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Explicit bases of motives over number fields with application to Feynman integrals

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by

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Table of Contents

Page

List of Figures

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Dedicated to My Mom, Xinyu Ma.

ABSTRACT OF THE DISSERTATION

Explicit Bases of Motives over Number Fields with Application to Feynman Integrals

by

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Professor Matthew Kerr, Chair

Let $K_*(k)$ be the algebraic K-theory of a number field k and $\mathcal{MT}(k)$ the Tannakian category of mixed Tate motives over k. Then $\mathrm{Ext}^1_{\mathcal{MT}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) = K_{2n-1}(k) \otimes \mathbb{Q}$. Periods of mixed Tate motives give zeta and multiple zeta values. These extension classes show up in settings like Feynman integral and Mahler measure.

Chapter 1 contains background material on higher Chow groups, KLM formula and Feynman integrals. In Chapter 2, we construct explicit bases for these extension classes mapping to $Li_n(\zeta_k)$ $(\forall n, k)$. In Chapter 4, we study the Feynman integral of the three spoke wheel graph, reinterpret it as an image of regulator using higher Abel-Jacobi maps and theoretically prove that it is a rational multiple of zeta three. In Chapter 5, a reflexive graph polytope based on the graph polynomial is constructed. In Chapter 6, to generalize the results beyond wheel with three spokes, a criterion is given on the vanishing of graph symbols. An essential blow-up construction is reinterpreted in toric language to reveal the ambient space's combinatorial structure.

1. Backgrounds.

1.1 Introduction

Let $\zeta_N \in \mathbb{C}^*$ be a primitive N^{th} root of 1 ($N \geq 2$). The seminal article [1] of A. Beilinson concludes with a construction of elements Ξ_b ($b \in (\mathbb{Z}/N\mathbb{Z})^*$) in motivic cohomology

$$H^1_{\mathcal{M}}\left(\mathrm{Spec}(\mathbb{Q}(\zeta_N)), \mathbb{Q}(n)\right) \cong K^{(n)}_{2n-1}\left(\mathbb{Q}(\zeta_N)\right) \otimes \mathbb{Q}$$

mapping to $Li_n(\zeta_N^b) = \sum_{k\geq 1} \frac{\zeta_N^{kb}}{k^n} \in \mathbb{C}/(2\pi\mathbf{i})^n\mathbb{R}$ under his regulator. Since by Borel's theorem [2] $\mathrm{rk}K^{(n)}_{2n-1}(\mathbb{Q}(\zeta_N))_{\mathbb{Q}} = \frac{1}{2}\phi(N)$ (for $N \geq 3$), an immediate consequence is that the $\{\Xi_b\}$ span $K^{(n)}_{2n-1}(\mathbb{Q}(\zeta_N))_{\mathbb{Q}}$; indeed, Beilinson's results anticipated the eventual proofs [3,4] of the equality (for number fields) of his regulator with that of Borel [5]. An expanded account of his construction was written up by Neukirch (with Rapoport and Schneider) in [6], up to a "crucial lemma" ((2.4) in [op. cit.]) which is required for the regulator computation but left unproved.

The intervening years have seen some improvements in technology, especially Bloch's introduction of higher Chow groups [7], which yield an *integral* definition of motivic cohomology for smooth schemes X. In particular, we have[1]

$$\begin{aligned} H^1_{\mathcal{M}}\left(\mathrm{Spec}(\mathbb{Q}(\zeta_N)), \mathbb{Z}(n)\right) &\cong CH^n\left(\mathbb{Q}(\zeta_N), 2n-1\right) \\ &:= H_{2n-1}\left\{Z^n(\mathbb{Q}(\zeta_N), \bullet), \partial\right\}, \end{aligned}$$

[1]We use the shorthand $CH^*(F,*)$ ($Z^*(F,*)$, etc.) for $CH^*(\mathrm{Spec}(F),*)$ (F a field).

and can ask for explicit cycles in $\ker(\partial) \subset Z^n(\mathbb{Q}(\zeta_N), 2n-1)$ representing (multiples of) Beilinson's elements Ξ_b. Another relevant development was the explicit realization of Beilinson's regulator in [8–10] as a morphism $\widetilde{AJ}$ of complexes, from a *rationally* quasi-isomorphic subcomplex $Z^n_{\mathbb{R}}(X, \bullet)$ of $Z^n(X, \bullet)$ to a complex computing the absolute Hodge cohomology of X. Here this "KLM morphism" yields an Abel-Jacobi mapping

$$AJ : CH^n(\mathbb{Q}(\zeta_N), 2n-1) \otimes \mathbb{Q} \to \mathbb{C}/(2\pi\mathbf{i})^n\mathbb{Q}, \tag{1.1}$$

and in the present note we shall construct (for all n) higher Chow cycles

$$\hat{\mathscr{Z}_b} \in \ker(\partial) \subset Z^n_{\mathbb{R}}(\mathbb{Q}(\zeta_N), 2n-1) \otimes \mathbb{Q}$$

with

$$(n-3)N^{n-1}\hat{\mathscr{Z}_b} \in Z^n_{\mathbb{R}}(\mathbb{Q}(\zeta_N), 2n-1)) \quad \text{and} \quad AJ(\hat{\mathscr{Z}_b}) = Li_n(\zeta_N^b).$$

(See Theorems 2.4.2, with $\hat{\mathscr{Z}} = \frac{(-1)^n}{N^{n-1}}\tilde{\mathscr{Z}}$.) This is entirely more explicit than the constructions in [1,6], and yields a transparent computation of the regulator, hence removing any confusion around the aforementioned lemma. Furthermore, in concert with the anticipated extension of $\widetilde{AJ}$ to the entire complex $Z^n(X, \bullet)$ (making (1.1) integral), we expect that our cycles will be useful for studying the torsion in $CH^n(\mathbb{Q}(\zeta_N), 2n-1)$ as begun in [11] (cf. Remark 2.4.1).

The Feynman integrals of certain families of graphs exhibit interesting patterns of zeta and multiple zeta values.They are periods of underlying mixed Tate motives. Applying the technique of Feynman parameters [12], the problem comes down to studying the motives associated to graph polynomials. By [13], general graph's motive are mixed tate. Therefore it is natural to ask the following question: for what kind of graphs, their graph polynomial are mixed tate, and how does this property relate to the combinatorial structure of a graph.

Broadhurst and Kreimer [14] computed Feynman amplitudes using numerical and analytic methods for certain families of graphs. The first attempt to give an algebro-geometric explanation to this problem is given by Bloch, Esnualt and Kreimer [15], where they interpreted Feynman integral as periods of mixed Hodge structure. In [16],Bloch, Kerr and Vanhove initiate a new method to interpret Feynman integrals as the pairing of an image of a higher Abel-Jacobi map and a rational homology class, applying this to the sunset graph. Their approach starts with completing a certain K-theoretic "graph symbol" and the ambient space is a toric variety arising from a reflexive polytope. In [17], rich tools are developed on the topics of symbols over toric varieties. This motivates our work to apply the symbol approach to other graphs.

1.2 Definition of Higher Chow cycle groups

Bloch defined higher Chow groups in [7], with the objective to define an integral cohomology theory to represent up to torsion the weighted-graded pieces of higher $K-$theory. Bloch's idea is to construct a chain complex the elements of which in every degree n are cycles of co-dimension q of $X \times \Delta^n$. One motivation would be as follows. For an algebraic variety X defined over an infinite field, one wants to have an algebro-geometric version of singular homology theory. This motivates to consider the morphisms from algebraic $n-$simplex Δ^n to X. And it turns out it is more appropriate to consider the correspondences, i.e. the sub-varieties in the product space $X \times \Delta^n$. Another motivation for higher Chow group is if one patches two copies of $X \times \Delta^n$ along $X \times \partial\Delta^n$, and call it S_X^m, then Karoubi-Villamay theory says $K_m(X)$ is a direct summand of $K_0(S_X^m)$.

Here we use a cubical version of higher Chow group, which is equivalent to the simplicial version above.

Let $\square^n := (\mathbb{P}^1 \setminus \{1\})^n$ with coordinates $(z_1, ..., z_n)$. For a multi-index $J \subset \{1, ..., n\}$ and function $f : J \to \{0, \infty\}$, define subsets $\partial_f^J \square^n := \bigcap_{j \in J}\{z_j = f(j)\}$, and put $\partial\square^n := \bigcup_j(\partial_0^j \square^n \cup \partial_\infty^j \square^n)$. One has inclusion and projection maps

$$\iota_{j,k} : \square^{n-1} \hookrightarrow \square^n \qquad (z_1, ..., z_{n-1}) \mapsto (z_1, ..., z_{j-1}, k, z_j..., z_{n-1})$$

($k = 0$ or ∞) and

$$\pi_j : \square^n \twoheadrightarrow \square^{n-1} \qquad (z_1, ..., z_j, ..., z_n) \mapsto (z_1, ..., \hat{z}_j, ..., z_n).$$

Now, let X be an algebraic variety defined over an infinite field k and $Z^p(X \times \square^n)$ the abelian group of codimension-p algebraic cycles defined over k.

Definition 1.2.1 *The admissible cycles are the formal sums of irreducible subvarieties $Z \in Z^P(X \times \square^n)$ such that Z intersects all faces properly. In another words, $Z \cap (X \times \partial_f^J \square^n)$ has codimension p in $X \times \partial_f^J \square^n$ $(\forall J, f)$. We denote the group of admissible cycles by $c^p(X, n) \subset Z^P(X \times \square^n)$.*

One can have a class of "trivial" admissible cycles by pulling back admissible cycles along projection to faces, called degenerate cycles:

$$d^P(X, n) := \sum_j \pi_j^* c^p(X, n-1) \subset c^p(X, n).$$

Definition 1.2.2 *The higher Chow precycles are the following group*

$$Z^p(X, n) := \frac{c^p(X, n)}{d^p(X, n)}.$$

We define the Bloch differential

$$\partial_{\mathcal{B}} : Z^p(X, n) \to Z^p(X, n-1)$$

$$Z \mapsto \sum_j (-1)^j (\partial_0^j - \partial_\infty^j) Z,$$

which satisfies $\partial_{\mathcal{B}} \circ \partial_{\mathcal{B}} = 0$. This makes $Z^p(X, \cdot)$ a chain complex. A higher Chow cycle is a closed precycle, and the higher Chow groups are the cohomology groups of the chain complex:

$$CH^p(X, n) := H^{-n}\{Z^p(X, \cdot), \partial_{\mathcal{B}}\}.$$

1.3 Properties of higher Chow groups

Let $f : X \to Y$ be a map of varieties. The pull-back and push-forward give rise to the pull-back map $f^* : CH^q(Y, p) \to CH^q(Y, p)$ if f is flat, and the push-forward map $f_* : CH^{q+d}(X, p) \to CH^q(Y, p)$ if f is a proper of relative dimension d. These are functorial, when the composition is defined.

Besides functoriality, we list several properties of higher Chow groups which will used later in this thesis.

1. Homotopy. Let $p_X : X \times \mathbb{A}^1 \to X$ be the projection. Then

$$p_X^* : CH^q(X, p) \to CH^q(X \times \mathbb{A}^1, p)$$

   is an isomorphism.

2. Localization. Let $i : Z \to X$ be a closed codimension d subscheme of a quasi-projective variety X, and $j : U \to X$ the complement. Then the sequence

$$z^{q-d}(Z, *) \xrightarrow{i_*} z^q(X, *) \xrightarrow{j^*} z^q(U, *)$$

   defines a quasi-isomorphism $z^{q-d}(Z, *) \to cone(j^*)[-1]$, giving rise to the long exact localization sequence

$$... \to CH^{q-d}(Z, p) \xrightarrow{i_*} CH^q(X, p) \xrightarrow{j^*} CH^q(U, p) \xrightarrow{\delta} CH^{q-d}(Z, p-1) \to ....$$

3. Products. There are functorial maps of complexes

$$\boxtimes_{X,Y} : z^q(X,*) \otimes_{\mathbb{Z}} z^{q'}(Y,*) \to z^{q+q'}(X \times_k Y, *)$$

which are associative and (graded) commutative. Following $\boxtimes_{X,X}$ by the pull-back via the diagonal makes the bi-graded group $\oplus_{p,q} CH^q(X,p)$ into a bi-graded ring, graded-commutative in p.

4. Comparison with K-theory. Let X be a smooth quasi-projective variety. There are natural isomorphisms

$$CH^q(X,p) \otimes \mathbb{Q} \cong K_p(X)^{(q)},$$

where $K_p(X)^{(q)}$ is the weight q subspace (for the Adams operations) in the rational higher algebraic $K-$theory of X, $K_p(X) \otimes \mathbb{Q}$.

Let X be a smooth variety, $Y \hookrightarrow X$ an open embedding. $D = X \setminus Y$ a codimension 1 divisor. A situation we encountered a lot is that of a precycle ξ in $Z^p(Y,n)$ which gives a higher Chow cycle on Y, but whose extension to X does not meet the faces of $X \times \square^n$ properly. So two questions naturally arise. One is : could we modify or move ξ such that it will meet the faces of $X \times \square^n$ properly? The second is : does the moved cycle give a higher Chow cycle in X?

The first question is answered by Bloch's moving lemma [5], which says

$$\frac{Z^p(X,\cdot)}{Z^{p-1}(D,\cdot)} \xrightarrow{\iota^*} Z^p(Y,\cdot)$$

is a quasi-isomorphism. Intuitively, this says we could "move" a closed precycle ξ on Y by adding an exact precycle, such that the result extends to an admissible cycle Ξ on X such that $\partial_{\mathcal{B}}(\Xi - \xi)$ is supported on D. This yields a residue map

$$\mathrm{Res} : CH^p(Y,n) \to CH^{p-1}(D,n-1).$$

See Matt Kerr's thesis [8]. The residue map fits in the long-exact sequence

$$\to CH^P(X,n) \xrightarrow{j^*} CH^P(Y,n) \xrightarrow{\mathrm{Res}} CH^{p-1}(D,n-1) \xrightarrow{\iota^*} CH^p(X,n-1),$$

which answers the second question. So when $\mathrm{Res}\,(\xi)$ vanishes, ξ could be lifted to a higher Chow cycle class Ξ in X.

In practice, D could be singular and it is hard to compute its higher Chow group. So we could break D into smooth strata and define higher residue maps from Y to stratas. The idea is we want to take successive higher residue maps and $\mathrm{Res}\,(\xi) = 0$ when all higher residue maps vanish. Although higher residue maps could be defined in broader cases, for our convenience, here we just let D be a normal crossing divisor. In [10] two ways to define the higher residue maps are given, here we only mention one that uses Gysin spectral sequence which take residues consecutively from highest codimension strata to $codimension - 1$ strata.

Let $D = \bigcup D_i$ be a NCD, $D_I := \bigcap_{i\in I} D_i$, $D^k = \bigcup_{|I|=k} D_I$ and $\widetilde{D^k} \cong \amalg_{|I|=k} D_I$.

Define(mostly 3rd quadrant) double complexes

$$\mathcal{K}_{X\backslash Y}(p)_0^{a,b} := Z^{p+a}(\widetilde{D^{-a}}, -b)$$

with vertical differential $\partial_{\mathcal{B}}$, horizontal differential

$$Gy := 2\pi i \sum_{|I|=-a} \sum_{i\in I} (-1)^{<i>_{I\backslash\{i\}}} (\iota_{D_I \subset D_{i\backslash\{i\}}})_*$$

($< i >_J :=$ position in which i occurs in J), and total differential $\mathbb{D} := \partial_{\mathcal{B}} + (-1)^b Gy$.

In the first sheet E_1, we have $\mathcal{K}(p)_1^{a,b} \cong H_{\mathcal{K}}^{2p+2a+b}(\widetilde{D^{-a}}, \mathbb{Q}(p+a))$. Consider the quotient double complex

$$\mathcal{K}^{\leq -k}(p)^{a,b} := \begin{cases} 0 & a > -k \\ \mathcal{K}^{a,b} & a \leq -k \end{cases}.$$

Since residue is the dual of Gysin morphism, the map of double complexes

$$\mathcal{K}_{X\setminus Y}(p)_0^{a,b} \twoheadrightarrow \mathcal{K}^{\leq -k}_{X\setminus Y}(p)_0^{a,b},$$

induces higher residue maps.

We would use the higher residue maps on $CH^n(Y,n)$. We will first take the Res^n to the highest codimension stratas, and on $\ker(\mathrm{Res}^n)$ we take Res^{n-1} dimension 1 stratas, so on so force as follows

$$CH^n(Y,n) \xrightarrow{\mathrm{Res}^n} CH^0(\widetilde{D^n}),$$

$$\ker(\mathrm{Res}^n) \xrightarrow{\mathrm{Res}^{n-1}} CH^1(\widetilde{D^{n-1}},1),$$

$$\ker(\mathrm{Res}^{n-1}) \xrightarrow{\mathrm{Res}^{n-2}} CH^2(\widetilde{D^{n-2}},2),$$

.

.

.

$$\ker(\mathrm{Res}^2) \xrightarrow{\mathrm{Res}^1} CH^{n-1}(\widetilde{D^1},n-1),$$

and $\ker(\mathrm{Rex}^1) \twoheadleftarrow CH^n(X,n)$.

Especially, we have $\ker(\mathrm{Res}^1) = \bigcap \ker(\hat{Res}^i)$. Here $\hat{Res}^i$ are the iterated Poincáre residues,

$$CH^n(Y,n) \xrightarrow{\hat{Res}^i} CH^{n-i}(\widetilde{D^i} \setminus \widetilde{D^{i+1}}, n-i).$$

Definition 1.3.1 *Let F be a field. For $n \geq 2$ let $K_n^M(F)$ denote the quotient of the abelian group $\otimes^n \mathbb{Z}[\mathbb{P}^1_F \setminus \{0,\infty\}]$ by the Steinberg relations: the subgroup generated by all permutations of*

$$\alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_n + \beta_1 \otimes \alpha_2 \otimes ... \otimes \alpha_n - \alpha_1\beta_1 \otimes \alpha 2 \otimes ... \otimes \alpha_n,$$

$$\alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_n + \alpha_2 \otimes \alpha_1 \otimes ... \otimes \alpha_n, \qquad \alpha_1 \otimes (1-\alpha_1) \otimes ... \otimes \alpha_n.$$

For $n < 2$, put $K_1^M(F) := F^*$ and $K_0^M(F) := \mathbb{Z}$.

We shall write elements of $\otimes^n \mathbb{Z}[\mathbb{P}^1_F \setminus \{0, \infty\}]$ additively and elements of the quotient multiplicatively(eg. $\{\alpha_1, ..., \alpha_n\}$).

Milnor symbols satisfy Steinberg relations:

Lemma 1.3.2 *if $(\cdot, \cdot)$ is a symbol then (assuming all terms are defined)*

- $(a, -a) = 1$;
- $(b, a) = (a, b)^{-1}$;
- $(a, a) = (a, -1)$ *is an element of order* 1 *or* 2;
- $(a, b) = (a + b, -\frac{b}{a})$.

1.4 Iterated residues in toric setting

In the case when X is toric variety, [17] develops powerful techniques to compute the iterated residues. Here we briefly review their construction related to our cases.

Let $\mathbb{P}_\Delta$ be a $n-$dimensional toric variety with toric coordinates $x_1, ..., x_n$ and $\mathbb{P}_{\tilde{\Delta}}$ be a smooth toric variety lying over it. We know that a toric variety can be decomposed as disjoint union of torus orbits of different dimensions. For convenience, we apply [17]'s notation, but with slight differences. Use $\Delta(i)$ to denote the set of closures of codimension-i orbits. For example, for a $\sigma \in \Delta(i)$, $\mathbb{D}^*_\sigma \subset \mathbb{P}_\Delta$ represents the corresponding orbit and $\mathbb{D}_\sigma$ represents it closure. And

$$\mathbb{D} := \cup_{\sigma \in \Delta(1)} \mathbb{D}_\sigma = \sqcup_{i=1}^n (\sqcup_{\sigma \in \Delta(i)} \mathbb{D}^*_\sigma)$$

is the complement of $(\mathbb{C}^*)^n$ in $\mathbb{P}_\Delta$.

Actually, we can always construct a polytope whose combinatorial structure(its face structure) describes the intersection behavior of $\mathbb{D}$, and this is the polytope Δ to which $\mathbb{P}_\Delta$ is associated via the normal fan construction.

Now suppose we could find a change of toric coordinates to x_j^σ such that

$$\mathbb{D}_\sigma^* = \{x_1^\sigma, ..., x_{n-i}^\sigma \in \mathbb{C}^*\} \cap \{x_{n-i+1}^\sigma = \cdots = x_n^\sigma = 0\} \subset \mathbb{P}_\Delta.$$

This new set of coordinates makes a new symbol $\{x_1^\sigma, ..., x_{n-i}^\sigma\}$. It will facilitate our computation of higher residue maps.

Lemma 1.4.1 *the diagram*

$$\begin{tikzcd}
CH^n(\mathbb{P}_{\tilde{\Delta}} \setminus \tilde{\mathbb{D}}, n) \arrow[r, "\mathrm{Res}^i_{\tilde{\sigma}}"] \arrow[d, "I^*"'] & CH^{n-i}(\mathbb{D}^*_{\tilde{\sigma}}, n-i) \arrow[d, "I^*_{\tilde{\sigma}}"] \\
CH^n(\tilde{X}^*, n) \arrow[r, "\mathrm{Res}^i_{\tilde{\sigma}}"] & CH^{n-i}(D^*_{\tilde{\sigma}}, n-i)
\end{tikzcd}$$

commutes for any $\tilde{\sigma} \in \Delta(i)$, as does a similar diagram with all tildes removed.

The lemma above enable us to compute the $\mathrm{Res}^i_{\sigma \mathrm{or} \tilde{\sigma}}$ (bottom row) on ξ.

Proposition 1.4.2 *For $\sigma \in \Delta(i)$, and $\tilde{\sigma} \in \tilde{\Delta}(i-k)$ lying over σ in the above sense,*

$$\mathrm{Res}^i_\sigma \xi = (I^*_\sigma)\langle \pm \{x_1^\sigma, ..., x_{n-i}^\sigma\}\rangle$$

$$\mathrm{Res}^{i-k}_{\tilde{\sigma}} \xi = (I^*_{\tilde{\sigma}})\langle \pm \{x_1^\sigma, ..., x_{n-i}^\sigma, y_1^{\tilde{\sigma}}, ..., y_k^{\tilde{\sigma}}\}\rangle,$$

where the parenthetical expressions are optional.

1.5 Deligne cohomology complex

Deligne cohomology is introduced in [18], designed to be the target of Beilinson regulator from motivic cohomology. As KLM formula is a lift of the Beilinson regulator to Bloch's higher Chow complex, it is natural to see Deligne cohomology is also the target of KLM formula. Deligne cohomology could be defined for any field of character 0, here we only consider $\mathbb{Q}$.

Definition 1.5.1 *For smooth quasi-projective($d-$dimensional)X, the $a-$currents $D^a(X)$ are functionals on the space of compactly supported C^∞ forms of degree $2d-a$. A current $\eta \in F^bD^a(X)$ if it kills $\Gamma_c(F^{d-b+1}\Omega^{2d-a}_{X^\infty})$.*

Example 1.5.2 *The integration against a real-codimension a $C^\infty-$Borel-Moore chain Γ on X, denoted by δ_Γ.*

Example 1.5.3 *Differential $a-$forms with log poles along subvarieties of X.*

Example 1.5.4 *On $\square^n$, we define the following,*

$$R_n := \sum_{j=1}^{n}((-1)^n 2\pi i)^{j-1} log(z_j)\frac{dz_{j+1}}{z_{j+1}} \wedge ... \wedge \frac{dz_n}{z_n}\delta_{T_{z_1}\cap ... \cap T_{z_{j-1}}}$$
$$\Omega_n := \wedge_{j=1}^{n}\frac{dz_j}{z_j}, \quad T_n := \bigcap_{j=1}^{n} T_{z_j} := \bigcap_{j=1}^{n} z_j^{-1}(\mathbb{R}_{\leq 0 \cup \{\infty\}}).$$

Lemma 1.5.5 *R_n, Ω_n, and T_n satisfy the following residue formula*

$$d[R_n] - \Omega_n + (2\pi\sqrt{-1})^n\delta_{T_n} = 2\pi\sqrt{(}-1)\sum_{i=1}^{n} R(z_1, ..., \hat{z_i}, ..., z_n)\delta(z_i = 0).$$

Proof See the proof in [17]. ∎

For a precycle $\xi \in Z^p(X, n)$, let $R_\xi := (\pi_X)_*(\pi_\square)^*R_n$, and we define Ω_ξ and T_ξ similarly. Then we have

$$d[R_\xi] = \Omega_\xi - (2\pi\sqrt{-1})^n\delta_{T_\xi} + 2\pi\sqrt{-1}R_{\partial\xi}. \tag{1.2}$$

For convenience, we follow the definition of Deligne cohomology in [9].

Definition 1.5.6 *For smooth projective variety X, Deligne cohomology group $H^{2p-n}_{\mathcal{D}}(X, \mathbb{Q}(p))$ is the $(2p-n)$-th cohomology of the following complex:*

$$C^{\cdot}_{\mathcal{D}}(X, \mathbb{Q}(p)) := (C^{\cdot+1}_{top}(X; \mathbb{Q}(p)) \oplus F^p\mathcal{D}^{\cdot+1}(X) \oplus \mathcal{D}^{\cdot}(X))[-1]$$

where $D(T, \Omega, R) = (-\partial_{top}T, -d[\Omega], d[R] - \Omega + \delta_T)$.

$H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p))$ sits in a short-exact sequence

$$0 \to J^{p,n}(X)_{\mathbb{Q}} \to H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p)) \to Hg^{p,n}(X)_{\mathbb{Q}} \to 0,$$

where

$$\begin{aligned} Hg^{p,n}(X)_{\mathbb{Q}} &:= Hom_{\mathbb{Q}-MHS}(\mathbb{Q}(0), H^{2p-n}(X), \mathbb{Q}(p)) \\ &= F^pH^{2p-n}(X,\mathbb{C}) \cap H^{2p-n}(X,\mathbb{Q}(p)), \\ J^{p,n}(X)_{\mathbb{Q}} &:= Ext^1_{\mathbb{Q}-MHS}(\mathbb{Q}(0), H^{2p-n-1}(X,\mathbb{Q}(p))) \\ &= \frac{H^{2p-n-1}(X,\mathbb{C})}{F^pH^{2p-n-1}(X,\mathbb{C}) + H^{2p-n-1}(X,\mathbb{Q}(p))}. \end{aligned}$$

When X is smooth projective and $n \geq 1$, it is easy to see $Hg^{p,n}(X)_{\mathbb{Q}} = 0$. In a triple (Γ, Ω, K) which is D closed, (Ω, K) define the map to $Hg^{p,n}(X)_{\mathbb{Q}}$.

1.6 KLM formula

The KLM formula says for $Z \in Z^p_{\mathbb{R}}(X,n)$ $\partial_{\mathcal{B}}$-closed, its image in $H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p))$ is represented by

$$Z \mapsto ((2\pi i)^pT_Z, (2\pi i)^{p-n}\Omega_Z, (2\pi i)^{p-n}R_Z). \tag{1.3}$$

Here $Z^p_{\mathbb{R}}(X,n)$ is a complex quasi-isomorphic to $Z^p(X,n)$, elements of which are required to intersect some real analytic subset properly, see [8]. R_Z(respectively for T_Z, Ω_Z) $= (\pi_X)_*(\pi_{\square})^*R_n(T_n,\Omega_n)$.

Now suppose for $\xi \in Z^p_{\mathbb{R}}(X,n)$ $\partial_{\mathcal{B}}$-closed, and $n \geq 1$. Since $[T_\xi]$ and $[\Omega_\xi]$ define the map to $Hg^{p,n}(X)_{\mathbb{Q}}$ (which is zero for $n \geq 1$), there exist $K \in F^p\mathcal{D}^{2p-n-1}(X)$ and $\Gamma \in C^{2p-n-1}_{top}(X;\mathbb{Q}(p))$ such that $\Omega_\xi = d[K]$ and $T_\xi = \partial\Gamma$. Therefore if we define

$$\tilde{R}_\xi := R_\xi - K + (2\pi i)^n\delta_\Gamma, \tag{1.4}$$

then by (1.5.5), $\tilde{R}_\xi$ is a closed current and defines a class $[\tilde{R}_\xi] \in H^{2p-n-1}(X, \mathbb{C})$ projecting to

$$AJ_X^{p,n}(\xi) \in J^{p,n}(X)_{\mathbb{Q}} \cong \frac{H^{2p-n-1}(X, \mathbb{C})}{F^p H^{2p-n-1}(X, \mathbb{C}) + H^{2p-n-1}(X, \mathbb{Q}(p))}.$$

The maps $\{AJ^{p,n}(\xi)\}$ are called regulator maps because they are lifts of Beilinson regulators. Assume X is projective and defined over a number field F. By composing structure morphism $X \to \mathrm{Spec}(F) \to \mathrm{Spec}(\mathbb{Q})$ we could view X as a variety over $\mathbb{Q}$. Applying $AJ^{p,n}$ for this X to the cycle-classes which are defined over $\mathrm{Spec}(\mathcal{O}_F)$ and composing with the projection to real Deligne cohomology will give Beilinson's regulator.

2. An explicit basis for the rational higher Chow groups of abelian number fields.

2.1 Beilinson's Construction

2.1.1 Notation.

Let ζ be a primitive N-th root of unity, i.e. $\zeta = exp^(2\pi i(\frac{a}{N}))$, where a is coprime to N. Each such a yields an embedding σ of $\mathbb{F} := \frac{\mathbb{Q}[\omega]}{(\omega^N - 1)}$ into $\mathbb{C}$ (by sending $\omega \mapsto \zeta$). (If $N = 2$, then $\mathbb{F} = \mathbb{Q}$ and $\omega = \zeta = -1$.)

Working over any subfield of $\mathbb{C}$ containing ζ, write

$$\square^n := (\mathbb{P}^1 \setminus \{1\})^n \supset \mathbb{T}^n := (\mathbb{P}^1 \setminus \{0, 1\})^n$$

with coordinates $(z_1, ..., z_n)$. We have isomorphism from $\mathbb{T}^n$ to $\mathbb{G}_m^n$ with coordinates $(t_1, ..., t_n)$, given by $t_i = \frac{z_i}{z_i - 1}$. Define a function

$$f(z_1, \cdots, z_n) := 1 - \zeta t_1 \cdots t_n$$

on $\mathbb{T}^n$(with b coprime to N), and normal crossing subschemes

$$S^n = \{\underline{z} \in \mathbb{T}^n | \text{ some } z_i = \infty\} \supset S^n \cup |(f)_0| =: \tilde{S}^n \subset \mathbb{T}^n.$$

(Alternatively, we may view these schemes as defined over $\mathbb{F}$ by replacing ζ^b with ω^b.)

Now consider the morphism

$$\iota_n : \mathbb{T}^{n-1} \to \mathbb{T}^n$$

$$(t_1, ..., t_{n-1}) \mapsto (t_1, ..., t_{n-1}, (\zeta^b t_1, ..., t_{n-1})^{-1}).$$

We record the following:

Lemma 2.1.1 *ι_n sends $\mathbb{T}^{n-1}$ isomorphically onto $|(f_n)_0|$, with $\iota_n(\tilde{S}^{n-1}) = |(f_n)_0| \cap S^n$.*

Proof To clarify, when we define $\tilde{S}^{n-1} = S^{n-1} \cup |(f)_0|$, we mean $f(z_1, \cdots, z_{n-1}) = 1 - \zeta t_1 \cdots t_{n-1}$. Then the lemma follows straight forward. ■

We also remark that the Zariski closure of $\iota_n(\mathbb{T}^{n-1})$ in $\square^n$ is just $\iota_n(\mathbb{T}^{n-1})$.

2.1.2 Results for Betti cohomology.

The construction just described has quite pleasant cohomological properties, as we shall now see.

Lemma 2.1.2 *As a $\mathbb{Q} - MHS$,*

$$H^q(\mathbb{T}^n; S^n) \cong \begin{cases} \mathbb{Q}(-n) & ,q = n \\ 0 & ,q \neq n \end{cases}.$$

Lemma 2.1.3 *As a $\mathbb{Q} - MHS$,*

$$H^q(\mathbb{T}^n, \tilde{S}^n) \cong \begin{cases} \mathbb{Q}(0) \oplus \mathbb{Q}(-1) \oplus \ldots \oplus \mathbb{Q}(-n) & ,q = n \\ 0 & ,q \neq n \end{cases}.$$

Proof This is clear for $(\mathbb{T}^1, \tilde{S}^1) \cong (\mathbb{G}_m, \{1, \bar{\zeta}\})$. Now consider the exact sequence

$$\underline{H^{*-1}(\mathbb{T}^n, S^n)} \xrightarrow{\iota_n^*} H^{*-1}(\mathbb{T}^{n-1}, \tilde{S}^{n-1}) \xrightarrow{\delta}$$

$$H^*(\mathbb{T}^n, \tilde{S}^n) \to \underline{H^*(\mathbb{T}^n, S^n)} \xrightarrow{\iota_n^*} H^*(\mathbb{T}^{n-1}, \tilde{S}^{n-1})$$

of $\mathbb{Q}$-MHS, associated to the inclusion $(\mathbb{T}^{n-1}, \tilde{S}^{n-1}) \hookrightarrow (\mathbb{T}^n, S^n)$. (This is just the relative cohomology sequence, once one notes that the pair $((\mathbb{T}^n, S^n), \iota_n(\mathbb{T}^{n-1}, \tilde{S}^{n-1})) = (\mathbb{T}^n, S^n \cup \iota_n(\mathbb{T}^{n-1})) = (\mathbb{T}^n, \tilde{S}^n)$ by lemma (2.1.1)). If $* \neq n$, then the underlined terms are 0 via Lemma (2.1.2) and induction. If $* = n$, then the end terms are 0 via Lemma (2.1.2) and induction, and

$$0 \to H^{n-1}(\mathbb{T}^{n-1}, \tilde{S}^{n-1}) \xrightarrow{\delta} H^n(\mathbb{T}^n, \tilde{S}^n) \to H^n(\mathbb{T}^n, S^n) \to 0 \tag{2.1}$$

is a short-exact sequence.

Now observe that:

- $H^n(\mathbb{T}^n, S^n; \mathbb{C}) = F^n H^n(\mathbb{T}^n, S^n; \mathbb{C})$ is generated by the holomorphic form

  $\eta := \frac{1}{(2\pi\mathbf{i})^n} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}$;

- $H_{n-1}(\mathbb{T}^{n-1}, \tilde{S}^{n-1}; \mathbb{Q})$ is generated by images $\underline{e}(U_i)$ of the cells $\cup_{i=0}^n U_i = [0,1]^n \setminus \cup_{l=1}^n \{\sum x_i = l - \frac{a}{N}$, where $\underline{e} : [0,1]^n \to \mathbb{T}^n$ is defined by $(x_1, \cdots, x_n) \mapsto (e^{2\pi\mathbf{i}x_1}, \cdots, e^{2\pi\mathbf{i}x_n}) = (t_1, \cdots, t_n)$;

and

- $\int_{\underline{e}(U_i)} dx_1 \wedge \cdots dx_n \in \mathbb{Q}$.

(Writing $\mathcal{S}^1$ for the unit circle, $\left((\mathcal{S}^1)^n, (\mathcal{S}^1)^n \cap \tilde{S}^n\right)$ is a deformation retract of $(\mathbb{T}^n, \tilde{S}^n)$. The $\underline{e}(U_i)$ visibly yield all the raltive cycles in the former, justifying the second observation.) Together these immediately imply that 2.1 is split, completing the proof.

■

2.1.3 Results for Deligne cohomology

Recall that Beilinson's absolute Hodge cohomology [1] of an analytic scheme Y over $\mathbb{C}$ sits in an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{r-1}(Y, \mathbb{A}(p))) \to H^r_{\mathcal{D}}(Y, \mathbb{A}(p))$$
$$\to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^r(Y, \mathbb{A}(p))) \to 0.$$

(Here we use a subscript "$\mathcal{D}$" since the construction after all is a "weight-corrected" version of Deligne cohomolgy; the subscript "MHS" of course means "$\mathbb{A}$-MHS".) We shall not have any use for details of its construction here.

Lemma 2.1.4 *The map $\iota_n^* : H^n_{\mathcal{D}}(\mathbb{T}^n, S^n; \mathbb{A}(n)) \to H^n_{\mathcal{D}}(\mathbb{T}^n, \tilde{S}^n; \mathbb{A}(n))$ is zero ($\mathbb{A} = \mathbb{Q}$ or $\mathbb{R}$).*

Proof Consider the exact sequence

$$\to H^n_{\mathcal{D}}(\mathbb{T}^n, S^n; \mathbb{Q}(n)) \xrightarrow{\iota_n^*} H^n_{\mathcal{D}}(\mathbb{T}^{n-1}, \tilde{S}^{n-1}; \mathbb{Q}(n)) \xrightarrow{\delta_{\mathcal{D}}} H^{n+1}_{\mathcal{D}}(\mathbb{T}^n, \tilde{S}^n; \mathbb{Q}(n)) \to .$$

It suffices to show that $\delta_{\mathcal{D}}$ is injective. Now

$$\mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^n(\mathbb{T}^{n-1}, \tilde{S}^{n-1}; \mathbb{Q}(n))\big) = \{0\}$$
$$\mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{n+1}(\mathbb{T}^n, \tilde{S}^n; \mathbb{Q}(n))\big) = \{0\}$$

by lemma 2.1.3, and so $\delta_{\mathcal{D}}$ is given by

$$\mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{n-1}(\mathbb{T}^{n-1}, \tilde{S}^{n-1}; \mathbb{Q}(n))\big) \xrightarrow{\delta_{\mathcal{D}}} \mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^n(\mathbb{T}^n, \tilde{S}^n; \mathbb{Q}(n))\big).$$

Since (2.1) is split, the corresponding sequence of Ext^1-groups is exact, and $\delta_{\mathcal{D}}$ is injective.

■

2.1.4 Results for motivic cohomology

Let X be any smooth simplicial scheme (of finite type), defined over a subfield of $\mathbb{C}$. We have Deligne class maps ($\mathbb{A} = \mathbb{Q}$ or $\mathbb{R}$)

$$c_{\mathcal{D},\mathbb{A}} : H^r_{\mathcal{M}}(X, \mathbb{Q}(p)) \to H^r_{\mathcal{D}}(X^{an}_{\mathbb{C}}, \mathbb{A}(p)).$$

The case of particular interest here is where $r = 1$, X is a point, and

$$c_{\mathcal{D},\mathbb{A}}(Z) = \frac{1}{(2\pi\mathbf{i})^{p-1}} \int_{Z^{\mathrm{an}}_{\mathbb{C}}} R_{2p-1} \in \mathbb{C}/\mathbb{A}(p), \tag{2.2}$$

where(interpreting $\log(z)$ as the 0-current with branch cut along $T_z := z^{-1}(\mathbb{R}_-)$)

$$\begin{aligned} R_{2p-1} &:= \sum_{k=1}^{2p-1} (2\pi\mathbf{i})^{k-1} R^{(k)}_{2p-1} \\ &= \sum_{k=1}^{2p-1} (2\pi\mathbf{i})^{k-1} \log(z_k) \frac{dz_{k+1}}{z_{k+1}} \wedge \cdots \frac{dz_{2p-1}}{z_{2p-1}} \cdot \delta_{T_{z_1} \cap \cdots T_{z_{k-1}}} \end{aligned} \tag{2.3}$$

is the regulator current of KLM formula, belonging to $D^{2p-2}\big((\mathbb{P}^1)^{\times(2p-1)}\big)$.

Now take a number field K, $[K : \mathbb{Q}] = d = r_1 + 2r_2$, and set

$$d_m = d_m(K) := \begin{cases} r_1 + r_2 - 1, & m = 1, \\ r_1 + r_2, & m > 1, \text{odd} \\ r_2, & m > 0, \text{even} \end{cases} .$$

For X defined over K, write $\widetilde{X}^{an}_{\mathbb{C}} := \amalg_{\sigma \in Hom(K,\mathbb{C})} (X^{\sigma})^{an}_{\mathbb{C}}$ and

$$\begin{tikzcd} H^r_{\mathcal{M}}(X, \mathbb{Q}(p)) \arrow[rr, "\tilde{c}_{\mathcal{D},\mathbb{R}}"] \arrow[dr, "\tilde{c}^{+}_{\mathcal{D},\mathbb{R}}"'] & & H^r(\widetilde{X^{an}_{\mathbb{C}}}, \mathbb{R}(p)) \\ & H^r_{\mathcal{D}}\big(\widetilde{X^{an}_{\mathbb{C}}}, \mathbb{R}(p)\big)^{+} \arrow[ur] & \end{tikzcd}$$

for the map sending $Z \mapsto (c_{\mathcal{D},\mathbb{R}}({}^{\sigma}Z))_{\sigma}$, which factors through the invariants under de Rham conjugation. If $X = \mathrm{Spec}(K)$, then we have $H^r(\widetilde{X^{an}_{\mathbb{C}}}, \mathbb{R}(p)) \cong \mathbb{R}(p-1)^{\oplus d}$ and $H^r(\widetilde{X^{an}_{\mathbb{C}}}, \mathbb{R}(p))^{+} \cong \mathbb{R}(p-1)^{\oplus d_p}$. Write $H^r_{\mathcal{M}}(X, \mathbb{R}(p)) = H^r_{\mathcal{M}}(X, \mathbb{R}(p)) \otimes_{\mathbb{Q}} \mathbb{R}$.

Lemma 2.1.5 *For $X = Spec(K), \mathbb{G}_{m,K}^{\times n}, (\mathbb{T}_K^n, S_K^n)$, or $(\mathbb{T}_K^n, \tilde{S}_K^n)$,*

$$\tilde{c}_{\mathcal{D},\mathbb{R}}^{+} \otimes \mathbb{R} : H_{\mathcal{M}}^{r}(X, \mathbb{R}(p)) \to H_{\mathcal{D}}^{r}(\widetilde{X_{\mathbb{C}}^{an}}, \mathbb{R}(p))^{+}$$

is an isomorphism $(\forall r, p)$.

Proof By [4], the composition

$$K_{2p-1}(\mathcal{O}_K) \otimes \mathbb{Q} \xrightarrow{\cong} H_{\mathcal{M}}^{1}(\mathrm{Spec(K)}, \mathbb{Q}(p)) \xrightarrow{\tilde{c}_{\mathcal{D},\mathbb{R}}^{+}} \mathbb{R}(p-1)^{\oplus d_p} \xrightarrow{\cdot \frac{2}{(2\pi\mathbf{i})^{p-1}}} \mathbb{R}^{d_p}$$

is exactly the Borel regulator (and the groups are zero for $r \neq 1$). The lemma follows for $X = Spec(K)$.

Let Y be a smooth quasi-projective variety, defined over K, and pick $p \in \mathbb{G}_m(K)$. Write $Y \overset{\imath}{\hookrightarrow} \mathbb{G}_{m,Y} \overset{\jmath}{\hookrightarrow} \mathbb{A}_Y^1 \overset{\kappa}{\hookleftarrow} Y$ for the Cartesian products with Y of the morphisms $\mathrm{Spec}(K) \overset{\imath_p}{\hookrightarrow} \mathbb{G}_{m,K} \overset{\jmath}{\hookrightarrow} \mathbb{A}_K^1 \overset{\imath_0}{\hookleftarrow} \mathrm{Spec}(K)$. Then by the homotopy property, $\imath^* : H_{\mathcal{K}}^r(\mathbb{G}_{m,Y}, \mathbb{R}(p)) \to H_{\mathcal{K}}^r(Y, \mathbb{R}(p)) \cong H_{\mathcal{K}}^r(\mathbb{A}_Y^1, \mathbb{R}(p))$ splits the localization sequence

$$\xrightarrow{\kappa_*} H_{\mathcal{K}}^r(\mathbb{A}_Y^1, \mathbb{R}(p)) \xrightarrow{\jmath^*} H_{\mathcal{K}}^r(\mathbb{G}_{m,Y}, \mathbb{R}(p)) \xrightarrow{\mathrm{Rex}} H_{\mathcal{K}}^{r-1}(Y, \mathbb{R}(p-1)) \xrightarrow{\kappa_*}$$

for $\mathcal{K} = \mathcal{M}, \mathcal{D}$ (in particular, $\kappa_* = 0$). It follows that

$$H_{\mathcal{K}}^r(\mathbb{G}_{m,Y}, \mathbb{R}(p)) \cong H_{\mathcal{K}}^r(Y, \mathbb{R}(p)) \oplus H_{\mathcal{K}}^{r-1}(Y, \mathbb{R}(p-1)),$$

compatibly with $c_{\mathcal{D},\mathbb{R}}$; applying this iteratively gives the lemma for $\mathbb{G}_{m,K}^{\times n}$.

Finally, both $(\mathbb{T}^n, S_K^n)$ and $(\mathbb{T}_K^n, \tilde{S}_K^n)$ may be regarded as (co)simplicial normal crossing schemes $X^{\cdot}$. (That is, writing $\tilde{S}_K^n = \cup Y_i$), we take $X^0 = \mathbb{T}_K^n, X^1 = \coprod_i Y_i, X^2 = \coprod_{i<j} Y_i \cap Y_j$, etc.) We have spectra sequence $E_1^{i,j} = H_{\mathcal{K}}^{2p+j}(X^i, \mathbb{R}(p)) \Rightarrow H_{\mathcal{K}}^{2p+i+j}(X^{\cdot}, \mathbb{R}(p))$, compatible with $c_{\mathcal{D},\mathbb{R}}$, and all X^i are disjoint unions of powers of $\mathbb{G}_{m,K}$. Lemma is proved.

■

Lemma 2.1.6 *The map $\iota_n^* : H_{\mathcal{M}}^n(\mathbb{T}^n, S^n; \mathbb{A}(n)) \to H_{\mathcal{M}}^n(\mathbb{T}^{n-1}, S^{n-1}; \mathbb{A}(n))$ is zero ($\mathbb{A} = \mathbb{Q}$ or $\mathbb{R}$) .*

2.1.5 The Beilinson elements.

For any smooth quasi-projective variety X, we have

$$CH^p(X,r) \cong H_{\mathcal{M}}^{2p-r}(X,\mathbb{Z}(p)). \tag{2.4}$$

This isomorphism does not apply for singular varieties(e.g. our simplicial schemes above), and for our purposes in this paper it is the right-hand side of (2.4) that provides the correct generalization. In particular, we have

$$H_{\mathcal{M}}^r(X\times(\square^a,\partial\square^a),\mathbb{Q}(p)) \cong H_{\mathcal{M}}^{r-a}(X,\mathbb{Q}(p))$$

where $\partial\square^a$ is defined as in Chpater 1. We note here that the (rational) motivic cohomology of a cosimplicial normal-crossing scheme $X^{\cdot}$ can be computed via(the simple complex associated to)a double complex:

$$E_0^{a,b} := Z^p(X^a,-b)^{\sharp}_{\mathbb{Q}} \Rightarrow H_{\mathcal{M}}^{2p+a+b}(X^{\cdot},\mathbb{Q}(p)), \tag{2.5}$$

where "$\sharp$" denotes cycles meeting all compoenents of all $X^{q>a}\times\partial_I^{\epsilon}\square^{-b}$ properly.

Continuing to write t_i for $\frac{z_i}{z_i-1}$, we shall now consider

$$f(\underline{z}) = f_{n-1}(z_1,...,z_{n-1}) := 1-\omega^b t_1\cdots t_{n-1}$$

as a regular function on $\square_{\mathbb{F}}^{n-1}$, and

$$\mathcal{Z} := \left\{(\underline{z};f(\underline{z})),t_1^N,...,t_{n-1}^N|\underline{z}\in\square^{n-1}\setminus|(f)_0|\right\}$$

as an element of

$$\ker\Big\{Z^n(\square^{n-1}\setminus|(f)_0|,n)^{\sharp}_{\mathbb{Q}} \xrightarrow{\partial\oplus\sum(\rho_i^{\epsilon})^*}$$
$$Z^n(\square^{n-1}\setminus|(f)_0|,n-1)\oplus(\oplus_{i,\epsilon}Z^n(\square^{n-2}\setminus|(f|_{z_i=\epsilon})_0|),n)_{\mathbb{Q}}\Big\}$$

hence of $H^n_{\mathcal{M}}\big(\square^{n-1}_{\mathbb{F}} \setminus |(f)_0|), \partial\square^{n-1} \setminus \partial|(f)_0|; \mathbb{Q}(n)\big)$ (where $\partial|(f)_0| := \partial\square^{n-1} \cap |(f)_0| = \cup_{i,\epsilon}|(f|_{z_i=\epsilon})_0|$), and $\sharp$ indicates cycles meeting faces of $\partial\square^{n-1} \setminus \partial|(f)_0|$ properly). For simplicity, we write the class of Z in this group as a symbol $\{f_{n-1}, t_1^N, ..., t_{n-1}^N\}$.

We have a vertical localization exact sequence

$$\begin{tikzcd}
{} \arrow[d] & & \\
H^n_{\mathcal{M}}(\square^{n-1}, \partial\square^{n-1}; \mathbb{Q}(n)) \arrow[rr, "\cong"] \arrow[d] & & CH^n(\mathbb{F}, 2n-1) \\
H^n_{\mathcal{M}}(\square^{n-1} \setminus |(f)_0|; \partial\square^{n-1} \setminus |(f)_0|; \mathbb{Q}(n)) \arrow[d, "Res_{|(f)_0|}"'] & \ni & \{f, t_1^l, \cdots, t_{n-1}^l\} \\
H^q_{\mathcal{M}}(\mathbb{T}^{n-2}; \tilde{S}^{n-2}; \mathbb{Q}(n-1)) & & H^{n-1}_{\mathcal{M}}(\mathbb{T}^{n-1}, S^{n-1}; \mathbb{Q}(n-1)) \arrow[ll, "i^*"'] \arrow[d, phantom, sloped, "\ni"] \\
& & l^{n-1}\{t_1, \cdots t_{n-1}\}
\end{tikzcd}.$$

in which eveidently

$$\mathrm{Res}_{|(f)_0|}\{f_{n-1}, t_1^N, ..., t_{n-1}^N\} = \imath^*_{n-1}\{t_1^N, ..., t_{n-1}^N\}.$$

Proposition 2.1.7 *$\mathcal{Z}$ lifts to a class $\tilde{\Xi} \in CH^n(\mathbb{F}, 2n-1)_{\mathbb{Q}}$.*

Proof Apply (2.1.5) and Lemma 2.1.6. ∎

This is essentially Beilinson's construction; we normalize the class by

$$\Xi := \frac{(-1)^n}{N^{n-1}}\tilde{\Xi}.$$

2.2 The Higher Chow cycles

2.2.1 Representing Beilinson's elements.

We first describe (2.5) more explicitly in the relevant cases. As above, write $\partial : Z^n(\square^r, s)^\sharp_{\mathbb{Q}} \to Z^n(\square^r, s-1)^\sharp_{\mathbb{Q}}$ for the higher Chow differential, and

$$\delta : Z^n(\square^r, s)^\sharp_{\mathbb{Q}} \to \oplus_{i,\epsilon} Z^n(\square^{r-1}, s)^\sharp_{\mathbb{Q}}$$

for the cosimplicial differential $\sum_{i=1}^r (-1)^{i-1}((\rho_i^0 \times id_{\square^s})^* - (\rho_i^\infty \times \mathrm{id}_{\square^s})^*)$. A complex of cocycles for the top motivic cohomology group in (2.1.5) is given by $\mathcal{J}^n_\square(k) :=$

$$Z^n_{\mathcal{M}}\big((\square^{n-1}_{\mathbb{F}}, \partial\square^{n-1}_{\mathbb{F}}), k\big)_{\mathbb{Q}} := \bigoplus_{a=0}^{n-1} \bigoplus_{\substack{(I,\epsilon) \\ |I|=a}} Z^n\big(\square^{n-a-1}_{\mathbb{F}}, a+k\big)^\sharp_{\mathbb{Q}} \tag{2.6}$$

with differential $\mathbb{D} := \partial + (-1)^{n-a-1}\delta$. These are, of course, the simple complex resp. total differential associated to the natural double complex $E_0^{a,b} = \bigoplus_{\substack{(I,\epsilon) \\ |I|=a}} Z^n\big(\square^{n-a-1}_{\mathbb{F}}, -b\big)^\sharp_{\mathbb{Q}}$. Analogously one defines $\mathcal{J}^n_{\square\backslash f}(k) := Z^n_{\mathcal{M}}\big((\square^{n-1}_{\mathbb{F}} \backslash |(f)_0|, \partial\square^{n-1}_{\mathbb{F}} \backslash \partial|(f)_0|), k\big)_{\mathbb{Q}}$ and $\mathcal{J}^{n-1}_f(k) := Z^{n-1}_{\mathcal{M}}\big((\mathbb{T}^{n-2}, \tilde{S}^{n-2}), k\big)_{\mathbb{Q}}$ so that $\mathcal{J}^{n-1}_f(\bullet) \xrightarrow{\imath_*} \mathcal{J}^n_\square(\bullet) \to \mathcal{J}^n_{\square\backslash f}(\bullet)$ are morphisms of (homological) complexes.

Now define

$$\theta : \mathcal{J}^n_\square(k) \to Z^n(\mathbb{F}, n+k-1)_{\mathbb{Q}}$$

by simply adding up the cycles (with no signs) on the right-hand side of (2.6). (Use the natural maps $\square^{n-a-1} \times \square^{a+k} \to \square^{n+k-1}$ obtained by concatenating coordinates.) Then we have

Lemma 2.2.1 *θ is a quasi-isomorphism of complexes.*

Proof Checking that θ is a morphism of complexes is easy and left to reader. The $a = n-1, (I,\epsilon) = (\{1,...,n-1\}, \underline{0})$ term of (2.6) is a copy of $Z^n(\mathbb{F}, n+k-1)$ in $\mathcal{J}^n_\square(k)$

which leads to a morphism $\psi : Z^n(\mathbb{F}, n + \bullet - 1) \to \mathcal{J}^n_\square(\bullet)$ with $\theta \circ \psi = \mathrm{id}$. Moreover, it is elementary that ψ is a quasi-isomorphism: taking $d_0 = \partial$ gives

$$E_1^{a,b} = \bigoplus_{\substack{(I,\epsilon) \\ |I|=a}} CH^n(\square_{\mathbb{F}}^{n-a-1}, -b)_{\mathbb{Q}} \cong CH^n(\mathbb{F}, -b)^{\oplus 2^a \binom{n-1}{a}},$$

hence $E_2^{a,b} = 0$ except for $E_2^{n-1,b} \cong CH^n(\mathbb{F}, -b)$, which is exactly the image of $\psi(\ker\partial)$.

$\blacksquare$

By the moving lemmas of Bloch [19] and [20], we have another quasi-isomorphism

$$\frac{\mathcal{J}^n_\square(\bullet)}{\imath_* \mathcal{J}^{n-1}_f(\bullet)} \xrightarrow{\simeq} \mathcal{J}^n_{\square\backslash f}(\bullet),$$

which enables us to replace any $\mathcal{Y}_{\square\backslash f} \in \ker(\mathbb{D}) \subset \mathcal{J}^n_{\square\backslash f}(n)$ by a homologous $\mathcal{Y}'_{\square\backslash f}$ arising as the restriction of some $\mathcal{Y}'_\square(n)$ with $\mathbb{D}\mathcal{Y}'_\square = \imath_*(\mathcal{Y}''_f)$, $\mathcal{Y}''_f \in \ker(\mathbb{D}) \in \mathcal{J}^{n-1}_f(n-1)$. This gives an "explicit" prescription for computing $\mathrm{Res}_{|(f)_0|}$ in (2.1.5).

Now we come to our central point: the cycle $\mathcal{Z} = \{f_{n-1}, t_1^N, ..., t_{n-1}^N\}$ already belongs to $\big(Z^n(\square_{\mathbb{F}}^{n-1}, n)^\sharp_{\mathbb{Q}} \subset \big)\mathcal{J}^n_\square(n)$, without "moving" it by a boundary. Its restriction to $\mathcal{J}^n_{\square\backslash f}(n)$ is clearly $\mathbb{D}$-closed, and $\mathbb{D}\mathcal{Z} = \imath_*\{t_1^N, ..., t_{n-1}^N\} =: \imath_*\mathcal{T}$. By Proposition 2.1.7, the class of $\mathcal{T}$ in homology of $\mathcal{J}^{n-1}_f(\bullet)$ is trivial, and so there exists $\mathcal{T}' \in \mathcal{J}^{n-1}_f(n)$ with $\mathbb{D}\mathcal{T}' = -\mathcal{T}$. Writing

$$\mathcal{W} := \imath_*\mathcal{T}', \quad \tilde{Z} := \mathcal{Z} + \mathcal{W},$$

we now have $\mathbb{D}\tilde{\mathcal{Z}} = 0$. This allows us to make a rather precise statement about the lift in Proposition 2.1.7. Writing $\pi_i : \square^{2n-1} \twoheadrightarrow \square^{n-i}$ for the projection $(z_1, ..., z_{2n-1}) \mapsto (z_1, ..., z_{n-i})$, we have

Theorem 2.2.1 *$\tilde{\Xi}$ has a representative in $Z^n(\mathbb{F}, 2n-1)_{\mathbb{Q}}$ of the form*

$$\tilde{\mathscr{L}} = \mathscr{L} + \mathscr{W} = \mathscr{L} + \mathscr{W}_1 + \mathscr{W}_2 + \cdots + \mathscr{W}_{n-1}, \tag{2.7}$$

where $\mathscr{L} = \theta(\mathcal{Z})$ (i.e., $\mathcal{Z}$ interpreted as an element of $Z^n(\mathbb{F}, 2n-1)_{\mathbb{Q}}$) and $\mathscr{W}_i$ is supported on $\pi_i^{-1}|(f_{n-i})_0|$.

Proof Viewing $(|(f_{n-1})_0|, \partial|(f_{n-1})_0|) \cong (\mathbb{T}^{n-2}, \tilde{S}^{n-2})$ as a simplicial subscheme $\mathfrak{X}^\bullet$ of $(\square^{n-1}, \partial\square^{n-1}) =: X^\bullet$, $\mathfrak{X}^{i-1} \subset X^{i-1}$ comprises $2^{i-1}\binom{n-1}{i-1}$ copies of $|(f_{n-i})_0| \subset \square^{n-1}$. We may decompose

$$\mathcal{W} \in \bigoplus_{i=1}^{n} \bigoplus_{\substack{(I,\epsilon)\\ |I|=i-1}} \imath_* Z^{n-1}(|(f_{n-i})_0|, n+i-1)^\sharp_{\mathbb{Q}} \subset \bigoplus_{i=1}^{n-1} E_0^{i-1,-n-i+1}$$

into its constituent pieces $\mathcal{W}_i \in E_0^{i-1,-n-i+1}$, and define $\mathscr{W}_i := \theta(\mathcal{W}_i)$ and $\mathscr{W} := \theta(\mathcal{W})$. Clearly $\mathrm{supp}(\mathscr{W}_i) \subset \pi_i^{-1}|(f_{n-i})_0|$, and $\tilde{\mathscr{L}} := \theta(\tilde{\mathcal{Z}})$ is ∂-closed, giving the desired representation. ■

Remark 2.2.2 *In fact, $\sigma(\mathscr{L})$ belongs to $Z^n_{\mathbb{R}}(Spec(\mathbb{C}), 2n-1)_{\mathbb{Q}}$ for $\sigma \in Hom(\mathbb{F}, \mathbb{C})$: the intersections $T_{z_1} \cap \cdots T_{z_k} (\rho_I^\epsilon)^* \sigma(\mathscr{L})$ are empty excepting $T_{z_1} \cap \cdots T_{z_k} \cap \sigma(\mathscr{L})$ for $k \leq n-1$ and $T_{z_1} \cap \cdots T_{z_k} \cap (\rho_n^0)^* \sigma(\mathscr{L})$ for $k \leq n-2$, which are both of the expected real codimension. A trivial modification of the above argument them shows that the $\mathscr{W}_i$ may be chosen so that the $\sigma(\mathscr{W}_i)$ (and hence $\sigma(\tilde{\mathscr{L}})$)) are in $Z^n_{\mathbb{R}}(Spec(\mathbb{C}, 2n-1)_{\mathbb{Q}}$ as well. We shall henceforth assume that this has been done.*

2.3 Computing the KLM map.

We begin by simplifying the formula (2.2) for the regulator map.

Lemma 2.3.1 *Let $K \subset \mathbb{C}$ and suppose $Z \in ker(\partial) \subset Z^n_{\mathbb{R}}(Spec(K), 2n-1)_{\mathbb{Q}}$ satisfies*

$$T_{z_1} \cap \cdots T_{z_n} \cap Z^{an}_{\mathbb{C}} = \emptyset. \tag{2.8}$$

Then

$$c_{\mathcal{D},\mathbb{Q}}(Z) = \int_{Z^{an}_{\mathbb{C}} \cap T_{z_1} \cdots \cap T_{z_{n-1}}} log(z_n) \frac{dz_{n+1}}{z_{n+1}} \wedge \cdots \wedge \frac{dz_{2n-1}}{z_{2n-1}}$$

in $\mathbb{C}/\mathbb{Q}(n)$.

Proof We have $c_{\mathcal{D},\mathbb{Q}}(Z) =$

$$\sum_{k=1}^{n-1}(2\pi\mathbf{i})^{k-n}\int_{Z_{\mathbb{C}}^{an}} R_{2n-1}^{(k)} + \int_{Z_{\mathbb{C}}^{an}} R_{2n-1}^{(n)} + \sum_{k=1}^{n-1}(2\pi\mathbf{i})^{k-n}\int_{Z_{\mathbb{C}}^{an}} R_{2n-1}^{(n+k)}.$$

The terms $\int_{Z_{\mathbb{C}}^{an}} R_{2n-1}^{k}$ are zero by type, since $\dim_{\mathbb{C}} Z_{\mathbb{C}} = n-1$, and the $\int_{Z_{\mathbb{C}}^{an}} R_{2n-1}^{(n+k)}$ are integrals over $Z_{\mathbb{C}}^{an} \cap T_{z_1} \cap \cdots T_{z_{n+k-1}} = \emptyset$. So only the middle term remains. ■

Lemma 2.3.2 *For any* $\sigma \in Hom(\mathbb{F}, \mathbb{C})$, $T_{z_1} \cap \cdots T_{z_n} \cap \sigma(\tilde{\mathscr{L}}) = \emptyset$.

Proof From Theorem 2.2.1, $\sigma(\mathscr{W}_i)$ is supported over $\pi_i^{-1}(|(f_{n-i})_0|)$; that is, on $\sigma(\mathscr{W}_i)$ we have $z_1 \cdots z_{n-i} = \bar{\zeta}^b$, and so $T_{z_1} \cap \cdots \cap T_{z_{n-i}} \cap \sigma(\mathscr{W}) = \emptyset$ since $\bar{\zeta}^b \notin (-1)^{n-i}\mathbb{R}_+$. On $\sigma(\mathscr{L})$, $z_n = f_{n-1}(z_1, \cdots z_{n-1}) = 1 - \zeta^b t_1 \cdots t_{n-1}$ (where $t_i = \frac{z_i}{z_i - 1}$); and on T_{z_i}, $t_i \in [0,1]$. It follows that on $T_{z_1} \cap \cdots T_{z_n} \cap \sigma(\mathscr{L})$, z_n belongs to $\mathbb{R}_- \cap \big(1 - \zeta^b[0,1]\big)$, which is empty.

■

We may now compute the regulator on the cycle of $Theorem 2.2.1$, independently of the choice of the $\mathscr{W}_i$:

Theorem 2.3.1 $c_{\mathcal{D},\mathbb{Q}}(\sigma(\Xi)) = Li_n(\zeta^b) \in \mathbb{C}/\mathbb{Q}(n)$.

Proof By Lemmas 2.3.1 and 2.3.2, we obtain $c_{\mathcal{D},\mathbb{Q}}(\sigma(\tilde{\mathscr{L}})) =$

$$\int_{\sigma(\mathscr{L})_{\mathbb{C}}^{an} \cap T_{z_1} \cap \cdots \cap T_{z_{n-1}}} \log(z_n)\frac{dz_{n+1}}{z_{n+1}} \wedge \cdots \wedge \frac{dz_{2n-1}}{z_{2n-1}}$$
$$+ \sum_{i=1}^{n-1}\int_{\sigma(\mathscr{W}_i)_{\mathbb{C}}^{an} \cap T_{z_1} \cap \cdots \cap T_{z_{n-1}}} \log(z_n)\frac{dz_{n+1}}{z_{n+1}} \wedge \cdots \wedge \frac{dz_{2n-1}}{z_{2n-1}},$$

in which the second term is 0 since $\sigma(\mathscr{W}_i)^{an}_{\mathbb{C}} \cap T_{z_1} \cap \cdots \cap T_{z_{n-1}} = \emptyset$ $(\forall i)$. The remaining (first) term becomes

$$\begin{gathered}\int_{\bar{z}\in\mathbb{R}_-^{\times(n-1)}} \log(f_{n-1}(\bar{z}))\frac{dt_1^N}{t_1^N}\wedge\cdots\wedge\frac{dt_{n-1}^N}{t_{n-1}^N} = \\ (-N)^{n-1}\int_{\bar{t}\in[0,1]^{\times(n-1)}} \log(1-\zeta^b t_1\cdots t_{n-1})\frac{dt_1}{t_1}\wedge\cdots\frac{dt_{n-1}}{t_{n-1}} = \\ (-N)^{n-1}\int_0^{\zeta^b}\int_0^{u_{n-1}}\cdots\int_0^{u_2}\log(1-u_1)\frac{du_1}{u_1}\wedge\cdots\frac{du_{n-1}}{u_{n-1}} = \\ (-1)^n N^{n-1} Li_n(\zeta^b),\end{gathered} \tag{2.9}$$

where $u_{n-1} = \zeta^b t_{n-1}, u_{n-2} = \zeta^b t_{n-2}t_{n-1}, ..., u_1 = \zeta^b t_1\cdots t_{n-1}$. ■

To write the image of our cycles under the Borel regulator, we refine notation by writing σ_a (for $\sigma : \omega \mapsto e^{\frac{2\pi \mathbf{i} a}{N}}$), $f_{n-1,b} = 1-\omega^b t_1\cdots t_{n-1}$, Ξ_b, $\tilde{\mathscr{L}}_b$, $\mathscr{L}_b$, etc. So Theorem 2.3.1 reads $c_{\mathcal{D},\mathbb{Q}}(\sigma_a(\Xi_b)) = (-1)^n N^{n-1} Li_n(e^{\frac{2\pi \mathbf{i} ab}{N}})$, and one has the

Corollary 2.3.3 *Let $N \geq 3$ and set*

$$A := \left\{a \in \mathbb{N} \,\middle|\, (a,N) = 1 \text{ and } 1 \leq a \leq \left\lfloor \tfrac{N}{2} \right\rfloor \right\};$$

then for any $b \in A$,

$$\tilde{c}^+_{\mathcal{D},\mathbb{R}}(\Xi_b) = \left(\pi_n(Li_n(e^{\frac{2\pi \mathbf{i} ab}{N}}))\right)_{a\in A} \in \mathbb{R}(n-1)^{\oplus\frac{1}{2}\phi(N)},$$

where $\pi_n : \mathbb{C} \to \mathbb{R}(n-1)$ is $i\mathrm{Im}$ [resp. Re] for n even [resp. odd]. If $N = 2$, then $\tilde{c}^+_{\mathcal{D},\mathbb{R}} = 0$ for n even and $\tilde{c}^+_{\mathcal{D},\mathbb{R}}(\Xi_1) = \zeta(n) \in \mathbb{R}(n-1)$ for n odd.

As an immediate consequence, we get a (rational) basis for the higher Chow cycles on a point over any abelian extension of $\mathbb{Q}$:

Corollary 2.3.4 *The $\{\Xi_b\}_{b\in A}$ span $CH^n(\mathbb{F}, 2n-1)_{\mathbb{Q}}$. Moreover, for any subfield $\mathbb{E} \subset \mathbb{F}$, with $\Gamma = Gal(\mathbb{F}/\mathbb{E})$, there exists a subset $B \subset A$ (with $|B| = d_n(\mathbb{E})$) such that the $\left\{\sum_{\gamma\in\Gamma} {}^{\gamma}\Xi_b\right\}_{b\in B}$ span $CH^n(\mathbb{E}, 2n-1)_{\mathbb{Q}}$.*

Proof In view of Lemma 2.1.5, for the first statement we need only check the linear independence of the vectors $\underline{v}^{(b)}$ in Corollary 2.3.3. Let χ be one of the $\frac{1}{2}\phi(N)$ Dirichlet characters modulo N with $\chi(-1) = (-1)^{n-1}$; and let $\rho_\alpha : \mathbb{C}^{|A|} \to \mathbb{C}^{|A|}$ be the permutation operator defined by $\mu(\underline{v})_j = v_{\alpha \cdot j}$, where $\alpha \in (\mathbb{Z}/N\mathbb{Z})^*$ is a generator. Then the linear combinations

$$\underline{v}^\chi := \sum_{b \in A} \chi(b) \underline{v}^{(b)} = \left(\frac{1}{2} \sum_{b=1}^{N} \chi(b) \pi_n \left(Li_n(e^{\frac{2\pi \mathbf{i} ab}{N}}) \right) \right)_{a \in A}$$

are independent (over $\mathbb{C}$) provided they are nonzero, since their eigenvalues $\overline{\chi(\alpha)}$ under ρ_α are distinct. By the computation in [21], if χ is induced from a primitive character χ_0 modulo $N_0 = N/M$, then (with $\mu =$ Möbius function, $\tau(\cdot) =$ Gauss sum)

$$v_1^\chi = \frac{1}{2M^{n-1}} \left\{ \sum_{d|M} \mu(d) \chi_0(d) d^{n-1} \right\} \tau(\chi_0) L(\overline{\chi_0}, n),$$

the last two factors of which are nonzero by primitivity of χ_0; the bracketed term is $\prod_{\substack{p > 1 \text{ prime} \\ p|M}} (1 - \chi_0(p) p^{n-1})$, hence also nonzero.

The second statement follows at once, since the composition of $\sum_{\gamma \in \Gamma}$ with $CH^n(\mathbb{E}, 2n-1)_\mathbb{Q} \hookrightarrow CH^n(\mathbb{F}, 2n-1)_\mathbb{Q}$ is a multiple of the identity. ■

2.4 Explicit representatives

We finally turn to the construction of the cycles described by Theorem 2.2.1. Here the benefit of using t_i^N (at least, if one is happy to work rationally) comes to the fore: it allows us to obtain uniform formulas for all N, and to use as few terms as possible; in fact, it turns out that *for all* n it is possible to take $\mathscr{W}_3 = \cdots = \mathscr{W}_{n-1} = 0$. (While it is easy to argue abstractly that $\mathscr{W}_{n-1}$ can always be taken to be zero, this stronger statement surprised us.) For brevity, we shall use the notation $(f_1(\underline{t},u,v),\ldots,f_m(\underline{t},u,v))$ for

$$\left\{(f_1(\underline{t},u,v),\ldots,f_m(\underline{t},u,v))|\,t_i,u,v\in\mathbb{P}^1\right\}\cap\square^m;$$

all precycles are defined over $\mathbb{F}=\mathbb{Q}(\omega)$, and we write $\xi:=\omega^b$.

2.4.1 K_3 case $(n=2)$

Let $\mathscr{Z}=\left(\frac{t}{t-1},1-\xi t,t^N\right)$, as dictated by Theorem **??**; then all $\partial_i^\varepsilon\mathscr{Z}=0$. In particular,

$$\partial_1^0\mathscr{Z}=\left(1-\xi t,t^N\right)|_{\frac{t}{t-1}=0}=(1,0)=0$$

and

$$\partial_2^0\mathscr{Z}=\left(\frac{\xi^{-1}}{\xi^{-1}-1},\xi^{-N}\right)=\left(\frac{1}{1-\xi},1\right)=0.$$

So we may take $\mathscr{W}=0$ and $\tilde{\mathscr{Z}}=\mathscr{Z}$.

In contrast, if we took $\mathscr{Z}=\left(\frac{t}{t-1},1-\xi t,t\right)$, then $\partial_2^0\mathscr{Z}=\left(\frac{1}{1-\xi},\xi^{-1}\right)$ and a nonzero $\mathscr{W}$-term is required. (Of course, if one wants to treat torsion, this approach becomes necessary.)

2.4.2 K_5 case $(n=3)$

Of course $\mathscr{Z}=\left(\frac{t_1}{t_1-1},\frac{t_2}{t_2-1},1-\xi t_1t_2,t_1^N,t_2^N\right)$. Taking

$$\mathscr{W}_1=\frac{1}{2}\left(\frac{t_1}{t_1-1},\frac{1}{1-\xi t_1},\frac{(u-t_1^N)(u-t_1^{-N})}{(u-1)^2},t_1^Nu,\frac{u}{t_1^N}\right),$$

we note that $z_2=\frac{1}{1-\xi t_1}\implies t_2=\frac{(1-\xi t_1)^{-1}}{(1-\xi t_1)^{-1}-1}=\frac{1}{\xi t_1}\implies f_2(t_1,t_2)=0$. Now we have

$$\partial\mathscr{Z}=\partial_3^0\mathscr{Z}=\left(\frac{t_1}{t_1-1},\frac{t_2}{t_2-1},t_1^N,t_2^N\right)\Big|_{1-\xi t_1t_2=0}=\left(\frac{t_1}{t_1-1},\frac{1}{1-\xi t_1},t_1^N,\frac{1}{t_1^N}\right)$$

and

$$\partial\mathscr{W}_1=-\partial_3^\infty\mathscr{W}_1=-2\cdot\frac{1}{2}\left(\frac{t_1}{t_1-1},\frac{1}{1-\xi t_1},t_1^N,\frac{1}{t_1^N}\right)=-\partial\mathscr{Z}.$$

Therefore $\tilde{\mathscr{Z}} = \mathscr{Z} + \mathscr{W}_1$ is closed.

Remark 2.4.1 *See [11, §3.1] for a detailed discussion of properties of these cycles, esp. the (integral!) distribution relations of [loc. cit., Prop. 3.1.26].*

In particular, we can specialize to $N = 2$ to obtain

$$2\tilde{\mathscr{Z}} = 2\left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, 1 + t_1t_2, t_1^2, t_2^2\right) + \left(\tfrac{t_1}{t_1-1}, \tfrac{1}{1-t_1}, \tfrac{(u-t_1^2)(u-t_1^{-2})}{(u-1)^2}, t_1^2 u, \tfrac{u}{t_1^2}\right)$$

in $Z^3_{\mathbb{R}}(\mathbb{Q}, 5)$, spanning $CH^3(\mathbb{Q}, 5)_{\mathbb{Q}} \cong K_5(\mathbb{Q})_{\mathbb{Q}}$, with

$$c_{\mathcal{D},\mathbb{Q}}(2\tilde{\mathscr{Z}}) = -8Li_3(-1) = 6\zeta(3) \in \mathbb{C}/\mathbb{Q}(3).$$

2.4.3 K_7 case ($n = 4$)

Set

$$\mathscr{Z} = \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{t_3}{t_3-1}, 1 - \xi t_1t_2t_3, t_1^N, t_2^N, t_3^N\right), \quad \mathscr{W}_1 = \tfrac{1}{2}\left(\mathscr{W}_1^{(1)} + \mathscr{W}_1^{(2)}\right),$$

$$\mathscr{W}_1^{(1)} = \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{1}{1-\xi t_1t_2}, \tfrac{(u-t_1^N)(u-t_2^N)}{(u-1)(u-t_1^Nt_2^N)}, \tfrac{u}{t_1^N}, \tfrac{u}{t_2^N}, \tfrac{1}{u}\right),$$

$$\mathscr{W}_1^{(2)} = \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{1}{1-\xi t_1t_2}, \tfrac{(u-t_1^N)(u-t_2^N)}{(u-1)(u-t_1^Nt_2^N)}, \tfrac{t_1^N}{u}, \tfrac{t_2^N}{u}, \tfrac{u}{t_1^Nt_2^N}\right),$$

$$\mathscr{W}_2 = -\tfrac{1}{2}\left(\tfrac{t_1}{t_1-1}, \tfrac{1}{1-\xi t_1}, \tfrac{(v-t_1^Nu)(v-ut_1^{-N})}{(v-u^2)(v-1)}, \tfrac{(u-t_1^N)(u-vt_1^{-N})}{(u-v)^2}, \tfrac{vt_1^N}{u}, \tfrac{v}{t_1^Nu}, \tfrac{u}{v}\right).$$

Direct computation shows

$$\partial\mathscr{Z} = -\partial_4^0\mathscr{Z} = -\partial_4^\infty\mathscr{W}_1^{(1)} = -\partial_4^\infty\mathscr{W}_1^{(2)},$$

$$\partial\mathscr{W}_1 = -\tfrac{1}{2}\partial_3^\infty\mathscr{W}_1^{(1)} + \tfrac{1}{2}\partial_4^\infty\mathscr{W}_1^{(1)} - \tfrac{1}{2}\partial_3^\infty\mathscr{W}_1^{(2)} + \tfrac{1}{2}\partial_4^\infty\mathscr{W}_1^{(2)},$$

$$\partial\mathscr{W}_2 = -\partial_3^\infty\mathscr{W}_2 = \tfrac{1}{2}\partial_3^\infty\mathscr{W}_1^{(1)} + \tfrac{1}{2}\partial_3^\infty\mathscr{W}_1^{(2)},$$

which sum to zero.

Alternately, we can take

$$\mathscr{W}_1 = \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{1}{1-\xi t_1 t_2}, \tfrac{(u-t_1^N)(u-t_2^N)}{(u-1)(u-t_1^N t_2^N)}, \tfrac{t_1^N}{u}, \tfrac{t_2^N}{u}, \tfrac{u}{t_1^N t_2^N} \right),$$

$$\mathscr{W}_2 = \left(\tfrac{t_1}{t_1-1}, \tfrac{1}{1-\xi t_1}, \tfrac{(u-vt_1^N)(u-vt_1^{-N})}{(u-v)^2}, \tfrac{vt_1^N}{u}, \tfrac{v}{t_1^N u}, \tfrac{u}{v}, v-1 \right).$$

Writing

$$\mathscr{V}_1 = \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{1}{1-\xi t_1 t_2}, t_1^N, t_2^N, \tfrac{1}{t_1^N t_2^N} \right),$$

$$\mathscr{V}_2 = \left(\tfrac{t_1}{t_1-1}, \tfrac{1}{1-\xi t_1}, \tfrac{(u-t_1^N)(u-t_1^{-N})}{(u-1)^2}, \tfrac{t_1^N}{u}, \tfrac{1}{t_1^N u}, u \right),$$

one has $\partial\mathscr{Z} = -\mathscr{V}_1$, $\partial\mathscr{W}_1 = -\mathscr{V}_2 + \mathscr{V}_1$, $\partial\mathscr{W}_2 = \mathscr{V}_2$; so again $\tilde{\mathscr{Z}}$ is a closed cycle.

For the general n construction to appear natural, we need to write out one more case.

2.4.4 K_9 case ($n = 5$)

Begin by writing

$$\mathscr{Z} = \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{t_3}{t_3-1}, \tfrac{t_4}{t_4-1}, 1-\xi t_1 t_2 t_3 t_4, t_1^N, t_2^N, t_3^N, t_4^N \right),$$

$$\mathscr{W}_1 = \tfrac{1}{2} \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{t_3}{t_3-1}, \tfrac{1}{1-\xi t_1 t_2 t_3}, \tfrac{(u-t_1^l)(u-t_2^l)(u-t_3^l)}{(u-1)^2(u-t_1^l t_2^l t_3^l)}, \tfrac{t_1^N}{u}, \tfrac{t_2^N}{u}, \tfrac{t_3^N}{u}, \tfrac{u}{t_1^N t_2^N t_3^N} \right),$$

$$\mathscr{W}_2^{(1)} = \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{1}{1-\xi t_1 t_2}, \tfrac{(u-t_1^N v)(u-t_2^N v)}{(u-t_1^N t_2^N v)(u-v)}, \tfrac{v}{u}, \tfrac{t_1^N v}{u}, \tfrac{t_2^N v}{u}, \tfrac{u}{v t_1^N t_2^N}, v-1 \right),$$

$$\mathscr{W}_2^{(2)} = \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{1}{1-\xi t_1 t_2}, \tfrac{(u-t_1^N v)(u-t_2^N v)}{(u-t_1^N t_2^N v)(u-v)}, \tfrac{v t_1^N}{u}, \tfrac{v}{u}, \tfrac{t_2^N v}{u}, \tfrac{u}{v t_1^N t_2^N}, v-1 \right),$$

$$\mathscr{W}_2^{(3)} = \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{1}{1-\xi t_1 t_2}, \tfrac{(u-t_1^N v)(u-t_2^N v)}{(u-t_1^N t_2^N v)(u-v)}, \tfrac{v t_1^N}{u}, \tfrac{v t_2^N}{u}, \tfrac{v}{u}, \tfrac{u}{v t_1^N t_2^N}, v-1 \right),$$

$$\mathscr{W}_2^{(4)} = \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{1}{1-\xi t_1 t_2}, \tfrac{(u-t_1^N v)(u-t_2^N v)}{(u-vt_1^{-N} t_2^{-N})^{-1}(u-v)^3}, \tfrac{v t_1^N}{u}, \tfrac{v t_2^N}{u}, \tfrac{v}{u t_1^N t_2^N}, \tfrac{u}{v}, v-1 \right),$$

$$\mathscr{W}_2 = \tfrac{1}{2} \left(\mathscr{W}_2^{(1)} - \mathscr{W}_2^{(2)} + \mathscr{W}_2^{(3)} - \mathscr{W}_2^{(4)} \right).$$

To compute the boundaries, introduce

$$\mathscr{U}_1 = \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{t_3}{t_3-1}, \tfrac{1}{1-\xi t_1 t_2 t_3}, t_1^N, t_2^N, t_3^N, \tfrac{1}{t_1^N t_2^N t_3^N}\right),$$

$$\mathscr{U}_2 = \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{1}{1-\xi t_1 t_2}, \tfrac{(u-t_1^N)(u-t_2^N)}{(u-t_1^N t_2^N)(u-1)}, \tfrac{1}{u}, \tfrac{t_1^N}{u}, \tfrac{t_2^N}{u}, \tfrac{u}{t_1^N t_2^N}\right),$$

$$\mathscr{U}_3 = \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{1}{1-\xi t_1 t_2}, \tfrac{(u-t_1^N)(u-t_2^N)}{(u-t_1^N t_2^N)(u-1)}, \tfrac{t_1^N}{u}, \tfrac{1}{u}, \tfrac{t_3^N}{u}, \tfrac{u}{t_1^N t_2^N}\right),$$

$$\mathscr{U}_4 = \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{1}{1-\xi t_1 t_2}, \tfrac{(u-t_1^N)(u-t_2^N)}{(u-t_1^N t_2^N)(u-1)}, \tfrac{t_1^N}{u}, \tfrac{t_2^N}{u}, \tfrac{1}{u}, \tfrac{u}{t_1^N t_2^N}\right),$$

$$\mathscr{U}_5 = \left(\tfrac{t_1}{t_1-1}, \tfrac{t_2}{t_2-1}, \tfrac{1}{1-\xi t_1 t_2}, \tfrac{(u-t_1^N)(u-t_2^N)(u-t_1^{-N}t_2^{-N})}{(u-1)^3}, \tfrac{t_1^N}{u}, \tfrac{t_2^N}{u}, \tfrac{1}{u t_1^N t_2^N}, u\right)$$

and

$$\mathscr{V}_1 = \left(\tfrac{t_1}{t_1-1}, \tfrac{1}{1-\xi t_1}, \tfrac{(u-t_1^N v)(u-t_1^{-N}v)}{(u-v)^2}, \tfrac{v}{u}, \tfrac{t_1^N v}{u}, \tfrac{v}{u t_1^N}, \tfrac{u}{v}, v-1\right),$$

$$\mathscr{V}_2 = \left(\tfrac{t_1}{t_1-1}, \tfrac{1}{1-\xi t_1}, \tfrac{(u-t_1^N v)(u-t_1^{-N}v)}{(u-v)^2}, \tfrac{v t_1^N}{u}, \tfrac{v}{u}, \tfrac{v}{t_1^N u}, \tfrac{u}{v}, v-1\right),$$

$$\mathscr{V}_3 = \left(\tfrac{t_1}{t_1-1}, \tfrac{1}{1-\xi t_1}, \tfrac{(u-t_1^N v)(u-t_1^{-N}v)}{(u-v)^2}, \tfrac{v t_1^N}{u}, \tfrac{v}{u t_1^N}, \tfrac{v}{u}, \tfrac{u}{v}, v-1\right).$$

Then $\partial\mathscr{Z} = \mathscr{U}_1$, $\partial\mathscr{W}_1 = -\mathscr{U}_1 + \frac{1}{2}(-\mathscr{U}_2 + \mathscr{U}_3 - \mathscr{U}_4 + \mathscr{U}_5)$, $\partial\mathscr{W}_2^{(1)} = -\mathscr{V}_1 + \mathscr{U}_2$, $\partial\mathscr{W}_2^{(2)} = -\mathscr{V}_2 + \mathscr{U}_3$, $\partial\mathscr{W}_2^{(3)} = -\mathscr{V}_3 + \mathscr{U}_4$, and $\partial\mathscr{W}_2^{(4)} = \mathscr{U}_5 - \mathscr{V}_1 + \mathscr{V}_2 - \mathscr{V}_3$; and so $\tilde{\mathscr{Z}}$ is closed.

As for $n = 3$, we obtain a generator for $CH^5(\mathbb{Q}, 9)_{\mathbb{Q}} \cong K_9(\mathbb{Q})_{\mathbb{Q}}$ by setting $N = 2$ and $\xi = -1$; the integral cycle $2\tilde{\mathscr{Z}}$ has $c_{\mathcal{D},\mathbb{Q}}(2\tilde{\mathscr{Z}}) = 15\zeta(5)$.

2.4.5 General n construction $(n \geq 4)$

To state the final result, we define

$$\mathscr{Z} := \left(\tfrac{t_1}{t_1-1}, \ldots, \tfrac{t_{n-1}}{t_{n-1}-1}, 1-\xi t_1 \cdots t_{n-1}, t_1^N, \ldots, t_{n-1}^N\right),$$

$$\mathscr{W}_1 := \tfrac{1}{n-3}\tilde{\mathscr{W}}_1 := \tfrac{(-1)^{n-1}}{n-3} \times$$

$$\left(\tfrac{t_1}{t_1-1}, \ldots, \tfrac{t_{n-2}}{t_{n-2}-1}, \tfrac{1}{1-\xi t_1 \cdots t_{n-2}}, \tfrac{(u-t_1^N)\cdots(u-t_{n-2}^N)}{(u-t_1^N\cdots t_{n-2}^N)(u-1)^{n-3}}, \tfrac{t_1^N}{u}, \ldots, \tfrac{t_{n-2}^N}{u}, \tfrac{u}{t_1^N\cdots t_{n-2}^N}\right),$$

and

$$\mathscr{W}_2 := \tfrac{1}{n-3}\sum_{i=1}^{n-1}(-1)^{i-1}\mathscr{W}_2^{(i)},$$

where for $1 \le i \le n-2$, $\mathscr{W}_2^{(i)} :=$

$$\Big(\tfrac{t_1}{t_1-1}, \ldots, \tfrac{t_{n-3}}{t_{n-3}-1}, \tfrac{1}{1-\xi t_1\cdots t_{n-3}}, \tfrac{(u-t_1^N v)\cdots(u-t_{n-3}^N v)}{(u-t_1^N\cdots t_{n-3}^N v)(u-v)^{n-4}},$$
$$\tfrac{vt_1^N}{u}, \ldots, \tfrac{v}{u}, \ldots, \tfrac{vt_{n-3}^N}{u}, \tfrac{u}{vt_1^N\cdots t_{n-3}^N}, v-1\Big)$$

(with $\frac{v}{u}$ occurring in the $(n+i-1)^{\text{th}}$ entry[1]) and $\mathscr{W}_2^{(n-1)} :=$

$$\Big(\tfrac{t_1}{t_1-1}, \ldots, \tfrac{t_{n-3}}{t_{n-3}-1}, \tfrac{1}{1-\xi t_1\cdots t_{n-3}}, \tfrac{(u-t_1^N v)\cdots(u-t_{n-3}^N v)}{(u-t_1^{-N}\cdots t_{n-3}^{-N} v)^{-1}(u-v)^{n-2}},$$
$$\tfrac{vt_1^N}{u}, \ldots, \tfrac{vt_{n-3}^N}{u}, \tfrac{v}{ut_1^N\cdots t_{n-3}^N}, \tfrac{u}{v}, v-1\Big).$$

Theorem 2.4.2 *$\tilde{\mathscr{Z}} = \mathscr{Z} + \mathscr{W}_1 + \mathscr{W}_2$ yields a closed cycle, with the properties described in Theorem 2.3.1. (In particular, this recovers the second K_7 construction and the K_9 construction above, for $n = 4$ and 5.)*

Proof Writing

$$\mathscr{Y}_0 := \partial_n^0 \mathscr{Z} = \left(\tfrac{t_1}{t_1-1}, \ldots, \tfrac{t_{n-2}}{t_{n-2}-1}, \tfrac{1}{1-\xi t_1\cdots t_{n-2}}, t_1^N, \ldots, t_{n-2}^N, \tfrac{1}{t_1^N\cdots t_{n-2}^N}\right),$$

$\mathscr{Y}_i := \partial_{2n-1}^0 \mathscr{W}_2^{(i)}$ $(i = 1, \ldots, n-1)$, and $\mathscr{X}_{i,j} := \partial_j^\infty \mathscr{W}_2^{(i)}$ $(j = 1, \ldots, n-2)$, one computes that $\partial\mathscr{Z} = (-1)^{n-1}\mathscr{Y}_0$,

$$\partial\tilde{\mathscr{W}}_1 = (-1)^n\partial_n^\infty\tilde{\mathscr{W}}_1 + \sum_{i=1}^{n-1}(-1)^i\partial_i^\infty\tilde{\mathscr{W}}_1 = (-1)^n(n-3)\mathscr{Y}_0 + \sum_{i=1}^{n-1}(-1)^i\mathscr{Y}_i,$$

and $\partial\mathscr{W}_2^{(i)} = \mathscr{Y}_i + \sum_{j=1}^{n-2}(-1)^j\mathscr{X}_{i,j}$. We have therefore

$$\partial\tilde{\mathscr{Z}} = \frac{1}{n-3}\sum_{i=1}^{n-1}\sum_{j=1}^{n-2}(-1)^{i+j-1}\mathscr{X}_{i,j}, \tag{2.10}$$

and for each $i > j$ the reader will verify that $\mathscr{X}_{i,j} = \mathscr{X}_{j,i-1}$, so that the terms on the right-hand side of (2.10) cancel in pairs. ■

[1]That is, either before $(i = 1)$, after $(i = n-2)$, or in the middle of the sequence $\frac{vt_1^N}{u}, \frac{vt_2^N}{u}, \ldots, \frac{vt_{n-3}^N}{u}$.

2.5 Beyond abelian field extensions.

In algebraic number theory, it can be shown that every cyclotomic field is an abelian extension of the rational number field $\mathbb{Q}$ [22]. The Kronecker-Weber theorem provides a partial converse: every abelian extension of $\mathbb{Q}$ is contained within some cyclotomic field. Since we have constructed the explicit basises of motives for all cyclotomic fields, all the higher Chow groups over abelian extensions are availble now.

We want to extend our work to general number field. Francis Brown [23] constructed Dedekind zeta motives over totally real number fields applying hyperbolic geometry. Rob de Jeu [24] uses relative K-theory to construct symbols which are mapped to Borel's regulators. Motivated by Rob's work, we construct the following higher Chow cycle over a non-abelian field.

Consider the field $F = \mathbb{Q}[\sqrt[3]{2}, \zeta_3]$, ($\zeta_3$ is the third root of unity). Its Galois group is S_3, so F is a non-abelian field extension.

We give the following higher Chow cycle Z

$$
\begin{aligned}
&Z_1 + Z_2 + Z_3 + Z_4 + Z_5 = \\
&\big(t, \frac{(t-\sqrt[3]{2})^3}{t^3-2}, 1+\frac{t}{\sqrt[3]{2}}\big) - \big(t, 2, (\frac{t+\sqrt[3]{2}}{t-\sqrt[3]{2}})^2\big) + \big(-\sqrt[3]{2}, t, \frac{(t-2)^2}{(t-4)(t-1)}\big) \\
&+ (\sqrt[3]{2}\zeta_3, t, \frac{(t+\zeta_3)^3}{(t-1)^3}) + (\sqrt[3]{2}\bar{\zeta_3}, t, \frac{(t+\bar{\zeta_3})^3}{(t-1)^3}).
\end{aligned}
$$

Compute the Bloch boundary,

$$
\begin{aligned}
\partial_{\mathcal{B}} Z_1 &= 2(\sqrt[3]{2}, 2) - (\sqrt[3]{2}\zeta_3, 1+\zeta_3) - (\sqrt[3]{\zeta_3^{2}}, 1+{\zeta_3}^2) + (-\sqrt[3]{2}, 4) \\
&= 2(\sqrt[3]{2}, 2) - (\sqrt[3]{2}\zeta_3, -\bar{\zeta_3}) - (\sqrt[3]{\zeta_3^{2}}, -\zeta_3) + (-\sqrt[3]{2}, 4),
\end{aligned}
$$

$$
\partial_{\mathcal{B}} Z_2 = -2(-\sqrt[3]{2}, 2) + 2(\sqrt[3]{2}, 2),
$$

$$
\partial_{\mathcal{B}} Z_3 = -(-\sqrt[3]{2}, 4) + 2(-\sqrt[3]{2}, 2),
$$

$$
\partial_{\mathcal{B}} Z_4 = (\sqrt[3]{2}\zeta_3, -\bar{\zeta_3}),
$$

$$\partial_{\mathcal{B}} Z_5 = (\sqrt[3]{2}\zeta_3^2, -\zeta_3).$$

Therefore Z is $\partial_{\mathcal{B}}$ closed.

3. Preliminaries and Definitions.

3.1 Graph hypersurface and Graph symbol

We start by defining the graph symbol for a graph.

Definition 3.1.1 *For a graph Γ with e edges, let $A_1, ..., A_e$ represent coordinates in $\mathbb{P}^{e-1}$. Its coordinate symbol is the Milnor symbol $\{-\frac{A_1}{A_e}, ..., -\frac{A_{e-1}}{A_e}\} \subset K^M_{e-1}(\mathbb{C}^*)^{e-1}$.*

Here $K^M_{e-1}(\mathbb{C}^*)^{e-1}$ is the Milnor-K group associated to the field of rational functions over $\mathbb{P}^{e-1}$.

The definition of coordinate symbols is actually independent from the choice of edge whose corresponding coordinate appear in the denominator.

Lemma 3.1.2 *Coordinate symbols are well defined up to a possible negative sign.*

Proof We just need to show that

$$\{-\frac{A_1}{A_e}, ..., -\frac{A_{e-1}}{A_e}\} = \{-\frac{A_2}{A_1}, ..., -\frac{A_e}{A_1}\}^{-1^l}$$

in $K^M_{e-1}(\mathbb{C}(\mathbb{P}^{e-1}))$ for some power l. By steinberg relation 1.3.2,

$$\{-\frac{A_1}{A_e}, -\frac{A_1}{A_e} ..., -\frac{A_{e-1}}{A_e}\} = E = \{-\frac{A_1}{A_e}, -\frac{A_e}{A_1} ..., -\frac{A_{e-1}}{A_e}\}, \tag{3.1}$$

E is the identity of Milnor group.

$$
\begin{aligned}
\{-\frac{A_1}{A_e}, -\frac{A_2}{A_e}..., -\frac{A_{e-1}}{A_e}\} &= \{-\frac{A_1}{A_e}, -\frac{A_2}{A_e}..., -\frac{A_{e-1}}{A_e}\}\{-\frac{A_1}{A_e}, -\frac{A_e}{A_1}..., -\frac{A_{e-1}}{A_e}\} \\
&= \{-\frac{A_1}{A_e}, -\frac{A_2}{A_e} \times -\frac{A_e}{A_1}..., -\frac{A_{e-1}}{A_e}\} \\
&= \{-\frac{A_1}{A_e}, -\frac{A_2}{A_1}..., -\frac{A_{e-1}}{A_e}\}. \\
&= \{-\frac{A_1}{A_e}, -\frac{A_2}{A_1}..., -\frac{A_{e-1}}{A_1}\} \\
&= \{-\frac{A_2}{A_1}, ..., -\frac{A_e}{A_1}\}^{(-1)^{(e-1)}}
\end{aligned}
$$

Since what we finally care about is the regulator of this Milnor symbol, we will see this negative sign will not make a difference. ■

Definition 3.1.3 *For a graph Γ with k edges we assign to every edge e_i a variable x_i, then its graph polynomial, which is also known as the first Symanzik polynomial, is defined by*

$$\Phi(x_1, ..., x_k) = \sum_{T \in \mathcal{T}} \prod_{e_i \notin T} x_i, \tag{3.2}$$

The graph hypersurface X_Γ is hypersurface defined by the zero locus of graph polynomial in projective space $\mathbb{P}^{k-1}$

Definition 3.1.4 *For a graph Γ, its graph symbol ξ_Γ is the restriction of its coordinate symbol along the X^*_Γ, here $X^*_\Gamma = X_\Gamma \cap (\mathbb{C}^*)^{e-1}$.*

For the graph Γ, the Feynman integral we concerned about is

$$I = \int_\sigma \frac{\Omega}{\Phi^2}$$

where $\sigma = \{(A_1, ..., A_k) | A_i \geq 0\}$ is the real simplex in projective space $\mathbb{P}^{k-1}$, $\Omega = \sum_1^6 (-1)^i A_i dA_1 \wedge ...\widehat{dA_i}...dA_k$.

Here we recall several important combinatorial properties of graph hypersurface from [25].

3.2 BEK's blow up

In this part we review a sequential blow-ups constructed in [15], which gives a toric boundary whose combinatorial properties is essential to our final goal.

Proposition 3.2.1 *Let* Γ *be as above. Define*

$$\eta = \eta_\Gamma = \frac{\Omega_{2n-1}(A)}{\Phi_\Gamma^2}.$$

Then there exists a tower

$$P = P_r \xrightarrow{\pi_{r,r-1}} P_{r-1} \xrightarrow{\pi_{r-1,r-2}} \ldots \xrightarrow{\pi_{2,1}} P_1 \xrightarrow{\pi_{1,0}} \mathbb{P}^{2n-1};$$

$$\pi = \pi_{1,0} \circ \ldots \circ \pi_{r,r-1}$$

where P_i *is obtained from* P_{i-1} *by blowing up the strict transform of a coordinate linear space* $L_i \subset X_\Gamma$ *and such that*

(i)$\pi^*\eta_\Gamma$ *has no poles along the exceptional divisors associated to the blowups.*

(ii) Let $B \subset P$ *be the total transform in* P *of the union of coordinate hyperplanes* Δ^{2n-2} : $A_1 A_2 \cdot \ldots \cdot A_{2n} = 0$ *in* $\mathbb{P}^{2n-1}$*. Then* B *is a normal crossings divisor in* P*. No face(= non-empty intersection of components) of* B *is contained in the strict transform* Y *of* X_Γ *in* P.

(iii)the strict transform of $\sigma^{2n-1}(\mathbb{R})$ *in* P *does not meet* Y.

The proof in in [15]. The algorithm to construct the blowups will be following. In the first place, we always consider the linear spaces contained in the graph hypersurface. We have a lemma which gives a characterization of these spaces.

Lemma 3.2.2 *A coordinate linear space* $L : A_{e_1} = \ldots = A_{e_p} = 0$ *is contained in* X_Γ *if and only if the union of the edges* $e_1 \cup \ldots \cup e_p$ *supports a loop(i.e. writing* Γ_L *for this subgraph, including all endpoints of the* e_i*, we have* $h_1(\Gamma_L) > 0$*).*

Let S denote the set of coordinate linear spaces which are maximal and contained in the graph hypersurface , i.e. $L \in S, L \subset L^{'} \subset X_\Gamma \Rightarrow L = L^{'}$. Notice these linear spaces correspond to subgraphs with no external edges and supporting one loop. Define

$$\mathcal{F} = \{L \subset X_\Gamma \text{ coordinate linear space } |L = \bigcap L^{(i)}, L^{(i)} \in S\}.$$

Let $\mathcal{F}_{\min} \subset \mathcal{F}$ be the set of minimal elements in $\mathcal{F}$. Define $P_1 \xrightarrow{\pi_{1,0}} \mathbb{P}^{2n-1}$ to be the blowup of elements of $\mathcal{F}_{min}$. Now define $\mathcal{F}_1$ to be the collection of strict transforms in P_1 of elements in $\mathcal{F} \setminus \mathcal{F}_{min}$. Again elements in $\mathcal{F}_{1,min}$ are disjoint, and we define P_2 by blowing up elements in $\mathcal{F}_{1,min}$.

The exceptional locus we get from this blow up have nice geometric structure and graphical interpretation.

Definition 3.2.3 *Define the modified quotient graph $\Gamma//\Gamma^{'}$ to be the graph obtained from Γ by contracting each connected component of $\Gamma^{'}$ to a point. Do not identify the points associated to different components.*

Proposition 3.2.4 *Let $G \subset \Gamma$ be a subgraph, and suppose $h_1(G) > 0$. Then $L(G) : A_e = 0, e \in G$ is contained in X_Γ. Let $P \to \mathbb{P}(E_\Gamma)$ be the blow up of $L(G) \subset \mathbb{P}(E_\Gamma)$, and let $F \subset P$ be the exceptional locus. Let $Y \subset P$ be the strict transform of X_Γ in P. Then we have canonical identifications*

$$F \cong \mathbb{P}(E_G) \times \mathbb{P}(E_{\Gamma//G}) \tag{3.3}$$

$$Y \cap F = \big(X_G \times \mathbb{P}(E_{\Gamma//G})\big) \cup \big(\mathbb{P}(E_G) \times X_{\Gamma//G}\big). \tag{3.4}$$

See proof in [BEK]. BEK's construction will yield a toric variety. We will come back to this point in a later chapter. Now we want to show how to compute the residue maps of graph symbol ξ_Γ restricted on the graph hypersurface in this setup.

First we need an important lemma about graph hypersurfaces.

Proposition 3.2.5 *Let $\Gamma' \subset \Gamma$ be a subgraph, and assume $h_1(\Gamma) > 0$. We have $Edge(\Gamma) = Edge(\Gamma') \amalg Edge(\Gamma//\Gamma')$. Suppose edge variables $A_1, ..., A_r$ are associated to Γ' and $A_{r+1}, ..., A_m$ to $\Gamma//\Gamma'$. Then the graph polynomials satisfy*

$$\Phi_\Gamma = \Phi_{\Gamma'}(A_1, ..., A_r) \cdot \Phi_{\Gamma//\Gamma'}(A_{r+1}, ..., A_m) + F(A_1, ..., A_m) \tag{3.5}$$

where the degree of F in $A_1, ..., A_r$ is strictly greater than the degree of $\Phi_{\Gamma'}$.

Now apply the notation from section 1.4, let $\mathbb{D}_\sigma$ be an exceptional divisor got from blowing up a subgraph Γ'. By (3.3), we immediately get a set of local toric coordinates on $\mathbb{D}^*_\sigma$. More specifically, by blowing up linear space $L(\Gamma')$, we do a change of coordinates

$$A_2 = B_2 \cdot A_1, ..., A_r = B_r \cdot A_1,$$

and F is defined by $A_1 = 0$. Therefore

$$\{B_2, ..., B_r, A_{r+1}, ..., A_n\} \cap \{A_1 = 0\}$$

gives the local toric coordinates we need. Then by lemma 1.4.2

$$(I^*_\sigma)\langle\{B_2, ..., B_r, A_{r+1}, ..., A_n\}\rangle$$

gives $\mathrm{Res}^1_{\mathbb{D}^*_\sigma}(\xi)$.

Notice by (3.5), the degree of F in $A_1, ..., A_r$ is strictly greater than the degree of $\Phi_{\Gamma'}$. So on $\mathbb{D}^*_\sigma$, F will vanish. The reason is as follows. After change of coordinates, the graph polynomial becomes:

$$\Phi_\Gamma = {A_1}^q\big(\Phi_{\Gamma'}(1, ..., B_{r-1}, B_r) \cdot \Phi_{\Gamma//\Gamma'}(A_{r+1}, ..., A_m) + {A_1}^p F(1, B_2, ..., A_m)\big).$$

Therefore we have the strict transformation of graph hypersurface as

$$\Phi_\Gamma|_{\mathbb{D}^*_\sigma} = \Phi_{\Gamma'}(1, ..., B_r) \cdot A^m_{r+1}\Phi_{\Gamma//\Gamma'}(1, ..., \frac{A_m}{A_{r+1}}). \tag{3.6}$$

The upshot is if the graph polynomials of Γ' and $\Gamma//\Gamma'$ define trivial symbols, then $\mathrm{Res}^1_{\mathbb{D}^*_\sigma}(\xi)$ is trivial.

For higher codimension stratas, we could do a similar analysis of local coordinates for the blowup. We give a criteria here to determine whether the graph symbol ξ can be extended.

Proposition 3.2.6 *For a graph Γ, if for all its subgraphs Γ' and corresponding modified quotient graph $\Gamma//\Gamma'$, their graph polynomial defines a trivial symbol, i.e. $\xi_{\Gamma'} = 0$ and $\xi_{\Gamma//\Gamma'} = 0$, then ξ_Γ could be completed to a higher Chow cycle on Y, Y is the strict transformation of X_Γ in BEK's blow up.*

4. Motivic interpretation of Feynman integration for three wheels spoke graph.

4.1 Set up of Three-wheel spokes

4.1.1 Three wheels spokes

From now on, we will focus on three-wheels spoke graph Γ_3.

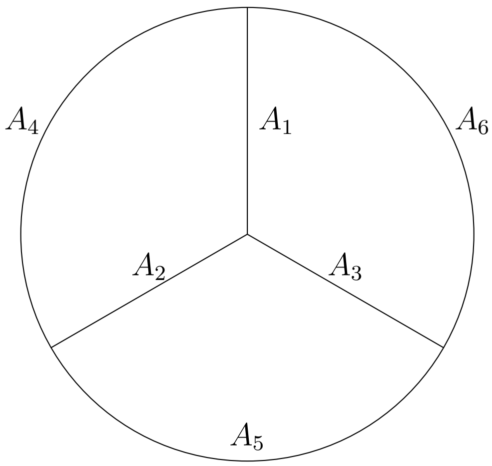

Figure 4.1. Three wheel spokes

For three-wheel spoke graph, the Feynman integral is

$$I = \int_\sigma \frac{\Omega}{\Phi^2}$$

where $\sigma = \{(A_1, ..., A_6) | A_i \geq 0\}$ is the simplex in projective space $\mathbb{P}^5$, $\Omega = \sum_1^6 (-1)^i A_i dA_1 \wedge ... \widehat{dA_i} ... dA_6$. We let $\eta := \frac{\Omega}{\Phi^2}$. From section 11 [15], $H^5(\mathbb{P}^5 \setminus X_\Gamma) \cong \mathbb{Q}(-3)$, and η generates this group.

We have graph symbol $\xi = \{-\frac{A_1}{A_6}, ..., -\frac{A_5}{A_6}\}$. Notice $T_5 = \bigcap_{i=1}^{5}\{-\frac{A_i}{A_6} \leq 0\} \cup \{\frac{A_i}{A_6} = \infty\} = \sigma$. Φ is the graph polynomial defined as in (3.1.3). From(BEK)

$$X_{\Gamma_3} \cong \mathrm{Sym}^2\mathbb{P}^2.$$

Therefore its singular locus Sing_{X_Γ} the diagnoal $\Delta \cong \mathbb{P}^2$. And by [26], $H^4(X_\Gamma, \mathbb{Q}) = \mathbb{Q}(-2) \oplus \mathbb{Q}(-2)$, and is generated by Δ and $\{0\} \otimes \mathbb{P}^2 \oplus \mathbb{P}^2 \otimes \{0\}$. The point is these classes are defined over $\mathbb{Q}$.

Lemma 4.1.1 *$[\eta]$ is defined over $\mathbb{Q}$.*

Proof We have long exact sequence

$$... \to H_6(\mathbb{P}^5) \to H_6(\mathbb{P}^5, \mathbb{P}^5 \setminus X_\Gamma) \to H_5(\mathbb{P}^5 \setminus X_\Gamma) \to H_5(\mathbb{P}^5) \to ...$$

By duality, we have $H^5(\mathbb{P}\backslash X_\Gamma) \cong H^4(X_\Gamma)/\mathrm{im}(H^4(\mathbb{P}^5))$. Therefore, $H^5(\mathbb{P}\backslash X_\Gamma)$ is generated by a class defined over $\mathbb{Q}$. Notice η is fixed under Galois actions, $[\eta]$ is also defined over $\mathbb{Q}$. ∎

4.1.2 Construction of the toric variety

We will first do a BEK's blow up

$$\pi_1 : P \to \mathbb{P}^5$$

to separate σ and X_{Γ_3}. Γ_3 has 6 5-edge subgraphs, 3 4-edge subgraphs, and 4 3-edge subgraphs. According to BEK's blow up procedure, we will first blow up the 6 points corresponding to 5-edge subgraphs, and then blow up the linear spaces corresponding to 3-edge and 4-edge subgraphs. We get a smooth toric variety, and $P \setminus (\mathbb{C}^*)^5$ is a NCD which has 19 components. We take $D_{A_{i_1},...,A_{i_k}}$ to represent the component corresponding to subgraph with edges $A_{i_1}, ..., A_{i_k}$.

Notice the strict transformation Y of X_{Γ_3} is still singular [15], we need to do one more blow up

$$\tilde{\pi} : \tilde{P} \to P,$$

along the singular locus. Let $\mathbb{D} \subset \tilde{P}$ be the strict transformation of $P \setminus (\mathbb{C}^*)^5$, and we call it the boundary.

4.2 Reinterpretation of Feynman Integral

We pull back the integration I along the composition of blowups

$$\pi_1 \circ \tilde{\pi} : \tilde{P} \to P \to \mathbb{P}^5$$

For convenience, we let in $\tilde{P}$ we still use σ, η and ξ to denote the strict transformation of σ and pull back of η and ξ (η and ξ are the same as defined in 4.1). η can be viewed as a $5-$current on $\tilde{P}$ and $\eta \in F^5\mathcal{D}^5(\tilde{P})$. Then in $\tilde{P}$

$$I = \int_\sigma \eta = \int_{\tilde{P}} \delta_\sigma \wedge \eta.$$

Notice that in $\tilde{P} \setminus \mathbb{D}$,

$$d[R_\xi] = \Omega_\xi - (2\pi\sqrt{-1})^n \delta_{T_\xi} + 2\pi\sqrt{-1} R_{\partial\xi},$$

and $\sigma = T_\xi$, so we have

$$I = \frac{-1}{(2\pi i)^n} \int_{\tilde{P}} (d[R_\xi] - \Omega_\xi - K_{\mathbb{D}}) \wedge \eta$$

where $K_{\mathbb{D}}$ represents a current supported on $\mathbb{D}$. Since $\Omega_\xi \wedge \eta$ and $K_{\mathbb{D}} \wedge \eta$ are 0 by type, I becomes

$$\int_{\tilde{P}} d[R_\xi] \wedge \eta.$$

We want to apply integration by parts and consider the residue of η. However, η has double poles along Y and E. By lemma 4.5.1, $\exists\ \hat{\eta} = \eta + d\tau$, such that $\hat{\eta}$ and τ have log poles along Y and E.

In $\mathbb{P}^5$, we know the singular locus of the graph hypersurface $\mathrm{Sing}_{X_\Gamma} \cong \mathbb{P}^2$. Let $\{L(\Gamma')\}$ be the linear spaces along which the BEK's blow ups take place. When $\dim(\mathrm{Sing}_{X_\Gamma} \cap L(\Gamma')) = 1$ or 2, $Bl_{\left(\mathrm{Sing}_{X_\Gamma} \cap L(\Gamma')\right)}(\mathrm{Sing}_{X_\Gamma}) = \mathrm{Sing}_{X_\Gamma}$. So after the BEK's blow up procedure has been applied to $\mathbb{P}^5$, the strict transformation of Sing_{X_Γ} in Y is $\mathbb{P}^2$ blown up at several points. Therefore the exceptional divisor E in $\tilde{P}$ is a $\mathbb{P}^2$ bundle over $\mathbb{P}^2$ blown up at several points and $Y \cap E$ is a $\mathbb{P}^1$ bundle. By projective bundle formula, its third cohomology group is trivial. The class of double residue along $Y \cap E$ is 0.

Consider in the double complex of Gysin spectral sequence with NCD defined by Y and E. Since the class of double residue along $Y \cap E$ is 0, we could modify $\hat{\eta}$ by an exact form such that the double residue along $Y \cap E$ is vanishing.

Therefore we could take residue along Y and E. Then

$$\begin{aligned} I &= \int_{\tilde{P}} d[R_\xi] \wedge \hat{\eta} + \int_{\tilde{P}} d[R_\xi] \wedge d\tau. \\ &= \int_{\tilde{P}} [R_\xi]|_Y \wedge Res_Y \hat{\eta} + \int_{\tilde{P}} [R_\xi]|_E \wedge Res_E \hat{\eta} \end{aligned}$$

Applying argument in (4.3), we are able to substitute $[R_\xi]$ by a closed current $[R_\Xi]$ which could be lifted to $AJ(\Xi)$,

$$\int_{Y \cup E} [R_\xi] \wedge Res_Y \hat{\eta} = \int_{Y \cup E} [R_\Xi] \wedge Res_Y \hat{\eta}$$

By lemma (4.1.1), $Res_{Y \cup E} \hat{\eta}$ is defined over $\mathbb{Q}$, by Poincare duality corresponding to a rational homology class $Z \to \mathrm{Spec}(\mathbb{Q})$ on $Y \cup E$. So the Feynman integral becomes

$$\int_Z [R_\Xi].$$

Pushing this regulator forward to Spec($\mathbb{Q}$) rewrites Feynman integral as an image of Borel regulator $CH^3(\mathrm{Spec}\mathbb{Q}, 5) \longrightarrow \mathbb{C}/\mathbb{Q}(3)$. In this way, we theoretically proved Feynman integral is a rational multiple of $\zeta(3)$.

4.3 Computation of $H_3(\mathbb{D}^+)$

In this section, we want to show that we could find $[R_\Xi]$ such that

$$R_\Xi|_Y = R_\xi|_Y + \mathcal{R}|_Y,$$

where $\mathcal{R}$ is some current which will be killed by pairing with residue of $\hat{\eta}$, and $[R_\Xi]$ is a closed current which could be lifted to the Abel-Jacobi image of Ξ.

Now let us reconsider the graph symbol ξ. ξ represents a class in $CH^5((\mathbb{C}^*)^5, 5)$, to be able to extend it to a class in $CH^5(\tilde{P}, 5)$, we need the higher residue map on all stratas to vanish, which is checked in section 6.1.1. But we also need the extension of ξ to $\tilde{P}$ to intersect all faces properly.

For the $\{D_{A_i}\}$, when $i = 6$, $\xi = \{\infty, ..., \infty\}$ on $D^*_{A_6}$, so ξ is not intersecting properly there. On the other hand, when $i \neq 6$, ξ consists of 5 regular functions only one of which is constantly 0.

For the $\{D_I\}$, where $|I| = 5$, i.e. the boundary components corresponding to 5 edge subgraphs, again, when $6 \notin I$, $\xi = \{0, ..., 0\}$. So ξ is not behaving properly there. When $6 \in I$, notice D_I is the exceptional divisor obtained by blowing up a point, and ξ consists of 5 regular functions on $(D_I)^*$ except one is constantly ∞.

For the $\{D_I\}$, where $|I| = 3$, when $6 \notin I$, ξ has 3 entries which is constantly 0 on $(D_I)^*$. When $6 \in I$, ξ has 3 entries which is constantly ∞ on $(D_I)^*$. Similarly, ξ does not intersect properly on $(D_I)^*$ when $|I| = 4$. Now the point is, we need to apply Bloch's moving lemma to ξ obtain a precycle on $\tilde{P}$. And the moving process only happens where ξ

does not intersect properly. We call the locus where ξ does not intersect properly the bad locus and denote it by $\mathbb{D}^+$. As we will see later, it is important to know the combinatorial structure of $\mathbb{D}^+$.

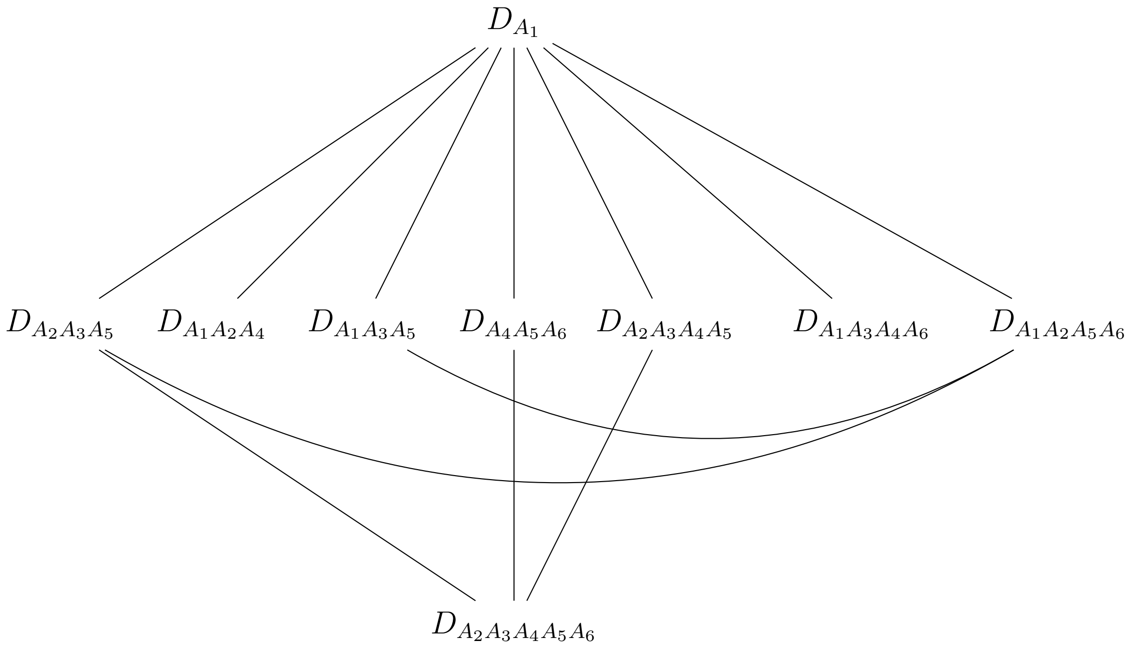

Figure 4.2. Structure of $\mathbb{D}^+$

Each node in the graph represents a divisor of boundary component and a line connecting them represents that their intersection is nonempty.

We have shown that the residue maps of ξ to all smooth substrata will vanish. So by Bloch's moving lemma on higher Chow groups,

$$\xi = j^*\Xi + \partial\mu$$

for $\Xi \in CH^5(Y, 5)$ and $\mu \in CH^5(Y \setminus (\mathbb{D}^+ \cap Y), 6)$. The point here is that since the locus where ξ does not intersect faces properly is $\mathbb{D}^+$, the moving lemma takes place exactly on the complement of $\mathbb{D}^+$.

Now modulo currents supported on $\mathbb{D}^+$ and working on Y, we could apply (1.3) to Ξ and $\xi + \partial\mu$ and have

$$((2\pi i)^5 T_\xi, 0, R_\xi) + D((2\pi i)^5 T_\mu, 0, \frac{1}{2\pi i} R_\mu) \equiv ((2\pi i)^5 T_\Xi, 0, R_\Xi). \tag{4.1}$$

By proposition(3.2.1), we know $T_\xi \cap Y = \emptyset$. So we get

$$T_\Xi = -\partial T_\mu + S_{\mathbb{D}^+}, \tag{4.2}$$

where $S_{\mathbb{D}^+}$ is a chain supported on $\mathbb{D}^+ \cap Y$. Y is a smooth projective variety, so T_Ξ is an exact current. Let $T_\Xi = \partial\Gamma$. Therefore $S_{\mathbb{D}^+}$ is also an exact current. Let $S_{\mathbb{D}^+} = \partial\gamma$. Up to current supported on $\mathbb{D}^+$

$$\Gamma = -T_\mu + \gamma.$$

By (1.5.6), (4.1) shows

$$R_\xi + d[\frac{1}{2\pi i} R_\mu] + (2\pi i)^5 \delta_{T_\mu} \equiv R_\Xi. \tag{4.3}$$

So we have

$$R_\xi + d[\frac{1}{2\pi i} R_\mu] + (2\pi i)^5 \delta_\gamma \equiv R_\Xi - (2\pi i)^5 \delta_\Gamma. \tag{4.4}$$

Notice by (1.4),

$$R_\Xi - (2\pi i)^5 \delta_\Gamma \tag{4.5}$$

defines a closed current representing the regulator class of Ξ in $H^4(Y, \mathbb{C}/\mathbb{Q}(5))$.

Now notice $S_{\mathbb{D}^+}$ is a three chain supported on $\mathbb{D}^+ \cap Y$. If $H_3(\mathbb{D}^+ \cap Y) = 0$, we could choose γ such that γ is also supported on $\mathbb{D}^+ \cap Y$. The upshot is if this is true, then R_ξ will equal to $\tilde{R}_\Xi$ up to currents supported on $\mathbb{D}^+$.

Proposition 4.3.1 $H_3(\mathbb{D}^+ \cap Y) = \{0\}$.

Proof We will either compute through Mayer-Vietoris spectral sequence or directly compute Mayer-Vietoris spectral sequence.

Part of the first page of Mayer-Vietoris spectral sequence is as follows:

$$
\begin{aligned}
0 \longrightarrow H_1(\mathcal{D}^{[2]}) \longrightarrow & H_1(\mathcal{D}^{[1]}) \\
H_2(\mathcal{D}^{[2]}) \longrightarrow & H_2(\mathcal{D}^{[1]}) \longrightarrow H_2(\mathcal{D}^{[0]}) \\
& H_3(\mathcal{D}^{[1]}) \longrightarrow H_3(\mathcal{D}^{[0]}).
\end{aligned}
$$

Notice by the intersection behavior of components of $\mathbb{D}^{+} \cap Y$, $\mathcal{D}^{[2]} = \emptyset$. Therefore it is enough to show the target spaces $H_2(\mathcal{D}^{[1]})$ and $H_3(\mathcal{D}^{[0]})$ are trivial.

For subgraph G with edges $\{A_1, A_2, A_5, A_6\}$,

$$
\begin{aligned}
Y \cap D_{A_1A_2A_5A_6} &= (X_G \times \mathbb{P}^1) \cup (\mathbb{P}^3 \times X_{\Gamma//G}) \\
&= (\{A_2 + A_3 + A_4 + A_5 = 0\} \times \mathbb{P}^1) \cup (\mathbb{P}^3 \times \{A_1A_6 = 0\}).
\end{aligned}
\tag{4.6}
$$

So $Y \cap D_{A_1A_2A_5A_6}$ is a copy of $\mathbb{P}^2 \times \mathbb{P}^1$ intersecting two copies of $\mathbb{P}^3$. By applying MV

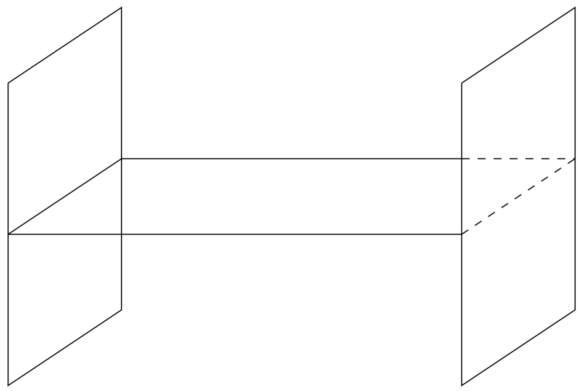

Figure 4.3. $Y \cap D_{A_1A_2A_5A_6}$

sequence to it, we know H_3 of it is $\{0\}$.

For subgraph with edges $\{A_2, A_3, A_6\}$

$$
\begin{aligned}
Y \cap D_{A_2A_3A_6} &= (X_G \times \mathbb{P}^1) \cup (\mathbb{P}^2 \times X_{\Gamma//G}) \\
&= (\{A_2 + A_3 + A_6 = 0\} \times \mathbb{P}^1) \cup (\mathbb{P}^2 \times \{A_3A_5 + A_3A_6 + A_5A_6 = 0\}).
\end{aligned}
\tag{4.7}
$$

Computation of MV sequence shows H_3 of it is $\{0\}$. ■

For subgraph with edges $\{A_2, A_3, A_4, A_5, A_6\}$, the graph polynomial is

$$A_5A_6 + A_4A_6 + A_3A_6 + A_3A_5 + A_3A_4 + A_2A_6 + A_2A_5 + A_2A_4 = 0.$$

Notice it could be transformed into

$$\frac{1}{A_6} = \frac{1}{A_2 + A_3} + \frac{1}{A_4 + A_5}.$$

So it is a cone over a cone over $\mathbb{P}^1$. Therefore its 3rd homology group is $\{0\}$.

For subgraph with edge A_1, the graph polynomial Φ is

$$A_4A_5A_6 + A_3A_4A_5 + A_2A_4A_5 + A_2A_5A_6 + A_2A_4A_5 + A_2A_3A_5 + A_2A_3A_4.$$

Let $A_2 + A_4 = B_0$, $A_2 + A_3 + A_5 = B_1$, $A_3 + A_4 + A_6 = B_2$, $-A_2 = C_0, -A_3 = C_1$, $-A_1 = C_2$, then the graph polynomial is transformed into

$$\Phi = B_0(B_1B_2 - C_1^2) - C_1^2B_2.$$

Let X be the hypersurface defined by $\Phi = 0$. For convenience, let $Q_3 = B_1B_2 - A_1^2$, $K_3 = C_1^2B_2$. Let $p = (1, 0, 0, 0, 0)$, then the projection from p induces isomorphism B_0, B_1, B_2, C_0, C_1 in $\mathbb{P}^4$ from $X \setminus (X \cap \mathcal{V}(Q_3))$ to $\mathbb{P}^3 \setminus \mathcal{V}(Q_3)$. Also $X \cap \mathcal{V}(Q_3) = \mathcal{V}(Q_3, K_3)$. We have

$$\begin{aligned} \cdots \to & H_c^3(X \setminus (X \cap \mathcal{V}(Q_3))) \to H^3(X) \to H^3(X \cap Q_3) \\ & \to H_c^4(X \setminus (X \cap \mathcal{V}(Q_3))) \to H^4(X) \to \cdots. \end{aligned}$$

For $H_c^3(\mathbb{P}^3 \setminus \mathcal{V}(Q_3))$, notice $\mathbb{P}^3 \setminus \mathcal{V}(Q_3)$ in $\mathbb{P}^4$ is a cone over $\mathbb{P}^2 \setminus \mathcal{V}(Q_3)$ in $\mathbb{P}^3$. So $H_c^3(\mathbb{P}^3 \setminus \mathcal{V}(Q_3)) = 0$(by Artin's vanishing Theorem). On the other hand, for $\mathcal{V}(Q_3, K_3)$, it is defined by

$$\begin{cases} B_1B_2 - C_1^2 & = 0, \\ C_0^2B_2 & = 0. \end{cases}$$

whose solution sets are

$$\begin{cases} B_2 & = 0, \\ C_1 & = 0. \end{cases}$$

or

$$\begin{cases} C_0 & = 0, \\ B_1 B_2 - C_1^2 & = 0. \end{cases}$$

So it is a $\mathbb{P}^2$ normal crossing $\mathbb{P}^2$. Therefore

$$H^3(X \cap \mathcal{V}(Q_3)) = H^3(\mathcal{V}(Q_3, K_3)) = \{0\}.$$

Now for $D_{A_2A_3A_6} \cap D_{A_1}$, notice it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^2$, and $(\mathbb{P}^1 \cap Y) \times (\mathbb{P}^2 \cap Y)$is given by

$$\{A_2 + A_4 = 0, A_1 = 0\} \times \mathbb{P}^2 \cup \mathbb{P}^1 \times \{A_3A_5 + A_3A_6 + A_5A_6 = 0\}.$$

Therefore we know the corresponding map in spectral sequence is 0.

For $D_{A_1A_2A_4} \cap D_{A_1}$ and $D_{A_1} \cap D_{A_2A_3A_4A_5}$, for same argument as above, we know the corresponding map in spectral sequence is 0.

For $D_{A_2A_3A_4A_5} \cap D_{A_2A_3A_4A_5A_6}$, $D_{A_2A_3A_4A_5A_6}$ is contained in a $\mathbb{P}^4$ parameterizing tangents to point $A_2 = A_3 = A_4 = A_5 = A_6 = 0$. $D_{A_2A_3A_4A_5} = \mathbb{P}^3 \times \mathbb{P}^1$, where $\mathbb{P}^3$ parametrizes the tangents $A_2 = A_3 = A_4 = A_5 = 0$. $\mathbb{P}^4$ and $\mathbb{P}^3 \times \mathbb{P}^1$ normal crosses in a copy of $\mathbb{P}^3$. And $D_{A_2A_3A_4A_5} \cap D_{A_2A_3A_4A_5A_6}$ is given by

$$\begin{cases} A_2 + A_3 & = 0, \\ A_4 + A_5 & = 0. \end{cases}$$

we know the corresponding map in spectral sequence is 0.

Consider $D_{A_2A_3A_4A_5A_6}$ and $D_{A_4A_5A_6}$, similarly we get the definition

$$A_6(A_4 + A_5) = 0.$$

This is just a point.

$D_{A_2A_3A_6} \cap D_{A_1A_2A_5A_6}$ is a point.

4.4 Vanishing of graph symbol along all stratas

In section 4.3, we want to complete the graph symbol ξ to a higher Chow cycle over Y. (Here we use the notation in 1.3.) To be able to do this, we need to make sure all its higher residues maps Res^i on all stratas vanishes. First notice in three spoke wheel case, all the stratas $\widetilde{D^k}$ are defined over $\mathbb{Q}$. By Beilinson-Soulé vanishing conjecture, which says that $CH^p(k,n) = \{0\}$ for $p < \frac{n+1}{2}$, the target spaces of Res^i for $i > 1$ are all $\{0\}$. So we only need to check Res^1. For this, $\hat{Res}^i$ have to vanish for all i. By proposition 3.2.6, we only need to check the vanishing of graph symbols on all subgraphs and modified quotient graphs.

Remark 4.4.1 *In general cases, it is very difficult to compute the higher residue maps.*

The subgraph Γ_{124} with edges $\{A_1, A_2, A_4\}$ and its modified quotient graph $\Gamma//\Gamma_{124}$ are as follows. The graph polynomial of Γ_{124} is

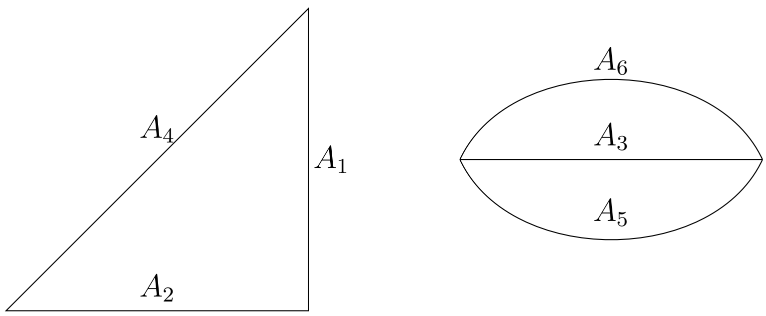

Figure 4.4. Subgraph with three edges and its modified quotient graph

$$A_1 + A_2 + A_4 = 0,$$

which gives $-\frac{A_1}{A_4} + (-\frac{A_2}{A_4}) = 1$. So the symbol $\{-\frac{A_1}{A_4}, -\frac{A_2}{A_4}\}$ is trivial. For $\Gamma//\Gamma_{124}$, its garph polynomial is

$$A_3A_5 + A_3A_6 + A_5A_6 = 0,$$

$$\Longleftrightarrow \frac{1}{-\frac{A_3}{A_6}} + \frac{1}{-\frac{A_5}{A_6}} = 1.$$

therefore its graph symbol $\{-\frac{A_3}{A_6}, -\frac{A_5}{A_6}\}$ is trivial.

The subgraph Γ_{1256} with edges $\{A_1, A_2, A_5, A_6\}$ and its modified quotient graph $\Gamma//\Gamma_{1256}$ are as follows. The graph polynomial of Γ_{1256} is

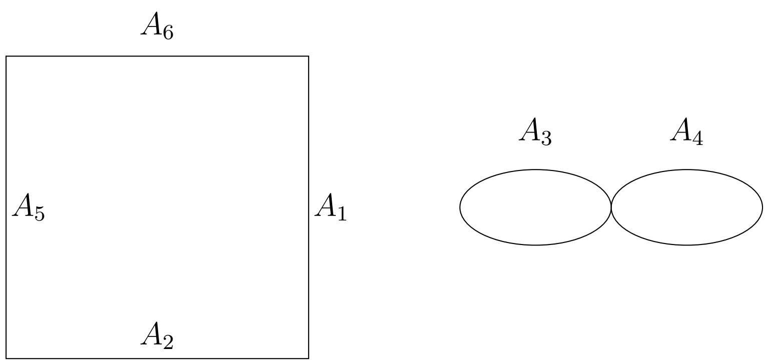

Figure 4.5. Subgraph with four edges and its modified quotient subgraph.

$$A_1 + A_2 + A_5 + A_6 = 0,$$

$$\Longleftrightarrow -\frac{A_1}{A_6} + (-\frac{A_2}{A_6}) + (-\frac{A_5}{A_6}) = 1.$$

So the symbol

$$\begin{aligned}
&\{-\frac{A_1}{A_6}, -\frac{A_2}{A_6}, -\frac{A_5}{A_6}\}\\
=&\{-\frac{A_1}{A_6}, -\frac{A_2}{A_6} + -\frac{A_5}{A_6}, -\frac{A_2}{A_5}\}\\
=&\{-\frac{A_1}{A_6}, 1 - (-\frac{A_1}{A_6}), -\frac{A_2}{A_5}\}\\
\\
=&0.
\end{aligned}$$

For $\Gamma//\Gamma_{124}$, its garph polynomial is

$$A_3A_4 = 0,$$

therefore its graph symbol is trivial.

The subgraph Γ_{12345} with edges $\{A_1, A_2, A_3, A_4, A_5\}$ and its modified quotient graph $\Gamma//\Gamma_{12345}$, the graph symbol will be shown to be 0 in lemma 6.1.1 .

Therefore we have shown that the higher residues of ξ vanish on all substratas.

4.5 Lemma about reducing the order of the double pole

Let Ω_Γ is a differential form defined on $\mathbb{P}^{2n-1} \setminus X_\Gamma$, with double poles along X_Γ. We want to take residue of Ω_Γ, which only makes sense for differential forms with log poles. Fortunately, we can modify this while fixing the integral value.

Lemma 4.5.1 *Let P be a projective variety, and D be a normal crossing divisor in P, X be a hypersurface in P, and $X_D = X \cap D$. Let $\eta \in H^k(P \setminus X, D \setminus X)$ having double pole along X. Then $\exists \tau \in A^{k-1}(P \setminus X)$ and $\hat{\eta} \in A^k(P) < logX >$ such that, $\hat{\eta} = \eta + d\tau, \iota^*\tau = d\kappa$.*

Consider the following two complexes:

$$\begin{tikzcd} {} & {} \\ A^k(P \setminus X) \arrow[r] \arrow[u] & A^k(D \setminus X_D) \arrow[u] \\ A^{k-1}(P \setminus X) \arrow[r] \arrow[u] & A^{k-1}(D \setminus X_D) \arrow[u] \\ {} \arrow[u] & {} \arrow[u] \end{tikzcd}$$

and

$$\begin{tikzcd}
{} & {} \\
A^k(P) < logX > \arrow[r] \arrow[u] & A^k(D) < logX_D > \arrow[u] \\
A^{k-1}(P) < logX > \arrow[r] \arrow[u] & A^{k-1}(D) < logX_D > \arrow[u] \\
{} \arrow[u] & {} \arrow[u]
\end{tikzcd}$$

which are quasi-isomorphic, and both compute $H^k(P \setminus X, D \setminus X)$. $[(\eta, 0)]$ lives in first complex and represent the class $[\eta]$. Hence $\exists\ \tilde{\varepsilon} \in A^{k-1}(P) < logX >$, s.t. $-D[\tilde{\varepsilon}, 0] + [\hat{\eta}, \varepsilon] = [\eta, 0]$. Notice we can always extend ε to whole P, we actually have $-D[\tau, \kappa] + [\hat{\eta}, 0] = [\eta, 0]$. Proved.

5. Graph polytope of three wheels.

5.1 Newton polytope associated to graph polynomial

For reasons, we changed our labeling of the edges of three wheels. Sorry for the inconvenience for the readers.

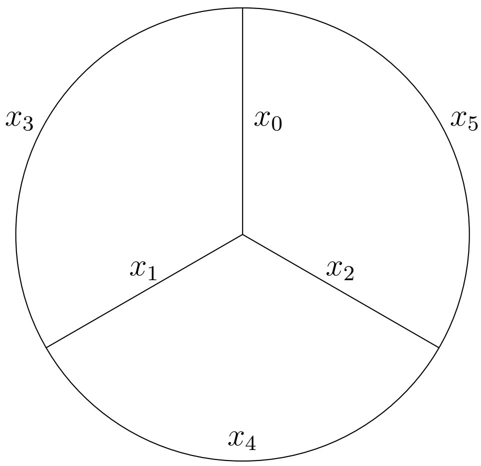

Figure 5.1. Three wheel spokes 2

Consider the first Symanzik polynomial of three wheels. To this polynomial, we associate a Newton polytope as follows. Firstly consider the set $\mathfrak{M} \subset \mathbb{Z}^n$ corresponding to exponents of monomials appearing in $\mathcal{U}$ with nonzero coefficients and secondly the convex hull

$$\Delta_{\mathcal{U}} := \{\sum_{m \in \mathfrak{M}} a_m m | a_m \leq 0, \sum a_m = 1\} \tag{5.1}$$

of these points.

Notice $\Delta_{\mathcal{U}}$ is a 5 dimensional $0-1$ polytope in $\mathbb{R}^6$. So to get a full dimensional polytope we consider the Newton Polytope of $\mathcal{U}(1, x_1, ..., x_5)$, which is the projection of $\Delta_{\mathcal{U}}$ to hyperplane $X_0 = 0$.

Also we want our polytope to be reflexive. $\Delta_{2\mathcal{U}} := 2\Delta_{\mathcal{U}(1,...,x_5)} + (-1,-1,-1,-1,-1)$ is a reflexive polytope. Its 16 vertices are as follows:

$$
\begin{array}{cccc}
[-1,-1,1,1,1] & [-1,1,1,-1,1] & [-1,1,1,1,-1] & [1,-1,-1,1,1] \\
[1,-1,1,1,-1] & [1,1,-1,-1,1] & [1,1,-1,1,-1] & [1,1,1,-1,-1] \\
[-1,-1,-1,1,1] & [-1,-1,1,-1,1] & [-1,1,-1,-1,1] & [-1,1,-1,1,-1] \\
[-1,1,1,-1,-1] & [1,-1,-1,-1,1] & [1,-1,-1,1,-1] & [1,-1,-1,1,-1].
\end{array}
$$

Let$P_{\Delta_{2\mathcal{U}}}$be the toric variety associated to $\Delta_{2\mathcal{U}}$. We compute $\Delta^{\circ}_{2\mathcal{U}}$, the dual polytope of $\Delta_{2\mathcal{U}}$, whose vertices are :

$$
\begin{array}{cccc}
[-1,0,-1,0,-1] & [0,0,-1,0,0] & [0,1,0,0,0] & [1,1,0,0,0] \\
[0,0,0,1,0] & [-1,0,0,0,0] & [1,1,1,1,1] & [0,0,0,0,1] \\
[0,0,1,1,1] & [0,0,0,0,-1] & [0,-1,0,-1,-1] & [-1,-1,-1,-1,-1] \\
[1,0,0,0,0] & [0,0,0,-1,0] & [0,-1,0,0,0] & [0,0,1,0,0].
\end{array}
$$

Later on, we will use $[i]$ to denote the i-th vertex of $\Delta_{2\mathcal{U}}$ and $\Delta^{\circ}_{2\mathcal{U}}$, depending on the context.

I put all the other combinatorial data of this polytope in the appendix.

There is an interesting graphical interpretation of this polytope. In $\mathbb{P}^5$, we let hyperplane $x_i = 0$ to represent the edges x_i in the graph. Then the linear subspace defined by equations $x_{j_k} = 0, k = 1, ..., n$ corresponds to the subgraph consisting edges x_{j_k}s. $P_{2\Delta}$ is closed related to the toric variety obtained by blowing up the linear subspaces corresponding to 3 and 5 edge sub graphs containing a loop. Each vertex in $\Delta^{\circ}_{2\mathcal{U}}$ canonically corresponds to a subgraph as follows. If we let $[-1,-1,-1,-1,-1]$ to represent edge

x_0, $[1,0,0,0,0]$ to represent edge x_1, so on and so forth. Then $[0,0,1,1,1]$ will represent the subgraph consisting edges x_4, x_5, x_6 since it is the sum of $[0,0,0,0,1]$, $[0,0,0,1,0]$, $[0,0,1,0,0]$. Similarly, $[0,-1,0,-1,-1]$ will represent the subgraph x_0, x_1, x_3 since it is the sum of $[-1,-1,-1,-1,-1]$, $[1,0,0,0,0]$, $[0,0,1,0,0]$.

One may ask "why is this correspondence canonical ?" The answer is we have the following interesting observation of $\Delta_{\mathcal{U}}$ and $\Delta_{2\mathcal{U}}$: we can factor the restriction of graph polynomial to facets with a graphical interpretation. Each vertex of the above two polytopes corresponds to a monomial in edge variables. Therefore, a facet of them will correspond to a subpolynomial of Φ_Γ. For example, one facet $F = [0,1,3,4,5,7,9,13,15]$. It corresponds to subpolynomial

$$\begin{aligned}&x_3x_4x_5 + x_2x_3x_5 + x_1x_4x_5 + x_1x_3x_4\\&\quad + x_1x_2x_5 + x_1x_2x_3 + x_3x_5x_0 + x_1x_5x_0 + x_1x_3x_0\\&=(x_1x_3 + x_1x_5 + x_3x_5)(x_0 + x_2 + x_4).\end{aligned}$$

Notice here $x_0+x_2+x_4$ is the graph polynomial of the subgraph Γ' with edges x_0, x_2 and x_4. $x_1x_3 + x_1x_5 + x_3x_5$ is also a graph polynomial of $\Gamma_{K_3}//\Gamma'$.

The upshot is according to the bijection [27] between faces of a polytope and its dual, F corresponds to $[-1,0,-1,0,-1]$, which by our method corresponds to Γ'.

Actually, all facet polynomial of $\Delta_{\mathcal{U}}$ can factor as $\Phi_{\Gamma^*}\Phi_{\Gamma//\Gamma^*}$, where Γ^* is the corresponding subgraph assciated to the vertex of $\Delta^{\circ}_{2\mathcal{U}}$ dual to the facet.

We do not see such patterns on higher codimension faces. Since $\Phi_{\Gamma^*}\Phi_{\Gamma//\Gamma^*}$ could be viewed as the graph polynomial of the graph constructed by connecting Φ_{Γ^*} and $\Phi_{\Gamma//\Gamma^*}$ using one edge, it is obvious not all graphs will posses this property.

Question 5.1.1 *Give a criteria for graph Γ, such that we could construct a reflexive polytope Δ_Γ, all of whose facet polynomials could be factored as product of subgraph and modified quotient graph's graph polynomial.*

Example 5.1.2 *For all $2-$connected graphs with less or equal than 7 edges, this property holds.*

I put some of the examples I computed in the appendix.

Also notice that not all subgarphs of threespoke graph appears in the set of facets of polytope. So it is natural to ask

Question 5.1.3 *What kind of subgraphs by the correspondence rule we mentioned will appear in the reflexive polytope's list of facets?*

In the three wheel spoke graph case, subgraphs with 3 and 5 edges appears. No obvious pattern is discovered for more general graphs so far.

5.2 An explanation based on face polynomial in toric variety setting

Notice question 5.1.1 is purely combinatorial. In this section, we give an explanation which could possibly be generalized to a large class of graphs.

First we give an introduction to face polynomial from [17] and [28].

Let $\Delta \subset \mathbb{R}^n$ be a reflexive polytope, and

$$F = \sum_{\underline{m} \in \Delta \cap \mathbb{Z}^n} \alpha_{\underline{m}} \underline{x}^{\underline{m}} \in \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$$

a nonzero Laurent polynomial with support $\mathfrak{M}_F := \{\underline{m} \in \mathbb{Z}^n | \alpha_{\underline{m}} \neq 0\}$ contained in Δ. Let P_Δ be the toric variety associated to Δ [29], let $X_F \subset P_\Delta$ be the zero-locus of the section of $-K_{\mathbb{P}_\Delta}$ given by F. Let $\mathbb{D}^*_\sigma$ be the torus orbits, here we use $\sigma \in \Delta(i)$ to represent the faces of Δ. $\mathbb{D}^*_\sigma$ is defined by a set of toric coordinates

$$\mathbb{D}^*_\sigma := \{x^\sigma_1, ..., x^\sigma_{n-i} \in \mathbb{C}^*\} \cap \{x^\sigma_{n-i+1} = \cdots = x^\sigma_n = 0\}.$$

The the face polynomials of F attached to $\sigma \in \Delta(i)$ is obtained by rewriting $\underline{x}^{-\underline{o}_\sigma} F(\underline{x})$ in the $\{x^{\sigma_j}\}^n_{j=1}$ and setting $x^\sigma_{n-i+1} = \cdots = x^\sigma_n = 0$ to get a Laurent polynomial in $x^\sigma_1, \cdots, x^\sigma_{n-i}$. The support $\mathfrak{M}_{F_\sigma}$ of F_σ lies in $\sigma - \underline{o}_\sigma$, and its vanishing locus is $D^*_{F,\sigma} = X_F \cap \mathbb{D}^*_\sigma$.

Now we do a BEK style blow up but only along the linear subspaces corresponding to three and five edges subgraphs and get a toric variety P_{BEK}. Let $P_{2\Delta}$ be the toric variety associated to Δ, $\sum_{P_{\text{BEK}}}$ and $\sum_{P_{2\Delta}}$ be the fan of theses two varieties. Notice $\sum_{P_{\text{BEK}}}$ and $\sum_{P_{2\Delta}}$ have same set of one rays, there exists a fan $\sum_m$ which subdivides both of them. This implies a smooth toric $P_{\sum_m}$ variety lying over $P_{2\Delta}$ and P_{BEK} [30].

$$\begin{tikzcd}
 & P_{\sum_m} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
P_{2\Delta} \arrow[dr] & & P_{\text{BEK}} \arrow[dl] \\
 & \mathbb{P}^5 &
\end{tikzcd}$$

Figure 5.2. Relationships between toric varieties

Notice the subdivides in fans corresponds to blow up of locus with co-dimension greater or equal than two. Then $\pi_1 \circ \pi_2^{-1}$ is isomorphism when restricted to co-dimension one orbits.

The upshot is by 3.6, we know the graph polynomial on $\mathbb{D}_\sigma^*$ in P_{BEK} has the form $\Phi_{\Gamma'}(1, ..., B_r) \cdot \Phi_{\Gamma//\Gamma'}(A_{r+1}, ..., A_m)$, then the face polynomial on $\mathbb{D}_\sigma^*$ in $P_{2\Delta}$ also has the form $\Phi_{\Gamma'}(1, ..., B_r) \cdot \Phi_{\Gamma//\Gamma'}(A_{r+1}, ..., A_m)$. Then by definition of face polynomial, proved.

Remark 5.2.1 *Notice the above proof did not answer the second question. And it could be generalized to the case where a reflexive polytope could be constructed for the graph. While we do have cases where a canonical reflexive polytope does not exist but the face polynomials of the graph polytope still factor as product of subgraph and modified quotient graphs' graph polynomials.*

5.3 smoothing of the graph toric variety

$P_{2\Delta}$ is not smooth. As we could see in Chapter 4, we would like the ambient space we work in to be smooth. Partial desingularizations of $P_{2\Delta}$ can be obtained by subdividing faces of Δ° and replacing $\sum(\Delta^\circ)$ by the refinement from the fan on the subdivision. In particular, a maximal triangulation of $\partial\Delta^\circ$ if a collection of simplices $\underline{\theta}$

- $\cup_\alpha \theta_\alpha = \partial\Delta^\circ$
- the union of vertices of the $\{\theta_\alpha\}$ is $\partial\Delta^{circ} \cap \mathbb{Z}^n$
- $\theta_\alpha \cap \theta_\beta$(if nonempty) is a common face of θ_α and θ_β $(\forall \alpha, \beta)$.

When $\underline{\theta}$ has a projective support [28], is is called a maximal projective triangulation and a theorem of Batyrev [28] asserts that after the refinement, the toric variety we get is projective , with at worst singularities in codimension ≥ 4.

For three wheel spokes, the general theory only guarantee the less than co-dimension1 singularities. It turns out actully the MPCP process actually give a smooth variety. For the refined toric variety to be smooth, we need all faces of Δ° after subdivision to be a simplex [29]. Notice there are rectangular two faces of Δ°, we need subdivide them in to two triangles and check these subdivisions induces subdivisions on higher dimension faces.

Notice every edge of Δ° could be associated to a subgraph. Based on this association, there are three kinds of rectangles. Here the label of the vertices is the same as in the appendix.

- $[14, 10, 2, 15]$, where [14] represents the subgraph with edges x_0, x_2, x_5, [10] represents the subgraph with edges x_0, x_1, x_2, x_4, x_5, [2] represents edge $x_2 = 0$, where [15] represents the subgraph with edges x_1, x_2, x_4

- $[14, 11, 8, 5]$,where [11] represents the subgraph without edge x_3, where [8] represents the subgraph without edge x_1, and [5] represents the edge x_5

- $[14, 2, 6, 7]$ where [6] represent edge x_4, and [7] represent the subgraph without edge x_0.

Notice $\pi_1 : P_{\sum_m} \to P_{2\Delta}$ gives a way to subdivide these faces, and since $P_{\sum_m}$ is smooth, it turns out toric variety get from MPCP process is also smooth. $P_{\sum_m}$ is a choice of MPCP-desingularization. Since $P_{\sum_m}$ is a blow-up of P_{BEK}, and P_{BEK} is smooth, so is $P_{\sum_m}$.

6. Beyond wheel with three spokes.

We know the $n-$spoke wheel gives $\zeta(2n-3)$, so it is reasonable to expect that there is a uniform way to explain at least for this family of graphs, why their Feynman integral gives zeta values. So in this chapter we try to generalize the approaches we applied to three wheel spokes graphs to more general graphs.

Our general plan is to construct a toric variety P(where $D = P \setminus (\mathbb{C}^*)^\bullet$), with Y be the strict transformation of graph hypersurface in P. We hope to complete graph symbol ξ defined on $Y^* := (P \setminus D) \cap (\mathbb{C}^*)^\bullet$, to Ξ, then rewrite the Feynman integral as a pairing of $AJ(\Xi)$ and a rational homology class. As we will see in this chapter, some steps go through, while some other steps do not, which indicates interesting questions.

6.1 Further discussion on vanishing of graph symbols

As we saw in chapter 4, the graph symbol $\xi = \{-\frac{A_1}{A_e}, ..., -\frac{A_{e-1}}{A_e}\}$ is $\partial_{\mathcal{B}}$-closed on $(\mathbb{C}^*)^\bullet$ and hence $\partial_{\mathcal{B}}$-closed on Y^*. Thus, to be able to extend it to Y, we need all of its higher residues to vanish. And we know that if all of a graph's subgraphs and modified quotient graphs' graph symbol vanish, Res^1 will vanish. Hence it is natural to ask the question for which graphs their graph symbol will vanish.

We immediately have several simple observations

Lemma 6.1.1 *For a graph Γ, let Γ' be a graph obtained by adding on external edge to Γ($h_1(\Gamma) = h_1(\Gamma')$), then ξ_Γ vanishes if and only if $\xi_{\Gamma'}$ does.*

Proof Since such an external edge must be contained in any spanning tree, therefore Γ and Γ' have the same graph polynomial. ■

Lemma 6.1.2 *For a given graph Γ and an edge e, suppose ξ_Γ vanishes. Let e be substituted by two parallel edges e_1, e_2, then for the new graph Γ', its graph symbol $\xi_{\Gamma'}$ still vanishes.*

Proof Write Γ's graph polynomial as

$$\Phi_\Gamma = P_1 x_e + P_2,$$

where x_e is the variable corresponding to edge e, P_1 and P_2 are two polynomials. Notice $P_1 x_e$ corresponds to the set of spanning trees which do not contain edge e, denoted as $\mathcal{T}_1$, and P_2 corresponds to the set which do, denoted as $\mathcal{T}_2$. Define $\mathcal{T}_1'$ to be the set of spanning trees of Γ' which do not contain e_1 and e_2. $\mathcal{T}_2'$ is defined similarly. There is a one to one map from $\mathcal{T}_1$ to $\mathcal{T}_1'$ and one to two correspondence from $\mathcal{T}_2$ to $\mathcal{T}_2'$. Therefore, we know

$$\Phi_{\Gamma'} = P_1 x_{e_1} x_{e_2} + P_2(x_{e_1} + x_{e_2}),$$

and

$$\Phi_{\Gamma'} = 0 \Longleftrightarrow 0 = P_1 \frac{1}{\frac{1}{x_{e_1}} + \frac{1}{x_{e_2}}} + P_2.$$

$\xi_\Gamma = \{\frac{x_e}{x_n}, \sigma\}$. Then $\xi_{\Gamma'} = \{\frac{x_{e_1}}{x_n}, \frac{x_{e_2}}{x_n}, \sigma\}$. Notice if we working $\otimes\mathbb{Q}$, then

$$\begin{aligned}
\xi_{\Gamma'} &= \{(\frac{x_{e_1}}{x_n})^{-1}, (\frac{x_{e_2}}{x_n})^{-1}, \sigma\} \\
&= \{\frac{1}{\frac{x_{e_1}}{x_n}} + \frac{1}{\frac{x_{e_2}}{x_n}}, -(\frac{x_{e_1}}{x_{e_2}}), \sigma\} \\
&= \{(\frac{1}{\frac{x_{e_1}}{x_n}} + \frac{1}{\frac{x_{e_2}}{x_n}})^{-1}, -(\frac{x_{e_1}}{x_{e_2}}), \sigma\}^{-1} \\
&= \{(\frac{1}{\frac{x_{e_1}}{x_n}} + \frac{1}{\frac{x_{e_2}}{x_n}})^{-1}, \sigma, -(\frac{x_{e_1}}{x_{e_2}})\}.
\end{aligned}$$

Vanishing of ξ_Γ implies vanishing of $\{(\frac{1}{\frac{x_{e_1}}{x_n}} + \frac{1}{\frac{x_{e_2}}{x_n}})^{-1}, \sigma\}$. ■

Lemma 6.1.3 *For a given graph* Γ *and an edge* e*, suppose* ξ_Γ *vanishes. Subdivide* e *to two edges* e_1 *and* e_2 *to get* Γ'*, then for the new graph* Γ'*, its graph symbol* $\xi_{\Gamma'}$ *still vanishes.*

Figure 6.1. Subdividing an edge

Proof Write Γ's graph polynomial as

$$\Phi_\Gamma = P_1 x_e + P_2.$$

Notice $P_1 x_e$ corresponds to the set of spanning trees which do not contain edge e, denoted as $\mathcal{T}_1$, and P_2 corresponds to the set which do, denoted as $\mathcal{T}_2$. Define $\mathcal{T}_1{}'$ to be the set of spanning trees of Γ' which do not contain e_1 or e_2. $\mathcal{T}_2{}'$ is the set of spanning trees which contain both of e_1 and e_2. There is a one to two map from $\mathcal{T}_1$ to $\mathcal{T}_1{}'$ and one to one correspondence from $\mathcal{T}_2$ to $\mathcal{T}_2{}'$. Therefore, we know

$$\Phi_{\Gamma'} = P_1(x_{e_1} + x_{e_2}) + P_2.$$

By a similar reasoning as in lemma 6.1.2, $\xi_{\Gamma'}$ vanishes. ■

We give a criterion addressing the problem of vanishing of graph symbol.

For a planar graph Γ, its modified dual graph is defined as follows.

Definition 6.1.4 $\Gamma^\vee$ *has a vertex* v_F *for each face* F *of* Γ*,* v_{F_1} *and* v_{F_2} *in* $\Gamma^\vee$ *are connected if and only if* F_1 *and* F_2 *share at least one edge.*

Theorem 6.1.1 *For a planar graph* Γ*, consider it modified dual graph* $\Gamma^\vee$*. Let* $v \in \Gamma^\vee$ *be the vertex corresponding the unbounded face . Then* ξ_Γ *vanishes if* $h_1(\Gamma^\vee \setminus v) = 0$*.*

Proof We know ξ_{K_3} vanishes. Any Γ satisfying $h_1(\Gamma^\vee \setminus v) = 0$ could be obtained by adding double edge and subdividing edge from K_3. Then by lemma 6.1.2 and lemma 6.1.3, we proved the result. ■

We are also interested for what kind of graphs, their graph symbol vanishes.

Lemma 6.1.5 *For a graph Γ and its subgraph $\Gamma' \subset \Gamma$, if $\xi_{\Gamma'}$ is non vanishing, then ξ_Γ is not vanishing either.*

Proof We could do a BEK style blow up along the linear subspace corresponding to Γ' and let F be the exceptional divisor. By (3.3),

$$F \cong \mathbb{P}(E_G) \times \mathbb{P}(E_{\Gamma//G}).$$

So $Res_F(\xi_\Gamma)$ could be factored as product of $\xi_{\Gamma'}$ and $\xi_{\Gamma//\Gamma'}$. Since ξ_Γ vanishes. Both $\xi_{\Gamma'}$ and $\xi_{\Gamma//\Gamma'}$ should equal identity elements in their Milnor-K group. Then we get a contradiction. ■

For three wheels spoke graph K_4, ξ_{K_4} does not vanish. Otherwise $AJ(\xi_{K_4})$ would be torsion and does not yield zeta value. Notice if $h_1(\Gamma^\vee \setminus v) \neq 0$, then there always exists an subgraph Γ', which could be obtained from K_4 by subdividing edges. Therefore by the above lemma for such graph their graph symbols do not vanish.

Proposition 6.1.6 *For a planar graph Γ without multiple edges, let $v \in \Gamma^\vee$ be the vertex corresponding the face $F \subset \Gamma$ which has the maximal number of edges. ξ_Γ does not vanish if $h_1(\Gamma^\vee \setminus v) \neq 0$.*

Based on the conclusions, we immediately know for all one-loop graphs, all sunset family graphs, all subgraphs of three wheel spokes, their graph symbols will vanish. On the other hand, all wheel spokes and zig-zag graphs, their graph symbols will not vanish.

6.2 Toric interpretation of BEK's blow up.

In [BEK], section 7, the authors applied a sequence of blow ups along linear subspace in $\mathbb{P}^n$. These subspaces correspond to subgraphs whose first homology group is nonzero. Actually, we get a toric variety out of this, which we denote as P_{BEK}. For three wheel spokes graph, we are going to blow up linear subspaces corresponding to $3, 4$ and 5 edges subgraphs with nonzero first homology group.

Lemma 6.2.1 *Let $N_{\mathbb{R}} = \mathbb{R}$, where $N = \mathbb{Z}^n$ is a lattice. N has standard basis $e_1, ..., e_n$. Set $e_0 = -e_1 - e_2 - ... - e_n$. Each subset of $\{e_0, e_1, ..., e_n\}$ can generate a cone. All such cones form a fan Σ. Then the toric variety associated with Σ, denoted by X_Σ is isomorphic to $\mathbb{P}^n$.*

So $\mathbb{P}^n$ itself is a toric variety determined by the fan Σ.

Definition 6.2.1 *Let $\sigma = Cone(u_1, ..., u_n)$ be a smooth cone in a lattice N, where $u_1, ..., u_n$ is a basis for N. σ is contained in a fan Σ. Let $u_0 = u_1 + u_2 + ... + u_n$ and $\Sigma'(\sigma)$ be the set of cones generated by subsets of $\{u_0, u_1, ..., u_n\}$, not containing $\{u_1, ..., u_n\}$. Then $\Sigma^*(\sigma) = \Sigma'(\sigma) \cup \Sigma \backslash \{\sigma\}$ is called the star subdivision of Σ along σ.*

Lemma 6.2.2 *$\Sigma^*(\sigma)$ is a refinement of Σ, and the induced toric morphism $\phi : X_{\Sigma^*(\sigma)} \longrightarrow X_\Sigma$ makes $X_{\Sigma^*(\sigma)}$ the blow up of X_Σ at the distinguished point γ_σ corresponding to the cone σ.*

In another words, to blow up a affine toric variety along its distinguished point, it is enough to add the ray which is the sum of all the generators of the corresponding cone and perform the $*-$subdivision.

Lemma 6.2.3 *For a cone σ generated by standard basis $(e_1, e_2, ..., e_n)$ in $\mathbb{Z}^n$, $X_\sigma \cong \mathbb{C}^n$. And its distinguished point is $(0, 0, ..., 0)$.*

Consider the fan Σ such that $X_\Sigma \simeq \mathbb{P}^n$. $\mathbb{P}^n$ can be covered by $n+1$ affine varieties which are all isomorphic to $\mathbb{C}^n$. Under the homogeneous coordinate, the $n+1$ distinguished points for these affine varieties are exactly $(1,0,...,0),(0,1,0,...,0),...(0,0,\ ...,1),(-1,...,-1)$. Therefore by 6.2.1 and 6.2.2, blowing up these points corresponds to adding rays generated by $(1,1,...,1),(-1,0,0,\ ...,0)$(0, -1 ,0 , ..., 0), ...,$(0,0,...,-1)$ to Σ.

Similarly, blowing up of larger dimensional linear subspaces also has nice descriptions. Let σ and $e_1,e_2,...,e_n$ be the same as in 6.2.2. Let $\tau = Cone(e_1,...,e_r)$, $2 \leq r \leq n$ be a face of σ. τ corresponds to an open orbit $O(\tau)$ whose closure $V(\tau)$ is isomorphic to $\{0\}\times\mathbb{C}^{n-r}$, which is actually the subspace $S_r = \{(\underbrace{0,...,0,}_{r\ of\ them} *,*,...,*)\}$. Now let $e_0 = e_1 + ... + e_r$ and consider the fan

$$\Sigma^*(\tau) = \{Cone(A)|A \subseteq \{e_0,e_1,...,e_n\}, \{e_1,...,e_r\}\ is\ not\ subset\ of\ A\ \}. \tag{6.1}$$

Then $X_{\Sigma^*(\tau)}$ is the blow up of $\mathbb{C}^n$ along the $n-r$ dimensional linear subspace S_r.

We deduce from the above discussion that in $\mathbb{P}^n$, to blow up the linear subspace defined by $X_{i_1} = 0,...,X_{i_k} = 0$ (X_js are homogeneous coordinates)corresponds to add the ray generated by $e_{i_1} + e_{i_2} + ... + e_{i_k}$ in Σ.

More generally, let Σ be a fan in $N_\mathbb{R} \simeq \mathbb{R}^n$ and assume that $\tau \in \Sigma$ has the property that all cones of Σ containing τ are smooth. Let $u_\tau = \Sigma_{\rho\in\tau(1)} u_\rho$ and for each cone $\sigma \in \Sigma$ containing τ, set

$$\Sigma^*_\sigma(\tau) = \{Cone(A)|A \subset \{u_\tau \cup \sigma(1), \tau(1) \nsubseteq A\}. \tag{6.2}$$

Then the star subdivision of Σ relative to τ is the fan

$$\Sigma^*(\tau) = \{\sigma \in \Sigma|\tau \nsubseteq \sigma\} \cup \bigcup_{\tau\subset\sigma} \Sigma^*_{(\sigma)}(\tau). \tag{6.3}$$

The fan $\Sigma^*(\tau)$ is a refinement of Σ and hence induces a toric morphism

$$\phi : X_{\Sigma^*(\tau)} \to X_\Sigma. \tag{6.4}$$

$X_{\Sigma^*(\tau)}$ is the blow up of X_Σ along V_τ.

The following fact helps us to keep tract of the strict transformations of linear subspaces.

Lemma 6.2.4 *τ and σ are two cones in Σ satisfying $\tau \subseteq \sigma$. We do a star-subdivision for σ. Let $V(\tau)$ and $\widetilde{V(\tau)}$ be the closure of orbits in the interior of τ before and after the blowup. Then $\widetilde{V(\tau)}$ is the strict transformation of $V(\tau)$.*

Based on above facts, we can describe BEK$'s$ blow up in toric language. First we constructed a fan Σ as in Lemma 2.1. We let each ray e_i to represent the $i-$th edge E_i in the Feymann Graph. A subgraph with edges $E_{i_1}, ..., E_{i_k}$ will corresponds to a linear subspace L defined by $X_{i_1} = 0, ..., X_{i_k} = 0$. Then blowing up a linear subspace L corresponds to add the ray which is the sum of corresponding rays $e_{i_1}, ..., e_{i_k}$ in Σ.

6.3 Newton Polytope of Graph Polynomial

For a graph Γ, its graph polynomial Φ_Γ gives rise a Newton Polytope Δ_Γ. On one hand, the combinatorics of Δ_Γ totally determines the algebra-geometric properties of P_Δ, the toric variety we associate to the graph. On the other hand, Δ_Γ should also encodes info about the graph Γ. So we believe these Newton Polytopes from graphs deserve careful studys.

Lemma 6.3.1 *Let $n = \#E(\Gamma)$, then Δ_Γ is $n-1$ dimensional polytope in $\mathbb{R}^n$.*

Because the vertices of Δ_Γ all have $\#E(\Gamma) - h_1(\Gamma)$ many 1s as their entries, Δ_Γ is contained in the hyperplane $\{(x_0, ..., x_{n-1}) | \Sigma x_i = \#E(\Gamma) - h_1(\Gamma)\}$. This proves Δ_Γ has dimension less or equal than $n-1$. On the other hand, if we pick an edge $e \in \Gamma$ which has distinct endpoints, and consider the graph Γ/e. All spanning trees T in Γ/e can be lifted to a spanning tree $T \cup \{e\}$ in Γ. Vertices corresponding to these spanning trees will have 0 in e's place. And Γ also have spanning trees which does not contain e. Vertices corresponding to the second kind of spanning tress will have 1 in e's place. By induction, we know the first set of vertices have rank $n-2$. Therefore, the rank of Δ_Γ should be larger than $n-2$.

We project to this polytope to hyperplane by just let one of the graph variable to be 1, which is the same as ignore one of the entries of Δ_Γ's vertices. Notice that, we actually don't lose any graphical information by doing this, since the sum of entries is $\#E(\Gamma) - h_1(\Gamma)$, we can deduce the deleted term is 1 or 0 by summing the rest of the terms. For convenience, we still use Δ_Γ to denote the resulting polytope.

For a graph Γ, let $V(\Gamma)$ be the set of vertices of Γ. Consider the set of graphs, $\mathfrak{S}_1^3$, such that for $\Gamma \in \mathfrak{S}_1^3$, $h_1(\Gamma^\vee \setminus v_{\text{un}}) = 1$, and $\exists \Gamma' \subset \Gamma$, such that all faces of Γ' are triangles except the unbounded face. v_{un} corresponds to the unbounded face. The reader may wonder why we consider such graphs. Actually, two important families of graphs : wheels spokes and zig-zag graphs belong to this set and it is generalization of three spoke wheel graph.

For $\Gamma \in \mathfrak{S}_1^3$, we have a canonical way to translate Δ_Γ into reflexive polytope. Consider the polytope $\Delta_{2\Gamma} := 2\Delta_\Gamma + (-1, ..., -1)$.

Lemma 6.3.2 $\Delta_{2\Gamma}$ *is a reflexive polytope.*

First, we notice that $\Delta_{2\Gamma}$ always contains origin. For $\Gamma \in \mathfrak{S}_1^3$, Γ can be decomposed into a maximal chain of faces and an extra edge. For a certain edge e belongs to some face, we can extend e into a spanning tree T_1 by adding adjacent edges in each face. We can also just extend e into another spanning tree T_2 by always adding the other edge in the faces. The upshot of this is this amounts to say there always exit two vertices in $\Delta_{2\Gamma}$,

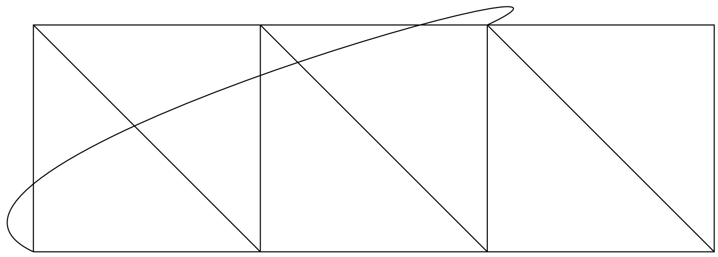

Figure 6.2. Maximal chain of faces

whose sum is $(0, 0, ..., -2, ..., 0)$, with -2 in the place corresponding to e. Similarly, we can construct two spanning trees who do not both contain a edge e. We can choose e to be the edge shared by two triangular faces. Separate the remaining four edges in two sets; each of them is connected. Then we can extend them into spanning trees having different edges in the triangle faces. In this way, we show that there exits two vertices in $\Delta_{2\Gamma}$, their sum is $(0, 0, ..., 2, ..., 0)$, with 2 in the place corresponding to e.

We also have the following lemma about reflexive polytope.

Lemma 6.3.3 *A lattice polytope P is reflexive. $\Longleftrightarrow$ For any facet F of P, there is no lattice point between the hyperplane spanned by F and its parallel through origin.*

$\Delta_{2\Gamma}$ can be viewed as cutting the polytope $[-1, 1]^n$ with rational hyperplanes. Since origin is the only internal lattice point in $[-1, 1]^n$, it is easy to see there is no lattice point between these rational hyperplanes and their parallel through origin. Therefore, we proved lemma 1.3.

Therefore can associate a Gorenstein Fano toric variety P_Γ to it.

We have a description of what $\Delta_{2\Gamma}^\circ$ looks like. By combinatorics, we know that for a reflexive polytope P, its dual polytope P° is defined as follows:

$$P^\circ = \{\vec{x} | \vec{x} \cdot \vec{y} \geq -1, \forall \vec{y} \in P\}. \tag{6.5}$$

In our situation, since we know all the vertices of $\Delta_{2\Gamma}$ have $(n+1)/2$ "-1" entries and $(n-1)/2$ "1" entries or $(n+1)/2$ "1" entries and $(n-1)/2$ "-1" entries. Immediately, we know the special points $(1, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, ..., 1)$, and $(-1, -1, ..., -1)$ are in $\Delta^{\circ}_{2\Gamma}$. Because $\Delta_{2\Gamma}$ defines an embedding into projective space, these points have to be vertices of $\Delta^{\circ}_{2\Gamma}$.

7. Appendix.

7.1 Data of the graph polytope

The set of points of $\Delta_{\mathcal{U}}$ are

$$\begin{array}{cccc}
[-1,-1,1,1,1] & [-1,1,1,-1,1] & [-1,1,1,1,-1] & [1,-1,-1,1,1] \\
[1,-1,1,1,-1] & [1,1,-1,-1,1] & [1,1,-1,1,-1] & [1,1,1,-1,-1] \\
[-1,-1,-1,1,1] & [-1,-1,1,-1,1] & [-1,1,-1,-1,1] & [-1,1,-1,1,-1] \\
[-1,1,1,-1,-1] & [1,-1,-1,-1,1] & [1,-1,-1,1,-1] & [1,-1,-1,1,-1].
\end{array}$$

We compute its dual polytope $\Delta^{\circ}_{\mathcal{U}}$. Its points are

$$\begin{array}{cccc}
(-1,-1,-1,-1,-1) & (0,1,0,0,0) & (0,0,1,0,0) & (1,0,0,0,0) \\
(0,0,0,0,1) & (0,0,0,1,0) & (1,1,1,1,1) & (-1,0,0,0,0) \\
(0,-1,0,0,0) & (0,0,-1,0,0) & (0,0,0,-1,0) & (0,0,0,0,-1) \\
(0,-1,0,-1,-1) & (-1,0,-1,0,-1) & (1,1,0,0,1) & (0,0,1,1,1),
\end{array}$$

which are labeled as

$$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16.
\end{array}$$

Then $\Delta_{\mathcal{U}}^{\circ}$'s facets are

$$[0,1,3,9,10,11,12,13,14],[0,1,4,7,9,10,13,14],[0,1,2,7,10,11,12,13],$$
$$[0,2,5,7,8,11,12,13,15],[0,4,5,7,8,9,13,15],[0,2,4,7,8,10,12,15],$$
$$[0,3,5,8,9,11,12,13],[0,3,4,8,9,10,12,14],[1,4,5,6,7,9,13,14,15],$$
$$[1,2,5,6,7,11,13,15],[1,2,4,6,7,10,14,15],[1,3,5,6,9,11,13,14],$$
$$[1,2,3,6,10,11,12,14],[3,4,5,6,8,9,14,15],$$
$$[2,3,5,6,8,11,12,15],[2,3,4,6,8,10,12,14,15].$$

Its codimension two faces are

$$[0,1,9,10,13,14],[0,1,10,11,12,13],[0,3,9,11,12,13],[0,3,9,10,12,14],$$
$$[1,3,9,11,13,14],[1,3,10,11,12,14],[0,1,7,10,13],[0,4,7,9,13],$$
$$[0,4,7,10],[0,4,9,10,14],[1,4,7,9,13,14],[1,4,7,10,14],$$
$$[0,2,7,11,12,13],[0,2,7,10,12],[1,2,7,11,13],[1,2,7,10],$$
$$[1,2,10,11,12],[0,5,7,8,13,15],[0,2,7,8,12,15],[0,5,8,11,12,13],$$
$$[2,5,7,11,13,15],[2,5,8,11,12,15],[0,4,7,8,15],[0,5,8,9,13],$$
$$[0,4,8,9],[4,5,7,9,13,15],[4,5,8,9,15],[0,4,8,10,12],$$
$$[2,4,7,10,15],[2,4,8,10,12,15],[0,3,8,9,12],[3,5,9,11,13],$$
$$[3,5,8,9],[3,5,8,11,12],[3,4,8,9,14],[3,4,8,10,12,14],$$
$$[1,5,6,7,13,15],[1,4,6,7,14,15],[1,5,6,9,13,14],[4,5,6,9,14,15],$$
$$[1,2,6,7,15],[1,5,6,11,13],[1,2,6,11],[2,5,6,11,15],$$
$$[1,2,6,10,14],[2,4,6,10,14,15],[1,3,6,11,14],[3,5,6,9,14],$$
$$[3,5,6,11],[2,3,6,11,12],[2,3,6,10,12,14],$$
$$[3,5,6,8,15],[3,4,6,8,14,15],[2,3,6,8,12,15]$$

Its codimension three faces are

$$[0,1,10,13],[0,9,13],[0,9,10,14],[1,9,13,14],[1,10,14],[0,11,12,13],$$
$$[0,10,12],[1,11,13],[1,10,11,12],[0,3,9,12],[3,9,11,13],[3,11,12],$$
$$[3,9,14],[3,10,12,14],[1,3,11,14],[0,7,13],[0,7,10],[1,7,13],$$
$$[1,7,10],[0,4,7],[0,4,9],[4,7,9,13],[0,4,10],[4,7,10],$$
$$[4,9,14],[4,10,14],[1,4,7,14],[0,2,7,12],[2,7,11,13],[2,11,12],$$
$$[2,7,10],[2,10,12],[1,2,7],[1,2,11],[1,2,10],[0,7,8,15],$$
$$[0,5,8,13],[5,7,13,15],[5,8,15],[0,8,12],[2,7,15],[2,8,12,15],$$
$$[5,11,13],[5,8,11,12],[2,5,11,15],[0,4,8],[4,7,15],[4,8,15],$$
$$[0,8,9],[5,9,13],[5,8,9],[4,8,9],[4,5,9,15],[4,8,10,12],$$
$$[2,4,10,15],[3,8,9],[3,8,12],[3,5,9],[3,5,11],[3,5,8],$$
$$[3,4,8,14],[1,6,7,15],[1,5,6,13],[5,6,15],[1,6,14],[4,6,14,15],$$
$$[5,6,9,14],[1,2,6],[2,6,15],[1,6,11],[5,6,11],[2,6,11],$$
$$[2,6,10,14],[3,6,14],[3,6,11],[3,5,6],[2,3,6,12],[3,6,8,15]$$

Its one-dimension edges are

$$[0,13],[0,10],[1,13],[1,10],[0,9],[9,13],[9,14],[10,14],[1,14],[0,12],$$
$$[11,13],[11,12],[10,12],[1,11],[3,9],[3,12],[3,11],[3,14],[0,7],[7,13],$$
$$[7,10],[1,7],[0,4],[4,7],[4,9],[4,10],[4,14],[2,7],[2,12],[2,11],$$
$$[2,10],[1,2],[0,8],[7,15],[8,15],[5,13],[5,8],[5,15],[8,12],[2,15],$$
$$[5,11],[4,8],[4,15],[8,9],[5,9],[3,8],[3,5],[1,6],[6,15],[5,6],$$
$$[6,14],[2,6],[6,11],[3,6]$$

7.2 Data of BEK's toric variety's normal fan

The one rays of the normal fan are

$$(-1,-1,-1,-1,-1),(0,1,0,0,0),(0,0,1,0,0),(1,0,0,0,0),(0,0,0,0,1),(0,0,0,1,0),$$

$$(1,1,1,1,1),(-1,0,0,0,0),(0,-1,0,0,0),(0,0,-1,0,0),(0,0,0,-1,0),$$

$$(0,0,0,0,-1),(0,-1,0,-1,-1),(-1,0,-1,0,-1),(1,1,0,0,1),(0,0,1,1,1).$$

Label them from 0 to 15. The two fans are

$$(0,1),(0,2),(1,2),(1,3),(2,3),(0,3),(3,4),(0,4),(1,4),(2,4),(2,5),(0,5),$$

$$(1,5),(3,5),(4,5),(4,6),(1,6),(5,6),(2,6),(3,6),(1,7),(2,7),(4,7),(5,7),$$

$$(0,7),(3,8),(5,8),(4,8),(2,8),(0,8),(1,9),(5,9),(0,9),(3,9),(4,9),(3,10),$$

$$(0,10),(1,10),(4,10),(2,10),(3,11),(5,11),(0,11),(1,11),(2,11),(5,12),$$

$$(3,12),(11,12),(10,12),(8,12),(1,12),(4,12),(0,12),(2,12),(5,13),(9,13),$$

$$(7,13),(11,13),(2,13),(0,13),(1,13),(4,13),(3,13),(3,14),$$

$$(5,14),(6,14),(0,14),(9,14),(10,14),(2,14),(4,14),(1,14),$$

$$(3,15),(4,15),(8,15),(7,15),(6,15),(1,15),(0,15),(2,15),(5,15).$$

three fans are

$$
\begin{gathered}
(0,1,2),(1,2,3),(0,1,3),(0,3,4),(0,1,4),(2,3,4),(0,2,4),(1,2,4),\\
(0,2,5),(1,2,5),(0,3,5),(2,3,5),(1,3,5),(1,4,5),(3,4,5),(0,4,5),\\
(1,4,6),(4,5,6),(1,5,6),(1,2,6),(2,4,6),(2,5,6),(1,3,6),(3,4,6),\\
(2,3,6),(3,5,6),(1,2,7),(1,4,7),(2,4,7),(4,5,7),(2,5,7),(1,5,7),\\
(0,1,7),(0,2,7),(0,4,7),(0,5,7),(3,5,8),(3,4,8),(4,5,8),(2,3,8),\\
(2,5,8),(2,4,8),(0,4,8),(0,5,8),(0,3,8),(0,2,8),(1,5,9),(0,1,9),\\
(0,5,9),(0,3,9),(3,5,9),(1,3,9),(0,4,9),(4,5,9),(3,4,9),(1,4,9),\\
(0,3,10),(1,3,10),(0,1,10),(3,4,10),(0,4,10),(1,4,10),(2,4,10),(2,3,10),\\
(0,2,10),(1,2,10),(3,5,11),(0,3,11),(0,5,11),(1,3,11),(1,5,11),(0,1,11),\\
(2,3,11),(0,2,11),(2,5,11),(1,2,11),(3,5,12),(3,11,12),(5,11,12),(3,10,12),\\
(5,8,12),(3,8,12),(1,10,12),(1,11,12),(1,3,12),(4,10,12),(3,4,12),(4,8,12),\\
(0,5,12),(0,4,12),(0,11,12),(0,10,12),(0,8,12),(0,1,12),(0,3,12),(1,2,12),\\
(0,2,12),(2,5,12),(2,3,12),(2,10,12),(2,11,12),(2,4,12),(2,8,12),(5,9,13),\\
(5,7,13),(5,11,13),(2,7,13),(2,11,13),(2,5,13),(0,9,13),(0,11,13),(0,5,13),\\
(0,2,13),(0,7,13),(1,11,13),(1,9,13),(1,7,13),(1,5,13),(1,2,13),(0,1,13),\\
(4,9,13),(4,5,13),(0,4,13),(1,4,13),(4,7,13),(3,9,13),(3,5,13),(3,11,13),\\
(0,3,13),(1,3,13),(3,5,14),(3,6,14),(5,6,14),(0,3,14),(5,9,14),(0,9,14),\\
(3,9,14),(0,10,14),(3,10,14),(2,6,14),(2,3,14),(2,10,14),(4,5,14),(3,4,14),\\
(0,4,14),(4,10,14),(4,6,14),(4,9,14),(2,4,14),(1,3,14),(1,5,14),(1,6,14),\\
(0,1,14),(1,10,14),(1,9,14),(1,2,14),(1,4,14),(3,4,15),(4,8,15),(3,8,15),\\
(4,7,15),(3,6,15),(4,6,15),(1,6,15),(1,7,15),(1,4,15),(0,7,15),(0,8,15),\\
(0,4,15),(2,3,15),(2,4,15),(2,8,15),(2,7,15),(2,6,15),(1,2,15),(0,2,15),\\
(5,8,15),(5,7,15),(2,5,15),(1,5,15),(0,5,15),(5,6,15),(3,5,15),(4,5,15).
\end{gathered}
$$

7.3 Examples of factorization of facet graph polynomials

7.3.1 Example 1

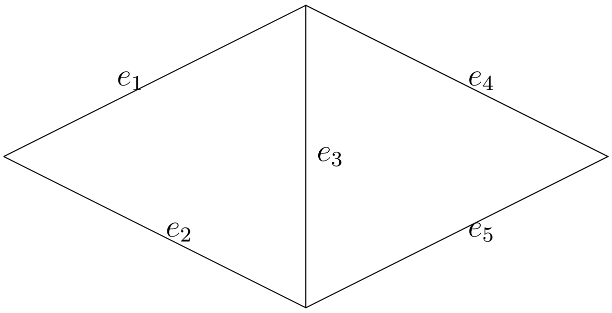

Figure 7.1. Example 1

The combinatorial info of graph polytope is

spanning trees	monomials	coordinate of vertices	label
$e_1e_2e_4$	e_3e_5	$(0,0,1,0,1)$	0
$e_1e_2e_5$	e_3e_4	$(0,0,1,1,0)$	1
$e_1e_3e_4$	e_2e_5	$(0,1,0,0,1)$	2
$e_1e_3e_5$	e_2e_4	$(0,1,0,1,0)$	3
$e_1e_4e_5$	e_2e_3	$(0,1,1,0,0)$	4
$e_2e_3e_4$	e_1e_5	$(1,0,0,0,1)$	5
$e_2e_3e_5$	e_1e_4	$(1,0,0,1,0)$	6
$e_2e_4e_5$	e_1e_3	$(1,0,1,0,0)$	7

.

The factorization of its facets are as follows

facet	face polynomial	factorization
$[0,1,2,3,5,6]$	$e_3e_5+e_3e_4+e_2e_5+e_2e_4+e_2e_5+e_1e_4$	$(e_1+e_2+e_3)(e_4+e_5)$
[2,3,4,5,6,7]	$e_2e_5+e_2e_4+e_2e_3+e_1e_5+e_1e_4+e_1e_3$	$(e_3+e_4+e_5)(e_1+e_2)$
$[0,1,2,3,4]$	$e_3e_5+e_3e_4+e_2e_3+e_2e_4+e_2e_3$	$1\cdot(e_3e_5+e_3e_4+e_2e_3+e_2e_4+e_2e_3)$
$[0,1,5,6,7]$	$e_3e_5+e_3e_4+e_1e_5+e_1e_4+e_1e_3$	$1\cdot(e_3e_5+e_3e_4+e_1e_5+e_1e_4+e_1e_3)$
$[0,2,4,5,7]$	$e_3e_5+e_2e_5+e_2e_3+e_1e_5+e_1e_3$	$1\cdot(e_3e_5+e_2e_5+e_2e_3+e_1e_5+e_1e_3)$
$[1,3,4,6,7]$	$e_3e_4+e_2e_4+e_2e_3+e_1e_4+e_1e_3$	$1\cdot(e_3e_4+e_2e_4+e_2e_3+e_1e_4+e_1e_3)$
$[0,1,4,7]$	$e_3e_5+e_3e_4+e_2e_3+e_1e_3$	$e_3(e_1+e_2+e_4+e_5)$

.

7.3.2 Example 2

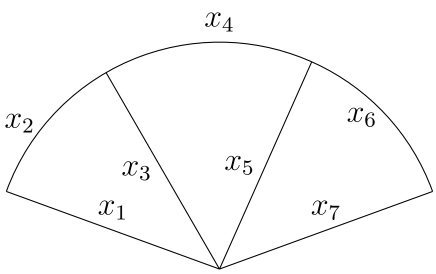

Figure 7.2. Example 2

The combinatorial info of graph polytope is

spanning trees	monomials	coordinate of vertices	label
$x_1x_2x_4x_6$	$x_3x_5x_7$	$(0,0,1,0,1,0,1)$	0
$x_1x_2x_4x_7$	$x_3x_5x_6$	$(0,0,1,0,1,1,0)$	1
$x_1x_2x_5x_6$	$x_3x_4x_7$	$(0,0,1,1,0,0,1)$	2
$x_1x_2x_5x_7$	$x_3x_4x_6$	$(0,0,1,1,0,1,0)$	3
$x_1x_2x_6x_7$	$x_3x_4x_5$	$(0,0,1,1,1,0,0)$	4
$x_1x_3x_4x_6$	$x_2x_5x_7$	$(0,1,0,0,1,0,1)$	5
$x_1x_3x_4x_7$	$x_2x_5x_6$	$(0,1,0,0,1,1,0)$	6
$x_1x_3x_5x_6$	$x_2x_4x_7$	$(0,1,0,1,0,0,1)$	7
$x_1x_3x_5x_7$	$x_2x_4x_6$	$(0,1,0,1,0,1,0)$	8
$x_1x_3x_6x_7$	$x_2x_4x_5$	$(0,1,0,1,1,0,0)$	9
$x_1x_4x_5x_6$	$x_2x_3x_7$	$(0,1,1,0,0,0,1)$	10
$x_1x_4x_5x_6$	$x_2x_3x_7$	$(0,1,1,0,0,0,1)$	10
$x_1x_4x_5x_7$	$x_2x_3x_6$	$(0,1,1,0,0,1,0)$	11
$x_1x_4x_6x_7$	$x_2x_3x_5$	$(0,1,1,0,1,0,0)$	12
$x_2x_3x_4x_6$	$x_1x_5x_7$	$(1,0,0,0,1,0,1)$	13
$x_2x_3x_4x_7$	$x_1x_5x_6$	$(1,0,0,0,1,1,0)$	14
$x_2x_3x_5x_6$	$x_1x_4x_7$	$(1,0,0,1,0,0,1)$	15
$x_2x_3x_5x_7$	$x_1x_4x_6$	$(1,0,0,1,0,1,0)$	16
$x_2x_3x_6x_7$	$x_1x_4x_5$	$(1,0,0,1,1,0,0)$	17
$x_2x_4x_5x_6$	$x_1x_3x_7$	$(1,0,1,0,0,0,1)$	18
$x_2x_4x_5x_7$	$x_1x_3x_6$	$(1,0,1,0,0,1,0)$	19
$x_2x_4x_6x_7$	$x_1x_3x_5$	$(1,0,1,0,1,0,0)$	20

One of the facet is $[0, 1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 18, 19]$. The factorization is as follows

$$
\begin{aligned}
&x_3x_5x_7 + x_3x_5x_6 + x_3x_4x_7 + x_3x_4x_6 + x_2x_5x_7 + x_2x_5x_6 + x_2x_4x_7 + x_2x_4x_6+\\
&x_2x_3x_7 + x_2x_3x_6 + x_1x_5x_7 + x_1x_5x_6 + x_1x_4x_7 + x_1x_4x_6 + x_1x_3x_7 + x_1x_3x_6\\
&= (x_7 + x_6)(x_3x_5 + x_3x_4 + x_2x_5 + x_2x_4 + x_2x_3 + x_1x_5 + x_1x_4 + x_1x_3).
\end{aligned}
$$

Notice $x_3x_5 + x_3x_4 + x_2x_5 + x_2x_4 + x_2x_3 + x_1x_5 + x_1x_4 + x_1x_3$ is the graph polynomial of the subgraph consists of edges $\{x_1, x_2, x_3, x_4, x_5\}$, and $x_7 + x_6$ is the graph polynomial of the modified quotient graph consists of edges $\{x_7, x_6\}$.

Another facet is $[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 14, 15, 16, 17]$. The factorization is as follows

$$
\begin{aligned}
&x_3x_5x_7 + x_3x_5x_6 + x_3x_4x_7 + x_3x_4x_6 + x_2x_5x_7 + x_2x_5x_6 + x_2x_4x_6+\\
&x_2x_4x_5 + x_1x_5x_7 + x_1x_5x_6 + x_1x_4x_7 + x_1x_4x_6 + x_1x_4x_5\\
&= (x_1 + x_2 + x_3)(x_5x_7 + x_5x_6 + x_4x_7 + x_4x_6 + x_4x_5).
\end{aligned}
$$

Notice $x_1 + x_2 + x_3$ is the graph polynomial of the subgraph consists of edges $\{x_1, x_2, x_3\}$, and $x_5x_7 + x_5x_6 + x_4x_7 + x_4x_6 + x_4x_5$ is the graph polynomial of the modified quotient graph consists of edges $\{x_4, x_5, x_6, x_7\}$.

The third non-isomorphic facet is $[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]$. The corresponding polynomial is graph polynomial of modified quotient graph by deleting edge x_1.

All facets are isomorphic to one of the above case and are listed below:

$$
\begin{gathered}
[0, 1, 2, 3, 5, 6, 7, 8, 10, 11, 13, 14, 15, 16, 18, 19]\\
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 14, 15, 16, 17]\\
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] \qquad .\\
[0, 2, 4, 5, 7, 9, 10, 12, 13, 15, 17, 18, 20]\\
[0, 1, 4, 5, 6, 9, 12, 13, 14, 17, 20]
\end{gathered}
$$

$$[1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 20]$$

$$[0, 1, 2, 3, 4, 13, 14, 15, 16, 17, 18, 19, 20]$$

$$[0, 1, 5, 6, 10, 11, 12, 13, 14, 18, 19, 20]$$

$$[2, 3, 4, 7, 8, 9, 10, 11, 12, 15, 16, 17, 18, 19, 20]$$

$$[5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]$$

.

7.4 Other examples

- All one loop graphs.
- Sunset family graphs(Grpah with two vertices and edges linking them).
- Dunce's Cap

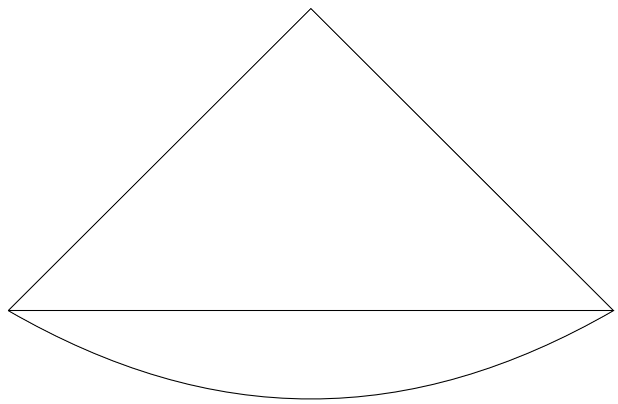

Figure 7.3. Dunce's Cap

- Sunglasses' graph

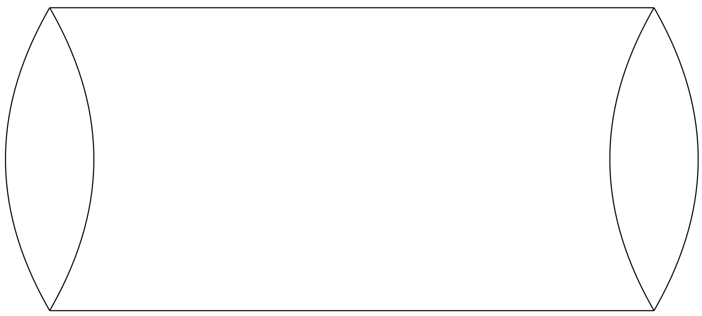

Figure 7.4. Sunglasses' graph

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