{A}_\theta^{alg}) \to C_0(\mathcal{A}_\theta^{alg})$ is the identity map, the $\mathbb{Z}_2$ action on the bar complex is translated to the Kozul complex with no alteration. Hence we get that

$$HH_0((\mathcal{A}_{\theta,-1}^{alg\bullet})^{\mathbb{Z}_2}) = \langle a_{\bar{0},0}, a_{\bar{1},0}, a_{\bar{0},1}, a_{\bar{1},1} \rangle$$

.

□

Lemma 3.9. $H_1(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg}) \cong 0.$

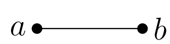
Proof. From the previous calculations we do have an explicit formula for the kernel and the image equations of $H_1(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg})$.

$$a^1_{n,m-1} - a^2_{n-1,m-2} = \lambda^{m-1} a^1_{n-2,m-1} - \lambda^{1-n} a^2_{n-1,m} \text{ (Kernel condition)}$$

$$a^1_{n,m-1} = a_{n.m-2} - \lambda^{1-n} a_{n,m}; a^2_{n-1,m-1} = a_{n,m-1} - \lambda^m a_{n-2,m-1} \text{(Image of } a \in \mathcal{A}_\theta^{alg}\text{)}$$

We introduce a combinatorial method of plotting these equations on the $\mathbb{Z}^2$ plane to study their solutions.

Definition 3.10. For two elements a, b in the lattice plane $\mathbb{Z}^2$ are said to be *kernel-connected* and drawn like

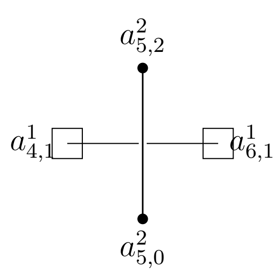


if there exists a kernel equation containing a and b.

For example consider the following kernel equation.

$$a^1_{6,1} - a^2_{5,0} = \lambda a^1_{4,1} - \lambda^{-5} a^2_{5,2}.$$

Then, we have the diagram below as the corresponding *kernel-diagram*, with the boxes representing elements of $a^1_{\bullet,\bullet}$ and the filled circles are elements of $a^2_{\bullet,\bullet}$.



For a given kernel solution S, we can draw all kernel diagrams for kernel equations containing its non-zero elements. This way we will have $Dgm(S)$ as a lattice diagram in which the non-zero lattice points of S are connected to one another by line segments on account of their relations to other points through kernel equations.

We say that two non-zero points are *primarily kernel connected* if there exists a kernel equation containing both of them. A kernel diagram $Dgm(\varphi)$ is *connected* if given any two points $f, g \in Dgm(\varphi)$, there exists points $(h_i)_{i=0}^{n} \in Dgm(\varphi)$ such that any two adjacent points in the sequence $f = h_0, h_1, ..., h_n = g$ are primarily connected.

We notice that for a given kernel element $\varphi = (\varphi^1, \varphi^2)$, at the point (n, m) we have the element $(\varphi^1_{n,m}, \varphi^2_{n,m})$. Hence we see that in a connected component we have three possibility at a point (n, m). These are

1. $\varphi^1_{n,m}$
2. $\varphi^2_{n,m}$
3. 0.

Hence we conclude that the kernel diagram of φ, $Dgm(\varphi)$ is a subset of $\mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2$. If we show that each of these connected component is a boundary then we are done proving our lemma. Hence, we shall concentrate on any connected component say, $\pi_i(Dgm(\varphi))$ and call it the diagram of the kernel element. This assumption causes no loss in the generality of the cases we need to consider.

Lemma 3.11. *All kernel solutions are disjoint unions of closed graphs with no open edges as drawn below.*

Proof. If such point b was to exist in any of the kernel diagrams, which is *kernel-connected* to a non-zero lattice point $b = w_{r,s}$ in the manner pictured above, then consider the kernel equation that contains this non-zero point $w_{r,s}$ and the other three zero points located at $(r+2, s), (r+1, s+1)$ and $(r+1, s-1)$. We derive a contradiction on the kernel condition. □

Now, the proof will proceed with induction over the *number* of non-zero elements in a given kernel solution.

After going through a simple yet tedious process, one figures out that there are no kernel solution with the number of non-zero entries less than or equal to 3. This can be seen as follows:

Consider three non-zero points $a^2_{n_1,m_1}, a^2_{n_2,m_2}, a^1_{n_3,m_3}$ constituting a kernel solution. It is easy to check that $m_3 \neq m_1$. Let $m_3 < m_1$, then consider the following equation at $(n, m) = (n_1 + 1, m_1 + 2)$,

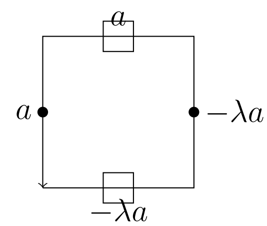
$$a^1_{n,m-1} - a^2_{n-1,m-2} = \lambda^{m-1} a^1_{n-2,m-1} - \lambda^{1-n} a^2_{n-1,m}.$$

We derive that for $a^2_{n_1,m_1} \neq 0$, it is imperative that $n_2 = n_1$ and $m_2 = m_1 + 2$. Again consider the above equation at $(n, m) = (n_1 + 1, m_1 + 4)$. We find that $a^2_{n_2,m_2} = 0$. This is a contradiction.

For simplicity assume that A_4, a solution containing 4 non-zero points be centered at $(0, 0)$. Now we study the four equations which contain these points. Hence we obtain the following relation over its non-zero points,

$$b = -\lambda a = -\lambda c = d.$$

The above relations mean that now A_1 can be written in terms of $Image(b_2)$ and the following diagram illustrates this.

$b_2(a') = A_1 =$ 

,

where $a'_{0,0} = a$ and $a'_{r,s} = 0$ for $(r,s) \neq (0,0)$.

Assume that all kernel solutions having the number of non-zero elements less than or equal to $(x-1)$ come from image. Then consider a kernel solution S_0 with x non-zero elements in it. Since this solution is finitely supported over the lattice plane, there exists a closed square region β over which S_0 is supported.

Inside β consider the left most column at least one point of which is non-zero. Choose the bottom point μ of this column. It is clear that $\mu = a^2_{r,s}$ for some $(r,s) \in \mathbb{Z}^2$. As if it were $a^1_{r,s}$ then consider the following kernel equation,

$$a^1_{r,s} - a^2_{r-1,s-1} = \lambda^s a^1_{r-2,s} - \lambda^{1-r} a^2_{r-1,s+1} \ .$$

All but $a^1_{r,s}$ are zero. This is a contradiction.

We shall now construct a new solution S_1 from S_0, with the number of non-zero elements in S_1 at most equal to x. Consider the following map $\wedge$

$$\wedge : \mathbb{Z}^2 \to \mathbb{Z}^2, \text{ such that}$$

$$(a^1_{r+1,s-1})^\wedge = 0, (a^2_{r,s})^\wedge = 0, (a^1_{r+1,s+1})^\wedge = a^1_{r+1,s+1} - a^2_{r,s},$$
$$(a^2_{r+2,s})^\wedge = a^2_{r+2,s} + \lambda^{s+1} a^2_{r,s} \text{ and } (a^j_{p,q})^\wedge = a^j_{p,q} \text{ for all other lattice points.}$$

Lemma 3.12. *Let $S_1 := \wedge(S_0)$. Then S_1 is a kernel solution such that $|S_1| \leq |S_0|$. If for some $u \in {}_{-1}\mathcal{A}_\theta^{alg}$, $b_2(u) = S_1$ then $b_2(u') = S_0$, where $u' = u + {}_{r,s}g$,*

$$
{}_{r,s}g_{p,q} = \begin{cases} a_{r,s}^2 & \text{if } (p,q) = (r+1, s) \\ 0 & \text{else.} \end{cases}
$$

Proof. Second part is clear that the cardinality of non-zero entries is reduced by 2 when it is imposed that $(a_{r,s+2}^1)^\wedge = (a_{r,s}^2)^\wedge = 0$ but then the cardinality may increase by two in case both $a_{r+1,s}^1$ and $a_{r+1,s}^2$ are zero in S_0.

To check that S_1 is a solution it is enough to check that all the kernel equations containing any of these four altered elements hold. In other words we want to check that the following equation

$$
(a_{n,m}^1)^\wedge - (a_{n-1,m-1}^2)^\wedge - \lambda^m (a_{n-2,m}^1)^\wedge + \lambda^{1-n}(a_{n-1,m+1}^2)^\wedge = 0
$$

holds for $(n,m) = (r+3, s+1), (r+1, s+1), (r+1, s-1),$ and $(r+3, s-1)$.

<u>Case 1: $(n,m) = (r+1, s+1)$</u>

$(a_{r+1,s+1}^1)^\wedge - (a_{r,s}^2)^\wedge - \lambda^{s+1}(a_{r-1,s+1}^1)^\wedge + \lambda^{-r}(a_{r,s+2}^2)^\wedge = 0$

$\implies a_{r+1,s+1}^1 - a_{r,s}^2 - 0 - \lambda^{s+1} a_{r-1,s+1}^1 + \lambda^{-r} a_{r,s+2}^2 = 0$ This holds as in S_0 .

<u>Case 2 : $(n,m) = (r+1, s-1)$</u>

$(a_{r+1,s-1}^1)^\wedge - (a_{r,s-2}^2)^\wedge - \lambda^{s-1}(a_{r-1,s-1}^1)^\wedge + \lambda^{-r}(a_{r,s}^2)^\wedge = 0$

$\implies 0 - 0 - 0 + 0 = 0$.

<u>Case 3 : $(n,m) = (r+3, s-1)$</u>

$(a_{r+3,s-1}^1)^\wedge - (a_{r+2,s-2}^2)^\wedge - \lambda^{s-1}(a_{r+1,s-1}^1)^\wedge + \lambda^{-2-r}(a_{r+2,s}^2)^\wedge = 0$

$\implies a_{r+3,s-1}^1 - a_{r+2,s-2}^2 - 0 + \lambda^{-2-r}(a_{r+2,s}^2 + \lambda^{s+1} a_{r,s}^2) = 0$

$\implies a_{r+3,s-1}^1 - a_{r+2,s-2}^2 + \lambda^{-2-r}(a_{r+2,s}^2 + \lambda^{s+1} a_{r,s}^2) = 0.$

An easy exercise on S_0 reveals that $a_{r,s}^2 = \lambda^r a_{r+1,s-1}^1$, hence we get the required relation.

Case 4 : $(n, m) = (r + 3, s + 1)$

$(a^1_{r+3,s+1})^\wedge - (a^2_{r+2,s})^\wedge - \lambda^{s+1}(a^1_{r+1,s+1})^\wedge + \lambda^{-2-r}(a^2_{r+2,s+2})^\wedge = 0$

$\implies a^1_{r+3,s+1} - a^2_{r+2,s} - \lambda^{s+1}a^2_{r,s} - \lambda^{s+1}(a^1_{r+1,s+1} - a^2_{r,s}) + \lambda^{-2-r}a^2_{r+2,s+2} = 0$

$\implies a^1_{r+3,s+1} - a^2_{r+2,s} - \lambda^{s+1}a^1_{r+1,s+1} + \lambda^{-2-r}a^2_{r+2,s+2} = 0.$

Above relation is indeed satisfied in S_0.

It is easy to see that like kernel diagram we can construct image diagrams. These diagrams are obtained by looking at how b_2-image of a non-zero lattice point look like as an element of $\mathcal{A}_\theta^{alg} \oplus \mathcal{A}_\theta^{alg}$. Given an element at $(0, 0)$, its image are the four elements, two among a^1 and other two a^2. The smallest image diagram is the diagram of the kernel solution A_4, as described in previous pages.

Hence for each non-zero lattice element we have its image diagram similar to A_4, and hence for an element $a \in \mathcal{A}_\theta^{alg}$ we have its image diagram obtained by superimposing the image diagrams for each lattice points. For $H_1(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg})$ to be zero it is a necessary condition that for each kernel element $\varphi = (\varphi^1, \varphi^2)$ there exists an image diagram similar to the construction of $Dgm(\varphi)$. Our induction process explicitly constructs the image candidate itself.

With this pictorial realization, it is clear from the diagram as well as from the explicit equations that if we verify that

$$b_2(u')_{p,q} = (a^1_{p,q}, a^2_{p,q}) \text{ for } (p, q) = (r + 1, s + 1), (r, s), (r + 1, s - 1) \text{ and } (r + 2, s)$$

then we have proved that $b_2(u') = S_0$.

Observe that

$(b_2(u'))_{r+1,s+1} = (b_2(u + {}_{rs}g))_{r+1,s+1}$

$= (b_2(u))_{r+1,s+1} + (b_2({}_{rs}g))_{r+1,s+1} = ((a^1_{r+1,s+1})^\wedge, (a^2_{r+1,s+1})^\wedge) + (a^2_{r,s}, 0) = (a^1_{r+1,s+1}, a^2_{r+1,s+1})$,

$(b_2(u'))_{r,s} = (b_2(u + {}_{rs}g))_{r,s}$

$= (b_2(u))_{r,s} + (b_2({}_{rs}g))_{r,s} = ((a^1_{r,s})^\wedge, (a^2_{r,s})^\wedge) + (0, a^2_{r,s}) = (a^1_{r,s}, a^2_{r,s}),$

$(b_2(u'))_{r+1,s-1} = (b_2(u + {}_{rs}g))_{r+1,s-1}$

$= (b_2(u))_{r+1,s-1} + (b_2({}_{rs}g))_{r+1,s-1} = ((a^1_{r+1,s-1})^\wedge, (a^2_{r+1,s-1})^\wedge) + (-\lambda^r a^2_{r,s}, 0) = (a^1_{r+1,s-1}, a^2_{r+1,s-1})$

and

$(b_2(u'))_{r+2,s} = (b_2(u + {}_{rs}g))_{r+2,s}$

$= (b_2(u))_{r+2,s} + (b_2({}_{rs}g))_{r+2,s} = ((a^1_{r+2,s})^\wedge, (a^1_{r+2,s})^\wedge) + (0, -\lambda^{s+1} a^2_{r,s}) = (a^1_{r+2,s}, a^2_{r+2,s}).$ □

Lemma 3.13. $\wedge^N(S_0) = 0$ *for any solution* S *for a sufficiently large number* N*.*

Proof. The lemma states that $|S_1| \leq |S_0|$ is strict after sufficiently many iterations. Notice that the above process reduces the number of non-zero entries in the left most column by 2. And when this process is iterated this will lead to shifting of the left most non-zero column rightwards. Pictorially if it is assumed that at no stage the iteration reduces the number of non-zero elements, then these many elements are compressed by the shifting of the left most non-zero column towards right by this process as the right edge of β is not changed by this process unless all the non-zero elements are collected on this right edge of β.

Hence, all the non-zero elements of $\wedge^d(S_0)$ will at some stage be bounded to the right edge of β. But, then, consider the lowest element of $\wedge^d(S_0)$ on this edge. It has to be only $a^2_{w,t}$ for some (w, t). Consider the following kernel equation. Every term in the equation

$$a^1_{w+1,t-1} - a^2_{w,t-2} - \lambda^{t-1} a^1_{w-1,t-1} + \lambda^{-w} a^2_{w,t} = 0$$

has all but $a^2_{w,t}$ equal zero, which is a contradiction.

Hence the induction process is complete. □

Theorem 3.14. *For* $\mathbb{Z}_2$ *action on* $\mathcal{A}^{alg}_\theta$ *we have the Hochschild homology groups as follows:*

$$HH_0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) \cong \mathbb{C}^5, HH_1(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) \cong 0, HH_2(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) \cong \mathbb{C}.$$

Proof. We already know that $HH_0(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_2} = \mathcal{A}_\theta^{alg}/im(1 \otimes b_1) \cong \mathbb{C}$. We also know that $H_0(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg})^{\mathbb{Z}_2} \cong \mathbb{C}^4$. Hence we have $HH_0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) \cong \mathbb{C}^5$.

We also know from the previous section that $HH_2(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_2} \cong \mathbb{C}$. As we know that $H_2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta) = 0$, hence, we derive the desired dimension for $HH_2(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)$.

From the previous section we also know that $HH_1(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) \cong 0$. Combining this result with $H_1(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg}) = 0$, we get that $H_1(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) = 0$. □

3.4.2 Cyclic homology of $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2$

A. Connes introduced[C] an S, B, I long exact sequence relating the Hochschild and cyclic homology of an algebra A,

$$... \xrightarrow{B} HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} ....$$

Since the $\mathbb{Z}_2$ action on $\mathcal{A}_{\theta,-1}^{alg}$ commutes with the map ${}_{-1}b$, we obtain the following exact sequence

$$... \xrightarrow{B} (HH_n(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} \xrightarrow{I} (HC_n(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} \xrightarrow{S} (HC_{n-2}(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} \xrightarrow{B} (HH_{n-1}(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} \xrightarrow{I} ....$$

We know that $(HH_2(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} = (HH_1(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} = 0$. Hence we obtain that

$$(HC_2(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} \cong (HC_0(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2}.$$

But, a preliminary result shows that $HH_0(\mathcal{A}_{\theta,-1}^{alg}) = HC_0(\mathcal{A}_{\theta,-1}^{alg})$. Hence, we obtain that

$$(HC_2(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} = (HC_0(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} \cong \mathbb{C}^4.$$

Also, since $(HH_2(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} = (HH_3(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} = 0$, we obtain that $(HC_3(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} \cong (HC_1(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2}$. Since $HH_1(\mathcal{A}_{\theta,-1}^{alg})$ is trivial, we have $(HC_3(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} \cong (HC_1(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} = 0$. □

3.4.3 Periodic cyclic homology

Theorem 3.15. *$HC_{even}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) \cong \mathbb{C}^6$ while $HC_{odd}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) = 0$.*

Proof. It is clear from the above calculations that we have

$$(HC_\bullet(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} \cong (HC_{\bullet-2}(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2}$$

Hence we conclude that $HC_{even}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) \cong \mathbb{C}^6$

As for the odd cyclic homology, we have $(HC_3(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} \cong (HC_1(\mathcal{A}_{\theta,-1}^{alg}))^{\mathbb{Z}_2} = 0$, and we also have $HC_3(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_2} = HC_1(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_2} = HH_1(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_2} = 0$. Combining these two results, we obtain $HC_{odd}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) = 0$. So we have computed the Hochschild and cyclic homology of the noncommutative $\mathbb{Z}_2$ orbifold. □

3.5 $\mathbb{Z}_3$ action on $\mathcal{A}_\theta^{alg}$

The group $\mathbb{Z}_3$ is embedded in $SL(2,\mathbb{Z})$ through its generator $g = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \in SL(2,\mathbb{Z})$. The generator acts on $\mathcal{A}_\theta^{alg}$ in the following way

$$U_1 \mapsto U_2^{-1}, U_2 \mapsto \frac{U_1 U_2^{-1}}{\sqrt{\lambda}}.$$

3.5.1 Hochschild homology

Theorem 3.16. *$HH_0((\mathcal{A}_{\theta,\omega^{\pm1}}^{alg\bullet})^{\mathbb{Z}_3}) \cong \mathbb{C}^3$, while $HH_k((\mathcal{A}_{\theta,\omega^{\pm1}}^{alg\bullet})^{\mathbb{Z}_3})$ are trivial groups for $k \geq 1$.*

Lemma 3.17. *Consider the following chain complex $J_{*,\omega\pm1}^{\mathbb{Z}_3}$*

$$J_{*,\omega^{\pm1}}^{\mathbb{Z}_3} := 0 \xleftarrow{\omega^{\pm1}b} (\mathcal{A}_{\theta,\omega^{\pm1}}^{alg})^{\mathbb{Z}_3} \xleftarrow{\omega^{\pm1}b} ((\mathcal{A}_{\theta,\omega^{\pm1}}^{alg})^{\otimes 2})^{\mathbb{Z}_3} \xleftarrow{\omega^{\pm1}b} ((\mathcal{A}_{\theta,\omega^{\pm1}}^{alg})^{\otimes 3})^{\mathbb{Z}_3} \ldots$$

where,

$${}_{\omega^{\pm1}}b(a_0 \otimes a_1 \otimes .... \otimes a_n) = b'(a_0 \otimes a_1 \otimes .... \otimes a_n) + (-1)^n((\omega^{\mp1} \cdot a_n)a_0 \otimes a_1 \otimes .... \otimes a_{n-1}).$$

Then,

$$H_\bullet(J^{\mathbb{Z}_3}_{*,\omega^{\pm 1}}, {}_{\omega^{\pm 1}}b) = (H_\bullet(J_*(\mathcal{A}_\theta^{alg}, {}_{\omega^{\pm 1}}\mathcal{A}_\theta^{alg})), b)^{\mathbb{Z}_3}$$

Hence using the adjusted Connes' complex for the algebraic case we can now calculate the homology groups.

- $H_0(\mathcal{A}_\theta^{alg}, {}_{\omega^{\pm 1}}\mathcal{A}_\theta^{alg}) = {}_{\omega^{\pm 1}}\mathcal{A}_\theta^{alg} \otimes_{\mathfrak{B}_\theta^{alg}} \mathfrak{B}_\theta^{alg}/Image(1 \otimes b_1)$,
- $H_1(\mathcal{A}_\theta^{alg}, {}_{\omega^{\pm 1}}\mathcal{A}_\theta^{alg}) = Ker(1 \otimes b_1)/Image(1 \otimes b_2)$,
- $H_2(\mathcal{A}_\theta^{alg}, {}_{\omega^{\pm 1}}\mathcal{A}_\theta^{alg}) = Ker(1 \otimes b_2)$.

Lemma 3.18. $H_2(\mathcal{A}_\theta^{alg}, {}_{\omega^{\pm 1}}\mathcal{A}_\theta^{alg}) \cong 0$.

Proof. We prove here for the case $g = \omega$, the proof for $g = \omega^2$ is similar. Consider the map $(1 \otimes b_2)$ in the tensor complex. To calculate the kernel of this map we have a closer look at the differential.

$$(1 \otimes b_2)(a \otimes I) = a \otimes_{\mathfrak{B}_\theta^{alg}} (U_2 \otimes I - \lambda \otimes U_2) \otimes e_1 - a \otimes_{\mathfrak{B}_\theta^{alg}} (\lambda U_1 \otimes I - I \otimes U_1) \otimes e_2.$$

Using the twisted bimodule structure of ${}_\omega\mathcal{A}_\theta^{alg}$ over $\mathcal{A}_\theta^{alg}$, the equation can be simplified to the following,

$$(1 \otimes b_2)(a \otimes I) = (aU_2 - \sqrt{\lambda} U_1 U_2^{-1} a, U_2^{-1} a - \lambda a U_1).$$

Hence we obtain the following relation over an element $(a \otimes 1)$ to reside in $ker(1 \otimes b_2)$.

$$H_2(\mathcal{A}_\theta^{alg}, {}_\omega\mathcal{A}_\theta^{alg}) = \Big\{ a \in {}_\omega\mathcal{A}_\theta^{alg} | a_{n,m-1} = \lambda^{1.5-m} a_{n-1,m+1}; \lambda^{m+1} a_{n-1,m} = \lambda^n a_{n,m+1} \Big\}.$$

Since, no such nontrivial element exists in ${}_\omega\mathcal{A}_\theta^{alg}$. If such a non-trivial element existed, then all $a_{m,n}$ have to be same up to multiple of λ. But since the algebra consists of finitely supported elements, they are reduced to zero. So, we have the desired result.

$$H_2(\mathcal{A}_\theta^{alg}, {}_\omega\mathcal{A}_\theta^{alg}) = 0.$$

□

Lemma 3.19. $H_0(\mathcal{A}_\theta^{alg}, {}_\omega\mathcal{A}_\theta^{alg})^{\mathbb{Z}_2} \cong \mathbb{C}^3$.

Proof. We have the map $(1 \otimes b_1)$ in the tensor complex defined below,

$$(1 \otimes b_1)(a \otimes I \otimes e_j) = a \otimes_{\mathfrak{B}_\theta^{alg}} (I \otimes U_j - U_j \otimes I) = aU_j - U_j^{-1}a.$$

As before if we use the twisted bicomplex structure of ${}_\omega\mathcal{A}_\theta^{alg}$ over the algebra $\mathcal{A}_\theta^{alg}$, the map$(1 \otimes b_1)$ can be simplified as follows,

$$b_1(a_1, 0) = a_1U_1 - U_2^{-1}a_1 \text{ and } b_1(0, a_2) = a_2U_2 - \frac{U_1U_2^{-1}}{\sqrt{\lambda}}a_2.$$

Observe that $b_1(a_1, a_2) = b_1(a_1, 0)+b_1(0, a_2)$. Furthermore we can simplify the calculation by considering only elements of the type $b_1(U_1^nU_2^m, 0)$, and similarly for $b_1(0, a_2)$ we can consider the elements of the type $b_1(0, U_1^nU_2^m)$. We observe that

$$b_1(U_1^nU_2^m, 0) = ({U_1}^n{U_2}^m)U_1 - U_2^{-1}({U_1}^n{U_2}^m) = \lambda^{-n}(\lambda^mU_1U_2 - 1)U_1^nU_2^{m-1}$$

and

$$b_1(0, U_1^nU_2^{m-1}) = ({U_1}^n{U_2}^{m-1})U_2 - \tfrac{U_1U_2^{-1}}{\sqrt{\lambda}}({U_1}^n{U_2}^{m-1}) = (\lambda^{-2n}U_2^2 - U_1)U_1^nU_2^{m-2}.$$

From the above relations we see that in this quotient space $\bar{a_{1,0}} = \bar{a_{0,-1}} = \bar{a_{0,2}}$. Observe that $\langle \bar{a_{0,0}}, \bar{a_{1,0}}, \bar{a_{0,1}} \rangle$ is a basis of this quotient space.

Now to obtain $(H_0(\mathcal{A}_\theta^{alg}, {}_\omega\mathcal{A}_\theta^{alg}))^{\mathbb{Z}_3}$ we consider complex map $h : J_* \to C_*$, since the map $h_0 : J_0(\mathcal{A}_\theta^{alg}) \to C_0(\mathcal{A}_\theta^{alg})$ is the identity map, the $\mathbb{Z}_3$ action on the bar complex is translated to the Kozul complex with no alteration. Hence we get that

$$H_0(\mathcal{A}_\theta^{alg}, {}_\omega\mathcal{A}_\theta^{alg})^{\mathbb{Z}_3} = \langle \bar{a_{0,0}}, \bar{a_{1,0}}, \bar{a_{0,1}} \rangle.$$

□

Lemma 3.20. $H_1(\mathcal{A}_\theta^{alg}, {}_\omega\mathcal{A}_\theta^{alg}) \cong 0$.

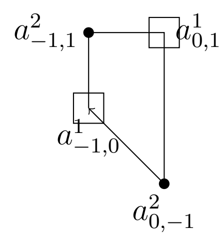
Proof. From the previous calculations we do have an explicit formula for the kernel and the image equations of $H_1(\mathcal{A}_\theta^{alg}, {}_\omega\mathcal{A}_\theta^{alg})$.

$$\lambda^{m-1}a^1_{n-1,m-1} - \lambda^{-n}a^1_{n,m} = \lambda^{0.5-n}a^2_{n-1,m} - a^2_{n,m-2} \text{ (Kernel condition)},$$

$$a^1_{n,m-1} = \lambda^{1.5-n}a_{n-1,m} - a_{n.m-2};\ a^2_{n-1,m-1} = \lambda^m a_{n-2,m-1} - \lambda^{1-n}a_{n-1,m}\text{(Image of } a \in \mathcal{A}_\theta^{alg}).$$

As before, *a,b,c, and d* are said to be *kernel-connected* if there exists a kernel equation containing them. So, any given kernel solution A we have a diagram associated to it, extended by plotting all the kernel equations that contain any of its non zero points, which for simplicity we denote by A.

e.g. equation $\lambda^0 a^1_{-1,0} - \lambda^{-0}a^1_{0,1} = \sqrt{\lambda}a^2_{-1,1} - a^2_{0,-1}$ is drawn below:



.

Now, the proof will proceed with induction over the *number* of non-zero elements in a given kernel solution.

After going through a simple yet tedious process, one figures out that there are no kernel solution with the number of non-zero entries less than or equal to 3. This can be seen as follows:

Consider three non-zero points $a^2_{n_1,m_1}, a^2_{n_2,m_2}, a^1_{n_3,m_3}$ constituting a kernel solution. It is easy to check that $m_2 \neq m_1$. Let $m_1 < m_2$, then consider the following equation at

$(p, q) = (n_1 + 1, m_1)$,

$$\lambda^{q-1} a^1_{p-1,q-1} - \lambda^{-p} a^1_{p,q} = \lambda^{0.5-p} a^2_{p-1,q} - a^2_{p,q-2}.$$

then we see that $m_3 = m_1$. Considering the above equation at $(p, q) = (n_2, m_2)$ we get, $n_3 = n_2$ and $m_3 = m_2 - 1$. Now finally we consider the above kernel equation at $(p, q) = (n_1 - 1, m_1 + 2)$, on which we get that $a^2_{n_1,m_1} = 0$, which is a contradiction.

Assume that all kernel solutions having number of non-zero elements less than or equal to $(x - 1)$ come from image. Then consider a kernel solution S_0 with x non-zero elements in it. Since this solution is finitely supported over the lattice plane, there exists a closed square region β over which S_0 is supported.

Inside β consider the left most column at least one point of which is not zero. Choose the bottom point μ of this column. It is clear that $\mu = a^2_{r,s}$ for some $(r, s) \in \mathbb{Z}^2$. As if it were $a^1_{r,s}$ then consider the following kernel equation.

$$\lambda^{s-1} a^1_{r-1,s-1} - \lambda^{-r} a^1_{r,s} = \lambda^{0.5-r} a^2_{r-1,s} - a^2_{r,s-2}.$$

All but $a^1_{r,s}$ are zero. This is a contradiction.

We shall now construct a new solution S_1 from S_0, with the number of non-zero elements in S_1 at most equal to x. Consider the following map $\wedge$

$$\wedge : \mathbb{Z}^2 \to \mathbb{Z}^2, \text{ such that}$$

$$(a^1_{r,s+2})^\wedge = 0, (a^2_{r,s})^\wedge = 0, (a^1_{r+1,s})^\wedge = a^1_{r+1,s} + \sqrt{\lambda} a^2_{r,s},$$
$$(a^2_{r+1,s+1})^\wedge = a^2_{r+1,s+1} + \lambda^{s+2} a^1_{r,s+2} \text{ and } (a^j_{p,q})^\wedge = a^j_{p,q} \text{ for all other lattice points.}$$

Lemma 3.21. *Let $S_1 := \wedge(S_0)$. Then S_1 is a kernel solution such that $|S_1| \leq |S_0|$. If for some $u \in {}_\omega\mathcal{A}^{alg}_\theta$, $b_2(u) = S_1$ then $b_2(u') = S_0$, where $u' = u + {}_{r,s}g$,*

$$
{}_{r,s}g_{p,q} = \begin{cases} -a^1_{r,s+2} & \textit{if } (p,q) = (r,s+1) \\ 0 & \textit{else.} \end{cases}
$$

Proof. The second part is clear that the cardinality of non-zero entries is reduced by 2 when it is asked that $(a^1_{r,s+2})^\wedge = (a^2_{r,s})^\wedge = 0$ but then the cardinality may increase by two in case both $a^1_{r+1,s}$ and $a^2_{r+1,s}$ are zero in S_0.

To check that S_1 is a solution it is enough to check that all the kernel equations containing any of these four altered elements hold. In other words we want to check that the following equation

$$
\lambda^m (a^1_{n-1,m})^\wedge - \lambda^{-n}(a^1_{n,m+1})^\wedge = \lambda^{0.5-n}(a^2_{n-1,m+1})^\wedge - (a^2_{n,m-1})^\wedge
$$

holds for $(n,m) = (r,s+1), (r+1,s+2), (r+1,s-1)$ and $(r+2,s+1)$.

Case 1: $(n,m) = (r,s+1)$

$\lambda^{s+1}(a^1_{r-1,s+1})^\wedge - \lambda^{-r}(a^1_{r,s+2})^\wedge = \lambda^{0.5-r}(a^2_{r-1,s+2})^\wedge - (a^2_{r,s})^\wedge$

$\Longrightarrow \lambda^{s+1}(a^1_{r-1,s+1})^\wedge = \lambda^{0.5-r}(a^2_{r-1,s+2})^\wedge$, which holds as : $a^1_{r-1,s+1} = a^2_{r-1,s+1} = 0$.

Case 2 : $(n,m) = (r+1,s+2)$

$\lambda^{s+2}(a^1_{r,s+2})^\wedge - \lambda^{-1-r}(a^1_{r+1,s+3})^\wedge - \lambda^{-0.5-r}(a^2_{r,s+3})^\wedge + (a^2_{r+1,s+1})^\wedge$

$= 0 - \lambda^{-1-r}a^1_{r+1,s+3} - \lambda^{-0.5-r}a^2_{r,s+3} + a^2_{r+1,s+1} + \lambda^{s+2}a^1_{r,s+2}.$

The above equation holds in S_0.

Case 3 : $(n,m) = (r+2,s)$

$\lambda^{s}(a^1_{r+1,s})^\wedge - \lambda^{-r-2}(a^1_{r+2,s+1})^\wedge - \lambda^{-1.5-r}(a^2_{r+1,s+1})^\wedge + (a^2_{r+2,s-1})^\wedge = 0$

$\Longrightarrow \lambda^{s}a^1_{r+1,s} + \lambda^{s+0.5}a^2_{r,s} - \lambda^{-r-1}a^2_{r+2,s+1} - \lambda^{-0.5-r}a^2_{r+1,s+1} - \lambda^{s-r+0.5}a^1_{r,s+2} - a^2_{r+2,s-1} = 0$

$\Longrightarrow \lambda^{s+1.5}a^2_{r,s} = \lambda^{s-r+1.5}a^1_{r,s+2}$

The above is true if one considers the kernel equation.

$$\lambda^{s+1}a^1_{r-1,s+1} - \lambda^{-r}a^1_{r,s+2} + a^2_{r,s} - \lambda^{0.5-r}a^2_{r-1,s+2} = 0$$

$a^2_{r-1,s+2} = a^2_{r-1,s+1} = 0$

$\implies a^2_{r,s} = \lambda^{-r}a^1_{r,s+2}$, which is what we required.

<u>Case 4 : $(n, m) = (r + 1, s - 1)$</u>

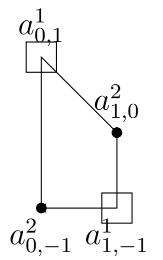
$\lambda^{s-1}(a^1_{r,s-1})^\wedge - \lambda^{-1-r}(a^1_{r+1,s})^\wedge - \lambda^{-0.5-r}(a^2_{r,s})^\wedge + (a^2_{r+1,s-2})^\wedge$

$= \lambda^{s-1}a^1_{r,s-1} - \lambda^{-1-r}a^1_{r+1,s} - \lambda^{-1-r}(\sqrt{\lambda}a^2_{r,s}) - 0 + a^2_{r+1,s-2}$

$= \lambda^{s-1}a^1_{r,s-1} - \lambda^{-1-r}a^1_{r+1,s} - \lambda^{-0.5-r}a^2_{r,s} + a^2_{r+1,s-2}.$

The above is a kernel equation in S_0, and we have shown that $S_1 := \wedge(S_0)$ is a kernel solution.

It is easy to see that like kernel equation diagram there is also an image solution diagram indicating how image elements are formed given an element at $(0, 0)$,



So, it is clear from the diagram as well as from the explicit equations that we have for the map b_2, an image element $a_{0,0}$ induces kernel solution elements to its right-below and up for a^1, or right and below for a^2 elements. Hence if we verify that

$$b_2(u')_{p,q} = (a^1_{p,q}, a^2_{p,q}) \text{ for } (p, q) = (r, s + 2), (r, s), (r + 1, s + 1), (r + 1, s)$$

then we have proved that $b_2(u') = S_0$.

Observe that

$(b_2(u'))_{r,s+2} = (b_2(u + {}_{rs}g))_{r,s+2}$

$= (b_2(u))_{r,s+2} + (b_2({}_{rs}g))_{r,s+2} = ((a^1_{r,s+2})^\wedge, (a^2_{r,s+2})^\wedge) + (a^1_{r,s+2}, 0) = (a^1_{r,s+2}, a^2_{r,s+2})$,

$(b_2(u'))_{r,s} = (b_2(u + {}_{rs}g))_{r,s}$

$= (b_2(u))_{r,s} + (b_2({}_{rs}g))_{r,s} = ((a^1_{r,s})^\wedge, (a^2_{r,s})^\wedge) + (0, \lambda^r a^1_{r,s+2}) = (a^1_{r,s}, a^2_{r,s}),$

$(b_2(u'))_{r+1,s+1} = (b_2(u + {}_{rs}g))_{r+1,s+1}$

$= (b_2(u))_{r+1,s+1} + (b_2({}_{rs}g))_{r+1,s+1} = ((a^1_{r+1,s+1})^\wedge, (a^2_{r+1,s+1})^\wedge) + (0, -\lambda^{s+2} a^1_{r,s+2}) = (a^1_{r+1,s+1}, a^2_{r+1,s+1})$

and

$(b_2(u'))_{r+1,s} = (b_2(u + {}_{rs}g))_{r+1,s}$

$= (b_2(u))_{r+1,s} + (b_2({}_{rs}g))_{r+1,s} = ((a^1_{r+1,s})^\wedge, (a^1_{r+1,s})^\wedge) + (\lambda^{0.5-r} a^1_{r,s+2}, 0) = (a^1_{r+1,s}, a^2_{r+1,s}).$ □

Lemma 3.22. $\wedge^N(S_0) = 0$ *for any solution* S *for a sufficiently large number* N.

Proof. The lemma states that $|S_1| \leq |S_0|$ is strict after sufficiently many iterations. Notice that the above process reduces the number of non-zero entries in the left most column by 2. And when this process is iterated this will lead to shifting of the left most non-zero column rightwards. Pictorially if it is assumed that at no stage the iteration reduces the number of non-zero elements, then these many elements are compressed by the shifting of the left most non-zero column towards right. As the right edge of β is not changed by this process unless all the non-zero elements are collected on this right edge of β, all the non-zero elements of $\wedge^d(S_0)$ will at some stage be bounded to the right edge of β. But, then, consider the lowest element of $\wedge^d(S_0)$ on this edge. It has to be only $a^2_{w,t}$ for some (w,t). Consider the following kernel equation,

$$\lambda^{1+t} a^1_{w-1,t+1} - \lambda^{-w} a^1_{w,t+2} = -a^2_{w,t} + \lambda^{0.5-w} a^2_{w-1,t+1} \ .$$

This equation has all but $a^2_{w,t}$ as zero, which is a contradiction.

Hence the induction process is complete. □

□

Theorem 3.23. *For $\mathbb{Z}_3$ action on $\mathcal{A}_\theta^{alg}$ we have the Hochschild homology groups as follows:*

$$HH_0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_3) \cong \mathbb{C}^7, HH_1(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_3) \cong 0, HH_2(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_3) \cong \mathbb{C}.$$

Proof. From the calculations in the previous sections regarding the $\mathbb{Z}_3$ invariance we know that $HH_0(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_3} \cong \mathbb{C}$ and similarly one can check that $H_0(\mathcal{A}_\theta^{alg}, {}_{\pm\omega}\mathcal{A}_\theta^{alg})^{\mathbb{Z}_3} \cong \mathbb{C}^3$. Hence we have the following result

$$HH_0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_3) \cong \mathbb{C}^7.$$

As $HH_2(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_3} \cong \mathbb{C}$. And we have $H_2(\mathcal{A}_\theta^{alg}, {}_{\pm\omega}\mathcal{A}_\theta^{alg}) \cong 0$, we conclude that

$$HH_2(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) \cong \mathbb{C}.$$

Since $HH_1(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_3} = H_1(\mathcal{A}_\theta^{alg}, {}_{\omega^{\pm 1}}\mathcal{A}_\theta^{alg}) = 0$, we obtain that $HH_1(\mathcal{A}_\theta \rtimes \mathbb{Z}_3) = 0$. $\square$

3.5.2 Cyclic homology of $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_3$

Theorem 3.24. $HC_{even}(\mathcal{A}_{\theta,\omega^{\pm 1}}^{alg}) \cong \mathbb{C}^3$, *while* $HC_{odd}(\mathcal{A}_{\theta,\omega^{\pm 1}}^{alg}) = 0$.

Proof. We apply the S, B, I long exact sequence relating the Hochschild and cyclic homology of an algebra A,

$$... \xrightarrow{B} HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} ....$$

Since the $\mathbb{Z}_3$ action on $\mathcal{A}_{\theta,\omega^{\pm 1}}^{alg}$ commutes with the map ${}_{\omega^{\pm 1}}b$, we obtain the following exact sequence

$$... \xrightarrow{B} (HH_n(\mathcal{A}_{\theta,\omega^{\pm 1}}^{alg}))^{\mathbb{Z}_3} \xrightarrow{I} (HC_n(\mathcal{A}_{\theta,\omega^{\pm 1}}^{alg}))^{\mathbb{Z}_3} \xrightarrow{S} (HC_{n-2}(\mathcal{A}_{\theta,\omega^{\pm 1}}^{alg}))^{\mathbb{Z}_3} \xrightarrow{B} (HH_{n-1}(\mathcal{A}_{\theta,\omega^{\pm 1}}^{alg}))^{\mathbb{Z}_3} \xrightarrow{I} ....$$

We know that $(HH_2(\mathcal{A}_{\theta,\omega^{\pm 1}}^{alg})) = (HH_1(\mathcal{A}_{\theta,\omega^{\pm 1}}^{alg})) = 0$. Hence we obtain that

$$(HC_2(\mathcal{A}_{\theta,\omega^{\pm 1}}^{alg}))^{\mathbb{Z}_3} \cong (HC_0(\mathcal{A}_{\theta,\omega^{\pm 1}}^{alg}))^{\mathbb{Z}_3}.$$

But, a preliminary result shows that $HH_0(\mathcal{A}^{alg}_{\theta,\omega^{\pm 1}}) = HC_0(\mathcal{A}^{alg}_{\theta,\omega^{\pm 1}})$. Hence, we obtain that

$$(HC_2(\mathcal{A}^{alg}_{\theta,\omega^{\pm 1}}))^{\mathbb{Z}_3} = (HC_0(\mathcal{A}^{alg}_{\theta,\omega^{\pm 1}}))^{\mathbb{Z}_3} \cong \mathbb{C}^3.$$

Also, since $(HH_2(\mathcal{A}^{alg}_{\theta,\omega^{\pm 1}}))^{\mathbb{Z}_3} = (HH_3(\mathcal{A}^{alg}_{\theta,\omega^{\pm 1}}))^{\mathbb{Z}_3} = 0$, we obtain that

$$(HC_3(\mathcal{A}^{alg}_{\theta,\omega^{\pm 1}}))^{\mathbb{Z}_3} \cong (HC_1(\mathcal{A}^{alg}_{\theta,\omega^{\pm 1}}))^{\mathbb{Z}_3}.$$

Since $HH_1(\mathcal{A}^{alg}_{\theta,\omega^{\pm 1}})$ is trivial, we have

$$(HC_3(\mathcal{A}^{alg}_{\theta,\omega^{\pm 1}}))^{\mathbb{Z}_3} \cong (HC_1(\mathcal{A}^{alg}_{\theta,\omega^{\pm 1}}))^{\mathbb{Z}_3} = 0.$$

□

3.5.3 Periodic cyclic homology

Theorem 3.25. *$HC_{even}(\mathcal{A}^{alg}_{\theta} \rtimes \mathbb{Z}_3) \cong \mathbb{C}^8$ while $HC_{odd}(\mathcal{A}^{alg}_{\theta} \rtimes \mathbb{Z}_3) = 0$.*

Proof. We have

$$... \xrightarrow{B} (HH_2(\mathcal{A}^{alg}_{\theta}))^{\mathbb{Z}_3} \xrightarrow{I} (HC_2(\mathcal{A}^{alg}_{\theta}))^{\mathbb{Z}_3} \xrightarrow{S} (HC_0(\mathcal{A}^{alg}_{\theta}))^{\mathbb{Z}_3} \xrightarrow{B} (HH_1(\mathcal{A}^{alg}_{\theta,}))^{\mathbb{Z}_3} \xrightarrow{I} ....$$

Since $HH_2(\mathcal{A}^{alg}_{\theta})^{\mathbb{Z}_3} \cong HC_0(\mathcal{A}^{alg}_{\theta})^{\mathbb{Z}_3} \cong \mathbb{C}$. Also we have the following isomorphism of the homology groups $(HC_{even}(\mathcal{A}^{alg}_{\theta,\omega^{\pm 1}}))^{\mathbb{Z}_3} \cong (HC_2(\mathcal{A}^{alg}_{\theta,\omega^{\pm 1}}))^{\mathbb{Z}_3} \cong \mathbb{C}^3$. Hence we conclude that

$$HC_{even}(\mathcal{A}^{alg}_{\theta} \rtimes \mathbb{Z}_3) \cong \mathbb{C}^8.$$

As for the odd cyclic homology, we have $(HC_3(\mathcal{A}^{alg}_{\theta,\omega^{\pm 1}}))^{\mathbb{Z}_3} \cong (HC_1(\mathcal{A}^{alg}_{\theta,\omega^{\pm 1}}))^{\mathbb{Z}_3} = 0$, and we also have $HC_3(\mathcal{A}^{alg}_{\theta})^{\mathbb{Z}_3} = HC_1(\mathcal{A}^{alg}_{\theta})^{\mathbb{Z}_3} = HH_1(\mathcal{A}^{alg}_{\theta})^{\mathbb{Z}_3} = 0$. Combining these two results, we obtain that

$$HC_{odd}(\mathcal{A}^{alg}_{\theta} \rtimes \mathbb{Z}_3) = 0.$$

So we have computed the Hochschild and cyclic homology of the $\mathbb{Z}_3$ orbifold. □

3.6 $\mathbb{Z}_4$ action on $\mathcal{A}_\theta^{alg}$

The group $\mathbb{Z}_4$ is embedded in $SL(2,\mathbb{Z})$ through its generator $g = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \in SL(2,\mathbb{Z})$.
The generator acts of $\mathcal{A}_\theta^{alg}$ in the following way

$$U_1 \mapsto U_2^{-1}, U_2 \mapsto U_1.$$

3.6.1 Hochschild homology

Theorem 3.26. *Let i be the generator of $\mathbb{Z}_4$, we have $HH_0((\mathcal{A}_{\theta,\pm i}^{alg\bullet})^{\mathbb{Z}_4}) \cong \mathbb{C}^2$, while $HH_k((\mathcal{A}_{\theta,\pm i}^{alg\bullet})^{\mathbb{Z}_4})$ are trivial groups for $k \geq 1$.*

Lemma 3.27. *Consider the following chain complex $J_{*,\pm i}^{\mathbb{Z}_4}$*

$$J_{*,\pm i}^{\mathbb{Z}_4} := 0 \xleftarrow{\pm_i b} (\mathcal{A}_{\theta,\pm i}^{alg})^{\mathbb{Z}_4} \xleftarrow{\pm_i b} ((\mathcal{A}_{\theta,\pm i}^{alg})^{\otimes 2})^{\mathbb{Z}_4} \xleftarrow{\pm_i b} ((\mathcal{A}_{\theta,\pm i}^{alg})^{\otimes 3})^{\mathbb{Z}_4} \ldots$$

where,

$$_{\pm i}b(a_0 \otimes a_1 \otimes .... \otimes a_n) = b'(a_0 \otimes a_1 \otimes .... \otimes a_n) + (-1)^n((\mp i \cdot a_n)a_0 \otimes a_1 \otimes .... \otimes a_{n-1}).$$

Then,

$$H_\bullet(J_{*,\pm i}^{\mathbb{Z}_4}, {}_{\pm i}b) = (H_\bullet(J_*(\mathcal{A}_\theta^{alg}, {}_{\pm i}\mathcal{A}_\theta^{alg})), b)^{\mathbb{Z}_4}.$$

Hence using the adjusted Connes' complex for the algebraic case we can now calculate the homology groups.

- $H_0(\mathcal{A}_\theta^{alg}, {}_{\pm i}\mathcal{A}_\theta^{alg}) = {}_{\pm i}\mathcal{A}_\theta^{alg} \otimes_{\mathfrak{B}_\theta^{alg}} \mathfrak{B}_\theta^{alg}/Image(1 \otimes b_1)$,
- $H_1(\mathcal{A}_\theta^{alg}, {}_{\pm i}\mathcal{A}_\theta^{alg}) = Ker(1 \otimes b_1)/Image(1 \otimes b_2)$,
- $H_2(\mathcal{A}_\theta^{alg}, {}_{\pm i}\mathcal{A}_\theta^{alg}) = Ker(1 \otimes b_2)$.

Lemma 3.28. $H_2(\mathcal{A}_\theta^{alg}, {}_i\mathcal{A}_\theta^{alg}) \cong 0$.

Proof. We prove here for case $g = i$, the proof for $g = -i$ is similar. Consider the map $(1 \otimes b_2)$ in the tensor complex. To calculate the kernel of this map we have a closer look at this map.

$$(1 \otimes b_2)(a \otimes I) = a \otimes_{\mathfrak{B}_\theta^{alg}} (U_2 \otimes I - \lambda \otimes U_2) \otimes e_1 - a \otimes_{\mathfrak{B}_\theta^{alg}} (\lambda U_1 \otimes I - I \otimes U_1) \otimes e_2$$

Using the twisted bimodule structure of ${}_i\mathcal{A}_\theta^{alg}$ over $\mathcal{A}_\theta^{alg}$, we simplify the above equation to the following,

$$(1 \otimes b_2)(a \otimes I) = (aU_2 - \lambda U_1 a, U_2^{-1}a - \lambda a U_1).$$

Hence we obtain the following relation over an element $(a \otimes 1)$ to reside in $ker(1 \otimes b_2)$.

$$H_2(\mathcal{A}_\theta^{alg}, {}_\omega\mathcal{A}_\theta^{alg}) = \left\{ a \in {}_i\mathcal{A}_\theta^{alg} | \lambda a_{n-1,m} = a_{n,m-1}; \lambda^{m-n} a_{n-1,m} = \lambda^2 a_{n,m+1} \right\}.$$

Since, no such nontrivial element exists in ${}_i\mathcal{A}_\theta^{alg}$ because if it did then $|a_{n,m}| = |a_{n,m+2}|$. So, we have the desired result,

$$H_2(\mathcal{A}_\theta^{alg}, {}_i\mathcal{A}_\theta^{alg}) = 0.$$

□

Lemma 3.29. $H_0(\mathcal{A}_\theta^{alg}, {}_i\mathcal{A}_\theta^{alg})^{\mathbb{Z}_4} \cong \mathbb{C}^2$.

Proof. We have the map $(1 \otimes b_1)$ in the tensor complex defined below,

$$(1 \otimes b_1)(a \otimes I \otimes e_j) = a \otimes_{\mathfrak{B}_\theta^{alg}} (I \otimes U_j - U_j \otimes I) = U_j^{-1}a - aU_j.$$

As before if we use the twisted bicomplex structure of ${}_i\mathcal{A}_\theta^{alg}$ over the algebra $\mathcal{A}_\theta^{alg}$, the map$(1 \otimes b_1)$ can be simplified as follows,

$$b_1(a_1, 0) = a_1 U_1 - U_2^{-1} a_1 \text{ and } b_1(0, a_2) = a_2 U_2 - U_1 a_2.$$

Observe that $b_1(a_1, a_2) = b_1(a_1, 0) + b_1(0, a_2)$. Further we can simplify the calculation by considering only elements of the type $b_1(U_1^n U_2^m, 0)$, and similarly for $b_1(0, a_2)$ we can consider the elements of the type $b_1(0, U_1^n U_2^m)$. We observe that

$$b_1(U_1^n U_2^m, 0) = ({U_1}^n {U_2}^m)U_1 - U_2^{-1}({U_1}^n {U_2}^m) = \lambda^{-n}(\lambda^m U_1 U_2 - 1)U_1^n U_2^{m-1}$$

and

$$b_1(0, U_1^n U_2^{m-1}) = ({U_1}^n {U_2}^{m-1})U_2 - U_1({U_1}^n {U_2}^{m-1}) = U_1^n U_2^m (U_2 - \lambda^{-n} U_1).$$

From the above relations it is clear that in the quotient space, we have only the coefficients $a_{0,0}$ and $a_{1,0}$ remaining. Hence we now have the following result.

$$H_0(\mathcal{A}_\theta^{alg}, {}_i\mathcal{A}_\theta^{alg}) = {}_i\mathcal{A}_\theta^{alg} / \langle \lambda^{-n}(\lambda^m U_1 U_2 - 1)U_1^n U_2^{m-1}, U_1^n U_2^m (U_2 - \lambda^{-n} U_1) \rangle$$

Now to obtain $(H_0(\mathcal{A}_\theta^{alg}, {}_i\mathcal{A}_\theta^{alg}))^{\mathbb{Z}_4}$ we consider complex map $h : J_* \to C_*$, since the map $h_0 : J_0(\mathcal{A}_\theta^{alg}) \to C_0(\mathcal{A}_\theta^{alg})$ is the identity map, the $\mathbb{Z}_4$ action on the bar complex is translated to the Kozul complex with no alteration. Hence we get that

$$HH_0((\mathcal{A}_{\theta,i}^{alg\bullet})^{\mathbb{Z}_4}) = \langle \bar{a_{0,0}}, \bar{a_{1,0}} \rangle.$$

□

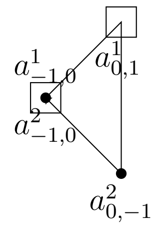
Lemma 3.30. *$H_1(\mathcal{A}_\theta^{alg}, {}_i\mathcal{A}_\theta^{alg}) \cong 0$.*

Proof. From the previous calculations we do have an explicit formula for the kernel and the image equations of $H_1(\mathcal{A}_\theta^{alg}, {}_i\mathcal{A}_\theta^{alg})$.

$$\lambda^{-m} a^1_{n-1,m} - \lambda^n a^1_{n,m+1} = a^2_{n-1,m} - a^2_{n,m-1} \text{ (Kernel Equation)}$$

$$a^1_{n,m} = a_{n-1,m} - \lambda a_{n,m-1} \text{ and } a^2_{n,m} = \lambda^{-m} a_{n-1,m} - \lambda^{1+n} a_{n,m+1} \text{ (Image Solution)}$$

As before, *a,b,c, and d* are said to be connected if there exists a kernel equation containing them. So, any given kernel solution will have its elements connected by the kernel diagram. e.g. equation $\lambda^{-0}a^1_{-1,0} - \lambda^0 a^1_{0,1} = a^2_{-1,0} - a^2_{0,-1}$ is drawn below:



Now, the proof will proceed with induction over the *number* of non-zero elements in a given kernel solution.

After going through the process as in the previous cases, one figures out that there is no kernel solution with the number of non-zero entries less than or equal to 3. Assume that all kernel solutions having number of non-zero elements less than or equal to $(x-1)$ come from image. Then consider a kernel solution S_0 with x non-zero elements in it. Since this solution is finitely supported over the lattice plane, there exists a closed square region β over which S_0 is supported. We shall now construct a new solution S_1 from S_0, with the number of non-zero elements in S_1 at most equal to x.

Inside β, consider the left most column at least one point of which is non-zero. Choose the bottom point μ of this column. It is clear that $\mu = a^2_{r,s}$ for some $(r,s) \in \mathbb{Z}^2$. As if it were $a^1_{r,s}$ then the following kernel solution,

$$\lambda^{1-s}a^1_{r-1,s-1} - \lambda^r a^1_{r,s} = a^2_{r-1,s-1} - a^2_{r,s-2}$$

As, in it all but $a^1_{r,s}$ are zero. This is a contradiction.

Now a new solution S_1 shall be constructed from S_0, with the number of elements at most equal to x. To do so, consider the following map $\wedge$

$$\wedge : \mathbb{Z}^2 \to \mathbb{Z}^2, \text{ such that}$$

$(a^1_{r,s+2})^\wedge = (a^2_{r,s})^\wedge = 0.$

And let $(a^1_{r+1,s+1})^\wedge = a^1_{r+1,s+1} + \lambda^{-1-r} a^2_{r,s}; (a^2_{r+1,s+1})^\wedge = a^2_{r+1,s+1} + \lambda^{-2-s} a^2_{r,s}$ and $(a^j_{p,q})^\wedge = a^j_{p,q}$ for other lattice points.

Lemma 3.31. *Let $S_1 := \wedge(S_0)$. Then S_1 is a kernel solution such that $|S_1| \leq |S_0|$. If for some $u \in {}_\omega\mathcal{A}^{alg}_\theta$, $b_2(u) = S_1$ then $b_2(u') = S_0$, where $u' = u + {}_{r,s}g$,*

$$_{r,s}g_{p,q} = \begin{cases} \lambda^{-1} a^1_{r,s+2} & \textit{if } (p,q) = (r, s+1) \\ 0 & \textit{else.} \end{cases}$$

Proof. Second part is clear that the cardinality of non-zero entries is reduced by 2 when it is asked that $a'^1_{r,s+2} = a'^2_{r,s} = 0$ but then the cardinality will then increase by two in case both $a^1_{r+1,s}$ and $a^2_{r+1,s}$ are zero in S_0.

To check that S_1 is a solution, it is enough to check that the kernel equations containing any of these four altered elements hold in S_1. That is,

$$\lambda^{-m} a^1_{n-1,m} - \lambda^n a^1_{n,m+1} = a^2_{n-1,m} - a^2_{n,m-1} \text{ holds}$$

for $(n,m) = (r, s+1), (r+1, s+2), (r+1, s)$ and $(r+2, s+1)$.

<u>Case 1: $(n,m) = (r, s+1)$</u>

$\lambda^{-1-s}(a^1_{r-1,s+1})^\wedge - \lambda^r (a^1_{r,s+2})^\wedge = (a^2_{r-1,s+1})^\wedge - (a^2_{r,s})^\wedge$

$\implies \lambda^{-1-s} a^1_{r-1,s+1} - 0 = a^2_{r-1,s+1} - 0.$

This holds as : $a^1_{r-1,s+1} = a^2_{r-1,s+1} = 0.$

<u>Case 2: $(n,m) = (r+1, s+2)$</u>

$\lambda^{-2-s}(a^1_{r,s+2})^\wedge - \lambda^{r+1}(a^1_{r+1,s+3})^\wedge = (a^2_{r,s+2})^\wedge - (a^2_{r+1,s+1})^\wedge$

$\implies 0 - \lambda^{r+1} a^1_{r+1,s+3} = a^2_{r,s+2} - a^2_{r+1,s+1} - \lambda^{-2-s} a^1_{r,s+2}$

$\implies \lambda^{-2-s} a^1_{r,s+2} - \lambda^{r+1} a^1_{r+1,s+3} = a^2_{r,s+2} - a^2_{r+1,s+1}.$

This holds as S_0 was a kernel solution.

<u>Case 3: $(n, m) = (r+1, s)$</u>

$\lambda^{-s}(a^1_{r,s})^\wedge - \lambda^{r+1}(a^1_{r+1,s+1})^\wedge = (a^2_{r,s})^\wedge - (a^2_{r+1,s-1})^\wedge$

$\implies \lambda^{-s} a^1_{r,s} - \lambda^{r+1}(a^1_{r+1,s+1} + \lambda^{-1-r} a^2_{r,s}) = 0 - a^2_{r+1,s-1}$

$\implies \lambda^{-s} a^1_{r,s} - \lambda^{r+1} a^1_{r+1,s+1} = a^2_{r,s} - a^2_{r+1,s-1}.$

This holds as S_0 was a kernel solution.

<u>Case 4: $(n, m) = (r+2, s+1)$</u>

$\lambda^{-1-s}(a^1_{r+1,s+1})^\wedge - \lambda^{r+2}(a^1_{r+2,s+2})^\wedge = (a^2_{r+1,s+1})^\wedge - (a^2_{r+2,s})^\wedge$

$\lambda^{-1-s}(a^1_{r+1,s+1} + \lambda^{-1-r} a^2_{r,s}) - \lambda^{r+2} a^1_{r+2,s+2} = a^2_{r+1,s+1} + \lambda^{-2-s} a^1_{r,s+2} - a^2_{r+2,s}$

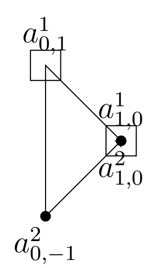
$\lambda^{-1-s} a^1_{r+1,s+1} + \lambda^{-2-s-r} a^2_{r,s} - \lambda^{r+2} a^1_{r+2,s+2} = a^2_{r+1,s+1} + \lambda^{-2-s} a^1_{r,s+2} - a^2_{r+2,s}$

Using $a^1_{r,s+2} = \lambda^{-r} a^2_{r,s}$, we can reduce the above equation to the following equality.

$$\lambda^{-1-s} a^1_{r+1,s+1} - \lambda^{r+2} a^1_{r+2,s+2} = a^2_{r+1,s+1} - a^2_{r+2,s}$$

This holds as S_0 was a kernel solution.

It is easy to see that like kernel diagram we can construct image diagrams. These diagrams are obtained by looking at how b_2-image of a non-zero lattice point look like as an element of $\mathcal{A}^{alg}_\theta \oplus \mathcal{A}^{alg}_\theta$.



So, it is clear from the diagram as well as from the image solution equations an image element induces kernel solution elements to its right and up for a^1, or right and below for a^2 elements.

Hence the plausible image elements for the elements of S_1 other than the four altered ones are not altered and hence (a_1, a_2) at these lattice point come from image elements by induction.

So, it is clear from the diagram as well as from the explicit equations that we have for the map b_2, an image element $a_{0,0}$ induces kernel solution elements to its right-below and up for a^1, or right and below for a^2 elements. Hence if we verify that

$$b_2(u')_{p,q} = (a^1_{p,q}, a^2_{p,q}) \text{ for } (p,q) = (r, s+2), (r,s), (r+1, s+1)$$

then we have proved that $b_2(u') = S_0$.

Observe that

$(b_2(u'))_{r,s+2} = (b_2(u + {}_{rs}g))_{r,s+2} = (b_2(u))_{r,s+2} + (b_2({}_{rs}g))_{r,s+2} = ((a^1_{r,s+2})^\wedge, (a^2_{r,s+2})^\wedge) + (a^1_{r,s+2}, 0) = (a^1_{r,s+2}, a^2_{r,s+2})$,

$(b_2(u'))_{r,s} = (b_2(u + {}_{rs}g))_{r,s} = (b_2(u))_{r,s} + (b_2({}_{rs}g))_{r,s} = ((a^1_{r,s})^\wedge, (a^2_{r,s})^\wedge) + (0, \lambda^r a^1_{r,s+2}) = (a^1_{r,s}, a^2_{r,s})$, and

$(b_2(u'))_{r+1,s+1} = (b_2(u + {}_{rs}g))_{r+1,s+1} = (b_2(u))_{r+1,s+1} + (b_2({}_{rs}g))_{r+1,s+1}$
$= ((a^1_{r+1,s+1})^\wedge, (a^2_{r+1,s+1})^\wedge) + (-\lambda^{-1-r} a^2_{r,s}, -\lambda^{-s-2} a^1_{r,s}) = (a^1_{r+1,s+1}, a^2_{r+1,s+1})$.

□

Lemma 3.32. $\wedge^N(S_0) = 0$ *for any solution* S *for a sufficiently large number* N.

Proof. The lemma states that $|S_1| \leq |S_0|$ is strict after sufficiently many iterations. Notice that the above process reduces the number of non-zero entries in the left most column by 2. And when this process is iterated this will lead to shifting of the left most non-zero column rightwards. Pictorially if it is assumed that at no stage the iteration reduces the number of non-zero elements, then these many elements are compressed by the shifting of the left most non-zero column towards right by this process as the right edge of β is not changed by this process unless all the non-zero elements are collected on this right edge of β.

Hence, all the non-zero elements of $\wedge^d(S_0)$ will at some stage be bounded to the right edge of β. But, then, consider the lowest element of $\wedge^d(S_0)$ on this edge. It has to be only $a^2_{w,t}$ for some (w,t). Consider the following kernel equation,

$$\lambda^{-t}a^1_{w,t} - \lambda^{w+1}a^1_{w+1,t+1} = a^2_{w,t} - a^2_{w+1,t-1}\ .$$

The above equation has all but $a^2_{w,t}$ as zero, which is a contradiction. □

Hence, with the above lemmas we conclude that

$$HH_1(\mathcal{A}^{alg}_{\theta,i}) = 0.$$

Above computations were for $g = i \in \mathbb{Z}_4$. Action of $-i \in \mathbb{Z}_4$ differs from that of i by swapping U_1 and U_2. That is action of i on torus generated by $U_2U_1 = \lambda U_1U_2$ is same as $-i$ acts on torus generated by $U_1U_2 = \lambda U_2U_1$. Now, since, the results are independent of λ, so considering λ^{-1} would give the same result. Hence we have the following result,

$$HH_0(\mathcal{A}^{alg\bullet}_{\theta,\pm i})^{\mathbb{Z}_4} \cong \mathbb{C}^2,\ HH_1(\mathcal{A}^{alg\bullet}_{\theta,\pm i}) = 0,\ \text{and}\ HH_2(\mathcal{A}^{alg\bullet}_{\theta,\pm i}) = 0.$$

□

Theorem 3.33. *The Hochschild homology groups for $\mathcal{A}^{alg}_\theta \rtimes \mathbb{Z}_4$ are as follows*

$$HH_0(\mathcal{A}^{alg}_\theta \rtimes \mathbb{Z}_4) \cong \mathbb{C}^8;\ HH_1(\mathcal{A}^{alg}_\theta \rtimes \mathbb{Z}_4) \cong 0;\ HH_2(\mathcal{A}^{alg}_\theta \rtimes \mathbb{Z}_4) \cong \mathbb{C}.$$

Proof. We know that $HH_0(\mathcal{A}^{alg}_{\theta,\pm i}) \cong \mathbb{C}^2$, following the process of checking the invariance of a cycle, we obtain that $HH_0(\mathcal{A}^{alg}_{\theta,\pm i})^{\mathbb{Z}_4} \cong \mathbb{C}^2$. Similarly we compute that $HH_0(\mathcal{A}^{alg}_{\theta,-1})^{\mathbb{Z}_4} \cong \mathbb{C}^3$. To see this we consider the action of i on the elements of $HH_0(\mathcal{A}^{alg}_{\theta,-1})$, we observe that under this action

- $1 \mapsto 1$,
- $U_1 \mapsto U_2^{-1} \sim U_2$,

- $U_2 \mapsto U_1$,
- $U_1U_2 \mapsto U_2^{-1}U_1 \sim U_1U_2$..

Hence we see that

$$\varphi = a1 + bU_1 + cU_2 + dU_1U_2 \mapsto a1 + bU_2 + cU_1 + dU_1U_2.$$

The above element is invariant iff $b = c$ hence we have a 3 dimensional invariant sub-space of $HH_0(\mathcal{A}_{\theta,-1}^{alg})$. Hence, we obtain that

$$HH_0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_4) \cong \mathbb{C}^8.$$

Since we have $HH_1(\mathcal{A}_\theta^{alg}, {}_{\pm i}\mathcal{A}_\theta^{alg}) = HH_1(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg}) = HH_1(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg})^{\mathbb{Z}_4} = 0$, we conclude that

$$HH_1(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_4) \cong 0.$$

As, $HH_2(\mathcal{A}_{\theta,\pm i}^{alg}) = HH_2(\mathcal{A}_{\theta,-1}^{alg}) = 0$, while $HH_2(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_4} \cong \mathbb{C}$. We have

$$HH_2(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_4) \cong \mathbb{C}.$$

□

3.6.2 Cyclic homology of $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_4$

Theorem 3.34. *$HC_{even}(\mathcal{A}_{\theta,\pm i}^{alg})^{\mathbb{Z}_4} \cong \mathbb{C}^2$, while $HC_{odd}(\mathcal{A}_{\theta,\pm i}^{alg})^{\mathbb{Z}_4} = 0$.*

Proof. We apply the S, B, I long exact sequence relating the Hochschild and cyclic homology of an algebra A.

$$... \xrightarrow{B} HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} ....$$

Since the $\mathbb{Z}_4$ action on $\mathcal{A}_{\theta,\pm i}^{alg}$ commutes with the map ${}_{\pm i}b$, we obtain the following exact sequence

$$\dots \xrightarrow{B} (HH_n(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4} \xrightarrow{I} (HC_n(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4} \xrightarrow{S} (HC_{n-2}(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4} \xrightarrow{B} (HH_{n-1}(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4} \xrightarrow{I} \dots.$$

We know that $(HH_2(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4} = (HH_1(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4} = 0$. Hence we obtain that

$$(HC_2(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4} \cong (HC_0(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4}.$$

But, a preliminary result shows that $HH_0(\mathcal{A}_{\theta,\pm i}^{alg}) = HC_0(\mathcal{A}_{\theta,\pm i}^{alg})$. Hence, we obtain that

$$(HC_2(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4} = (HC_0(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4} \cong \mathbb{C}^2.$$

Also, since $(HH_2(\mathcal{A}_{\theta,\omega^{\pm 1}}^{alg}))^{\mathbb{Z}_4} = (HH_3(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4} = 0$, we obtain that

$$(HC_3(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4} \cong (HC_1(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4}.$$

Since $HH_1(\mathcal{A}_{\theta,\pm i}^{alg}) = 0$, we have $(HC_1(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4} = 0$. Hence, we have

$$HC_{odd}(\mathcal{A}_{\theta,\pm i}^{alg})^{\mathbb{Z}_4} = 0.$$

□

3.6.3 Periodic cyclic homology

Theorem 3.35. *$HC_{even}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_4) \cong \mathbb{C}^9$ while $HC_{odd}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_4) = 0$.*

Proof. We have.

$$\dots \xrightarrow{B} (HH_2(\mathcal{A}_\theta^{alg}))^{\mathbb{Z}_4} \xrightarrow{I} (HC_2(\mathcal{A}_\theta^{alg}))^{\mathbb{Z}_4} \xrightarrow{S} (HC_0(\mathcal{A}_\theta^{alg}))^{\mathbb{Z}_4} \xrightarrow{B} (HH_1(\mathcal{A}_{\theta,}^{alg}))^{\mathbb{Z}_4} \xrightarrow{I} \dots$$

Since $HH_2(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_4} \cong HC_0(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_4} \cong \mathbb{C}$, we have $(HC_2(\mathcal{A}_\theta^{alg}))^{\mathbb{Z}_4} \cong \mathbb{C}^2$. Also we notice here that the $\mathbb{Z}_4$ invariant sub-space of $HC_{even}(\mathcal{A}_{\theta,-1}^{alg})$ is 3 dimensional. Hence we conclude that

$$HC_{even}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_4) \cong \mathbb{C}^9.$$

As for the odd cyclic homology, we have $(HC_3(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4} \cong (HC_1(\mathcal{A}_{\theta,\pm i}^{alg}))^{\mathbb{Z}_4} = 0$, and we also have $HC_3(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_4} = HC_1(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_4} = HH_1(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_4} = 0$. Combining these two results, we obtain that

$$HC_{odd}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_4) = 0.$$

So we have computed the Hochschild and cyclic homology of the $\mathbb{Z}_4$ orbifold. $\square$

3.7 $\mathbb{Z}_6$ action on $\mathcal{A}_\theta^{alg}$

The group $\mathbb{Z}_6$ is embedded in $SL(2,\mathbb{Z})$ through its generator $g = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \in SL(2,\mathbb{Z})$. The generator acts of $\mathcal{A}_\theta^{alg}$ in the following way

$$U_1 \mapsto U_2, U_2 \mapsto \frac{U_1^{-1}U_2}{\sqrt{\lambda}}.$$

3.7.1 Hochschild homology

We will use $-\omega$ to stand for the generator of $\mathbb{Z}_6$.

Theorem 3.36. *$HH_0((\mathcal{A}_{\theta,-\omega}^{alg\bullet})^{\mathbb{Z}_6}) \cong \mathbb{C}$, while $HH_k((\mathcal{A}_{\theta,-\omega}^{alg\bullet})^{\mathbb{Z}_6})$ are trivial groups for $k \geq 1$.*

Lemma 3.37. *Consider the following chain complex $J_{*,-\omega}^{\mathbb{Z}_6}$*

$$J_{*,-\omega}^{\mathbb{Z}_6} := 0 \xleftarrow{-\omega b} (\mathcal{A}_{\theta,-\omega}^{alg})^{\mathbb{Z}_6} \xleftarrow{-\omega b} ((\mathcal{A}_{\theta,-\omega}^{alg})^{\otimes 2})^{\mathbb{Z}_6} \xleftarrow{-\omega b} ((\mathcal{A}_{\theta,-\omega}^{alg})^{\otimes 3})^{\mathbb{Z}_6} \ldots$$

where,

$${}_{-\omega}b(a_0 \otimes a_1 \otimes \ldots . \otimes a_n) = b'(a_0 \otimes a_1 \otimes \ldots . \otimes a_n) + (-1)^n((-\omega^2 \cdot a_n)a_0 \otimes a_1 \otimes \ldots . \otimes a_{n-1}).$$

Then,

$$H_\bullet(J_{*,-\omega}^{\mathbb{Z}_6}, {}_{-\omega}b) = (H_\bullet(J_*(\mathcal{A}_\theta^{alg}, {}_{-\omega}\mathcal{A}_\theta^{alg})), b)^{\mathbb{Z}_6}$$

Hence using the adjusted Connes' complex for the algebraic case we can now calculate the homology groups.

- $H_0(\mathcal{A}_\theta^{alg}, {}_{-\omega}\mathcal{A}_\theta^{alg}) = {}_{-\omega}\mathcal{A}_\theta^{alg} \otimes_{\mathfrak{B}_\theta^{alg}} \mathfrak{B}_\theta^{alg}/Image(1 \otimes b_1)$,
- $H_1(\mathcal{A}_\theta^{alg}, {}_{-\omega}\mathcal{A}_\theta^{alg}) = Ker(1 \otimes b_1)/Image(1 \otimes b_2)$,

- $H_2(\mathcal{A}_\theta^{alg}, {}_{-\omega}\mathcal{A}_\theta^{alg}) = Ker(1 \otimes b_2)$.

Lemma 3.38. $H_2(\mathcal{A}_\theta^{alg}, {}_{-\omega}\mathcal{A}_\theta^{alg}) \cong 0$.

Proof. We prove here for case $g = -\omega$, the proof for $g = -\omega^2$ is similar. Consider the map $(1 \otimes b_2)$ in the tensor complex. To calculate the kernel of this map we have a closer look at this map,

$$(1 \otimes b_2)(a \otimes I) = a \otimes_{\mathfrak{B}_\theta^{alg}} (U_2 \otimes I - \lambda \otimes U_2) \otimes e_1 - a \otimes_{\mathfrak{B}_\theta^{alg}} (\lambda U_1 \otimes I - I \otimes U_1) \otimes e_2.$$

Using the twisted bimodule structure of ${}_{-\omega}\mathcal{A}_\theta^{alg}$ over $\mathcal{A}_\theta^{alg}$, we simplify the equation to the following,

$$(1 \otimes b_2)(a \otimes I) = (aU_2 - \frac{U_1^{-1}U_2}{\sqrt{\lambda}}a, U_2 a - \lambda a U_1).$$

Hence we obtain the following relation over an element $(a \otimes 1)$ to reside in $ker(1 \otimes b_2)$.

$$H_2(\mathcal{A}_\theta^{alg}, {}_{-\omega}\mathcal{A}_\theta^{alg}) = \left\{ a \in {}_{-\omega}\mathcal{A}_\theta^{alg} | \lambda^{n+1.5} a_{n+1,m} = a_{n,m-1}; \lambda^{m-n} a_{n-1,m} = \lambda a_{n,m-1} \right\}.$$

Since, no such nontrivial element exists in ${}_{-\omega}\mathcal{A}_\theta^{alg}$ because if it did then $|a_{n,m}| = |a_{n+2,m}|$. So, we have the desired result.

$$H_2(\mathcal{A}_\theta^{alg}, {}_{-\omega}\mathcal{A}_\theta^{alg}) = 0.$$

□

Lemma 3.39. $H_0(\mathcal{A}_\theta^{alg}, {}_{-\omega}\mathcal{A}_\theta^{alg})^{\mathbb{Z}_6} \cong \mathbb{C}$.

Proof. We have the map $(1 \otimes b_1)$ in the tensor complex defined below,

$$(1 \otimes b_1)(a \otimes I \otimes e_j) = a \otimes_{\mathfrak{B}_\theta^{alg}} (I \otimes U_j - U_j \otimes I) = U_j^{-1} a - a U_j.$$

As before if we use the twisted bicomplex structure of ${}_{-\omega}\mathcal{A}_\theta^{alg}$ over the algebra $\mathcal{A}_\theta^{alg}$, the map$(1 \otimes b_1)$ can be simplified as follows,

$$b_1(a_1,0) = a_1U_1 - U_2a_1 \text{ and } b_1(0,a_2) = a_2U_2 - \lambda^{-0.5}U_1^{-1}U_2a_2.$$

Observe that $b_1(a_1,a_2) = b_1(a_1,0) + b_1(0,a_2)$. Further we can simplify the calculation by considering only elements of the type $b_1(U_1^nU_2^m,0)$, and similarly for $b_2(0,a_2)$ we can consider the elements of the type $b_2(0,U_1^nU_2^m)$. We observe that

$$b_1(U_1^nU_2^m,0) = ({U_1}^n{U_2}^m)U_1 - U_2({U_1}^n{U_2}^m) = (\lambda^mU_1 - U_2)U_1^nU_2^m$$

and

$$b_2(0,U_1^nU_2^{m-1}) = ({U_1}^n{U_2}^{m-1})U_2 - \lambda^{-0.5}U_1^{-1}U_2({U_1}^n{U_2}^{m-1}) = (U_1 - \lambda^{n-0.5})U_1^{n-1}U_2^m.$$

From the above relations it is clear that in the quotient space, we have only the coefficients $a_{0,0}$ remains. Hence we now have the following result.

$$H_0(\mathcal{A}_\theta^{alg}, {}_{-\omega}\mathcal{A}_\theta^{alg}) = {}_{-\omega}\mathcal{A}_\theta^{alg}/\langle\lambda^{-n}(\lambda^mU_1 - \lambda^nU_2)U_1^nU_2^m, (U_1 - \lambda^{n-0.5})U_1^{n-1}U_2^m\rangle$$

Now to obtain $(H_0(\mathcal{A}_\theta^{alg}, {}_{-\omega}\mathcal{A}_\theta^{alg}))^{\mathbb{Z}_6}$ we consider complex map $h : J_* \to C_*$, since the map $h_0 : J_0(\mathcal{A}_\theta^{alg}) \to C_0(\mathcal{A}_\theta^{alg})$ is the identity map, the $\mathbb{Z}_6$ action on the bar complex is translated to the Kozul complex with no alteration. Hence the equivalence class of $\bar{a_{0,0}}$ is invariant under $\mathbb{Z}_6$ action. So we have

$$HH_0((\mathcal{A}_{\theta,-\omega}^{alg})^{\mathbb{Z}_6}) = \langle\bar{a_{0,0}}\rangle$$

□

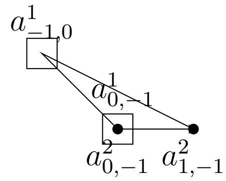
Lemma 3.40. $H_1(\mathcal{A}_\theta^{alg}, {}_{-\omega}\mathcal{A}_\theta^{alg}) \cong 0.$

Proof. From the previous calculations we do have an explicit formula for the kernel and the image equations of $H_1(\mathcal{A}_\theta^{alg}, {}_{-\omega}\mathcal{A}_\theta^{alg})$,

$$\lambda^m a_{n-1,m}^1 - \lambda^n a_{n,m-1}^1 - \lambda^{n+0.5}a_{n+1,m-1}^2 + a_{n,m-1}^2 = 0 \text{ (Kernel Equation)},$$

$$a_{n,m}^1 = \lambda^{n+1.5}a_{n+1,m-1} - a_{n,m-1} \text{ and } a_{n,m}^2 = \lambda^{1+m}a_{n-1,m} - \lambda^n a_{n,m-1} \text{ (Image Solution)}.$$

As before, *a,b,c, and d* are said to be connected if there exists a kernel equation containing them. So, any given kernel solution will have its elements connected by the kernel diagram. e.g. equation $\lambda^0 a^1_{-1,0} - \lambda^0 a^1_{0,-1} - \sqrt{\lambda} a^2_{1,-1} + a^2_{0,-1}$ is drawn below:



Now, the proof will proceed with induction over the *number* of non-zero elements in a given kernel solution.

After going through the process, explicitly described in previous cases, we conclude that there are no kernel solution with the number of non-zero entries less than or equal to 3. Assume that all kernel solutions having number of non-zero elements less than or equal to $(x-1)$ come from image. Then consider a kernel solution S_0 with x non-zero elements in it. Since this solution is finitely supported over the lattice plane, there exists a closed square region β over which S_0 is supported. We shall now construct a new solution S_1 from S_0, with the number of non-zero elements in S_1 at most equal to x.

Inside β, consider the left most column at least one point of which is non-zero. Choose the bottom point μ of this column. It is clear that $\mu = a^1_{r,s}$ for some $(r,s) \in \mathbb{Z}^2$. As if it were $a^2_{r,s}$ then the following kernel solution.

$$\lambda^{1+s} a^1_{r-2,s+1} - \lambda^{r-1} a^1_{r-1,s} - \lambda^{r-0.5} a^2_{r,s} + a^2_{r-1,s} = 0$$

As all terms in it but $a^2_{r,s}$ are zero, we derive a contradiction.

Now a new solution S_1 shall be constructed from S_0, with the number of elements at most equal to x. To do so, consider the following map $\wedge$

$$\wedge : \mathbb{Z}^2 \to \mathbb{Z}^2, \text{ such that}$$

$(a^1_{r,s})^\wedge = 0;$

$(a^1_{r+1,s})^\wedge = a^1_{r+1,s} + \lambda^{-r-1.5} a^1_{r,s};$

$(a^2_{r+1,s})^\wedge = a^2_{r+1,s} + \lambda^{-0.5} a^1_{r,s};$

$(a^2_{r+2,s-1})^\wedge = a^2_{r+2,s-1} - \lambda^{s-r-1.5} a^1_{r,s}$ and $(a^j_{p,q})^\wedge = a^j_{p,q}$ for other lattice points.

Lemma 3.41. *Let $S_1 := \wedge(S_0)$. Then S_1 is a kernel solution. If for some $u \in {}_{-\omega}\mathcal{A}^{alg}_\theta$, $b_2(u) = S_1$ then $b_2(u') = S_0$, where $u' = u + {}_{r,s}g$,*

$$ {}_{r,s}g_{p,q} = \begin{cases} \lambda^{-r-1.5} a^1_{r,s} & \text{if } (p,q) = (r+1, s-1), \\ 0 & \text{else.} \end{cases} $$

Proof. To check that S_1 is a solution, it is enough to check that the kernel equations containing any of these four altered elements hold in S_1. That is,

$$\lambda^m a^1_{n-1,m} - \lambda^n a^1_{n,m-1} - \lambda^{n+0.5} a^2_{n+1,m-1} + a^2_{n,m-1} = 0 \text{ holds}$$

for $(n,m) = (r, s+1), (r+1, s+1), (r+2, s)$ and $(r+1, s)$

Case 1: $(n,m) = (r, s+1)$

$\lambda^{s+1}(a^1_{r-1,s+1})^\wedge - \lambda^r (a^1_{r,s})^\wedge - \lambda^{r+0.5}(a^2_{r+1,s})^\wedge + (a^2_{r,s})^\wedge = 0$

$\implies \lambda^{s+1} a^1_{r-1,s+1} - 0 - \lambda^{r+0.5}(a^2_{r+1,s} + \lambda^{-0.5} a^1_{r,s}) + a^2_{r,s} = 0$

The above equation holds in S_0.

Case 2: $(n,m) = (r+1, s+1)$

$\lambda^{s+1}(a^1_{r,s+1})^\wedge - \lambda^{r+1}(a^1_{r+1,s})^\wedge - \lambda^{r+1.5}(a^2_{r+2,s})^\wedge + (a^2_{r+1,s})^\wedge = 0$

$\implies \lambda^{s+1} a^1_{r-1,s+1} - \lambda^{r+1}(a^1_{r+1,s} + \lambda^{-r-1.5} a^1_{r,s}) - \lambda^{r+1.5}(a^2_{r+2,s}) + a^2_{r+1,s} + \lambda^{-0.5} a^1_{r,s} = 0$

This holds in S_0.

Case 3: $(n,m) = (r+2, s)$

$\lambda^s (a^1_{r+1,s})^\wedge - \lambda^{r+2}(a^1_{r+2,s-1})^\wedge - \lambda^{r+2.5}(a^2_{r+3,s-1})^\wedge + (a^2_{r+2,s-1})^\wedge = 0$

$\implies \lambda^s(a^1_{r+1,s} + \lambda^{-r-1.5}a^1_{r,s}) - \lambda^{r+2}(a^1_{r+2,s-1}) - \lambda^{r+2.5}(a^2_{r+3,s-1}) + (a^2_{r+2,s-1} - \lambda^{s-r-1.5}a^1_{r,s}) = 0$

Here also the above condition holds as S_0 is a kernel solution.

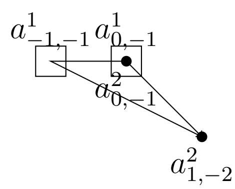
Case 4: $(n, m) = (r+1, s)$

$\lambda^s(a^1_{r,s})^\wedge - \lambda^{r+1}(a^1_{r+1,s-1})^\wedge - \lambda^{r+1.5}(a^2_{r+2,s-1})^\wedge + (a^2_{r+1,s-1})^\wedge = 0$

$\implies 0 - \lambda^{r+1}(a^1_{r+1,s-1}) - \lambda^{r+1.5}(a^2_{r+2,s-1}) + (a^2_{r+2,s-1} + \lambda^s a^1_{r,s}) = 0$

This is a kernel condition in S_0. Hence we have checked all the possible cases and found the relations to be coherent with the kernel condition.

It is easy to see that like kernel diagram we can construct image diagrams. These diagrams are obtained by looking at how b_2-image of a non-zero lattice point look like as an element of $\mathcal{A}^{alg}_\theta \oplus \mathcal{A}^{alg}_\theta$.



So, it is clear from the diagram as well as from the explicit equations that we have for the map b_2, an image element $a_{0,0}$ induces kernel solution elements to its left and up for a^1, and up and right-up for a^2 elements. Hence if we verify that

$$b_2(u')_{p,q} = (a^1_{p,q}, a^2_{p,q}) \text{ for } (p,q) = (r+1, s), (r, s), (r+2, s-1)$$

then we have proved that $b_2(u') = S_0$.

Observe that

$(b_2(u'))_{r,s} = (b_2(u + {}_{rs}g))_{r,s}$

$= (b_2(u))_{r,s} + (b_2({}_{rs}g))_{r,s} = ((a^1_{r,s+2})^\wedge, (a^2_{r,s+2})^\wedge) + (\lambda^{r+1.5}\lambda^{-r-1.5}a^1_{r,s+2}, 0) = (a^1_{r,s+2}, a^2_{r,s+2})$,

$(b_2(u'))_{r+1,s} = (b_2(u + {}_{rs}g))_{r+1,s}$

$= (b_2(u))_{r+1,s} + (b_2({}_{rs}g))_{r+1,s}$

$= ((a^1_{r+1,s})^\wedge, (a^2_{r+1,s})^\wedge) + (-\lambda^{-r-1.5}a^1_{r,s}, \lambda^{1+r}\lambda^{-r-1.5}a^1_{r,s}) = (a^1_{r+1,s}, a^2_{r+1,s})$, and

$(b_2(u'))_{r+2,s-1} = (b_2(u + {}_{rs}g))_{r+2,s-1}$

$= (b_2(u))_{r+2,s-1} + (b_2({}_{rs}g))_{r+2,s-1}$

$= ((a^1_{r+2,s-1})^\wedge, (a^2_{r+2,s-1})^\wedge) + (0, \lambda^s\lambda^{-r-1.5}a^1_{r,s}) = (a^1_{r+2,s-1}, a^2_{r+2,s-1})$.

□

Lemma 3.42. $\wedge^N(S_0) = 0$ *for any solution* S *for a sufficiently large number* N.

Proof. We know that from the above lemma that the left-most non-zero column of S_0 move rightwards as $\wedge$ transforms the lattice plane $\mathbb{Z}^2$. While $\wedge$ transforms the lattice plane. We see that the right most column in the region within which lies all the non-zero lattice points of S_0, C_η does not move rightwards unless $C_{\eta-2} = 0$. Consider the solution $S_r := \wedge^r(S_0)$ such that $C_{\eta-2} = C_{\eta+i} = 0$ for $i \geq 1$. Clearly such an r exists. Consider the non-zero element $a^1_{\eta-1,w} \in C_{\eta-1}$, such that $a^j_{\eta-1,t} = 0$ for $\forall t < w$ and $j = 1, 2$.

Now consider the following kernel equation

$$\lambda^w a^1_{\eta-2,w} - \lambda^{\eta-1}a^1_{\eta-1,w-1} - \lambda^{\eta-0.5}a^2_{\eta,w-1} + a^2_{\eta-1,w-1} = 0$$

We see that $a^2_{\eta,w-1} = 0$.

Also similar computation involving elements $a^1_{\eta+1,w-3}, a^2_{\eta+1,w-3}, a^2_{\eta+1,w-3}$, and $a^1_{\eta,w-1}$ implies that $a^1_{\eta,w-1} = 0$. Now we consider the equation,

$$\lambda^w a^1_{\eta-1,w} - \lambda^\eta a^1_{\eta,w-1} - \lambda^{\eta+0.5}a^2_{\eta+1,w-1} + a^2_{\eta,w-1} = 0$$

We see now that $a^1_{\eta-1,w} = 0$. This is a contradiction. Hence we have $C_{\eta-1} = 0$.

Now with a brief glance on the equation,

$$\lambda^v a^1_{\eta,v} - \lambda^{\eta+1}a^1_{\eta+1,v-1} - \lambda^{\eta+1.5}a^2_{\eta+2,v-1} + a^2_{\eta+1,v-1} = 0.$$

With this we conclude that $C_\eta = 0$. Hence we arrive at the conclusion that $\wedge^r(S_0) = 0$. Hence, with the above lemmas we conclude that

$$HH_1(\mathcal{A}^{alg}_{\theta,-\omega}) = 0.$$

□

Above computations were for $g = -\omega \in \mathbb{Z}_6$. Action of $-\omega^2 \in \mathbb{Z}_6$ is similar and we leave it to the reader to check that,

$$HH_0(\mathcal{A}^{alg}_{\theta,-\omega^{\pm1}})^{\mathbb{Z}_6} \cong \mathbb{C},\ HH_1(\mathcal{A}^{alg}_{\theta,-\omega^{\pm1}}) = 0,\ \text{and } HH_2(\mathcal{A}^{alg}_{\theta,-\omega^{\pm1}}) = 0.$$

□

Theorem 3.43. *The Hochschild homology groups for $\mathcal{A}^{alg}_\theta \rtimes \mathbb{Z}_6$ are as follows*

$$HH_0(\mathcal{A}^{alg}_\theta \rtimes \mathbb{Z}_6) \cong \mathbb{C}^9;\ HH_1(\mathcal{A}^{alg}_\theta \rtimes \mathbb{Z}_6) \cong 0;\ HH_2(\mathcal{A}^{alg}_\theta \rtimes \mathbb{Z}_6) \cong \mathbb{C}.$$

Proof. We know that $HH_0(\mathcal{A}^{alg\bullet}_{\theta,\omega^{\pm1}}) \cong \mathbb{C}^3$, $HH_0(\mathcal{A}^{alg\bullet}_{\theta,-\omega^{\pm1}}) \cong \mathbb{C}$ and $HH_0(\mathcal{A}^{alg\bullet}_{\theta,-1}) \cong \mathbb{C}^4$, as we have notice earlier, the $\mathbb{Z}_6$ action on the zeroth homology is same in both the complexes. Hence we follow the procedure detailed in previous sub-sections to conclude that $HH_0(\mathcal{A}^{alg\bullet}_{\theta,\omega^{\pm1}})^{\mathbb{Z}_6} \cong C^3$. To see this we consider the action of $-\omega$ on the elements of $HH_0(\mathcal{A}^{alg}_{\theta,\omega^{\pm1}})$, we observe that under this action

- $1 \mapsto 1$,
- $U_1 \mapsto U_2$,
- $U_2 \mapsto \dfrac{U_1^{-1}U_2}{\sqrt{\lambda}} \sim U_2^{-1} \sim U_1$,

Hence we see that

$$\varphi = a1 + bU_1 + cU_2 \mapsto a1 + bU_2 + cU_1.$$

The above element is invariant iff $b = c$ hence we have a 2 dimensional invariant sub-space of $HH_0(\mathcal{A}^{alg}_{\theta,\omega})$.

Now we consider the elements of $HH_0(\mathcal{A}^{alg}_{\theta,-1})$ invariant under the action of $g = -\omega$, we observe that under this action

- $1 \mapsto 1$,
- $U_1 \mapsto U_2$,
- $U_2 \mapsto \dfrac{U_2 U_1}{\sqrt{\lambda}} \sim \sqrt{\lambda} U_1 U_2$,
- $U_1 U_2 \mapsto \dfrac{U_2 U_1^{-1} U_2}{\sqrt{\lambda}} \sim \dfrac{U_1^{-1}}{\sqrt{\lambda}} \sim \dfrac{U_1}{\sqrt{\lambda}}$.

Hence we see that

$$\varphi = a1 + bU_1 + cU_2 + dU_1U_2 \mapsto a1 + bU_2 + c\sqrt{\lambda}U_1U_2 + d\frac{U_1}{\sqrt{\lambda}}.$$

The above element is invariant iff $b = c = \dfrac{d}{\sqrt{\lambda}}$ hence we have a 2 dimensional invariant sub-space of $HH_0(\mathcal{A}^{alg}_{\theta,-1})$. Summing up all the calculated sub-spaces above we conclude that

$$HH_0(\mathcal{A}^{alg}_{\theta} \rtimes \mathbb{Z}_4) \cong \mathbb{C}^9.$$

Previous calculations imply that $HH_1(\mathcal{A}^{alg\bullet}_{\theta,g})^{\mathbb{Z}_6} = 0$ for all $g \in \mathbb{Z}_6$, hence we have

$$HH_1(\mathcal{A}^{alg}_{\theta} \rtimes \mathbb{Z}_6) \cong 0.$$

We notice that for $1 \neq g \in \mathbb{Z}_6$, $HH_2(\mathcal{A}^{alg\bullet}_{\theta,g}) = 0$, while $HH_2(\mathcal{A}^{alg\bullet}_{\theta})^{\mathbb{Z}_6} \cong \mathbb{C}$. Hence we have

$$HH_2(\mathcal{A}^{alg}_{\theta} \rtimes \mathbb{Z}_6) \cong \mathbb{C}.$$

□

3.7.2 Cyclic homology of $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_6$

Theorem 3.44. *$HC_{even}(\mathcal{A}_{\theta,-\omega}^{alg}) \cong \mathbb{C}$, while $HC_{odd}(\mathcal{A}_{\theta,-\omega}^{alg}) = 0$.*

Proof. We apply the S, B, I long exact sequence relating the Hochschild and cyclic homology of an algebra A.

$$... \xrightarrow{B} HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} ...$$

.

Since the $\mathbb{Z}_6$ action on $\mathcal{A}_{\theta,-\omega}^{alg}$ commutes with the map ${}_{-\omega}b$, we obtain the following exact sequence

$$... \xrightarrow{B} (HH_n(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} \xrightarrow{I} (HC_n(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} \xrightarrow{S} (HC_{n-2}(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} \xrightarrow{B} (HH_{n-1}(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} \xrightarrow{I} ....$$

We know that $(HH_2(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} = (HH_1(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} = 0$. Hence we obtain that

$$(HC_2(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} \cong (HC_0(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6}.$$

But, a preliminary result shows that $HH_0(\mathcal{A}_{\theta,-\omega}^{alg}) = HC_0(\mathcal{A}_{\theta,-\omega}^{alg})$. Hence, we obtain that

$$(HC_2(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} = (HC_0(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} \cong \mathbb{C}.$$

Also, since $(HH_2(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} = (HH_3(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} = 0$, we obtain that

$$(HC_3(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} \cong (HC_1(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6}.$$

Since, $HH_1(\mathcal{A}_{\theta,-\omega}^{alg}) = 0$, we have $(HC_1(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} = 0$. Hence we conclude that

$$HC_{odd}(\mathcal{A}_{\theta,-\omega}^{alg}) = 0.$$

□

3.7.3 Periodic cyclic homology

Theorem 3.45. *$HC_{even}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_6) \cong \mathbb{C}^{10}$ while $HC_{odd}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_6) = 0$.*

Proof. We have the S, B, I sequence relating the Hochschild and the cyclic homology.

$$... \xrightarrow{B} (HH_2(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} \xrightarrow{I} (HC_2(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} \xrightarrow{S} (HC_0(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} \xrightarrow{B} (HH_1(\mathcal{A}_{\theta,-\omega}^{alg}))^{\mathbb{Z}_6} \xrightarrow{I} ...$$

Hence, $(HC_2(\mathcal{A}_\theta^{alg}))^{\mathbb{Z}_6} \cong \mathbb{C}^2$. Also we notice here that in this case, the $\mathbb{Z}_6$ invariant sub-space of $HC_{even}(\mathcal{A}_{\theta,-1}^{alg})$ is 2 dimensional and the $\mathbb{Z}_6$ invariant sub-space of $HC_{even}(\mathcal{A}_{\theta,\pm\omega}^{alg})$ is 2 dimensional. Hence we conclude that

$$HC_{even}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_6) \cong \mathbb{C}^{10}.$$

As for the odd cyclic homology, we have $(HC_3(\mathcal{A}_{\theta,\pm\omega}^{alg}))^{\mathbb{Z}_6} \cong (HC_1(\mathcal{A}_{\theta,\pm\omega}^{alg}))^{\mathbb{Z}_6} = 0$, and we also have $HC_3(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_6} = HC_1(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_6} = HH_1(\mathcal{A}_\theta^{alg})^{\mathbb{Z}_6} = 0$. Combining these two results, we obtain that

$$HC_{odd}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_6) = 0.$$

So we have computed the Hochschild and cyclic homology of the $\mathbb{Z}_6$ orbifold. □

CHAPTER IV

Hochschild and cyclic cohomology and Chern-Connes pairing

4.1 Hochschild cohomology of $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2$

We see that the dual of the algebraic non-commutative torus $\mathcal{A}_\theta^{alg}$ is

$$\mathcal{A}_\theta^{alg*} = \left\{ \sum_{(n,m)\in\mathbb{Z}^2} a_{n,m} U_1^n U_2^m | U_1^* = U_1^{-1}, U_2^* = U_2^{-1}, U_2U_1 = e^{2\pi i\theta} U_1U_2 \right\}.$$

With an explicit description as above we now construct the cohomology complex for the non-commutative torus algebra and as before use the paracyclic spectral decomposition to calculate the cohomology of the corresponding orbifold.

As before, the cohomology $H^\bullet(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*)$ decomposes into two groups each corresponding to the group elements of $\mathbb{Z}_2$.

$$H^\bullet(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*) = H^\bullet(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \oplus H^\bullet(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2}$$

Theorem 4.1. *We have*

$$H^0(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \cong \mathbb{C}^4,\ H^1(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*}) = 0 \text{ and } H^2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} = 0$$

We can identify $Hom_{\mathcal{B}_\theta^{alg}}(\mathcal{B}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})$ with ${}_{-1}\mathcal{A}_\theta^{alg*}$. ${}_{-1}\mathcal{A}_\theta^{alg*}$ can be identified with $\mathcal{A}_\theta^{alg*}$ with $\mathcal{A}_\theta^{alg*}$ bi-module structure as below

$$\alpha \cdot a = (-1 \cdot \alpha)a \text{ while } a \cdot \alpha = a\alpha, \text{ for } a \in {}_{-1}\mathcal{A}_\theta^{alg*} \text{ and } \alpha \in \mathcal{A}_\theta^{alg*}.$$

Also we notice the $\mathbb{Z}_2$ action on $\mathcal{A}_\theta^{alg*}$ which acts by involution. It maps $U_j \mapsto U_j^{-1}$ for $j = 1, 2$.

Using the adjusted Connes' resolution and the notations as in previous chapters, we get the following complex.

$$_{-1}\mathcal{A}_\theta^{alg*} \xrightarrow{{}_{-1}\alpha_1} {}_{-1}\mathcal{A}_\theta^{alg*} \oplus {}_{-1}\mathcal{A}_\theta^{alg*} \xrightarrow{{}_{-1}\alpha_2} {}_{-1}\mathcal{A}_\theta^{alg*} \to 0$$

where the map ${}_{-1}\alpha_1(\varphi) = (U_1^{-1}\varphi - \varphi U_1, U_2^{-1}\varphi - \varphi U_2)$ and the map ${}_{-1}\alpha_2$ is described as ${}_{-1}\alpha_2(\varphi_1, \varphi_2) = (U_2^{-1}\varphi_1 - \lambda\varphi_1 U_2 - \lambda U_1^{-1}\varphi_2 + \varphi_2 U_1)$.

Lemma 4.2. $H^0(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \cong \mathbb{C}^4$.

Proof. Let $\varphi = \sum \varphi_{n,m} U_1^n U_2^m \in {}_{-1}\mathcal{A}_\theta^{alg*}$ satisfy $U_1^{-1}\varphi - \varphi U_1 = U_2^{-1}\varphi - \varphi U_2 = 0$
$\implies \varphi_{n+1,m} = \lambda^m \varphi_{n-1,m} = \lambda^{m+n-1}\varphi_{n-1,m-2}$.
Hence we see that $H^0(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*}) \cong \mathbb{C}^4$. The generators of this group are the cocycles generated by $\varphi_{0,0}, \varphi_{0,1}, \varphi_{1,0}$ and $\varphi_{1,1}$.

Let us consider the cocycle $\mathcal{D}_{0,0}$, generated by $\varphi_{0,0}$. The above condition on the coefficients of a cocycle relates $\varphi_{n,m} = \varphi_{0,0}$ for all $(n, m) \in \mathbb{Z}^2$. Hence the group action of $\mathbb{Z}_2$ on $\mathcal{A}_\theta^{alg*}$ leaves $\mathcal{D}_{0,0}$ invariant.

Similarly for $\mathcal{D}_{0,1}$, we get that $\varphi_{2k.2l+1} = \lambda^{2kl+k}\varphi_{0,1}$, while
$\varphi_{-2k.-2l-1} = \lambda^{2(-k)(-l-1)+(-k)}\varphi_{0,1} = \lambda^{2kl+k}\varphi_{0,1}$. Hence $\mathcal{D}_{0,1} \in H^0(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2}$.

Similarly for $\mathcal{D}_{1,0}$, we get that $\varphi_{2k+1.2l} = \lambda^{2kl+l}\varphi_{1,0}$, while
$\varphi_{-2k.-2l-1} = \lambda^{2(-k-1)(-l)+(-l)}\varphi_{1,0} = \lambda^{2kl+l}\varphi_{1,0}$. Hence $\mathcal{D}_{1,0} \in H^0(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2}$.

Similarly for $\mathcal{D}_{1,1}$, we get that $\varphi_{2k+1.2l+1} = \lambda^{2kl+k+l}\varphi_{1,1}$, while
$\varphi_{-2k-1.-2l-1} = \lambda^{2kl+k+l}\varphi_{1,1}$. Hence $\mathcal{D}_{1,1} \in H^0(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2}$. □

Lemma 4.3. $H^2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} = 0$.

Proof. Let $\varphi \in \mathcal{A}_\theta^{alg*}$ and let $\tilde{\varphi}$ be the corresponding element of $Hom_{\mathfrak{B}_\theta^{alg}}(J_2, \mathcal{A}_\theta^{alg*})$:

$$\tilde{\varphi}(a \otimes b \otimes e_1 \wedge e_2)(x) = \varphi((-1 \cdot b)xa) \ \forall a, b, x \in \mathcal{A}_\theta^{alg}$$

Let $\psi = k_2^* \tilde{\varphi} = \tilde{\varphi} \circ k_2$. One has

$$\psi(x_0, x_1, x_2) = \tilde{\varphi}(k_2(I \otimes x_1 \otimes x_2))(x_0) \ \forall x_0, x_1, x_2 \in \mathcal{A}_\theta^{alg}.$$

The group $\mathbb{Z}_2$ acts on $\mathcal{A}_\theta^{alg}$ in the bar complex as $-1 \cdot \chi(x_0, x_1, x_2) = \chi(-1 \cdot x_0, -1 \cdot x_1, -1 \cdot x_2)$. Further we pullback the map ${}_{-1}\psi = -1 \cdot \psi$ back on to the Kozul complex via the map h_2^*. Let $w = h_2^*({}_{-1}\psi)$ denote the pull back of ${}_{-1}\psi$ on the Kozul complex. We have

$w(x_0) = {}_{-1}\psi(x_0, U_2, U_1) - \lambda {}_{-1}\psi(x_0, U_1, U_2) = \psi(-1 \cdot x_0, U_2^{-1}, U_1^{-1}) - \lambda \psi(-1 \cdot x_0, U_1^{-1}, U_2^{-1})$

$= \tilde{\varphi}(k_2(I \otimes U_2^{-1} \otimes U_1^{-1}))(-1 \cdot x_0) - \lambda \tilde{\varphi}(k_2(I \otimes U_1^{-1} \otimes U_2^{-1}))(-1 \cdot x_0)$.

We know that from the calculation in chapter 3 that

$k_2(I \otimes U_2^{-1} \otimes U_1^{-1}) - \lambda k_2(I \otimes U_1^{-1} \otimes U_2^{-1}) = (U_1^{-1} U_2^{-1} \otimes U_1^{-1} U_2^{-1})$. Hence we have

$$\tilde{\varphi}(k_2(I \otimes U_2^{-1} \otimes U_1^{-1}))(-1 \cdot x_0) - \lambda \tilde{\varphi}(k_2(I \otimes U_1^{-1} \otimes U_2^{-1}))(-1 \cdot x_0) =$$
$$\tilde{\varphi}(U_1^{-1} U_2^{-1} \otimes U_1^{-1} U_2^{-1})(-1 \cdot x_0) = \varphi(U_1 U_2 \cdot (-1 \cdot x_0) \cdot U_1^{-1} U_2^{-1})$$

Hence we need to compare $\varphi(x)$ with $\varphi(U_1 U_2 \cdot (-1 \cdot x) \cdot U_1^{-1} U_2^{-1})$.

Using the Kozul complex, we see that $H^2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} = {}_{-1}\mathcal{A}_\theta^{alg*}/Im({}_{-1}\alpha_2)$. Since ${}_{-1}\alpha_2(U_2, 0) = (1 - \lambda U_2^2)$ and ${}_{-1}\alpha_2(0, U_1) = (U_1^2 - \lambda)$, we have $H^2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*}) \cong \mathbb{C}^4$ generated by the cocycles supported at $\varphi_{0,0}, \varphi_{1,0}, \varphi_{0.1}$ and $\varphi_{1,1}$.

Case 1:

We check the invariance of $\varphi_{0,0}$,

$\varphi_{0,0}(x) = x_{0,0}$ while $\varphi(U_1 U_2 \cdot (-1 \cdot x) \cdot U_1^{-1} U_2^{-1}) = \lambda^{-1} x_{0,0}$.

Hence it is *not* invariant under the $\mathbb{Z}_2$ action.

Case 2:

We check the invariance of $\varphi_{1,0}$,

$\varphi_{1,0}(x) = x_{1,0}$ while $\varphi_{1,0}(U_1 U_2 \cdot (-1 \cdot x) \cdot U_1^{-1} U_2^{-1}) = x_{-1,0}$

Since the cocycle $\varphi_{1,0}$ is equivalent to $\lambda\varphi_{-1,0}$, we have it *not* invariant under the $\mathbb{Z}_2$ action.

Case 3:

We check the invariance of $\varphi_{0,1}$,

$\varphi_{0,1}(x) = x_{0,1}$ while $\varphi_{0,1}(U_1 U_2 \cdot (-1 \cdot x) \cdot U_1^{-1} U_2^{-1}) = x_{0,-1}$

Since the cocycle $\varphi_{0,1}$ is equivalent to $\lambda^{-1}\varphi_{0,-1}$, we have it *not* invariant under the $\mathbb{Z}_2$ action.

Case 4:

We check the invariance of $\varphi_{1,1}$,

$\varphi_{1,1}(x) = x_{1,1}$ while $\varphi_{1,1}(U_1 U_2 \cdot (-1 \cdot x) \cdot U_1^{-1} U_2^{-1}) = \lambda^{-1} x_{-1,-1}$

Since $\varphi_{1,1}$ is a cocycle equivalent to $\varphi_{-1,-1}$, we have this cocycle is *not* invariant under the $\mathbb{Z}_2$ action.

For $\psi = a\varphi_{0,0} + b\varphi_{1,0} + c\varphi_{0,1} + d\varphi_{1,1}$, the pull back of the corresponding cocycle after $\mathbb{Z}_2$ action on the Kozul complex is Ψ, which can be described as follows,

$$\Psi = a\lambda^{-1}\varphi_{0,0} + b\lambda^{-1}\varphi_{1,0} + c\lambda^{-1}\varphi_{0,1} + d\lambda^{-1}\varphi_{1,1}.$$

Hence we see that the coefficients of this pull back is different from those of the original cocycle. And the only invariant cocycle is the zero cocycle.

Therefore we conclude that

$$H^2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} = 0.$$

We remark in this computation that $H^2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})$ is of 4 dimension. But there is no nontrivial $\mathbb{Z}_2$ invariant cocycle. $\square$

Definition 4.4 (Lines)**.** For $s_0 \in \mathbb{Z}$ by H_{s_0} define a $\mathbb{Z}^2$ lattice such that

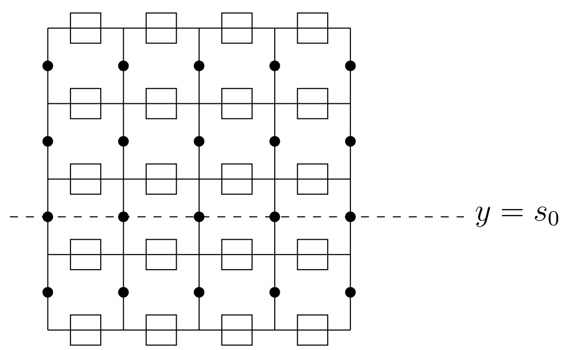
$$(H_{s_0})_{w,s} := \begin{cases} (\pi_1(Dgm(\varphi)))_{w,s} & \text{for } s = s_0 \\ 0 & else. \end{cases}$$

Lemma 4.5. *For $s_0 \in \mathbb{Z}$ there exists $\gamma_{s_0} \in \mathcal{A}_\theta^{alg*}$ such that for all $w \in \mathbb{Z}$ we have* $({}_{-1}\alpha_1(\gamma_{s_0}))_{w,s_0} - (H_{s_0})_{w,s_0} = 0$.

Proof. We know that ${}_{-1}\alpha_1(\varphi) = (U_1^{-1}\varphi - \varphi U_1, U_2^{-1}\varphi - \varphi U_2)$, hence if ${}_{-1}\alpha_1(\varphi) = (\varphi_1, \varphi_2)$ then,

$$\varphi^1_{n,m} = \varphi_{n+1,m} - \lambda^m \varphi_{n-1,m};\ \varphi^2_{n,m} = \lambda^{-n}\varphi_{n,m+1} - \varphi_{n,m-1}.$$

Since we know that a connected component of the kernel diagram looks like :



Assume that $\varphi^1_{0,s_0} \neq 0$, from the diagram it is clear that in the row $y = s_0$ in $\pi_1(Dgm(\varphi))$, $\varphi^2_{w,s_0} = 0$ for all $w \in \mathbb{Z}$.

Define $$(\gamma'_{s_0})_{w,s} = \begin{cases} -\lambda^{-s_0}\varphi^1_{0,s} & \text{for } (w,s) = (-1, s_0) \\ 0 & else. \end{cases}$$

Hence we have, $({}_{-1}\alpha_1(\gamma'_{s_0}))_{0,s_0} - (H_{s_0})_{0,s_0} = 0$. Since $\pi_1(Dgm(\varphi))_{-1,s_0} = (0,0)$, we define

$$(\gamma''_{s_0})_{w,s} = \begin{cases} -\lambda^{-s_0}(\varphi^1_{-2,s} - \lambda^{-s_0}\varphi^1_{0,s}) & \text{for } (w,s) = (-3, s_0) \\ (\gamma'_{s_0})_{w,s} & else. \end{cases}$$

Hence we have, $({}_{-1}\alpha_1(\gamma''_{s_0}))_{-2,s_0} - (H_{s_0})_{-2,s_0} = ({}_{-1}\alpha_1(\gamma''_{s_0}))_{0,s_0} - (H_{s_0})_{0,s_0} = 0$.

We can hence construct a sequence $\gamma^{(n)}_{s_0}$ which satisfies the required condition for finitely many lattice points. Define $\gamma^{\leq}_{s_0} := \lim_{n\to\infty} \gamma^{(n)}_{s_0}$. Since $\gamma^{\leq}_{s_0,w} \in \mathcal{A}^{alg*}_\theta$, we have

$$({}_{-1}\alpha_1(\gamma^{\leq}_{s_0}))_{\bullet,s_0} - (H_{s_0})_{\bullet,s_0} = 0 \text{ for } \bullet \leq 0.$$

We can similarly define $\gamma^{>}_{s_0}$ such that

$$({}_{-1}\alpha_1(\gamma^{>}_{s_0}))_{\bullet,s_0} - (H_{s_0})_{\bullet,s_0} = 0 \text{ for } \bullet > 0.$$

Then $\gamma_{s_0} := \gamma^{\leq}_{s_0} + \gamma^{>}_{s_0}$, satisfies the following equation.

$$({}_{-1}\alpha_1(\gamma_{s_0}))_{\bullet,s_0} - (H_{s_0})_{\bullet,s_0} = 0 \text{ for } \bullet \in \mathbb{Z}.$$

□

It may be surprising to note the degree of freedom we had while constructing γ_{s_0}. This can be traced back to the fact that the kernel of ${}_{-1}\alpha_1$ is a 4 dimensional vector space. That is ${}_{-1}\alpha_1$ is way away from being injective. As we shall prove an arbitrary cocycle to be a coboundary, we shall notice the various possibility we have in doing so; hence reveal the nature of map ${}_{-1}\alpha_1$.

Lemma 4.6. *$H^1(\mathcal{A}^{alg}_\theta, {}_{-1}\mathcal{A}^{alg*}_\theta) \cong 0$.*

Proof. Let $\varphi \in ker({}_{-1}\alpha_2)$. Recall that with the diagram representation of the kernel element $\varphi = (\varphi^1, \varphi^2)$, we have a lattice $Dgm(\varphi) \subset \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2$. It is important to recall that each $\pi_i(Dgm(\varphi))$ are lattices in $\mathbb{Z}^2$ with alternate rows of φ^1 and φ^2. Here $\pi_i : \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2 \to \mathbb{Z}^2$ are the projection maps.

Let us understand the construction of a $\pi_i(Dgm(\varphi))$. The $\pi_i(Dgm(\varphi))$ consists of alternate non-zero entries, meaning one considering a row/column will find zeros at least at alternate positions. Each $\pi_i(Dgm(\varphi))$ has rows/columns of φ^2's and φ^1's alternately placed.

Applying the lemma 4.5 above for H_{s_0} for each s_0 we get a sequence $\gamma_{s_0} \in \mathcal{A}_\theta^{alg*}$. Define $\gamma = \gamma_0 + \gamma_2 + \gamma_{-2} + ...$, then the lattice

$$_{-1}\alpha_1(\gamma) - \pi_1(Dgm(\varphi))$$

has alternate rows of zero. We need not worry about convergence as we are adding disjointly supported lattices. These are the rows where φ^1 is used to reside in $\pi_1(Dgm(\varphi))$. The other alternate set consists of the modified φ'^2's. We state that what remains belongs to $im(_{-1}\alpha_1)$.

Firstly notice that each of these φ'^2 rows belong to $ker(_{-1}\alpha_2)$ separately. This is easy to see as there is no kernel equation that relates $\varphi'_{p,q}$ with $\varphi_{l,w}$ for $q \neq w$. Now also note that if there is even a single zero entry in any of these rows, then, the whole row ought to be a *zero row*. Again, this is also seen through the repetitive application of the kernel equation to the row starting with the kernel equation containing the zero entry. With the following lemma we prove that $H^1(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*}) = 0$.

Lemma 4.7. *For $s_0 \in \mathbb{Z}$, let $(\eta)_w$ be a sequence satisfying $\eta_{w+1} = \lambda^{s_0-1}\eta_{w-1}$. Then there exist ρ^-, ρ^+ such that $_{-1}\alpha_1(\rho^-_{s_0}) = {}_{-1}\alpha_1(\rho^+_{s_0}) = \psi$, where,*

$$(\psi)_{w,s} := \begin{cases} \eta_w & \textit{for } (w,s) = (w,s_0) \\ 0 & \textit{else.} \end{cases} .$$

Proof. We define

$$(\rho'^-_{s_0})_{w,s} := \begin{cases} \eta_w & \text{for } (w,s) = (w,s_0-1) \\ 0 & \textit{else.} \end{cases} .$$

Then, ${}_{-1}\alpha_1(\rho'^-_{s_0})-\psi$ has non-zero entries only in the row at $s = s_0-2$. Similarly we can define $\rho''^-_{s_0}$ such that ${}_{-1}\alpha_1(\rho'^-_{s_0}) + {}_{-1}\alpha_1(\rho''^-_{s_0}) - \psi$ has non-zero entries only in the row at $s = s_0 - 4$. Define $\rho^-_{s_0} = \rho'^-_{s_0} + \rho''^-_{s_0} + ...$, then ${}_{-1}\alpha_1(\rho^-_{s_0}) - \psi = 0$. Since each $\rho^{(n)-}_{s_0}$ are disjointly supported, $\rho^-_{s_0}$ is well defined.

Similarly we can define $\rho^+_{s_0}$. □

Hence we consider a given row of φ^2 in ${}_{-1}\alpha_1(\gamma)-\pi_1(Dgm(\varphi))$, for each of its rows below the x-axis we get corresponding ρ^-_s and each of the φ^2 rows above the x-axis we get corresponding ρ^+_s, then $\rho = \sum_{s\in\mathbb{N}} \rho^-_s + \rho^+_{-s}$ has the property that ${}_{-1}\alpha_1(\rho) = {}_{-1}\alpha_1(\gamma) - \pi_1(Dgm(\varphi))$, hence, $H^1(\mathcal{A}^{alg}_\theta, {}_{-1}\mathcal{A}^{alg*}_\theta) = 0$. □

4.2 $H^\bullet(\mathcal{A}^{alg}_\theta, \mathcal{A}^{alg*}_\theta)^{\mathbb{Z}_2}$

Lemma 4.8. $H^0(\mathcal{A}^{alg}_\theta, \mathcal{A}^{alg*}_\theta)^{\mathbb{Z}_2} \cong \mathbb{C}$.

Proof. Let $\varphi = \sum \varphi_{n,m} U_1^n U_2^m \in {}_{-1}\mathcal{A}^{alg*}_\theta$ satisfying $U_1\varphi - \varphi U_1 = U_2\varphi - \varphi U_2 = 0$.
$\implies \varphi_{n-1,m} = \lambda^m \varphi_{n-1,m} = \lambda^{m+n-1}\varphi_{n-1,m}$.
We see that $m = n - 1 = 0$ satisfies the above condition. Hence, we have

$$H^0(\mathcal{A}^{alg}_\theta, \mathcal{A}^{alg*}_\theta) \cong \mathbb{C}$$

and is generated by $\varphi_{0,0}$.

Since the action of $\mathbb{Z}_2$ on bar complex translates as it is on to the Kozul complex using the maps $h_0 = k_0 = id$, we get that

$$H^0(\mathcal{A}^{alg}_\theta, \mathcal{A}^{alg*}_\theta)^{\mathbb{Z}_2} \cong \mathbb{C}.$$

□

Lemma 4.9. $H^2(\mathcal{A}^{alg}_\theta, \mathcal{A}^{alg*}_\theta)^{\mathbb{Z}_2} \cong \mathbb{C}$.

Proof. We see from the calculations in [C] that $H^2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} = \mathcal{A}_\theta^{alg*}/Im\alpha_2$.

Since $\alpha_2(U_2, 0) = (1-\lambda)(U_2^2)$ and $\alpha_2(0, U_1) = (1-\lambda)(U_1^2)$. We have $H^2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg*}) \cong \mathbb{C}$ generated by the cocycle equivalent to $\varphi_{-\bar{1},-1}$. To check the invariance of this cocycle we need to check that,

$$\tilde{\varphi}(k_2(I \otimes U_2^{-1} \otimes U_1^{-1}))(-1 \cdot x_0) - \lambda\tilde{\varphi}(k_2(I \otimes U_1^{-1} \otimes U_2^{-1}))(-1 \cdot x_0) =$$
$$\tilde{\varphi}(U_1^{-1}U_2^{-1} \otimes U_2^{-1}U_1^{-1})(-1 \cdot x_0) = \varphi(U_1^{-1}U_2^{-1} \cdot (-1 \cdot x_0) \cdot U_2^{-1}U_1^{-1}).$$

Consider the cocycle $\varphi_{-\bar{1},-1} \in H^2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg*})$, we see that

$\varphi_{-\bar{1},-1}(x) = x_{-1,-1}$ while $\bar{\varphi}(U_1^{-1}U_2^{-1} \cdot (-1 \cdot x) \cdot U_2^{-1}U_1^{-1}) = x_{-1,-1}$. We conclude that $\varphi_{-\bar{1},-1}$ is invariant under the $\mathbb{Z}_2$ action.

□

Lemma 4.10. $H^1(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} = 0$.

Proof. We firstly look at the kernel set of α_2.

$$\alpha_2(\varphi_1, \varphi_2) = (U_2^{-1}\varphi_1 - \lambda\varphi_1 U_2 - \lambda U_1^{-1}\varphi_2 + \varphi_2 U_1).$$

We get that for $(\varphi^1, \varphi^2) \in ker(\alpha_2)$, we need to ensure that

$$(\lambda^n - \lambda)\varphi^1_{n,m-1} = (\lambda - \lambda^m)\varphi^2_{n-1,m}.$$

We define an element $\varphi \in \mathcal{A}_\theta^{alg*}$ as follows

$$\varphi_{n-1,m} = (1-\lambda^m)^{-1}\varphi^1_{n,m}.$$

It is easy to check that $\alpha_1(\varphi)_{n,m} = (\varphi^1, \varphi^2)_{n,m}$ for all $(n,m) \neq (0,m); (n,0)$. Hence we have

$$H^1(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg*}) \cong \mathbb{C}^2.$$

Which is generated by $\tilde{\varphi^1_{-1,0}}$ and $\tilde{\varphi^2_{0,-1}}$. To calculate the invariance we use the complex maps h_1 and k_1. We recall the technique deployed in previous lemma. We consider the element $(a\varphi^1_{-1,0}, b\varphi^2_{0,-1}) \in \mathcal{A}^{alg}_\theta \oplus \mathcal{A}^{alg}_\theta$ and let $(a\tilde{\varphi^1_{-1,0}}, b\tilde{\varphi^2_{0,-1}}) \in Hom_{\mathfrak{B}^{alg}_\theta}(J_1, \mathcal{A}^{alg}_\theta)$:

$$\tilde{\varphi^1_{-1,0}}(a \otimes b \otimes e_1)(x) = \varphi^1_{-1,0}(bxa), \; \forall a, b, x \in \mathcal{A}^{alg}_\theta.$$

Let $\psi = k_1^*(a\tilde{\varphi^1_{-1,0}}, b\tilde{\varphi^2_{0,-1}}) = (a\tilde{\varphi^1_{-1,0}}, b\tilde{\varphi^2_{0,-1}}) \circ k_1$. One has

$$\psi(x_0, x_1) = (a\tilde{\varphi^1_{-1,0}}, b\tilde{\varphi^2_{0,-1}})(k_1(I \otimes x_1))(x_0), \; \forall x_0, x_1 \in \mathcal{A}^{alg}_\theta.$$

Thereafter we have the action of $\mathbb{Z}_2$ on ψ. Define ${}_{-1}\psi(x_0, x_1) = \psi(-1 \cdot x_0, -1 \cdot x_1)$. We now pull back ${}_{-1}\psi$ on to the Kozul complex to compare with the original cocycle. The pull back w can be described as follows:

$$(w_1, w_2) = h_1^*({}_{-1}\psi), \text{ consider } w_1(x) = {}_{-1}\psi(x, U_1).$$

We observe that $w_1(x) = {}_{-1}\psi(x, U_1) = \psi(-1 \cdot x, U_1^{-1}) = a\tilde{\varphi^1_{1,0}}(k_1(I \otimes U_1^{-1}))(-1 \cdot x)$.
We know from our computations in chapter 3 that $k_1(I \otimes U_1^{-1}) = -(U_1^{-1} \otimes U_1^{-1})$, using this relation we compute that

$$\begin{aligned} &a\tilde{\varphi^1_{-1,0}}(k_1(I \otimes U_1^{-1}))(-1 \cdot x) = -\tilde{\varphi^1_{-1,0}}(U_1^{-1} \otimes U_1^{-1})(-1 \cdot x) \\ &= -\varphi^1_{-1,0}(U_1^{-1} \otimes U_1^{-1})(-1 \cdot x) = -\varphi^1_{-1,0}(U_1^{-1} \cdot (-1 \cdot x) \cdot U_1^{-1}) = -x_{-1,0}. \end{aligned}$$

Similarly, we can calculate w_2 and hence we finally conclude that

$$h_1^*(-1 \cdot (k_1^*(a\tilde{\varphi^1_{-1,0}}, b\tilde{\varphi^2_{0,-1}}))) = -(a\tilde{\varphi^1_{-1,0}}, b\tilde{\varphi^2_{0,-1}}).$$

So, we have

$$H^1(\mathcal{A}^{alg}_\theta, \mathcal{A}^{alg}_\theta)^{\mathbb{Z}_2} = 0.$$

□

Theorem 4.11. *The Hochschild cohomology groups for $\mathcal{A}^{alg}_\theta \rtimes \mathbb{Z}_2$ are as follows*

$$H^0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*) \cong \mathbb{C}^5;\ H^1(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*) \cong 0;$$

$$H^2(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*) \cong \mathbb{C}.$$

Proof. We know that $H^0(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \cong \mathbb{C}^4$ and $H^0(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \cong \mathbb{C}$. Hence, we obtain that $H^0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*) \cong \mathbb{C}^5$.

Now, $H^1(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*) = H^1(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \oplus H^1(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \cong 0$ is clear as each of these summands is zero. As for $H^2(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*)$, observe that $H^2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \cong \mathbb{C}$ and $H^2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} = 0$. Hence, we have $H^2(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*)^{\mathbb{Z}_2} \cong \mathbb{C}$.

□

4.3 Cyclic cohomology of $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2$

We consider the S, B, I sequence for cohomology exact sequence.

$$\cdots \to H^1(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \xrightarrow{B} HC^0(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \xrightarrow{I} HC^2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \xrightarrow{S}$$
$$H^2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \xrightarrow{B} HC^1(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \xrightarrow{I} \dots$$

Since, $HC^1(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*}) = H^1(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*}) = 0$. We get $HC^2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*}) \cong \mathbb{C}^4$.

Theorem 4.12. *For the algebraic non-commutative toroidal orbifold $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2$, we have,*

$$HC^0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) \cong \mathbb{C}^5; HC^1(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) \cong 0;$$

$$HC^2(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) \cong \mathbb{C}^6$$

Proof. Since $H^0(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \cong \mathbb{C}^4$, while, $H^0(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \cong \mathbb{C}$, we have,

$$HC^0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) \cong \mathbb{C}^5.$$

We see that $HC^1(\mathcal{A}_\theta^{alg}, {}_{\pm 1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} = 0$, and we have

$$HC^1(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*) \cong 0.$$

Also since $HC^2(\mathcal{A}_\theta^{alg}, \mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \cong \mathbb{C}^2$ and $HC^2(\mathcal{A}_\theta^{alg}, {}_{-1}\mathcal{A}_\theta^{alg*})^{\mathbb{Z}_2} \cong \mathbb{C}^4$, we have

$$HC^2(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) \cong \mathbb{C}^6.$$

□

Theorem 4.13 (Periodic Cyclic Cohomology). *$HC^{even}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) \cong \mathbb{C}^6$ and*

$$HC^{odd}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) = 0.$$

Proof. Since $H^\bullet(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*) = 0$ for $\bullet \geq 3$, we have the isomorphism $HC^\bullet(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*) \cong HC^{\bullet+2}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2, (\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)^*)$ for $\bullet > 1$. Hence the results. □

4.4 Chern-Connes pairing for $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2$

In this section we calculate the Chern-Connes pairing associated with this toroidal orbifold $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2$. We see that among the six projections generating $K_0(\mathcal{A}_\theta \rtimes \mathbb{Z}_2)$, described in [ELPH], five belong to the algebra $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2$. They are the following

- $[1]$
- $[p^\theta]$, where $p^\theta = \dfrac{1}{2}(1+t)$.
- $[q_0^\theta]$, where $q_0^\theta = \dfrac{1}{2}(1-U_1 t)$.
- $[q_1^\theta]$, where $q_1^\theta = \dfrac{1}{2}(1-U_2 t)$.
- $[r^\theta]$, where $r^\theta = \dfrac{1}{2}(1-\sqrt{\lambda}U_1 U_2 t)$.

A complete description of the group $K_0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)$ is unknown, with the Chern-Connes pairing of these five generators we will have some understanding of its non-commutative index theory. We describe pairing of the elements of $HC^{even}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)$ with these five projections. Using the fact that $\langle [e], [S\phi] \rangle = \langle [e], [\phi] \rangle$, we have the following computations

Pairing of $[S\tau]$

The following are the pairings with the element $[S\tau] \in HC^{even}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)$.

1. $\langle [1], [\tau] \rangle = 1$
2. $\langle [p^\theta], [\tau] \rangle = \frac{1}{2}$
3. $\langle [q_0^\theta], [\tau] \rangle = \frac{1}{2}$
4. $\langle [q_1^\theta], [\tau] \rangle = \frac{1}{2}$
5. $\langle [r^\theta], [\tau] \rangle = \frac{1}{2}$.

Pairing of $[S\mathcal{D}_{0,0}]$

The following are the pairings with the element $[S\mathcal{D}_{0,0}] \in HC^{even}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)$.

1. $\langle [1], [\mathcal{D}_{0,0}] \rangle = 0$
2. $\langle [p^\theta], [\mathcal{D}_{0,0}] \rangle = \frac{1}{2}$
3. $\langle [q_0^\theta], [\mathcal{D}_{0,0}] \rangle = 0$
4. $\langle [q_1^\theta], [\mathcal{D}_{0,0}] \rangle = 0$
5. $\langle [r^\theta], [\mathcal{D}_{0,0}] \rangle = 0$.

Pairing of $[S\mathcal{D}_{1,0}]$

The following are the pairings with the element $[S\mathcal{D}_{1,0}] \in HC^{even}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)$.

1. $\langle [1], [\mathcal{D}_{1,0}] \rangle = 0$
2. $\langle [p^\theta], [\mathcal{D}_{1,0}] \rangle = 0$
3. $\langle [q_0^\theta], [\mathcal{D}_{1,0}] \rangle = -\frac{1}{2}$
4. $\langle [q_1^\theta], [\mathcal{D}_{1,0}] \rangle = 0$
5. $\langle [r^\theta], [\mathcal{D}_{1,0}] \rangle = 0$.

<u>Pairing of $[S\mathcal{D}_{0,1}]$</u>

The following are the pairings with the element $[S\mathcal{D}_{0,1}] \in HC^{even}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)$.

1. $\langle [1], [\mathcal{D}_{0,1}] \rangle = 0$
2. $\langle [p^\theta], [\mathcal{D}_{0,1}] \rangle = 0$
3. $\langle [q_0^\theta], [\mathcal{D}_{0,1}] \rangle = 0$
4. $\langle [q_1^\theta], [\mathcal{D}_{0,1}] \rangle = -\dfrac{1}{2}$
5. $\langle [r^\theta], [\mathcal{D}_{0,1}] \rangle = 0$.

<u>Pairing of $[S\mathcal{D}_{1,1}]$</u>

The following are the pairings with the element $[S\mathcal{D}_{1,1}] \in HC^{even}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)$.

1. $\langle [1], [\mathcal{D}_{1,1}] \rangle = 0$
2. $\langle [p^\theta], [\mathcal{D}_{1,1}] \rangle = 0$
3. $\langle [q_0^\theta], [\mathcal{D}_{1,1}] \rangle = 0$
4. $\langle [q_1^\theta], [\mathcal{D}_{1,1}] \rangle = 0$
5. $\langle [r^\theta], [\mathcal{D}_{1,1}] \rangle = -\dfrac{\sqrt{\lambda}}{2}$.

Pairing of $[\varphi]$

The following are the pairings with the element $[\varphi] \in HC^{even}(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)$, where φ is the even cocycle computed in the paper of A.Connes [C]

1. $\langle [1], [\varphi] \rangle = 0$
2. $\langle [p^\theta], [\varphi] \rangle = 0$
3. $\langle [q_0^\theta], [\varphi] \rangle = 0$
4. $\langle [q_1^\theta], [\varphi] \rangle = 0$
5. $\langle [r^\theta], [\varphi] \rangle = 0.$

We observe that since these five projections of the algebraic noncommutative toroidal orbifold $\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2$ are projections of the smooth orbifold, $\mathcal{A}_\theta \rtimes \mathbb{Z}_2$; their linear independence in $K_0(\mathcal{A}_\theta \rtimes \mathbb{Z}_2)$ implies that they are linearly independent in $K_0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)$. We conjecture that these five projections span the group $K_0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2)$.

Conjecture 4.14. *$K_0(\mathcal{A}_\theta^{alg} \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^5$.*

CHAPTER V

Hochschild homology of smooth orbifold

5.1 Hochschild homology of $\mathcal{A}_\theta \rtimes \mathbb{Z}_2$

In this section we give partial results regarding the Hochschild homology of the smooth non-commutative $\mathbb{Z}_2$ toroidal orbifold, $\mathcal{A}_\theta \rtimes \mathbb{Z}_2$. We also present a lemma which characterises a class of elements of the group whose dimension is an open problem as I write this chapter.

I also have strong conviction that the method we used to calculate the (co)homology dimensions for the algebraic non-commutative orbifold will be useful and instrumental in computing the $HH_1(\mathcal{A}_\theta \rtimes \mathbb{Z}_2)$ whose dimension remains uncalculated.

Theorem 5.1. *: For $\theta \notin \mathbb{Q}$, we have $HH_2(\mathcal{A}_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{C}$.*

Proof. We consider the map $(1 \otimes b_2)$ in the tensored complex.

$$0 \leftarrow {}_{-1}\mathcal{A}_\theta \xleftarrow{b_1} {}_{-1}\mathcal{A}_\theta \oplus {}_{-1}\mathcal{A}_\theta \xleftarrow{b_2} {}_{-1}\mathcal{A}_\theta$$

To calculate the kernel of this map we have a closer look at this map,

$$(1 \otimes b_2)(a \otimes I) = a \otimes_{\mathfrak{B}_\theta} (U_2 \otimes I - \lambda \otimes U_2) \otimes e_1 - a \otimes_{\mathfrak{B}_\theta} (\lambda U_1 \otimes I - I \otimes U_1) \otimes e_2.$$

Using the twisted bimodule structure of ${}_{-1}\mathcal{A}_\theta$ over $\mathcal{A}_\theta$, we can simplify the equation to the following,

$$(1 \otimes b_2)(a \otimes I) = (\lambda U_2^{-1} a - aU_2, \lambda a U_1 - U_1^{-1} a).$$

Hence we obtain the following relation for an element $(a \otimes 1)$ to reside in $ker(1 \otimes b_2)$.

$$H_2(\mathcal{A}_\theta, {}_{-1}\mathcal{A}_\theta) = \{a \in {}_{-1}\mathcal{A}_\theta | a_{n,m} = \lambda^{m-1} a_{n-2,m}; a_{n-1,m} = \lambda^n a_{n-1,m-2}\} .$$

If an element $\varphi \in {}_{-1}\mathcal{A}_\theta$ were to satisfy these relations. Then for a fixed $n_0 \in \mathbb{Z}$ the sequence $(\varphi_{n_0,2m})_m \notin \mathcal{S}(\mathbb{Z})$. This is a contradiction. Hence there are no such nontrivial elements in $\mathcal{A}_\theta$. Hence we get that:

$$H_2(\mathcal{A}_\theta, {}_{-1}\mathcal{A}_\theta) = 0.$$

Now we consider the homology for $g = 1$ part in the paracyclic decomposition. We notice through computations similar to Chapter 3, that, for $\theta \notin \mathbb{Q}$, $H_2(\mathcal{A}_\theta, \mathcal{A}_\theta) \cong \mathbb{C}$. Here the generator of the homology group $H_2(\mathcal{A}_\theta, \mathcal{A}_\theta)$ is the element $a_{-1,-1} U_1^{-1} U_2^{-1} \in \mathcal{A}_\theta$, which we see is invariant under the $\mathbb{Z}_2$ in a similar way as we have demonstrated in earlier calculations pertaining to the algebraic noncommutative orbifold. Hence using the paracyclic decomposition, we have

$$HH_2(\mathcal{A}_\theta \rtimes \mathbb{Z}_2) = H_2(\mathcal{A}_\theta, \mathcal{A}_\theta)^{\mathbb{Z}_2} \oplus H_2(\mathcal{A}_\theta, {}_{-1}\mathcal{A}_\theta)^{\mathbb{Z}_2}.$$

Using the above formula we conclude that

$$HH_2(\mathcal{A}_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{C}.$$

□

Theorem 5.2. *For $\theta \notin \mathbb{Q}$, satisfying the Diophantine condition, we have, $H_0(\mathcal{A}_\theta, \mathcal{A}_\theta)^{\mathbb{Z}_2} \cong \mathbb{C}$*

Proof. As we know from [C] that for $\theta \notin \mathbb{Q}$ satisfying the Diophantine condition,

$$H_0(\mathcal{A}_\theta, \mathcal{A}_\theta) \cong \mathbb{C}.$$

This group is generated by $\varphi_{0,0}$. To compute the invariance we need to deploy the method we used in Chapter 3. We need to push the cycle to the bar complex, $C_*(\mathcal{A}_\theta)$ and then consider the natural action that exists on the bar complex.

Using the map $h_0 : J_0(\mathcal{A}_\theta) \to C_0(\mathcal{A}_\theta)$, and the map $k_0 : C_(\mathcal{A}_\theta) \to J_0(\mathcal{A}_\theta)$, we notice that $h_0 = k_0 = id$. Hence we observe that the $\mathbb{Z}_2$ action on the bar complex is induced on to the Kozul complex without any alteration. Hence in the Kozul complex $-1 \cdot a_{\bar{0},0} = a_{\bar{0},0}$. Hence we conclude that

$$H_0(\mathcal{A}_\theta, \mathcal{A}_\theta)^{\mathbb{Z}_2} \cong \mathbb{C}$$

$\square$

Theorem 5.3. *For $\theta \notin \mathbb{Q}$, satisfying the Diophantine condition and $\Gamma \subset SL(2,\mathbb{Z})$ a finite group. We have*

$$H_1(\mathcal{A}_\theta, \mathcal{A}_\theta)^{\mathbb{Z}_2} = 0.$$

Proof. Connes proved [C] that for $\theta \notin \mathbb{Q}$ satisfying the Diophantine condition,

$$H_1(\mathcal{A}_\theta, \mathcal{A}_\theta) \cong \mathbb{C}^2.$$

This group is generated by $\varphi^1_{-1,0}$ and $\varphi^2_{0,-1}$. To compute the invariant sub-space we need to deploy the method we used in Chapter 3. We need to push the cycle to the bar complex, $C_*(\mathcal{A}_\theta)$ using the map $h_1 : J_1(\mathcal{A}_\theta) \to C_1(\mathcal{A}_\theta)$,

$$h_1(I \otimes e_i) = I \otimes U_j$$

and then consider the natural action that exists on the bar complex. Thereafter compare the pull back of the twisted element with the original cycle.

We can represent a general cycle $\varphi \in H_1(\mathcal{A}_\theta, {}_{-1}\mathcal{A}_\theta)$ in terms of generators $\varphi^{\bar{1}}_{-1,0}$ and $\varphi^{\bar{2}}_{0,-1}$ as follows,

$$\varphi = aU_1^{-1} \otimes e_1 + bU_2^{-1} \otimes e_2$$

The twisted pull back of φ is the cycle

$$(1 \otimes k_1)(-1 \cdot (1 \otimes h_1)(\varphi)) = (1 \otimes k_1)(-1 \cdot (aU_1^{-1} \otimes U_1 + bU_2 \otimes U_2^{-1})) =$$

$$(1 \otimes k_1)(aU_1 \otimes U_1^{-1} + bU_2 \otimes U_2^{-1}) = aU_1 A \otimes e_1 + aU_1 B \otimes e_2 + bU_2 A' \otimes e_1 + bU_2 B' \otimes e_2 =$$

$$(aU_1 A + bU_2 A') \otimes e_1 + (aU_1 B + bU_2 B') \otimes e_2.$$

With further simplification using the fact that $\mathcal{A}_\theta$ is a $\mathfrak{B}_\theta$ module, we have $U_1 A = -U_1^{-1}$ and $U_2 B' = -U_2^{-1}$. We also notice that $A' = B = 0$, hence we have

$$(1 \otimes k_1)(-1 \cdot (1 \otimes h_1)(aU_1^{-1} \otimes e_1 + bU_2^{-1} \otimes e_2)) = -(aU_1^{-1} \otimes e_1 + bU_2^{-1} \otimes e_2).$$

Whence,

$$H_1(\mathcal{A}_\theta, \mathcal{A}_\theta)^{\mathbb{Z}_2} = 0.$$

$\square$

Conjecture 5.4. *For $\theta \notin \mathbb{Q}$, we conjecture that*

$$HC_{even}(\mathcal{A}_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{C}^6, HC_{odd}(\mathcal{A}_\theta \rtimes \mathbb{Z}_2) = 0.$$

This conjecture when true would mean that the dimension of the $K_0(\mathcal{A}_\theta \rtimes \mathbb{Z}_2)$ and $HC_{even}(\mathcal{A}_\theta \rtimes \mathbb{Z}_2)$ are same, which is very interesting. Using this we can try for a possible postulation of the Poincare duality.

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