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Several Problems Concerning Multivariate Functions and Associated Operators

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Chapter 1

Introduction

For many years, mathematicians have exploited the deep connections between operator theory and function theory to obtain results in both areas. In particular, operator-theoretic techniques and results have been quite useful in proving results about functions and function spaces. For example, mathematicians have made interpolation problems – specifically, the Pick problem – more tractable by rewriting the problem in terms of multiplier algebras of Hilbert function spaces. Furthermore, many results about analytic functions on the unit disk have been generalized to functions on the bidisk or polydisk by way of operator-based representations of key function spaces. Results about function spaces also have implications for the analysis and characterization of certain classes of operators. For instance, many spectral properties of a contractive operator on a Hilbert space can be obtained by studying a related shift-invariant subspace of a vector-valued Hardy space [47].

This thesis concerns two distinct problems appearing in this overlap between operator theory and function theory. Chapter 2, called *Fundamental Agler Decompositions*, addresses the first problem, and Chapter 3, called *Differentiating Matrix Functions*, addresses the second problem. These chapters are self-contained and possess their own comprehensive introductions, which include discussions of relevant definitions, associated developments, and related literature and summaries of the main results. Most results in *Fundamental*

Agler Decompositions have been published in [20] and the results in *Differentiating Matrix Functions* have been published in [21]. For the ease of the reader, we include brief discussions here as well.

In *Fundamental Agler Decompositions*, we discuss results motivated by the Pick Interpolation problem on the bidisk, denoted $\mathbb{D}^2$. The two-variable Pick problem asks:

Given points $\lambda^1, \dots, \lambda^n \in \mathbb{D}^2$ and $\mu^1, \dots, \mu^n \in \mathbb{D}$ when is there a holomorphic $\phi : \mathbb{D}^2 \to \overline{\mathbb{D}}$ with $\phi(\lambda^i) = \mu^i$ for $1 \leq i \leq n$?

The answer rests on a representation formula of J. Agler from [2], who showed that for each holomorphic $\phi : \mathbb{D}^2 \to \overline{\mathbb{D}}$, there are positive kernels $K_1, K_2 : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$ satisfying

$$1 - \phi(z)\overline{\phi(w)} = (1 - z_1 \bar{w}_1) K_2(z, w) + (1 - z_2 \bar{w}_2) K_1(z, w) \qquad \forall\, z, w \in \mathbb{D}^2.$$

This representation using kernels (K_1, K_2) is called an *Agler decomposition of* ϕ and has been used to generalize many results about bounded analytic functions on the disk to bounded analytic functions on the bidisk.

Agler's original proof of the existence of such (K_1, K_2) was nonconstructive, and the structure of such decompositions and their associated Hilbert spaces remained mysterious for many years. In *Fundamental Agler Decompositions*, we introduce specific shift-invariant subspaces of the Hardy space on the bidisk and use them to give an elementary proof of the existence of Agler decompositions, which is constructive for inner functions. These shift-invariant subspaces are actually specific cases of Hilbert spaces that can be defined from Agler decompositions, and we analyze the properties of these Hilbert spaces. We then restrict attention to rational inner functions and show that these shift-invariant subspaces simplify the current theory surrounding Agler decompositions of rational inner functions. The chapter ends with an application of the analysis, which yields a characterization of stable polynomials on the polydisk.

In *Differentiating Matrix Functions*, we discuss results motivated by the study of one-variable matrix functions, specifically functions $F : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$. Such matrix functions are defined using real-valued functions and appear frequently in both science and engineering models, especially those involving systems of linear differential equations. Matrix functions also play a key role in spectral theory; for instance, the matrix sign function provides insight into the location of the eigenvalues of a given matrix [30]. Derivatives of such matrix functions are also quite important. They both provide ways of measuring the sensitivity of a matrix solution to changes in input data and provide simple characterizations of monotone and convex matrix functions [30, 19].

We begin the analysis by generalizing the construction of one-variable matrix functions to d-variable matrix functions, which are defined on the set of d-tuples of pairwise-commuting $n \times n$ self-adjoint matrices, denoted CS^d_n. We focus on differentiability properties of such matrix functions. First, we analyze the geometry of CS^d_n and conclude that a suitable notion of differentiability for functions on this space is differentiation along curves. We then analyze the properties of differentiable curves in CS^d_n. Our main results show that an m-times continuously differentiable real-valued function defined on $\mathbb{R}^d$ can be used to define a d-variable matrix-valued function that can be m-times continuously differentiated along C^m curves in CS^d_n. The chapter also includes formulas for the derivatives and ends with a discussion of how these derivatives imply characterizations of d-variable monotone and convex matrix functions.

Chapter 2

Fundamental Agler Decompositions

2.1 Introduction

Solving an interpolation problem involves constructing a function using a set of data points so that the function possesses additional desired properties. The work in this chapter is motivated by the two-variable generalization of a one-variable interpolation problem, called the Pick problem. The characterization of the two-variable Pick problem's solvability is related to a particular decomposition of two-variable holomorphic functions. This decomposition and associated objects play an important role in two-variable, analytic function theory.

In this introductory section, we discuss the one-variable Pick Interpolation problem, its two-variable generalization, and the associated theory of reproducing kernel Hilbert spaces. We also discuss realization formulas motivated by the Pick problem and along the way, introduce other important definitions and known results. The introduction ends with a summary of the results appearing in this chapter.

2.1.1 Pick Interpolation on the Disk and Bidisk

Recall that the unit disk $\mathbb{D}$ is the set $\{z \in \mathbb{C} : |z| < 1\}$ and the torus $\mathbb{T}$ is the set $\{z \in \mathbb{C} : |z| = 1\}$. In 1916, Pick considered the following interpolation problem on $\mathbb{D}$:

Pick's Interpolation Question: *Given points* $\lambda^1, \ldots, \lambda^n \in \mathbb{D}$ *and* $\mu^1, \ldots, \mu^n \in \mathbb{D}$*, when is there a holomorphic* $\phi : \mathbb{D} \to \overline{\mathbb{D}}$ *such that* $\phi(\lambda^i) = \mu^i$ *for* $i = 1, \ldots, n$*?*

The answer to this question requires the following definition:

Definition 2.1.1. Let $\Omega \subseteq \mathbb{C}^d$. Then, a function $K : \Omega \times \Omega \to \mathbb{C}$ is called a *positive kernel on* Ω if for all finite sets $\{\lambda^1, \ldots, \lambda^m\} \subseteq \Omega$, the matrix

$$\Big(K(\lambda^i, \lambda^j)\Big)_{i,j=1}^{m}$$

is positive semidefinite. A positive kernel is called *holomorphic* if it is holomorphic in the first variable and conjugate-holomorphic in the second variable.

Using the language of positive kernels, we can now state:

Pick's Interpolation Answer: *A holomorphic interpolating function* $\phi : \mathbb{D} \to \overline{\mathbb{D}}$ *exists if and only if there is a positive kernel* $K : \{1, \ldots, n\} \times \{1, \ldots, n\} \to \mathbb{C}$ *such that*

$$1 - \mu^i \bar{\mu}^j = (1 - \lambda^i \bar{\lambda}^j) K(i,j) \qquad \forall\ i, j \in \{1, \ldots, n\}. \tag{2.1.1}$$

It is not hard to show that Pick's condition (2.1.1) is necessary. However, the proof does require knowledge about the basic theory of reproducing kernel Hilbert spaces and their multipliers. Since this theory is key in later sections, we introduce it now and then show that Pick's condition is necessary. First, consider the following definitions:

Definition 2.1.2. A *reproducing kernel Hilbert space* $\mathcal{H}$ on $\Omega \subseteq \mathbb{C}^d$ is a Hilbert space of functions $f : \Omega \to \mathbb{C}$ such that for each $w \in \Omega$, point evaluation at w is a continuous linear functional. Thus, for each $w \in \Omega$, there is an element $K_w \in \mathcal{H}$ such that

$$\langle f, K_w \rangle_{\mathcal{H}} = f(w) \qquad \forall\ f \in \mathcal{H}.$$

It makes sense to define $K(z,w) := K_w(z)$ and regard K as a function on $\Omega \times \Omega$. Such a K is a positive kernel, and the space $\mathcal{H}$ with reproducing kernel K is denoted $\mathcal{H}(K)$. If $\mathcal{H}(K)$ is a space of holomorphic functions, then K is a holomorphic kernel.

The following well-known result, which appears as Theorem 2.23 in [4], shows that reproducing kernel Hilbert spaces can be uniquely identified with positive kernels. This result was originally proven for Hilbert spaces $\mathcal{H}$ such that evaluation at each point in Ω is a *nonzero* continuous linear functional and kernels K such that $K(z,z) > 0$ for all $z \in \Omega$. Nevertheless, the arguments still hold for our more general setting.

Theorem 2.1.3. *Given a positive kernel K on Ω, there is a unique reproducing kernel Hilbert space $\mathcal{H}(K)$ on Ω with reproducing kernel K.*

Several facts about the Hilbert space $\mathcal{H}(K)$ are quite important. Define $\mathcal{L}$ to be the set of finite linear combinations of functions of the form $K(\cdot, w)$, where w is any fixed point in Ω. Then $\mathcal{L}$ is dense in $\mathcal{H}(K)$. Moreover, the inner product of $\mathcal{H}(K)$ is defined by

$$\langle\, K(\cdot, w), K(\cdot, z)\, \rangle_{\mathcal{H}(K)} := K(z,w)$$

on $\{K(\cdot, w)\}_{w \in \Omega}$ and extends to $\mathcal{L}$ by linearity. Basically, $\mathcal{H}(K)$ is the completion of $\mathcal{L}$ with respect to this inner product. The following property follows from Parseval's identity and appears as Proposition 2.18 in [4]:

Theorem 2.1.4. *Let $\mathcal{H}(K)$ be a reproducing kernel Hilbert space on Ω and let $\{f_i\}_{i \in I}$ be an orthonormal basis for $\mathcal{H}(K)$. Then*

$$K(z,w) = \sum_{i \in I} f_i(z)\overline{f_i(w)}.$$

Several later proofs will require information about multipliers and hence, the following definition is included for clarity:

Definition 2.1.5. A function ψ on Ω is a *multiplier* of a Hilbert space $\mathcal{H}$ of functions on Ω if for all $f \in \mathcal{H}$, the function $\psi f \in \mathcal{H}$ as well. Denote the operator of multiplication by ψ on $\mathcal{H}$ by M_ψ. If $\mathcal{H} = \mathcal{H}(K)$, then the closed graph theorem implies that M_ψ is bounded; the resultant operator norm is denoted by $\|M_\psi\|_{\mathcal{H}}$.

The following well-known result characterizes the multipliers of reproducing kernel Hilbert spaces and is a special case of Theorem 2.3.9 in [10]:

Theorem 2.1.6. *Let $\mathcal{H}(K)$ be a reproducing kernel Hilbert space on Ω, and let ψ be a function on Ω. Then, M_ψ is a bounded linear operator on $\mathcal{H}(K)$ with $\|M_\psi\|_{\mathcal{H}(K)} \leq b$ if and only if*

$$\left(b^2 - \psi(z)\overline{\psi(w)}\right)K(z,w) \text{ is a positive kernel on } \Omega.$$

To illustrate these objects and obtain definitions needed to show Pick's condition (2.1.1) is necessary, consider the following reproducing kernel Hilbert space and its multipliers.

Example 2.1.7. The *Hardy space* on $\mathbb{D}$, denoted $H^2(\mathbb{D})$, is the space of holomorphic functions defined on $\mathbb{D}$ satisfying

$$\|f\|_{H^2} := \lim_{r \nearrow 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{\frac{1}{2}} < \infty. \tag{2.1.2}$$

Write $f(z) = \sum a_n z^n$ using its power series expansion at zero. Then, the H^2 norm can be equivalently expressed as

$$\|f\|_{H^2} = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{\frac{1}{2}}.$$

It is well-known that $H^2(\mathbb{D})$ is actually a Hilbert space with inner product

$$\langle f, g \rangle_{H^2} := \lim_{r \nearrow 1} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})\overline{g(re^{i\theta})}\, d\theta = \sum_{n=0}^{\infty} a_n \bar{b}_n,$$

where $g(z) = \sum b_n z^n$. The Cauchy integral formula shows that $H^2(\mathbb{D})$ is actually a reproducing kernel Hilbert space with reproducing kernel K given by

$$K(z,w) = \frac{1}{1-z\bar{w}} \qquad \forall\ z,w \in \mathbb{D}.$$

Now, let $H^\infty(\mathbb{D})$ be the Banach space of bounded holomorphic functions on $\mathbb{D}$ with norm

$$\|\phi\|_\infty := \sup_{z\in\mathbb{D}} |\phi(z)|.$$

Using (2.1.2), it is easy to see that each $\phi \in H^\infty(\mathbb{D})$ is a multiplier of $H^2(\mathbb{D})$ and

$$\|M_\phi\|_{H^2} \leq \|\phi\|_\infty. \tag{2.1.3}$$

It is not hard to show that equality occurs in (2.1.3). For the proof, see page 10 of [4].

Necessity of Pick's Condition (2.1.1): Given these definitions and well-known results, the necessity of Pick's condition is basically immediate. To see why, assume there is an interpolating function $\phi : \mathbb{D} \to \overline{\mathbb{D}}$ with $\phi(\lambda^i) = \mu^i$ for $i = 1, \dots, n$. Then, the function ϕ is a multiplier of the Hardy space with multiplier norm $\|M_\phi\|_{H^2} \leq 1$. It follows immediately from Theorem 2.1.6 that the function $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ defined by

$$K(z,w) := \frac{1-\phi(z)\overline{\phi(w)}}{1-z\bar{w}} \tag{2.1.4}$$

is a positive kernel. Restricting (2.1.4) to the set $\{\lambda^1, \dots, \lambda^n\}$ gives precisely (2.1.1).

Now, consider the situation in several variables. Recall that the polydisk $\mathbb{D}^d$ is the set $\{(z_1, \dots, z_d) : z_1, \dots, z_d \in \mathbb{D}\}$ and the d-torus $\mathbb{T}^d$ is the set $\{(z_1, \dots, z_d) : z_1, \dots, z_d \in \mathbb{T}\}$. The bidisk $\mathbb{D}^2$ is particularly interesting because in 1989, J. Agler generalized Pick's result to the bidisk. Specifically in [1], he proved:

Theorem 2.1.8. ***Pick Interpolation on the Bidisk.*** *Let $\lambda^1, \dots, \lambda^n \in \mathbb{D}^2$ and $\mu^1, \dots, \mu^n \in$*

$\mathbb{D}$, where each $\lambda^i = (\lambda_1^i, \lambda_2^i)$. Then, there is a holomorphic $\phi : \mathbb{D}^2 \to \overline{\mathbb{D}}$ with

$$\phi(\lambda^i) = \mu^i \qquad \forall\ i \in \{1, \ldots, n\}$$

if and only if there are positive kernels $K_1, K_2 : \{1, \ldots, n\} \times \{1, \ldots, n\} \to \mathbb{C}$ such that

$$1 - \mu^i \bar{\mu}^j = (1 - \lambda_1^i \bar{\lambda}_1^j) K_2(i,j) + (1 - \lambda_2^i \bar{\lambda}_2^j) K_1(i,j) \qquad \forall\ i, j \in \{1, \ldots, n\}. \tag{2.1.5}$$

Unlike the one-variable case, the necessity of (2.1.5) does not follow immediately from the theory of reproducing kernel Hilbert spaces. Rather, this is the context of the Agler decomposition theorem. Specifically in [2], Agler proved the following theorem:

Theorem 2.1.9. ***Agler Decomposition Theorem.*** *Let $\phi : \mathbb{D}^2 \to \overline{\mathbb{D}}$ be holomorphic. Then, there are positive holomorphic kernels $K_1, K_2 : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$ such that*

$$1 - \phi(z)\overline{\phi(w)} = (1 - z_1 \bar{w}_1) K_2(z,w) + (1 - z_2 \bar{w}_2) K_1(z,w) \qquad \forall\ z, w \in \mathbb{D}^2. \tag{2.1.6}$$

The terms in Theorem 2.1.9 are important enough to warrant their own definition.

Definition 2.1.10. Let $\phi : \mathbb{D}^2 \to \overline{\mathbb{D}}$ be holomorphic. Then, (2.1.6) is called an *Agler decomposition of* ϕ, and the kernels (K_1, K_2) are called *Agler kernels of* ϕ. To make future calculations easier, the ordering of the kernels in (2.1.6) is opposite of the order that typically appears in the literature.

This definition extends to holomorphic $\phi : \mathbb{D}^d \to \overline{\mathbb{D}}$. Specifically, assume there exist d positive holomorphic kernels $K_1, \ldots, K_d : \mathbb{D}^d \times \mathbb{D}^d \to \mathbb{C}$ such that

$$1 - \phi(z)\overline{\phi(w)} = (1 - z_1 \bar{w}_1) K_1(z,w) + \cdots + (1 - z_d \bar{w}_d) K_d(z,w) \qquad \forall\ z, w \in \mathbb{D}^d. \tag{2.1.7}$$

Then (2.1.7) is called an *Agler decomposition of* ϕ, and the kernels $(K_1, \ldots, K_d)$ are called *Agler kernels of* ϕ.

2.1.2 Associated Results, History, and Literature

Agler's proof of the existence of Agler kernels in [2] used a nonconstructive separation argument. This argument hinged on the fact that holomorphic $\phi : \mathbb{D}^2 \to \overline{\mathbb{D}}$ satisfy von Neumann's inequality, defined as follows:

Definition 2.1.11. A holomorphic function $\phi : \mathbb{D}^d \to \overline{\mathbb{D}}$ satisfies *von Neumann's inequality* if for all d-tuples of commuting contractions $(T_1, \dots, T_d)$ on any Hilbert space $\mathcal{H}$, the operator $\phi(T_1, \dots, T_d)$ is also a contraction on $\mathcal{H}$, i.e.

$$\|\phi(T_1, \dots, T_d)\|_{\mathcal{H}} \leq 1.$$

It was pointed out in [23] – and details also appear in [5] using [31] – that functions on the polydisk $\mathbb{D}^d$ possess an Agler decomposition as in (2.1.7) if and only if they satisfy von Neumann's inequality. In [2], Agler also showed that holomorphic functions on the polydisk $\mathbb{D}^d$ possess Agler decompositions if and only if they have a *coisometric transfer function realization*, defined as follows:

Definition 2.1.12. A holomorphic $\phi : \mathbb{D}^d \to \overline{\mathbb{D}}$ has a *coisometric transfer function realization* if there is a Hilbert space $\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_d$ and a coisometric operator $U : \mathbb{C} \oplus \mathcal{M} \to \mathbb{C} \oplus \mathcal{M}$ such that if we define the operators

$$E_z := z_1 I_{\mathcal{M}_1} + \cdots + z_d I_{\mathcal{M}_d} \qquad \forall\ z = (z_1, \dots, z_d) \in \mathbb{D}^d,$$

where each $I_{\mathcal{M}_r}$ is the identity on $\mathcal{M}_r$ and write U in block form as follows:

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathbb{C} \\ \mathcal{M} \end{pmatrix} \to \begin{pmatrix} \mathbb{C} \\ \mathcal{M} \end{pmatrix}, \tag{2.1.8}$$

then:

$$\phi(z) = A + BE_z \left(I_{\mathcal{M}} - DE_z\right)^{-1} C \quad \text{for } z \in \mathbb{D}^d.$$

This realization formula has proven quite useful in analytic function theory on the bidisk and polydisk. Still, the associated Hilbert space $\mathcal{M}$ is a bit mysterious and shedding light on the structure of $\mathcal{M}$ may open doors to additional applications. Further, as implied by Theorem 2.1.14 below, the study of Agler kernels of ϕ is closely related to the study of $\mathcal{M}$.

Remark 2.1.13. To see the connection between Agler kernels and coisometric transfer function realizations, assume (K_1, K_2) are Agler kernels of ϕ. Then we can define an isometry

$$V : \mathbb{C} \oplus \mathcal{H}(K_1) \oplus \mathcal{H}(K_2) \oplus \mathcal{H} \ \rightarrow \ \mathbb{C} \oplus \mathcal{H}(K_1) \oplus \mathcal{H}(K_2) \oplus \mathcal{H},$$

where $\mathcal{H}$ is an arbitrary Hilbert space. To obtain V, first define it by

$$V \begin{bmatrix} 1 \\ \bar{w}_2 K_1(\cdot, w) \\ \bar{w}_1 K_2(\cdot, w) \end{bmatrix} = \begin{bmatrix} \overline{\phi(w)} \\ K_1(\cdot, w) \\ K_2(\cdot, w) \end{bmatrix} \qquad \text{for each } w \in \mathbb{D}^2$$

and extend it by linearity. Then V is isometric on the initial domain and can be extended to an isometry on $\mathcal{H}(K_1) \oplus \mathcal{H}(K_2)$; this last extension might require the addition of an arbitrary infinite dimensional Hilbert space $\mathcal{H}$. If we set $U := V^*$ and write U in block form as in (2.1.8), then $\phi(z) = A + BE_z \left(I_{\mathcal{M}} - DE_z\right)^{-1} C \quad \text{for } z \in \mathbb{D}^d$.

Combining these results, which are mostly due to J. Agler, yields the following theorem:

Theorem 2.1.14. *Let $\phi : \mathbb{D}^d \rightarrow \overline{\mathbb{D}}$ be holomorphic. Then, the following are equivalent:*

(1) *ϕ has an Agler decomposition with Agler kernels given by $(K_1, \ldots, K_d)$.*

(2) *ϕ has a coisometric transfer function realization on $\mathbb{D}^d$ with the associated Hilbert space $\mathcal{M}$ given by:*

$$\mathcal{M} = \mathcal{H}(K_1) \oplus \cdots \oplus \mathcal{H}(K_d) \oplus \mathcal{H},$$

where $\mathcal{H}$ is an arbitrary and often unnecessary infinite-dimensional Hilbert space.

(3) *ϕ satisfies von Neumann's inequality.*

The importance of these classes of functions motivates the following definition:

Definition 2.1.15. The set of holomorphic functions $\phi : \mathbb{D}^d \to \overline{\mathbb{D}}$ is called the *Schur class* on $\mathbb{D}^d$ and is denoted $\mathcal{S}(\mathbb{D}^d)$. If $\phi \in \mathcal{S}(\mathbb{D}^d)$, then ϕ is called a *Schur function.* It was shown by N. Th. Varopoloulos, M. Crabb, and A. Davis in [58, 25] that for $d \geq 3$, only a strict subset of functions in $\mathcal{S}(\mathbb{D}^d)$ possess Agler decompositions. This subset of functions is called the *Schur-Agler class* on $\mathbb{D}^d$.

Since Agler's seminal work, there has been much interest in both analyzing Agler decompositions on the bidisk as in [16, 24, 42, 38] and better understanding the Schur-Agler class on the polydisk as in [12, 14, 15, 41, 43]. Researchers have also used these Agler kernels and realization formulas to solve function theory questions with operator theory techniques and as a method to craft analytic functions with desired properties as in [3, 5, 6, 9, 17, 37, 39, 46]. Nevertheless, many properties of Agler kernels and their associated reproducing kernel Hilbert spaces have remained fairly mysterious.

2.1.3 Basic Definitions and Summary of Main Results

In this chapter, we study the structure and origin of Agler kernels on the bidisk using the theory of reproducing kernel Hilbert spaces. Specifically, for Agler kernels (K_1, K_2) of a Schur function ϕ, we analyze the Hilbert spaces $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$. We also consider the positive holomorphic kernel

$$K_\phi(z,w) := \frac{1-\phi(z)\overline{\phi(w)}}{(1-z_1\bar{w}_1)(1-z_2\bar{w}_2)} \qquad \forall\ z,w \in \mathbb{D}^2. \tag{2.1.9}$$

The Hilbert space with the reproducing kernel K_ϕ is denoted $\mathcal{H}_\phi$. For every $\phi \in \mathcal{S}(\mathbb{D}^2)$, the space $\mathcal{H}_\phi$ is contained in the two-variable Hardy space $H^2(\mathbb{D}^2)$, which will be defined momentarily. The Hardy spaces on both the bidisk and polydisk play an important role throughout this chapter and can be defined in a way analogous to the one-variable case. Specifically:

Definition 2.1.16. The *Hardy space* on the polydisk $\mathbb{D}^d$, denoted $H^2(\mathbb{D}^d)$, is the space of holomorphic functions defined on $\mathbb{D}^d$ satisfying

$$\|f\|_{H^2} := \lim_{r\nearrow 1}\left(\frac{1}{(2\pi)^d}\int_0^{2\pi}\cdots\int_0^{2\pi}|f(re^{i\theta_1},\dots,re^{i\theta_d})|^2 d\theta_1\dots d\theta_d\right)^{\frac{1}{2}} < \infty. \tag{2.1.10}$$

Then, $H^2(\mathbb{D}^d)$ is a Hilbert space, and its inner product is the obvious generalization of $H^2(\mathbb{D})$'s inner product. Write $f(z) = \sum_{n\in\mathbb{N}^d} a_n z^n$ using its power series expansion at zero with mutli-index notation, i.e. $n = (n_1,\dots,n_d)$ and $z^n = z_1^{n_1}\cdots z_d^{n_d}$. Then $\|f\|_{H^2}^2 = \sum |a_n|^2$. As before, $H^2(\mathbb{D}^d)$ is a reproducing kernel Hilbert space with kernel K given by

$$K(z,w) = \frac{1}{\prod_{i=1}^d (1-z_i\bar{w}_i)} \qquad \forall\ z,w\in\mathbb{D}^d.$$

If $H^\infty(\mathbb{D}^d)$ is the Banach space of bounded holomorphic functions on $\mathbb{D}^d$ with norm $\|\phi\|_\infty := \sup_{z\in\mathbb{D}^d}|\phi(z)|$, then each $\phi\in H^\infty(\mathbb{D}^d)$ is a multiplier of $H^2(\mathbb{D}^d)$ and $\|M_\phi\|_{H^2} = \|\phi\|_\infty$.

Both the general theory and the results proved in this chapter are especially nice for *inner functions*, defined as follows:

Definition 2.1.17. A function $\phi \in \mathcal{S}(\mathbb{D}^d)$ is *inner* if its radial boundary values satisfy

$$\lim_{r \nearrow 1} \left|\phi\big(re^{i\theta_1}, \ldots, re^{i\theta_d}\big)\right| = \left|\phi\big(e^{i\theta_1}, \ldots, e^{i\theta_d}\big)\right| = 1 \text{ a.e. on } \mathbb{T}^d.$$

Inner functions play a primary role in the study of Agler kernels because if ϕ is inner, then $\mathcal{H}_\phi$ has a nice structure; it is equal isometrically to $H^2(\mathbb{D}^2) \ominus \phi H^2(\mathbb{D}^2)$. Moreover, inner functions are in some sense quite general because they are locally, uniformly dense in $\mathcal{S}(\mathbb{D}^d)$. The follow result of Rudin appears as Theorem 5.5.1 in [53]:

Theorem 2.1.18. *Every $\phi \in \mathcal{S}(\mathbb{D}^d)$ is a limit (uniformly on compact subsets of $\mathbb{D}^d$) of a sequence of inner functions on $\mathbb{D}^d$ that are continuous on $\overline{\mathbb{D}^d}$.*

Summary of Results

Given those definitions, we can now discuss the main results of the chapter. Most of the results concern properties of Agler decompositions of Schur functions on the bidisk. Here is a summary of the main results by section:

Section 2.2

In Section 2.2, we consider inner ϕ and introduce fundamental shift-invariant subspaces of $\mathcal{H}_\phi$ and hence, of $H^2(\mathbb{D}^2)$. These subspaces are special cases of spaces that appear naturally in the theory of scattering systems and scattering-minimal unitary colligations; such subspaces are discussed extensively by Ball-Sadosky-Vinnikov in [16]. Specifically, for $r = 1, 2$, we let Z_r denote the coordinate function $Z_r(z_1, z_2) = z_r$. We then let S_1^{max} denote the largest subspace in $\mathcal{H}_\phi$ invariant under multiplication by Z_1 and let $S_2^{min} = \mathcal{H}_\phi \ominus S_1^{max}$. We define S_2^{max} and S_1^{min} analogously.

We show that these subspaces yield an elementary proof of the Agler decomposition theorem, which is constructive for inner functions. The result is implied by analyses in [16], and related arguments appear in a recent paper by Grinshpan-Kaliuzhnyi-Verbovetskyi-Vinnikov-

Woerdeman in [28], who prove a generalization of the Agler decomposition theorem. Their arguments use the theory of scattering systems and shift-invariant subspaces of scattering subspaces. This proof is independently interesting and important because it removes the need for scattering systems and provides concrete decompositions. We also develop a uniqueness criterion for Agler decompositions of inner functions and show that non-extreme functions never have unique Agler decompositions. We end with an algorithm for constructing Agler decompositions for particularly well-behaved polynomials.

Section 2.3

In Section 2.3, we observe that the spaces S_r^{max} and S_r^{min} are special cases of more general objects. Specifically, if $\phi \in \mathcal{S}(\mathbb{D}^2)$ with Agler kernels (K_1, K_2), we define the following Hilbert spaces:

$$S_r^K := \mathcal{H}\left(\frac{K_r(z,w)}{1 - z_r \bar{w}_r}\right),$$

for $r = 1, 2$. It is not hard to show that for any inner $\phi \in \mathcal{S}(\mathbb{D}^2)$, the spaces S_r^{max} and S_r^{min} satisfy backward-shift invariant properties, and S_r^{min} is in some sense a minimal S_r^K space. In Propositions 2.3.4 and 2.3.7, we show that these properties extend to general S_r^K spaces. In particular, we prove that for general ϕ, the associated S_r^K spaces also possess backward-shift invariant properties and contain minimal sets. In Theorem 2.3.10, we characterize the Schur functions ϕ possessing Agler kernels arising from orthogonal decompositions of $\mathcal{H}_\phi$.

Section 2.4

In Section 2.4, we use the subspaces S_r^{max} to examine Agler decompositions of rational inner functions. Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ be rational inner, and let the degree of ϕ in the variable z_r be k_r for $r = 1, 2$. We denote this by $\deg \phi = (k_1, k_2)$ and $\deg_r \phi = k_r$. It is known that for all

Agler kernels (K_1, K_2) of ϕ, each $\mathcal{H}(K_r)$ is finite dimensional. Specifically,

$$\dim(\mathcal{H}(K_1)) \leq k_2(k_1+1) \quad \text{and} \quad \dim(\mathcal{H}(K_2)) \leq k_1(k_2+1).$$

The finiteness condition was proved by Cole and Wermer in [24], and the specific dimension bounds were found by Knese in [41]. We provide a simple short proof using S_1^{max} and S_2^{max}.

We then consider rational inner functions ϕ continuous on $\overline{\mathbb{D}^2}$. In Proposition 2.4.6, we consider and slightly extend analyses from [16] about the Hilbert spaces S_r^{max} and $\mathcal{H}_\phi$ associated to ϕ. We use those results to show that such ϕ have unique Agler decompositions if and only if they are functions of one variable. This result was originally proven by Knese in [40] using alternate methods. In Proposition 2.4.8, we show that this property does not extend to all rational inner functions and construct rational inner functions of arbitrarily high degree with unique Agler decompositions.

Section 2.5

In the concluding section, we provide an application of the analysis of $\mathcal{H}_\phi$ in Proposition 2.4.6. Specifically, recall that a polynomial in d variables is called *stable* if it has no zeros on $\overline{\mathbb{D}^d}$. We first generalize Proposition 2.4.6 to the polydisk in Proposition 2.5.1. We then use it to generalize a result of Knese in [38] characterizing stable polynomials on $\mathbb{D}^2$ to polynomials on $\mathbb{D}^d$.

2.2 The Agler Decomposition Theorem

In this section, we consider the origins of Agler decompositions for Schur functions on the bidisk. In Subsection 2.2.1, we introduce necessary notation and additional definitions. In Subsection 2.2.2, we consider inner ϕ and introduce the subspaces S_r^{max} and S_r^{min}, which will be used to construct Agler decompositions. Then in Subsection 2.2.3, we provide an elementary proof of the Agler decomposition theorem, which is explicitly constructive for inner functions. We also discuss several related results about the uniqueness of Agler decompositions. Lastly in Subsection 2.2.4, we present an algorithm for constructing Agler kernels of particularly well-behaved polynomials.

2.2.1 Notation and Definitions

For clarity, we include the following well-known definition:

Definition 2.2.1. The space $L^2(\mathbb{T}^d)$ is the space of a.e. defined, Lebesgue-measurable functions on $\mathbb{T}^d$ satisfying

$$\|f\|_{L^2} := \left(\frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} |f(e^{i\theta_1}, \ldots, e^{i\theta_d})|^2 \, d\theta_1 \ldots d\theta_d \right)^{\frac{1}{2}} < \infty.$$

Write $f(z) \sim \sum_{n \in \mathbb{Z}^d} \hat{f}(n) z^n$ using its Fourier series with mutli-index notation, i.e. $n = (n_1, \ldots, n_d)$ and $z^n = z_1^{n_1} \cdots z_d^{n_d}$. Then $\|f\|_{L^2}^2 = \sum_{n \in \mathbb{Z}^d} |\hat{f}(n)|^2$. Moreover, $L^2(\mathbb{T}^d)$ is a Hilbert space with inner product given by

$$\langle f, g \rangle_{L^2} := \frac{1}{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} f(e^{i\theta_1}, \ldots, e^{i\theta_d}) \overline{g(e^{i\theta_1}, \ldots, e^{i\theta_d})} \, d\theta_1 \ldots d\theta_d = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) \overline{\hat{g}(n)},$$

Let $L^\infty(\mathbb{T}^d)$ denote the Banach space of bounded a.e.-defined, Lebesgue-measurable functions on $\mathbb{T}^d$ with norm defined by $\|\phi\|_\infty := ess \ sup \ \phi$. Then each $\phi \in L^\infty(\mathbb{T}^d)$ is a multiplier of $L^2(\mathbb{T}^d)$ and $\|M_\phi\|_{L^2} \leq \|\phi\|_\infty$.

In this section, we deal exclusively with the bidisk and so denote $H^\infty(\mathbb{D}^2)$, $H^2(\mathbb{D}^2)$, $L^\infty(\mathbb{T}^2)$, and $L^2(\mathbb{T}^2)$ by H^∞, H^2, L^∞, and L^2. By a subspace of a Hilbert space $\mathcal{H}$, we mean a linear subspace. Given such a subspace U of $\mathcal{H}$, we let $\overline{U}$ denote the closure of U in $\mathcal{H}$. Then, $\overline{U}$ is a Hilbert space that inherits the inner product of $\mathcal{H}$. We also let P_V denote the projection operator onto a closed subspace V of $\mathcal{H}$. For $r = 1, 2$, let z_r denote the r^{th} component of the independent variable z and Z_r denote the coordinate function defined by $Z_r(z_1, z_2) = z_r$. Moreover, let X_r denotes the backward shift operator on H^2 in the z_r coordinate. Specifically X_1 and X_2 are defined by

$$(X_1 g)(z) := \frac{g(z) - g(0, z_2)}{z_1} \quad \text{and} \quad (X_2 g)(z) := \frac{g(z) - g(z_1, 0)}{z_2}$$

for each $g \in H^2$. Let each $X_r^m g$ denote the function obtained by applying the backward shift operator m times to g. Define the following closed subspaces of L^2:

$$\begin{aligned}
L^2_{*-} &:= \big\{ f \in L^2 : \hat{f}(n_1, n_2) = 0 \text{ for } n_2 \geq 0 \big\}, \\
L^2_{-*} &:= \big\{ f \in L^2 : \hat{f}(n_1, n_2) = 0 \text{ for } n_1 \geq 0 \big\}, \\
L^2_{+-} &:= \big\{ f \in L^2 : \hat{f}(n_1, n_2) = 0 \text{ for } n_1 < 0 \text{ or } n_2 \geq 0 \big\}, \\
L^2_{-+} &:= \big\{ f \in L^2 : \hat{f}(n_1, n_2) = 0 \text{ for } n_1 \geq 0 \text{ or } n_2 < 0 \big\}, \\
L^2_{--} &:= \big\{ f \in L^2 : \hat{f}(n_1, n_2) = 0 \text{ for } n_1 \geq 0 \text{ or } n_2 \geq 0 \big\}.
\end{aligned}$$

We will often treat H^2 as a closed subspace of L^2 in the usual way. In particular, each function $f \in H^2$ is associated to the L^2 function whose Fourier coefficients equal the Taylor coefficients of f around zero. For details, see [53]. This associated L^2 function is also denoted by f. Then H^2 can be viewed as the space of functions:

$$\big\{ f \in L^2 : \hat{f}(n_1, n_2) = 0 \text{ for } n_1 < 0 \text{ or } n_2 < 0 \big\} \subset L^2. \tag{2.2.1}$$

This identification is equivalent to associating a function in H^2 with its a.e.-defined radial boundary value function on $\mathbb{T}^2$. For $n_1, n_2 \in \mathbb{N}$ and $f \in H^2$, let $\hat{f}(n_1, n_2)$ denote both the Taylor coefficient of f and the Fourier coefficient of the associated L^2 function f.

2.2.2 Important Hilbert Spaces

In this subsection, we analyze the mathematical objects key in proving the Agler decomposition theorem. However, before considering the important Hilbert spaces, we need one additional result about reproducing kernel Hilbert spaces, which appears as Theorem 5 in [18].

Theorem 2.2.2. *Let $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ be reproducing kernel Hilbert spaces on Ω. Then $K := K_1 + K_2$ is a positive kernel on Ω and the Hilbert space $\mathcal{H}(K)$ is precisely the vector space of functions $\mathcal{H}(K_1) + \mathcal{H}(K_2)$ equipped with the norm*

$$\|f\|^2_{\mathcal{H}(K)} := \min_{\substack{f=f_1+f_2 \\ f_1 \in \mathcal{H}(K_1),\ f_2 \in \mathcal{H}(K_2)}} \|f_1\|^2_{\mathcal{H}(K_1)} + \|f_2\|^2_{\mathcal{H}(K_2)} \qquad \forall\ f \in \mathcal{H}(K_1) + \mathcal{H}(K_2).$$

We first examine the structure of the space $\mathcal{H}_\phi$ when ϕ is inner.

Remark 2.2.3. Structure of $\mathcal{H}_\phi$. Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ be inner, and recall that $\mathcal{H}_\phi$ is the Hilbert space with reproducing kernel given by (2.1.9). First consider its complementary subspace defined by

$$\phi H^2 := \mathcal{H}\left(\frac{\phi(z)\overline{\phi(w)}}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)} \right).$$

Recall that the set of linear combinations of its kernel functions

$$\mathcal{L} := \left\{ \sum_{l=1}^{L} c_l \frac{\phi(z)\overline{\phi(w^l)}}{(1 - z_1\bar{w}_1^l)(1 - z_2\bar{w}_2^l)} : L \in \mathbb{N},\ \text{each } l \in \mathbb{C},\ \text{and } w^l = (w_1^l, w_2^l) \in \mathbb{D}^2 \right\}$$

is dense in ϕH^2. Using the integral form of the H^2 norm and the definition of the inner product on reproducing kernel Hilbert spaces, one can easily show that

$$\|f\|_{\phi H^2} = \|f\|_{H^2} \qquad \forall\ f \in \mathcal{L}.$$

It follows that ϕH^2 is a closed subspace of H^2. An examination of the linear combinations of kernel functions of $\mathcal{H}_\phi$ and ϕH^2 also implies that $\mathcal{H}_\phi \perp \phi H^2$ in the H^2 inner product. Thus, $\mathcal{H}_\phi \subseteq H^2 \ominus \phi H^2$. Moreover, Theorem 2.2.2 implies that $H^2 = \mathcal{H}_\phi + \phi H^2$ which means $(H^2 \ominus \phi H^2) \subseteq \mathcal{H}_\phi$. Thus, $\mathcal{H}_\phi$ is a closed subspace of H^2 and $H^2 = \mathcal{H}_\phi \oplus \phi H^2$. Moreover, multiplication by ϕ is isometric on H^2 and unitary on L^2. We can use that fact to obtain the following sequence, which results in a useful, alternate definition of $\mathcal{H}_\phi$:

$$\begin{aligned} \mathcal{H}_\phi &= H^2 \ominus \phi H^2 \\ &= H^2 \cap (\phi H^2)^\perp \\ &= H^2 \cap \phi[L^2 \ominus H^2] \\ &= \big\{\phi f \in H^2 : f \in L^2_{*-} \oplus L^2_{-+}\big\}. \end{aligned} \tag{2.2.2}$$

Now we define the primary subspaces of interest, which will be used to construct Agler decompositions.

Definition 2.2.4. Maximal and Minimal Shift-Invariant Subspaces. Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ be inner. Define S_1^{max} to be the largest subspace in $\mathcal{H}_\phi$ invariant under M_{Z_1}, i.e. invariant under multiplication by the coordinate function Z_1. Lemma 2.2.5 shows such a subspace must exist. It is immediate that S_1^{max} is a closed subspace of $\mathcal{H}_\phi$ and hence, of H^2. Define $S_2^{min} := \mathcal{H}_\phi \ominus S_1^{max}$, and define S_2^{max} and S_1^{min} analogously.

Lemma 2.2.5. *Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ be inner. Then there exists a maximal M_{Z_1}-invariant subspace S_1^{max} of $\mathcal{H}_\phi$ such that if S_1 is also an M_{Z_1}-invariant subspace of $\mathcal{H}_\phi$, then $S_1 \subseteq S_1^{max}$.*

Proof. The proof is an easy application of Zorn's Lemma. Let $\mathcal{L}$ denote the set of M_{Z_1}-

invariant subspaces of $\mathcal{H}_\phi$ partially ordered by set inclusion. Assume

$$S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots \subseteq S_n \subseteq \ldots$$

is a totally ordered chain in $\mathcal{L}$. Now set $S = \cup_{n=1}^{\infty} S_n$. Then S is an M_{Z_1}-invariant subspace of $\mathcal{H}_\phi$, and each $S_n \subseteq S$. Thus, S is an upper bound of the totally ordered chain. By Zorn's Lemma, $\mathcal{L}$ has a maximal element, which we denote S_1^{max}. Assume S_1 is any other M_{Z_1} invariant subspace of $\mathcal{H}_\phi$. Then, the set $S = S_1 + S_1^{max}$ is also an M_{Z_1}-invariant subspace. If $S_1 \not\subseteq S_1^{max}$, then $S_1^{max} \subsetneq S$, which contradicts the fact that S_1^{max} is maximal. Thus, $S_1 \subseteq S_1^{max}$. $\square$

In Remark 2.2.3, we identified the space $\mathcal{H}_\phi$ of functions on $\mathbb{D}^2$ with the following space of L^2 functions

$$\big\{\phi f \in H^2 : f \in L^2_{*-} \oplus L^2_{-+}\big\}. \tag{2.2.3}$$

Other closed subspaces of H^2 such as S_r^{max} and S_r^{min} can also be identified with closed subspaces of L^2 by associating the H^2 functions with their radial boundary value functions. In particular, each S_r^{max} can be viewed as the maximal subspace of (2.2.3) invariant under M_{Z_r}. Moreover, establishing M_{Z_r}-invariance of a subspace of H^2 is equivalent to establishing M_{Z_r}-invariance of the associated subspace of L^2. The following lemma characterizes the S_r^{max} and S_r^{min} spaces as subspaces of L^2 and establishes the M_{Z_r}-invariance of each S_r^{min}. This lemma is a special case of results that appear in Theorem 5.5 and Proposition 5.11 of Ball-Sadosky-Vinnikov in [16]. We include simple proofs. Some of the arguments originate in [16], while others are our own.

Lemma 2.2.6. *Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ be inner. Then*

$$S_1^{max} = H^2 \cap \phi L^2_{*-} \qquad S_1^{min} = \overline{P_{H^2}\phi L^2_{+-}}$$
$$S_2^{max} = H^2 \cap \phi L^2_{-*} \qquad S_2^{min} = \overline{P_{H^2}\phi L^2_{-+}},$$

and S_r^{max} and S_r^{min} are invariant under M_{Z_r} for $r = 1, 2$.

Proof. We prove the results for S_1^{max} and S_2^{min}. By definition,

$$S_1^{max} = \big\{ f \in \mathcal{H}_\phi : Z_1^k f \in \mathcal{H}_\phi, \ \forall\, k \in \mathbb{N} \big\}.$$

Let S_1 denote the set $H^2 \cap \phi L^2_{*-}$. By the characterization of $\mathcal{H}_\phi$ in (2.2.3), S_1 is a subspace of $\mathcal{H}_\phi$. Since $Z_1 S_1 \subseteq S_1$, we have $S_1 \subseteq S_1^{max}$. Now assume $g \in S_1^{max}$. Then $g \in \mathcal{H}_\phi$, and (2.2.3) implies that $g = \phi f$, for $f \in L^2_{*-} \oplus L^2_{-+}$. Proceeding towards a contradiction, assume $g \notin S_1$. Then there is some $(n_1, n_2) \in \mathbb{Z}^2$ such that $\hat{f}(n_1, n_2) \neq 0$ and $n_2 \geq 0$. The characterization of $\mathcal{H}_\phi$ in (2.2.3) implies that

$$Z_1^{|n_1|} g \notin \mathcal{H}_\phi,$$

which contradicts the definition of S_1^{max}. Thus, $S_1^{max} = H^2 \cap \phi L^2_{*-}$ and so S_1^{max} is precisely the space of L^2 functions orthogonal to the closure of

$$(L^2 \ominus H^2) + \phi(H^2 \oplus L^2_{-+})$$

in L^2. Then we can calculate

$$\begin{aligned} S_2^{min} &:= \mathcal{H}_\phi \ominus S_1^{max} \\ &= P_{\mathcal{H}_\phi}\big[(S_1^{max})^\perp\big] \\ &= \overline{P_{\mathcal{H}_\phi}\big[(L^2 \ominus H^2) + \phi(H^2 \oplus L^2_{-+})\big]} \\ &= \overline{P_{\mathcal{H}_\phi} \phi L^2_{-+}} \\ &= \overline{P_{H^2} \phi L^2_{-+}}, \end{aligned}$$

where the last equality follows because $\phi L^2_{-+} \perp \phi H^2$. Now, define the set

$$\mathcal{L} := \{ f \in L^2_{-+} : \hat{f}(n_1, n_2) = 0 \text{ for all but finitely many } n_1 \}.$$

Then, $\mathcal{L}$ is dense in L^2_{-+}. Define $V = P_{H^2}\phi\mathcal{L}$, and let $f \in \mathcal{L}$. Then, there is some $M \in \mathbb{N}$ such that we can write $f(z) = \sum_{m=1}^{M} f_m(z_2) z_1^{-m}$ a.e. on $\mathbb{T}^2$, where each $f_m \in L^2(\mathbb{T})$ and satisfies

$$f_m(z_2) \sim \sum_{n=0}^{\infty} \hat{f}(-m,n) z_2^n.$$

Then, $P_{H^2}(\phi f) = \sum_{m=1}^{M} P_{H^2}(\phi f_m Z_1^{-m})$. By explicit calculation of Fourier coefficients, one can obtain

$$P_{H^2}\big(\phi f_m Z_1^{-m}\big)(z) \sim \sum_{j,k\geq 0} \widehat{\phi f_m}(j+m,k) z_1^j z_2^k.$$

Viewing $P_{H^2}\big(\phi f_m Z_1^{-m}\big)$ as a holomorphic function on $\mathbb{D}^2$ and analyzing Taylor coefficients shows:

$$P_{H^2}\big(\phi f_m Z_1^{-m}\big)(z) = (X_1^m \phi f_m)(z) = (X_1^m \phi)(z) f_m(z_2),$$

for $z \in \mathbb{D}^2$, where X_1 denotes the backward shift operator on H^2 in the z_1 coordinate. By examining $P_{H^2}\phi f$, it is immediate that:

$$V \subseteq \Big\{ \sum_{m=1}^{M} \big(X_1^m \phi\big)(z) f_m(z_2) : M \in \mathbb{N},\ f_m \in H^2(\mathbb{D}) \Big\}. \tag{2.2.4}$$

By selecting specific $f \in \mathcal{L}$ and doing analogous calculations, containment in the other direction is basically immediate. Thus, as a space of holomorphic functions, V equals the set in (2.2.4). This characterization implies V is invariant under M_{Z_2}. As

$$S_2^{min} = \overline{P_{H^2}\phi L^2_{-+}} = \overline{P_{H^2}\phi\mathcal{L}} = \overline{V},$$

S_2^{min} must be invariant under M_{Z_2}. The results for S_2^{max} and S_1^{min} follow by symmetry. $\square$

2.2.3 Proof of the Existence of Agler Decompositions

In this subsection, we provide an elementary proof of the Agler decomposition theorem. Before proceeding, we need several additional results about reproducing kernel Hilbert spaces. This first result appears in [18] as Theorem 11.

Theorem 2.2.7. *Let M be a closed subspace of a reproducing kernel Hilbert space $\mathcal{H}(K)$ on Ω. Then M is a reproducing kernel Hilbert space on Ω with reproducing kernel*

$$L_M(z,w) := P_M\big[K(\cdot,w)\big](z) \qquad \forall\ z,w \in \Omega,$$

where P_M denotes the orthogonal projection onto M.

The following result appears as Theorem 2.3.13 in [10]:

Theorem 2.2.8. *Let $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ be reproducing kernel Hilbert spaces on Ω. Then $\mathcal{H}(K_1)$ is contained in $\mathcal{H}(K_2)$ if and only if there is some constant $b > 0$ such that difference*

$$K_2(z,w) - \frac{1}{b^2}K_1(z,w) \tag{2.2.5}$$

is a positive kernel on Ω. Moreover, (2.2.5) *holds for $b = 1$ if and only if the containment is contractive.*

We can now prove the Agler decomposition theorem using the subspaces from the previous subsection. J. Agler first proved this result as Theorem 2.6 in [2].

Theorem 2.2.9. ***Agler Decomposition Theorem.*** *Let $\phi \in \mathcal{S}(\mathbb{D}^2)$. Then there are positive holomorphic kernels $K_1, K_2 : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$ satisfying*

$$1 - \phi(z)\overline{\phi(w)} = (1 - z_1\bar{w}_1)K_2(z,w) + (1 - z_2\bar{w}_2)K_1(z,w) \qquad \forall\ z,w \in \mathbb{D}^2.$$

Proof. Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ be inner, and let S_1 and S_2 denote the subspaces S_1^{max} and S_2^{min} from Lemma 2.2.6. Since S_1 and S_2 are closed subspaces of $\mathcal{H}_\phi$, it follows from Theorem 2.2.7

that they are reproducing kernel Hilbert spaces that inherit the $\mathcal{H}_\phi$ inner product and have reproducing kernels given by

$$L_{S_r}(z,w) := P_{S_r}\left[\frac{1-\phi(\cdot)\overline{\phi(w)}}{(1-\cdot\ \bar{w}_1)(1-\cdot\ \bar{w}_2)}\right](z) \qquad \forall\ z,w\in\mathbb{D}^2$$

and for $r=1,2$. By Lemma 2.2.6, each S_r is invariant under M_{Z_r}. As each S_r inherits the $\mathcal{H}_\phi$ norm and $\mathcal{H}_\phi$ inherits the H^2 norm, we have $\|M_{Z_r}\|_{S_r}=1$. Theorem 2.1.6 implies

$$K_r(z,w) := (1-z_r\bar{w}_r)L_{S_r}(z,w)$$

is a positive kernel on $\mathbb{D}^2$ for $r=1,2$. As the S_r are Hilbert spaces of holomorphic functions, it follows that the K_r are holomorphic kernels. Since $\mathcal{H}_\phi = S_1\oplus S_2$, we have

$$\begin{aligned}\frac{1-\phi(z)\overline{\phi(w)}}{(1-z_1\bar{w}_1)(1-z_2\bar{w}_2)} &= L_{S_1}(z,w)+L_{S_2}(z,w)\\ &= \frac{K_1(z,w)}{1-z_1\bar{w}_1}+\frac{K_2(z,w)}{1-z_2\bar{w}_2}.\end{aligned} \tag{2.2.6}$$

Rearranging terms shows that (K_1,K_2) are Agler kernels of ϕ.

Now, let $\phi\in\mathcal{S}(\mathbb{D}^2)$ be arbitrary. Then, Theorem 2.1.18 gives a sequence of inner functions $\{\phi^n\}$ converging locally, uniformly to ϕ. Let $\{K_1^n\}$ and $\{K_2^n\}$ denote the sequences of Agler kernels for the $\{\phi^n\}$ that are guaranteed by our previous arguments. Basic manipulations of (2.2.6) show that

$$\frac{1-\phi^n(z)\overline{\phi^n(w)}}{(1-z_1\bar{w}_1)(1-z_2\bar{w}_2)} - K_r^n(z,w)$$

is a positive kernel for $r=1,2$ and $n\in\mathbb{N}$. Thus,

$$\frac{1}{(1-z_1\bar{w}_1)(1-z_2\bar{w}_2)} - K_r^n(z,w) \tag{2.2.7}$$

is also a positive kernel and so Theorem 2.2.8 implies that each $\mathcal{H}(K_r^n) \subseteq H^2$ contractively. Now the Cauchy-Schwarz inequality coupled with (2.2.7) restricted to the set $\{(z,w) \in \mathbb{D}^2 : z = w\}$ can be used to show that

$$|K_r^n(z,w)|^2 \leq |K_r^n(z,z)||K_r^n(w,w)| \leq \frac{1}{(1-|z_1|^2)(1-|z_2|^2)} \frac{1}{(1-|w_1|^2)(1-|w_2|^2)},$$

for all $z, w \in \mathbb{D}^2$ and $n \in \mathbb{N}$. Since the sequences $\{K_r^n\}$ are locally, uniformly bounded, they form a normal family. By Montel's theorem, there is a subsequence $\{\phi^{n_k}\}$ such that the associated kernel subsequences $\{K_1^{n_k}\}$ and $\{K_2^{n_k}\}$ converge locally uniformly to positive holomorphic kernels K_1 and K_2 satisfying

$$1 - \phi(z)\overline{\phi(w)} = (1 - z_1\bar{w}_1)K_2(z,w) + (1 - z_2\bar{w}_2)K_1(z,w),$$

for all $z, w \in \mathbb{D}^2$. □

The previous proof is particularly nice because it does not use von Neumann's inequality. Then we can deduce von Neumann's inequality on $\mathbb{D}^2$ as a corollary of Theorem 2.2.9 using the arguments appearing in Theorem 1.2 of [23] or in [5], which relies on results from [31].

Corollary 2.2.10. ***von Neumann's Inequality.*** *Let $\phi \in \mathcal{S}(\mathbb{D}^2)$, and let (T_1, T_2) be any pair of commuting contractions on a Hilbert space $\mathcal{H}$. Then, $\phi(T_1, T_2)$ is also a contraction on $\mathcal{H}$.*

The proof of Theorem 2.2.9 provides simple Agler kernels for inner functions. For ease of notation, positive kernels $K(z,w)$ on $\mathbb{D}^2 \times \mathbb{D}^2$ will be denoted by simply K.

Remark 2.2.11. Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ be inner. By the arguments in the proof of Theorem 2.2.9, there are positive holomorphic kernels on $\mathbb{D}^2$, now denoted K_r^{max} and K_r^{min}, such that

$$S_r^{max} = \mathcal{H}\left(\frac{K_r^{max}}{1 - z_r\bar{w}_r}\right) \quad \text{and} \quad S_r^{min} = \mathcal{H}\left(\frac{K_r^{min}}{1 - z_r\bar{w}_r}\right), \tag{2.2.8}$$

for $r = 1, 2$. Moreover, (K_1^{max}, K_2^{min}) and (K_1^{min}, K_2^{max}) are pairs of Agler kernels of ϕ.

This proof of Theorem 2.2.9 provides insight into the uniqueness of Agler decompositions for inner functions. The following result generalizes part of Theorem 5.10 in [16].

Theorem 2.2.12. *Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ be inner. Then ϕ has a unique Agler decomposition if and only if*

$$\phi L^2_{--} \cap H^2 = \{0\}.$$

Proof. Using the definitions of S_r^{max} and S_r^{min} and their characterizations in Lemma 2.2.6, it is easy to show that each S_r^{min} is a closed subspace of S_r^{max} and

$$S_1^{max} \ominus S_1^{min} = S_2^{max} \ominus S_2^{min} = \phi L^2_{--} \cap H^2. \tag{2.2.9}$$

$(\Rightarrow)$ Assume ϕ has a unique Agler decomposition. By Remark 2.2.11, this implies

$$(K_1^{max}, K_2^{min}) = (K_1^{min}, K_2^{max}).$$

By the representations of S_r^{max} and S_r^{min} in Remark 2.2.11, we must have $S_r^{max} = S_r^{min}$. Using (2.2.9), this implies $\phi L^2_{--} \cap H^2 = \{0\}$.

$(\Leftarrow)$ Assume $\phi L^2_{--} \cap H^2 = \{0\}$. Then it follows from (2.2.9) that each $S_r^{max} = S_r^{min}$ and so each $K_r^{max} = K_r^{min}$. In particular, (K_1^{min}, K_2^{min}) is a pair of Agler kernels of ϕ. Let (L_1, L_2) be any pair of Agler kernels of ϕ. By Theorem 2.1.6 and the maximality of S_r^{max} established in Lemma 2.2.5,

$$\mathcal{H}\left(\frac{L_r(z,w)}{1 - z_r \bar{w}_r}\right) \subseteq S_r^{max}$$

for $r = 1, 2$. In particular, for each fixed $w \in \mathbb{D}^2$ and $r = 1, 2$, the functions

$$\frac{K_r^{min}(\cdot, w)}{1 - Z_r \bar{w}_r}, \ \frac{L_r(\cdot, w)}{1 - Z_r \bar{w}_r} \in S_r^{max} = S_r^{min}.$$

By the definition of Agler kernels, we have

$$\begin{aligned}\frac{1-\phi(z)\overline{\phi(w)}}{(1-z_1\bar{w}_1)(1-z_2\bar{w}_2)} &= \frac{L_1(z,w)}{1-z_1\bar{w}_1}+\frac{L_2(z,w)}{1-z_2\bar{w}_2}\\ &= \frac{K_1^{min}(z,w)}{1-z_1\bar{w}_1}+\frac{K_2^{min}(z,w)}{1-z_2\bar{w}_2}.\end{aligned}\tag{2.2.10}$$

As $S_1^{min} \perp S_2^{min}$ in $\mathcal{H}_\phi$, the decomposition in (2.2.10) is unique for each fixed $w \in \mathbb{D}^2$. It follows that for $r = 1, 2$,

$$\frac{L_r(\cdot,w)}{1-Z_r\bar{w}_r} = \frac{K_r^{min}(\cdot,w)}{1-Z_r\bar{w}_r} \qquad \forall\ w \in \mathbb{D}^2.$$

Then $L_1 = K_1^{min}$ and $L_2 = K_2^{min}$, and since (L_1, L_2) were arbitrary, ϕ has a unique Agler decomposition. $\square$

We also observe that certain functions have extremely non-unique Agler decompositions. Recall that a function ϕ is an *extreme point* of $\mathcal{S}(\mathbb{D}^2)$ if and only if there is *no* $f \in \mathcal{S}(\mathbb{D}^2)$ such that $\phi \pm f \in \mathcal{S}(\mathbb{D}^2)$.

Theorem 2.2.13. *If $\phi \in \mathcal{S}(\mathbb{D}^2)$ is not an extreme point of $\mathcal{S}(\mathbb{D}^2)$, then ϕ does not have a unique Agler decomposition.*

Proof. Assume ϕ is not extreme. Then, there is some $f \in \mathcal{S}(\mathbb{D}^2)$ such that $\phi \pm f \in \mathcal{S}(\mathbb{D}^2)$ and so there are pairs of Agler kernels (K_1, K_2) and (L_1, L_2) satisfying

$$1-(\phi+f)(z)\overline{(\phi+f)(w)} = (1-z_1\bar{w}_1)K_2 + (1-z_2\bar{w}_2)K_1, \tag{2.2.11}$$

$$1-(\phi-f)(z)\overline{(\phi-f)(w)} = (1-z_1\bar{w}_1)L_2 + (1-z_2\bar{w}_2)L_1, \tag{2.2.12}$$

where L_r and K_r are functions of $z, w \in \mathbb{D}^2$. Adding (2.2.11) and (2.2.12) and dividing the resultant equation by 2 yields

$$1 - \phi(z)\overline{\phi(w)} - f(z)\overline{f(w)} = (1 - z_1\bar{w}_1)\frac{K_2+L_2}{2} + (1 - z_2\bar{w}_2)\frac{K_1+L_1}{2},$$

which implies

$$\begin{aligned}1 - \phi(z)\overline{\phi(w)} =& (1 - z_1\bar{w}_1)\left(\frac{K_2 + L_2}{2} + t\frac{f(z)\overline{f(w)}}{1 - z_1\bar{w}_1}\right) \\ &+ (1 - z_2\bar{w}_2)\left(\frac{K_1 + L_1}{2} + (1-t)\frac{f(z)\overline{f(w)}}{1 - z_2\bar{w}_2}\right),\end{aligned}$$

for any $t \in [0, 1]$. Hence, ϕ has infinitely many pairs of Agler kernels. $\square$

2.2.4 Construction of Polynomial Agler Decompositions

In this subsection, we give an algebraic algorithm for constructing Agler decompositions for a special class of polynomials.The algorithm is motivated by the arguments appearing in the proof of Theorem 2.2.13. Specifically, let

$$p(z) = \sum_{m=0}^{M}\sum_{n=0}^{N} a_{mn} z_1^m z_2^n$$

be any polynomial such that $p \in \mathcal{S}(\mathbb{D}^2)$ and

$$\|p\|_\infty = \sum_{m=0}^{M}\sum_{n=0}^{N} |a_{mn}|. \tag{2.2.13}$$

We will describe how to construct Agler kernels of such polynomials. We first reduce the problem to a simpler situation:

Remark 2.2.14. A Simple Reduction. Let $p \in \mathcal{S}(\mathbb{D}^2)$ satisfy (2.2.13). Then by the maximum modulus principle, there is some $\tau = (\tau_1, \tau_2) \in \mathbb{T}^2$ such that

$$|p(\tau)| = \left| \sum_{m=0}^{M} \sum_{n=0}^{N} a_{mn} \tau_1^m \tau_2^n \right| = \sum_{m=0}^{M} \sum_{n=0}^{N} |a_{mn}|,$$

which implies that there is some $\mu \in \mathbb{T}$ such that

$$p(\tau) = \mu \sum_{m=0}^{M} \sum_{n=0}^{N} |a_{mn}|.$$

Define $q(z) := \frac{1}{\mu} p(\tau z)$, and write

$$q(z) = \sum_{m=0}^{M} \sum_{n=0}^{N} b_{mn} z_1^m z_2^n.$$

Working through the definitions makes it clear that

$$q(1,1) = \sum_{m=0}^{M} \sum_{n=0}^{N} b_{mn} = \sum_{m=0}^{M} \sum_{n=0}^{N} |a_{mn}| = \sum_{m=0}^{M} \sum_{n=0}^{N} |b_{mn}|.$$

Thus, each b_{mn} is real and nonnegative. Now assume that (K_1, K_2) are Agler kernels of q. Then, since

$$1 - p(z)\overline{p(w)} = 1 - q(\tfrac{z}{\tau})\overline{q(\tfrac{w}{\tau})} = (1 - z_1 \bar{w}_1) K_2(\tfrac{z}{\tau}, \tfrac{w}{\tau}) + (1 - z_2 \bar{w}_2) K_1(\tfrac{z}{\tau}, \tfrac{w}{\tau}),$$

the kernels $(K_1(\frac{z}{\tau}, \frac{w}{\tau}), K_2(\frac{z}{\tau}, \frac{w}{\tau}))$ are Agler kernels of p. Thus, when constructing Agler kernels of such polynomials, we can assume the polynomial's coefficients are real and nonnegative.

Before considering the general algorithm for constructing Agler kernels, we address the cases where the polynomial has only one or two terms. We omit the proofs of the following lemmas because they are simple algebraic calculations.

Lemma 2.2.15. ***Monomial Case.*** *Let $p(z) = az_1^n z_2^m \in \mathcal{S}(\mathbb{D}^2)$ with $a \geq 0$. Define*

$$K_2(z,w) := a^2 \sum_{k=0}^{m-1} z_1^k \bar{w}_1^k + \frac{1-a^2}{1-z_1\bar{w}_1} \quad \text{and} \quad K_1(z,w) := a^2 z_1^m \bar{w}_1^m \sum_{k=0}^{n-1} z_2^k \bar{w}_2^k.$$

Then, (K_1, K_2) are Agler kernels of p.

Lemma 2.2.16. ***Binomial Case.*** *Let $p(z) = az_1^j z_2^l + bz_1^m z_2^n \in \mathcal{S}(\mathbb{D}^2)$ with $a, b \geq 0$. Define*

$$K_2(z,w) := \frac{ab(z_1^j z_2^l - z_1^m z_2^n)\overline{(w_1^j w_2^l - w_1^m w_2^n)}}{1-z_1\bar{w}_1} + (a^2+ab)\sum_{k=0}^{j-1} z_1^k \bar{w}_1^k + (b^2+ab)\sum_{k=0}^{m-1} z_1^k \bar{w}_1^k$$

$$K_1(z,w) := (a^2+ab) z_1^j \bar{w}_1^j \sum_{k=0}^{l-1} z_2^k \bar{w}_2^k + (b^2+ab) z_1^m \bar{w}_1^m \sum_{k=0}^{n-1} z_2^k \bar{w}_2^k + \frac{1-a^2-b^2-2ab}{1-z_2\bar{w}_2}.$$

Then, (K_1, K_2) are Agler kernels of p.

Remark 2.2.17. Another Simple Reduction. Let $L \geq 3$ and let $p \in \mathcal{S}(\mathbb{D}^2)$ be a polynomial with L terms and with nonnegative, real coefficients. In this remark, we show how to construct Agler kernels of p using known Agler kernels of two polynomials q_1, q_2, where $q_1, q_2 \in \mathcal{S}(\mathbb{D}^2)$ are polynomials with L-1 terms and nonnegative, real coefficients.

To begin, write $p(z) = p_1(z) + p_2(z)$, where $p_1(z) = az_1^j z_2^l + bz_1^m z_2^n + cz_1^s z_2^t$ has precisely three terms, satisfies $c \geq a$ and $c \geq b$, and does not contain any terms of the same degree in each variable as p_2. Now define:

$$\begin{aligned}
q(z) &:= -az_1^j z_2^l + bz_1^m z_2^n + (a-b)z_1^s z_2^t \\
q_1(z) &:= p(z) + q(z) = 2bz_1^j z_2^n + (a-b+c)z_1^s z_2^t + p_2(z) \\
q_2(z) &:= p(z) - q(z) = 2az_1^m z_2^n + (b-a+c)z_1^s z_2^t + p_2(z).
\end{aligned}$$

Then $p \pm q \in \mathcal{S}(\mathbb{D}^2)$ and as in the proof of Theorem 2.2.13, it follows that

$$1 - p(z)\overline{p(w)} = \frac{1}{2}\Big(1 - q_1(z)\overline{q_1(w)}\Big) + \frac{1}{2}\Big(1 - q_2(z)\overline{q_2(w)}\Big) + q(z)\overline{q(w)}.$$

Then, if (M_1, M_2) and (L_1, L_2) are Agler kernels of q_1 and q_2 respectively then

$$K_2(z,w) := \frac{1}{2}\Big(M_2(z,w) + L_2(z,w)\Big) \quad \text{and} \quad K_1(z,w) := \frac{1}{2}\Big(M_1(z,w) + L_1(z,w)\Big) + \frac{q(z)\overline{q(w)}}{1 - z_2\bar{w}_2}$$

are Agler kernels of p.

The following result is immediate:

Theorem 2.2.18. *Let $p \in \mathcal{S}(\mathbb{D}^2)$ be a polynomial satisfying (2.2.13) with precisely L terms. If $L = 1$, one can obtain Agler kernels of ϕ by reducing to the case where p has a positive coefficient and applying Lemma 2.2.15. If $L \geq 2$, one can obtain Agler kernels of p using the following steps:*

1. *Using the arguments in Remark 2.2.14, reduce p to a polynomial p' with L terms and nonnegative, real coefficients.*

2. *Using the arguments in Remark 2.2.17 L-2 times, reduce the construction of Agler kernels of p' to the construction of Agler kernels of 2^{L-2} binomials $\{q_1, \dots, q_{2^{L-2}}\}$.*

3. *Using Lemma 2.2.16, obtain Agler kernels of $\{q_1, \dots, q_{2^{L-2}}\}$. Working backwards, use these to construct Agler kernels of p' and then p.*

To illustrate this method, let's consider the following simple example:

Example 2.2.19. Let $p(z) = \frac{1}{3} + \frac{1}{6}z_2^2 + \frac{1}{2}z_1^2 z_2$. It is clear that $p \in \mathcal{S}(\mathbb{D}^2)$ and satisfies (2.2.13). Since p already has positive coefficients, we can proceed to Step 2 of the algorithm. As p has only three terms, we will only use the reduction argument from Remark 2.2.17 once. Define

$$q(z) := -\tfrac{1}{3} + \tfrac{1}{6}z_2^2 + \tfrac{1}{6}z_1^2 z_2.$$

$$q_1(z) := p(z) + q(z) = \tfrac{1}{3}z_2^2 + \tfrac{2}{3}z_1^2 z_2.$$

$$q_2(z) := p(z) - q(z) = \tfrac{2}{3} + \tfrac{1}{3}z_1^2 z_2.$$

Using Lemma 2.2.16, we obtain the following Agler kernels (M_1, M_2) for q_1 :

$$\begin{aligned} M_2(z,w) &= \frac{\frac{2}{9}(z_2^2 - z_1^2 z_2)(\bar{w}_2^2 - \bar{w}_1^2 \bar{w}_2)}{1 - z_1 \bar{w}_1} + \tfrac{2}{3}\big(1 + z_1 \bar{w}_1\big) \\ M_1(z,w) &= \tfrac{1}{3}\big(1 + z_2 \bar{w}_2\big) + \tfrac{2}{3} z_1^2 \bar{w}_1^2. \end{aligned}$$

Similarly, we obtain the following Agler kernels (L_1, L_2) for q_2 :

$$\begin{aligned} L_2(z,w) &= \frac{\frac{2}{9}(1 - z_1^2 z_2)(1 - \bar{w}_1^2 \bar{w}_2)}{1 - z_1 \bar{w}_1} + \tfrac{1}{3}\big(1 + z_1 \bar{w}_1\big) \\ L_1(z,w) &= \tfrac{1}{3} z_1^2 \bar{w}_1^2. \end{aligned}$$

Then as in Remark 2.2.17,

$$1 - p(z)\overline{p(w)} = \frac{1}{2}\Big(1 - q_1(z)\overline{q_1(w)}\Big) + \frac{1}{2}\Big(1 - q_2(z)\overline{q_2(w)}\Big) + q(z)\overline{q(w)}.$$

This implies that we have the following Agler kernels of p:

$$\begin{aligned} K_2(z,w) &= \tfrac{1}{2}\Big(M_2(z,w) + L_2(z,w)\Big) \\ &= \frac{\frac{1}{9}(z_2^2 - z_1^2 z_2)(\bar{w}_2^2 - \bar{w}_1^2 \bar{w}_2)}{1 - z_1 \bar{w}_1} + \frac{\frac{1}{9}(1 - z_1^2 z_2)(1 - \bar{w}_1^2 \bar{w}_2)}{1 - z_1 \bar{w}_1} + \tfrac{1}{2}\big(1 + z_1 \bar{w}_1\big), \\ K_1(z,w) &= \tfrac{1}{2}\Big(M_1(z,w) + L_1(z,w)\Big) + \frac{q(z)\overline{q(w)}}{1 - z_2 \bar{w}_2} \\ &= \tfrac{1}{6}\big(1 + z_2 \bar{w}_2\big) + \tfrac{1}{2} z_1^2 \bar{w}_1^2 + \frac{(-\frac{1}{3} + \frac{1}{6} z_2^2 + \frac{1}{6} z_1^2 z_2)(-\frac{1}{3} + \frac{1}{6}\bar{w}_2^2 + \frac{1}{6}\bar{w}_1^2 \bar{w}_2)}{1 - z_2 \bar{w}_2}, \end{aligned}$$

as desired.

2.3 The Structure of Agler Spaces

In the previous section, we showed that for ϕ inner, the subspaces S_r^{max} and S_r^{min} of $\mathcal{H}_\phi$ yield simple Agler decompositions. In this section, we first introduce natural analogues of these spaces for general Schur functions. As before, we often denote kernels defined on the bidisk simply by K instead of by $K(z,w)$.

Definition 2.3.1. Let $\phi \in \mathcal{S}(\mathbb{D}^2)$, and let (K_1, K_2) denote a pair of Agler kernels of ϕ. Define the Hilbert spaces

$$S_1^K := \mathcal{H}\left(\frac{K_1}{1 - z_1\bar{w}_1}\right) \quad \text{and} \quad S_2^K := \mathcal{H}\left(\frac{K_2}{1 - z_2\bar{w}_2}\right).$$

We call S_1^K and S_2^K *Agler spaces of* ϕ*.* By definition, (K_1, K_2) satisfy

$$1 - \phi(z)\overline{\phi(w)} = (1 - z_1\bar{w}_1)K_2 + (1 - z_2\bar{w}_2)K_1, \tag{2.3.1}$$

which immediately implies

$$\frac{1 - \phi(z)\overline{\phi(w)}}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)} = \frac{K_1}{1 - z_1\bar{w}_1} + \frac{K_2}{1 - z_2\bar{w}_2}.$$

Arithmetic and an application of Theorem 2.2.8 can be used to show that S_1^K, S_2^K, $\mathcal{H}(K_1)$, and $\mathcal{H}(K_2)$ are all contractively contained in $\mathcal{H}_\phi$ and H^2. Moreover, it follows from Theorem 2.1.6 that each S_r is invariant under M_{Z_r}, and $\|M_{Z_r}\|_{S_r} \leq 1$ for $r = 1, 2$.

In this section, we use the S_r^{max} and S_r^{min} spaces to analyze the properties of Agler spaces. Specifically, in Subsection 2.3.1, we consider two properties of S_r^{max} and S_r^{min} and show that they hold for general Agler spaces as well. In Subsection 2.3.2, we use those two properties to characterize the Schur functions which have Agler decompositions arising from an orthogonal decomposition of $\mathcal{H}_\phi$.

2.3.1 Two Properties of Agler Spaces

By Remark 2.2.11, S_r^{max} and S_r^{min} are special cases of the S_r^K spaces. We will show that two properties of S_r^{max} and S_r^{min} extend to general Agler spaces. First, recall that X_r denotes the backward shift operator in the z_r coordinate for $r = 1, 2$. Specifically X_1 and X_2 are defined by

$$(X_1 g)(z) = \frac{g(z) - g(0, z_2)}{z_1} \quad \text{and} \quad (X_2 g)(z) = \frac{g(z) - g(z_1, 0)}{z_2}$$

for $g \in H^2$. We will use the following result of Alpay-Bolotnikov-Dijksma-Sadosky, which appears as Theorem 2.5 in [11]:

Theorem 2.3.2. *Let $\phi \in \mathcal{S}(\mathbb{D}^2)$. Then $\mathcal{H}_\phi$ is invariant under each X_r and*

$$\begin{aligned} \|X_1 f\|^2_{\mathcal{H}_\phi} &\leq \|f\|^2_{\mathcal{H}_\phi} - \|f(0, z_2)\|^2_{H^2} \\ \|X_2 f\|^2_{\mathcal{H}_\phi} &\leq \|f\|^2_{\mathcal{H}_\phi} - \|f(z_1, 0)\|^2_{H^2} \qquad \forall\ f \in \mathcal{H}_\phi. \end{aligned}$$

Observe the following fact:

Lemma 2.3.3. *Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ be inner. Then, S_1^{max} and S_1^{min} are invariant under X_2, and S_2^{max} and S_2^{min} are invariant under X_1.*

Proof. It follows from the arguments in Lemma 2.2.6 that

$$S_1^{max} = \ \big\{ f \in \mathcal{H}_\phi : Z_1^k f \in \mathcal{H}_\phi,\ \forall\ k \in \mathbb{N} \big\} \tag{2.3.2}$$

$$S_1^{min} = \ clos_{H^2} \Big\{ \sum_{m=1}^{M} \big(X_2^m \phi\big)(z) f_m(z_1) : M \in \mathbb{N},\ f_m \in H^2(\mathbb{D}) \Big\}, \tag{2.3.3}$$

where $clos_{H^2}$ indicates that we are taking the closure of the set in $H^2(\mathbb{D}^2)$. It follows from (2.3.2) and the X_2-invariance of $\mathcal{H}_\phi$ that S_1^{max} is invariant under X_2. In particular, if $f \in S_1^{max}$, then $X_2 f \in S_1^{max}$ because

$$z_1^k (X_2 f)(z) = \big(X_2 Z_1^k f\big)(z) \in \mathcal{H}_\phi \ \ \forall\ k \in \mathbb{N}.$$

It is clear from (2.3.3) and the fact that X_2 is a contraction on H^2 that S_1^{min} is invariant under X_2. The result follows for S_2^{max} and S_2^{min} by symmetry. □

We will show that the properties listed in Lemma 2.3.3 also hold for general Agler spaces. First, for $r = 1, 2$, let H_r^2 denote the space $H^2(\mathbb{D})$ with independent variable z_r. Specifically, we have

$$H_r^2 = \mathcal{H}\left(\frac{1}{1 - z_r \bar{w}_r}\right).$$

Proposition 2.3.4. *Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ and let (K_1, K_2) be Agler kernels of ϕ. Then S_1^K is invariant under X_2, and S_2^K is invariant under X_1. Moreover, for all $f \in S_2^K$ and $g \in S_1^K$,*

$$\|X_1 f\|_{S_2^K}^2 \leq \|f\|_{S_2^K}^2 - \|f(0, z_2)\|_{H^2}^2,$$
$$\|X_2 g\|_{S_1^K}^2 \leq \|g\|_{S_1^K}^2 - \|g(z_1, 0)\|_{H^2}^2.$$

Proof. Let (K_1, K_2) be a pair of Agler kernels of ϕ. Solving (2.3.1) for K_1 yields

$$K_1 = \frac{1 + z_1 \bar{w}_1 K_2}{1 - z_2 \bar{w}_2} - \frac{\phi(z)\overline{\phi(w)} + K_2}{1 - z_2 \bar{w}_2}. \tag{2.3.4}$$

Since the left-hand-side of (2.3.4) is a positive kernel, it follows from Theorem 2.2.8 that

$$\mathcal{H}\left(\frac{\phi(z)\overline{\phi(w)} + K_2}{1 - z_2 \bar{w}_2}\right) \subseteq \mathcal{H}\left(\frac{1 + z_1 \bar{w}_1 K_2}{1 - z_2 \bar{w}_2}\right), \tag{2.3.5}$$

and the embedding operator is a contraction. Now, consider the vector space of functions

$$Z_1 S_2^K := \{Z_1 f : f \in S_2^K\}.$$

We can define the following inner product on $Z_1 S_2^K$:

$$\langle Z_1 g_1, Z_1 g_2 \rangle_{Z_1 S_2^K} := \langle g_1, g_2 \rangle_{S_2^K},$$

for $Z_1g_1, Z_1g_2 \in Z_1S_2^K$. It is easy to show that $Z_1S_2^K$ is complete with respect to this inner product. Specifically, if $\{Z_1g_m\}$ is Cauchy in $Z_1S_2^K$, then $\{g_m\}$ is Cauchy in S_2^K and thus, converges to some $g \in S_2^K$. Then $Z_1g \in Z_1S_2^K$ and $\{Z_1g_m\}$ converges to Z_1g. Now, fix $w \in \mathbb{D}^2$. Then, $\frac{Z_1\bar{w}_1K_2(\cdot,w)}{1-Z_2\bar{w}_2} \in Z_1S_2^K$ and

$$\left\langle Z_1g, \frac{Z_1\bar{w}_1K_2(\cdot,w)}{1-Z_2\bar{w}_2}\right\rangle_{Z_1S_2^K} = \left\langle g, \frac{\bar{w}_1K_2(\cdot,w)}{1-Z_2\bar{w}_2}\right\rangle_{S_2^K} = w_1g(w),$$

for all $Z_1g \in Z_1S_2^K$. Thus, by definition, $Z_1S_2^K$ with this inner product is the following reproducing kernel Hilbert space:

$$\mathcal{H}\left(\frac{z_1\bar{w}_1K_2}{1-z_2\bar{w}_2}\right).$$

Now, let $f \in S_2^K$. By Theorem 2.2.2,

$$f \in \mathcal{H}\left(\frac{\phi(z)\overline{\phi(w)}+K_2}{1-z_2\bar{w}_2}\right).$$

Then, (2.3.5) paired with Theorem 2.2.2 guarantees that we can write

$$f(z) = f_1(z_2) + z_1f_2(z),$$

for $f_1 \in H_2^2$ and $f_2 \in S_2^K$. Now, observe that $f_1 \in H_2^2$ and $Z_1f_2 \in Z_1H^2$. Since $H_2^2 \perp Z_1H^2$ in H^2, there is a unique f_1 and Z_1f_2 from those two sets satisfying $f = f_1 + Z_1f_2$. In particular, we must have

$$f_1(z_2) = f(0,z_2) \quad \text{and} \quad f_2(z) = (X_1f)(z).$$

Thus, $X_1f \in S_2^K$ and so S_2^K is invariant under the backward shift X_1. Now, as the containment in (2.3.5) is contractive and the decomposition of f into f_1 and Z_1f_2 is unique, it follows from Theorem 2.2.2 that for $f \in S_2^K$,

$$\begin{aligned}
\|f\|^2_{S_2^K} &\geq \|f\|^2_{\mathcal{H}\left(\frac{\phi(z)\overline{\phi(w)}+K_2}{1-z_2\bar{w}_2}\right)} \\
&\geq \|f\|^2_{\mathcal{H}\left(\frac{1+z_1\bar{w}_1K_2}{1-z_2\bar{w}_2}\right)} \\
&= \|f_1\|^2_{H^2} + \|Z_1f_2\|^2_{Z_1S_2^K} \\
&= \|f_1\|^2_{H^2} + \|f_2\|^2_{S_2^K} \\
&= \|f(0,z_2)\|^2_{H^2} + \|X_1f\|^2_{S_2^K},
\end{aligned}$$

which establishes the norm inequality. Analogous arguments give the result for S_1^K. $\square$

Now, we establish another property of the S_r^{max} and S_r^{min} spaces. The remark below is a special case of part of Theorem 5.5 in Ball-Sadosky-Vinnikov in [16]. We include a simple proof.

Remark 2.3.5. Minimality of S_1^{min} and S_2^{min}. Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ be inner and assume there is an orthogonal decomposition $\mathcal{H}_\phi = S_1 \oplus S_2$, with $Z_rS_r \subseteq S_r$ for $r = 1, 2$. Then, $S_r^{min} \subseteq S_r$. To see this, let $f \in S_1^{min}$ and write $f = f_1 + f_2$ where $f_r \in S_r$. By the maximality of S_2^{max} established in Lemma 2.2.5, we have $f_2 \in S_2^{max}$, which implies $f \perp f_2$. By assumption, $f_1 \perp f_2$, so that

$$\|f_2\|^2_\phi = \langle f_2, f_1 + f_2\rangle_\phi = \langle f_2, f\rangle_\phi = 0.$$

Thus, $f = f_1 \in S_1$, which implies $S_1^{min} \subseteq S_1$. Similarly, $S_2^{min} \subseteq S_2$.

When ϕ is a general Schur function, there are similar minimal sets. But, before considering those sets, we need a bit of notation.

Definition 2.3.6. For $r = 1, 2$ and a holomorphic function ψ on $\mathbb{D}^2$, define the set

$$\psi H_r^2 := \{\psi g : g \in H_r^2\}.$$

In analogous way to the proof of Proposition 2.3.4, one can show that the set ψH_r^2 contains the same functions as the reproducing kernel Hilbert space given by:

$$\mathcal{H}\left(\frac{\psi(z)\overline{\psi(w)}}{1 - z_r\bar{w}_r}\right).$$

Now, we can state the following result:

Proposition 2.3.7. *Let $\phi \in \mathcal{S}(\mathbb{D}^2)$, and let (K_1, K_2) be Agler kernels of ϕ. Then the following set containments hold:*

$$(X_1\phi)H_2^2 \subseteq S_2^K \quad \text{and} \quad (X_2\phi)H_1^2 \subseteq S_1^K.$$

Proof. Recall from the proof of Proposition 2.3.4 that

$$\mathcal{H}\left(\frac{\phi(z)\overline{\phi(w)} + K_2}{1 - z_2\bar{w}_2}\right) \subseteq \mathcal{H}\left(\frac{1 + z_1\bar{w}_1 K_2}{1 - z_2\bar{w}_2}\right),$$

and the embedding operator is a contraction. Then by Theorem 2.2.2 and Definition 2.3.6, we have the following set relationships:

$$\phi H_2^2 = \mathcal{H}\left(\frac{\phi(z)\overline{\phi(w)}}{1 - z_2\bar{w}_2}\right) \subseteq \mathcal{H}\left(\frac{1 + z_1\bar{w}_1 K_2}{1 - z_2\bar{w}_2}\right).$$

Let $g \in H_2^2$, so that $f := \phi g \in \phi H_2^2$. As in the proof of Proposition 2.3.4, we can write

$$f(z) = f_1(z_2) + z_1 f_2(z),$$

for $f_1 \in H_2^2$ and $f_2 \in S_2^K$. As before, the properties of H^2 imply that f_1 must equal $f(0, z_2)$ and f_2 must equal $X_1 f$. Thus, $X_1 f \in S_2^K$. Since

$$\big(X_1 f\big)(z) = \big(X_1\phi\big)(z) g(z_2),$$

the inclusion $(X_1\phi)H_2^2 \subseteq S_2^K$ follows. Analogous arguments give the result for S_1^K. $\square$

Remark 2.3.8. The arguments in Propositions 2.3.4 and 2.3.7 generalize to the case where ϕ is in the Schur-Agler class of $\mathbb{D}^d$. Specifically given positive holomorphic kernels $(K_1, \ldots, K_d)$ such that

$$1 - \phi(z)\overline{\phi(w)} = (1 - z_1\bar{w}_1)K_1 + \cdots + (1 - z_d\bar{w}_d)K_d,$$

and $r \in \{1, \ldots, d\}$, and similar arguments can be used to show that

$$\text{(1)} \qquad \mathcal{H}\left(\frac{K_r(z,w)}{\prod_{j\neq r}(1 - z_j\bar{w}_j)}\right) \text{ is invariant under } X_r.$$

$$\text{(2)} \qquad X_r f \in \mathcal{H}\left(\frac{K_r(z,w)}{\prod_{j\neq r}(1 - z_j\bar{w}_j)}\right) \text{ for all } f \in \mathcal{H}\left(\frac{\phi(z)\overline{\phi(w)}}{\prod_{j\neq r}(1 - z_j\bar{w}_j)}\right).$$

These results look slightly different from Propositions 2.3.4 and 2.3.7 because on $\mathbb{D}^d$, it makes sense to number the kernels differently.

2.3.2 Agler Spaces via Orthogonal Decompositions

Recall that the Agler decompositions constructed in Section 2.2 for inner functions were obtained via an orthogonal decomposition

$$\mathcal{H}_\phi = S_1 \oplus S_2,$$

where $Z_rS_r \subseteq S_r$ and $\|M_{Z_r}\|_{S_r} \leq 1$ for $r = 1, 2$. It thus makes sense to ask:

"For which Schur functions ϕ does there exist such an orthogonal decomposition of $\mathcal{H}_\phi$?"

Such orthogonal decompositions will yield Agler decompositions as in the proof of Theorem 2.2.9. The previous propositions allow us to characterize the Schur functions with such decompositions. We first use those propositions to generalize the minimal sets S_r^{min} as follows:

Definition 2.3.9. General Minimal Sets. Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ and define

$$V_1 := \Big\{ \sum_{m=1}^{M} (X_2^m \phi)(z) f_m(z_1) : M \in \mathbb{N},\ f_m \in H^2(\mathbb{D}) \Big\}, \tag{2.3.6}$$

$$V_2 := \Big\{ \sum_{m=1}^{M} (X_1^m \phi)(z) f_m(z_2) : \ M \in \mathbb{N},\ f_m \in H^2(\mathbb{D}) \Big\}, \tag{2.3.7}$$

and define the closed subspaces $S_1^{min} := clos_{\mathcal{H}_\phi} V_1$ and $S_2^{min} := clos_{\mathcal{H}_\phi} V_2$. It follows from the proof of Lemma 2.2.6 that for ϕ inner, this definition of S_r^{min} agrees with the one given in Section 2.2. Then, Propositions 2.3.4 and 2.3.7 imply that each $V_r \subseteq S_r^K$ for any Agler spaces (S_1^K, S_2^K) of ϕ.

Now, we can characterize the Schur functions with the desired orthogonal decompositions of $\mathcal{H}_\phi$ as follows:

Theorem 2.3.10. *Let $\phi \in \mathcal{S}(\mathbb{D}^2)$. Then $\mathcal{H}_\phi$ has an orthogonal decomposition*

$$\mathcal{H}_\phi = S_1 \oplus S_2,$$

into closed subspaces S_1 and S_2 such that $Z_r S_r \subseteq S_r$ and $\|M_{Z_r}\|_{S_r} \le 1$ for $r = 1, 2$ if and only if $S_1^{min} \perp S_2^{min}$ in $\mathcal{H}_\phi$.

Proof. ($\Rightarrow$) Assume such an orthogonal decomposition of $\mathcal{H}_\phi$ exists. Then by arguments identical to those in the proof of Theorem 2.2.9, there are positive holomorphic kernels K_1 and K_2 such that

$$S_r = \mathcal{H}\left(\frac{K_r(z,w)}{1 - z_r \bar{w}_r} \right),$$

for $r = 1, 2$. Now since $\mathcal{H}_\phi = S_1 \oplus S_2$, we have

$$\frac{1 - \phi(z)\overline{\phi(w)}}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)} = \frac{K_1(z,w)}{1 - z_1\bar{w}_1} + \frac{K_2(z,w)}{1 - z_2\bar{w}_2}.$$

Thus, (K_1, K_2) are Agler kernels of ϕ, and Propositions 2.3.4 and 2.3.7 imply that each

$V_r \subseteq S_r$. As each S_r is closed in $\mathcal{H}_\phi$, it is clear that $S_r^{min} \subseteq S_r$. Since $S_1 \perp S_2$ in $\mathcal{H}_\phi$, we obtain $S_1^{min} \perp S_2^{min}$ in $\mathcal{H}_\phi$.

($\Leftarrow$) Assume $S_1^{min} \perp S_2^{min}$. Define $S_2^{max} := \mathcal{H}_\phi \ominus S_1^{min}$. We will show $S_1^{min} \oplus S_2^{max}$ gives the desired orthogonal decomposition of $\mathcal{H}_\phi$. First, for a fixed $w \in \mathbb{D}^2$, write the kernel $K_\phi(z, w)$ from (2.1.9) as $K_{\phi,w}(z)$. Then $K_{\phi,w}(z) \in \mathcal{H}_\phi$ and applying the backward shift X_1 to $K_{\phi,w}$ yields:

$$(X_1 K_{\phi,w})(z) = \bar{w}_1 K_{\phi,w}(z) - \overline{\phi(w)} \frac{(X_1\phi)(z)}{1 - z_2\bar{w}_2}.$$

Now we can calculate the adjoint of X_1 in $\mathcal{H}_\phi$, which we denote by X_1^*. Let $f \in \mathcal{H}_\phi$ and $w \in \mathbb{D}^2$. Then

$$\begin{aligned}
(X_1^* f)(w) &= \langle X_1^* f, K_{\phi,w}\rangle_{\mathcal{H}_\phi} \\
&= \langle f, X_1 K_{\phi,w}\rangle_{\mathcal{H}_\phi} \\
&= \langle f, \bar{w}_1 K_{\phi,w} - \overline{\phi(w)} \tfrac{X_1\phi}{1-Z_2\bar{w}_2}\rangle_{\mathcal{H}_\phi} \\
&= w_1 f(w) - \langle f, \tfrac{X_1\phi}{1-Z_2\bar{w}_2}\rangle_{\mathcal{H}_\phi} \phi(w).
\end{aligned}$$

Similarly, we have

$$(X_2^* f)(w) = w_2 f(w) - \langle f, \tfrac{X_2\phi}{1-Z_1\bar{w}_1}\rangle_{\mathcal{H}_\phi} \phi(w).$$

Observe that

$$\frac{X_2\phi}{1 - Z_1\bar{w}_1} \in S_1^{min} \text{ and } \frac{X_1\phi}{1 - Z_2\bar{w}_2} \in S_2^{min},$$

for each $w \in \mathbb{D}^2$. Then, for $f \in S_1^{min}$ and $g \in S_2^{max}$, the orthogonality assumptions imply that

$$(X_1^* f)(z) = z_1 f(z), \tag{2.3.8}$$

$$(X_2^* g)(z) = z_2 g(z). \tag{2.3.9}$$

Now, we will show that the desired properties hold for S_1^{min}. First let $f \in V_1$. Then $Z_1 f \in V_1$. As S_1^{min} is a closed subspace of $\mathcal{H}_\phi$, we can use (2.3.8) and Theorem 2.3.2 to calculate

$$\begin{aligned}
\|Z_1 f\|_{S_1^{min}} &= \|Z_1 f\|_{\mathcal{H}_\phi} \\
&= \|X_1^* f\|_{\mathcal{H}_\phi} \\
&\leq \|X_1\|_{\mathcal{H}_\phi} \|f\|_{\mathcal{H}_\phi} \\
&\leq \|f\|_{S_1^{min}}.
\end{aligned}$$

Now, let $f \in S_1^{min}$. Then there is a sequence $\{f_n\} \subseteq V$ that converges to f in $\mathcal{H}_\phi$. Then, as $\{Z_1 f_n\}$ satisfies

$$\|Z_1 f_n - Z_1 f_m\|_{S_1^{min}} \leq \|f_n - f_m\|_{S_1^{min}},$$

for $m, n \in \mathbb{N}$, the sequence $\{Z_1 f_n\}$ is Cauchy in S_1^{min}. Thus, $\{Z_1 f_n\}$ converges in S_1^{min} and in H^2, since S_1^{min} is contained contractively in H^2. As the limit in H^2 must be $Z_1 f$, the sequence converges to $Z_1 f$ in S_1^{min} as well, and

$$\|Z_1 f\|_{S_1^{min}} = \lim_{n\to\infty} \|Z_1 f_n\|_{S_1^{min}} \leq \lim_{n\to\infty} \|f_n\|_{S_1^{min}} = \|f\|_{S_1^{min}}.$$

Thus, $Z_1 S_1^{min} \subseteq S_1^{min}$, and $\|M_{Z_1}\|_{S_1^{min}} \leq 1$.

Now, consider S_2^{max}. Let $g \in S_2^{max}$. By the formula for X_2^*, we know $Z_2 g = X_2^* g \in \mathcal{H}_\phi$. Let

$$f(z) = \sum_{m=1}^{M} (X_2^m \phi)(z) f_m(z_1)$$

be an arbitrary element in V_1. It is clear that $X_2 f \in V_1 \subseteq S_1^{min}$ as well. Then, as $g \perp S_1^{min}$ in $\mathcal{H}_\phi$, we can calculate

$$\begin{aligned}\langle Z_2 g, f\rangle_{\mathcal{H}_\phi} &= \langle X_2^* g, f\rangle_{\mathcal{H}_\phi}\\ &= \langle g, X_2 f\rangle_{\mathcal{H}_\phi}\\ &= 0.\end{aligned}$$

As f was arbitrary, $Z_2 g \perp V_1$. Since V_1 is dense in S_1^{min}, it follows that $Z_2 g \perp S_1^{min}$, and so $Z_2 g \in S_2^{max}$. Thus, S_2^{max} is invariant under M_{Z_2}, and for $g \in S_2^{max}$, we have

$$\begin{aligned}\|Z_2 g\|_{S_2^{max}} &= \|X_2^* g\|_{\mathcal{H}_\phi}\\ &\leq \|X_2\|_{\mathcal{H}_\phi}\|g\|_{\mathcal{H}_\phi}\\ &\leq \|g\|_{S_2^{max}}.\end{aligned}$$

Thus, $\|M_{Z_2}\|_{S_2^{max}} \leq 1$ and as $Z_2 S_2^{max} \subseteq S_2^{max}$, the theorem is proved. □

We will provide several examples to illustrate both the uses and limitations of Theorem 2.3.10, but first we need an alternate definition of $\mathcal{H}_\phi$. If $A : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator between two Hilbert spaces, let $\mathcal{M}(A)$ denote the range of A with inner product defined by

$$\langle Ax, Ay\rangle_{\mathcal{M}(A)} = \langle x, y\rangle_{\mathcal{H}_1},$$

for all $x, y \in \mathcal{H}_1$ orthogonal to the kernel of A. It is well-known and discussed at length in [55] that if $\phi \in \mathcal{S}(\mathbb{D})$, then

$$\mathcal{H}\left(\frac{1-\phi(z)\overline{\phi(w)}}{1-z\bar{w}}\right) = \mathcal{M}\big((1-T_\phi T_{\bar{\phi}})^{\frac{1}{2}}\big),$$

where $T_\phi := P_{H^2} M_\phi$ is the Toeplitz operator with symbol ϕ. The analysis generalizes immediately for $\phi \in \mathcal{S}(\mathbb{D}^2)$. In particular,

$$\mathcal{H}_\phi = \mathcal{M}\big((1 - T_\phi T_{\bar{\phi}})^{\frac{1}{2}}\big),$$

where $T_\phi := P_{H^2} M_\phi$ is the Toeplitz operator with symbol ϕ. Now define $\mathcal{H}_{\bar{\phi}}$ to be $\mathcal{M}\big((1 - T_{\bar{\phi}} T_\phi)^{\frac{1}{2}}\big)$, and observe that $\mathcal{H}_{\bar{\phi}}$ is trivial for ϕ inner. Moreover, it follows from (I-8) in [55] that $f \in \mathcal{H}_\phi$ if and only if $T_{\bar{\phi}} f \in \mathcal{H}_{\bar{\phi}}$, and for all $f, g \in \mathcal{H}_\phi$,

$$\langle f, g \rangle_{\mathcal{H}_\phi} = \langle f, g \rangle_{H^2} + \langle T_{\bar{\phi}} f, T_{\bar{\phi}} g \rangle_{\mathcal{H}_{\bar{\phi}}}.$$

If $\phi \in \mathcal{S}(\mathbb{D}^2)$ is inner, $T_{\bar{\phi}} f \equiv 0$ for each $f \in \mathcal{H}_\phi$.

Example 2.3.11. Let ϕ be inner and consider $\psi := t\phi$, where $0 < t < 1$. Then, the V_1 and V_2 spaces for ϕ and ψ are identical, so there is no confusion if we just refer to them as V_1 and V_2. Let $f_r \in V_r$ for $r = 1, 2$. As $T_{\bar{\phi}} f_r = 0$, we have $T_{\bar{\psi}} f_r = 0$ for each r. Since ϕ is inner, $V_1 \perp V_2$ in $\mathcal{H}_\phi$ and so

$$\langle f_1, f_2 \rangle_{H^2} = \langle f_1, f_2 \rangle_{\mathcal{H}_\phi} = 0,$$

which immediately implies

$$\langle f_1, f_2 \rangle_{\mathcal{H}_\psi} = \langle f_1, f_2 \rangle_{H^2} + \langle T_{\bar{\psi}} f_1, T_{\bar{\psi}} f_2 \rangle_{\mathcal{H}_{\bar{\psi}}} = 0.$$

Since $V_1 \perp V_2$ in $\mathcal{H}_\psi$, we get $S_1^{min} \perp S_2^{min}$ in $\mathcal{H}_\psi$. Theorem 2.3.10 then implies that there is an orthogonal decomposition of $\mathcal{H}_\psi$ yielding an Agler decomposition of ψ.

As demonstrated by the following function, not all examples arise from inner functions or one-variable functions.

Example 2.3.12. Consider $\phi(z) = \frac{1}{2}(z_1 + z_1 z_2)$. Then, we can calculate

$$V_1 = \{z_1 f(z_1) : f \in H^2(\mathbb{D})\}$$
$$V_2 = \{(1 + z_2) f(z_2) : f \in H^2(\mathbb{D})\}.$$

Moreover, for every $f_2 \in V_2$, we have

$$T_{\bar{\phi}} f_2 = P_{H^2}\big(\tfrac{1}{2}\bar{z}_1(1 + \bar{z}_2) f_2(z_2)\big) = 0.$$

As $V_1 \perp V_2$ in H^2, for any $f_1 \in V_1,\ f_2 \in V_2$, we have

$$\langle f_1, f_2 \rangle_{\mathcal{H}_\phi} = \langle f_1, f_2 \rangle_{H^2} + \langle T_{\bar{\phi}} f_1, T_{\bar{\phi}} f_2 \rangle_{\mathcal{H}_{\bar{\phi}}} = 0.$$

Thus, $V_1 \perp V_2$ in $\mathcal{H}_\phi$ and so $S_1^{min} \perp S_2^{min}$ in $\mathcal{H}_\phi$. This same argument holds for any ϕ such that $V_1 \perp V_2$ in H^2, and $T_{\bar{\phi}} V_r = \{0\}$ for $r = 1$ or $r = 2$.

It is also quite easy to find functions for which the assumptions of Theorem 2.3.10 fail.

Example 2.3.13. Set $\phi(z) = \frac{1}{2}(z_1 + z_2)$. Then, for $r = 1, 2$, the set V_r contains precisely the functions in H_r^2. As $1 \in V_1 \cap V_2$, we cannot have $V_1 \perp V_2$ in $\mathcal{H}_\phi$. Thus, there is no orthogonal decomposition of $\mathcal{H}_\phi$ that yields Agler kernels.

2.4 Agler Decompositions of Rational Inner Functions

In this section, we restrict attention to rational inner functions. We use the framework of the maximal and minimal subspaces S_r^{max} and S_r^{min} to simplify the known theory of Agler kernels for rational inner functions. In particular, we use the subspaces to obtain simple proofs for several known, important results.

In Subsection 2.4.1, we let (K_1, K_2) be Agler kernels of ϕ and provide a new proof showing that $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ have finite dimensions with bounds dependent on $\deg \phi$. In Subsection 2.4.2, we consider the uniqueness of Agler decompositions for rational inner functions. In particular, we provide a new proof of the fact that rational inner ϕ continuous on $\overline{\mathbb{D}^2}$ have unique Agler kernels if and only if they are functions of one variable. We also obtain several new results in Propositions 2.4.6 and 2.4.8.

Before beginning, let us review the structure of rational inner functions on the bidisk.

Definition 2.4.1. A set $X \subseteq \mathbb{C}^d$ is called *determining* for an algebraic set $A \subseteq \mathbb{C}^d$ if $f \equiv 0$ whenever f is holomorphic on A and $f|_{X \cap A} = 0$. A d-variable polynomial p is called *atoral* if $\mathbb{T}^d$ is not determining for any of the irreducible components of the zero set of p.

For more information about determining sets and atoral polynomials see [7]. Now, we establish notation and characterize the rational inner functions on $\mathbb{D}^2$.

Definition 2.4.2. Let p be a polynomial on $\mathbb{C}^2$. Assume the degree of p in the z_r variable is j_r for $r = 1, 2$. Then we write $\deg p = (j_1, j_2)$ and $\deg_r p = j_r$ for $r = 1, 2$. We also define the polynomial's reflection $\tilde{p}$ as

$$\tilde{p}(z) := z_1^{j_1} z_2^{j_2} \overline{p\left(\tfrac{1}{\bar{z}}\right)}.$$

Remark 2.4.3. Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ be rational inner. By the atoral-toral factorization of Agler-McCarthy-Stankus in [7], there are functions m and p, which are unique up to multiplication by unimodular constants, such that

$$\phi(z) = m(z)\frac{\tilde{p}(z)}{p(z)}, \tag{2.4.1}$$

where m is a monomial and p is an atoral polynomial with no zeros in $\mathbb{D}^2$ and finitely many zeros on $\mathbb{T}^2$. Then, $\deg \phi = (k_1, k_2)$, where $k_r = \deg_r m + \deg_r p$ for $r = 1, 2$. Also, every function of the form (2.4.1) is rational inner.

2.4.1 Dimension Bounds for Associated Hilbert Spaces

In this subsection, we provide a simple proof of a known result about the dimensions of $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ when ϕ is rational inner. The finiteness result was proved by Cole-Wermer as Corollary 2.2 in [24], the specific dimension bounds were shown by Knese in Theorem 2.10 of [41]. In [16], Ball-Sadosky-Vinnikov gave an alternate proof of the Cole-Wermer result for a subset of the Agler kernels of ϕ. We use the S_r^{max} subspaces to provide a very simple proof of the Cole-Wermer result, which is distinct from the arguments in [16].

Recall that S_r^{max} can be viewed equivalently as a space of holomorphic functions on $\mathbb{D}^2$ contained in H^2 and a space of L^2 functions contained in (2.2.1). Then the following result about S_r^{max} for $r = 1, 2$ can be viewed as both a statement about the analytic functions and a statement about their radial boundary value functions.

Lemma 2.4.4. *Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ be rational inner with representation* (2.4.1). *Then*

$$S_1^{max} \subseteq \left\{ \tfrac{f}{p} \in H^2 : f \in H^2 \textit{ and } \hat{f}(n_1, n_2) = 0 \textit{ for } n_2 \geq k_2 \right\},$$
$$S_2^{max} \subseteq \left\{ \tfrac{f}{p} \in H^2 : f \in H^2 \textit{ and } \hat{f}(n_1, n_2) = 0 \textit{ for } n_1 \geq k_1 \right\}.$$

Proof. Let $g \in S_1^{max}$. By Lemma 2.2.6, there is an $h \in L^2_{*-}$ such that $g = \phi h = \frac{m\tilde{p}}{p}h$, by representation (2.4.1). Then

$$m\tilde{p}h = pg \in H^2.$$

Since $h \in L^2_{*-}$ and $\deg_2(m\tilde{p}) = k_2$, if we set $f := m\tilde{p}h$, it follows immediately from the definition of Fourier coefficients that $\hat{f}(n_1, n_2) = 0$ whenever $n_2 \geq k_2$. The result follows similarly for S_2^{max}. $\square$

Now we provide a simple proof of the Cole-Wermer result. In the following proof, we index Taylor coefficients by m, n instead of n_1, n_2 to simplify notation.

Theorem 2.4.5. *Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ be rational inner with representation* (2.4.1), *and let (K_1, K_2) be Agler kernels of ϕ. Then,*

$$dim(\mathcal{H}(K_1)) \leq k_2(k_1+1) \text{ and } dim(\mathcal{H}(K_2)) \leq k_1(k_2+1).$$

Setting $m_1 := dim(\mathcal{H}(K_1))$ and $m_2 := dim(\mathcal{H}(K_2))$, we can write

$$K_1(z,w) = \frac{1}{p(z)\overline{p(w)}} \sum_{i=1}^{m_1} q_i(z)\overline{q_i(w)} \text{ and } K_2(z,w) = \frac{1}{p(z)\overline{p(w)}} \sum_{j=1}^{m_2} r_j(z)\overline{r_j(w)},$$

for polynomials $\{q_i\}$ with $\deg q_i \leq (k_1, k_2 - 1)$ for $1 \leq i \leq m_1$, and polynomials $\{r_j\}$ with $\deg r_j \leq (k_1 - 1, k_2)$ for $1 \leq j \leq m_2$.

Proof. Let ϕ be rational inner, and let (K_1, K_2) be Agler kernels of ϕ. Fix $w \in \mathbb{D}^2$. Then for $r = 1, 2$, the function $K_r(\cdot, w) \in \mathcal{H}(K_r)$. Moreover, since

$$\frac{K_r(z,w)}{1 - z_r\bar{w}_r} - K_r(z,w) = \frac{z_r\bar{w}_r K_r(z,w)}{1 - z_r\bar{w}_r}$$

is a positive kernel, Theorem 2.2.8 implies $K_r(\cdot, w) \in S_r^K$. By the maximality of S_r^{max}, it then follows that $K_r(\cdot, w) \in S_r^{max}$. By Lemma 2.4.4, we can write

$$K_1(z,w) = \frac{1}{p(z)} \sum_{\substack{m \geq 0 \\ 0 \leq n < k_2}} a_{mn}(w) z_1^m z_2^n, \tag{2.4.2}$$

$$K_2(z,w) = \frac{1}{p(z)} \sum_{\substack{0 \leq m < k_1 \\ n \geq 0}} b_{mn}(w) z_1^m z_2^n, \tag{2.4.3}$$

for $z \in \mathbb{D}^2$ and coefficients $a_{mn}(w)$ and $b_{mn}(w)$ in $l^2(\mathbb{N}^2)$. Now, substituting (2.4.1), (2.4.2), and (2.4.3) into

$$1 - \phi(z)\overline{\phi(w)} = (1 - z_1\bar{w}_1)K_2(z,w) + (1 - z_2\bar{w}_2)K_1(z,w),$$

and canceling the denominator $p(z)$ yields:

$$\begin{aligned} &p(z) - \overline{\phi(w)}(m\tilde{p})(z) \\ &\quad = (1 - z_1\bar{w}_1) \sum_{\substack{0 \leq m < k_1 \\ n \geq 0}} b_{mn}(w) z_1^m z_2^n + (1 - z_2\bar{w}_2) \sum_{\substack{m \geq 0 \\ 0 \leq n < k_2}} a_{mn}(w) z_1^m z_2^n. \end{aligned}$$

Algebraic manipulation implies that

$$\sum_{\substack{0 \leq m < k_1 \\ n \geq 0}} b_{mn}(w) z_1^m z_2^n = \frac{-1}{(1 - z_1\bar{w}_1)} \left((1 - z_2\bar{w}_2) \sum_{\substack{m \geq 0 \\ 0 \leq n < k_2}} a_{mn}(w) z_1^m z_2^n - p(z) + \overline{\phi(w)}(m\tilde{p})(z) \right).$$

Since the right-hand-side of the above equation has no term with a power of z_2 larger than k_2, we can conclude:

$$K_2(z,w) = \frac{1}{p(z)} \sum_{\substack{0 \leq m < k_1 \\ 0 \leq n \leq k_2}} b_{mn}(w) z_1^m z_2^n.$$

Similar arguments imply

$$K_1(z,w) = \frac{1}{p(z)} \sum_{\substack{0\le m\le k_1 \\ 0\le n<k_2}} a_{mn}(w) z_1^m z_2^n.$$

Recall that the linear span of the set of functions $\{K_1(\cdot,w)\}_{w\in\mathbb{D}^2}$ is dense in $\mathcal{H}(K_1)$. Fix $g \in \mathcal{H}(K_1)$, and let $\{f_n/p\}$ be a sequence with elements in the linear span of $\{K_1(\cdot,w)\}_{w\in\mathbb{D}^2}$ that converges to g. Then for each n, $\deg f_n \le (k_1, k_2-1)$. As $\mathcal{H}(K_1)$ is contractively contained in H^2, we know $\{f_n/p\}$ also converges to g in H^2. Since

$$\|f_n - pg\|_{H^2} \le \|p\|_\infty \|f_n/p - g\|_{H^2},$$

$\{f_n\}$ converges to gp in H^2. Then, $\deg gp \le (k_1, k_2-1)$. If we set $f = gp$, then $g = f/p$, and it follows that

$$\mathcal{H}(K_1) \subseteq \left\{ \tfrac{f}{p} : f(z) = \sum_{\substack{0\le m\le k_1 \\ 0\le n<k_2}} c_{mn} z_1^m z_2^n \right\}, \text{ and } \dim(\mathcal{H}(K_1)) \le k_2(k_1+1).$$

Let $m_1 = \dim(\mathcal{H}(K_1))$, and let $\{f_i\}_{i=1}^{m_1}$ be an orthonormal basis for $\mathcal{H}(K_1)$. For each i, we have $f_i = \frac{q_i}{p}$, where $\deg q_i \le (k_1, k_2-1)$. By Theorem 2.1.4,

$$K_1(z,w) = \frac{1}{p(z)\overline{p(w)}} \sum_{i=1}^{m_1} q_i(z)\overline{q_i(w)}.$$

An analogous argument gives the result for $\mathcal{H}(K_2)$. □

Given a rational inner ϕ with $\deg \phi = (k_1, k_2)$, one can actually choose (K_1, K_2) so that $\dim(\mathcal{H}(K_1)) = k_2$ and $\dim(\mathcal{H}(K_2)) = k_1$. Such decompositions are discussed by Kummert in [45] and Knese in [40].

2.4.2 Uniqueness of Agler Decompositions

In this subsection, we examine when rational inner functions have unique pairs of Agler kernels. We first restrict attention to rational inner functions continuous on $\overline{\mathbb{D}^2}$. We will need the following results about S_1^{max} and S_2^{max}, which are proven by Ball-Sadosky-Vinnikov in Proposition 6.9 of [16]. Here, we also consider a related result for $\mathcal{H}_\phi$, which simplifies the proofs for S_1^{max} and S_2^{max}.

Proposition 2.4.6. *Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ be rational inner and continuous on $\overline{\mathbb{D}^2}$ with representation* (2.4.1). *Then*

$$\begin{aligned}
\mathcal{H}_\phi &= \left\{ \tfrac{f}{p} : f \in H^2 \text{ and } \hat{f}(n_1, n_2) = 0 \text{ if } n_1 \geq k_1 \text{ and } n_2 \geq k_2 \right\} \\
S_1^{max} &= \left\{ \tfrac{f}{p} : f \in H^2 \text{ and } \hat{f}(n_1, n_2) = 0 \text{ if } n_2 \geq k_2 \right\} \\
S_2^{max} &= \left\{ \tfrac{f}{p} : f \in H^2 \text{ and } \hat{f}(n_1, n_2) = 0 \text{ if } n_1 \geq k_1 \right\}.
\end{aligned}$$

Proof. Because ϕ is continuous on $\overline{\mathbb{D}^2}$, the polynomial p from representation (2.4.1) has no zeros on $\overline{\mathbb{D}^2}$. It follows that $p, \frac{1}{p} \in H^\infty(\mathbb{D}^2)$ and so, $\frac{1}{p}H^2 = H^2$. Now, set

$$q(z) := \overline{p\left(\tfrac{1}{\bar{z}}\right)}.$$

By the related properties of p, it is clear that $q, \frac{1}{q} \in L^\infty(\mathbb{T}^2)$ and so, these functions multiply L^2 into L^2. Let $f \in H^2$ and $g \in L^2 \ominus H^2$. As $q \equiv \bar{p}$ on $\mathbb{T}^2$, we have

$$\langle qg, f \rangle_{L^2} = \langle g, pf \rangle_{L^2} = 0,$$

$$\langle \tfrac{1}{q} g, f \rangle_{L^2} = \langle g, \tfrac{1}{p} f \rangle_{L^2} = 0.$$

Then, it is immediate that

$$q[L^2 \ominus H^2] \subseteq L^2 \ominus H^2 \text{ and } \tfrac{1}{q}[L^2 \ominus H^2] \subseteq L^2 \ominus H^2.$$

Thus, $q[L^2 \ominus H^2] = L^2 \ominus H^2$. By the characterization of $\mathcal{H}_\phi$ in Remark 2.2.3, we have

$$\begin{aligned}
\mathcal{H}_\phi &= \phi[L^2 \ominus H^2] \cap H^2 \\
&= \Big[\tfrac{m\tilde{p}}{p}[L^2 \ominus H^2] \cap \tfrac{1}{p}H^2\Big] \\
&= \tfrac{1}{p}\Big[m\tilde{p}[L^2 \ominus^2] \cap H^2\Big] \\
&= \tfrac{1}{p}\Big[Z_1^{k_1}Z_2^{k_2}q[L^2 \ominus H^2] \cap H^2\Big] \\
&= \tfrac{1}{p}\Big[Z_1^{k_1}Z_2^{k_2}[L^2 \ominus H^2] \cap H^2\Big] \\
&= \Big\{\tfrac{f}{p} : f \in H^2 \text{ and } \hat{f}(n_1, n_2) = 0 \text{ if } n_1 \geq k_1 \text{ and } n_2 \geq k_2\Big\},
\end{aligned}$$

as desired. We now prove the result for S_1^{max}. Set

$$S_1 := \Big\{\tfrac{f}{p} : f \in H^2 \text{ and } \hat{f}(n_1, n_2) = 0 \text{ if } n_2 \geq k_2\Big\}.$$

From Lemma 2.4.4, we know $S_1^{max} \subseteq S_1$. Moreover, S_1 is invariant under M_{Z_1} and by the characterization of $\mathcal{H}_\phi$, we have $S_1 \subseteq \mathcal{H}_\phi$. By the maximality of S_1^{max}, we have $S_1 \subseteq S_1^{max}$ and so, the two sets are equal. The result follows similarly for S_2^{max}. □

Now we can characterize when such rational inner functions have unique Agler decompositions. The following corollary follows from Theorem 2.2.12 and was originally proved by Knese as Corollary 1.16 in [40].

Corollary 2.4.7. *Let $\phi \in \mathcal{S}(\mathbb{D}^2)$ be rational inner and continuous on $\overline{\mathbb{D}^2}$ with representation* (2.4.1). *Then ϕ has a unique Agler decomposition if and only if ϕ is a function of one variable.*

Proof. By Proposition 2.4.6,

$$S_1^{max} \cap S_2^{max} = \Big\{\tfrac{f}{p} : f \in H^2 \text{ and } \hat{f}(n_1, n_2) = 0 \text{ if } n_1 \geq k_1 \text{ or } n_2 \geq k_2\Big\}. \tag{2.4.4}$$

As $S_1^{max} \cap S_2^{max} = H^2 \cap \phi L^2_{--}$, it follows from Theorem 2.2.12 that ϕ has a unique Agler decomposition if and only if (2.4.4) $= \{0\}$, which occurs if and only if k_1 or k_2 is zero. $\square$

Corollary 2.4.7 does not hold for general rational inner functions. Rather, we can construct rational inner functions with arbitrarily high degree and unique Agler decompositions.

Proposition 2.4.8. *Let* $(k_1, k_2) \in \mathbb{N}^2$. *Then there exists a rational inner function* ϕ *such that* $\deg \phi = (k_1, k_2)$, *and* ϕ *has a unique Agler decomposition.*

Proof. Let $(k_1, k_2) \in \mathbb{N}^2$. By Theorem 2.2.12, an inner function ϕ has a unique Agler decomposition if and only if $H^2 \cap \phi L^2_{--} = \{0\}$. Let p be an atoral polynomial with $\deg p = (k_1, k_2)$ and with no zeros on $\mathbb{D}^2$. Then, $\phi = \frac{\tilde{p}}{p}$ is rational inner with $\deg \phi = (k_1, k_2)$. As $S_1^{max} \cap S_2^{max} = H^2 \cap \phi L^2_{--}$, we can use Lemma 2.4.4 to conclude that

$$H^2 \cap \phi L^2_{--} \subseteq \left\{ \tfrac{q}{p} \in H^2 : q \in H^2 \text{ and } \hat{q}(n_1, n_2) = 0 \text{ if } n_1 \geq k_1 \text{ or } n_2 \geq k_2 \right\}. \tag{2.4.5}$$

Let $\mathcal{L}$ denote the set on the right-hand-side of (2.4.5). We will construct a ϕ such that $\mathcal{L}$ is trivial. Let p be an atoral polynomial with $\deg p = (k_1, k_2)$ and with zeros at the following k_1^{th} and k_2^{th} roots of unity:

$$\left(e^{\frac{2\pi i k}{k_1}}, e^{\frac{2\pi i j}{k_2}} \right), \tag{2.4.6}$$

where $1 \leq k \leq k_1,\ 1 \leq j \leq k_2$. Using the power series representation of p centered at each root of unity, one can use basic estimates to show

$$\tfrac{1}{|p|^2} \text{ is not integrable near each } \left(e^{\frac{2\pi i k}{k_1}}, e^{\frac{2\pi i j}{k_2}} \right).$$

By symmetry, it actually suffices to consider the situation at the point $(1,1)$. First, write

$$p(z) = \sum_{\substack{m+n\geq 1\\ 0\leq m\leq k_1\\ 0\leq n\leq k_2}} c_{mn}(1-z_1)^m(1-z_2)^n,$$

for some constants c_{mn}. Then, near $(1,1)$, we have

$$\begin{aligned}
\left|p(e^{i\theta_1},e^{i\theta_2})\right|^2 &\leq c_1\Big(\left|1-e^{i\theta_1}\right|^2+\left|1-e^{i\theta_2}\right|^2\Big)\\
&= c_1\Big((1-\cos\theta_1)^2+\sin^2\theta_1+(1-\cos\theta_1)^2+\sin^2\theta_2\Big)\\
&\leq c_2\big(\theta_1^2+\theta_2^2\big),
\end{aligned}$$

for some positive real constants c_1 and c_2. Therefore, for some fixed $\epsilon>0$, there is a positive constant C such that

$$\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}\frac{1}{|p(e^{i\theta_1},e^{i\theta_2})|^2}d\theta_1 d\theta_2 \geq C\int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon}\frac{1}{\theta_1^2+\theta_2^2}d\theta_1 d\theta_2,$$

which diverges. Therefore, if there is a function q with $\frac{q}{p}\in H^2$, then q vanishes at each root of unity in (2.4.6). To be explicit, we will take $p(z)=3-z_1^{k_1}-z_2^{k_2}-z_1^{k_1}z_2^{k_2}$ and consider $\phi=\frac{\tilde{p}}{p}$. Observe that if $\frac{q}{p}\in\mathcal{L}$, then q is a polynomial with $\deg_r q<k_r$ for $r=1,2$. We can write

$$q(z)=\sum_{\substack{0\leq m<k_1\\ 0\leq n<k_2}} a_{mn}z_1^m z_2^n, \text{ where } q\Big(e^{\frac{2\pi ik}{k_1}},e^{\frac{2\pi ij}{k_2}}\Big)=0,$$

for all k,j with $1\leq k\leq k_1$ and $1\leq j\leq k_2$. We will show that such a q must be identically zero. For each k, where $1\leq k\leq k_1$, define

$$q_k(z_2):=q\Big(e^{\frac{2\pi ik}{k_1}},z_2\Big)=\sum_{0\leq n<k_2}\Big(\sum_{0\leq m<k_1}a_{mn}e^{\frac{2\pi ikm}{k_1}}\Big)z_2^n.$$

As $\deg q_k \le k_2 - 1$ and q_k has k_2 zeros, $q_k \equiv 0$. That implies

$$\sum_{0 \le m < k_1} a_{mn} e^{\frac{2\pi i k m}{k_1}} = 0, \tag{2.4.7}$$

for all k and n with $1 \le k \le k_1,\ 0 \le n \le k_2 - 1$. Fix n with $0 \le n \le k_2 - 1$. It follows from (2.4.7) that we have the following matrix equation:

$$\begin{bmatrix} 1 & e^{\frac{2\pi i}{k_1}} & \cdots & \left(e^{\frac{2\pi i}{k_1}}\right)^{k_1-1} \\ 1 & e^{\frac{4\pi i}{k_1}} & \cdots & \left(e^{\frac{4\pi i}{k_1}}\right)^{k_1-1} \\ \vdots & \vdots & & \vdots \\ 1 & e^{\frac{2k_1\pi i}{k_1}} & \cdots & \left(e^{\frac{2k_1\pi i}{k_1}}\right)^{k_1-1} \end{bmatrix} \cdot \begin{bmatrix} a_{0n} \\ a_{1n} \\ \vdots \\ \\ a_{(k_1-1)n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ 0 \end{bmatrix}.$$

Observe that the matrix is a Vandermonde matrix. It then has determinant given by

$$\prod_{1 \le s < t \le k_1} \left(e^{\frac{2\pi i s}{k_1}} - e^{\frac{2\pi i t}{k_1}}\right) \ne 0.$$

As the matrix is nonsingular, each $a_{mn} = 0$, and so $q \equiv 0$. Thus,

$$\phi(z) = \frac{\tilde{p}(z)}{p(z)} = \frac{3z_1^{k_1} z_2^{k_2} - z_1^{k_1} - z_2^{k_2} - 1}{3 - z_1^{k_1} - z_2^{k_2} - z_1^{k_1} z_2^{k_2}}$$

has a trivial $\mathcal{L}$ set. Thus, $H^2 \cap \phi L^2_{--} = \{0\}$, and ϕ has a unique Agler decomposition. $\square$

2.5 Application: Characterizing Stable Polynomials

In this section, we generalize some of the analysis from Section 2.4 to the polydisk $\mathbb{D}^d$ and use it to obtain a result about stable polynomials. First, let $d \geq 2$ and let ϕ be rational inner on $\mathbb{D}^d$ with $\deg \phi = (k_1, \dots, k_d)$. Again by the analysis of Agler-McCarthy-Stankus in [7], ϕ has an almost unique representation as

$$\phi(z) = m(z)\frac{\tilde{p}(z)}{p(z)}, \tag{2.5.1}$$

for a monomial m and an atoral polynomial p with no zeros on $\mathbb{D}^d$, such that $\deg_r \phi = \deg_r m + \deg_r p$ for each r. Moreover, any function of the form (2.5.1) is rational inner. We also define the reproducing kernel Hilbert space

$$\mathcal{H}_\phi := \mathcal{H}\left(\frac{1 - \phi(z)\overline{\phi(w)}}{\prod_{i=1}^d (1 - z_i \bar{w}_i)} \right).$$

For a fixed d, we define the notation $H^2 := H^2(\mathbb{D}^d)$ and $L^2 := L^2(\mathbb{T}^d)$. Then, as in the two-variable case, $\mathcal{H}_\phi = \phi(L^2 \ominus H^2) \cap H^2$. The arguments in Proposition 2.4.6 generalize immediately to yield the following result:

Proposition 2.5.1. *Let $\phi \in \mathcal{S}(\mathbb{D}^d)$ be rational inner and continuous on $\overline{\mathbb{D}^d}$ with $\deg \phi = (k_1, \dots, k_d)$ and representation (2.5.1). Then*

$$\mathcal{H}_\phi = \tfrac{1}{p}\Big[Z_1^{k_1} \cdots Z_d^{k_d} \big[L^2 \ominus H^2 \big] \cap H^2 \Big].$$

A polynomial p in d complex variables is called *stable* if p has no zeros on $\overline{\mathbb{D}^d}$. We can now generalize a result of Knese [38, Theorem 1.1] about stable polynomials in two complex variables to polynomials in d complex variables and simultaneously, provide a simple proof of the original result.

Theorem 2.5.2. *Let p be a non-constant polynomial in d complex variables. Then p is* *stable if and only if there is a constant $c > 0$ such that for all $z \in \mathbb{D}^d$,*

$$|p(z)|^d - |\tilde{p}(z)|^d \geq c \prod_{i=1}^{d} \big(1 - |z_i|^2\big). \tag{2.5.2}$$

Proof. ($\Rightarrow$) Assume p is a non-constant stable polynomial in d complex variables. Then, p is immediately atoral since p has no zeros on $\overline{\mathbb{D}^d}$. Thus, the function $\phi := \frac{\tilde{p}}{p}$ is inner and continuous on $\overline{\mathbb{D}^d}$. By Proposition 2.5.1,

$$\mathcal{H}_\phi = \tfrac{1}{p}\Big[Z_1^{k_1} \cdots Z_d^{k_d}\big[L^2 \ominus H^2\big] \cap H^2\Big].$$

It is immediate that $\frac{1}{p} \in \mathcal{H}_\phi$, and by Theorem 2.2.8, there is a constant $c_1 > 0$ such that

$$\frac{1 - \phi(z)\overline{\phi(w)}}{\prod_{i=1}^d (1 - z_i \bar{w}_i)} - \frac{c_1}{p(z)\overline{p(w)}} \tag{2.5.3}$$

is a positive kernel. Setting $w = z$ in (2.5.3) gives

$$\frac{1 - |\phi(z)|^2}{\prod_{i=1}^d (1 - |z_i|^2)} - \frac{c_1}{|p(z)|^2} \geq 0,$$

and rearranging terms yields

$$|p(z)|^2 - |\tilde{p}(z)|^2 \geq c_1 \prod_{i=1}^{d} \big(1 - |z_i|^2\big).$$

As p has no zeros on $\overline{\mathbb{D}^d}$ and since $p, \tilde{p}$ are clearly bounded on $\overline{\mathbb{D}^d}$, there is a constant $c_2 > 0$ such that

$$\begin{aligned}|p(z)| - |\tilde{p}(z)| &\geq c_1 \frac{1}{|p(z)| + |\tilde{p}(z)|} \prod_{i=1}^{d} \big(1 - |z_i|^2\big) \\ &\geq c_2 \prod_{i=1}^{d} \big(1 - |z_i|^2\big).\end{aligned}$$

Again, as p does not vanish on $\overline{\mathbb{D}^d}$, there is a constant $c_3 > 0$ such that

$$\begin{aligned}
|p(z)|^d - |\tilde{p}(z)|^d &= \big(|p(z)| - |\tilde{p}(z)|\big)\left(\sum_{j=1}^{d} |p(z)|^{j-1}|\tilde{p}(z)|^{d-j}\right) \\
&\geq \big(|p(z)| - |\tilde{p}(z)|\big)|p(z)|^{d-1} \\
&\geq c_3\big(|p(z)| - |\tilde{p}(z)|\big) \\
&\geq c\prod_{i=1}^{d}\big(1 - |z_i|^2\big),
\end{aligned}$$

where $c = c_2c_3 > 0$.

($\Leftarrow$) Assume p satisfies equation (2.5.2). Proceeding towards a contradiction, assume p has a zero on $\partial\mathbb{D}^d$. Since $\tilde{p}/p$ is bounded, $\tilde{p}$ must have a zero at the same point. Without loss of generality, we can assume the zero occurs at a point $(\tau_1, \dots, \tau_d) \in \mathbb{D}^{n_1} \times \mathbb{T}^{n_2}$, where $n_1 + n_2 = d$. Assume $n_2 < d$. As $p(r\tau_1, \dots, r\tau_d) = O(1-r)$ and $\tilde{p}(r\tau_1, \dots, r\tau_d) = O(1-r)$, it is immediate that

$$|p(r\tau_1, \dots, r\tau_d)|^d - |\tilde{p}(r\tau_1, \dots, r\tau_d)|^d = O(1-r)^d. \tag{2.5.4}$$

Combining (2.5.2) and (3.5.2) and using the fact that $n_2 < d$, we obtain a contradiction as $r \nearrow 1$.

Assume $n_2 = d$. For some constant a, we have $p(r\tau_1, \dots, r\tau_d) = a(1-r) + O(1-r)^2$, and

$$\begin{aligned}
\tilde{p}(r\tau_1, \dots, r\tau_d) &= r^{k_1+\dots+k_d}\tau_1^{k_1}\cdots\tau_d^{k_d}\bar{p}(\tfrac{\tau_1}{r}, \dots, \tfrac{\tau_d}{r}) \\
&= r^{k_1+\dots+k_d}\tau_1^{k_1}\cdots\tau_d^{k_d}\big[\bar{a}(1-\tfrac{1}{r}) + O(1-r)^2\big].
\end{aligned}$$

Using our equations for $p(r\tau_1, \ldots, r\tau_d)$ and $\tilde{p}(r\tau_1, \ldots, r\tau_d)$, we have

$$\begin{aligned}
&|p(r\tau_1, \ldots, r\tau_d)|^d - |\tilde{p}(r\tau_1, \ldots, r\tau_d)|^d \\
&\qquad = \left|a(1-r) + O(1-r)^2\right|^d - r^{d(k_1+\cdots+k_d)}\left|\bar{a}(1-\tfrac{1}{r}) + O(1-r)^2\right|^d \\
&\qquad = |a|^d(1-r)^d\left[1 - r^{d(k_1+\cdots+k_d-1)}\right] + O(1-r)^{d+1} \\
&\qquad = O(1-r)^{d+1}.
\end{aligned} \tag{2.5.5}$$

Combining (2.5.2) and (2.5.5), we get a contradiction as $r \nearrow 1$. $\square$

Chapter 3

Differentiating Matrix Functions

3.1 Introduction

Over the last century, one-variable matrix functions $F : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, have played important roles in many areas of mathematics and engineering. For example, matrix functions such as e^A, $\log(A)$ and $A^{\frac{1}{2}}$ are key in solving both systems of differential equations and nonlinear matrix equations [30]. Matrix functions also have applications in diverse areas including control theory, theoretical particle physics, and Markov models [30, 57, 44]. Sometimes, the definition of a matrix function can be extended to yield an operator function defined on well-behaved linear operators of a given Hilbert space. Such operator functions play a primary role in spectral theory and are closely related to deep results such as the spectral mapping theorem [27].

Derivatives of one-variable matrix functions are also quite important. If a matrix function $F(A)$ provides the solution to a modeling problem, an immediate question is: How sensitive is this solution to perturbations in the input data? Such questions are typically answered using condition numbers, which are calculated using the matrix function's derivative [30]. Derivatives are also used to obtain tractable characterizations of monotone and convex matrix functions; understanding such classes of functions is quite valuable because they have immediate applications to electrical networking theory and small particle physics [13, 19, 60].

In this chapter, we consider questions motivated by the study of such one-variable matrix functions, which are generally defined using real-valued functions. Specifically, if f is a real-valued function defined on $\mathbb{R}$, then there is a canonical way to use f to define a matrix-valued function F on the space of $n \times n$ self-adjoint matrices. This construction can sometimes be extended to general $n \times n$ matrices, but in this chapter, we restrict attention to the self-adjoint case. In particular, let S be an $n \times n$ self-adjoint matrix diagonalized by a unitary U as follows

$$S = U \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} U^*.$$

Then, the eigenvalues $\{x_1, \dots, x_n\}$ of S are real and it makes sense to define:

$$F(S) := U \begin{pmatrix} f(x_1) & & \\ & \ddots & \\ & & f(x_n) \end{pmatrix} U^*.$$

One question of interest is:

Which properties of the original function f are inherited by the matrix function F?

In this chapter, we generalize this matrix function construction to multivariate functions and consider whether differentiability properties of the original function pass to the matrix function. In this introduction, we introduce basic definitions and notation, discuss the question of interest, review the related literature, and end with a summary of the main ideas and results in this chapter.

3.1.1 Basic Definitions

Let us first establish the mathematical objects of interest.

Definition 3.1.1. Multivariate Matrix Functions. Let f be a real-valued function defined on an open set $\Omega \subseteq \mathbb{R}^d$. Then f induces a matrix-valued function F on the space of d-tuples of $n \times n$ pairwise-commuting self-adjoint matrices with joint spectrum in Ω. Specifically, let $S = (S^1, ..., S^d)$ be such a d-tuple and let U be a unitary matrix diagonalizing S as follows:

$$S^r = U \begin{pmatrix} x_1^r & & \\ & \ddots & \\ & & x_n^r \end{pmatrix} U^*, \tag{3.1.1}$$

for $1 \leq r \leq d$. Denote the joint spectrum of S, also called the set of joint eigenvalues of S, by $\sigma(S) := \left\{x_i = (x_i^1, ..., x_i^d) : 1 \leq i \leq n\right\}$ and define

$$F(S) := U \begin{pmatrix} f(x_1) & & \\ & \ddots & \\ & & f(x_n) \end{pmatrix} U^*. \tag{3.1.2}$$

It is easy to see that $F(S)$ is independent of the unitary U chosen to diagonalize S. For clarity, we require some additional notation. In particular, the space of d-tuples of pairwise-commuting $n \times n$ self-adjoint matrices with joint spectrum in $\Omega \subsetneq \mathbb{R}^d$ is denoted $CS_n^d(\Omega)$. If $\Omega = \mathbb{R}^d$, the matrix space is denoted CS_n^d. For $d > 1$, the space of d-tuples of $n \times n$ self-adjoint matrices is denoted S_n^d and for $d = 1$, is denoted S_n. The set of $n \times n$ self-adjoint matrices with spectrum in $\Omega \subsetneq \mathbb{R}$ is denoted $S_n(\Omega)$.

3.1.2 The Question of Interest

In this chapter, we will answer the question:

Do the differentiability properties of the original function f pass to the matrix function F?

This question sounds deceptively simple, but even for a one-variable function, it is nontrivial. To see one complication, let $f \in C^1(\mathbb{R}, \mathbb{R})$ and consider the simple case of differentiating the associated matrix function F along a C^1 curve $S(t)$ of $n \times n$ self-adjoint matrices. At first glance, it seems reasonable to write $S(t) = U(t)D(t)U^*(t)$, for $U(t)$ unitary and $D(t)$ diagonal. Then $F(S(t)) = U(t)F(D(t))U^*(t)$, and we can differentiate using the product rule.

However, there is no guarantee that we can decompose $S(t)$ into its eigenvector and eigenvalue matrices so that the eigenvectors are even continuous. In particular, eigenvector behavior at points where distinct eigenvalues coalesce can be unpredictable.

Example 3.1.2. To illustrate, consider the following example of Rellich from [51]:

$$S(t) = e^{-\frac{1}{t^2}} \begin{pmatrix} \cos\left(\frac{2}{t}\right) & \sin\left(\frac{2}{t}\right) \\ \\ \sin\left(\frac{2}{t}\right) & -\cos\left(\frac{2}{t}\right) \end{pmatrix} \text{ for t } \neq 0, \text{ and } S(0) = 0.$$

For $t \neq 0$, the eigenvalues of $S(t)$ are $\pm\, e^{-\frac{1}{t^2}}$ and their associated eigenvectors are

$$\begin{pmatrix} \cos\left(\frac{1}{t}\right) \\ \\ \sin(\frac{1}{t}) \end{pmatrix} \text{ and } \begin{pmatrix} \sin(\frac{1}{t}) \\ \\ -\cos(\frac{1}{t}) \end{pmatrix},$$

and there is a singularity in the eigenvectors at $t = 0$. Thus, even an infinitely differentiable curve can have singularities in its eigenvectors.

3.1.3 Relevant Literature and History

The differentiability of matrix functions defined from one-variable functions is discussed frequently in the literature. For examples, see [19, 26, 32]. The most comprehensive result is by A.L. Brown and H.L. Vasudeva in [22], who proved that an m-times continuously differentiable real-valued function induces an m-times continuously Fréchet differentiable matrix-valued function.

A one-variable function also induces an operator function defined on the set of bounded self-adjoint operators on any separable, infinite-dimensional Hilbert space. There is some subtlety in the differentiability question because in this situation, a C^1 function does not always induce a C^1 operator-valued function. In [48], V.V. Peller showed that if the operator function is continuously differentiable, then the original function belongs locally to the Besov space $B^1_{11}(\mathbb{R})$. A survey of such necessary and sufficient conditions for differentiability is provided in [49].

It should be noted that there is an alternate approach for inducing a matrix function from a multivariate function; the d matrices $S^1, ..., S^d$ are viewed as operators on Hilbert spaces $H^1, ..., H^d$ and $F(S)$ is viewed as an operator on $H^1 \otimes ... \otimes H^d$. Brown and Vasudeva generalized their one-variable differentiability result to these matrix functions in [22].

3.1.4 Summary of Main Results

In this chapter, we focus on matrix functions defined as in (3.1.2).

Section 3.2

In Section 3.2, we analyze the geometry of CS^d_n and conclude that a suitable notion of differentiability for functions on this space is differentiation along curves. If we fix S in CS^d_n, Theorem 3.2.5 characterizes the directions Δ in S^d_n such that there is a C^1 curve $S(t)$ in CS^d_n with $S(0) = S$ and $S'(0) = \Delta$. In Theorem 3.2.8, we show that the joint eigenvalues of locally Lipschitz curves in CS_n^d can be represented by locally Lipschitz functions.

Section 3.3

In Section 3.3, we examine the differentiability properties of the induced matrix functions. Specifically, in Theorem 3.3.2, we show that a C^1 function always induces a matrix function that can be differentiated along C^1 curves in CS_n^d. We then calculate a formula for the derivative of a matrix function along curves and in Theorem 3.3.8, prove that the formula is continuous.

Section 3.4

In Section 3.4, we consider higher-order differentiation. With additional domain restrictions, in Theorem 3.4.2, we show that a real-valued C^m function induces a matrix function that can be m-times continuously differentiated along C^m curves. We also calculate a formula for the derivatives and in Theorem 3.4.7, show the derivatives are continuous.

Section 3.5

In Section 3.5, we highlight several applications of the differentiability results. In particular, we discuss how the derivatives of matrix functions play a role in the characterization of monotone and convex matrix functions.

3.2 The Geometry of CS_n^d

In this section, we examine the structure of the space CS_n^d with the goal of determining which concepts of differentiation are most appropriate for functions defined on CS_n^d. In Subsection 3.2.1, we first study the basic properties of CS_n^d and show that CS_n^d is a stratified space. In Subsection 3.2.2 we characterize the C^1 curves in CS_n^d; this characterization implies that the differential map associated to the stratification of CS_n^d into smooth submanifolds of $\mathbb{R}^m$ is only defined on a subset of the vectors tangent to CS_n^d. Thus, we primarily study differentiation along curves. In Subsection 3.2.3, we show that locally Lipschitz curves in CS_n^d have joint eigenvalues given by locally Lipschitz functions.

3.2.1 Basic Properties of CS_n^d

To begin, observe that CS_n^d is not even a linear space; if A and B are pairwise-commuting d-tuples, the sum $A+B$ need not pairwise commute. Thus, neither the Fréchet nor Gâteaux derivatives can be defined for functions on CS_n^d because both require the function to be defined on linear sets around each point.

Now, let us impose the following norm on CS_n^d :

Definition 3.2.1. Let $S = (S^1, ..., S^d)$ be in CS_n^d (or S_n^d) and let $x_i = (x_i^1, ..., x_i^d)$ be in $\sigma(S)$. Define

$$\|S\| := \max_{1\leq r\leq d} \|S^r\| \quad \text{and} \quad \|x_i\| := \max_{1\leq r\leq d} |x_i^r|, \tag{3.2.1}$$

where $\|S^r\|$ is the usual operator norm.

Recall that each $S \in S_n$ is uniquely determined by its upper triangular part, which has n^2 degrees of freedom. Then we can define a bijective map $J : S_n \to \mathbb{R}^{n^2}$ by

$$J\big(S\big)_{i,j=1}^{n} := \big(S_{11}, \ldots, S_{nn}, \big(Re(S_{ij}), Im(S_{ij})\big)_{j>i}\big)$$

where this is interpreted as an n^2-tuple. By this identification, CS_n^d can be viewed as a subset of $\mathbb{R}^m$, where $m = dn^2$, and inherits the following nom:

$$\|S\|_{\mathbb{R}^m}^2 := \sum_{r=1}^{d} \sum_{j \geq i} |S_{ij}^r|^2 \qquad \forall\ S \in CS_n^d. \tag{3.2.2}$$

Using basic facts about self-adjoint matrices, it is easy to show that the norm on CS_n^d defined by (3.2.2) and the norm defined in (3.2.1) are equivalent norms. Moreover, in both of these norms, CS_n^d is a closed subset of S_n^d or equivalently, of $\mathbb{R}^m$.

Remark 3.2.2. Recall that CS_n^d is precisely the set of elements $S \in S_n^d$ with

$$[S^r, S^s] = S^r S^s - S^s S^r = 0 \qquad \forall\ 1 \leq r, s \leq d.$$

Thus, CS_n^d is the zero set of the polynomials associated with $d(d-1)/2$ commutator operations and so is a real algebraic variety. These polynomials are defined on exactly $m = dn^2$ real variables.

A result by Whitney in [59] and discussed by Kaloshin in [33] says every algebraic variety defined by polynomials on m real variables can be decomposed into smooth submanifolds of $\mathbb{R}^m$ that fit together 'regularly' and whose tangent spaces fit together 'regularly.' For a manifold N, let TN denote the tangent space of N and let $T_S N$ denote the tangent space based at a point S in N. Let X be a closed subset of $\mathbb{R}^m$. Before further discussing Whitney's result, we need the following definition:

Definition 3.2.3. A *stratification* of X is a locally finite partition Z of X into locally closed pieces $\{M_\alpha\}$ such that

1. Each piece $M_\alpha \in Z$ is a smooth submanifold of $\mathbb{R}^m$.

2. (*Condition of frontier*) If $M_\alpha \cap \overline{M}_\beta \neq \emptyset$ for pieces M_α, M_β, then $M_\alpha \subset \overline{M}_\beta$.

Example 3.2.4. Consider CS_2^2, the space of pairs of self-adjoint, commuting 2×2 matrices. In the following definitions, $a, b, c, d \in \mathbb{R}$. Define

$$M_1 := \left\{ \left(U \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} U^*, U \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} U^* \right) : U \text{ is } 2\times 2 \text{ unitary, } a \neq b, c \neq d \right\},$$

$$M_2 := \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, U \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} U^* \right) : U \text{ is } 2\times 2 \text{ unitary, } c \neq d \right\},$$

$$M_3 := \left\{ \left(U \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} U^*, \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \right) : U \text{ is } 2\times 2 \text{ unitary, } a \neq b \right\},$$

$$M_4 := \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \right) \right\}.$$

It is easy to see that $CS_2^2 = \cup M_i$ and each M_i is locally closed. With some work, one can show each M_i is a smooth submanifold of $\mathbb{R}^8$. As this example clearly satisfies the condition of frontier, this partition $\{M_i\}$ is a stratification of CS_2^2. As in this example, one should generally expect a stratification of CS_n^d to be related to the number and multiplicity of the repeated eigenvalues of the elements of CS_n^d.

Whitney's result says CS_n^d has a specific decomposition Z into smooth submanifolds of $\mathbb{R}^m$ where $m = dn^2$, called a *Whitney stratification*. This stratification has further regularity involving the tangent spaces of the pieces of Z. As we do not need those details here, they are omitted. The interested reader should see [33] or [50] for the specifics. We let $\{M_\alpha\}$ denote the pieces of Z and define the tangent space $TCS_n^d := \cup TM_\alpha$. Given a function $F : CS_n^d \to S_n$, one type of derivative is a map $DF : TCS_n^d \to TS_n$ such that

$$DF|_{TM_\alpha} : TM_\alpha \to TS_n$$

is the usual differential map for each manifold M_α. In Theorem 3.3.12, we analyze such maps. However, these differential maps cannot be easily generalized to analyze higher-order differentiation. Furthermore, for each $S \in CS_n^d$ and piece M_α containing S, the tangent space $T_S M_\alpha$ might only contain a subset of the vectors tangent to CS_n^d at S. Example 3.2.6 will show that strict containment often occurs.

3.2.2 Continuously Differentiable Curves in CS_n^d

To retain information about all tangent vectors, we mostly study differentiation along differentiable curves. In this section, we determine which $\Delta \in S_n^d$ are vectors tangent to CS_n^d at a given point S. Specifically, for any $\Delta \in S_n^d$ and $S \in CS_n^d$, we ask

Is there a C^1 curve $S(t)$ in CS_n^d with $S(0) = S$ and $S'(0) = \Delta$?

For an element $S \in CS_n^d$ with distinct joint eigenvalues, Agler, McCarthy, and Young in [8] gave necessary and sufficient conditions on S and Δ for such a C^1 curve to exist. We extend their result to an arbitrary element S but first need additional notation. Fix $S \in CS_n^d$ and $\Delta \in S_n^d$. Let U be a unitary matrix diagonalizing each component of S such that the repeated joint eigenvalues of S appear consecutively. Numbering the x_i's appropriately, define

$$D^r := U^* S^r U = \begin{pmatrix} x_1^r & & \\ & \ddots & \\ & & x_n^r \end{pmatrix}, \tag{3.2.3}$$

for each $1 \le r \le d$. Then, for each r, define the two matrices

$$\Gamma^r := U^* \Delta^r U$$

$$\tilde{\Gamma}_{ij}^r := \begin{cases} \Gamma_{ij}^r & \text{if } x_i = x_j \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.4}$$

Then $\tilde{\Gamma}^r$ is a block diagonal matrix. Each block corresponds to a distinct joint eigenvalue of S and has dimension equal to the multiplicity of that eigenvalue. We now characterize the differentiable curves in CS_n^d as follows:

Theorem 3.2.5. *Let $S \in CS_n^d$ and $\Delta \in S_n^d$. Then there exists a C^1 curve $S(t)$ in CS_n^d with $S(0) = S$ and $S'(0) = \Delta$ if and only if for all $1 \leq s, r \leq d$,*

$$[D^r, \Gamma^s] = [D^s, \Gamma^r] \ \textit{ and } \left[\tilde{\Gamma}^r, \tilde{\Gamma}^s\right] = 0.$$

Proof. ($\Rightarrow$) Assume $S(t)$ is a C^1 curve in CS_n^d with $S(0) = S$ and $S'(0) = \Delta$. Define

$$R(t) := U^* S(t) U,$$

where U diagonalizes S as in (3.2.3). Then $R(t)$ is a C^1 curve in CS_n^d with $R(0) = D$ and $R'(0) = \Gamma$. We will first prove that

$$[D^r, \Gamma^s] = [D^s, \Gamma^r] \ \text{ and } \ [\Gamma^r, \Gamma^s]_{ij} = 0,$$

for all pairs $1 \leq r, s \leq d$ and (i, j) such that $x_i = x_j$. We will use those results to conclude

$$\left[\tilde{\Gamma}^r, \tilde{\Gamma}^s\right] = 0,$$

for each pair $1 \leq r, s \leq d$. Since $R(t)$ is C^1 in a neighborhood of $t = 0$, we can write

$$R^r(t) = D^r + \Gamma^r t + h^r(t),$$

for each $1 \leq r \leq d$, where $|h^r(t)_{ij}| = o(|t|)$ for $1 \leq i, j \leq n$. For each pair r and s, the pairwise-commutativity of $R(t)$ implies

$$\begin{aligned}
0 &= [R^r(t), R^s(t)] \\
&= [D^r + \Gamma^r t + h^r(t), D^s + \Gamma^s t + h^s(t)] \\
&= \big([D^r, h^s(t)] + [h^r(t), D^s] + [h^r(t), h^s(t)]\big) \\
&+ \big([D^r, \Gamma^s] + [\Gamma^r, D^s] + [\Gamma^r, h^s(t)] + [h^r(t), \Gamma^s]\big)t \\
&+ [\Gamma^r, \Gamma^s]t^2, \qquad (3.2.5)
\end{aligned}$$

where the term $[D^r, D^s]$ was omitted because it vanishes. Fix $t \neq 0$ and divide each term in (3.2.5) by t. Letting t tend towards zero yields

$$0 = [D^r, \Gamma^s] - [D^s, \Gamma^r]. \qquad (3.2.6)$$

Choose i and j such that $x_i = x_j$. Then, the ij^{th} entry of (3.2.5) reduces to

$$0 = [h^r(t), h^s(t)]_{ij} + (\ [\Gamma^r, h^s(t)]_{ij} - [\Gamma^s, h^r(t)]_{ij}\ )\, t + [\Gamma^r, \Gamma^s]_{ij} t^2.$$

Fix $t \neq 0$ and divide both sides by t^2. Letting t tend towards zero yields

$$0 = [\Gamma^r, \Gamma^s]_{ij}, \qquad (3.2.7)$$

as desired. Now, fix r and s with $1 \leq r, s \leq d$. Recall that $\tilde{\Gamma}^r$ and $\tilde{\Gamma}^s$ are block diagonal matrices with blocks corresponding to the distinct joint eigenvalues of S; specifically, $\tilde{\Gamma}^r_{ij} = \tilde{\Gamma}^s_{ij} = 0$ whenever $x_i \neq x_j$. It follows that $\tilde{\Gamma}^r\tilde{\Gamma}^s$ and $\tilde{\Gamma}^s\tilde{\Gamma}^r$ are also such block diagonal matrices. Thus, if i and j are such that $x_i \neq x_j$,

$$\left[\ \tilde{\Gamma}^r,\ \tilde{\Gamma}^s\right]_{ij} = \left(\ \tilde{\Gamma}^r\tilde{\Gamma}^s - \tilde{\Gamma}^s\tilde{\Gamma}^r\ \right)_{ij} = 0.$$

Now, fix i and j such that $x_i = x_j$. By the definition of $\tilde{\Gamma}$,

$$\begin{aligned}
\left[\tilde{\Gamma}^r, \tilde{\Gamma}^s\right]_{ij} &= \sum_{k=1}^{n} \tilde{\Gamma}^r_{ik}\tilde{\Gamma}^s_{kj} - \tilde{\Gamma}^s_{ik}\tilde{\Gamma}^r_{kj} \\
&= \sum_{\{k: x_k = x_i\}} \Gamma^r_{ik}\Gamma^s_{kj} - \Gamma^s_{ik}\Gamma^r_{kj} \\
&= [\Gamma^r, \Gamma^s]_{ij} - \sum_{\{k: x_k \neq x_i\}} \Gamma^r_{ik}\Gamma^s_{kj} - \Gamma^s_{ik}\Gamma^r_{kj} \\
&= - \sum_{\{k: x_k \neq x_i\}} \Gamma^r_{ik}\Gamma^s_{kj} - \Gamma^s_{ik}\Gamma^r_{kj},
\end{aligned}$$

where the last equality uses (3.2.7). Thus, it suffices to show that if $x_k \neq x_i$,

$$\Gamma^r_{ik}\Gamma^s_{kj} - \Gamma^s_{ik}\Gamma^r_{kj} = 0.$$

Assume $x_k \neq x_i$, and fix q with $x^q_k \neq x^q_i$. Apply (3.2.6) to pairs r, q and s, q to get

$$[D^q, \Gamma^r] = [D^r, \Gamma^q] \quad \text{and} \quad [D^q, \Gamma^s] = [D^s, \Gamma^q].$$

Restricting to the ik^{th} and kj^{th} entries of the previous two equations yields

$$\begin{aligned}
\Gamma^r_{ik}(x^q_i - x^q_k) &= \Gamma^q_{ik}(x^r_i - x^r_k), \\
\Gamma^r_{kj}(x^q_k - x^q_j) &= \Gamma^q_{kj}(x^r_k - x^r_j), \\
\Gamma^s_{ik}(x^q_i - x^q_k) &= \Gamma^q_{ik}(x^s_i - x^s_k), \\
\Gamma^s_{kj}(x^q_k - x^q_j) &= \Gamma^q_{kj}(x^s_k - x^s_j).
\end{aligned}$$

Since $x_i = x_j$ and $x^q_k \neq x^q_i$, we can replace all the x_j's with x_i's in the above set of equations and solve for the Γ^r and Γ^s entries. Using these relations gives

$$\Gamma^r_{ik}\Gamma^s_{kj} - \Gamma^s_{ik}\Gamma^r_{kj} = \frac{\Gamma^q_{ik}(x^r_i - x^r_k)\Gamma^q_{kj}(x^s_i - x^s_k)}{(x^q_i - x^q_k)^2} - \frac{\Gamma^q_{ik}(x^s_i - x^s_k)\Gamma^q_{kj}(x^r_i - x^r_k)}{(x^q_i - x^q_k)^2} = 0,$$

as desired. Thus, $[\tilde{\Gamma}^r, \tilde{\Gamma}^s] = 0$.

($\Leftarrow$) Fix S in CS_n^d and Δ in S_n^d and let U, D, Γ, and $\tilde{\Gamma}$ be as in the discussion preceding Theorem 3.2.5. Assume

$$[D^r, \Gamma^s] = [D^s, \Gamma^r] \text{ and } \left[\tilde{\Gamma}^r, \tilde{\Gamma}^s\right] = 0, \tag{3.2.8}$$

for $1 \leq r, s \leq d$. Define a skew-Hermitian matrix Y as follows:

$$Y_{ij} := \begin{cases} \frac{\Gamma_{ij}^q}{x_j^q - x_i^q} & \text{if } x_i \neq x_j \\ 0 & \text{otherwise,} \end{cases}$$

where q is chosen so that $x_i^q - x_j^q \neq 0$. Observe that Y is independent of q because the ij^{th} entry of the first equation in (3.2.8) is

$$\Gamma_{ij}^s(x_i^r - x_j^r) = \Gamma_{ij}^r(x_i^s - x_j^s).$$

Now, define the curve $S(t)$ by

$$S^r(t) := Ue^{Yt}\big[D^r + t\tilde{\Gamma}^r\big]e^{-Yt}U^*,$$

for each $1 \leq r \leq d$. Then, $S(t)$ is continuously differentiable. Because Y is skew-Hermitian, e^{Yt} is unitary. Since D^r and $\tilde{\Gamma}^r$ are self-adjoint, $S(t)$ is in S_n^d. By a simple calculation using (3.2.8),

$$[S^r(t), S^s(t)] = 0,$$

for each pair $1 \leq r, s \leq d$. Thus, $S(t)$ is in CS_n^d. By definition, $S(0) = S$. For each r,

$$(S^r)'(t) = U\left(Ye^{Yt}\big[D^r + t\tilde{\Gamma}^r\big]e^{-Yt} + e^{Yt}\big[\tilde{\Gamma}^r\big]e^{-Yt} - e^{Yt}\big[D^r + t\tilde{\Gamma}^r\big]Ye^{-Yt}\right)U^*,$$

so that

$$(S^r)'(0) = U\left([Y, D^r] + \tilde{\Gamma}^r\right)U^* = \Delta^r.$$

Thus, $S'(0) = \Delta$, and $S(t)$ is the desired curve. $\square$

Observe that by the construction in Theorem 3.2.5, if there is a C^1 curve $S(t)$ in CS_n^d with $S(0) = S$ and $S'(0) = \Delta$, there is actually a smooth curve $R(t)$ in CS_n^d with $R(0) = S$ and $R'(0) = \Delta$.

Example 3.2.6. Let $I \in CS_n^d$ be the identity element. By Theorem 3.2.5, there is a smooth curve $S(t)$ in CS_n^d with

$$S(0) = I \text{ and } S'(0) = \Delta \text{ if and only if } \Delta \in CS_n^d.$$

Thus, the set of vectors tangent to CS_n^d at I is CS_n^d. However, for a Whitney stratification of CS_n^d and piece M_α containing I, the tangent space $T_I M_\alpha$ is linear. Since CS_n^d is not linear, $T_I M_\alpha$ is a strict subset of the set of tangent vectors at I.

The conditions of Theorem 3.2.5 actually imply that if $S \in CS_n^d$ has any repeated joint eigenvalues, the set of vectors tangent to CS_n^d at S is not a linear set. Then, for any Whitney stratification of CS_n^d and piece M_α containing S, the tangent space $T_S M_\alpha$ is a strict subset of the vectors tangent to CS_n^d at S. We will thus focus on differentiation along curves rather than differential maps.

3.2.3 Joint Eigenvalues of Curves in CS_n^d

Let $S(t)$ be a differentiable curve in CS_n^d and let F be a matrix function on CS_n^d induced from a real-valued function f. Understanding differentiation of $F(S(t))$ requires additional information about $F(S(t))$. By definition, $F(S(t))$ is obtained by applying the original function f to the joint eigenvalues of $S(t)$. Thus, the behavior of the joint eigenvalues of curves in CS_n^d is of immediate importance.

If $S(t)$ is a continuous curve in S_n, a result by Rellich from [51, 52] states that the eigenvalues of $S(t)$ can be represented by n continuous functions. A proof is given by Kato in [36, pg 107-10]. With some modifications, the arguments show that the eigenvalues of a locally Lipschitz curve in S_n can be represented by locally Lipschitz functions. For clarity, we include the following definition:

Definition 3.2.7. Let X be a metric space and let I be an interval in $\mathbb{R}$. A function $g : I \to X$ is *Lipschitz with constant C* if

$$\|g(t_1) - g(t_2)\|_X \leq C|t_1 - t_2| \qquad \forall\ t_1, t_2 \in I.$$

Similarly, a function $g : I \to X$ is *locally Lipschitz* if for every $t_0 \in I$, there is a constant C_0 and neighborhood $N_0 \subseteq I$ of t_0 such that

$$\|g(t_1) - g(t_2)\|_X \leq C_0|t_1 - t_2| \qquad \forall\ t_1, t_2 \in N_0.$$

If g is locally Lipschitz on I, the local compactness of $\mathbb{R}$ implies that g is actually Lipschitz on any bounded interval J, with $\bar{J} \subset I$.

Then, the one-variable results generalize as follows:

Theorem 3.2.8. *Let $S(t)$ be a locally Lipschitz curve in CS_n^d defined on an open interval I. Then, there exist locally Lipschitz functions $x_1(t), \dots, x_n(t) : I \to \mathbb{R}^d$ such that $\sigma(S(t)) = \{x_i(t) : 1 \leq i \leq n\}$.*

Theorem 3.2.8 is an immediate consequence of Lemmas 3.2.11 and 3.2.14, which are proved below. Their proofs are technical but semi-straightforward modifications of the one-variable case. For the ease of the reader, we include the proofs below. The proofs require the following definition:

Definition 3.2.9. An *unordered n-d tuple* is an unordered tuple of n vectors, each with d

components. If G and H are two unordered n-d tuples given by

$$G = \left(\begin{pmatrix} x_1^1 \\ \vdots \\ x_1^d \end{pmatrix}, \ldots, \begin{pmatrix} x_n^1 \\ \vdots \\ x_n^d \end{pmatrix} \right) \quad \text{and} \quad H = \left(\begin{pmatrix} y_1^1 \\ \vdots \\ y_1^d \end{pmatrix}, \ldots, \begin{pmatrix} y_n^1 \\ \vdots \\ y_n^d \end{pmatrix} \right),$$

the distance between G and H is defined by

$$\|G - H\|_{n-d} := \min \left(\max_{1 \le i \le n} \|x_i - y_i\| \right),$$

where the minimum is taking over all reorderings of the vectors in G and

$$\|x_i - y_i\| = \max_{1 \le r \le d} |x_i^r - y_i^r| \qquad \forall\ 1 \le i \le n.$$

It is not difficult to see that this operation gives a metric on the set of unordered n-d tuples.

Remark 3.2.10. Let $S \in CS_n^d$. Then, the set of joint eigenvalues of S is an unordered n-d tuple. If $S(t)$ is a locally Lipschitz curve in CS_n^d defined on an open interval I, then Theorem 3.2.8 provides a specific ordering of the joint eigenvalues of $S(t)$ at each $t \in I$. This ordering may differ from the one in (3.2.3), where repeated joint eigenvalues appear consecutively. If we require that eigenvalues be ordered as in (3.2.3), Lemma 3.2.11 implies that the joint spectrum of $S(t)$ is at least locally Lipschitz in the unordered n-d tuple metric.

Lemma 3.2.11. *Let $S(t)$ be a locally Lipschitz curve in CS_n^d defined on an open interval I. Then, the joint spectrum of $S(t)$ is locally Lipschitz as an unordered n-d tuple.*

Proof. Let J be a bounded interval with $\bar{J} \subset I$. Then, there is a constant $C > 0$ such that

$$\|S^r(t_1) - S^r(t_2)\| \le \|S(t_1) - S(t_2)\| \le C|t_1 - t_2| \qquad \forall\ 1 \le r \le d \text{ and } t_1, t_2 \in \bar{J}. \tag{3.2.9}$$

To prove the lemma, it is sufficient to show that the joint spectrum $\sigma(S(t))$ is Lipschitz on

J as an unordered n-d tuple with Lipschitz constant $2C$. This proof has two main steps:

1. Show that for each fixed $t_1 \in J$, there is an interval $J_{t_1} \subseteq J$ such that

$$\|\sigma(S(t_1)) - \sigma(S(t_2))\|_{n-d} \leq 2C|t_1 - t_2| \qquad \forall\, t_2 \in J_{t_1}. \tag{3.2.10}$$

2. Use (3.2.10) to show that

$$\|\sigma(S(t_1)) - \sigma(S(t_2))\|_{n-d} \leq 2C|t_1 - t_2| \qquad \forall\, t_1, t_2 \in J.$$

Step 1.

Fix $t_1 \in J$. Let $\{x_i = (x_i^1, \ldots, x_i^d) : 1 \leq i \leq n\}$ denote the joint spectrum of $S(t_1)$. For each $t_2 \in J$, set $\delta_{t_2} := 2C|t_1 - t_2|$. For each $1 \leq i \leq n$ and $1 \leq r \leq d$, let $\partial D(x_i^r, \delta_{t_2})$ be the circle centered at x_i^r with radius δ_{t_2}. Shrink J to an interval J_{t_1} containing t_1 such that the following hold for all $t_2 \in J_{t_1}$, all $1 \leq i \leq n$, and all $1 \leq r \leq d$:

(1) x_i^r is the only eigenvalue of $S^r(t_1)$ inside $\partial D(x_i^r, \delta_{t_2})$ up to multiplicity.

(2) For each $\zeta \in \partial D(x_i^r, \delta_{t_2})$, the following holds: $\min\limits_{1 \leq j \leq n} |x_j^r - \zeta| = |x_i^r - \zeta| = \delta_{t_2}$.

(3) If $x_i \neq x_j$, then $\delta_{t_2} < \frac{1}{2}\|x_i - x_j\|$.

Now, fix $t_2 \in J_{t_1}$ with $t_2 \neq t_1$. The immediate goal is to find an interval $J_2 \subseteq J_{t_1}$ containing t_1 and t_2 such that for each r, the operator $\big(S^r(t) - \zeta I\big)^{-1}$ exists for each $t \in J_2$, $\zeta \in \partial D(x_i^r, \delta_{t_2})$, and $1 \leq i \leq n$. By (2), each operator $(S^r(t_1) - \zeta I)^{-1}$ exists and

$$\big\|\big(S^r(t_1) - \zeta I\big)^{-1}\big\| = \max_{1 \leq j \leq n} \frac{1}{|\zeta - x_j^r|} = \frac{1}{\delta_{t_2}}.$$

Then, a simple calculation gives

$$S^r(t) - \zeta I = \Big[I - \big(S^r(t_1) - S^r(t)\big)\big(S^r(t_1) - \zeta I\big)^{-1}\Big]\,\big(S^r(t_1) - \zeta I\big).$$

It follows that $(S^r(t)-\zeta I)^{-1}$ will exist if $\big(S^r(t_1)-\zeta I\big)^{-1}\big[I-\big(S^r(t_1)-S^r(t)\big)\big(S^r(t_1)-\zeta I\big)^{-1}\big]^{-1}$ exists. Thus, a sufficient condition for $(S^r(t)-\zeta I)^{-1}$ to exist is for

$$\big\|\big(S^r(t_1)-S^r(t)\big)\big(S^r(t_1)-\zeta I\big)^{-1}\big\| < 1.$$

By (3.2.9), the following holds for $t \in J$

$$\begin{aligned}\big\|\big(S^r(t_1)-S^r(t)\big)\big(S^r(t_1)-\zeta I\big)^{-1}\big\| &\le \big\|S^r(t_1)-S^r(t)\big\|\big\|\big(S^r(t_1)-\zeta I\big)^{-1}\big\| \\ &\le \frac{C|t-t_1|}{\delta_{t_2}} \\ &= \frac{|t-t_1|}{2|t_2-t_1|}.\end{aligned} \tag{3.2.11}$$

It is clear that there is some interval $J_2 \subseteq J_{t_1}$ containing t_1 and t_2 with $|t-t_1| < 2|t_2-t_1|$ for each $t \in J_2$. Then for each $1 \le i \le n$ and and $1 \le r \le d$, the operator $(S^r(t)-\zeta I)^{-1}$ exists for $\zeta \in \partial D(x_i^r, \delta_{t_2})$ and $t \in J_2$. Thus, each operator:

$$P_i^r(t) := \frac{1}{2\pi i}\int_{\partial D(x_i^r,\delta_{t_2})} \big(S^r(t)-\zeta I\big)^{-1} d\zeta$$

exists and is easily shown to be continuous on J_2 using the relationship

$$\big(S^r(t^*)-\zeta I\big)^{-1} - \big(S^r(t)-\zeta I\big)^{-1} = \big(S^r(t)-\zeta I\big)^{-1}\big(S^r(t)-S^r(t^*)\big)\big(S^r(t^*)-\zeta I\big)^{-1},$$

which holds as long as ζ is in the resolvent sets of both $S^r(t)$ and $S^r(t^*)$. A classical result from perturbation theory, which first appeared in [34, 35, 56], states that $P_i^r(t)$ is the total eigenprojection associated with the eigenvalues of $S^r(t)$ enclosed by $\partial D(x_i^r, \delta_{t_2})$. For each i, define

$$P_i(t) := P_i^1(t)\cdots P_i^d(t).$$

Then for each fixed i, the operator $P_i(t)$ is the total eigenprojection of the joint eigenvalues of

$S(t)$ enclosed by $\partial D(x_i^1, \delta_{t_2}) \times \cdots \times \partial D(x_i^d, \delta_{t_2})$ and is continuous on J_2. Thus, Rank $P_i(t)$ is the number of joint eigenvalues of $S(t)$ enclosed by $\partial D(x_i^1, \delta_{t_2}) \times \cdots \times \partial D(x_i^d, \delta_{t_2})$ including multiplicity. Since $P_i(t)$ is idempotent, Rank $P_i(t) =$ Trace $P_i(t)$, which is continuous on J_2. Let m_i denote the multiplicity of the eigenvalue x_i. Then by (1), Rank $P_i(t_1) = m_i$. By continuity, Rank $P(t_2) = m_i$ as well. Thus, $S(t_2)$ has m_i joint eigenvalues, denoted $y_k = (y_k^1, \dots, y_k^d)$ for $1 \leq k \leq m_i$, enclosed by $\partial D(x_i^1, \delta_{t_2}) \times \cdots \times \partial D(x_i^d, \delta_{t_2})$. Thus

$$\|x_i - y_k\| = \max_{1 \leq r \leq d} |x_i^r - y_k^r| < \delta_{t_2} = 2C\ |t_1 - t_2| \qquad \forall\ 1 \leq k \leq m_i. \tag{3.2.12}$$

Moreover, by (3), we can actually pair each distinct $x_i \in \sigma(S(t_1))$ with a set $\mathcal{L}_i$ of m_i joint eigenvalues of $S(t_2)$, such that the $\mathcal{L}_i$ sets are disjoint and (3.2.12) holds for each i. Then

$$\|\sigma(S(t_1)) - \sigma(S(t_2))\|_{n-d} < 2C\ |t_1 - t_2|. \tag{3.2.13}$$

As $t_2 \in J_{t_1}$ was arbitrary, (3.2.13) holds for each $t_2 \in J_{t_1}$. As t_1 was arbitrary, for each $t_1 \in J$ there is an interval J_{t_1}, which can be shrunk to be centered at t_1, such that (3.2.13) holds for each $t_2 \in J_{t_1}$.

Step 2.

Now, proceed to Step 2 and fix $t_1, t_2 \in J$. Without loss of generality, assume $t_1 < t_2$. Then $[t_1, t_2]$ is a compact set in J and hence can be covered by a finite set of open intervals $\{J_{a_1}, \dots, J_{a_K}\}$ with each $a_k \in [t_1, t_2]$ and

$$\|\sigma(S(a_k)) - \sigma(S(t))\|_{n-d} < 2C\ |a_k - t| \qquad \forall\ t \in J_{a_k} \text{ and } 1 \leq k \leq K. \tag{3.2.14}$$

Moreover, we can assume each J_{a_k} is centered at a_k and $t_1 \leq a_1 < a_2 < \cdots < a_K \leq t_2$. Now Lemma 3.2.12 can be used to obtain a subset of of those intervals, denoted $\{J_{b_m}\}_{m=1}^M$, covering $[t_1, t_2]$ such that $t_1 \leq b_1 < b_2 < \cdots < b_M \leq t_1$, each $J_{b_m} \cap [t_1, t_2]$ is not contained in a union of other intervals, and each J_{b_m} intersects precisely $J_{b_{m-1}}$ and $J_{b_{m+1}}$, as long as those two intervals are defined. Those conditions imply that $t_1 \in J_{b_1}$ and $t_2 \in J_{b_M}$. Using the properties of the $\{J_{b_m}\}$ intervals, choose points $t_{(m-1)m} \in J_{b_{m-1}} \cap J_{b_m}$ for $m = 2, \ldots, M$ such that

$$t_1 \leq b_1 < t_{12} < b_2 < t_{23} < \cdots < b_{M-1} < t_{(M-1)M} < b_M \leq t_2.$$

Note that the intersections $J_{b_{m-1}} \cap J_{b_m}$ cannot be nonempty because $\{J_{b_m}\}_{m=1}^{M}$ is an open cover of $[t_1, t_2]$. The intersection properties of the $\{J_{b_m}\}$ also imply that $t_{(m-1)m}$ can always be chosen to satisfy $b_{m-1} < t_{(m-1)m} < b_m$. Now using property (3.2.14) and the triangle inequality, we can calculate:

$$\begin{aligned}
\|\sigma(S(t_1)) - \sigma(S(t_2))\|_{n-d} &\leq \|\sigma(S(t_1)) - \sigma(S(b_1))\|_{n-d} + \|\sigma(S(b_1)) - \sigma(S(t_{12}))\|_{n-d} \\
&\quad + \sum_{m=2}^{M-1} \big(\|\sigma(S(t_{(m-1)m})) - \sigma(S(b_m))\|_{n-d} + \|\sigma(S(b_m)) - \sigma(S(t_{m(m+1)}))\|_{n-d}\big) \\
&\quad + \|\sigma(S(t_{(M-1)M})) - \sigma(S(b_M))\|_{n-d} + \|\sigma(S(b_M)) - \sigma(S(t_2))\|_{n-d} \\
&\leq 2C|t_1 - b_1| + 2C|b_1 - t_{12}| + 2C \sum_{m=2}^{M-1} \big(|t_{(m-1)m} - b_m| + |b_m - t_{m(m+1)}|\big) \\
&\quad + 2C|t_{(M-1)M} - b_M| + 2C|b_M - t_2| \\
&= 2C|t_1 - t_2|,
\end{aligned}$$

as desired. Thus, $\sigma(S(t))$ is Lipschitz as an unordered n-d tuple on J with constant $2C$. Since J was an arbitrary interval with $\bar{J} \subset I$, $\sigma(S(t))$ is locally Lipschitz on I. $\square$

The proof of Lemma 3.2.11 required the following result about interval coverings:

Lemma 3.2.12. *Let I be a finite interval and let $\{J_{a_k}\}_{k=1}^{K}$ be a finite set of open intervals covering I such that each J_{a_k} is centered at a point $a_k \in I$ and $a_1 < a_2 < \cdots < a_K$. Then, there is a subset $\{b_1, \ldots, b_M\} \subseteq \{a_1, \ldots, a_K\}$ such that:*

(1) *$\{J_{b_m}\}_{m=1}^{M}$ covers I.*

(2) *The centers are increasing:* $b_1 < b_2 < \cdots < b_M$.

(3) *For each* $1 \leq m \leq M$*, the interval* $J_{b_m} \cap I$ *is not contained in a union of other intervals and it intersects precisely* $J_{b_{m-1}}$ *and* $J_{b_{m+1}}$*, as long as those two intervals are defined.*

Proof. The proof is by induction on K, the number of intervals in the original cover. Consider the base case $K = 1$. If J_{a_1} covers I, define $b_1 := a_1$. Then $\{b_1\}$ trivially satisfies (1)-(3).

Proceeding via induction, assume the result holds for all finite intervals and coverings with K intervals. Let I be a finite interval and let $\{J_{a_k}\}_{k=1}^{K+1}$ be a covering of I by open intervals J_{a_k} centered at a_k with $a_1 < \cdots < a_{K+1}$. Without loss of generality, assume $I = (c, d)$. Identical arguments handle the case where I has closed endpoints. Define $I' := (c, d^*)$ to be the largest interval beginning at c and contained in I and $\cup_{k=1}^{K} J_{a_k}$. It is possible that $I' = \emptyset$.

By the inductive hypothesis, there is some subset $\{b_1, \ldots, b_M\} \subseteq \{a_1, \ldots, a_K\}$ satisfying (1)-(3) on I'. If $d^* = d$, this set also satisfies (1)-(3) on I. Now, assume $d^* < d$. Since (c, d^*) is the largest interval in I beginning at c and covered by $\cup_{k=1}^{K} J_{a_k}$, there must be a gap in the the covering after d^*. Since $\{J_{a_k}\}_{k=1}^{K+1}$ cover I, the interval $J_{a_{K+1}}$ begins before d^*, say at $d^* - \epsilon$. There must also be some J_{a_j} centered at $a_j \leq a_{K+1}$ containing (s, d) for some $s \in I$. Since $J_{a_{K+1}}$ extends at least as far left as J_{a_j} and their centers satisfy $a_j \leq a_{K+1}$, it follows that $J_{a_j} \subseteq J_{a_{K+1}}$. Thus, $(d^* - \epsilon, d) \subset J_{a_{K+1}}$, which implies that $\{J_{b_m}\}_{m=1}^{M} \cup J_{a_{K+1}}$ covers I.

Now, let N be the smallest integer such that $J_{a_{K+1}} \cap J_{b_N} \neq \emptyset$. Consider the set $\{b_1, \ldots, b_N, a_{K+1}\} \subseteq \{a_1, \ldots, a_{K+1}\}$ and define $b_{N+1} := a_{K+1}$. It is easy to show that $\{b_1, \ldots, b_{N+1}\}$ satisfies (1)-(3) on I. Specifically, by property (3) of the inductive hypothesis, if t_N is the endpoint of J_{b_N}, then $(c, t_N) \subseteq \cup_{m=1}^{N} J_{b_m}$. That implies $\{J_{b_m}\}_{m=1}^{N+1}$ covers I. Property (2) is clear and property (3) follows from the inductive hypothesis and the way we selected N. $\square$

Now, before proving the second lemma needed for Theorem 3.2.8, we require an auxiliary lemma. This lemma establishes the analogous global Lipschitz result. In Lemma 3.2.14, we relax to the desired locally Lipschitz conditions.

Lemma 3.2.13. *Let I be an open interval and for each $t \in I$, let $G(t)$ be an unordered n-d tuple of real numbers. If $G(t)$ is Lipschitz with constant C as an unordered n-d tuple, then there exist functions $x_1(t), \ldots, x_n(t) : I \to \mathbb{R}^d$ such that each $x_i(t)$ is Lipschitz with constant C and the functions satisfy $G(t) = \{x_i(t) : 1 \leq i \leq n\}$.*

Proof. **Auxillary Property:**

Before proving the main result, we show that if I_1 and I_2 are subintervals of I with $I_1 \cap I_2 \neq \emptyset$ and the Lipschitz result holds for I_1 and I_2, then it holds for $I_3 = I_1 \cup I_2$. Without loss of generality, we can assume neither interval is contained in the other and that I_1 lies to the left of I_2. Assume the result holds on I_1 and I_2. Let $\{x_i^1(t)\}$ and $\{x_i^2(t)\}$ be the respective representations of $G(t)$ on I_1 and I_2 that are Lipschitz with constant C. Choose $t_0 \in I_1 \cap I_2$. After a suitable reordering of $\{x_i^2(t)\}$, we have $x_i^1(t_0) = x_i^2(t_0)$ for $1 \leq i \leq n$. Define

$$x_i^3(t) = \begin{cases} x_i^1(t) & t \leq t_0 \\ x_i^2(t) & t \geq t_0 \end{cases}$$

for $i = 1, \ldots, n$. Then, $\{x_i^3(t)\}$ is a representation of $G(t)$ by Lipschitz functions on I_3 with constant C. To see this, fix $t_1, t_2 \in I$ and i with $1 \leq i \leq n$. The Lipschitz inequality for $x_i(t)$ is immediate if $t_1, t_2 \leq t_0$ or $t_1, t_2 \geq t_0$. Similarly, if $t_1 \leq t_0 \leq t_2$ or $t_2 \leq t_0 \leq t_1$, then

$$\begin{aligned} \left\|x_i^3(t_1) - x_i^3(t_2)\right\| &< \left\|x_i^3(t_1) - x_i^3(t_0)\right\| + \left\|x_i^3(t_0) - x_i^3(t_2)\right\| \\ &< C\Big(|t_1 - t_0| + |t_0 - t_2|\Big) = C|t_1 - t_2|. \end{aligned}$$

Main Result:

Now, we prove the main result. It follows by induction on n, the number of vectors of $G(t)$. The $n = 1$ case is immediate because if $G(t)$ has only one vector $x_1(t)$, the Lipschitz property of $G(t)$ implies:

$$\|x_1(t_1) - x_1(t_2)\| = \|G(t_1) - G(t_2)\|_{1-d} < C|t_1 - t_2| \qquad \forall\ t_1, t_2 \in I.$$

Proceeding via induction, assume the result holds for all $m < n$ and all intervals $I' \subseteq I$. Define the set:

$$F := \Big\{ t \in I : G(t) \text{ consists of } n \text{ identical vectors} \Big\}$$

and define $O := I \setminus F$. Then, the continuity of $G(t)$ as an unordered n-d tuple implies that F is locally closed in I and O is open. Fix $t_0 \in O$. Since the n vectors of $G(t_0)$ are not all identical, we can write $G(t_0)$ as two separate tuples: $G_1(t_0)$ of n_1 vectors, denoted $x_1, \ldots x_{n_1}$, and $G_2(t_0)$ of n_2 vectors, denoted $x_{n_1+1}, \ldots x_n$. Here, 'separate' means that no vector in one tuple appears in the other tuple, so that there is some $\delta > 0$ with $\|x_i - x_j\| > \delta$ for $i \leq n_1$ and $j > n_1$. Since $G(t)$ is Lipschitz with constant C, we can define:

$$G_1(t) := \Big\{ n_1 \text{ vectors } y_1, \ldots, y_{n_1} \text{ of } G(t) \text{ s.t. } \min_{\substack{\text{reorderings} \\ \text{of the } y_i's}} \big(\max_{1 \leq i \leq n_1} \|y_i - x_i\| \big) \leq C|t - t_0| \Big\},$$

for each $t \in I$. Since $G_1(t_0)$ and $G_2(t_0)$ are separated by δ, there is an interval I_0 centered at t_0 such that for $t \in I_0$, $G_1(t)$ is uniquely determined. Then for $t \in I_0$, define $G_2(t) := G(t) \setminus G_1(t)$. By shrinking I_0 around t_0 if necessary and using the Lipschitz property of $G(t)$, one can show $G_1(t)$ and $G_2(t)$ are Lipschitz as unordered n_1-d and n_2-d tuples with constant C on I_0. By the inductive hypothesis, there are functions $\{x_1(t), \ldots, x_{n_1}(t)\}$ and $\{x_{n_1+1}(t), \ldots, x_n(t)\}$ that satisfy

$$\|x_i(t_1) - x_i(t_2)\| < C|t_1 - t_2| \qquad \forall\, t_1, t_2 \in I_0 \text{ and } 1 \leq i \leq n,$$

and represent $G_1(t)$ and $G_2(t)$ in I_0. Thus, these n functions represent $G(t)$ in I_0.

Since O is open in I, it consists of at most countably many disjoint open subintervals $I_1, I_2, \ldots,$ etc. Since each point in O is contained in an interval where the result holds, the Auxilliary Property implies that the result holds on each compact subset of each I_k. To obtain the result on each I_k, the Lipschitz functions on compact subsets of I_k need to be glued together properly. This is easy but technical. Such a gluing is constructed in the proof of Lemma 3.2.14 and we refer the interested reader to that proof. It this situation, unlike in Lemma 3.2.14, the Lipschitz constant does not change.

Now, the result holds on each disjoint subinterval I_k of O. Let $\{x_i^k(t)\}$ be the n functions representing $G(t)$ on I_k for $k = 1, 2, \ldots$, etc. For $t \in F$, $G(t)$ consists of n identical vectors, which we call $x(t)$. For i with $1 \leq i \leq n$, define the function $x_i(t)$ on I as follows:

$$x_i(t) = \begin{cases} x_i^k(t) & t \in I_k \quad k = 1, 2, \ldots \\ x(t) & t \in F. \end{cases}$$

These n functions represent $G(t)$ on I. To see that these functions are Lipschitz with constant C, first observe that if $t_1 \in F$, then for each $t_2 \in I$,

$$\|x_i(t_1) - x_i(t_2)\| \leq \max_{1 \leq j \leq n} \|x_i(t_1) - x_j(t_2)\| = \|G(t_1) - G(t_2)\| < C|t_1 - t_2| \qquad \forall\, 1 \leq i \leq n.$$

Similarly, if $t_1 \in I_k$, then for each $t_2 \in I_k$,

$$\|x_i(t_1) - x_i(t_2)\| < C|t_1 - t_2| \qquad \forall\, 1 \leq i \leq n.$$

This shows that $\forall\, t_1 \in I$, there is an open interval I_{t_1} containing t_1 such that for each i,

$$\|x_i(t_1) - x_i(t_2)\| < C|t_1 - t_2| \qquad \forall\, t_2 \in I_{t_1}. \tag{3.2.15}$$

We are now in precisely the same situation that we encountered in Step 2 of the proof of Lemma 3.2.11. Using identical arguments involving (3.2.15) and Lemma 3.2.12, one can show that for $1 \leq i \leq n$ and any fixed $t_1, t_2 \in I$

$$\|x_i(t_1) - x_i(t_2)\| < C|t_1 - t_2|,$$

as desired. $\square$

Now we relax the global Lipschitz assumption to the desired locally Lipschitz assumption:

Lemma 3.2.14. *Let I be an open interval and for each $t \in I$, let $G(t)$ be an unordered n-d tuple of real numbers. If $G(t)$ is locally Lipschitz as an unordered n-d tuple, then there exist locally Lipschitz functions $x_1(t), \dots, x_n(t) : I \to \mathbb{R}^d$ such that $G(t) = \{x_i(t) : 1 \leq i \leq n\}$.*

Proof. Assume $I = (a, b)$, for $a, b \in \mathbb{R}$. The cases $a = -\infty$ and $b = \infty$ can be handled with straightforward modifications of these arguments. By composing $G(t)$ with a dilation, we can also assume $a + 1 \leq b - 1$, which implies $[a + 1, b - 1] \subset I$. Define the intervals:

$$I_k := \left[a + \tfrac{1}{k},\ b - \tfrac{1}{k}\right] \subseteq I \qquad \forall\, m \in \mathbb{N} \setminus \{0\}.$$

Since $G(t)$ is locally Lipschitz on I, $G(t)$ is Lipschitz on every compact subset of I and in particular, on each I_k. By Lemma 3.2.13, there are Lipschitz functions $\{x_i^k(t)\}$ that represent $G(t)$ on each I_k with Lipschitz constant M_k. Now, we glue these functions together in a nice way to obtain locally Lipschitz functions $\{y_i(t)\}$ that represent $G(t)$ on I. First, we recursively define Lipschitz functions $\{y_i^k(t)\}$ on I_k with Lipschitz constant C_k. For the initial case $k = 2$ and for $1 \leq i \leq n$, define

$$y_i^2(t) := x_i^2(t) \qquad \forall\, t \in I_2,$$

and set $C_2 := M_2$. For $k \geq 3$, choose points a_k and b_k such that

$$a_k \in \left(a + \tfrac{1}{k-1}, a + \tfrac{1}{k-2}\right) \quad \text{and} \quad b_k \in \left(b - \tfrac{1}{k-2}, b - \tfrac{1}{k-1}\right).$$

For $r = 1, 2$, let $\{x_i^{k_r}(t)\}$ denote the functions $\{x_i^k(t)\}$ renumbered to ensure

$$x_i^{k_1}(a_k) = y_i^{k-1}(a_k) \quad \text{and} \quad x_i^{k_2}(b_k) = y_i^{k-1}(b_k).$$

Then for $k \geq 3$ and $1 \leq i \leq n$, define $C_k := \max(M_k, C_{k-1})$ and

$$y_i^k(t) := \begin{cases} x_i^{k_1}(t) & t \in (a + \frac{1}{k},\ a_k] \\ y_i^{k-1}(t) & t \in [\ a_k,\ b_k\ ] \\ x_i^{k_2}(t) & t \in [b_k,\ b - \frac{1}{k}). \end{cases}$$

Induction implies that each $y_i^k(t)$ is Lipschitz on I_k with constant C_k. The base case $k = 2$ follows from the assumptions about $x_i^2(t)$. Proceeding by induction, assume the functions $\{y_i^k(t)\}$ are Lipschitz with constant C_k. Then, the definition of $y_i^{k+1}(t)$ makes it clear then each $y_i^{k+1}(t)$ is Lipschitz with constant $C_{k+1} := \max(M_{k+1}, C_k)$ on I_{k+1}. Now for $1 \leq i \leq n$, define

$$y_i(t) := \lim_{k \to \infty} y_i^k(t).$$

To see that each $y_i(t)$ is well-defined, fix any $t_0 \in I$. Then $t_0 \in I_K$ for some $K \geq 2$. By definition,

$$a_{K+2} \in \left(a + \tfrac{1}{K+1}, a + \tfrac{1}{K}\right) \quad \text{and} \quad b_{K+2} \in \left(b - \tfrac{1}{K}, b - \tfrac{1}{K+1}\right).$$

In particular,

$$t_0 \in \big(a_{K+2}, b_{K+2}\big) \subseteq \big(a_k,\ b_k\big), \qquad \forall\ k \geq K+2.$$

Then, there is a neighborhood N_0 of t_0 such that $N_0 \subseteq (a_k,\ b_k)$ for each $k \geq K+2$. Then for each i and $t \in N_0$,

$$\lim_{k \to \infty} y_i^k(t) = y_i^{K+1}(t).$$

Therefore, each $y_i(t)$ is well-defined on N_0 and is Lipschitz there with constant C_{K+1}, and $\{y_i(t)\}$ represents $G(t)$ on N_0. Since t_0 was arbitrary, this shows that each $y_i(t)$ is well-defined and locally Lipschitz on I. $\square$

3.3 Derivatives of Matrix Functions

Recall that every real-valued function defined on an open set $\Omega \subseteq \mathbb{R}^d$ induces a matrix function defined on $CS_n^d(\Omega)$ as in (3.1.2). It is clear that some properties of the original function pass to the matrix function. For example, we have:

Remark 3.3.1. Continuity. If the original function f is continuous, the matrix function F is as well. Specifically, Horn and Johnson proved in [32, pg 387-9] that a one-variable polynomial induces a continuous matrix polynomial. Their arguments generalize easily to multivariate polynomials. Since every continuous function on a compact set can be approximated uniformly by polynomials, it is immediate that matrix functions induced by continuous functions are continuous.

In this section, we consider differentiability and prove:

Theorem 3.3.2. *Let $S(t)$ be a C^1 curve in CS_n^d defined on an open interval I, and let Ω be an open set in $\mathbb{R}^d$ with $\sigma(S(t)) \subset \Omega$. If $f \in C^1(\Omega, \mathbb{R})$, then*

(1) $\frac{d}{dt}F(S(t))|_{t=t^*}$ *exists for all $t^* \in I$.*

(2) *If $T(t)$ is any other C^1 curve in CS_n^d with $\sigma(T(t)) \subset \Omega$, $T(0) = S(t^*)$, and $T'(0) = S'(t^*)$, then*

$$\frac{d}{dt}F(T(t))|_{t=0} = \frac{d}{dt}F(S(t))|_{t=t^*}.$$

In Subsection 3.3.1, we restrict attention to analytic functions and their induced matrix functions. After establishing derivative results for those functions, we consider general matrix functions. Specifically, in Subsection 3.3.2, we prove Theorem 3.3.2, obtain a formula for the derivative $\frac{d}{dt}F(S(t))|_{t=t^*}$, and show the derivative is continuous as a function of t^*. In Subsection 3.3.3, we define a related differential map on the Whitney stratification of CS_n^d and show that this map is also continuous.

3.3.1 Derivatives of Analytic Matrix Functions

Before proving Theorem 3.3.2, we assume f is real-analytic and prove Proposition 3.3.4. See [32] for the one-variable case. We first need some notation.

Definition 3.3.3. An open set $\Omega \subseteq \mathbb{R}^d$ is called a *rectangle* if $\Omega = I^1 \times ... \times I^d$ or more specifically,

$$\Omega = \Big\{(x_1, ..., x_d) : x_r \in I^r \ \forall\ 1 \leq r \leq d\Big\},$$

where each I^r is an open interval in $\mathbb{R}$. An open set $\tilde{\Omega} \subseteq \mathbb{C}^d$ is called a *complex rectangle* if $\tilde{\Omega} = (I^1 + iJ^1) \times ... \times (I^d + iJ^d)$ or specifically,

$$\tilde{\Omega} = \Big\{(x_1 + iy_1, ..., x_d + iy_d) : x_r \in I_r, y_r \in J_r \ \forall\ 1 \leq r \leq d\Big\},$$

where for each r, I^r and J^r are open intervals in $\mathbb{R}$.

Proposition 3.3.4. *Let $S(t)$ be a C^1 curve in CS_n^d defined on an open interval I. Let Ω be a rectangle in $\mathbb{R}^d$ with $\sigma(S(t)) \subset \Omega$. If f is a real-analytic function on Ω, then*

$$\tfrac{d}{dt}F(S(t))|_{t=t^*} \text{ exists and is continuous as a function of } t^* \text{ on } I.$$

The proof of Proposition 3.3.4 requires the following two lemmas.

Lemma 3.3.5. *Let Ω be a rectangle in $\mathbb{R}^d$ and let S be in CS_n^d with $\sigma(S) \subset \Omega$. Each real-analytic function on Ω can be extended to an analytic function defined on a complex rectangle $\tilde{\Omega}$ such that $\sigma(S)$ is in $\tilde{\Omega}$.*

Proof. The proof follows from basic properties of analytic functions. We only consider $d = 1$; the proof for higher dimensions uses the same arguments but requires more complicated notation. Since $d = 1$, Ω is an interval. Let E be a precompact interval with $\overline{E} \subset \Omega$ and $\sigma(S) \subset E$. For each $x \in \Omega$, the basic properties of analytic functions imply that f extends to an analytic function defined on an open rectangle $I_x + iJ_x$ centered at x. Since each J_x is centered at zero, for each pair J_x and J_y, either:

$$J_x \subseteq J_y \text{ or } J_y \subseteq J_x.$$

This gives an ordering on the J_x intervals defined by $J_x \leq J_y$ if $J_x \subseteq J_y$. Because $\overline{E}$ is compact, it is covered by a finite number of intervals $\{I_{x_1}, \ldots I_{x_M}\}$. Define:

$$J_E = \min_{1\leq m\leq M} J_{x_m}.$$

By construction, f extends to an analytic function defined on $\tilde{\Omega} := E + iJ_E$, where $\sigma(S) \subset \tilde{\Omega}$. Notice that $\tilde{\Omega}$ will not contain Ω. $\square$

Lemma 3.3.6. *Let $\tilde{\Omega}$ be a complex rectangle in $\mathbb{C}^d$ and let S be in CS_n^d with $\sigma(S) \subset \tilde{\Omega}$. If f is an analytic function on $\tilde{\Omega}$, then*

$$F(S) = \frac{1}{(2\pi i)^d} \int_{C^d} \cdots \int_{C^1} f(\zeta^1, ..., \zeta^d)(\zeta^1 I - S^1)^{-1} ... (\zeta^d I - S^d)^{-1} \, d\zeta^1 ... d\zeta^d,$$

where each C^r is a simple closed rectifiable curve strictly containing $\sigma(S^r)$, and $C^1 \times ... \times C^d \subset \tilde{\Omega}$.

Proof. Horn and Johnson prove the formula for a one-variable function in [32]. Their proof generalizes as follows. Since $S \in CS_n^d$, there is a unitary matrix U that diagonalizes S as in (3.1.1). It follows immediately that:

$$(\zeta^r I - S^r)^{-1} = U \text{ Diag}\left(\frac{1}{\zeta^r - x_1^r}, \ldots, \frac{1}{\zeta^r - x_n^r}\right) U^* \qquad \forall\ 1 \leq r \leq d, \tag{3.3.1}$$

where the 'Diag' notation means the diagonal matrix with the given values along its diagonal. For $1 \leq r \leq d$, let C^r be any simple closed rectifiable curve strictly containing $\sigma(S^r)$ such that $C^1 \times ... \times C^d \subset \tilde{\Omega}$. Let $\text{Int}(C^r)$ denote the interior of each C^r curve. Since $\tilde{\Omega}$ is a complex rectangle, it follows that $\mathrm{Int}(C^1) \times \cdots \times \mathrm{Int}(C^d) \subset \tilde{\Omega}$ as well. The multivariable Cauchy integral formula and (3.3.1) can then be used to obtain the following sequence of equalities:

$$\frac{1}{(2\pi i)^d} \int_{C^d} \cdots \int_{C^1} f(\zeta^1, \ldots, \zeta^d)(\zeta^1 I - S^1)^{-1} \ldots (\zeta^d I - S^d)^{-1} \, d\zeta^1 \ldots d\zeta^d$$

$$= U \Bigg(\frac{1}{(2\pi i)^d} \int_{C^d} \cdots \int_{C^1} \mathrm{Diag} \left(\frac{f(\zeta^1, \ldots, \zeta^d)}{\Pi_{r=1}^d (\zeta^r - x_1^r)}, \ldots, \frac{f(\zeta^1, \ldots, \zeta^d)}{\Pi_{r=1}^d (\zeta^r - x_n^r)} \right) d\zeta^1 \ldots d\zeta^d \Bigg) U^*$$

$$= U \ \mathrm{Diag}\big(f(x_1^1, \ldots, x_1^d), \ldots, f(x_n^1, \ldots, x_n^d) \big) \ U^*$$

$$= F(S),$$

which gives the desired formula. $\square$

Now we can prove Proposition 3.3.4 as follows:

Proof. For ease of notation, assume $d = 2$. With more complicated notation, the same arguments work in higher dimensions. For $r = 1, 2$, define

$$R^r(t) := (\zeta^r I - S^r(t))^{-1},$$

where ζ^r is in the resolvent set of $S^r(t)$. Fix $t_0 \in I$ and using Lemma 3.3.5, extend f to an analytic function on a complex rectangle $\tilde{\Omega}$ containing $\sigma(S(t_0))$. Choose simple closed rectifiable curves C^1 and C^2 such that $C^1 \times C^2 \subset \tilde{\Omega}$ and C^r strictly encloses the eigenvalues of $S^r(t_0)$. By Theorem 3.2.8, the joint eigenvalues of $S(t)$ are continuous and so, C^r strictly encloses the eigenvalues of $S^r(t)$ for t close to t_0 and $r = 1, 2$. Thus, Lemma 3.3.6 implies:

$$F(S(t)) = \frac{1}{(2\pi i)^2} \int_{C^2} \int_{C^1} f(\zeta^1, \zeta^2) \ R^1(t) \ R^2(t) \ d\zeta^1 d\zeta^2,$$

for t sufficiently close to t_0. For t_1, t_2 near t_0, we also have

$$R^r(t_1) - R^r(t_2) = R^r(t_1)\left(S^r(t_1) - S^r(t_2)\right)R^r(t_2), \tag{3.3.2}$$

which implies $R^r(t)$ is differentiable near t_0 and direct calculation gives

$$\tfrac{d}{dt}R^r(t)|_{t=t^*} = R^r(t^*)(S^r)'(t^*)R^r(t^*),$$

for $r = 1, 2$ and t^* near t_0. It can be easily shown that, for t^* sufficiently close to t_0, we can interchange integration and differentiation to yield

$$\begin{aligned}
\tfrac{d}{dt}F(S(t))|_{t=t^*} &= \frac{1}{(2\pi i)^2}\int_{C_2}\int_{C_1} f(\zeta^1, \zeta^2)\tfrac{d}{dt}\big(R^1(t)R^2(t)\big)|_{t=t^*}\, d\zeta^1 d\zeta^2 \\
&= \frac{1}{(2\pi i)^2}\int_{C_2}\int_{C_1} f(\zeta^1, \zeta^2)\Bigg(R^1(t^*)(S^1)'(t^*)R^1(t^*)R^2(t^*) \\
&\quad + R^1(t^*)R^2(t^*)(S^2)'(t^*)R^2(t^*)\Bigg)d\zeta^1 d\zeta^2.
\end{aligned} \tag{3.3.3}$$

Observe that each $(S^r)'(t)$ is continuous in t and by (3.3.2), each $R^r(t)$ is continuous in t near t_0, uniformly in ζ for $\zeta \in C^1 \times C^2$. Thus, as $f(\zeta^1, \zeta^2)$ is uniformly bounded on $C^1 \times C^2$, we can conclude $\frac{d}{dt}F(S(t))|_{t=t^*}$ is continuous for t^* near t_0. □

3.3.2 Derivatives of General Matrix Functions

In this section, we prove Theorem 3.3.2, obtain a formula for the derivative of an induced matrix function along a curve $S(t)$, and show that such derivatives are continuous. We begin with the proof of Theorem 3.3.2:

Proof. Observe that the theorem holds for polynomials: (1) follows from Proposition 3.3.4, and (2) follows from the formula in (3.3.3). Fix $t^* \in I$. Let $f \in C^1(\Omega, \mathbb{R})$ and and let p be a polynomial that agrees with f to first order on $\sigma(S(t^*))$. By Theorem 3.2.8, there are locally Lipschitz maps $x_i(t) := (x_i^1(t), ..., x_i^d(t))$, for $1 \leq i \leq n$, representing $\sigma(S(t))$ on I. For t sufficiently close to t^*, we can use the multivariate mean value theorem to conclude

$$\begin{aligned}
\|(F-P)(S(t))\| &= \max_i |(f-p)(x_i(t))| \\
&= \max_i \big|(f-p)(x_i(t)) - (f-p)(x_i(t^*))\big| \\
&= \max_i \big|\nabla(f-p)(x_i^*(t)) \cdot (x_i(t) - x_i(t^*))\ \big| \\
&\leq \max_i \sum_{r=1}^d \big|\left(\tfrac{\partial f}{\partial x^r} - \tfrac{\partial p}{\partial x^r}\right)(x_i^*(t))\big|\big|x_i^r(t) - x_i^r(t^*)\big|, \qquad (3.3.4)
\end{aligned}$$

where $x_i^*(t)$ is on the line connecting $x_i(t)$ and $x_i(t^*)$ in $\mathbb{R}^d$ and is obtained from the multivariate mean value theorem. The theorem can be applied because by continuity, there is a convex set $U \subseteq \Omega$ such that $x_i(t^*)$, $x_i(t) \in U$, for t sufficiently close to t^*. As f and p agree to first order on $\sigma(S(t^*))$ and the $x_i(t)$ are locally Lipschitz, (3.3.4) implies

$$\|(F-P)(S(t))\| = o(|t-t^*|).$$

Hence, as $F(S(t^*)) = P(S(t^*))$, we have:

$$\Big\|\frac{F(S(t)) - F(S(t^*))}{t-t^*} - \frac{P(S(t)) - P(S(t^*))}{t-t^*}\Big\| \to 0,$$

as $t \to t^*$. Therefore,

$$\tfrac{d}{dt}F(S(t))|_{t=t^*} \text{ exists and equals } \tfrac{d}{dt}P(S(t))|_{t=t^*}.$$

Applying the same argument to $F(T(t))$ at $t = 0$ gives

$$\tfrac{d}{dt}F(T(t))|_{t=0} \text{ exists and equals } \tfrac{d}{dt}P(T(t))|_{t=0}.$$

As (2) holds for $P(t)$, we must have $\frac{d}{dt}F(T(t))|_{t=0} = \frac{d}{dt}F(S(t))|_{t=t^*}$. $\square$

In the following proposition, we calculate an explicit formula for the derivative.

Proposition 3.3.7. *Let $S(t)$ be a C^1 curve in CS_n^d defined on an open interval I, and let $t^* \in I$. Let Ω be an open set in $\mathbb{R}^d$ with $\sigma(S(t)) \subset \Omega$ and let $f \in C^1(\Omega, \mathbb{R})$. Then,*

$$\tfrac{d}{dt}F(S(t))|_{t=t*} = U\Bigg(\sum_{r=1}^{d} \tilde{\Gamma}^r \tfrac{\partial F}{\partial x^r}(D) + [Y, F(D)]\Bigg)U^*,$$

where U diagonalizes $S(t^*)$ as in (3.2.3), $\frac{\partial F}{\partial x^r}(D)$ is defined in (3.3.6), and the other matrices are as follows:

$$D^r := U^* S^r(t^*) U \qquad\qquad \Gamma^r := U^*(S^r)'(t^*)U$$

$$\tilde{\Gamma}^r_{ij} := \begin{cases} \Gamma^r_{ij} & \textit{if } x_i = x_j \\ 0 & \textit{otherwise} \end{cases} \qquad Y_{ij} := \begin{cases} \frac{\Gamma^q_{ij}}{x^q_j - x^q_i} & \textit{if } x_i \neq x_j \\ 0 & \textit{otherwise,} \end{cases}$$

where the joint eigenvalues of $S(t^*)$ are given by $\big\{x_i = (x_i^1, ..., x_i^d) : 1 \le i \le n\big\}$ and each q is chosen so $x_j^q - x_i^q \neq 0$.

Proof. Let $t^* \in I$ and define the C^1 curve $T(t)$ by

$$T^r(t) := U\ e^{Yt}\big[D^r + t\tilde{\Gamma}^r\big]e^{-Yt}\ U^*,$$

for $1 \le r \le d$. Then, $T(t)$ is the curve defined in the proof of Theorem 3.2.5 for $S := S(t^*)$ and $\Delta := S'(t^*)$. It is immediate that $T(t) \subset CS_n^d$, $T(0) = S(t^*)$, and $T'(0) = S'(t^*)$. By restricting the domain of $T(t)$ to a neighborhood of $t = 0$, we can assume $\sigma(T(t)) \subset \Omega$. By

Theorem 3.3.2, it now suffices to calculate $\frac{d}{dt}F(T(t))|_{t=0}$. The first step is to diagonalize each $D^r + t\tilde{\Gamma}^r$. Let p be the number of distinct joint eigenvalues of $S(t^*)$. By definition,

$$\tilde{\Gamma}^r = \begin{pmatrix} \Gamma_1^r & & \\ & \ddots & \\ & & \Gamma_p^r \end{pmatrix},$$

for $1 \le r \le d$, where each Γ_l^r is a $k_l \times k_l$ self-adjoint matrix corresponding to a distinct joint eigenvalue of S with multiplicity k_l. It follows from Theorem 3.2.5 that

$$\left[\tilde{\Gamma}^r, \tilde{\Gamma}^s\right] = 0, \text{ which implies: } \left[\Gamma_l^r, \Gamma_l^s\right] = 0,$$

for $1 \le r, s \le d$ and $1 \le l \le p$. Thus, for each l, there is a $k_l \times k_l$ unitary matrix V_l such that V_l diagonalizes Γ_l^r for each $1 \le r \le d$. Let V be the $n \times n$ block diagonal matrix with blocks given by $V_1, ..., V_p$. Then, V is a unitary matrix that diagonalizes each $\tilde{\Gamma}^r$. By the diagonalization in (3.2.3), the joint eigenvalues of D are positioned so that

$$D^r = \begin{pmatrix} c_1^r I_{k_1} & & \\ & \ddots & \\ & & c_p^r I_{k_p} \end{pmatrix}, \tag{3.3.5}$$

for $1 \le r \le d$, where I_{k_l} is the $k_l \times k_l$ identity matrix and each c_l^r is a constant. Equation (3.3.5) shows that V and V^* will commute with D^r. Define the diagonal matrix

$$\Lambda^r := V^* \tilde{\Gamma}^r V,$$

for $1 \le r \le d$ and rewrite $T(t)$ as follows:

$$T^r(t) = Ue^{Yt}V\left(D^r + t\Lambda^r\right)V^*e^{-Yt}U^*,$$

for $1 \le r \le d$. Directly calculate $F(T(t))$ and $\frac{d}{dt}F(T(t))|_{t=0}$ as follows:

$$F(T(t)) = Ue^{Yt}V\ F\left(D^1 + t\Lambda^1, ..., D^d + t\Lambda^d\right)\ V^* e^{-Yt}U^*$$
$$= Ue^{Yt}V\ \left(F(D) + t\sum_{r=1}^{d} \Lambda^r \tfrac{\partial F}{\partial x^r}(D) + o(|t|)\right) V^* e^{-Yt}U^*,$$

where $\frac{\partial F}{\partial x^r}(D)$ is defined by

$$\tfrac{\partial F}{\partial x^r}(D) := \begin{pmatrix} \frac{\partial f}{\partial x^r}(x_1) & & \\ & \ddots & \\ & & \frac{\partial f}{\partial x^r}(x_n) \end{pmatrix}, \tag{3.3.6}$$

for $1 \le r \le d$ and the first-order approximation of F follows from the first-order approximation of f. Differentiating $F(T(t))$ and setting $t = 0$ gives

$$\tfrac{d}{dt}F(T(t))|_{t=0} = U\Big(\sum_{r=1}^{d} V\ \Lambda^r \tfrac{\partial F}{\partial x^r}(D)V^* + [Y, VF(D)V^*]\Big)U^*$$
$$= U\Big(\sum_{r=1}^{d}\ \tilde{\Gamma}^r \tfrac{\partial F}{\partial x^r}(D) + [Y, F(D)]\Big)U^*,$$

where V and V^* commute with $F(D)$ and each $\frac{\partial F}{\partial x^r}(D)$ because those matrices have decompositions akin to that of D^r in (3.3.5). □

We now prove that the derivative calculated in Proposition 3.3.7 is continuous in t^*.

Theorem 3.3.8. *Let $S(t)$ be a C^1 curve in CS_n^d defined on an open interval I. Let Ω be an open set in $\mathbb{R}^d$ with $\sigma(S(t)) \subset \Omega$. If $f \in C^1(\Omega, \mathbb{R})$, then*

$$\tfrac{d}{dt}F(S(t))|_{t=t^*} \textit{ is continuous as a function of } t^* \textit{ on } I.$$

For the proof, we will require the following lemma:

Lemma 3.3.9. *Let $S(t)$ be a C^1 curve in CS_n^d defined on an open interval I. Let Ω be an open, convex set in $\mathbb{R}^d$ with $\sigma(S(t)) \subset \Omega$. Fix $t_0 \in I$. Then there is a neighborhood $I_0 \subseteq I$ of t_0, a constant C, and a convex, bounded open set E with $\bar{E} \subset \Omega$ such that*

$$\|\tfrac{d}{dt}F(S(t))|_{t=t^*}\| \leq C \max_{1\leq s\leq d; x\in\bar{E}} \left|\tfrac{\partial f}{\partial x^s}(x)\right|,$$

for all $f \in C^1(\Omega, \mathbb{R})$ and $t^* \in I_0$.

Proof. Let $t_0 \in I$ and fix a bounded interval I_0 around t_0 with $\bar{I_0} \subset I$. By Theorem 3.2.8, the joint eigenvalues of $S(t)$ can be represented by continuous functions $x_i(t) = (x_i^1(t), \ldots, x_i^d(t))$ for $1 \leq i \leq n$ on I. Thus, there exists an open bounded convex set $E \subset \mathbb{R}^d$ such that $\bar{E} \subset \Omega$ and for each $t^* \in I_0$, the joint spectrum $\sigma(S(t^*)) = \{x_i(t^*) : 1 \leq i \leq n\} \subset E$. Fix $t^* \in I_0$ and $f \in C^1(\Omega, \mathbb{R})$. Then by Proposition 3.3.7,

$$\tfrac{d}{dt}F(S(t))|_{t=t*} = U\Big(\sum_{r=1}^{d} \tilde{\Gamma}^r \tfrac{\partial F}{\partial x^r}(D) + [Y, F(D)]\Big)U^*, \tag{3.3.7}$$

where U, D^r, $\tilde{\Gamma}^r$, and Y are functions of t^* defined in Proposition 3.3.7, and the joint eigenvalues of $S(t^*)$ are denoted by x_i, for $1 \leq i \leq n$. Observe that the matrix in (3.3.7) can be rewritten as

$$\Big[\sum_{r=1}^{d} \tilde{\Gamma}^r \tfrac{\partial F}{\partial x^r}(D) + [Y, F(D)]\Big]_{ij} = \begin{cases} \sum_{r=1}^{d} \Gamma_{ij}^r \frac{\partial f}{\partial x^r}(x_i) & \text{if } x_i = x_j \\ \\ \Gamma_{ij}^q \frac{f(x_i)-f(x_j)}{x_i^q - x_j^q} & \text{if } x_i \neq x_j, \end{cases} \tag{3.3.8}$$

where q is such that $x_i^q \neq x_j^q$, and $\Gamma_{ij}^q/(x_i^q - x_j^q)$ is the same for any q with $x_i^q \neq x_j^q$. Recall that for a given $n \times n$ self-adjoint matrix A and an $n \times n$ unitary matrix U,

$$\max_{ij} |(UAU^*)_{ij}| \leq n\|UAU^*\| = n\|A\| \leq n^2 \max_{ij} |A_{ij}|. \tag{3.3.9}$$

It is immediate from (3.3.7), (3.3.8), and (3.3.9) that

$$\left|\left|\tfrac{d}{dt}F(S(t))\right|_{t=t^*}\right|\right| \leq n\max\left|\sum_{r=1}^{d}\Gamma_{ij}^{r}\tfrac{\partial f}{\partial x^r}(x_i)\right| + n\max\left|\Gamma_{ij}^{q}\frac{f(x_i)-f(x_j)}{x_i^q-x_j^q}\right|, \tag{3.3.10}$$

where the first maximum is taken over (i,j) with $x_i = x_j$, the second maximum is taken over (i,j) with $x_i \neq x_j$, and q is such that $x_i^q \neq x_j^q$. Fix (i,j) with $x_i \neq x_j$. Since $f \in C^1(E,\mathbb{R})$ and E is convex, the multivariate mean value theorem can be applied as follows:

$$\begin{aligned}\left|f(x_i)-f(x_j)\right| &= \left|\nabla f(x^*)\cdot(x_i-x_j)\right| \\ &\leq \max_{s;x\in\bar{E}}\left|\tfrac{\partial f}{\partial x^s}(x)\right|\sum_{r=1}^{d}\left|x_i^r-x_j^r\right|, \end{aligned}\tag{3.3.11}$$

where x^* is on the line in E connecting x_i and x_j. If $x_i^q \neq x_j^q$, then for each r with $x_i^r \neq x_j^r$,

$$\Gamma_{ij}^{q}\frac{x_i^r-x_j^r}{x_i^q-x_j^q} = \Gamma_{ij}^{r}.$$

It follows from (3.3.11) that, for each (i,j,q) with $x_i^q \neq x_j^q$,

$$\begin{aligned}\left|\Gamma_{ij}^{q}\frac{f(x_i)-f(x_j)}{x_i^q-x_j^q}\right| &\leq \left|\tfrac{\Gamma_{ij}^{q}}{x_i^q-x_j^q}\right|\max_{s;x\in\bar{E}}\left|\tfrac{\partial f}{\partial x^s}(x)\right|\sum_{r=1}^{d}\left|x_i^r-x_j^r\right| \\ &\leq \max_{s;x\in\bar{E}}\left|\tfrac{\partial f}{\partial x^s}(x)\right|\sum_{r=1}^{d}\left|\Gamma_{ij}^{r}\right| \\ &\leq dn^2\max_{s;x\in\bar{E}}\left|\tfrac{\partial f}{\partial x^s}(x)\right|\max_{i,j,r}\left|(S^r)'(t^*)_{ij}\right|, \end{aligned}\tag{3.3.12}$$

where we used (3.3.9). Likewise,

$$\left|\sum_{r=1}^{d}\Gamma_{ij}^{r}\tfrac{\partial f}{\partial x^r}(x_i)\right| \leq dn^2\max_{s;x\in\bar{E}}\left|\tfrac{\partial f}{\partial x^s}(x)\right|\max_{i,j,r}\left|(S^r)'(t^*)_{ij}\right|. \tag{3.3.13}$$

Let M be a constant bounding each $|(S^r)'(t^*)_{ij}|$ on $\bar{I}_0$ and let $C = 2dn^3M$. Substituting

(3.3.12) and (3.3.13) into (3.3.10) gives

$$\left|\left|\tfrac{d}{dt}F(S(t))\big|_{t=t^*}\right|\right| \leq 2dn^3 \max_{s;x\in\bar{E}} \left|\tfrac{\partial f}{\partial x^s}(x)\right| \max_{i,j,r} \left|(S^r)'(t^*)_{ij}\right| \leq C \max_{s;x\in\bar{E}} \left|\tfrac{\partial f}{\partial x^s}(x)\right|, \tag{3.3.14}$$

for all t^* in I_0. □

To prove Theorem 3.3.8, we need the following generalization of the Stone-Weierstrass Theorem, which is proved on page 55 of [29]:

Lemma 3.3.10. ***Stone-Weierstrass Generalization.*** *Let $\Omega \subseteq \mathbb{R}^d$ be an open set and let $f \in C^m(\Omega, \mathbb{R})$. Let $K \subset \Omega$ be compact. Then there exists a sequence $\{\phi_k\}$ of real-analytic functions on $\mathbb{R}^d$ such that*

$$\Big|\phi_k(x) - f(x)\Big| < \tfrac{1}{k} \quad \textit{and} \quad \Big|\Big(\tfrac{\partial^{l_1}\cdots\partial^{l_N}}{(\partial x^{r_1})^{l_1}\cdots(\partial x^{r_N})^{l_N}}\phi_k\Big)(x) - \Big(\tfrac{\partial^{l_1}\cdots\partial^{l_N}}{(\partial x^{r_1})^{l_1}\cdots(\partial x^{r_N})^{l_N}}f\Big)(x)\Big| < \tfrac{1}{k},$$

for all $k \in \mathbb{N}\setminus\{0\}$, $x \in K$, $1 \leq l_1 + \cdots + l_N \leq M$, and $1 \leq r_1, \ldots, r_N \leq d$.

Now we can prove Theorem 3.3.8:

Proof. First assume Ω is convex. Let $t_0 \in I$. Let I_0 be the interval around t_0 and E be the convex, bounded open set given in Lemma 3.3.9. Since f is a C^1 function and $\bar{E}$ is compact, Lemma 3.3.10 guarantees a sequence $\{\phi_k\}$ of functions analytic on $\mathbb{R}^d$ such that

$$|\phi_k(x) - f(x)| < \tfrac{1}{k} \text{ and } \left|\tfrac{\partial\phi_k}{\partial x^r}(x) - \tfrac{\partial f}{\partial x^r}(x)\right| < \tfrac{1}{k},$$

for all $k \in \mathbb{N}\setminus\{0\}$, $x \in \bar{E}$, and $1 \leq r \leq d$. Lemma 3.3.9 guarantees that for each $t^* \in I_0$,

$$\begin{aligned}\left|\left|\tfrac{d}{dt}\Phi_k(S(t))\big|_{t=t^*} - \tfrac{d}{dt}F(S(t))\big|_{t=t^*}\right|\right| &= \left|\left|\tfrac{d}{dt}(\Phi_k - F)(S(t))\big|_{t=t^*}\right|\right| \\ &\leq C \max_{1\leq s\leq d;\ x\in\bar{E}} \left|\tfrac{\partial(\phi_k - f)}{\partial x^s}(x)\right| \\ &\leq \tfrac{C}{k},\end{aligned}$$

where C is a constant given in Lemma 3.3.9. This implies

$$\left\{\frac{d}{dt}\Phi_k(S(t))\Big|_{t=t^*}\right\} \text{ converges uniformly to } \frac{d}{dt}F(S(t))\Big|_{t=t^*} \text{ on } I_0.$$

By Proposition 3.3.4, each $\frac{d}{dt}\Phi_k(S(t))|_{t=t^*}$ is continuous on I. Since the uniform limit of continuous functions is continuous, $\frac{d}{dt}F(S(t))|_{t=t^*}$ is continuous on I_0. Since $t_0 \in I$ was arbitrary, the result follows.

Now, let $\Omega \subseteq \mathbb{R}^d$ be an arbitrary open set. Fix $t_0 \in I$ and let I_0 be a bounded open interval of t_0 with $\bar{I}_0 \subset I$. Let $E \subset \mathbb{R}^d$ be a bounded open set such that $\bar{E} \subset \Omega$ and $\sigma(S(t^*)) \subset E$ for all $t^* \in I_0$. Let O be an open set and K be a compact set such that $\bar{E} \subset O \subset K \subset \Omega$ and define a C^∞ bump function $b(x)$ on $\mathbb{R}^d$ such that

$$b(x) := \begin{cases} 1 & \text{if } x \in \bar{E} \\ 0 & \text{if } x \in O^c. \end{cases}$$

Now define $g \in C^1(\mathbb{R}^d, \mathbb{R})$ by

$$g(x) := \begin{cases} b(x)f(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega^c. \end{cases}$$

It is clear that g is C^1 on Ω. To see that g is C^1 on Ω^c, observe that $g \equiv 0$ on K^c, which is an open set containing Ω^c. As $\mathbb{R}^d$ is convex, it follows from the previous result that $\frac{d}{dt}G(S(t))|_{t=t^*}$ is continuous on I_0. Since $f \equiv g$ on $\bar{E}$, it follows from the formula in Proposition 3.3.7 that

$$\frac{d}{dt}F(S(t))|_{t=t^*} = \frac{d}{dt}G(S(t))|_{t=t^*}$$

for all $t^* \in I_0$, and thus, is continuous in I_0. Since $t_0 \in I$ was arbitrary, the result follows. $\square$

3.3.3 Differential Maps of Matrix Functions

Recall that CS_n^d can be viewed as a closed subset of $\mathbb{R}^m$ for $m = dn^2$, and possesses a Whitney stratification with pieces $\{M_\alpha\}$ that are smooth submanifolds of $\mathbb{R}^m$. Let Ω be an open set in $\mathbb{R}^d$ and let $f \in C^1(\Omega, \mathbb{R})$. Let V be an open set in CS_n^d such that for all $S \in V$, $\sigma(S) \subset \Omega$. Then, each $M_\alpha \cap V$ can be viewed as a smooth submanifold of $\mathbb{R}^m$. Define $TV := \cup T(M_\alpha \cap V)$. Then, $F(S)$ exists for all $S \in V$, and we can use the derivative results to define a differential map $DF : TV \to TS_n$:

Definition 3.3.11. Fix an element in TV, which will consist of an $S \in V$ and $\Delta \in T_S M_\alpha$, where M_α is the piece containing S. Let $S(t)$ be a smooth curve in M_α such that $S(0) = S$ and $S'(0) = \Delta$. Define

$$DF(S,\Delta) := \left(F(S), \tfrac{d}{dt}F(S(t))|_{t=0} \right) = \left(F(S), U\Big(\sum_{r=1}^{d} \tilde{\Gamma}^r \tfrac{\partial F}{\partial x^r}(D) + [Y, F(D)] \Big) U^* \right),$$

where U, D, $\tilde{\Gamma}^r$, Y, and $\frac{\partial F}{\partial x^r}(D)$ are defined using S and Δ as in Proposition 3.3.7, and set

$$\|DF(S,\Delta)\| = \max\left(\|F(S)\|, \|\tfrac{d}{dt}F(S(t))|_{t=0}\| \right).$$

It is easy to see that the map is well-defined and that the second component of $DF(S,\cdot)$ is linear in Δ, for $\Delta \in T_S M_\alpha$. Specifically, assume Δ_1, Δ_2, and $\Delta_1 + \Delta_2 \in T_S M_\alpha$. Then there exist C^1 curves, $S_1(t), S_2(t), S_{12}(t) \subset CS_n^d$ satisfying $S_1(0) = S_2(0) = S_{12}(0) = S$ and $S_1'(0) = \Delta_1$, $S_2'(0) = \Delta_2$, and $S_{12}'(0) = \Delta_1 + \Delta_2$. Then, the formula for derivatives along curves implies

$$\tfrac{d}{dt}F(S_1(t))|_{t=0} + \tfrac{d}{dt}F(S_2(t))|_{t=0} = \tfrac{d}{dt}F(S_{12}(t))|_{t=0}.$$

In the following theorem, let S be in a piece M_α and let R be in a piece M_β of a Whitney stratification of CS_n^d.

Theorem 3.3.12. *Let Ω be an open set in $\mathbb{R}^d$ and V be an open set in CS_n^d with $\sigma(S) \subset \Omega$*

for all $S \in V$. If $f \in C^1(\Omega, \mathbb{R})$, then

$$DF : TV \to TS_n \text{ is continuous.}$$

Specifically, if $S \in V$ with $\Delta \in T_S M_\alpha$, then given $\epsilon > 0$, there exist δ_1, $\delta_2 > 0$ such that if $R \in V$ with $\Lambda \in T_R M_\beta$, $\|S - R\| < \delta_1$, and $\|\Delta - \Lambda\| < \delta_2$, then

$$\|DF(S, \Delta) - DF(R, \Lambda)\| < \epsilon.$$

Proof. First, let $d = 2$ and let f be a real-analytic function defined on a rectangle $\Omega \subseteq \mathbb{R}^2$. The argument for higher dimensions is similar but requires more complicated notation. Fix $S \in V$ so that $\sigma(S) \subset \Omega$, and extend f to be analytic on a complex rectangle $\tilde{\Omega}$ with $\sigma(S) \subset \tilde{\Omega}$. Then, (3.3.3) implies that for all $R = (R^1, R^2) \in V$ sufficiently close to S and $\Lambda = (\Lambda^1, \Lambda^2) \in T_R M_\beta$, the second component of $DF(R, \Lambda)$ equals:

$$\frac{1}{(2\pi i)^2} \int_{C_2} \int_{C_1} f(\zeta^1, \zeta^2) \Big((\zeta^1 I - R^1)^{-1} \Lambda^1 (\zeta^1 I - R^1)^{-1} (\zeta^2 I - R^2)^{-1}$$

$$+ (\zeta^1 I - R^1)^{-1} (\zeta^2 I - R^2)^{-1} \Lambda^2 (\zeta^2 I - R^2)^{-1} \Big) d\zeta^1 d\zeta^2,$$

where each C^r is a simple closed rectifiable curve strictly containing $\sigma(S^r)$ and $C^1 \times C^2 \subset \tilde{\Omega}$. This equation, coupled with the fact that the matrix function F defined using f is a continuous matrix function, immediately implies the continuity conclusion for F.

Now, assume $\Omega \subseteq \mathbb{R}^d$ is convex. Let E be a bounded, convex set with $\bar{E} \subseteq \Omega$ and $\sigma(S) \subset E$. Then (3.3.14) and the arguments used to obtain it imply that for R with $\sigma(R) \subset E$ and $\Lambda \in T_R M_\beta$:

$$\|DG(R, \Lambda)\| \leq \max \left(\max_{x \in \bar{E}} |g(x)|,\ 2dn^3 \max_{\substack{1 \leq s \leq d \\ x \in \bar{E}}} \left| \tfrac{\partial g}{\partial x^s}(x) \right| \max_{\substack{1 \leq i,j \leq n \\ 1 \leq r \leq d}} \left| \Lambda^r_{ij} \right| \right). \tag{3.3.15}$$

for every $g \in C^1(\Omega, \mathbb{R})$. Fix a particular $f \in C^1(\Omega, \mathbb{R})$. As in the proof of Theorem 3.3.8, we can approximate f uniformly to first order on $\bar{E}$ by a sequence $\{\phi_k\}$ of real-analytic functions on $\mathbb{R}^d$. Observe that

$$\begin{aligned}\|DF(S,\Delta) - DF(R,\Lambda)\| \leq \|DF(S,\Delta) - D\Phi_k(S,\Delta)\| + \|D\Phi_k(S,\Delta) - D\Phi_k(R,\Lambda)\| \\ + \|D\Phi_k(R,\Lambda) - DF(R,\Lambda)\|.\end{aligned}$$

Using (3.3.15) and the continuity result for each Φ_k, we can obtain the continuity result for F. To relax the convexity condition, use arguments identical to those in the proof of Theorem 3.3.8. $\square$

3.4 Higher-Order Derivatives of Matrix Functions

We now consider higher-order differentiation and for ease of notation, discuss only two-variable functions. In this section, we show that a matrix function also inherits higher-order derivatives from the original real-valued function. Specifically, in Subsection 3.4.1 we obtain higher-order derivative results for matrix functions induced from analytic functions. In Subsection 3.4.2, we show that a C^m function always induces a matrix function that can be m-times continuously differentiated along C^m curves and obtain formulas for the derivatives.

We first clarify some notation. In earlier sections, $(\zeta^1, ..., \zeta^d)$ referred to a point in $\mathbb{C}^d$. In this section, (ζ_1, ζ_2) denotes a point in $\mathbb{C}^2$. Previously, $S(t)$ and $T(t)$ denoted two separate curves in CS_n^d. Now, $S(t)$ and $T(t)$ denote the two components of a single curve in CS_n^2. Let $(S(t), T(t))$ be a C^m curve in CS_n^2 defined on an interval I. If $m \geq 1$, the curve is locally Lipschitz. By Theorem 3.2.8, for $1 \leq s \leq n$, there are locally Lipschitz curves

$$(x_s(t), y_s(t)) \tag{3.4.1}$$

defined on I representing the joint eigenvalues of $(S(t), T(t))$. Let $U(t)$ be a unitary matrix diagonalizing $(S(t), T(t))$ so that the joint eigenvalues are ordered as in (3.4.1). To simplify notation, we write $(S(t), T(t))$ as (S, T). For $l \in \mathbb{N}$ with $1 \leq l \leq m$, define

$$S^l := S^{(l)}(t) \text{ and } T^l := T^{(l)}(t) \tag{3.4.2}$$

and the set of pairs of index tuples

$$I_l := \{(i_1, ..., i_k) \cup (i_{k+1}, ..., i_j) : i_1 + ... + i_j = l, i_q \in \mathbb{N}, i_q \neq 0, \text{ for } 1 \leq q \leq j\}.$$

For example, $I_2 = \{(2) \cup \emptyset,\ (1,1) \cup \emptyset,\ (1) \cup (1),\ \emptyset \cup (1,1),\ \emptyset \cup (2)\}$. For notational ease, for

$1 \leq s \leq n$, define

$$U := U(t),$$

$$x_s := x_s(t),$$

$$y_s := y_s(t).$$

For some formulas, we will conjugate the derivatives in (3.4.2) by U^* and so define

$$\Gamma^l := U^* S^l U \quad \text{and} \quad \Delta^l := U^* T^l U,$$

for $1 \leq l \leq m$. We will use the integral formula given in Lemma 3.3.6 and simplify it by defining

$$R_1 := (\zeta_1 I - S(t))^{-1} \quad \text{and} \quad R_2 := (\zeta_2 I - T(t))^{-1},$$

where ζ_1 and ζ_2 are in the resolvent sets of $S(t)$ and $T(t)$ respectively. Now, let J_1 and J_2 be open intervals in $\mathbb{R}$ and let f be an element of $C^m(J_1 \times J_2, \mathbb{R})$. Fix j and k in $\mathbb{N}$ such that $k \leq j \leq m$. Fix $k+1$ points $x_1, ..., x_{k+1}$ in J_1 and $j-k+1$ points $y_1, ..., y_{j-k+1}$ in J_2. Then

$$f^{[k,j-k]}(x_1, ..., x_{k+1}; y_1, ..., y_{j-k+1})$$

denotes the divided difference of f taken in the first variable k times and the second variable $j-k$ times, evaluated at the given points. For clarity, we include the following definition:

Definition 3.4.1. Divided Differences. Let J_1 and J_2 be open intervals in $\mathbb{R}$, and let $f \in C^m(J_1 \times J_2, \mathbb{R})$. The *divided differences* of f, which are denoted $f^{[k,j-k]}$, can be defined whenever $j, k \in \mathbb{N}$ with $k \leq j \leq m$. First, fix $x_1, x_2 \in J_1$ and $y_1, y_2 \in J_2$. Define

$$f(x, y_1)^{[1,0]}(x_1, x_2) = f^{[1,0]}(x_1, x_2; y_1) := \begin{cases} \frac{f(x_1,y_1) - f(x_2,y_1)}{x_1 - x_2} & x_1 \neq x_2 \\ f_x(x_1, y_1) & x_1 = x_2, \end{cases}$$

and similarly define $f^{[0,1]}(x_1;y_1,y_2) = f(x_1,y)^{[0,1]}(y_1,y_2)$. Higher-order divided differences are defined inductively using the formula:

$$f^{[k+1,j-k]}(x_1,..,x_{k+2};y_1,..,y_{j-k+1}) := \left(f^{[k,j-k]}(x_1,..,x_k,x;y_1,..,y_{j-k+1})\right)^{[1,0]}(x_{k+1},x_{k+2})$$
$$f^{[k,j-k+1]}(x_1,..,x_{k+1};y_1,..,y_{j-k+2}) := \left(f^{[k,j-k]}(x_1,..,x_{k+1},x;y_1,..,y_{j-k},y)\right)^{[0,1]}(y_{j-k+1},y_{j-k+2})$$

This definition is well-defined because the order in which divided differences are taken in each variable does not change the value of the final divided difference. One-variable divided differences appear frequently in the literature. For an overview, see [26, 54]. In these books, integral and summation formulas for one-variable divided differences $f^{[k]}(x_1,\dots,x_{k+1})$ are proved. The formulas generalize immediately to two variables and imply the following facts:

(1) Fix $j,k \in \mathbb{N}$ with $k \le j \le m$. Then, the function $f^{[k,j-k]}(x_1,...,x_{k+1};y_1,...,y_{j-k+1})$ exists and is continuous in the variables $x_1,...,x_{k+1},y_1,...,y_{j-k+1}$ on $J_1^{k+1} \times J_2^{j-k+1}$.

(2) Fix $j,k \in \mathbb{N}$ with $k \le j \le m$. Fix $k+1$ points $x_1,...,x_{k+1}$ in J_1 and $j-k+1$ points $y_1,...,y_{j-k+1}$ in J_2. Then, the value of $f^{[k,j-k]}(x_1,...,x_{k+1};y_1,...,y_{j-k+1})$ is independent of the order of the x_q's and y_r's.

(3) Fix $j,k \in \mathbb{N}$ with $k \le j \le m$. Fix $k+1$ points $x_1,...,x_{k+1}$ in J_1 and $j-k+1$ points $y_1,...,y_{j-k+1}$ in J_2. Then, the value of $f^{[k,j-k]}(x_1,...,x_{k+1};y_1,...,y_{j-k+1})$ depends only on the partial derivatives of f up to order k in the first variable and $j-k$ in the second variable evaluated on the set $\{(x_q,y_r) : 1 \le q \le k+1, 1 \le r \le j-k+1\}$.

Finally, let $\odot$ denote the Schur or Hadamard product of two matrices. In this section, we prove the following differentiability result:

Theorem 3.4.2. *Let J_1 and J_2 be open intervals in $\mathbb{R}$, and let $f \in C^m(J_1 \times J_2, \mathbb{R})$. Let (S,T) be a C^m curve in CS_n^2 defined on an open interval I with joint eigenvalues in $J_1 \times J_2$.*

For $1 \le l \le m$ and $t^* \in I$, $\frac{d^l}{dt^l} F(S,T)|_{t=t^*}$ exists and

$$\frac{d^l}{dt^l} F(S,T)\Big|_{t=t^*} = U\left(\sum_{I_l} \sum_{s_2,..,s_j=1}^{n} \frac{l!}{i_1! \cdots i_j!} \left[f^{[k,j-k]}(x_{s_1},..,x_{s_{k+1}}; y_{s_{k+1}},..,y_{s_{j+1}})\right]_{s_1,s_{j+1}=1}^{n} \odot \left[\Gamma^{i_1}_{s_1 s_2} ... \Gamma^{i_k}_{s_k s_{k+1}} \Delta^{i_{k+1}}_{s_{k+1} s_{k+2}} ... \Delta^{i_j}_{s_j s_{j+1}}\right]_{s_1,s_{j+1}=1}^{n}\right) U^*, \tag{3.4.3}$$

where the U, U^*, Γ^i, Δ^j, x_q and y_r are evaluated at t^*.

Notice that the derivative formula in Theorem 3.4.2 requires f to be defined on pairs (x_q, y_r) for $1 \le r, q \le n$, rather than just at the joint eigenvalues (x_q, y_q) of (S,T). This condition was not needed in Theorem 3.3.2.

3.4.1 Higher-Order Derivatives of Analytic Matrix Functions

Before proving Theorem 3.4.2, we consider the case where f is real-analytic and show:

Proposition 3.4.3. *Let J_1 and J_2 be open intervals in $\mathbb{R}$, and let f be real-analytic on $J_1 \times J_2$. Fix $m \in \mathbb{N}$ and let (S,T) be a C^m curve in CS_n^2 defined on an open interval I with joint eigenvalues in $J_1 \times J_2$. Then $\frac{d^m}{dt^m} F(S,T)$ exists, has the form in Theorem 3.4.2, and $\frac{d^m}{dt^m} F(S,T)|_{t=t^*}$ is continuous as a function of t^* on I.*

The proof of Proposition 3.4.3 requires the following two technical lemmas:

Lemma 3.4.4. *Let (S,T) be a C^m curve in CS_n^2 defined on an open interval I. Let $t^* \in I$, and let ζ_1 and ζ_2 be in the resolvent sets of $S(t^*)$ and $T(t^*)$ respectively. Then*

$$\frac{d^l}{dt^l}\Big(R_1 R_2\Big)\Big|_{t=t^*} = \sum_{I_l} \frac{l!}{i_1! \cdots i_j!} R_1 S^{i_1} R_1 ... S^{i_k} R_1 R_2 T^{i_{k+1}} R_2 ... T^{i_j} R_2, \tag{3.4.4}$$

for $1 \le l \le m$, where each R_1, R_2, S^r, and T^q is evaluated at t^*.

Proof. The result follows via induction on l. Recall from Proposition 3.3.4 that:

$$\frac{d}{dt}R_1 = R_1 S^1 R_1 \quad \text{and} \quad \frac{d}{dt}R_2 = R_2 T^1 R_2.$$

Now consider the base case $l = 1$. Direct calculation yields:

$$\begin{aligned} \frac{d}{dt}\big(R_1 R_2\big) &= \frac{d}{dt}(R_1)R_2 + R_1\frac{d}{dt}(R_2) \\ &= R_1 S^1 R_1 R_2 + R_1 R_2 T^1 R_2, \end{aligned}$$

which shows (3.4.4) holds for the case $l = 1$, since $I_1 = \{(1) \cup \emptyset,\ \emptyset \cup (1)\}$. Now assume (3.4.4) is true for $l - 1$. Then:

$$\frac{d^l}{dt^l}\big(R_1 R_2\big) = \frac{d}{dt}\left(\sum_{I_{(l-1)}} \frac{(l-1)!}{i_1! \cdots i_j!} R_1 S^{i_1} R_1 \dots S^{i_k} R_1 R_2 T^{i_{k+1}} R_2 \dots T^{i_j} R_2\right)$$

$$= \sum_{I_{(l-1)}} \frac{(l-1)!}{i_1! \cdots i_j!}\ \frac{d}{dt}\big(R_1 S^{i_1} R_1 \dots S^{i_k} R_1 R_2 T^{i_{k+1}} R_2 \dots T^{i_j} R_2\big). \tag{3.4.5}$$

Take the derivative of each term in (3.4.5) using the product rule. Recall that taking the derivative of an R_1 or R_2 term introduces an $R_1 S^1 R_1$ or $R_2 T^1 R_2$ into the product and taking the derivative of an S^{i_q} or T^{i_q} yields an S^{i_q+1} or T^{i_q+1}. Thus, it is clear that (3.4.5) will be a sum of the form:

$$\sum_{I_l} C\big((i_1, \dots, i_k) \cup (i_{k+1}, \dots, i_j)\big)\ R_1 S^{i_1} R_1 \dots S^{i_k} R_1 R_2 T^{i_{k+1}} R_2 \dots T^{i_j} R_2.$$

To calculate a formula for the coefficients, fix an an element $(i_1, \dots, i_k) \cup (i_{k+1}, \dots, i_j)$ in I_l and consider the associated product:

$$R_1 S^{i_1} R_1 \dots S^{i_k} R_1 R_2 T^{i_{k+1}} R_2 \dots T^{i_j} R_2. \tag{3.4.6}$$

To see which terms in (3.4.5) will have (3.4.6) in their derivative, consider q with $1 \leq q \leq k \leq j$ and define the following element in I_{l-1} :

$$T_q := \begin{cases} (i_1, \ldots, i_{q-1}, i_{q+1}, \ldots, i_k) \cup (i_{k+1}, \ldots, i_j) & \text{if } i_q = 1 \\ (i_1, \ldots, i_{q-1}, i_q - 1, i_{q+1}, \ldots, i_k) \cup (i_{k+1}, \ldots, i_j) & \text{if } i_q > 1. \end{cases}$$

Analogous T_q elements in I_{l-1} can be defined for q with $1 \leq k \leq q \leq j$. It is easily to see that for each q, the term in (3.4.5) associated with the element T_q will have (3.4.6) in its derivative. Moreover, since taking the derivative of a term in (3.4.5) raises the index associated with exactly one S or T term by 1, those are the only products in (3.4.5) with (3.4.6) in their derivatives. Thus, summing the constants associated with each T_q term will yield $C((i_1, \ldots, i_k) \cup (i_{k+1}, \ldots, i_j))$. Specifically:

$$\begin{aligned} C\big((i_1, \ldots, i_k) \cup (i_{k+1}, \ldots, i_j)\big) &= \sum_{q=1}^{j} C(T_q) \\ &= \sum_{q=1}^{j} \frac{(l-1)!}{i_1! \cdots (i_q - 1)! \cdots i_j!} \\ &= (l-1)! \sum_{q=1}^{j} \frac{i_q}{i_1! \cdots i_j!} \\ &= \frac{(l-1)!}{i_1! \cdots i_j!} \sum_{q=1}^{j} i_q \\ &= \frac{l!}{i_1! \cdots i_j!}. \end{aligned}$$

Therefore,

$$\frac{d^l}{dt^l}\big(R_1 R_2\big) = \sum_{I_l} \frac{l!}{i_1! \cdots i_j!} R_1 S^{i_1} R_1 \ldots S^{i_k} R_1 R_2 T^{i_{k+1}} R_2 \ldots T^{i_j} R_2,$$

as desired. $\square$

Lemma 3.4.5. *Let J_1 and J_2 be open intervals in $\mathbb{R}$, and let f be real-analytic on $J_1 \times J_2$.*

Let $j \geq k \in \mathbb{N}$. Choose $k+1$ points $x_1, ..., x_{k+1} \in J_1$ and $j-k+1$ points $y_1, ..., y_{j-k+1} \in J_2$. Extend f to be analytic on a complex rectangle $\tilde{\Omega} \subset \mathbb{C}^2$ such that each $(x_q, y_r) \in \tilde{\Omega}$. Then $f^{[k,j-k]}(x_1, ..., x_{k+1}; y_1, ..., y_{j-k+1})$ exists and

$$f^{[k,j-k]}(x_1, .., x_{k+1}; y_1, .., y_{j-k+1}) = \frac{1}{(2\pi i)^2} \int_{C_2} \int_{C_1} \frac{f(\zeta_1, \zeta_2)}{\prod_{q=1}^{k+1}(\zeta_1 - x_q) \prod_{r=1}^{j-k+1}(\zeta_2 - y_r)} d\zeta_1 d\zeta_2,$$

where C_1 and C_2 are simple closed rectifiable curves strictly enclosing the points $x_1, ..., x_{k+1}$ and $y_1, ..., y_{j-k+1}$ respectively, such that $C_1 \times C_2 \subset \tilde{\Omega}$.

Proof. Since f is analytic, the divided difference $f^{[k,j-k]}$ exists. For a one-variable function, this formula is proven in [26] on page 2. The two-variable analogue follows from the one-variable case. The definition of the divided difference operator coupled with the one-variable result yields:

$$f^{[k,j-k]}(x_1, \ldots, x_{k+1}; y_1, \ldots, y_{j-k+1}) = \left(f^{[k,0]}(x_1, \ldots, x_{k+1}; y)\right)^{[j-k]}(y_1, \ldots, y_{j-k+1})$$

$$= \frac{1}{2\pi i} \int_{C_2} \frac{f^{[k,0]}(x_1, \ldots, x_{k+1}; \zeta_2)}{\prod_{r=1}^{j-k+1}(\zeta_2 - y_r)} d\zeta_2$$

$$= \frac{1}{(2\pi i)^2} \int_{C_2} \int_{C_1} \frac{f(\zeta_1, \zeta_2)}{\prod_{q=1}^{k+1}(\zeta_1 - x_q) \prod_{r=1}^{j-k+1}(\zeta_2 - y_r)} \, d\zeta_1 d\zeta_2,$$

as desired. □

Using these lemmas, we can now prove Proposition 3.4.3:

Proof. Fix $t^* \in I$, and extend f to an analytic function defined on a complex rectangle $\tilde{\Omega}$ containing the joint eigenvalues of $\big(S(t^*), T(t^*)\big)$. Choose simple closed rectifiable curves C_1 and C_2 strictly containing the eigenvalues of $S(t^*)$ and $T(t^*)$ respectively, such that

$C_1 \times C_2 \subset \tilde{\Omega}$. From Lemma 3.3.6,

$$F(S,T) = \frac{1}{(2\pi i)^2} \int_{C_2} \int_{C_1} f(\zeta_1, \zeta_2)\ R_1 R_2\ d\zeta_1 d\zeta_2,$$

for all t sufficiently close to t^*. As in Proposition 3.3.4, we can interchange differentiation and integration and then use Lemma 3.4.4 to obtain:

$$\frac{d^m}{dt^m} F(S,T) = \frac{1}{(2\pi i)^2} \int_{C_2} \int_{C_1} f(\zeta_1, \zeta_2)\ \frac{d^m}{dt^m}\big(R_1 R_2\big)\ d\zeta_1 d\zeta_2$$

$$= \frac{1}{(2\pi i)^2} \int_{C_2} \int_{C_1} f(\zeta_1, \zeta_2) \Bigg(\sum_{I_m} \frac{m!}{i_1! \cdots i_j!} R_1 S^{i_1} R_1 \ldots S^{i_k} R_1 R_2 T^{i_{k+1}} R_2 \ldots T^{i_j} R_2 \Bigg)\ d\zeta_1 d\zeta_2. \tag{3.4.7}$$

As in Proposition 3.3.4, this formula immediately implies that the derivatives are continuous as functions of t^*. Now we simplify (3.4.7). An easy calculation gives:

$$R_1 = U\left(\sum_{s=1}^{n} \frac{E_s}{\zeta_1 - x_s}\right) U^* \quad \text{and} \quad R_2 = U\left(\sum_{s=1}^{n} \frac{E_s}{\zeta_2 - y_s}\right) U^*, \tag{3.4.8}$$

where E_s is the matrix with 1 in the ss^{th} entry and zeros elsewhere. Recall the definitions of Γ^l and Δ^l for $1 \leq l \leq m$. Now, substituting (3.4.8) for each R_1 and R_2 in (3.4.7) and using Lemma 3.4.5 yields:

$$\frac{d^m}{dt^m} F(S,T) = \frac{1}{(2\pi i)^2} \int_{C_2} \int_{C_1} f(\zeta_1, \zeta_2)\ U$$
$$\cdot \left(\sum_{I_m} \sum_{s_1..s_{j+1}=1}^{n} \frac{m!}{i_1! \cdot\cdot i_j!} \frac{E_{s_1} \Gamma^{i_1} E_{s_2} . . \Gamma^{i_k} E_{s_{k+1}} \Delta^{i_{k+1}} E_{s_{k+2}} . . \Delta^{i_j} E_{s_{j+1}}}{\prod_{q=1}^{k+1} (\zeta_1 - x_{s_q}) \prod_{r=k+1}^{j+1} (\zeta_2 - y_{s_r})} \right) U^*\ d\zeta_1 d\zeta_2$$

$$= U\Bigg(\sum_{I_m}\sum_{s_1..s_{j+1}=1}^{n}\frac{m!}{i_1!\cdots i_j!}E_{s_1}\Gamma^{i_1}E_{s_2}\ldots\Gamma^{i_k}E_{s_{k+1}}\Delta^{i_{k+1}}E_{s_{k+2}}\ldots\Delta^{i_j}E_{s_{j+1}}$$

$$\cdot\ \frac{1}{(2\pi i)^2}\int_{C_2}\int_{C_1}\frac{f(\zeta_1,\zeta_2)}{\prod_{q=1}^{k+1}\ (\zeta_1-x_{s_q})\ \prod_{r=k+1}^{j+1}\ (\zeta_2-y_{s_r})}\ d\zeta_1 d\zeta_2\Bigg)U^*$$

$$= U\Bigg(\sum_{I_m}\sum_{s_1..s_{j+1}=1}^{n}\frac{m!}{i_1!\cdots i_j!}E_{s_1}\Gamma^{i_1}E_{s_2}\ldots\Gamma^{i_k}E_{s_{k+1}}\Delta^{i_{k+1}}E_{s_{k+2}}\ldots\Delta^{i_j}E_{s_{j+1}}$$

$$\cdot\ f^{[k,j-k]}\left(x_{s_1},\ldots,x_{s_{k+1}};y_{s_{k+1}},\ldots,y_{s_{j+1}}\right)\Bigg)U^*.$$

Direct calculation using the definition of the E_s matrices gives:

$$[E_{s_1}\Gamma^{i_1}E_{s_2}\ldots\Gamma^{i_k}E_{s_{k+1}}\Delta^{i_{k+1}}E_{s_{k+2}}\ldots\Delta^{i_j}E_{s_{j+1}}]_{qr}$$

$$= \begin{cases} \Gamma^{i_1}_{s_1s_2}\ldots\Gamma^{i_k}_{s_ks_{k+1}}\Delta^{i_{k+1}}_{s_{k+1}s_{k+2}}\ldots\Delta^{i_j}_{s_js_{j+1}} & \text{if } q=s_1 \text{ and } r=s_{j+1} \\ \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the $(s_1s_{j+1})^{th}$ entry is a product of entries of Γ and Δ matrices and the other entries are zero. Thus,

$$\frac{d^m}{dt^m}F(S,T) = U\Bigg(\sum_{I_m}\sum_{s_1..s_{j+1}=1}^{n}\frac{m!}{i_1!\cdots i_j!}E_{s_1}\Gamma^{i_1}E_{s_2}\ldots\Gamma^{i_k}E_{s_{k+1}}\Delta^{i_{k+1}}E_{s_{k+2}}\ldots\Delta^{i_j}E_{s_{j+1}}$$

$$\cdot\ f^{[k,j-k]}\left(x_{s_1},\ldots,x_{s_{k+1}};y_{s_{k+1}},\ldots,y_{s_{j+1}}\right)\Bigg)U^*$$

$$= U\Bigg(\sum_{I_m}\sum_{s_2..s_j=1}^{n}\frac{m!}{i_1!\cdots i_j!}\left[f^{[k,j-k]}(x_{s_1},\ldots,x_{s_{k+1}};y_{s_{k+1}},\ldots,y_{s_{j+1}})\right]_{s_1,s_{j+1}=1}^{n}$$

$$\odot\ \left[\Gamma^{i_1}_{s_1s_2}\ldots\Gamma^{i_k}_{s_ks_{k+1}}\Delta^{i_{k+1}}_{s_{k+1}s_{k+2}}\ldots\Delta^{i_j}_{s_js_{j+1}}\right]_{s_1,s_{j+1}=1}^{n}\Bigg)U^*,$$

for all t in I sufficiently close to t^*. $\square$

3.4.2 Higher-Order Derivatives of General Matrix Functions

To prove the general derivative result, we need the following technical lemma:

Lemma 3.4.6. *Let J_1 and J_2 be open intervals in $\mathbb{R}$ and let $f \in C^m(J_1 \times J_2, \mathbb{R})$. Let $k,\ j \in \mathbb{N}$ such that $0 \leq k \leq j < m$. Then,*

$$f^{[k,j-k]}(x_1, \ldots, x_{k+1}; y_1, \ldots, y_{j-k+1})$$

is a continuously differentiable function defined on $J_1^{k+1} \times J_2^{j-k+1}$.

Proof. First, the definition of the divided difference operator implies that $f^{[k,j-k]}$ is well-defined and continuous on $J_1^{k+1} \times J_2^{j-k+1}$. Now, fix $q \in \mathbb{N}$ such that $1 \leq q \leq k+1$ and calculate the partial derivative of $f^{[k,j-k]}$ with respect to the variable x_q. It follows from the properties of the divided difference operator that:

$$\begin{aligned}
&\frac{\partial}{\partial x_q} f^{[k,j-k]}(x_1, \ldots, x_{k+1}; y_1, \ldots, y_{j-k+1}) \\
&= \lim_{h \to 0} \frac{f^{[k,j-k]}(x_1, .., x_q + h, .., x_{k+1}; y_1, .., y_{j-k+1}) - f^{[k,j-k]}(x_1, .., x_q, .., x_{k+1}; y_1, .., y_{j-k+1})}{h} \\
&= \lim_{h \to 0} f^{[k+1,j-k]}(x_1, \ldots, x_q + h, x_q, \ldots, x_{k+1}; y_1, \ldots, y_{j-k+1}) \\
&= f^{[k+1,j-k]}(x_1, \ldots, x_q, x_q, \ldots, x_{k+1}; y_1, \ldots, y_{j-k+1}).
\end{aligned} \tag{3.4.9}$$

Likewise, given $r \in \mathbb{N}$ with $1 \leq r \leq j-k+1$, direct calculation gives:

$$\frac{\partial}{\partial y_r} f^{[k,j-k]}(x_1, \ldots, x_{k+1}; y_1, \ldots, y_{j-k+1}) = f^{[k,j-k+1]}(x_1, \ldots, x_{k+1}; y_1, \ldots, y_r, y_r, \ldots y_{j-k+1}). \tag{3.4.10}$$

Because $j+1 \leq m$, the divided differences $f^{[k+1,j-k]}$, and $f^{[k,j-k+1]}$ are continuous in each variable. Thus, the derivative formulas (3.4.9) and (3.4.10) are continuous as well. □

We are now in a position to prove Theorem 3.4.2:

Proof. The arguments in this proof are very similar to those in Theorem 3.3.2. For the interested reader, we include the details.

First, Proposition 3.4.3 established the result for analytic functions. For an arbitrary C^m function f, the result follows via induction on l. Fix $t^* \in I$. Let p be a polynomial defined on $\mathbb{R}^2$ such that p and its partial derivatives to m^{th} order agree with f at the points $(x_q(t^*), y_r(t^*))$ for $1 \leq q, r \leq n$. This implies:

$$p^{[k,j-k]}(x_{s_1}, \ldots, x_{s_{k+1}}; y_{s_{k+1}}, \ldots, y_{s_{j+1}})|_{t=t^*} = f^{[k,j-k]}(x_{s_1}, \ldots, x_{s_{k+1}}; y_{s_{k+1}}, \ldots, y_{s_{j+1}})|_{t=t^*},$$

for $k, j \in \mathbb{N}$ with $0 \leq k \leq j \leq m$ and $1 \leq s_1, \ldots, s_{j+1} \leq n$. Thus, the right-hand-side of (3.4.3) evaluated at t^* is the same for f and p. The proof of Theorem 3.3.2 immediately implies that:

$$\tfrac{d}{dt}F(S,T)|_{t=t^*} \text{ exists and equals } \tfrac{d}{dt}P(S,T)|_{t=t^*}.$$

Since (3.4.3) is true for p, it also holds for f when $l = 1$ and $t = t^*$. Since t^* is arbitrary, the result follows for the base case $l = 1$.

Proceeding by induction, assume the result holds for $l - 1$. Fix $t^* \in I$ and define p as before. It follows that

$$\tfrac{d^{l-1}}{dt^{l-1}}F(S,T)|_{t=t^*} \text{ exists and equals } \tfrac{d^{l-1}}{dt^{l-1}}P(S,T)|_{t=t^*}.$$

We will show:

$$\tfrac{d^l}{dt^l}F(S,T)|_{t=t^*} \text{ exists and equals } \tfrac{d^l}{dt^l}P(S,T)|_{t=t^*}.$$

Let I^* be a precompact neighborhood of t^* with $\overline{I^*} \subset I$. For $t \in I^*$, we use the inductive hypothesis and (3.4.3) to obtain:

$$\left|\left|\tfrac{d^{l-1}}{dt^{l-1}}F(S,T) - \tfrac{d^{l-1}}{dt^{l-1}}P(S,T)\right|\right| \leq C \ \max\left|(f-p)^{[k,j-k]}(x_{s_1},\ldots,x_{s_{k+1}};y_{s_{k+1}},\ldots,y_{s_{j+1}})\right|, \tag{3.4.11}$$

where the maximum is taken over $j,k \in \mathbb{N}$ with $0 \leq k \leq j \leq l-1$ and the set $\{s_1,\ldots,s_{j+1} : 1 \leq s_1,\ldots,s_{j+1} \leq n\}$ The constant C depends on n and the values of (S,T) and their derivatives to $(l-1)^{th}$ order on I^*. By Lemma 3.4.6, the function:

$$(f-p)^{[k,j-k]}(x_1,\ldots,x_{k+1};y_1,\ldots,y_{j-k+1})$$

is continuously differentiable. For t near t^*, the multivariable mean value theorem can then be used to conclude that:

$$\begin{aligned}\left|\left|\tfrac{d^{l-1}}{dt^{l-1}}F(S,T) - \tfrac{d^{l-1}}{dt^{l-1}}P(S,T)\right|\right| \leq C \max\Bigg(&\Big|(f-p)^{[k,j-k]}(x_{s_1},\ldots,x_{s_{k+1}};y_{s_{k+1}},\ldots,y_{s_{j+1}}) \\ &- (f-p)^{[k,j-k]}(x_{s_1},\ldots,x_{s_{k+1}};y_{s_{k+1}},\ldots,y_{s_{j+1}})|_{t=t^*} \ \Big|\Bigg)\end{aligned}$$

$$\begin{aligned}&\leq C \ \max\Bigg(\sum_{q=1}^{k+1}\Big|\tfrac{\partial(f-p)^{[k,j-k]}}{\partial x_q}(x^*_{s_1},\ldots,x^*_{s_{k+1}};y^*_{s_{k+1}},\ldots,y^*_{s_{j+1}})\Big|\cdot\left|x_{s_q}-x_{s_q}(t^*)\right| \\ &+ \sum_{r=1}^{j-k+1}\Big|\tfrac{\partial(f-p)^{[k,j-k]}}{\partial y_r}(x^*_{s_1},\ldots,x^*_{s_{k+1}};y^*_{s_{k+1}},\ldots,y^*_{s_{j+1}})\Big|\cdot\left|y_{s_{r+k}}-y_{s_{r+k}}(t^*)\right|\Bigg),\end{aligned} \tag{3.4.12}$$

where $(x^*_{s_1},\ldots,x^*_{s_{k+1}};y^*_{s_{k+1}},\ldots,y^*_{s_{j+1}})$ is on the line in $J_1^{k+1}\times J_2^{j-k+1}$ connecting the points:

$$(x_{s_1},\ldots,x_{s_{k+1}},y_{s_{k+1}},\ldots,y_{s_{j+1}}) \ \text{ and } \ (x_{s_1},\ldots,x_{s_{k+1}},y_{s_{k+1}},\ldots,y_{s_{j+1}})|_{t=t^*}.$$

Recall that the functions x_{s_q} and y_{s_r} are locally Lipschitz. Furthermore, the derivative formulas in Lemma 3.4.6 and our assumptions about p imply that the functions

$$\frac{\partial (f-p)^{[k,j-k]}}{\partial x_q} \quad \text{and} \quad \frac{\partial (f-p)^{[k,j-k]}}{\partial y_r}$$

are continuous and equal zero at $(x_{s_1}, \ldots, x_{s_{k+1}}, y_{s_{k+1}}, \ldots y_{s_{j+1}})|_{t=t^*}$. Thus, (3.4.12) implies:

$$\left|\left| \tfrac{d^{l-1}}{dt^{l-1}} F(S,T) - \tfrac{d^{l-1}}{dt^{l-1}} P(S,T) \right|\right| = o(\ |t - t^*|\ ).$$

It follows immediately that:

$$\left|\left| \frac{\frac{d^{l-1}}{dt^{l-1}} F(S,T) - \frac{d^{l-1}}{dt^{l-1}} F(S,T)|_{t=t^*}}{t - t^*} - \frac{\frac{d^{l-1}}{dt^{l-1}} P(S,T) - \frac{d^{l-1}}{dt^{l-1}} P(S,T)|_{t=t^*}}{t - t^*} \right|\right| \to 0 \ \text{ when } \ t \to t^*.$$

Thus:

$$\tfrac{d^l}{dt^l} F(S,T)|_{t=t^*} \text{ exists and equals } \tfrac{d^l}{dt^l} P(S,T)|_{t=t^*}.$$

Because t^* was arbitrary, the result holds all $t \in I$. $\square$

We now show that the formula in Theorem 3.4.2 is continuous.

Theorem 3.4.7. *Let J_1 and J_2 be open intervals in $\mathbb{R}$ and $f \in C^m(J_1 \times J_2, \mathbb{R})$. Let (S,T) be a C^m curve in CS_n^2 defined on an open interval I with joint eigenvalues in $J_1 \times J_2$. Then for all $l \in \mathbb{N}$ with $1 \leq l \leq m$,*

$$\tfrac{d^l}{dt^l} F(S,T)|_{t=t^*} \textit{ is continuous as a function of } t^* \textit{ on } I.$$

For the proof, we require the following lemma. The result is well-known for one-variable functions, and Brown and Vasudeva prove this two-variable analogue in [22]. For clarity, we include the proof.

Lemma 3.4.8. *Let J_1 and J_2 be open intervals in $\mathbb{R}$, and let $f \in C^m(J_1 \times J_2, \mathbb{R})$. Choose $j, k \in \mathbb{N}$ with $k \leq j \leq m$. Let $x_1, ..., x_{k+1} \in J_1$ and $y_1, ..., y_{j-k+1} \in J_2$, and choose closed* *subintervals $\tilde{J}_1$ and $\tilde{J}_2$ containing the x and y points respectively. Then, there exists $(x^*, y^*) \in \tilde{J}_1 \times \tilde{J}_2$ with*

$$f^{[k,j-k]}(x_1, ..., x_{k+1}; y_1, ..., y_{j-k+1}) = \frac{f^{(k,j-k)}(x^*, y^*)}{k!(j-k)!}.$$

Proof. This follows from the analogous one-variable result, which is proved in [26]. Using that result, there exists $x^* \in \tilde{J}_1$ and $y^* \in \tilde{J}_2$ such that:

$$\begin{aligned}
f^{[k,j-k]}(x_1, \ldots, x_{k+1}; y_1, \ldots, y_{j-k+1}) &= (f^{[k,0]}(x_1, \ldots, x_{k+1}; y))^{[j-k]}(y_1, \ldots, y_{j-k+1}) \\
&= \tfrac{1}{(j-k)!} \tfrac{\partial^{j-k}}{\partial y^{j-k}} f^{[k,0]}(x_1, \ldots, x_{k+1}; y)\Big|_{y=y^*} \\
&= \tfrac{1}{(j-k)!} \left(\tfrac{\partial^{j-k} f}{\partial y^{j-k}}\Big|_{y=y^*} \right)^{[k]} (x_1, \ldots, x_{k+1}) \\
&= \tfrac{1}{(j-k)!} \tfrac{1}{k!} \tfrac{\partial^k}{\partial x^k} \left(\tfrac{\partial^{j-k} f}{\partial y^{j-k}}\Big|_{y=y^*} \right)\Big|_{x=x^*} \\
&= \tfrac{f^{(k,j-k)}(x^*,y^*)}{k!(j-k)!},
\end{aligned}$$

which follows because the divided difference operator in the first variable commutes with the partial derivative taken in the second variable. □

Now we can prove Theorem 3.4.7:

Proof. For $l < m$, the result follows from Theorem 3.4.2, which implies that $\frac{d^l}{dt^l} F(S,T)$ is differentiable and hence, continuous.

For $l = m$, fix $t_0 \in I$ and let I_0 be a precompact neighborhood of t_0 with $\bar{I}_0 \subset I$. Let $\tilde{J}_1$ and $\tilde{J}_2$ be closed, bounded subintervals of J_1 and J_2 so that the joint eigenvalues of $(S(t^*), T(t^*))$ are in $J := \tilde{J}_1 \times \tilde{J}_2$ for $t^* \in I_0$. Using Theorem 3.4.2 and Lemma 3.4.8, for $t^* \in I_0$, we have:

$$\begin{aligned}
\left|\left| \tfrac{d^l}{dt^l} G(S,T)|_{t=t^*} \right|\right| &\leq C_1 \max_{\substack{1 \leq k \leq j \leq m \\ 1 \leq s_1 .. s_{j+1} \leq n}} |g^{[k,j-k]}(x_{s_1}, \ldots, x_{s_{k+1}}; y_{s_{k+1}}, \ldots, y_{s_{j+1}})|_{t=t^*}| \\
&\leq C \max_{\substack{1 \leq k \leq j \leq m \\ (x,y) \in J}} |g^{(k,j-k)}(x,y)|, \qquad (3.4.13)
\end{aligned}$$

where $g \in C^m(J_1 \times J_2, \mathbb{R})$ is arbitrary, and C is a constant depending on n and the values of (S, T) and their derivatives to m^{th} order on I_0. Now, let $f \in C^m(J_1, \times J_2, \mathbb{R})$. As in the proof of Theorem 3.3.8, use Lemma 3.3.10 to approximate f to m^{th} order uniformly on J by analytic functions $\{\phi_r\}$ and use (3.4.13) to show

$$\{\tfrac{d^m}{dt^m}\Phi_r(S,T)|_{t=t^*}\} \text{ converges uniformly to } \tfrac{d^m}{dt^m}F(S,T)|_{t=t^*}$$

for t^* in a neighborhood of t_0. Now, the desired result follows from the continuity part of Proposition 3.4.3. $\square$

3.5 Application: Monotone and Convex Multivariate Matrix Functions

The formulas in Proposition 3.3.7 and Theorem 3.4.2 can be used to analyze monotonicity and convexity of matrix functions defined from real-valued functions.

Definition 3.5.1. Let $\Omega \subseteq \mathbb{R}$ be open, and let $f \in C^1(\Omega, \mathbb{R})$. Then, the induced matrix-valued function $F : S_n(\Omega) \to S_n$ is called *n-matrix monotone* on Ω if

$$F(A) \leq F(B) \text{ whenever } A \leq B, \quad \forall\, A, B \in S_n(\Omega).$$

If Ω is connected, an equivalent condition is

$$\tfrac{d}{dt} F(S(t))|_{t=t^*} \geq 0 \text{ whenever } S'(t^*) \geq 0, \quad \forall\, C^1 \text{ curves } S(t) \subset S_n(\Omega). \tag{3.5.1}$$

The local monotonicity condition in (3.5.1) extends immediately to multivariate matrix functions; the only adjustment is that $S(t)$ is in CS_n^d. However, in several variables, it is not known whether the global and local monotonicity conditions are equivalent.

In [8], Agler, McCarthy, and Young characterized locally matrix monotone functions on CS_n^d using a special case of Theorem 3.3.2 and Proposition 3.3.7. Specifically, they had to assume that $S(t)$ had distinct joint eigenvalues at each t. Our results in Section 3.3 extend the derivative formula to general C^1 curves in CS_n^d and show that the resultant derivative formula is continuous.

Definition 3.5.2. Let $\Omega \subseteq \mathbb{R}$ be open, and let $f \in C^1(\Omega, \mathbb{R})$. Then, the induced matrix-valued function $F : S_n(\Omega) \to S_n$ is called *n-matrix convex* on Ω if

$$F(\lambda A + (1-\lambda)B) \leq \lambda F(A) + (1-\lambda)F(B) \quad \forall\, A, B \in S_n(\Omega) \text{ and } \lambda \in [0,1]. \tag{3.5.2}$$

This condition extends to multivariate matrix functions with an additional restriction on the pairs A, B in $CS_n^d(\Omega)$; we also require $\lambda A + (1-\lambda)B \in CS_n^d(\Omega)$ for $\lambda \in (0,1)$. Given such A, B, define the curve $S(t)$ on $[0,1]$ by

$$S^r(t) := tA^r + (1-t)B^r, \tag{3.5.3}$$

for $1 \le r \le d$. If F is twice continuously differentiable along C^2 curves, it can be shown that the multivariate generalization of (3.5.2) is equivalent to

$$\tfrac{d^2}{dt^2} F(S(t))|_{t=t^*} \ge 0$$

for all $S(t)$ as in (3.5.3) and $t^* \in (0,1)$.

For $d = 2$, Theorem 3.4.2 tells us that, up to conjugation by a unitary matrix U diagonalizing $S(t^*)$,

$$\begin{aligned}\left[\tfrac{d^2}{dt^2} F(S(t))|_{t=t^*}\right]_{ij} &= 2\sum_{k=1}^{n} f^{[2,0]}(x_i, x_k, x_j; y_j)\Gamma_{ik}\Gamma_{kj} + f^{[1,1]}(x_i, x_k; y_k, y_j)\Gamma_{ik}\Delta_{kj} \\ &\quad + f^{[0,2]}(x_i; y_i, y_k, y_j)\Delta_{ik}\Delta_{kj},\end{aligned} \tag{3.5.4}$$

where $\{(x_i, y_i) : 1 \le i \le n\}$ are the joint eigenvalues of $t^*A + (1-t^*)B$ ordered as in the diagonalization given by U, and

$$\Gamma := U^*\big(A^1 - B^1\big)U \quad \text{and} \quad \Delta := U^*\big(A^2 - B^2\big)U.$$

Theorem 3.2.5 can be used to obtain that $(x_i - x_j)\Delta_{ij} = (y_i - y_j)\Gamma_{ij}$ for $1 \le i, j \le n$, which further simplifies (3.5.4). It then seems possible that characterizing the positivity of (3.5.4) would give a useful characterization of convex matrix functions on CS_n^2.

Bibliography

[1] J. Agler. Some interpolation theorems of Nevanlinna-Pick type. Preprint, 1988.

[2] J. Agler. On the representation of certain holomorphic functions defined on a polydisc. In *Topics in operator theory: Ernst D. Hellinger memorial volume*, volume 48 of *Oper. Theory Adv. Appl.*, pages 47–66. Birkhäuser Verlag, Basel, 1990.

[3] J. Agler and J.E. McCarthy. Interpolating sequences on the bidisk. *Internat. J. Math.*, 12(9):1103–1114, 2001.

[4] J. Agler and J.E. McCarthy. *Pick Interpolation and Hilbert Function Spaces.* American Mathematical Society, Providence, 2002.

[5] J. Agler and J.E. McCarthy. Distinguished varieties. *Acta Math.*, 194:133–153, 2005.

[6] J. Agler and J.E. McCarthy. What can Hilbert spaces tell us about bounded functions in the bidisk? In *A glimpse at Hilbert space operators*, volume 207 of *Oper. Theory Adv. Appl.*, pages 81–97. Birkhäuser Verlag, Basel, 2010.

[7] J. Agler, J.E. McCarthy, and M. Stankus. Toral algebraic sets and function theory on polydisks. *J. Geom. Anal.*, 16(4):551–562, 2006.

[8] J. Agler, J.E. McCarthy, and N. J. Young. Operator monotone functions and Loewner functions of several variables. *Ann. of Math. (2)*, 176(3):1783–1826, 2012.

[9] J. Agler, J.E. McCarthy, and N.J. Young. A Carathéodory theorem for the bidisk via Hilbert space methods. *Math. Ann.*, 352(3):581–624, 2012.

[10] D. Alpay. *The Schur Algorithm and Reproducing Kernel Spaces and System Theory.* American Mathematical Society, Providence, RI, 2001.

[11] D. Alpay, V. Bolotnikov, A. Dijksma, and C. Sadosky. Hilbert spaces contractively included in the Hardy space of the bidisk. *Positivity*, 5:25–50, 2001.

[12] J.M. Anderson, M. Dritschel, and J. Rovnyak. Schwarz-Pick inequalities for the Schur-Agler class on the polydisk and unit ball. *Comput. Methods Funct. Theory*, 8:339–361, 2008.

[13] W.N. Anderson and G.E. Trapp. A class of monotone operator functions related to electrical network theory. *Linear Algebra and Appl.*, 15:53–67, 1975.

[14] J.A. Ball and V. Bolotnikov. A tangential interpolation problem on the distinguished boundary of the polydisk for the Schur-Agler class. *J. Math. Anal. Appl.*, 273:328–348, 2002.

[15] J.A. Ball and V. Bolotnikov. Canonical transfer-function realization for Schur-Agler-class functions of the polydisk. In *A Panorama of Modern Operator Theory and related Topics: The Israel Gohberg Memorial Volume*, volume 218 of *Oper. Theory Adv. Appl.*, pages 75–212. Birkhäuser, Basel, 2012.

[16] J.A. Ball, C. Sadosky, and V. Vinnikov. Scattering systems with several evolutions and multidimensional input/state/output systems. *Integral Equations Operator Theory*, 52:323–393, 2005.

[17] J.A. Ball and T.T. Trent. Unitary colligations, reproducing kernel Hilbert spaces, and Nevanlinna-Pick interpolation in several variables. *J. Funct. Anal.*, 197:1–61, 1998.

[18] A. Berlinet and Christine Thomas-Agnan. *Reproducing Kernel Hilbert Spaces in Probability and Statistics*. Kluwer Acad. Publ., Norwell, MA, 2004.

[19] R. Bhatia. *Matrix Analysis*. Springer, New York/Berlin, 1997.

[20] K. Bickel. Fundamental Agler decompositions. *Integral Equations Operator Theory*, 74(2):233–257, 2012.

[21] K. Bickel. Differentiating matrix functions. *Oper. Matrices*, 7(1):71–90, 2013.

[22] A.L. Brown and H.L. Vasudeva. The calculus of operator functions and operator convexity. *Dissertationes Math. (Rozprawy Mat.)*, 390, 2000.

[23] B.J. Cole and J. Wermer. Pick interpolation, von Neumann inequalities, and hyperconvex sets. In *Complex Potential Theory*, pages 89–129. Kluwer Acad. Publ., Dordrecht, 1994.

[24] B.J. Cole and J. Wermer. Andô's theorem and sums of squares. *Indiana Univ. Math. J.*, 48:767–791, 1999.

[25] M.J. Crabb and A.M. Davie. von Neumann's inequality for Hilbert space operators. *Bull. London Math. Soc.*, 7:49–50, 1975.

[26] W.F. Donoghue. *Monotone matrix functions and analytic continuation*. Springer, Berlin, 1974.

[27] N. Dunford and J. T. Schwartz. *Linear Operators Part 1: General Theory*. John Wiley and Sons, New York, 1988.

[28] A. Grinshpan, D. Kaliuzhnyi-Verbovetskyi, V. Vinnikov, and H. Woerdeman. Classes of tuples of commuting contractions satisfying the multivariable von Neumann inequality. *J. Funct. Anal.*, 256(9):3035–3054, 2009.

[29] M.P. Heble. *Approximation Problems in Analysis and Probability.* Elsevier Science Publishers B.V., New York, 1989.

[30] N.J. Higham. *Functions of Matrices.* SIAM, Philadelphia, 2008.

[31] J.A.R. Holbrook. Polynomials in a matrix and its commutant. *Linear Algebra and Appl.*, 48:293–301, 1982.

[32] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, Cambridge, 1991.

[33] V. Y. Kaloshin. A geometric proof of the existence of Whitney stratifications. *Mosc. Math. J.*, 5:125–133, 2005.

[34] T. Kato. On the convergence of the perturbation method, I. *Progress Theoret. Physics*, 4:514–523, 1949.

[35] T. Kato. On the convergence of the perturbation method, II. *Progress Theoret. Physics*, 5:95–101,207–212, 1950.

[36] T. Kato. *Perturbation Theory for Linear Operators.* Springer-Verlang, Berlin, 1966.

[37] G. Knese. A Schwarz lemma on the polydisk. *Proc. Amer. Math. Soc.*, 135:2759–2768, 2007.

[38] G. Knese. Bernstein-Szegö measures on the two dimensional torus. *Indiana Univ. Math. J.*, 57(3):1353–1376, 2008.

[39] G. Knese. Polynomials defining distinguished varieties. *Trans. Amer. Math. Soc.*, 362(11):5635–5655, 2010.

[40] G. Knese. Polynomials with no zeros on the bidisk. *Anal. PDE*, 3(2):109–149, 2010.

[41] G. Knese. Rational inner functions in the Schur-Agler class of the polydisk. *Publ. Mat.*, 55(2):343–357, 2011.

[42] G. Knese. A refined Agler decomposition and geometric applications. *Indiana Univ. Math. J.*, 60:1831–1842, 2011.

[43] G. Knese. Schur-Agler class rational inner functions on the tridisk. *Proc. Amer. Math. Soc.*, 139:4063–4072, 2011.

[44] A. Kreinin and M. Sidelnikova. Regularization algorithms for transition matrices. *Algo Research Quarterly*, 4:23–40, 2001.

[45] A. Kummert. Synthesis of two-dimensional lossless m-ports with prescribed scattering matrix. *Circuits Systems Signal Process*, 8(1):97–119, 1989.

[46] J. E. McCarthy. Shining a Hilbertian lamp on the bidisk. In *Topics in Complex Analysis and Operator Theory*, volume 561 of *Contemp. Math.*, pages 49–65. Amer. Math. Soc., Providence, RI, 2012.

[47] N. K. Nikol'skii. *Treatise on the Shift Operator: Spectral Function Theory*, volume 273 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1985.

[48] V.V. Peller. Hankel operators in the theory of perturbations of unitary and self-adjoint operators. *Funktsional. Anal. i Prilozhen*, 19:37–51, 1985. English translation: *Funct. Anal. Appl.* 19 (1985) 111-123.

[49] V.V. Peller. The behavior of functions of operators under perturbations. In *A Glimpse at Hilbert Space Operators*, volume 207 of *Oper. Theory Adv. Appl.*, pages 287–324. Birkhäuser, Basel, 2010.

[50] M. Pflaum. *Analytic and Geometric Study of Stratified Spaces*. Springer Lecture Notes in Mathematics, Berlin, 2001.

[51] F. Rellich. Störungstheorie der Spektralzerlegung, I. *Math. Ann.*, 113:600–619, 1937.

[52] F. Rellich. Störungstheorie der Spektralzerlegung, II. *Math. Ann.*, 113:677–685, 1937.

[53] W. Rudin. *Function Theory in Polydiscs*. Benjamin, New York, 1969.

[54] P. K. Sahoo and T. Riedel. *Mean Value Theorems and Functional Equations*. World Scientific Publishing Company Inc., River Edge, NJ, 1998.

[55] D. Sarason. *Sub-Hardy Hilbert Spaces in the Unit Disk*. University of Arkansas Lecture Notes, Wiley, New York, 1994.

[56] B. Szokefalvi-Nagy. Perturbations des transformations autoadjointes dans l'espace de Hilbert. *Comment. Math. Helv.*, 19:347–366, 1946-47.

[57] J. van den Eschof, A. Frommer, Th. Lippert, K. Schilling, and H. A. van der Vorst. Numerical methods for the QCD overlap operator. I. Sign-function and error bounds. *Computer Physics Communications*, 146:203–224, 2002.

[58] N.Th. Varopoulos. On an inequality of von Neumann and an application of the metric theory of tensor products to operators theory. *J. Funct. Anal.*, 16:83–100, 1974.

[59] H. Whitney. Tangents to an analytic variety. *Ann. of Math. (2)*, 81:496–549, 1965.

[60] E. Wigner and J. von Neumann. Significance of Loewner's theorem in the quantum theory of collisions. *Ann. of Math. (2)*, 59:418–433, 1954.