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Analysis of the Navier--Stokes-**αβ** equations

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Analysis of the Navier–Stokes-$\alpha\beta$ Equations

by

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Abstract

In this dissertation we prove various analytic results for the Navier–Stokes-$\alpha\beta$ equations. We establish well-posedness and regularity. In addition we determine estimates for the nodal distance and the number of determining modes. A method of averaging is developed and applied to derive a Kármán–Howarth type equation for the Navier–Stokes-$\alpha\beta$ equations. Finally we investigate an anisotropic generalization of the Navier–Stokes-$\alpha\beta$ equations. We show that the eigenvalues of the moment of inertia tensor convect with the flow and we derive energy type inequalities for the equations.

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Chapter 1

Background and summary of results

1.1 Introduction

This chapter serves to summarize the main results of this dissertation. The majority of the results apply to the Navier–Stokes–$\alpha\beta$ equations (NS-$\alpha\beta$). At the end of this chapter, we discuss additional results for an anisotropic generalization of the NS-$\alpha\beta$ equations. In order to give our results context, we present and discuss related results of the Navier–Stokes equations (NS). To make our contributions clear we label our results with (Brady). Results due to others are labeled with the appropriate reference. We only use this convention in the introduction, since in the main body of the dissertation it is clear from context who the results are attributed to.

The understanding of turbulence is one of the most outstanding problems in classical physics. Richard Feynman is quoted as saying,

> "I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrody-

> namics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic."

Many researches believe that the 3d Navier–Stokes equations (3d NS) are the proper equations to describe incompressible fluid flow, but after more than a century, "solving" the 3d NS remains one of the outstanding problems in mathematics. Specifically by "solving" we are referring to the problem of proving existence of smooth solutions. The existence of smooth solutions is currently one of the Millennium problems. The reader should consult Fefferman [7] for the details of the Millennium problem. For reference we present the Navier–Stokes equations as follows,

$$\begin{aligned} \rho\left(\tfrac{\partial \mathbf{u}}{\partial t} + (\operatorname{grad}\mathbf{u})\mathbf{u}\right) &= -\operatorname{grad} p + \mu\Delta\mathbf{u} + \rho\mathbf{f}, \\ \operatorname{div}\mathbf{u} &= 0, \end{aligned} \tag{1.1}$$

where $\mathbf{u}$ is the velocity field, ρ the mass-density is assumed to be constant, μ is the viscosity, p is the pressure, and $\rho\mathbf{f}$ is the force per volume. The condition $\operatorname{div}\mathbf{u} = 0$ is the incompressibility condition. In this thesis we are primarily concerned with the periodic domain. Thus, in our discussion of the NS equations we assume that the domain Ω is periodic.

Due to the existence issue and other issues, many alternative turbulence models have been proposed. The hope is that a model will have better functional properties than the 3d NS equations, while still capturing the essence of the NS equations. In this dissertation we focus on a model proposed by Fried and Gurtin [12], referred to as the Navier–Stokes-$\alpha\beta$ equations. This model differs from most turbulence models in that it arises out of a consistent continuum mechanical framework. While the reader should consult [12] for the full details, we very briefly mention the key ideas behind their model. Normally if one derives the Navier–Stokes equations in a continuum mechanical framework, one assumes that stress tensor is power conjugate to $\operatorname{grad}\mathbf{u}$, where $\mathbf{u}$ is the velocity field. In addition to this assumption, Fried and Gurtin hypothesized that there is a hyperstress that is power conjugate to $\operatorname{grad}\boldsymbol{\omega}$, where $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$ is the vorticity. The essential idea is that they allow for higher order tractions that depend on the curvature, whereas that traditional Cauchy stress tensor only depends on the normal stress. After applying the modern continuum mechanical approach as developed in Fried, Gurtin, and Anand [15] the resulting equations can be written in the following form,

$$\begin{aligned} \rho\left(\tfrac{\partial\mathbf{v}}{\partial t} + (\operatorname{grad}\mathbf{v})\mathbf{u} + (\operatorname{grad}\mathbf{u})^T\mathbf{v}\right) &= -\operatorname{grad}p + \mu\Delta\mathbf{w} + \rho\mathbf{f}, \\ \operatorname{div}\mathbf{u} &= 0, \\ \mathbf{v} &= (1-\alpha^2\Delta)\mathbf{u}, \\ \mathbf{w} &= (1-\beta^2\Delta)\mathbf{u}. \end{aligned} \tag{1.2}$$

We refer to these equations as the Navier–Stokes-$\alpha\beta$ (NS-$\alpha\beta$) equations. First we note that if we formally set $\alpha = \beta = 0$ we recover the Navier–Stokes equations. Note that the term $(\operatorname{grad}\mathbf{u})^T\mathbf{u} = \operatorname{grad}(\mathbf{u}\cdot\mathbf{u})/2$ and can be added to the pressure term. Thus, aside from the term $(\operatorname{grad}\mathbf{u})^T\mathbf{v}$ the NS-$\alpha\beta$ equations resembles the NS equations. The new term arises out of the necessity for the equations to satisfy the property of material frame indifference. I.e., the idea that the choice of a frame should not change the result of an experiment. In fact, Leray [21] was one of the first to propose a model of this type.

$$\begin{aligned} \rho\left(\tfrac{\partial\mathbf{v}}{\partial t} + (\operatorname{grad}\mathbf{v})\mathbf{u}\right) &= -\operatorname{grad}p + \mu\Delta\mathbf{v} + \rho\mathbf{f}, \\ \mathbf{v} &= (1-\alpha^2\Delta)\mathbf{u} \\ \operatorname{div}\mathbf{u} &= 0, \end{aligned} \tag{1.3}$$

It was then noted that this equation did not satisfy the property of material frame indifference [14]. Another similar model is the LANS-α equations developed by Holm and Marsden [17]. If we set $\beta = \alpha$ in the NS-$\alpha\beta$ equations, then mathematically we obtain the LANS-α equations. There are some key differences in the theory behind the models. First is that the LANS-α equations are developed using a Lagrangian approach that is only valid in a non-dissipative system. Thus, Holm and Marsden in [17] first derived a non dissipative version of what they refer to as the Euler-α equations. Then the assumption was made that the dissipation should take the form $\nu\Delta(1 - \alpha^2\Delta)$. While this choice is intuitive, it is important to note that the choice is arbitrary. Thus the development of the NS-$\alpha\beta$ equations is more that just a mathematical extension of the LANS-α equations, the NS-$\alpha\beta$ equations arise out of a consistent a physical framework that incorporates the dissipation from the outset. We should also add that the boundary conditions for the NS-$\alpha\beta$ and LANS-α equation differ from each other. The non-slip boundary conditions for the LANS-α equations are $\mathbf{u} = \mathbf{v} = \mathbf{0}$. The non-slip boundary conditions for the NS-$\alpha\beta$ equations are much more complicated. Although we assume that we are working on a periodic domain, we state the non-slip boundary conditions for completeness. The non-slip boundary conditions for the NS-$\alpha\beta$ equations are $\mathbf{u} = \mathbf{0}$ and $\mathbf{PGn} = -\mu\ell \operatorname{curl} \mathbf{u}$, where $\mathbf{n}$ is the unit normal to the surface, $P = I - \mathbf{n} \otimes \mathbf{n}$, $\mathbf{G} = \mu\beta^2(\operatorname{grad}\operatorname{curl}\mathbf{u} + (grad\operatorname{curl}\mathbf{u})^T)$ is the hyperstress, γ is a dimensionless parameter, and ℓ is a parameter referred to as the wall eddy length. Due to mathematical difficulties associated with this boundary condition we restrict to the periodic domain. Thus, for the rest of this dissertation we assume that we are working on a periodic domain Ω with periodic boundary conditions.

We now discuss the relevance of the parameters α and β in the NS-$\alpha\beta$ equations. Both of these parameters have dimensions of length and are thought of as filtering parameters. In fact, the inverse operator $(1 - \alpha\Delta)^{-1}$, referred to as a Helmholtz filter of filtering scale α, has the following form.

$$(1 - \alpha\Delta)^{-1}\mathbf{u}(\mathbf{x}) = \int_\Omega \frac{1}{4\pi\alpha^2|\mathbf{x}-\mathbf{y}|} e^{-|\mathbf{x}-\mathbf{y}|/\alpha}\mathbf{u}(\mathbf{y})\ d\mathbf{y}$$

Applying the operator $(1 - \alpha^2\Delta)$ is then referred to as unfiltering. Thus, we refer to $\mathbf{u}$ as the filtered velocity field and we refer to $\mathbf{v}$ and $\mathbf{w}$ as unfiltered velocity fields. We say that α is the convective filtering scale and β is the dissipative filtering scale.

Before proceeding further we present the functional setting that is necessary to state our results. We define the following functional spaces:

(i) $\mathcal{V} = \{\boldsymbol{\phi} \in C^\infty(\Omega)^3 : \operatorname{div} \boldsymbol{\phi} = 0 \quad \text{and} \quad \int_\Omega \boldsymbol{\phi}(\mathbf{x})d\mathbf{x} = \mathbf{0}\}$

(ii) H is the closure of $\mathcal{V}$ in $L^2(\Omega)^3$

(iii) V is the closure of $\mathcal{V}$ in $H^1(\Omega)^3$

The reason we define $\mathcal{V}$ to contain functions whose average is zero is given as follows. If we integrate the first equation in (1.3) over Ω, we find using integration by parts that

$$\frac{d}{dt}\int_\Omega \mathbf{u}(\mathbf{x},t)\ d\mathbf{x} = \int_\Omega \mathbf{f}(\mathbf{x},t)\ d\mathbf{x}. \tag{1.4}$$

Thus if we assume that

$$\int_\Omega \mathbf{u}(\mathbf{x},0)\,d\mathbf{x} = \mathbf{0} \qquad \text{and} \qquad \int_{\boldsymbol{\Omega}} \mathbf{f}(\mathbf{x},\mathbf{t})\,\mathbf{dx} = \mathbf{0},$$

then we can conclude that

$$\int_\Omega \mathbf{u}(\mathbf{x},t)\,d\mathbf{x} = \mathbf{0} \qquad \text{for all } t.$$

Our reason for making this assumption is in the simplification of the analysis while keeping the original nature of the problem. This assumption is commonly made for the Navier–Stokes equations, see [8] or [5]. Specifically it allows us to use a simpler form of Poincaré's inequality. If we did not make this assumption, the well-posedness results would still hold, although the analysis would be significantly more difficult. On the other hand it is not clear to us if the nodal and modal results would still hold.

We let P denote Helmholtz-Leray projection onto divergence free vector fields, *I.e.,* $P : L^2(\Omega)^3 \to H$. We define the stokes operator by $A = -P\Delta$. The domain of A is given by $D(A) = V \cap H^2(\Omega)^3$. The inverse of stokes operator is a self-adjoint compact operator, therefore there exists an orthonormal basis of eigenfunctions $\{\boldsymbol{\phi}_j\}$ and eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots$ such that $A\boldsymbol{\phi}_j = \lambda_j \boldsymbol{\phi}_j$. In fact as developed in [5] we may use the eigenvalues of A to define A^s for any $s \in \mathbb{R}$. For $s > 0$ the domain of A^s may be identified as $D(A^s) = V \cap H^{2s}(\Omega)^3$. For $s < 0$ we identify the duals spaces as $D(A^s) = D(A^{-s})'$. We use the shorthand notation $V^s = D(A^{s/2})$. Due to Poincaré's inequality the Sobolev norms on V^s may be characterized as $\|\mathbf{u}\|_{V^s} = \|A^{s/2}\mathbf{u}\|_H$. Lastly, since our vector fields are time dependent we define the following norms. Let u be a time dependent function on a banach space X. Then we write

$$\|u\|_{L^p([0,T];X)} = \left(\int_0^T \|u\|_X^p \, dt \right)^{1/p}$$

and

$$\|u\|_{L^\infty([0,T];X)} = \operatorname{ess\,sup}_{t\in[0,T]} \|u\|_X.$$

With the functional setting established we proceed by obtaining a more universal form of the NS-$\alpha\beta$ equations. We begin by dividing the first equation in (1.2) by the mass density ρ.

$$\frac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad} \mathbf{v})\mathbf{u} + (\operatorname{grad} \mathbf{u})^T \mathbf{v} = -\operatorname{grad}\left(\frac{p}{\rho}\right) + \nu\Delta\mathbf{w} + \mathbf{f}, \tag{1.5}$$

where ν is the kinematic viscosity. Next we transform the equations into a dimensionless form. We assume that our periodic domain has side length $2\pi\ell$. We define the following characteristic scales. A length scale $L = \ell$. A velocity scale U, which could be defined as

$$U^2 = \frac{1}{(2\pi\ell)^3} \int_\Omega |\mathbf{u}(\mathbf{x}, 0)|^2 \, d\mathbf{x}.$$

We then define the characteristic time scale $T = L/U$. Using these characteristic scales we define the following dimensionless quantities:

$$\mathbf{u}' = \frac{\mathbf{u}}{U} \qquad \mathbf{x}' = \frac{\mathbf{x}}{L} \qquad t' = \frac{t}{T} \tag{1.6}$$

Using these dimensionless quantities we obtain a non-dimensional form of the NS-$\alpha\beta$ equations.

$$\begin{gathered}
\tfrac{\partial \mathbf{v}'}{\partial t'} + (\operatorname{grad}' \mathbf{v}')\mathbf{u}' + (\operatorname{grad}' \mathbf{u}')^T \mathbf{v}' = -\operatorname{grad}' \left(\tfrac{p}{U^2\rho}\right) + \tfrac{\nu}{UL}\Delta' \mathbf{w}' + \mathbf{f}', \\
\operatorname{div}' \mathbf{u}' = 0, \\
\mathbf{v}' = (1 - \tfrac{\alpha^2}{L^2}\Delta')\mathbf{u}', \\
\mathbf{w}' = (1 - \tfrac{\beta^2}{L^2}\Delta')\mathbf{u}'
\end{gathered} \tag{1.7}$$

We see the appearance of the Reynold's number $Re = UL/\nu$ and we define $\varpi = p/(U^2\rho)$. We set $\epsilon = \alpha^2/L^2$ and $\gamma = \beta^2/\alpha^2$, where it is understood that $\epsilon > 0$ and $0 < \gamma \leq 1$. It is important to note that in terms of the new dimensionless variable $\mathbf{x}'$, the periodic box now has side length 2π. Finally we remove the primes with the understanding that the quantities considered are dimensionless.

$$\begin{aligned}
\tfrac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad} \mathbf{v})\mathbf{u} + (\operatorname{grad} \mathbf{u})^T \mathbf{v} &= -\operatorname{grad} \varpi + Re^{-1}\Delta \mathbf{w} + \mathbf{f}, \\
\operatorname{div} \mathbf{u} &= 0, \\
\mathbf{v} &= (1 - \epsilon\Delta)\mathbf{u}, \\
\mathbf{w} &= (1 - \epsilon\gamma\Delta)\mathbf{u}
\end{aligned} \tag{1.8}$$

We still refer to these equations as the Navier–Stokes-$\alpha\beta$ equations with the understanding that $\alpha^2 = \epsilon\ell^2$ and $\beta^2 = \epsilon\gamma\ell^2$. We use the identity

$$(\operatorname{grad} \mathbf{v})\mathbf{u} + (\operatorname{grad} \mathbf{u})^T \mathbf{v} = (\operatorname{curl} \mathbf{v}) \times \mathbf{u} + \operatorname{grad}(\mathbf{v} \cdot \mathbf{u})$$

to rewrite the first equation as

$$\frac{\partial \mathbf{v}}{\partial t} = -\operatorname{grad} \varpi + \mathbf{u} \times (\operatorname{curl} \mathbf{v}) + Re^{-1}\Delta \mathbf{w} + \mathbf{f} \tag{1.9}$$

Then we apply the Helmholtz-Leray projector P

$$\begin{aligned}
\tfrac{\partial \mathbf{v}}{\partial t} + Re^{-1} A\mathbf{w} &= \widetilde{B}(\mathbf{u}, \mathbf{v}) + \mathbf{f}, \\
\operatorname{div} \mathbf{u} &= 0, \\
\mathbf{v} &= (1 + \epsilon A)\mathbf{u}, \\
\mathbf{w} &= (1 + \epsilon\gamma A)\mathbf{u},
\end{aligned} \tag{1.10}$$

where $\widetilde{B}(\mathbf{u}, \mathbf{v}) = P\mathbf{u} \times (\operatorname{curl} \mathbf{v})$, and $A = -P\Delta$ is the Stokes operator. Note that we have assumed that $P\mathbf{f} = \mathbf{f}$, otherwise we would have added that gradient part of $\mathbf{f}$ to the modified pressure ϖ.

1.2 Well-posedness results for the NS-$\alpha\beta$ equations

Before we state our well-posedness results for the NS-$\alpha\beta$ equations we review the existence theorems for the Navier–Stokes equations. First using similar methods as outlined for the NS-$\alpha\beta$ equations we arrive at the following form for the Navier–Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + Re^{-1} A\mathbf{u} = -B(\mathbf{u}, \mathbf{u}) + \mathbf{f}, \tag{1.1}$$

where $B(\mathbf{u}, \mathbf{v}) = P(\operatorname{grad} \mathbf{v})\mathbf{u}$.

Definition 1.1. A weak solution of the Navier–Stokes equation on $[0, T]$ is a function $\mathbf{u} \in L^2([0, T]; V) \cap C_w([0, T]; H)$ that for all $\mathbf{u} \in V$ satisfies

$$\begin{gathered}\tfrac{\partial \mathbf{u}}{\partial t} \in L^1_{\text{loc}}([0, T]; V') \\ \langle \tfrac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\rangle_{V',V} + Re^{-1}\langle A^{1/2}\mathbf{u}, A^{1/2}\mathbf{v}\rangle + \langle B(\mathbf{u}, \mathbf{u}), \mathbf{v}\rangle_{V',V} = \langle \mathbf{f}, \mathbf{v}\rangle_{V',V} \\ \mathbf{u}(0) = \mathbf{u}_o,\end{gathered} \tag{1.2}$$

where $\mathbf{u}_o$ is the initial condition and C_w are the weakly continuous functions.

As shown in [5] for $\mathbf{u} \in H$ and $\mathbf{f} \in V'$ we have existence of weak solutions for dimensions $n = 2, 3$. The main outstanding issue is for the existence of strong solutions.

Definition 1.2. A strong solution of the Navier–Stokes equations on $[0, T]$ is a function $\mathbf{u} \in L^\infty([0, T]; V) \cap L^2([0, T]; V^2)$ that satisfies

$$\begin{gathered}\tfrac{\partial \mathbf{u}}{\partial t} + Re^{-1}A\mathbf{u} + B(\mathbf{u}, \mathbf{u}) = \mathbf{f} \\ \mathbf{u}(0) = \mathbf{u}_o.\end{gathered} \tag{1.3}$$

We should note that these are not classical solutions and the equation only makes sense in H. For $n = 2$ we have existence of strong solutions for all $T > 0$, but for

$n = 3$ (see [8]) we only have existence of strong solutions for $0 < T < T^* < \infty$, where T^* depends on the parameters of the system. After the formulation of the problem by Leray, proving or disproving the existence of strong solutions for 3d NS has remained elusive for over 80 years.

Based on the above definition of strong solutions for the Navier–Stokes equations, we make the following definition of strong solutions for the NS-$\alpha\beta$ equations.

Definition 1.3. (Strong Solution of the NS-$\alpha\beta$ equations.) Let $T > 0$. A function $\mathbf{u} \in C([0,T]; V^2) \cap L^2([0,T]; V^4)$ with $\partial \mathbf{v}/\partial t \in L^2([0,T]; H)$ is said to be a strong solution to the the NS-$\alpha\beta$ equations on the interval $[0,T]$ if it satisfies,

$$\begin{gathered} \tfrac{\partial \mathbf{v}}{\partial t} + Re^{-1} A\mathbf{w} = \widetilde{B}(\mathbf{u}, \mathbf{v}) + \mathbf{f} \\ \mathbf{u}(0) = \mathbf{u}_o. \end{gathered} \tag{1.4}$$

The reason we call the above solutions strong is that the above equation makes sense in H as opposed to V^r for $r < 0$. For the 3d NS-$\alpha\beta$ equations we do in fact have the existence of strong solutions.

Theorem 1.4. *(Brady) (Global existence of strong solutions to the NS-$\alpha\beta$ equations.) Let $\mathbf{f} \in L^2([0,T]; H) \cap L^\infty([0,\infty); H)$ and $\mathbf{u}_o \in V^3$. Then for any $T > 0$, equation* (1.4) *has a strong solution $\mathbf{u}$ on $[0,T]$.*

In addition, we have the following theorems for the NS-$\alpha\beta$ equations:

Theorem 1.5. *(Brady)(Uniqueness and continuous dependence on the initial data.) Strong solutions are unique and depend continuously on the initial data.*

Theorem 1.6. *(Brady)(Regularity) Let $s \geq 3$ and let $\mathbf{u}$ be a strong solution to the NS-$\alpha\beta$ equations with $\mathbf{u} \in L^2([0,T]; V^{s-1})$. Suppose the initial condition and forcing function satisfy $\mathbf{u}_o \in V^s$ and $\mathbf{f} \in L^\infty([0,\infty); V^{s-1})$. Then we conclude that the solution satisfies $\mathbf{u} \in L^\infty([0,T]; V^{s+1}) \cap L^2([0,T]; V^{s+2})$.*

This collection of theorems fully answers the well-posedness and regularity questions surrounding the NS-$\alpha\beta$ equations. It should be noted that in setting $\gamma = 1$ these results extend the LANS-α equations on the periodic domain. In [9] they show the existence of what they call regular solutions for the LANS-α equations. These regular solutions are not quite as "weak" as weak solutions to the Navier–Stokes equations, but they are not strong enough to be strong solutions. Uniqueness and continuous dependence on the initial data were already established in [9] for the LANS-α equations. Regularity and strong solutions are new for the LANS-α equations.

1.3 Determining Nodes and Modes

We now move on to the fascinating ideas of determining nodes and modes. We first present the definitions and results for the 2d NS equations, then we give similar definitions and results for the NS-$\alpha\beta$ equations. As with strong solutions these results do not yet exist for the 3d NS equations, whereas our results are for the 3d NS-$\alpha\beta$ equations.

Let $\mathcal{E} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ be a finite number of distinct points in the domain. We refer to $\mathcal{E}$ as a set of nodes. The nodal distance of a set of nodes is a measure of how far away a point could be from a node. So for each $\mathbf{x} \in \Omega$ we define

$$d(\mathbf{x}) = \min_{1 \le i \le N} \|\mathbf{x}_i - \mathbf{x}\| \tag{1.1}$$

and

$$d_{\mathcal{E}} = \sup_{x \in \Omega} d(\mathbf{x}). \tag{1.2}$$

We refer to $d_{\mathcal{E}}$ as the nodal distance or nodal value. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions to the Navier–Stokes equations with forcing functions $\mathbf{f}_1$ and $\mathbf{f}_2$, respectively. We assume that the two forcing functions have the same asymptotic behavior. I.e.,

$$\lim_{t\to\infty} \int_{\Omega} \|\mathbf{f}_1 - \mathbf{f}_2\|_H^2 \, d\mathbf{x} = 0$$

For convenience we also assume that

$$Gr := \|\mathbf{f}_1\|_{L^2([0,\infty);H)} = \|\mathbf{f}_2\|_{L^2([0,\infty);H)}$$

where we refer to Gr as the Grashof number. Traditionally the Grashof number is a non-dimensional parameter that represents the strength of the forcing. For the current discussion Gr serves this purpose. Suppose that the two solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ agree asymptotically on the set $\mathcal{E}$. I.e.,

$$\lim_{t\to\infty} |\mathbf{u}_1(\mathbf{x}_i, t) - \mathbf{u}_2(\mathbf{x}_i, t)| = 0 \quad \text{for } i = 1, 2, \ldots, N.$$

We say that $\mathcal{E}$ is a set of determining nodes for the Navier–Stokes equations if we can conclude that

$$\lim_{t\to\infty} \|\mathbf{u}_1 - \mathbf{u}_2\|_H = 0.$$

An amazing result is that a set of nodes are determining if their nodal distance is small enough. A bound has been given for the nodal distance of the 2d NS equations on a periodic domain with the above functional setting.

Theorem 1.1. *(Foias et al. [8]) Let $\mathcal{E}$ be a set a of nodes on Ω. Suppose the nodal distance satisfies*

$$d_{\mathcal{E}} \leq cRe^{-4}Gr^{-2}, \tag{1.3}$$

where c is an arbitrary constant. Then $\mathcal{E}$ is a set of determining nodes for the Navier–

Stokes equations.

Our results for the NS-$\alpha\beta$ equations are similar, but due to the presence of the higher order terms in NS-$\alpha\beta$ we slightly modify the assumptions that $\mathbf{u}$ (or $\mathbf{v}$) must satisfy.

Suppose for the two solutions $\mathbf{u}_1$ and $\mathbf{u}_2$, that in addition to $\mathbf{u}_1$ and $\mathbf{u}_2$ agreeing asymptotically on $\mathcal{E}$ we also have that $A^{1/2}\mathbf{u}_1$ and $A^{1/2}\mathbf{u}_2$ agree asymptotically on the set $\mathcal{E}$. I.e.,

$$\lim_{t\to\infty} |\mathbf{u}_1(\mathbf{x}_i,t) - \mathbf{u}_2(\mathbf{x}_i,t)| = \lim_{t\to\infty} |A^{1/2}\mathbf{u}_1(\mathbf{x}_i,t) - A^{1/2}\mathbf{u}_2(\mathbf{x}_i,t)| = 0 \quad \text{for } i = 1,2,\ldots,N. \tag{1.4}$$

The need to control both higher order values should not be surprising due to the presence of the term, $\partial/\partial t\mathbf{v} = \partial/\partial t(1+\epsilon A)\mathbf{u}$. We make the following definition for determining nodes for the NS-$\alpha\beta$ equations.

Definition 1.2. (Determining nodes for the Navier–Stokes-$\alpha\beta$ equations) Let $\mathcal{E}$ be a set of nodes and let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions of the Navier–Stokes-$\alpha\beta$ equations that satisfy condition (1.4). We say that $\mathcal{E}$ is a set of determining nodes for the NS-$\alpha\beta$ equations, if we can conclude that

$$\lim_{t\to\infty} \|\mathbf{u}_1 - \mathbf{u}_2\|_V = 0.$$

Recall that $\|\mathbf{u}\|_V = \|A^{1/2}\mathbf{u}\|_H$. The key point of Definition 2.1 is that we require the same order of convergence on the domain Ω as we assumed on $\mathcal{E}$. I.e., since we assume that $A^{1/2}\mathbf{u}_1$ and $A^{1/2}\mathbf{u}_2$ agree asymptotically on $\mathcal{E}$ we require that $\|A^{1/2}\mathbf{u}_1 - A^{1/2}\mathbf{u}_2\|_H \to 0$ as $t \to \infty$ in order for $\mathcal{E}$ to be a set of determining nodes.

As with the Navier–Stokes equation, the following theorem says that $\mathcal{E}$ is a set of determining nodes if the nodal distance is sufficiently small.

Theorem 1.3. *(Brady)(Determining Nodes) Suppose that we are given a set of nodes, $\mathcal{E} \in \Omega$ with nodal distance $d_{\mathcal{E}}$. We also assume that $\gamma \leq 1$ for the NS-$\alpha\beta$ equations. If the nodal distance $d_{\mathcal{E}}$ satisfies*

$$d_{\mathcal{E}} < \frac{c\gamma^6\epsilon^4}{(1+\epsilon)^4 Re^8 Gr^4}, \tag{1.5}$$

then we may conclude that $\mathcal{E}$ is a set of determining nodes for the NS-$\alpha\beta$ equations. Note that c is some constant that does not depend on any of present parameters.

Another way of discussing the finite dimensionality of the flow is through the modes (eigenfunctions) of the Stokes operator. Let $\{\mathbf{w}_j\}_1^\infty$ be the eigenfunctions of the Stokes operator A and let P_m be the projection operator onto the first m modes. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions to the Navier–Stokes equations with forcing functions $\mathbf{f}_1$ and $\mathbf{f}_2$ respectively. As with determining nodes we assume that the two forcing functions have the same asymptotic behavior.

$$\lim_{t\to\infty} \int_\Omega \|\mathbf{f}_1 - \mathbf{f}_2\|_H^2 \, d\mathbf{x} = 0$$

and that

$$Gr := \|\mathbf{f}_1\|_{L^2([0,\infty);H)} = \|\mathbf{f}_2\|_{L^2([0,\infty);H)}.$$

We say that m is the number of determining modes or the modal value if

$$\lim_{t\to\infty} \int_\Omega \|\mathrm{P}_m\mathbf{u}_1 - \mathrm{P}_m\mathbf{u}_2\|_H^2 \, d\mathbf{x} = 0$$

implies

$$\lim_{t\to\infty} \int_\Omega \|\mathbf{u}_1 - \mathbf{u}_2\|_H^2 \, d\mathbf{x} = 0.$$

Theorem 1.4. *(Foias et al. [8])(Determining Modes for the 2d Navier–Stokes equa-*

tions) An upper bound for the number of determining modes for the Navier–Stokes equations is given by the following estimate

$$\lambda_{m+1} \geq cRe^2 Gr. \tag{1.6}$$

where c is an arbitrary constant.

As is shown in Constantin and Foias [5], for large m we have $\lambda_m \sim m^{n/2}$, where n is the number of dimensions. Thus for large m we have

$$m \sim Re^2 Gr \tag{1.7}$$

For the NS-$\alpha\beta$ we say that m is the number of determining modes if whenever we have

$$\lim_{t\to\infty} \int_\Omega \|\mathbf{f}_1 - \mathbf{f}_2\|_H^2 \, d\mathbf{x} = 0$$

and

$$\lim_{t\to\infty} \int_\Omega \|\mathrm{P}_m \mathbf{v}_1 - \mathrm{P}_m \mathbf{v}_2\|_H^2 \, d\mathbf{x} = 0$$

we may conclude that

$$\lim_{t\to\infty} \int_\Omega |\mathbf{v}_1 - \mathbf{v}_2|^2 \, d\mathbf{x} = 0.$$

Theorem 1.5. *(Brady)(Determining Modes for the NS-$\alpha\beta$ equations) Let $\gamma \leq 1$ for the NS-$\alpha\beta$ equations. Let m be the least integer that satisfies*

$$\lambda_{m+1} \geq \frac{c(2+\epsilon)^2 Re^4 Gr^2}{\gamma^3 \epsilon}, \tag{1.8}$$

then m is an upper bound for the number of determining modes. Note that c is some constant that does do not depend on any of the present parameters.

Since $\lambda_m \sim m^{2/3}$ we have

$$m \sim \frac{c(2+\epsilon)^3 Re^6 Gr^3}{\gamma^{9/4}\epsilon^{3/2}}. \tag{1.9}$$

We briefly discuss our estimates for nodal distance and modal number. We mainly focus on the modal number since the parameter dependence behaves inversely for the nodal distance. It is very important to note that our estimates for the nodal distance and modal number are not sharp. Thus, the above estimates merely serve as bounds and the functional dependence could be very different. The other important comment we make before proceeding is that these results do not exist for the 3d Navier–Stokes equations. The same difficulties with establishing well-posedness for 3d NS also prevent the determination of the nodal distance or modal number.

We now discuss the functional behavior of our bound for λ_m in (1.8). For purposes of discussion we assume that we choose the smallest possible λ_m for the inequality, thus when we say that λ_m may decrease we mean smallest possible value. For the Reynolds number we see that as $Re \to \infty$ $\lambda_m \to \infty$ and as $Re \to 0^+$ $\lambda_m \to 0$. It is not surprising that one would need more modes to describe a more turbulent flow. As far as the decrease of modes as Re decreases, this is most likely due to a highly viscous flow being over damped.

Similar behavior is shown with the Grashof number. We see that as the Grashof decreases the number of modes needed decreases. This should not be surprising, since a Grashof number of zero indicates no forcing and thus any solution will decay to zero due to dissipation . As the Grashof number is increased the number of modes is suggested to increase. This seems reasonable as in increase in forcing would compete with the viscous damping.

Before discussing ϵ we investigate the influence of γ. Note that γ ranges over $0 < \gamma \leq 1$. In that case of $\gamma = 1$ we recover the LANS-α equations on the periodic domain. Korn [20] had given an estimate for the modal number of the LANS-α equations, but we believe that there is a serious flaw in their proof. Interestingly as $\gamma \to 0^+$, λ_m increases. In terms of the NS-$\alpha\beta$ equations having $\gamma \to 0^+$ reduces the filtering on the dissipative term and thus decreases the regularizing effect of dissipation. In the situation of $\gamma = 0$ determining a finite modal number would actually be more difficult than the case of 3d Navier–Stokes! This is due to the presence of higher order derivatives in the non-linear term with no corresponding higher order derivatives on the dissipative term.

For $\epsilon \to 0^+$ we have $\lambda_m \to \infty$. While not surprising, the result is disappointing. Had this result been finite we may have had a way of giving a bound for the modal number of 3d Navier–Stokes, but due to the difficulties of the 3d Navier–Stokes equations the result was not unexpected.

As mentioned before similar (inverse) comments hold for the nodal distance. We do comment that in the case of the LANS-α equations ($\gamma = 1$), Korn [20] derived the nodal distance of

$$d_{\mathcal{E}} = \frac{\alpha^6}{3c\nu(\lambda_1^{-1} + \alpha^2)^2(\alpha^2 + 1)\max\{1, \lambda_1^{-1/2}\}} \left[(2\lambda_1 + 4)Gr^2\lambda_1 + \frac{c(\lambda_1 + \alpha^2)^4 Gr^6}{\alpha^1 2\lambda_1^2}\right]^{-1}.$$

Note that we have not non-dimensionalized their result. To compare their result to ours we note that λ_1 is the first eigenvalue of the Stokes operator, ν is the kinematic viscosity, and $Gr = \|\mathbf{f}\|_{L^\infty([0,\infty);H)}$ has not been non-dimensionalized. We believe that there were a number of small errors in their derivation. First, is that the $[\dots]^{-1}$

is a typo and it should be to the -2 power. Plus, their nodal distance is inversely proportional to the viscosity, which seems non-physical. Aside from the small errors in their derivation we see by comparison that our result is a nice improvement in the case of the LANS-α equations.

1.4 A Kármán–Howarth type equation for the NS-$\alpha\beta$ equations

Even if the existence question is settled for the Navier–Stokes equations, there is still a practical issue of implementing the equations to simulate highly turbulent flows. In fact it has been shown [8] that the degrees of freedom for a highly turbulent flow scale as $Re^{9/4}$, where Re is the Reynolds number. Thus, the computational power needed increases non-linearly with the Reynolds number. Due to the analytic and computational difficulties associated with the Navier–Stokes equations, many researches have tried to understand the fundamental nature of turbulence. That is to try to find basic structures that are universal to turbulent flows. The literature on this subject is immense, the voluminous work by Monin and Yaglom [22] is a standard reference to this vast subject.

We are particularly interested in the idea of homogeneous isotropic turbulence on a periodic domain. Heuristically homogeneous isotropic flows are flows that on average do not have a preferred place or direction. The hope was that by stripping away as much of the problem as possible one would be able to gain an understanding of the fundamental nature of turbulence. As we summarize below, this turned out not to be the case and has led to the infamous closure problem for homogeneous isotropic turbulence. The first researchers to investigate homogeneous isotropic turbulence were Taylor [26] and Kármán and Howarth [19]. Later Robertson [25] gave the theory a solid mathematical foundation. In Chapter 3 we look in depth at these ideas, but we give a quick summary here. The idea of homogeneous isotropic turbulence is that the probabilities of the flow are invariant under translations and rotations. From a frequentist point of view this means if we performed an experiment many times, then the flow would show no preferential place or direction. Although for each experiment, the flow could be non-zero at any point. For the Navier–Stokes equations we study the correlation between the velocity fields at two nearby points. We let $\mathbf{x}$ and $\mathbf{x}' = \mathbf{x} + \mathbf{r}$ be two points in the domain. For short-hand we write $\mathbf{u} = \mathbf{u}(\mathbf{x})$ and $\mathbf{u}' = \mathbf{u}(\mathbf{x}')$ and we write $\langle \cdot \rangle$ for a generic averaging procedure. In the literature it is usually assumed that the average satisfies the needed properties, in chapter 3 we develop a method of averaging and rigorously develop the needed framework. Using the average $\langle \cdot \rangle$ we define the following correlation tensors. The tensor Q is referred to as the double correlation tensor.

$$\mathbf{Q}(\mathbf{r}, t) = \langle \mathbf{u} \otimes \mathbf{u}' \rangle \tag{1.1}$$

Similarly we have the triple correlation tensor $\mathbb{T}$ defined by

$$\mathbb{T}(\mathbf{r}, t) = \langle \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u}' \rangle + \langle \mathbf{u}' \otimes \mathbf{u} \otimes \mathbf{u} \rangle \tag{1.2}$$

As is shown by Robertson [25] there are scalar representatives $Q(r, t)$ and $T(r, t)$ of $\mathbf{Q}(r, t)$ and $\mathbb{T}(r, t)$ respectively, where $r = |\mathbf{r}|$ is referred to as the separation distance. Very importantly, the scalar representatives only depend on the separation distance! The process of how these scalar representatives arise and how they are implemented is very complicated (the reader should consult chapter 3 or [25] for the details), but to give the flavor we give the representation for $\mathbf{Q}$ below.

$$\mathbf{Q}(\mathbf{r}, t) = Q(r, t)\mathbf{e} \otimes \mathbf{e} + \left(\frac{r}{2} \frac{\partial Q(r, t)}{\partial t} + Q(r, t) \right) (I - \mathbf{e} \otimes \mathbf{e}), \tag{1.3}$$

where

$$\mathbf{e} = \frac{1}{|\mathbf{r}|}\mathbf{r}.$$

Then by applying the dynamics of the Navier–Stokes equations to $\mathbf{Q}$, Robertson [25] found that the scalar representatives Q and T satisfy the following partial differential equation.

$$\frac{\partial}{\partial t}Q(r,t) = \nu DQ(r,t) + T(r,t), \tag{1.4}$$

where D is the differential operator

$$D = \frac{\partial}{\partial r}\frac{1}{r^4}\frac{\partial}{\partial r}r^4.$$

This result is very remarkable in that it reduces the Navier–Stokes equations to a scalar-valued partial differential equation. The major problem is that there are two unknowns, Q and T. This is referred to as the closure problem for homogeneous isotropic turbulence. Many models have been proposed to link Q and T, but all of them fail severely when compared to experiments and simulations. Batchelor [2] is a good reference for the models that were proposed as of 1953. Amazingly, no significant progress has been made since.

Even though there is no full understanding of homogeneous isotropic turbulence for the Navier–Stokes equations, we were curious how such an approach would compare for the Navier–Stokes-$\alpha\beta$ equations. We quickly realized that the traditional ad hoc averaging procedures did not hold up under close scrutiny when applied to the higher order NS-$\alpha\beta$ equations. Looking at the work on statistical solutions by Vishik and Furiskov [29] and Foias et al. [8], we were inspired to develop our own method of averaging. Our method is similar to [29] and [8], but as we discuss in chapter 3 there are subtle differences and more importantly our method of application differs. The key idea of our method of averaging is that instead of averaging with respect to a probability density function as in [24], or taking the average over the domain as in [2], we average over the space of admissible solutions. That is, we consider a probability measure on the space of solutions to the equation in question. Armed with this method of averaging we were able to apply the ideas of Robertson [25] to the NS-$\alpha\beta$ equations.

As with the Navier–Stokes equations we define the following tensors:
The double correlation tensor $\mathbf{Q}$

$$\mathbf{Q}(\mathbf{r}, t) = \langle \mathbf{u} \otimes \mathbf{u}' \rangle, \tag{1.5}$$

the triple correlation tensor $\mathbb{T}$,

$$\mathbb{T}(\mathbf{r}, t) = \langle \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u}' \rangle + \langle \mathbf{u}' \otimes \mathbf{u} \otimes \mathbf{u} \rangle. \tag{1.6}$$

In addition we have a new third order tensor $\mathbb{S}$

$$\mathbb{S}(\mathbf{r}, t) = \langle \mathbf{u} \otimes \Delta\mathbf{u} \otimes \mathbf{u}' \rangle + \langle \mathbf{u} \otimes \Delta\mathbf{u} \otimes \mathbf{u}' \rangle + 2\langle \operatorname{grad}\mathbf{u} (\operatorname{grad}\mathbf{u})^T \otimes \mathbf{u}' \rangle + 2\langle \mathbf{u}' \otimes \operatorname{grad}\mathbf{u} (\operatorname{grad}\mathbf{u})^T \rangle. \tag{1.7}$$

Note that in the above definitions we choose $\mathbf{u}$ the unfiltered velocity field as opposed to $\mathbf{v}$ or $\mathbf{w}$, the filtered velocity fields. As with the Navier–Stokes equations there are scalar-valued functions that represent the above tensors and as before the scalar fields only depend on the separations distance $r = |\mathbf{r}|$ and t. We write these scalar-valued functions as Q, T, and S respectively. We show in chapter 3 that when we apply the dynamics of the NS-$\alpha\beta$ equations to $\mathbf{Q}$, that the scalar-valued functions satisfy a partial differential equation that we refer to as the Kámán–Howarth-$\alpha\beta$ equation. Note that in our development of the KH-$\alpha\beta$ equation we did not non-dimensionalize the NS-$\alpha\beta$ equations. Thus we see the appearance of α, β, and ν as opposed to ϵ, $\gamma\epsilon$, and Re^{-1} respectively.

Theorem 1.1. *(Brady)(The Kámán–Howarth-$\alpha\beta$ equation.) We have the following Kármán–Howarth type equation for the Navier–Stokes-$\alpha\beta$ equations.*

$$(1-\alpha^2 D)\frac{\partial Q(r,t)}{\partial t} = 2\nu D(1-\alpha^2 D)Q(r,t) + (1-\alpha^2 D)T(r,t) + \alpha^2 S(r,t), \quad (1.8)$$

where

$$D = \frac{\partial}{\partial r}\frac{1}{r^4}\frac{\partial}{\partial r}r^4.$$

We make a few comments about the KH-$\alpha\beta$ equation. As mentioned earlier if we set $\beta = \alpha$, we arrive at the LANS-α equations as a special case of the NS-$\alpha\beta$ equations. Thus in equation (1.8) by setting $\beta = \alpha$ we obtain the following Kármán–Howarth type equation for the LANS-α equations.

$$(1-\alpha^2 D)\frac{\partial Q}{\partial t} = 2\nu D(1-\alpha^2 D)Q + (1-\alpha^2 D)T + \alpha^2 S. \quad (1.9)$$

In [16] Holm derived a Kármán–Howarth type equation for the LANS-α equations. We state his version below.

$$\frac{\partial Q(r,t)}{\partial t} = \left(r\frac{\partial}{\partial r} + 5\right)(T(r,t) - \alpha^2 S(r,t)) + 2\nu D Q(r,t). \quad (1.10)$$

There are two main reasons for the difference in these two equations. One is that Holm investigated the velocity correlation tensor $\langle \mathbf{v} \otimes \mathbf{u}' \rangle$, so in his equation Q is the scalar defining function for $\mathbf{Q}(\mathbf{r}) = \langle \mathbf{v} \otimes \mathbf{u}' \rangle$. Similarly our T is the defining scalar function for the same triple correlation tensor $\langle \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u}' \rangle + \langle \mathbf{u}' \otimes \mathbf{u} \otimes \mathbf{u} \rangle$ that appears in the papers by Robertson [25] and Kármám and Howarth [19], whereas Holm's T arises from the triple correlation tensor $\langle (\mathbf{v} \otimes \mathbf{u}' + \mathbf{u}' \otimes \mathbf{v} + \mathbf{v}' \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v}') \otimes \mathbf{u} \rangle$. Also the scalar fields S arise from different tensors, but this is expected based on the afore mentioned differences.

The next observation is that with $\beta = \alpha$ if we take the limit as α goes to zero, we recover the Navier Stokes equations from the NS-$\alpha\beta$ equations. So in equation (1.9), taking the limit as α goes to zero we arrive at the Kármán–Howarth equation that Roberson [25] obtained for the Navier Stokes equations.

As we mentioned there is the major issue of the closure problem for the Kármán–Howarth equation. Our hope was that with the extra filtering parameters we may have had a way to control the terms and somehow solve the closure problem for the NS-$\alpha\beta$ equations. But looking at equation (1.8) we see that there are three unknowns in the KH-$\alpha\beta$ equation. Thus, the closure problem is even greater for KH-$\alpha\beta$ than the closure problem for the original Kármán–Howarth equation. One could attempt to apply the various closure hypotheses used for the Kármán–Howarth equation for $\mathbf{Q}$ and $\mathbf{T}$, but $\mathbf{S}$ is new and has not been treated before. In fact we see that the tensor $\mathbf{S}$ is very complicated compared to $\mathbf{Q}$ or $\mathbf{T}$. Thus, any physically inspired closure hypothesis for the Kármán–Howarth equation will almost certainly fail for the KH-$\alpha\beta$ equation.

1.5 Investigations into an anisotropic generalization of the NS-$\alpha\beta$ equations.

Capriz [13] proposed a new theory termed ephemeral continua. Stated crudely, the idea is that each place (point) contains a loculus of (local) subplaces that are not part of the macroscopic structure but yet still influence the macroscopic behavior. Capriz and Fried [4] applied this model to arrive at an anisotropic generalization of the Navier–Stokes-$\alpha\beta$ equations. In ephemeral continua the convention is to not use bold-faced type for vectors and tensors. Thus, in this section we drop the use of bold-face type. The anisotropic generalization is given below, we refer to these as the anisotropic equations.

$$\begin{aligned}\rho\left(\dot{u}-\tfrac{1}{2}\operatorname{div}(Y\dot{W}+\dot{W}Y)\right)&=-\operatorname{grad}p+\mu\Delta w+\rho f\\ \dot{Y}&=YW-WY\\ w&=(1-\beta^2\Delta)u\\ \operatorname{div}u&=0.\end{aligned}\tag{1.1}$$

Where u is the velocity of the flow, $W=\operatorname{skw}\operatorname{grad}u$, ρ is the mass density, μ is the viscosity, $\rho\mathbf{f}$ is the force per volume. The tensor Y is the most notable difference from Navier–Stokes. This tensor is a moment of inertia tensor for the ephemeral continua. For each (x,t), the tensor is assumed positive-definite. In addition we assume that we are working on the periodic domain of side length $2\pi\ell$. Our reason for working on a periodic domain is two-fold. One is that the boundary conditions have not been decided for the anisotropic equations. The second reason is that much like with the NS and NS-$\alpha\beta$ the assumption greatly simplifies the analysis. Since our domain is periodic we will use the same functional setting that we presented for the NS and NS-$\alpha\beta$ equations. Our first goal with these equations is to better understand the role of the moment of inertia tensor Y. Recall that since Y is a positive definite tensor for each (x,t), we have that there are three (not necessarily distinct) positive eigenvalues η_1, η_2, and η_3 at each (x,t). We have the following result characterizing the distribution of eigenvalues.

Theorem 1.1. *(Brady) Suppose that Y is a symmetric positive definite tensor field with $Y\in C^1(\Omega\times\mathbb{R})$. We assume that Y satisfies the evolution equation $\dot{Y}=YW-WY$ for some tensor field W and where the material time derivative $\dot{Y}$ is with respect to some given flow. Then, the eigenvalues are invariant under the flow. I.e., $\dot{\eta}_1=\dot{\eta}_2=\dot{\eta}_3=0$. In particular, if we are given an initial tensor field $Y(x,t_o)$, then we know the eigenvalue distribution for each later time by following the distribution along* *the flow.*

Thus, if we think of Y as characterizing a local material response at each (x,t), then the above theorem tells us that this material response convects with the flow! That is, we could think of each fluid element as being a spheroid that convects with the flow. This characterization fits nicely with the idea of an anisotropic incompressible fluid flow.

As with the NS and NS-$\alpha\beta$ equations we divide by ρ and apply the Helmholtz-Leray projector P to the first equation in (1.1) to obtain the following form of the anisotropic equations

$$\begin{aligned} \dot{u} + \nu A w &= \tfrac{1}{2} P \operatorname{div}(Y\dot{W} + \dot{W}Y) + f \\ \dot{Y} &= YW - WY \\ w &= (1+\beta^2 A)u. \end{aligned} \tag{1.2}$$

Our next goal is to find energy type inequalities and bounds for equation (1.2). The difficulty was with the non-linear term $\operatorname{div}(Y\dot{W} + \dot{W}Y)$. Using Theorem 1.1 we were able to prove the following theorem, which is instrumental in establishing bounds on u and Y.

Theorem 1.2. *(Brady) Let T be any tensor field in L^2 and let Y be given as above. In addition to Y, we consider all positive powers of Y. I.e., Y^s, where s is any positive number. Let η_{max} and η_{min} be the maximum and minimum eigenvalues of Y over Ω. Then we have the following inequality.*

$$\eta_{min}^{2s}|T|^2 \leq |Y^s T|^2 \leq \eta_{max}^{2s}|T|^2, \tag{1.3}$$

where η_{max} and η_{min} are independent of t.

We make a couple comments that explain why η_{max} and η_{min} do not depend on time. First, since we are working on a compact domain η_{max} and η_{min} exist and are bounded (from above and from below away from 0) for each t. Then by Theorem 1.1 η_{max} and η_{min} are constant with respect to time.

Using the above theorem we were able to establish the following bounds for u and Y.

Theorem 1.3. *(Brady) Let u and Y be solutions to the anisotropic equations* (1.2)*. Let $T > 0$ be fixed and suppose that $f \in L^2([0,T];H) \cap L^\infty([0,\infty);H)$, Then*

$$\begin{aligned} u &\in L^2([0,T];V^2) \cap L^\infty([0,\infty);V), \\ Y &\in L^2([0,T];L^2) \cap L^\infty([0,\infty);L^2), \\ \dot{Y} &\in L^2([0,T];L^2) \cap L^\infty([0,\infty);L^2), \end{aligned} \tag{1.4}$$

where $\|\cdot\|_{L^2}$ denotes the L^2 space for tensor fields over the domain Ω.

Chapter 2

Well-Posedness for the Navier–Stokes-$\alpha\beta$ equations

2.1 Introduction

Well-posedness for the 3d Navier-Stokes equations (NS) is a current outstanding problem in mathematics. Various models have been developed to address some of the difficulties encountered with the Navier-Stokes equations. Fried and Gurtin [12] developed such a turbulence model within a continuum mechanical framework. The equations are given below and we refer to them as the Navier–Stokes-$\alpha\beta$ equations (NS-$\alpha\beta$).

$$\begin{aligned} \rho\left(\tfrac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad}\mathbf{v})\mathbf{u} + (\operatorname{grad}\mathbf{u})^T\mathbf{v}\right) &= -\operatorname{grad} p + \mu\Delta\mathbf{w} + \rho\mathbf{f}, \\ \operatorname{div}\mathbf{u} &= 0, \\ \mathbf{v} &= (1-\alpha^2\Delta)\mathbf{u}, \\ \mathbf{w} &= (1-\beta^2\Delta)\mathbf{u} \end{aligned} \tag{2.1}$$

Mathematically the equations can be thought of as a generalization of the LANS-α equations, see [9] and references therein for an overview of the LANS-α equations.

In the literature the LANS-α equations are also referred to as the Navier–Stokes-α and the Camassa–Holm equations. In [17] the Euler-α equations are derived using a Lagrangian approach, but since this approach cannot account for dissipation the assumption is made that the dissipation takes the form of $\mu(1-\alpha^2\Delta)\mathbf{u}$. Thus the development of the NS-$\alpha\beta$ equations is more that just a mathematical extension, it gives a physical framework that incorporates the dissipation from the outset. We should also add that the boundary conditions for the NS-$\alpha\beta$ and LANS-α equation differ from each other. Due to mathematical difficulties associated with the boundary conditions of the NS-$\alpha\beta$ we restrict to the periodic domain in this chapter and so we consider periodic boundary conditions. In the NS-$\alpha\beta$ equations as given above (2.1) we see the presence of two parameters α and β. Both of these parameters has dimensions of length and are thought of as filtering parameters. In fact, the inverse operator $(1-\alpha\Delta)^{-1}$, referred to as a Helmholtz filter of filtering scale α, has the following form.

$$(1-\alpha\Delta)^{-1}\mathbf{u}(\mathbf{x}) = \int_\Omega \frac{1}{4\pi\alpha^2|\mathbf{x}-\mathbf{y}|} e^{-|\mathbf{x}-\mathbf{y}|/\alpha}\mathbf{u}(\mathbf{y})\, d\mathbf{y}$$

Applying the operator $(1-\alpha^2\Delta)$ is then referred to as unfiltering and we refer to $\mathbf{v}$ and $\mathbf{w}$ as unfiltered velocity fields. We say that α is the convective filtering scale and β is the dissipative filtering scale. Another illuminating way to look at the filter is through the Fourier transform. Letting $\hat{\mathbf{u}}$ be the Fourier transform of $\mathbf{u}$ we find

$$\begin{gathered} (1+\alpha^2|\mathbf{k}|^2)\hat{\mathbf{u}}(\mathbf{k}) = \hat{\mathbf{v}}(\mathbf{k}) \\ \text{or} \\ \hat{\mathbf{u}}(\mathbf{k}) = \tfrac{1}{(1+\alpha^2|\mathbf{k}|^2)}\hat{\mathbf{v}}(\mathbf{k}) \end{gathered} \tag{2.2}$$

Numerical evidence [12] suggests that setting $\beta \le \alpha$ recovers more of the energy spectrum (relative to $\beta > \alpha$). From a functional standpoint $\beta \le \alpha$ reduces the amount of regularization from the dissipation. Due to these observations, in this chapter we study the case of $\beta \leq \alpha$.

Our overall goal is this chapter is to investigate theorems that do not exists for the 3d Navier–Stokes equations. Specifically, we show the existence, uniqueness, continuous dependence of initial data, and regularity of global in time strong solutions to the NS-$\alpha\beta$ equations on the 3d periodic domain. In addition we find lower bounds for the nodal distance and upper bounds for the number of determining nodes. In particular, we are interested how choices of the parameters affects these bounds. As mentioned at the beginning of this paragraph, the remarkable finding of these results is that they do not yet exist for the 3d Navier-Stokes equations! In fact, the same difficulties in showing well-posedness for 3d Navier–Stokes, are the same difficulties that prevent the determination of the nodal distance and modal number for 3d Navier–Stokes.

2.2 Notation and Preliminaries

2.2.1 The NS-$\alpha\beta$ equations

We present the NS-$\alpha\beta$ equations in the following form.

$$\begin{aligned} \rho\left(\tfrac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad}\mathbf{v})\mathbf{u} + (\operatorname{grad}\mathbf{u})^T\mathbf{v}\right) &= -\operatorname{grad} p + \mu\Delta\mathbf{w} + \rho\mathbf{f}, \\ \operatorname{div}\mathbf{u} &= 0, \\ \mathbf{v} &= (1-\alpha^2\Delta)\mathbf{u}, \\ \mathbf{w} &= (1-\beta^2\Delta)\mathbf{u} \end{aligned} \tag{2.1}$$

To avoid issues related to the boundary conditions we choose the domain to be the periodic box with side length $2\pi\ell$, which we write as Ω. For the rest of the chapter, unless explicitly said otherwise, we will assume that the domain is a periodic box.

Next we divide by the mass density ρ to obtain

$$\frac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad} \mathbf{v})\mathbf{u} + (\operatorname{grad} \mathbf{u})^T \mathbf{v} = -\operatorname{grad}\left(\frac{p}{\rho}\right) + \nu \Delta \mathbf{w} + \mathbf{f}, \tag{2.2}$$

where ν is the kinematic viscosity. Next we transform the equations into a dimensionless form. We proceed by defining the following scales. The length $L = \ell$ associated to the side length of Ω. A characteristic velocity scale U, which could be defined as

$$U^2 = \frac{1}{(2\pi L)^3} \int_\Omega |\mathbf{u}(\mathbf{x}, 0)|^2 \, d\mathbf{x}.$$

We then define the characteristic time scale $T = L/U$. Using the characteristic quantities we define the following dimensionless quantities.

$$\mathbf{u}' = \frac{\mathbf{u}}{U} \qquad \mathbf{x}' = \frac{\mathbf{x}}{L} \qquad t' = \frac{t}{T} \tag{2.3}$$

Using these dimensionless quantities we obtain a non-dimensional form of the NS-$\alpha\beta$ equations.

$$\begin{gathered} \tfrac{\partial \mathbf{v}'}{\partial t'} + (\operatorname{grad}' \mathbf{v}')\mathbf{u}' + (\operatorname{grad}' \mathbf{u}')^T \mathbf{v}' = -\operatorname{grad}'\left(\tfrac{p}{U^2 \rho}\right) + \tfrac{\nu}{UL} \Delta' \mathbf{w}' + \mathbf{f}', \\ \operatorname{div}' \mathbf{u}' = 0, \\ \mathbf{v}' = (1 - \tfrac{\alpha^2}{L^2} \Delta')\mathbf{u}', \\ \mathbf{w}' = (1 - \tfrac{\beta^2}{L^2} \Delta')\mathbf{u}' \end{gathered} \tag{2.4}$$

We see the appearance of the Reynold's number $R = UL/\nu$ and we define $\varpi = p/(U^2 \rho)$. We set $\epsilon = \alpha^2/L^2$ and $\gamma = \beta^2/\alpha^2$, where it is understood that $\epsilon > 0$ and $0 < \gamma \leq 1$. It is important to note that in terms of the new dimensionless variable $\mathbf{x}'$, the periodic box now has side length 2π. Finally we remove the primes with the understanding that the quantities considered are dimensionless.

$$\begin{aligned}
\tfrac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad} \mathbf{v})\mathbf{u} + (\operatorname{grad} \mathbf{u})^T \mathbf{v} &= -\operatorname{grad} \varpi + Re^{-1} \Delta \mathbf{w} + \mathbf{f}, \\
\operatorname{div} \mathbf{u} &= 0, \\
\mathbf{v} &= (1 - \epsilon \Delta)\mathbf{u}, \\
\mathbf{w} &= (1 - \epsilon\gamma\Delta)\mathbf{u}
\end{aligned} \tag{2.5}$$

We will still refer to these equations as the Navier–Stokes-$\alpha\beta$ equations with the understanding that $\alpha^2 = \epsilon\ell^2$ and $\beta^2 = \epsilon\gamma\ell^2$. Integrating the first equation from (2.5) over Ω we find using integration by parts,

$$\frac{d}{dt} \int_\Omega \mathbf{u}(\mathbf{x}, t)\ d\mathbf{x} = \int_\Omega \mathbf{f}(\mathbf{x}, t)\ d\mathbf{x}. \tag{2.6}$$

Thus if we assume that

$$\int_\Omega \mathbf{u}(\mathbf{x}, 0)\, d\mathbf{x} = \mathbf{0} \qquad \text{and} \qquad \int_{\boldsymbol{\Omega}} \mathbf{f}(\mathbf{bfx}, \mathbf{t})\, \mathbf{dx} = \mathbf{0},$$

then we can conclude that

$$\int_\Omega \mathbf{u}(\mathbf{x}, t)\, d\mathbf{x} = \mathbf{0} \qquad \text{for all } t.$$

Our reason for making this assumption is in the simplification of the analysis while keeping the original nature of the problem. Most importantly it allows us to make frequent use of the simple form of Poincaré's inequality given in the next section.

2.2.2 Functional setting

First, we review some of the ideas and notations developed in Constantin and Foias [5]. We define the following functional spaces:

(i) $\mathcal{V} = \{\boldsymbol{\phi} \in C^\infty(\Omega)^3 : \operatorname{div} \boldsymbol{\phi} = 0 \quad \text{and} \quad \int_\Omega \boldsymbol{\phi}(\mathbf{x}) d\mathbf{x} = \mathbf{0}\}$

(ii) H is the closure of $\mathcal{V}$ in $L^2(\Omega)^3$

(iii) V is the closure of $\mathcal{V}$ in $H^1(\Omega)^3$

We let P denote Helmholtz-Leray projection onto divergence free vector fields, *I.e.,* $P : L^2(\Omega)^3 \to H$. We define the stokes operator by $A = -P\Delta$. The domain of A is given by $D(A) = V \cap H^2(\Omega)^3$. The inverse of stokes operator is a self-adjoint compact operator, therefore there exists an orthonormal basis of eigenfunctions $\{\boldsymbol{\phi}_j\}$ and eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ such that $A\boldsymbol{\phi}_j = \lambda_j \boldsymbol{\phi}_j$.

We make the following observations about the stokes operator from [5]. First is that on the periodic domain (and $\mathbb{R}^n$) the projection operator commutes with the laplacian. Thus we could have defined the stokes operator as $-\Delta$, but in keeping with tradition we define the stokes operator as $A = -P\Delta$. Using the eigenvalues of A we may define A^s for any $s \in \mathbb{R}$. For $s > 0$ the domain of A^s may be identified as $D(A^s) = V \cap H^{2s}(\Omega)^3$. For $s < 0$ we identify the duals spaces as $D(A^s) = D(A^{-s})'$. We use the shorthand notation $V^s = D(A^{s/2})$. Then, for $0 \leq r \leq s$ the embedding $V^s \hookrightarrow V^r$ is compact. For $r \leq s$, i.e., r and/or s could be negative, we have that $V^s \hookrightarrow V^r$ is a continuous embedding.

Let u be a time dependent function on a Banach space X. Then we write

$$\|u\|_{L^p([0,T];X)} = \left(\int_0^T \|u\|_X^p \, dt \right)^{1/p}$$

and

$$\|u\|_{L^\infty([0,T];X)} = \operatorname{ess\,sup}_{t\in[0,T]} \|u\|_X$$

The following generalization of Rellich's selection theorem is useful for showing existence of solutions. This theorem is also referred to as Aubin's Compactness Theorem in the Literature. See [5] or [27] for the proof and its application to the NS equations.

Theorem 2.1. *Let $X \hookrightarrow Y \hookrightarrow Z$, be reflexive Banach spaces with the first inclusion being compact and the second being continuous. Let $\mathbf{u}_m(t)$ be a bounded sequence in $L^p([0,T];X)$, such that $\partial/\partial t\ \mathbf{u}_m(t)$ is bounded in $L^p([0,T];Z)$, where $1 < p < \infty$. Then there is a subsequence $\mathbf{u}_{m'}(t)$ and $\mathbf{u}(t)$ in X such that*

$$\begin{aligned}
&\mathbf{u}_{m'} \rightharpoonup \mathbf{u} \quad \textit{weakly in} \quad L^p([0,T];X)\\
&\mathbf{u}_{m'} \to \mathbf{u} \quad \textit{strongly in} \quad L^p([0,T];Y)\\
&\mathbf{u}_{m'} \to \mathbf{u} \quad \textit{in} \quad C([0,T];Z)
\end{aligned}$$

We will make frequent use of the following inequalities.

1. (Poincaré's inequality) Let $\mathbf{u} \in V$ and

$$\int_\Omega \mathbf{u}\, d\mathbf{x} = \mathbf{0}.$$

   Then

$$\|\mathbf{u}\|_{L^2} \le \lambda_1^{-1/2} \|A^{1/2}\mathbf{u}\|_{L^2},$$

   where λ_1 is the first eigenvalue of the stokes operator and for the periodic box of side length L, $\lambda_1 = 4\pi^2/L^2$. So for our periodic box of side length 2π, we have $\lambda_1 = 1$. Thus for our choice of dimensionless parameters we have the following form of Poincaré's inequality

$$\|\mathbf{u}\|_{L^2} \leq \|A^{1/2}\mathbf{u}\|_{L^2}.$$

2. (Young's Inequality) Given $a, b, p, q > 0$ with $1/p + 1/q = 1$ we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

3. We will mostly use the following version of Young's Inequality where $p = q = 2$. Let $a, b, \epsilon > 0$, then

$$ab < \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$$

Let $\langle \cdot, \cdot \rangle$ represent the L^2 inner product and $|\cdot| = \|\cdot\|_{L^2}$ the L^2 norm. We also define $\|\cdot\| = |A^{1/2}\cdot|$, it is important to note that on V, we have $|A^{1/2}\cdot| = |\operatorname{grad}\cdot|$. Using Poincaré's inequality and the periodic boundary conditions it can be shown that the H^s norm is equivalent to the norm associated to $V^s = D(A^{s/2})$. I.e., for a given s there are constants c, c' such that $c|A^{s/2}\mathbf{u}| \leq \|\mathbf{u}\|_{H^s(\Omega)^3} \leq c'|A^{s/2}\mathbf{u}|$. Thus we will use $|A^{s/2}\cdot|$ for the $H^s(\Omega)^3$ norm on V^s. Finally we write $\|\cdot\|_\epsilon^2 = |\cdot|^2 + \epsilon\|\cdot\|^2$, and we refer to this as an H^1_ϵ norm. In addition, by Poincaré's inequality it is clear that $\|\cdot\|_\epsilon^2$ is equivalent to the norm on $V = D(A^{1/2})$. In the lemma below we give sharp constants for the equivalence different H^1_ϵ norms. We state the lemma in more generality than needed.

Lemma 2.2. *Let Ω be any bounded open sub-domain of $\mathbb{R}^3$ and let $\mathbf{u} \in H^1(\Omega)^3$. Then for $\gamma \leq 1$ we have the following sharp inequality.*

$$\|\mathbf{u}\|_{\gamma\epsilon}^2 \leq \|\mathbf{u}\|_\epsilon^2 \leq \gamma^{-1}\|\mathbf{u}\|_{\gamma\epsilon}^2$$

Note that we use the characterization of $\|\mathbf{u}\| = |\operatorname{grad}\mathbf{u}|$ in this lemma, then in *particular for* $\mathbf{u} \in V$ *we have* $\|\mathbf{u}\| = |A^{1/2}\mathbf{u}|$.

Proof. Since $\gamma \leq 1$ we have for the left hand side,

$$\|\mathbf{u}\|_{\gamma\epsilon}^2 = |\mathbf{u}|^2 + \gamma\epsilon\|\mathbf{u}\|^2 \leq |\mathbf{u}|^2 + \epsilon\|\mathbf{u}\|^2 = \|\mathbf{u}\|_\epsilon^2.$$

As for the right hand side we have

$$\|\mathbf{u}\|_\epsilon^2 = |\mathbf{u}|^2 + \epsilon\|\mathbf{u}\|^2 = \gamma^{-1}(\gamma|\mathbf{u}|^2 + \gamma\epsilon\|\mathbf{u}\|^2) \leq \gamma^{-1}(|\mathbf{u}|^2 + \gamma\epsilon\|\mathbf{u}\|^2)$$

Now we show sharpness. For the left-hand inequality it is sufficient to consider any non-zero constant vector field. For the right-hand inequality consider the vector field

$$\mathbf{f}_n(\mathbf{x}) = b_n\left(a_n - |\mathbf{r}|\right)_+ \mathbf{c} \qquad \text{for } n = 1, 2, 3, \ldots$$

where $\mathbf{r} = \mathbf{x} - o$, $(\cdot)_+ = max(\cdot, 0)$, c is any unit vector, $b_n = n^{3/2}$ and $a_n = 1/n$. By choosing n large enough and translating we may assume that the support of $\mathbf{f}_n$ is contained in Ω for all $n \geq N$. Since $\mathbf{f}_n$ is absolutely continuous and its derivatives exits almost everywhere we have that $\mathbf{f} \in H^1(\Omega)$. Now we compute the following norms.

$$\begin{aligned}
|\mathbf{f}_n|^2 &= \int_\Omega |\mathbf{f}_n(\mathbf{x})|^2 \, d\mathbf{x} \\
&= \int_\Omega b_n^2\left(a_n - |\mathbf{r}|\right)_+^2 \, d\mathbf{x} \\
&= 4\pi \int_0^{a_n} b_n^2\left(a_n - r\right)^2 r^2 \, dr \\
&= \frac{2}{15}\pi a_n^5 b_n^2 = \frac{2\pi}{15n^2}
\end{aligned}$$

A representation of $\nabla \mathbf{f}_n$ in $H^1(\Omega)$ is given by

$$\nabla \mathbf{f}_n(\mathbf{x}) = \begin{cases} -\mathbf{c} \otimes b_n \frac{\mathbf{r}}{|\mathbf{r}|} & \text{for } 0 \leq |\mathbf{r}| < a_n \\ 0 & \text{for } |\mathbf{r}| \geq a_n \end{cases}$$

and so we find

$$\begin{aligned} \|\mathbf{f}_n\|^2 &= \int_\Omega |\nabla \mathbf{f}_n(\mathbf{x})|^2 \; d\mathbf{x} \\ &= \int_{0 \leq |\mathbf{r}| < a_n} b_n^2 \; d\mathbf{x} + \int_{|\mathbf{r}| \geq a_n} 0 \; d\mathbf{x} \\ &= \frac{4}{3}\pi a_n^3 b_n^2 = \frac{4}{3}\pi. \end{aligned}$$

In taking the limit as $n \to \infty$ we find

$$\lim_{n \to \infty} \|\mathbf{f}_n\|_\epsilon = \epsilon \frac{4}{3}\pi$$

and

$$\lim_{n \to \infty} \gamma^{-1} \|\mathbf{f}_n\|_{\gamma\epsilon} = \epsilon \frac{4}{3}\pi.$$

$\square$

Lemma 2.3. *Let Ω be any domain with an open subdomain. Let $\mathbf{w} = (1 + \gamma\epsilon A)\mathbf{u}$ and $\mathbf{v} = (1 + \epsilon A)\mathbf{u}$ with $0 < \gamma \leq 1$. Then by letting $\langle \cdot, \cdot \rangle$ denote the L^2 inner product, we have the following inequalities.*

$$\begin{aligned} |\mathbf{w}|^2 \leq &\langle \mathbf{w}, \mathbf{v} \rangle \leq |\mathbf{v}|^2 \\ \gamma|\mathbf{v}|^2 \leq &\langle \mathbf{w}, \mathbf{v} \rangle \leq \gamma^{-1}|\mathbf{w}|^2 \end{aligned}$$

Proof. Since $0 < \gamma \leq 1$, we have

$$\begin{aligned}
|\mathbf{w}|^2 &= |\mathbf{u}|^2 + 2\gamma\epsilon\|\mathbf{u}\|^2 + \gamma^2\epsilon^2|A\mathbf{u}|^2 \\
&\leq |\mathbf{u}|^2 + (\gamma+1)\epsilon\|\mathbf{u}\|^2 + \gamma\epsilon^2|A\mathbf{u}|^2 = \langle \mathbf{w}, \mathbf{v}\rangle \\
&\leq |\mathbf{u}|^2 + 2\epsilon\|\mathbf{u}\|^2 + \epsilon^2|A\mathbf{u}|^2 = |\mathbf{v}|^2
\end{aligned}$$

and

$$\begin{aligned}
\gamma|\mathbf{v}|^2 &= \gamma|\mathbf{u}|^2 + 2\gamma\epsilon\|\mathbf{u}\|^2 + \gamma\epsilon^2|A\mathbf{u}|^2 \\
&\leq |\mathbf{u}|^2 + (\gamma+1)\epsilon\|\mathbf{u}\|^2 + \gamma\epsilon^2|A\mathbf{u}|^2 = \langle \mathbf{w}, \mathbf{v}\rangle \\
&\leq \gamma^{-1}\left(\gamma|\mathbf{u}|^2 + (\gamma^2+\gamma)\epsilon\|\mathbf{u}\|^2 + \gamma^2\epsilon^2|A\mathbf{u}|^2\right) \\
&\leq \gamma^{-1}\left(|\mathbf{u}|^2 + 2\gamma\epsilon\|\mathbf{u}\|^2 + \gamma^2\epsilon^2|A\mathbf{u}|^2\right) = \gamma^{-1}|\mathbf{w}|^2.
\end{aligned}$$

$\square$

Lemma 2.4. *For $\mathbf{v} = (1+\epsilon A)\mathbf{u}$ and $s \in \mathbb{R}$, then the following two inequalities hold.*

1. $|A^s\mathbf{v}| \leq (1+\epsilon)|A^{s+1}\mathbf{u}|$

2. $|A^s\mathbf{u}| \leq \epsilon^{-1}|A^{s-1}\mathbf{v}|$

Proof. Using Poincaré's inequality we find

$$\begin{aligned}
|A^s\mathbf{v}|^2 &= \langle A^s\mathbf{u} + \epsilon A^{s+1}\mathbf{u}, A^s\mathbf{u} + \epsilon A^{s+1}\mathbf{u}\rangle \\
&= |A^s\mathbf{u}|^2 + 2\epsilon|A^{s+1/2}\mathbf{u}|^2 + \epsilon^2|A^{s+1}\mathbf{u}|^2 \\
&\leq |A^{s+1}\mathbf{u}|^2 + 2\epsilon|A^{s+1}\mathbf{u}|^2 + \epsilon^2|A^{s+1}\mathbf{u}|^2 \\
&= (1+\epsilon)^2|A^{s+1}\mathbf{u}|^2.
\end{aligned}$$

Similarly

$$\begin{aligned}|A^{s-1}\mathbf{v}|^2 &= \langle A^{s-1}\mathbf{u} + \epsilon A^s\mathbf{u}, A^{s-1}\mathbf{u} + \epsilon A^s\mathbf{u}\rangle \\ &= |A^{s-1}\mathbf{u}|^2 + 2\epsilon|A^{s-1/2}\mathbf{u}|^2 + \epsilon^2|A^s\mathbf{u}|^2 \\ &\geq \epsilon^2|A^s\mathbf{u}|^2.\end{aligned}$$

□

The following lemma from [5] is used to show regularity of solutions to the Navier-Stokes equations. We state the Lemma below, then we give an extension that is used to show regularity for the NS-$\alpha\beta$ equations.

Lemma 2.5. *For $B(\mathbf{u}, \mathbf{v}) = (\operatorname{grad}\mathbf{v})\mathbf{u}$ and $r > 3/4$, we have*

$$|A^r B(\mathbf{u}, \mathbf{v})| \leq c|A^r\mathbf{u}|\ |A^{r+1/2}\mathbf{v}|. \tag{2.7}$$

Where c is a constant that does not depend on any of the remaining parameters in our dimensionless system.

We now extend the Lemma to $\langle \widetilde{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle$.

Lemma 2.6. *For $\mathbf{u} \in V^r$, $\mathbf{v} \in V^{(r+1)/2}$, $\mathbf{w} \in V^{s-r}$, $r > 3/4$, and $s > r$ we have*

$$\langle A^r B(\mathbf{u}, \mathbf{v}), A^{s-r}\mathbf{w}\rangle \leq c|A^{r/2}\mathbf{u}|\ |A^{r+1/2}\mathbf{v}|\ |A^{(s-r)}\mathbf{w}|, \tag{2.8}$$

where the constant c does not depend on any of the remaining parameters in our dimensionless system.

Proof. Since $\mathcal{V}$ is dense in V^s for any s, we consider $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$. We may relate B and $\widetilde{B}$ by $\langle \widetilde{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle = \langle B(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle + \langle B(\mathbf{w}, \mathbf{v}), \mathbf{u}\rangle = \langle B(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle + \langle (\operatorname{grad}\mathbf{v})^T\mathbf{u}, \mathbf{w}\rangle$.

Thus, $\langle A^r \widetilde{B}(\mathbf{u},\mathbf{v}), A^{(s-r)}\mathbf{w}\rangle = \langle \widetilde{B}(\mathbf{u},\mathbf{v}), A^s\mathbf{w}\rangle = \langle A^r B(\mathbf{u},\mathbf{v}), A^{(s-r)}\mathbf{w}\rangle + \langle A^r(\operatorname{grad}\mathbf{v})^T\mathbf{u}, A^{(s-r)}\mathbf{w}\rangle$. By the preceding Lemma and Cauchy-Schwarz we have

$$|\langle A^r B(\mathbf{u},\mathbf{v}), A^{(s-r)}\mathbf{w}\rangle| \leq |A^r B(\mathbf{u},\mathbf{v})|\ |A^{(s-r)}\mathbf{w}| \leq c|A^{r/2}\mathbf{u}|\ |A^{r+1/2}\mathbf{v}|\ |A^{(s-r)}\mathbf{w}|.$$

Thus, it is sufficient to show the same bound for $\langle A^r(\operatorname{grad}\mathbf{v})^T\mathbf{u}, A^{(s-r)}\mathbf{w}\rangle$.

Since we are working on the periodic domain we may write $\mathbf{u}$ and $\mathbf{v}$ in terms of their Fourier series. Our periodic box has side length 2π, so the series have the following form:

$$\mathbf{u} = \sum_{\mathbf{k}\in\mathbb{Z}^3} \mathbf{u}_{\mathbf{k}} e^{i\mathbf{x}\cdot\mathbf{k}} \quad \text{and} \quad \mathbf{v} = \sum_{\mathbf{k}\in\mathbb{Z}^3} \mathbf{v}_{\mathbf{k}} e^{i\mathbf{x}\cdot\mathbf{k}}.$$

We estimate $|A^{s/2}(\operatorname{grad}\mathbf{v})^T\mathbf{u}|$ by

$$\sup_{\boldsymbol{\phi}\in H, |\boldsymbol{\phi}|=1} \langle A^{s/2}(\operatorname{grad}\mathbf{v})^T\mathbf{u}, \boldsymbol{\phi}\rangle$$

Then in terms of the Fourier series we have

$$(\operatorname{grad}\mathbf{v})^T\mathbf{u} = \sum_{\mathbf{j}+\boldsymbol{\ell}=\mathbf{k},\, \mathbf{k}\in\mathbb{Z}^3} i e^{i\mathbf{x}\cdot\mathbf{k}} (\mathbf{v}_{\boldsymbol{\ell}}\cdot\mathbf{u}_{\mathbf{j}})\boldsymbol{\ell}.$$

On the periodic domain $A = -\Delta$, so

$$A^r(\operatorname{grad}\mathbf{v})^T\mathbf{u} = \sum_{\mathbf{j}+\boldsymbol{\ell}=\mathbf{k},\, \mathbf{k}\in\mathbb{Z}^3} i|\mathbf{k}|^{2r} e^{i\mathbf{x}\cdot\mathbf{k}} (\mathbf{v}_{\boldsymbol{\ell}}\cdot\mathbf{u}_{\mathbf{j}})\boldsymbol{\ell}$$

Taking the L^2 inner product with $\boldsymbol{\phi} = \sum_{\mathbf{k}} \boldsymbol{\phi}_{\mathbf{k}} exp(i\mathbf{x}\cdot\mathbf{k})$ we find

$$\langle A^r(\operatorname{grad}\mathbf{v})^T\mathbf{u}, \boldsymbol{\phi}\rangle = \sum_{\mathbf{j}+\boldsymbol{\ell}=\mathbf{k},\, \mathbf{k}\in\mathbb{Z}^3} i|\mathbf{k}|^{2r} e^{i\mathbf{x}\cdot\mathbf{k}} (\mathbf{v}_{\boldsymbol{\ell}}\cdot\mathbf{u}_{\mathbf{j}})(\boldsymbol{\ell}\cdot\boldsymbol{\phi}_{\mathbf{k}}).$$

Next we use the inequality $|\mathbf{k}| = |\mathbf{j}+\boldsymbol{\ell}|^{2r} \le c_r(|\mathbf{j}|^{2r} + |\boldsymbol{\ell}|^{2r})$ to obtain

$$
\begin{aligned}
|\langle A^r (\operatorname{grad} \mathbf{v})^T \mathbf{u}, \boldsymbol{\phi}\rangle| &\le \sum_{\mathbf{j}+\boldsymbol{\ell}=\mathbf{k},\, \mathbf{k}\in\mathbb{Z}^3} c_s(|\mathbf{j}|^{2r} + |\boldsymbol{\ell}|^{2r})|\mathbf{v}_{\boldsymbol{\ell}}|\ |\mathbf{u}_{\mathbf{j}}|\ |\boldsymbol{\ell}|\ |\boldsymbol{\phi}_{\mathbf{k}}| \\
&= c_s \left(\sum_{\mathbf{j}+\boldsymbol{\ell}=\mathbf{k},\, \mathbf{k}\in\mathbb{Z}^3} |\boldsymbol{\ell}|\ |\mathbf{v}_{\boldsymbol{\ell}}|\ (|\mathbf{j}|^{2r}|\mathbf{u}_{\mathbf{j}}|\ |\boldsymbol{\phi}_{\mathbf{k}}|) + \sum_{\mathbf{j}+\boldsymbol{\ell}=\mathbf{k},\, \mathbf{k}\in\mathbb{Z}^3} |\boldsymbol{\ell}|^{2r+1}|\mathbf{v}_{\boldsymbol{\ell}}|\ (|\mathbf{u}_{\mathbf{j}}|\ |\boldsymbol{\ell}|\ |\boldsymbol{\phi}_{\mathbf{k}}|) \right) \\
&\le c_s \left(|A^r \mathbf{u}|\ |\boldsymbol{\phi}| \left(\sum_{\boldsymbol{\ell}\in\mathbb{Z}^3} |\boldsymbol{\ell}|\ |\mathbf{v}_{\boldsymbol{\ell}}| \right) + |A^{r+1/2}\mathbf{v}|\ |\boldsymbol{\phi}| \left(\sum_{\mathbf{j}\in\mathbb{Z}^3} |\mathbf{u}_{\mathbf{j}}| \right) \right)
\end{aligned}
$$

Note for $r > 3/4$

$$
\begin{aligned}
\sum_{\mathbf{j}\in\mathbb{Z}^3} |\mathbf{u}_{\mathbf{j}}| &= \sum_{\mathbf{j}\in\mathbb{Z}^3} |\mathbf{j}|^{-2r}|\mathbf{j}|^{2s}|\mathbf{u}_{\mathbf{j}}| \\
&\le \left(\sum_{\mathbf{j}\in\mathbb{Z}^3} |\mathbf{j}|^{-4r} \right)^{1/2} \left(\sum_{\mathbf{j}\in\mathbb{Z}^3} |\mathbf{j}|^{4r}|\mathbf{u}_{\mathbf{j}}|^2 \right)^{1/2} \\
&\le c|A^r \mathbf{u}|
\end{aligned}
$$

and similarly

$$
\sum_{\boldsymbol{\ell}\in\mathbb{Z}^3} |\boldsymbol{\ell}|\ |\mathbf{v}_{\boldsymbol{\ell}}| \le c|A^{r+1/2}\mathbf{v}|
$$

Therefore we find

$$
|\langle A^r (\operatorname{grad} \mathbf{v})^T \mathbf{u}, \boldsymbol{\phi}\rangle| \le c'|A^r \mathbf{u}|\ A^{r+1/2}\mathbf{v}|\ |\boldsymbol{\phi}|
$$

and so

$$
|\langle A^r (\operatorname{grad} \mathbf{v})^T \mathbf{u}, A^{(s-r)}\mathbf{w}\rangle| \le c'|A^r \mathbf{u}|\ |A^{r+1/2}\mathbf{v}|\ |A^{(s-r)}\mathbf{w}|
$$

$\square$

2.2.3 Further discussion of the NS-$\alpha\beta$ equations

Using the identity

$$(\operatorname{grad}\mathbf{v})\mathbf{u} + (\operatorname{grad}\mathbf{u})^T\mathbf{v} = (\operatorname{curl}\mathbf{v}) \times \mathbf{u} + \operatorname{grad}(\mathbf{v}\cdot\mathbf{u})$$

we may rewrite our dimensionless NS-$\alpha\beta$ equations as

$$\begin{aligned}
\tfrac{\partial \mathbf{v}}{\partial t} + (\operatorname{curl}\mathbf{v}) \times \mathbf{u} &= -\operatorname{grad}\varpi + Re^{-1}\Delta\mathbf{w} + \mathbf{f}, \\
\operatorname{div}\mathbf{u} &= 0, \\
\mathbf{v} &= (1-\epsilon\Delta)\mathbf{u}, \\
\mathbf{w} &= (1-\gamma\epsilon\Delta)\mathbf{u}
\end{aligned} \tag{2.9}$$

where we have added $\operatorname{grad}(\mathbf{v}\cdot\mathbf{u})$ to the pressure term. Next we project onto divergence free vector fields using the projection P

$$\begin{aligned}
\tfrac{\partial \mathbf{v}}{\partial t} + Re^{-1}A\mathbf{w} &= \widetilde{B}(\mathbf{u},\mathbf{v}) + \mathbf{f}, \\
\mathbf{v} &= (1+\epsilon A)\mathbf{u}, \\
\mathbf{w} &= (1+\gamma\epsilon A)\mathbf{u} \\
\widetilde{B}(\mathbf{u},\mathbf{v}) &= P(\mathbf{u} \times (\operatorname{curl}\mathbf{v}))
\end{aligned} \tag{2.10}$$

We have assumed that $\mathbf{f} = P\mathbf{f}$, since otherwise we could add the gradient component to the pressure term. The operator $\widetilde{B}$ is defined as $\widetilde{B}(\mathbf{u},\mathbf{v}) = P(\mathbf{u} \times (\operatorname{curl}\mathbf{v}))$, and $\mathbf{v}$ and $\mathbf{w}$ are given as above with the laplacian replaced by the Stokes operator. We state the following lemma which is similar to Lemma 1 given in Foias et al. [9].

Lemma 2.7. *The trilinear term $\langle \widetilde{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle$ is antisymmetric in its first and third components. I.e.,*

$$\langle \widetilde{B}(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle = -\langle \widetilde{B}(\mathbf{w},\mathbf{v}),\mathbf{u}\rangle$$

We also have the following inequalities for the trilinear term $|\langle \widetilde{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle|$*:*

(i) $|\langle \widetilde{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle| \leq c_1|\mathbf{u}|\,\|\mathbf{v}\|\,\|\mathbf{w}\|^{1/2}|A\mathbf{w}|^{1/2}$,

(ii) $|\langle \widetilde{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle| \leq c_2\|\mathbf{u}\|^{1/2}|A\mathbf{u}|^{1/2}\|\mathbf{v}\|\,|\mathbf{w}|$,

(iii) $|\langle \widetilde{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle| \leq c_3\|\mathbf{u}\|\,\|\mathbf{v}\|\,|\mathbf{w}|^{1/2}\|\mathbf{w}\|^{1/2}$,

(iv) $|\langle \widetilde{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle| \leq c_4|\mathbf{u}|^{1/2}\|\mathbf{u}\|^{1/2}|\mathbf{v}|\;|A\mathbf{w}| + c_4\|\mathbf{u}\|\;|\mathbf{v}|\;\|\mathbf{w}\|^{1/2}|A\mathbf{w}|^{1/2}$,

(v) $|\langle \widetilde{B}(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle| \leq c_5\|\mathbf{u}\|^{1/2}|A\mathbf{u}|^{1/2}|\mathbf{v}|\;\|A\mathbf{w}\| + c_5|A\mathbf{u}|\;|\mathbf{v}|\;|\mathbf{w}|^{1/2}\|\mathbf{w}\|^{1/2}$.

Due to non-dimensionalizing the equations, the constants above do not depend on any of the parameters. I.e., they do not depend of ϵ, γ, $\mathbf{u}$, ϖ, $\mathbf{f}$, *and Re. The labeling of the constants in the above lemma is used only to illustrate that they differ from each other.*

Determination of the modal value and nodal distance hinges on the following lemma.

Lemma 2.8. *(Jones and Titi [18])(Generalized Gronwall) Let* $a = a(t)$ *and* $b = b(t)$ *be locally integrable real-valued functions on* $[0, \infty)$ *that satisfy the following condition for some* $T > 0$*:*

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} a(\tau)\,d\tau > 0 \tag{2.11}$$

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} a^-(\tau)\,d\tau < \infty \tag{2.12}$$

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} b^+(\tau)\,d\tau = 0 \tag{2.13}$$

where $a^-(t) = max\{-a(t), 0\}$ *and* $b^+(t) = max\{b(t), 0\}$*. Suppose that* $\xi = \xi(t)$ *is an absolutely continuous non-negative function on* $[0, \infty)$ *that satisfies the following inequality almost everywhere on* $[0, \infty)$*:*

$$\frac{d\xi}{dt} + a\xi \leq b \tag{2.14}$$

Then

$$\lim_{t\to\infty} \xi(t) = 0. \tag{2.15}$$

2.3 Estimates

In this section we develop bounds that are used to determine well-posedness, the nodal distance, and the number of determining modes. We have chosen to put these estimates in one section so that the reader may refer to them as necessary.

2.3.1 Estimates necessary for the determination of nodal distance and modal number

In this subsection we focus on finding the following asymptotic bounds.

$$R_k^2 = \limsup_{t\to\infty} \|A^{k/2}\mathbf{u}\|_\epsilon^2 = \limsup_{t\to\infty} \left(|A^{k/2}\mathbf{u}|^2 + \epsilon|A^{(k+1)/2}\mathbf{u}|^2\right) \tag{2.1}$$

For $k = 0, 1$.

$$\begin{aligned} S_k^2 = \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} & \|A^{k/2}\mathbf{u}(\tau)\|_\epsilon^2 d\tau \\ & = \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} (|A^{k/2}\mathbf{u}(\tau)|^2 + \epsilon|A^{(k+1)/2}\mathbf{u}(\tau)|^2) d\tau \end{aligned} \tag{2.2}$$

for $k = 0, 1$. In particular to obtain S_k^2 for $k = 1$ we make the choice of $T = Re\gamma^{-1}$. The reader should keep in mind that using these estimates in the generalized Gronwall Lemma only requires that (2.2) holds for some T, and this choice is the most convenient. We should also note the higher order estimates can be obtained, but the amount of work increases significantly. Since we do not need the higher order estimates, we have omitted them.

We take the L^2 inner product of equation (2.10) with $\mathbf{u}$ and obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_\epsilon^2 + Re^{-1}\|A^{1/2}\mathbf{u}\|_{\gamma\epsilon}^2 = \langle \mathbf{f}, \mathbf{u}\rangle$$

We bound the right-hand side using Cauchy-Schwarz and Young.

$$|\langle \mathbf{f}, \mathbf{u}\rangle| \leq |A^{-1/2}\mathbf{f}|\,\|\mathbf{u}\| \leq \frac{Re}{2}|A^{-1/2}\mathbf{f}|^2 + \frac{Re^{-1}}{2}\|\mathbf{u}\|^2$$

Using lemma 2.2 and Poicaré we obtain

$$\frac{d}{dt}\|\mathbf{u}\|_\epsilon^2 + Re^{-1}\gamma\|A^{1/2}\mathbf{u}\|_\epsilon^2 \leq Re|\mathbf{f}|^2 \tag{2.3}$$

In the coming equations we need to bound $|\mathbf{f}|^2$ over all values of t. We assume that $\mathbf{f} \in L^\infty([0,\infty); H)$, thus we have $|\mathbf{f}(t)|^2 \leq \|\mathbf{f}\|_{L^2([0,\infty);H)} = Gr^2$, where Gr is the Grasfhof number. Traditionally the Grashof number is a dimensionless quantity that measures the strength of the forcing. In our non-dimensional system, $\|\mathbf{f}\|_{L^2([0,\infty);H)}$ serves this purpose. Thus we find,

$$\frac{d}{dt}\|\mathbf{u}\|_\epsilon^2 + Re^{-1}\gamma\|A^{1/2}\mathbf{u}\|_\epsilon^2 \leq ReGr^2. \tag{2.4}$$

We apply Poicaré once again to obtain the following

$$\frac{d}{dt}\|\mathbf{u}\|_\epsilon^2 + Re^{-1}\gamma\|\mathbf{u}\|_\epsilon^2 \leq ReGr^2 \tag{2.5}$$

Now we apply Gronwall's inequality

$$\|\mathbf{u}(t)\|_\epsilon^2 \leq e^{-Re^{-1}\gamma t}\|\mathbf{u}_o\|_\epsilon^2 + e^{-Re^{-1}\gamma t}\int_0^t ReGr^2 e^{Re^{-1}\gamma s} ds$$

and simplifying

$$\|\mathbf{u}(t)\|_\epsilon^2 \le e^{-Re^{-1}\gamma t}\|\mathbf{u}_o\|_\epsilon^2 + \frac{Gr^2}{Re^{-2}\gamma}\left(1 - e^{-Re^{-1}\gamma t}\right) < e^{-Re^{-1}\gamma t}\|\mathbf{u}_o\|_\epsilon^2 + \frac{Gr^2}{Re^{-2}\gamma} \quad (2.6)$$

Where $\|\mathbf{u}_o\|_\epsilon = \|\mathbf{u}(0)\|_\epsilon$, the H_ϵ^1 norm of the initial condition. Taking the limit we find

$$R_0^2 = \limsup_{t\to\infty} \|\mathbf{u}\|_\epsilon^2 \le \frac{Gr^2}{Re^{-2}\gamma} \quad (2.7)$$

We integrate equation (2.6) from t to $t' = t + T$. We find

$$\int_t^{t'} \|\mathbf{u}(s)\|_\epsilon^2 \le \frac{1}{Re^{-1}\gamma}\|\mathbf{u}_o\|_\epsilon^2 e^{-Re^{-1}\gamma t}\left(1 - e^{-Re^{-1}\gamma T}\right) + \frac{Gr^2 T}{Re^{-2}\gamma} \quad (2.8)$$

Next we divide by T and take the limit as $t \to \infty$ to obtain.

$$S_0^2 = \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \|\mathbf{u}(\tau)\|_\epsilon^2 d\tau \le \frac{Gr^2}{Re^{-2}\gamma} \quad (2.9)$$

To obtain a bound for S_1^2, we begin by integrating (2.4) from t to $t' = t + T$.

$$\|\mathbf{u}(t')\|_\epsilon^2 - \|\mathbf{u}(t)\|_\epsilon^2 + Re^{-1}\gamma \int_t^{t'} \|A^{1/2}\mathbf{u}(s)\|_\epsilon^2 ds \le TReGr^2$$

Next we divide by $Re^{-1}\gamma T$ and neglect $\|\mathbf{u}(t')\|_\epsilon^2$.

$$\frac{1}{T}\int_t^{t'} \|A^{1/2}\mathbf{u}(s)\|_\epsilon^2 ds \le \frac{Gr^2}{Re^{-2}\gamma} + \frac{1}{Re^{-1}\gamma T}\|\mathbf{u}(t)\|_\epsilon^2 \quad (2.10)$$

We bound $\|\mathbf{u}(t)\|_\epsilon^2$ using equation (2.6).

$$\frac{1}{T}\int_t^{t'} \|A^{1/2}\mathbf{u}(s)\|_\epsilon^2 ds \le \frac{Gr^2}{Re^{-2}\gamma} + \frac{1}{Re^{-1}\gamma T} e^{-Re^{-1}\gamma t}\|\mathbf{u}_o\|_\epsilon^2 + \frac{Gr^2}{Re^{-3}\gamma^2 T} \quad (2.11)$$

Taking the limit we find

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t'} \|A^{1/2}\mathbf{u}(s)\|_\epsilon^2 ds \leq \frac{Gr^2}{Re^{-2}\gamma} + \frac{Gr^2}{Re^{-3}\gamma^2 T} \tag{2.12}$$

In particular by setting $T = Re\gamma^{-1}$ we find the following estimate for S_1^2.

$$S_1^2 = \limsup_{t\to\infty} \frac{1}{T}\int_t^{t'} \|A^{1/2}\mathbf{u}(s)\|_\epsilon ds \leq \frac{2Gr^2}{Re^{-2}\gamma} \tag{2.13}$$

Next we determine R_1^2. We proceed by taking the inner product of (2.10) with $A\mathbf{u}$ and obtain

$$\frac{1}{2}\frac{d}{dt}\|A^{1/2}\mathbf{u}\|_\epsilon^2 + Re^{-1}\|A\mathbf{u}\|_{\gamma\epsilon}^2 = \langle \widetilde{B}(\mathbf{u},\mathbf{v}), A\mathbf{u}\rangle + \langle \mathbf{f}, A\mathbf{u}\rangle.$$

We bound $\langle \mathbf{f}, A\mathbf{u}\rangle$ as

$$|\langle \mathbf{f}, \mathbf{u}\rangle| \leq |\mathbf{f}|\ |A\mathbf{u}| \leq \frac{Re}{2}|\mathbf{f}|^2 + \frac{Re^{-1}}{2}|A\mathbf{u}|^2.$$

We use lemma 2.7(iii) and Poincaré to bound the trilinear term.

$$\begin{aligned}
|\langle \widetilde{B}(\mathbf{u},\mathbf{v}), A\mathbf{u}\rangle| &\leq c_1\|\mathbf{u}\|\|\mathbf{v}\|\ |A\mathbf{u}|^{1/2}|A^{3/2}\mathbf{u}|^{1/2} \\
&\leq c_1\|\mathbf{u}\|(1+\epsilon)|A^{3/2}\mathbf{u}|\ |A\mathbf{u}|^{1/2}|A^{3/2}\mathbf{u}|^{1/2} \\
&= c_1(1+\epsilon)\|\mathbf{u}\|\ |A\mathbf{u}|^{1/2}|A^{3/2}\mathbf{u}|^{3/2}
\end{aligned}$$

We now apply Young's inequality with $p = 4$ and $q = 4/3$ to obtain

$$|\langle \widetilde{B}(\mathbf{u},\mathbf{v}), A\mathbf{u}\rangle| \leq \frac{c_2(1+\epsilon)^4}{Re^{-3}\gamma^3\epsilon^3}\|\mathbf{u}\|^4|A\mathbf{u}|^2 + \frac{Re^{-1}\gamma\epsilon}{2}|A^{3/2}\mathbf{u}|^2.$$

With Lemma 2.2 and by bounding $\mathbf{f}$, we find

$$\frac{d}{dt}\|A^{1/2}\mathbf{u}\|_\epsilon^2 + Re^{-1}\gamma\|A\mathbf{u}\|_\epsilon^2 \le \frac{c_2(1+\epsilon)^4}{Re^{-3}\gamma^3\epsilon^3}\|\mathbf{u}\|^4|A\mathbf{u}|^2 + ReGr^2 \tag{2.14}$$

Applying Poincaré on $\|A\mathbf{u}\|_\epsilon^2$ gives

$$\frac{d}{dt}\|A^{1/2}\mathbf{u}\|_\epsilon^2 + Re^{-1}\gamma\|A^{1/2}\mathbf{u}\|_\epsilon^2 \le \frac{c_2(1+\epsilon)^4}{Re^{-3}\gamma^3\epsilon^3}\|\mathbf{u}\|^4|A\mathbf{u}|^2 + ReGr^2 \tag{2.15}$$

In order to proceed we apply Gronwall's inequality in a non-traditional way. As in the proof of Gronwall we multiply both sides by $e^{Re^{-1}\gamma t}$, then we integrate from s to t. After the integration we multiply both sides by $e^{-Re^{-1}\gamma t}$.

$$\begin{aligned}\|A^{1/2}\mathbf{u}(t)\|_\epsilon^2 \le{}& \underbrace{e^{Re^{-1}\gamma(s-t)}\|A^{1/2}\mathbf{u}(s)\|_\epsilon^2}_{I_1} \\ &+ \underbrace{e^{-Re^{-1}\gamma t}\int_s^t e^{Re^{-1}\gamma\tau}\frac{c_2(1+\epsilon)^4}{Re^{-3}\gamma^3\epsilon^3}\|\mathbf{u}(\tau)\|^4|A\mathbf{u}(\tau)|^2 d\tau}_{I_2} + \underbrace{\frac{1}{Re^{-2}\gamma}Gr^2}_{I_3}\end{aligned} \tag{2.16}$$

First we work on bounding I_2. To start we bound $\|\mathbf{u}\|^4$ using inequality (2.6).

$$\begin{aligned}\|\mathbf{u}(\tau)\|^4 &\le \frac{1}{\epsilon^2}\left(e^{-Re^{-1}\gamma\tau}\|\mathbf{u}_o\|_\epsilon^2 + \frac{Gr^2}{Re^{-2}\gamma}\right)^2 \\ &\le \frac{2}{\epsilon^2}e^{-2Re^{-1}\gamma\tau}\|\mathbf{u}_o\|_\epsilon^4 + \frac{2Gr^4}{Re^{-4}\gamma^2\epsilon^2}\end{aligned}$$

Using this inequality and bounding the exponential terms by its maximum on $[s,t]$ we obtain

$$\begin{aligned}&e^{-Re^{-1}\gamma t}\int_s^t e^{Re^{-1}\gamma\tau}\|\mathbf{u}\|^4|A\mathbf{u}(\tau)|^2 d\tau \le \\ &\qquad\left(\frac{2}{\epsilon^2}e^{-Re^{-1}\gamma(t+s)}\|\mathbf{u}_o\|_\epsilon^4 + \frac{2Gr^4}{Re^{-4}\gamma^2\epsilon^2}\right)\int_s^t |A\mathbf{u}(\tau)|^2 d\tau\end{aligned}$$

Equation (2.11) serves to bound the integral of $|A\mathbf{u}(\tau)|^2$, and so we obtain

$$I_2 \leq \frac{2c_2(1+\epsilon)^4}{Re^{-3}\gamma^3\epsilon^3}\left(\frac{1}{\epsilon^2}e^{-Re^{-1}\gamma(t+s)}\|\mathbf{u}_o\|_\alpha^4 + \frac{Gr^4}{Re^{-4}\gamma^2\epsilon^2}\right) \\ \times\left(\frac{Gr^2(t-s)}{Re^{-2}\gamma\epsilon} + \frac{1}{Re^{-1}\gamma\epsilon}e^{-Re^{-1}\gamma s}\|\mathbf{u}_o\|_\epsilon^2 + \frac{Gr^2}{Re^{-3}\gamma^2\epsilon}\right) \quad (2.17)$$

In order to remove the dependence on s in equation (2.16), we will integrate the inequality over $[t-T, t]$ with respect to s. We first carry out the integration on I_2.

$$\begin{aligned}\left(\frac{2c_2(1+\epsilon)^4}{Re^{-3}\gamma^3\epsilon^3}\right)^{-1}\int_{t-T}^{t} I_2 ds \leq & \frac{TGr^2}{Re^{-3}\gamma^2\epsilon^3}\|\mathbf{u}_o\|_\epsilon^4 e^{-2Re^{-1}\gamma t + Re^{-1}\gamma T} \\ & + \frac{Gr^2}{Re^{-4}\gamma^3\epsilon^3}\|\mathbf{u}_o\|_\epsilon^4 e^{-2Re^{-1}\gamma t}\left(1 - e^{Re^{-1}\gamma T}\right) \\ & + \frac{T^2}{2Re^{-6}\gamma^3\epsilon^3}Gr^6 \\ & + e^{-3Re^{-1}\gamma t}\left(e^{Re^{-1}\gamma T} - 1\right)\frac{1}{2Re^{-2}\gamma^2\epsilon^3}\|\mathbf{u}_o\|_\epsilon^6 \\ & + e^{-Re^{-1}\gamma t}\left(e^{Re^{-1}\gamma T} - 1\right)\frac{1}{Re^{-6}\gamma^3\epsilon^3}\|\mathbf{u}_o\|_\epsilon^2 Gr^4 \\ & + e^{-2Re^{-1}\gamma t}\left(e^{Re^{-1}\gamma T} - 1\right)\frac{1}{Re^{-4}\gamma^3\epsilon^3}\|\mathbf{u}_o\|_\epsilon^4 Gr^2 \\ & + \frac{T}{Re^{-7}\gamma^4\epsilon^3}Gr^6\end{aligned}$$

The above expression is complicated, but notice that there are only two terms that do not decay exponentially in t. As such we write

$$\int_{t-T}^{t} I_2 ds \leq I_{2a} + I_{2b}$$

where

$$\begin{aligned}I_{2a} = & \frac{c_2(1+\epsilon)^4 T^2}{Re^{-9}\gamma^6\epsilon^6}Gr^6 + \frac{2c_2(1+\epsilon)^4 T}{Re^{-10}\gamma^7\epsilon^6}Gr^6 \\ & \left(= \frac{3c_2(1+\epsilon)^4}{Re^{-11}\gamma^8\epsilon^6}Gr^6 \qquad \text{for } T = Re\gamma^{-1}\right)\end{aligned}$$

Then, I_{2b} is the given by the remaining terms. It is important to note that $I_{2b} \to 0$ as $t \to \infty$.

Using equation (2.11) we find for I_1

$$\begin{aligned}
\int_{t-T}^{t} I_1 ds &\leq \int_{t-T}^{t} e^{Re^{-1}\gamma(s-t)} \|A^{1/2}\mathbf{u}(s)\|_\epsilon^2 ds \\
&\leq \int_{t-T}^{t} \|A^{1/2}\mathbf{u}(s)\|_\epsilon^2 ds \\
&\leq \frac{Gr^2 T}{Re^{-2}\gamma} + \frac{1}{Re^{-1}\gamma} e^{-Re^{-1}\gamma(t-T)} \|\mathbf{u}_o\|_\epsilon^2 + \frac{Gr^2}{Re^{-3}\gamma^2}
\end{aligned}$$

As was done with I_2 we write

$$I_1 = I_{1a} + I_{1b},$$

where

$$\begin{aligned}
I_{1a} &= \frac{Gr^2 T}{Re^{-2}\gamma} + \frac{Gr^2}{Re^{-3}\gamma^2} \\
&\left(= \frac{2Gr^2}{Re^{-3}\gamma^2} \qquad \text{for } T = Re\gamma^{-1} \right) \\
I_{1b} &= \frac{1}{Re^{-1}\gamma} e^{-Re^{-1}\gamma(t-T)} \|\mathbf{u}_o\|_\epsilon^2
\end{aligned}$$

and with I_{2b} we have $I_{1b} \to 0$ as $t \to \infty$.

Finally for the integral of I_3 we have

$$\int_{t-T}^{t} I_3 ds = \frac{Gr^2 T}{Re^{-2}\gamma} = I_{3a}$$

Thus we have

$$\|A^{1/2}\mathbf{u}(t)\|_\epsilon^2 \leq \frac{1}{T}\left(I_{1a} + I_{1b} + I_{2a} + I_{2b} + I_{3a}\right) \tag{2.18}$$

and so upon setting $T = Re\gamma^{-1}$ we obtain

$$\limsup_{t\to\infty} \|A^{1/2}\mathbf{u}(t)\|_\epsilon^2 \leq \frac{3Gr^2}{Re^{-2}\gamma} + \frac{3c_2(1+\epsilon)^4 Gr^6}{Re^{-10}\gamma^7\epsilon^6} \tag{2.19}$$

giving us a bound for R_1^1.

2.3.2 $L^2([0,T];V^s)$ and $L^\infty([0,T];V^s)$ estimates

In this section we derive estimates for $\|\mathbf{u}\|_{L^2([0,T];V^s)}$ $s = 0,1,2,3,4$ and $\|\mathbf{u}\|_{L^\infty([0,T];V^s)}$ $s = 0,1,2,3$. These estimates are important for showing the well-posedness results over the time interval $[0,T]$. Thus, in the following derivation we focus on finding generic bounds for the norms of $\mathbf{u}$ over a time interval $[0,T]$ where $T > 0$ is fixed.

First we note that equations (2.6) and (2.8) give us bounds for $\|\mathbf{u}\|_{L^2([0,T];V^s)}$ $s = 0,1$ and $\|\mathbf{u}\|_{L^\infty([0,T];V^s)}$ $s = 0,1$. Next we consider equation (2.15) and apply Gronwall over $[0,t]$.

$$\begin{aligned}\|A^{1/2}\mathbf{u}(t)\|_\epsilon^2 &\leq e^{-Re^{-1}\gamma t}\|A^{1/2}\mathbf{u}(0)\|_\epsilon^2 \\ &\quad + e^{-Re^{-1}\gamma t}\int_0^t e^{Re^{-1}\gamma\tau}\frac{c_2(1+\epsilon)^4}{Re^{-3}\gamma^3\epsilon^3}\|\mathbf{u}(\tau)\|^4|A\mathbf{u}(\tau)|^2 d\tau + \frac{1}{Re^{-2}\gamma}Gr^2\end{aligned} \tag{2.20}$$

Equation (2.6) implies that $\|\mathbf{u}(\tau)\|^4$ is bounded over $[0,T]$ and from equation (2.8) we have that $\int_0^t |A\mathbf{u}(\tau)|^2 d\tau$ is bounded over $[0,T]$. Therefore we have $\|A^{1/2}\mathbf{u}(t)\|_\epsilon^2$ is bounded on $[0,T]$ and we write

$$\|A^{1/2}\mathbf{u}(t)\|_\epsilon^2 \leq k_1(T) \quad \text{for} \quad 0 \leq t \leq T \tag{2.21}$$

Similarly we integrate equation (2.14) over $[0,T]$ to find

$$\int_0^T \|A\mathbf{u}(\tau)\|_\epsilon^2 \, d\tau \leq$$
$$\frac{1}{Re^{-1}\gamma}\|A^{1/2}\mathbf{u}(0)\|_\epsilon^2 + \frac{1}{Re^{-1}\gamma}\int_0^T \frac{c_2(1+\epsilon)^4}{Re^{-3}\gamma^3\epsilon^3}\|\mathbf{u}(s)\|^4|A\mathbf{u}(s)|^2 ds + \frac{Gr^2 T}{Re^{-2}\gamma} \quad (2.22)$$

We bound $\|\mathbf{u}(\tau)\|^4$ and $\int_0^t |A\mathbf{u}(\tau)|^2 d\tau$ as before to find

$$\int_0^T \|A\mathbf{u}(\tau)\|_\epsilon^2 \, d\tau \leq k_2(T), \quad (2.23)$$

for some $k_2(T)$ where $k_2 \to \infty$ as $T \to \infty$. We may now proceed with obtain H^3 estimates. We begin by taking the L^2 inner product of equation (2.10) with $A^2\mathbf{u}$.

$$\frac{1}{2}\frac{d}{dt}\|A\mathbf{u}\|_\epsilon^2 + Re^{-1}\|A^{3/2}\mathbf{u}\|_{\gamma\epsilon}^2 = \langle \widetilde{B}(\mathbf{u},\mathbf{v}), A^2\mathbf{u}\rangle + \langle \mathbf{f}, A^2\mathbf{u}\rangle$$

We bound the tri-linear term and the forcing terms as follows:

$$\langle \widetilde{B}(\mathbf{u},\mathbf{v}), A^2\mathbf{u}\rangle \leq c_6|A\mathbf{u}|\,\|\mathbf{v}\|\,|A^2\mathbf{u}| \leq \frac{Re^{-1}\gamma\epsilon}{2}|A^2\mathbf{u}|^2 + \frac{c_3}{2Re^{-1}\gamma\epsilon}|A\mathbf{u}|^2\|\mathbf{v}\|^2 \quad (2.24)$$
$$\leq \frac{\nu\gamma\epsilon}{2}|A^2\mathbf{u}|^2 + \frac{c_4(1+\epsilon)^2}{2\nu\gamma\epsilon}|A\mathbf{u}|^2|A^{3/2}\mathbf{u}|^2 \quad (2.25)$$

and

$$\langle \mathbf{f}, A^2\mathbf{u}\rangle = \langle A^{-1/2}\mathbf{f}, A^{3/2}\mathbf{u}\rangle \leq \frac{Gr^2}{2Re^{-1}} + \frac{Re^{-1}}{2}|A^{3/2}\mathbf{u}|^2. \quad (2.26)$$

Thus we have

$$\frac{d}{dt}\|A\mathbf{u}\|_\epsilon^2 + Re^{-1}\gamma\|A^{3/2}\mathbf{u}\|_\epsilon^2 \leq \frac{c_4(1+\epsilon)^2}{Re^{-1}\gamma\epsilon}|A\mathbf{u}|^2|A^{3/2}\mathbf{u}|^2 + \frac{Gr^2}{Re^{-1}}. \quad (2.27)$$

And with an another application of Poincaré's inequality on the viscous term we obtain

$$\frac{d}{dt}\|A\mathbf{u}\|_\epsilon^2 + Re^{-1}\gamma\|A\mathbf{u}\|_\epsilon^2 \leq \frac{c_4(1+\epsilon)^2}{Re^{-1}\gamma\epsilon}|A\mathbf{u}|^2|A^{3/2}\mathbf{u}|^2 + \frac{Gr^2}{Re^{-1}}. \tag{2.28}$$

We apply Gronwall over $[0,t]$ to obtain

$$\begin{aligned}\|A\mathbf{u}(t)\|_\epsilon^2 \leq{}& e^{-Re^{-1}\gamma t}\|A\mathbf{u}(0)\|_\epsilon^2 \\ &+ e^{-Re^{-1}\gamma t}\int_0^t e^{Re^{-1}\gamma\tau}\frac{c_4(1+\epsilon)^2}{Re^{-1}\gamma\epsilon}|A\mathbf{u}(\tau)|^2|A^{3/2}\mathbf{u}(\tau)|^2 d\tau + \frac{Gr^2}{Re^{-2}\gamma}\end{aligned} \tag{2.29}$$

Equation (2.21) implies that $|A\mathbf{u}(\tau)|^2 < k_1(t)/\epsilon$ and equation (2.6) implies that $|A^{3/2}\mathbf{u}(\tau)|^2 < k_2(t)/\epsilon$. Therefore we have

$$\|A\mathbf{u}(t)\|_\epsilon^2 \leq e^{-Re^{-1}\gamma t}\|A\mathbf{u}(0)\|_\epsilon^2 + \frac{c_4(1+\epsilon)^2}{Re^{-1}\gamma\epsilon^3}k_1(t)k_2(t) + \frac{Gr^2}{Re^{-2}\gamma} := k_3(t). \tag{2.30}$$

Thus we see that $\|A\mathbf{u}(t)\|_\epsilon^2$ is bounded on $[0,T]$. Finally we integrate (2.27) over $[0,T]$ and obtain the following inequality.

$$\begin{aligned}\int_0^T \|A^{3/2}\mathbf{u}(\tau)\|_\epsilon^2\, d\tau \leq{}& \\ &\|A\mathbf{u}(0)\|_\epsilon^2 + \int_0^T \frac{c_4(1+\epsilon)^2}{Re^{-1}\gamma\epsilon}|A\mathbf{u}(\tau)|^2|A^{3/2}\mathbf{u}(\tau)|^2 d\tau + \frac{Gr^2T}{Re^{-2}\gamma}\end{aligned} \tag{2.31}$$

And so with the same arguments as before we find that there is a function $k_4(T)$ such that

$$\int_0^T \|A^{3/2}\mathbf{u}(\tau)\|_\epsilon^2\, d\tau \leq k_4(T). \tag{2.32}$$

2.3.3 Estimate for $\frac{\partial \mathbf{v}}{\partial t}$

In order to apply Theorem 2.1, we must obtain a bound for $\frac{\partial \mathbf{u}}{\partial t}$ or an equivalent bound for $\frac{\partial \mathbf{v}}{\partial t}$. In particular we will find a bound for $\|\frac{\partial \mathbf{v}}{\partial t}\|^2_{L^2([0,T];H)}$. Beginning with equation (2.10) we have

$$\left\|\frac{\partial \mathbf{v}}{\partial t}\right\|_{L^2([0,T];H)} \leq \|\widetilde{B}(\mathbf{u},\mathbf{v})\|_{L^2([0,T];H)} + Re^{-1}\|A\mathbf{w}\|_{L^2([0,T];H)} + \|\mathbf{f}\|_{L^2([0,T];H)}$$

Lemma 2.7(ii) implies that

$$\begin{aligned}\|\widetilde{B}(\mathbf{u},\mathbf{v})\|^2_{L^2([0,T];H)} &= \int_0^T \|\widetilde{B}(\mathbf{u}(\tau),\mathbf{v}(\tau))\|^2_H \, d\tau \\ &\leq \int_0^T (1+\epsilon)|A\mathbf{u}(\tau)|\;|A^{3/2}\mathbf{u}(\tau)| \, d\tau \\ &\leq (1+\epsilon)\|\mathbf{u}\|^2_{L^2([0,T];V^2)}\|\mathbf{u}\|^2_{L^2([0,T];V^3)}\end{aligned}$$

Then, $\|A\mathbf{w}\|_{L^2([0,T];H)} \leq (1+\epsilon\gamma)\|\mathbf{u}\|_{L^2([0,T];V^4)}$ is bounded. Finally $\|\mathbf{f}\|_{L^2([0,T];H)}$ is bounded by our assumptions on $\mathbf{f}$. Therefore there is a function $k_5(T)$ such that

$$\left\|\frac{\partial \mathbf{v}}{\partial t}\right\|_{L^2([0,T];H)} \leq k_5(T) \tag{2.33}$$

and

$$\left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{L^2([0,T];V^2)} \leq \frac{k_5(T)}{\epsilon^2}. \tag{2.34}$$

2.4 Well-posedness results

We recall the standard definition of weak and strong solutions of the NS to motivate our definition of a strong solution to the NS-$\alpha\beta$ equations.

Definition 2.1. A weak solution of the NS on $[0,T]$ is a function $\mathbf{u} \in L^2([0,T];V) \cap C_w([0,T];H)$ that for all $\mathbf{u} \in V$ satisfies

$$\begin{gathered}\tfrac{\partial \mathbf{u}}{\partial t} \in L^1_{\text{loc}}([0,T];V') \\ \langle \tfrac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\rangle_{V',V} + Re^{-1}\langle A^{1/2}\mathbf{u}, A^{1/2}\mathbf{v}\rangle + \langle B(\mathbf{u},\mathbf{u}), \mathbf{v}\rangle_{V',V} = \langle \mathbf{f}, \mathbf{v}\rangle_{V',V} \\ \mathbf{u}(0) = \mathbf{u}_o,\end{gathered} \tag{2.1}$$

where $\mathbf{u}_o$ is the initial condition and C_w are the weakly continuous functions.

As shown in [5] for $\mathbf{u} \in H$ and $\mathbf{f} \in V'$ we have existence of weak solutions to the Navier–Stokes equations for dimensions $n = 2, 3$. The main outstanding issue is for the existence of strong solutions.

Definition 2.2. A strong solution of the Navier–Stokes equations on $[0,T]$ is a function $\mathbf{u} \in L^\infty([0,T];V) \cap L^2([0,T];V^2)$ that satisfies

$$\begin{gathered}\tfrac{\partial \mathbf{u}}{\partial t} + Re^{-1}A\mathbf{u} + B(\mathbf{u},\mathbf{u}) = \mathbf{f} \\ \mathbf{u}(0) = \mathbf{u}_o.\end{gathered} \tag{2.2}$$

We should note that these are not classical solutions and the equation only makes sense in H. For $n = 2$ we have existence of strong solutions to the NS for all $T > 0$, but for $n = 3$ we only have existence of strong solutions to the NS for $0 < T < T^* < \infty$, where T^* depends on the parameters of the system. Note that the key difference between weak and strong solutions is that for strong solutions the equation makes sense in H, whereas for weak solutions the equation only makes sense in V'. Thus with this context we make the following definition of strong solutions to the NS-$\alpha\beta$ equations.

Definition 2.3. (Strong Solution of the NS-$\alpha\beta$ equations.) Let $T > 0$. A function

$\mathbf{u} \in C([0,T];V^2) \cap L^2([0,T];V^4)$ with $\partial\mathbf{v}/\partial t \in L^2([0,T];H)$ is said to be a strong solution to the the NS-$\alpha\beta$ equations on the interval $[0,T]$ if it satisfies,

$$\begin{gathered} \tfrac{\partial \mathbf{v}}{\partial t} + Re^{-1} A\mathbf{w} = \widetilde{B}(\mathbf{u},\mathbf{v}) + \mathbf{f} \\ \mathbf{u}(0) = \mathbf{u}_o. \end{gathered} \tag{2.3}$$

Theorem 2.4. *(Global existence of strong solutions to the NS-$\alpha\beta$ equations). Let $\mathbf{f} \in L^2([0,T];H) \cap L^\infty([0,\infty);H)$ and $\mathbf{u}(0) \in V^3$. Then for any $T > 0$, equation* (2.3) *has a strong solution $\mathbf{u}$ on $[0,T]$.*

Proof. The proof follows closely the work of Foias et al. [9] and Constantin and Foias [5]. Reference [9] deals with the LANS-α equations, our methods differ from [9] in a few ways. First, we extend the results to the NS-$\alpha\beta$ equations. Second, we show the existence of strong solutions, in [9] for the LANS-α equations they show the existence of what they call regular solutions. Similar to the context given above these regular solutions are not quite as "weak" as weak solutions to NS, but they are not strong enough to be strong solutions. I.e., they do not exit in H. The third difference is that our method for obtaining bounds differ. The main reason for this difference is that we assumed that our initial condition is in V^3 as opposed to V.

In section 2.3 we derived various bounds for $\mathbf{u}$. Instead of considering $\mathbf{u}$ we could have considered the Galerkin system where we project onto the first m eigenfunctions of the Stokes operator. I.e., let P_m be the projection of the Stokes operator onto the first m modes, then $\mathbf{u}_m = P_m\mathbf{u}$. We form the Galerkin system

$$\frac{\partial}{\partial t}\mathbf{v}_m + Re^{-1} A\mathbf{w}_m = P_m\widetilde{B}(\mathbf{u}_m,\mathbf{v}_m) + P_m\mathbf{f}. \tag{2.4}$$

All of our estimates from section 2.3 still hold true for this Galerkin system, as long as we take the inner product with $A^s\mathbf{u}_m$, of course. Thus from equations (2.30), (2.31), and (2.34) we have that

$$\begin{aligned} \mathbf{u}_m & \quad \text{is bounded uniformly in } L^2([0,T];V^4) \\ \mathbf{u}_m & \quad \text{is bounded uniformly in } L^\infty([0,T];V^3) \\ \frac{\partial \mathbf{u}_m}{\partial t} & \quad \text{is bounded uniformly in } L^2([0,T];V^2) \end{aligned}$$

Then by Theorem 2.1 there exists a subsequence $\mathbf{u}_{m'}$ such that

$$\begin{aligned} \mathbf{u}_{m'} \rightharpoonup \mathbf{u} & \quad \text{weakly in} \quad L^2([0,T];V^4) \\ \mathbf{u}_{m'} \to \mathbf{u} & \quad \text{strongly in} \quad L^2([0,T];V^3) \\ \mathbf{u}_{m'} \to \mathbf{u} & \quad \text{in} \quad C([0,T];V^2) \end{aligned}$$

Equivalently we have

$$\begin{aligned} \mathbf{v}_{m'} \rightharpoonup \mathbf{v} & \quad \text{weakly in} \quad L^2([0,T];V^2) \\ \mathbf{v}_{m'} \to \mathbf{v} & \quad \text{strongly in} \quad L^2([0,T];V) \\ \mathbf{v}_{m'} \to \mathbf{v} & \quad \text{in} \quad C([0,T];H) \end{aligned}$$

and similar for $\mathbf{w}$. We relabel the subsequence as $\mathbf{u}_m$. Then for $\boldsymbol{\phi} \in H$ and for almost all $t_o, t \in [0,T]$, we want to show that $\mathbf{u}_m$ satisfies the following equation in the limit as $m \to \infty$.

$$\begin{aligned} &\langle \mathbf{v}_m(t), \boldsymbol{\phi} \rangle + Re^{-1} \int_{t_o}^{t} \langle A\mathbf{w}_m(\tau), \boldsymbol{\phi} \rangle \; d\tau \\ &\qquad = \int_{t_o}^{t} \left\langle P_m \widetilde{B}(\mathbf{u}_m(\tau), \mathbf{v}_m(\tau)), \boldsymbol{\phi} \right\rangle \; d\tau + \langle \mathbf{v}_m(t_o), \boldsymbol{\phi} \rangle + \int_{t_o}^{t} \langle P_m \mathbf{f}(\tau), \boldsymbol{\phi} \rangle \; d\tau. \quad (2.5) \end{aligned}$$

We have that $\mathbf{v}_m$ and $\mathbf{w}_m$ converge strongly to $\mathbf{v}$ and $\mathbf{w}$ in $L^2([0,T];V)$. Thus, except on a set E of measure zero, we have that $\mathbf{v}_m(\tau)$ and $\mathbf{w}_m(\tau)$ converge to $\mathbf{v}(\tau)$ and

$\mathbf{w}(\tau)$ in V. Plus $A\mathbf{w}_m$ converges weakly to $A\mathbf{w}_m$ in $L^2([0,T];H)$. Therefore,

$$lim_{m\to\infty}\langle v_m(t), \boldsymbol{\phi}\rangle = \langle v(t), \boldsymbol{\phi}\rangle$$

$$lim_{m\to\infty}\langle v_m(t_o), \boldsymbol{\phi}\rangle = \langle v(t_o), \boldsymbol{\phi}\rangle$$

and

$$lim_{m\to\infty}\int_{t_o}^{t}\langle A\mathbf{w}_m(\tau), \boldsymbol{\phi}\rangle\ d\tau = \int_{t_o}^{t}\langle A\mathbf{w}(\tau), \boldsymbol{\phi}\rangle\ d\tau.$$

In addition it is clear that

$$\lim_{m\to\infty}\int_{t_o}^{t}\langle P_m\mathbf{f}(\tau), \boldsymbol{\phi}\rangle = \int_{t_o}^{t}\langle \mathbf{f}(\tau), \boldsymbol{\phi}\rangle.$$

Now we show that the limit holds for the tri-linear term. We write

$$\left|\int_{t_o}^{t}\left\langle P_m\widetilde{B}(\mathbf{u}_m(\tau), \mathbf{v}_m(\tau)), \boldsymbol{\phi}\right\rangle\ d\tau - \int_{t_o}^{t}\left\langle \widetilde{B}(\mathbf{u}(\tau), \mathbf{v}(\tau)), \boldsymbol{\phi}\right\rangle\ d\tau\right| \leq$$
$$I_1(m) + I_2(m) + I_3(m),$$

where

$$\begin{aligned}
I_1(m) &= \left|\int_{t_o}^{t}\left\langle \widetilde{B}(\mathbf{u}_m(\tau), \mathbf{v}_m(\tau)), P_m\boldsymbol{\phi} - \boldsymbol{\phi}\right\rangle\ d\tau\right| \\
I_2(m) &= \left|\int_{t_o}^{t}\left\langle \widetilde{B}(\mathbf{u}_m(\tau) - \mathbf{u}(\tau), \mathbf{v}_m(\tau)), \boldsymbol{\phi}\right\rangle\ d\tau\right| \\
I_3(m) &= \left|\int_{t_o}^{t}\left\langle \widetilde{B}(\mathbf{u}(\tau), \mathbf{v}_m(\tau) - \mathbf{v}(\tau)), \boldsymbol{\phi}\right\rangle\ d\tau\right|
\end{aligned}$$

Thus,

$$\begin{aligned}
I_1(m) &\leq \int_{t_o}^{t} c|A\mathbf{u}_m(\tau)|\|\mathbf{v}_m(\tau)\||P_m\boldsymbol{\phi} - \boldsymbol{\phi}|\ d\tau \\
&\leq c\int_0^T |A\mathbf{u}_m(\tau)|^2\ d\tau \int_0^T \|\mathbf{v}_m(\tau)\|^2\ d\tau |P_m\boldsymbol{\phi} - \boldsymbol{\phi}|
\end{aligned}$$

Thus, $\lim_{m\to\infty} I_1(m) = 0$. Since $\mathbf{u}_m$ converges strongly in V^3 and $\mathbf{v}_m$ converges strongly in V the same argument used for $I_1(m)$ shows that $\lim_{m\to\infty} I_2(m) = \lim_{m\to\infty} I_3(m) = 0$. Therefore

$$\lim_{m\to\infty} \int_{t_o}^{t} \left\langle P_m \widetilde{B}(\mathbf{u}_m(\tau), \mathbf{v}_m(\tau)), \boldsymbol{\phi} \right\rangle \, d\tau = \int_{t_o}^{t} \left\langle \widetilde{B}(\mathbf{u}(\tau), \mathbf{v}(\tau)), \boldsymbol{\phi} \right\rangle \, d\tau,$$

And so for all $t_o, t \in [0,T] \backslash E$ and for all $\boldsymbol{\phi} \in V$ we have

$$\langle \mathbf{v}(t), \boldsymbol{\phi} \rangle + Re^{-1} \int_{t_o}^{t} \langle A\mathbf{w}(\tau), \boldsymbol{\phi} \rangle \, d\tau = \\ \int_{t_o}^{t} \left\langle \widetilde{B}(\mathbf{u}(\tau), \mathbf{v}(\tau)), \boldsymbol{\phi} \right\rangle \, d\tau + \langle \mathbf{v}(t_o), \boldsymbol{\phi} \rangle + \int_{t_o}^{t} \langle \mathbf{f}(\tau), \boldsymbol{\phi} \rangle \, d\tau. \quad (2.6)$$

Since $\mathbf{v}_m$ converges to $\mathbf{v}$ in $C([0,T]; H)$, we must have that (2.6) holds for all $t_o, t \in [0,T]$. Finally since $\boldsymbol{\phi} \in H$ is arbitrary we have that equation (2.3) is satisfied in H. Thus, we conclude that $\mathbf{u}$ is a strong solution of the NS-$\alpha\beta$ equations. □

Theorem 2.5. *Strong solutions are unique and depend continuously on the initial data.*

Proof. Note that this proof closely follows the similar theorem for the LANS-α equations in [9]. Essentially we use our lemmas to extend their result to the NS-$\alpha\beta$ equations. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions to the NS-$\alpha\beta$ equations with initial conditions $\mathbf{u}_1(0)$ and $\mathbf{u}_2(0)$. We write $\delta\mathbf{u} = \mathbf{u}_2 - \mathbf{u}_1$ and similarly for $\delta\mathbf{v}$ and $\delta\mathbf{w}$. Note that

$$\widetilde{B}(\mathbf{u}_2, \delta\mathbf{v}) + \widetilde{B}(\delta\mathbf{u}, \mathbf{v}_1) = \widetilde{B}(\mathbf{u}_2, \mathbf{v}_2) - \widetilde{B}(\mathbf{u}_1, \mathbf{v}_1).$$

And so by taking the difference of the two equations we arrive at

$$\frac{\partial \delta\mathbf{v}}{\partial t} + Re^{-1} A\delta\mathbf{w} + \widetilde{B}(\mathbf{u}_2, \delta\mathbf{v}) + \widetilde{B}(\delta\mathbf{u}, \mathbf{v}_1) = 0. \quad (2.7)$$

Since $\langle \widetilde{B}(\mathbf{u}, bfv), \mathbf{w} \rangle = -\langle \widetilde{B}(\mathbf{w}, bfv), \mathbf{u} \rangle$, we have after taking the inner-product of the above equation with $\delta\mathbf{u}$

$$\frac{1}{2}\|\delta\mathbf{u}\|_\epsilon^2 + Re^{-1}\|A^{1/2}\delta\mathbf{u}\|_{\gamma\epsilon}^2 + \langle\widetilde{B}(\mathbf{u}_2,\delta\mathbf{v}),\delta\mathbf{u}\rangle = 0. \tag{2.8}$$

We use Lemma 2.2 on the viscous term and Lemma 2.7(iv) on the trilinear term.

$$\frac{1}{2}\|\delta\mathbf{u}\|_\epsilon^2 + Re^{-1}\gamma\|A^{1/2}\delta\mathbf{u}\|_\epsilon^2 \leq$$
$$c\|\mathbf{u}_2\|^{1/2}|A\mathbf{u}_2|^{1/2}|\delta\mathbf{v}|\ \|\delta\mathbf{u}\| + c|A\mathbf{u}_2|\ |\delta\mathbf{v}|\ |\delta\mathbf{u}|^{1/2}\|\delta\mathbf{u}\|^{1/2} \tag{2.9}$$

Using manipulation similar to those in section 2.3 we find

$$\begin{aligned}\frac{1}{2}\|\delta\mathbf{u}\|_\epsilon^2 + Re^{-1}\gamma\|A^{1/2}\delta\mathbf{u}\|_\epsilon^2 &\leq c(1+\epsilon)\|\mathbf{u}_2\|^{1/2}|A\mathbf{u}_2|^{1/2}|A\delta\mathbf{u}|\ \|\delta\mathbf{u}\| \\ &\quad + c(1+\epsilon)|A\mathbf{u}_2|\ |A\delta\mathbf{u}|\ |\delta\mathbf{u}|^{1/2}\|\delta\mathbf{u}\|^{1/2} \\ &\leq \frac{c_1(1+\epsilon)}{Re^{-1}\gamma\epsilon}\left(|A\mathbf{u}_2|^2\|\delta\mathbf{u}\|^2 + |A\mathbf{u}_2|^2|\delta\mathbf{u}|\ \|\delta\mathbf{u}\|\right) \\ &\quad + \frac{Re^{-1}\gamma\epsilon}{2}|A\mathbf{u}|^2 \\ &\leq \frac{2c_1(1+\epsilon)}{Re^{-1}\gamma\epsilon}|A\mathbf{u}_2|^2\|\delta\mathbf{u}\|^2 + \frac{Re^{-1}\gamma\epsilon}{2}|A\mathbf{u}|^2\end{aligned}$$

We add

$$\frac{2c_1(1+\epsilon)}{Re^{-1}\gamma\epsilon^2}|A\mathbf{u}_2|^2|\delta\mathbf{u}|^2$$

to the left hand side. After simplifying we obtain

$$\frac{d}{dt}\|\delta\mathbf{u}\|_\epsilon^2 + Re^{-1}\gamma\|A^{1/2}\delta\mathbf{u}\|_\epsilon^2 \leq \frac{2c_1(1+\epsilon)}{Re^{-1}\gamma\epsilon^2}|A\mathbf{u}_2|^2\|\delta\mathbf{u}\|_\epsilon^2. \tag{2.10}$$

We drop the viscous term and apply Gronwall

$$\|\delta\mathbf{u}(t)\|_\epsilon^2 \leq \|\delta\mathbf{u}(0)\|_\epsilon^2 exp\left(\int_0^t \frac{2c_1(1+\epsilon)}{Re^{-1}\gamma\epsilon^2}|A\mathbf{u}_2(\tau)|^2\ d\tau\right). \tag{2.11}$$

Since $\mathbf{u} \in L^2([0,T], V^4)$ we conclude that the solution depend continuously on the initial data. If the initial conditions are equal, then the solutions are in fact the same. $\square$

We now prove a regularity theorem for strong solutions of the NS-$\alpha\beta$ equations.

Theorem 2.6. *(Regularity) Let $s \geq 3$ and let $\mathbf{u}$ be a strong solution to the NS-$\alpha\beta$ equations with $\mathbf{u} \in L^2([0,T]; V^{s-1})$. Suppose the initial condition and forcing function satisfy $\mathbf{u}_o \in V^s$ and $\mathbf{f} \in L^\infty([0,\infty); V^{s-1})$. Then we conclude that the solution satisfies $\mathbf{u} \in L^\infty([0,T]; V^{s+1}) \cap L^2([0,T]; V^{s+2})$.*

Proof. We began by taking the inner product of equation (2.10) with $A^s\mathbf{u}$, where $s \geq 3$. After some manipulations which are standard by now we obtain

$$\frac{1}{2}\frac{d}{dt}\|A^{s/2}\mathbf{u}\|_\epsilon^2 + Re^{-1}\|A^{(s+1)/2}\mathbf{u}\|_{\epsilon\gamma}^2 \leq |\langle \widetilde{B}(\mathbf{u},\mathbf{v}), A^s\mathbf{u}\rangle| + \langle \mathbf{f}, A^s\mathbf{u}\rangle \tag{2.12}$$

Appealing to Lemma's 2.6 and 2.4 we bound the tri-linear term as follows

$$\begin{aligned} |\langle \widetilde{B}(\mathbf{u},\mathbf{v}), A^s\mathbf{u}\rangle| &= |A^{(s-1)/2}\langle \widetilde{B}(\mathbf{u},\mathbf{v}), A^{(s+1)/2}\mathbf{u}\rangle| \\ &\leq c_1|A^{(s-1)/2}\mathbf{u}|\;|A^{s/2}\mathbf{v}|\;|A^{(s+1)/2}\mathbf{u}| \\ &\leq c_1(1+\epsilon)|A^{(s-1)/2}\mathbf{u}|\;|A^{s/2+1}\mathbf{u}|\;|A^{(s+1)/2}\mathbf{u}|. \end{aligned}$$

Then by Poincaré

$$|\langle \widetilde{B}(\mathbf{u},\mathbf{v}), A^s\mathbf{u}\rangle| \leq \frac{c_2(1+\epsilon)^2}{2Re^{-1}\gamma\epsilon}|A^{(s-1)/2}\mathbf{u}|^2|A^{(s+1)/2}\mathbf{u}|^2 + \frac{Re^{-1}\gamma\epsilon}{2}|A^{s/2+1}\mathbf{u}|^2$$

and

$$\langle \mathbf{f}, A^s\mathbf{u}\rangle = \langle A^{(s-1)/2}\mathbf{f}, A^{(s+1)/2}\mathbf{u}\rangle \leq \frac{Re}{2}|A^{(s-1)/2}\mathbf{f}|^2 + \frac{Re^{-1}}{2}|A^{(s+1)/2}\mathbf{u}|^2.$$

As in section 2.3 for Gr we write $Gr_s = \|\mathbf{f}\|_{L^\infty([0,\infty];V^s)}$ and so $|A^{(s-1)/2}\mathbf{f}|^2 \leq Gr_{s-1}$.

Thus we obtain

$$\frac{d}{dt}\|A^{s/2}\mathbf{u}\|_\epsilon^2 + Re^{-1}\|A^{(s+1)/2}\mathbf{u}\|_{\epsilon\gamma}^2 \leq \frac{c_2(1+\epsilon)^2}{Re^{-1}\gamma\epsilon}|A^{(s-1)/2}\mathbf{u}|^2|A^{(s+1)/2}\mathbf{u}|^2 + ReGr_{s-1}^2 \quad (2.13)$$

Next, we add $\frac{c_2(1+\epsilon)^2}{Re^{-1}\gamma\epsilon}|A^{(s-1)/2}\mathbf{u}|^2|A^{s/2}\mathbf{u}|^2$ to the left side and neglect the viscous term.

$$\frac{d}{dt}\|A^{s/2}\mathbf{u}\|_\epsilon^2 \leq \frac{c_2(1+\epsilon)^2}{Re^{-1}\gamma\epsilon}|A^{(s-1)/2}\mathbf{u}|^2\|A^{s/2}\mathbf{u}\|_\epsilon^2 + ReGr_{s-1}^2 \quad (2.14)$$

We will apply Gronwall with

$$g(t) = \int_0^t \frac{c_2(1+\epsilon)^2}{Re^{-1}\gamma\epsilon}|A^{(s-1)/2}\mathbf{u}(\tau)|^2 \, d\tau.$$

Note that by our assumption that $\mathbf{u} \in L^2([0,T];V^{s-1})$ and so $g(t)$ is finite for all $t < T$. Thus,

$$\begin{aligned}\|A^{s/2}\mathbf{u}(t)\|_\epsilon^2 &\leq e^{g(t)}\|A^{s/2}\mathbf{u}(0)\|_\epsilon^2 + e^{g(t)}\int_0^t e^{-g(\tau)}Gr_{s-1}^2 \, d\tau \\ &\leq e^{g(T)}\|A^{s/2}\mathbf{u}(0)\|_\epsilon^2 + e^{g(T)}Gr_{s-1}^2 \quad (2.15)\end{aligned}$$

Therefore we conclude that $\mathbf{u} \in L^\infty([0,T];V^{s+1})$ and $L^\infty([0,T];V^s)$. Now we return to equation (2.13). We integrate over $[0,T]$ and drop $\|A^{s/2}\mathbf{u}(T)\|_\epsilon^2$.

$$\begin{aligned}\int_0^T \|A^{(s+1)/2}\mathbf{u}(\tau)\|_{\epsilon\gamma}^2 \, d\tau &\leq Re\|A^{s/2}\mathbf{u}(0)\|_\epsilon^2 \\ &+ \frac{c_2(1+\epsilon)^2}{Re^{-2}\gamma\epsilon}\int_0^T |A^{(s-1)/2}\mathbf{u}(\tau)|^2|A^{(s+1)/2}\mathbf{u}(\tau)|^2 \, d\tau + Re^2Gr_{s-1}^2T \quad (2.16)\end{aligned}$$

From equation (2.15) we have shown that $|A^{(s+1)/2}\mathbf{u}(\tau)|^2$ is bounded over $[0,T]$. Thus,

$$\int_0^T \|A^{(s+1)/2}\mathbf{u}(\tau)\|_{\epsilon\gamma}^2\, d\tau \le Re\|A^{s/2}\mathbf{u}(0)\|_{\epsilon}^2 + \frac{c_2(1+\epsilon)^2}{Re^{-2}\gamma\epsilon}\|\mathbf{u}\|_{L^\infty([0,T];V^{s+1})}\|\mathbf{u}\|_{L^2([0,T];V^{s-1})} + Re^2 Gr_{s-1}^2 T \quad (2.17)$$

Therefore we have shown that $\mathbf{u} \in L^2([0,T];V^{s+2})$ a significant increase in regularity!

$\square$

It should be noted that in setting $\gamma = 1$ these results extend the LANS-α equations. As mentioned before in [9] they show the existence of what they call regular solutions for the LANS-α equations. These regular solutions are not quite as “weak” as weak solutions to NS, but they are not strong enough to be strong solutions. Uniqueness and continuous dependence on the initial data were already established in [9] for the LANS-α equations. Regularity and strong solutions are new for the LANS-α equations.

2.5 Determining Nodes and Nodal Distance

Let $\mathcal{E} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ be a finite number of distinct points in the domain. We refer to set $\mathcal{E}$ as a set of nodes. The nodal distance of a set of nodes is a measure how far away a point could be from a node. So for each $\mathbf{x} \in \Omega$ we define

$$d(\mathbf{x}) = \min_{1\le i\le N} \|\mathbf{x}_i - \mathbf{x}\| \quad (2.1)$$

and

$$d_{\mathcal{E}} = \sup_{x\in\Omega} d(\mathbf{x}). \quad (2.2)$$

We refer to $d_{\mathcal{E}}$ as the nodal distance or nodal value. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions to NS-$\alpha\beta$ with forcing functions $\mathbf{f}_1$ and $\mathbf{f}_2$ respectively. We assume that the two forcing functions have the same asymptotic behavior. I.e.,

$$\lim_{t\to\infty}\int_\Omega |\mathbf{f}_1 - \mathbf{f}_2|^2\, d\mathbf{x} = 0$$

For convenience we also assume that

$$Gr := \|\mathbf{f}_1\|_{L^2([0,\infty);H)} = \|\mathbf{f}_2\|_{L^2([0,\infty);H)}$$

where we refer to Gr as the Grashof number. Traditionally the Grashof number is a non-dimensional parameter that represents the strength of the forcing. For the current discussion Gr serves this purpose. Suppose for the two solutions $\mathbf{u}_1$ and $\mathbf{u}_2$, that $\mathbf{u}_1$ and $\mathbf{u}_2$ agree asymptotically on $\mathcal{E}$ and that $A^{1/2}\mathbf{u}_1$ and $A^{1/2}\mathbf{u}_2$ agree asymptotically on the set $\mathcal{E}$. I.e.,

$$\lim_{t\to\infty} |\mathbf{u}_1(\mathbf{x}_i, t) - \mathbf{u}_2(\mathbf{x}_i, t)| = \lim_{t\to\infty} |A^{1/2}\mathbf{u}_1(\mathbf{x}_i, t) - A^{1/2}\mathbf{u}_2(\mathbf{x}_i, t)| = 0 \quad \text{for } i = 1, 2, \ldots, N. \tag{2.3}$$

We make the following definition for determining nodes for the NS-$\alpha\beta$ equations.

Definition 2.1. (Determining nodes for the Navier–Stokes-$\alpha\beta$ equations) Let $\mathcal{E}$ be a set of nodes and let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions of the Navier–Stokes-$\alpha\beta$ equations that satisfy condition (2.3). We say that $\mathcal{E}$ is a set of determining nodes for the NS-$\alpha\beta$ equations, if we can conclude that

$$\lim_{t\to\infty} \|\mathbf{u}_1 - \mathbf{u}_2\|_V = 0.$$

Recall that $\|\mathbf{u}\|_V = \|A^{1/2}\mathbf{u}\|_H$. The key point of Definition 2.1 is that we require the same order of convergence on the domain Ω as we assumed on $\mathcal{E}$. I.e., since we assume that $A^{1/2}\mathbf{u}_1$ and $A^{1/2}\mathbf{u}_2$ agree asymptotically on $\mathcal{E}$ we require that $\|A^{1/2}\mathbf{u}_1 - A^{1/2}\mathbf{u}_2\|_H \to 0$ as $t \to \infty$ in order for $\mathcal{E}$ to be a set of determining nodes.

An amazing result is that a set of nodes are determining if their nodal distance is small enough. The proof hinges on two key lemmas, generalized Gronwall and the following lemma, which is proved in [10].

Lemma 2.2. *Let $\mathcal{E}$ be a finite set of of points in Ω and let $\mathbf{w} \in V^2$. We write $\eta(\mathbf{w})^2 = \max_{\mathbf{x}_i \in \mathcal{E}} |\mathbf{w}(\mathbf{x}_i)|^2$. Then we can conclude the following.*

$$\|\mathbf{w}\|^2 \leq c d_{\mathcal{E}}^{-1/2} \eta(\mathbf{w})^2 + c d_{\mathcal{E}}^{1/2} |A\mathbf{w}|^2$$

where the constant c does not depend on $\mathbf{u}$, $\mathbf{f}$, ϖ, ϵ, γ, or Re.

In fact in the following Theorem we prove a slightly stronger result, which is that for an appropriate nodal distance

$$\lim_{t \to \infty} |A\mathbf{u}_1 - A\mathbf{u}_2| = 0.$$

Theorem 2.3. *(Determining Nodes) Suppose that we are given a set of nodes, $\mathcal{E} \in \Omega$ with nodal distance $d_{\mathcal{E}}$. We also assume that $\gamma \leq 1$ for the NS-$\alpha\beta$ equations. If the nodal distance $d_{\mathcal{E}}$ satisfies*

$$d_{\mathcal{E}} < \frac{c\gamma^6 \epsilon^4}{(1+\epsilon)^4 Re^8 Gr^4}, \tag{2.4}$$

then we may conclude that $\mathcal{E}$ is a set of determining nodes for the NS-$\alpha\beta$ equations. Note that c is some constant that does not depend on any of present parameters.

Proof. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions to NS-$\alpha\beta$ with respective forcing functions $\mathbf{f}_1$ and $\mathbf{f}_2$. We write $\delta\mathbf{u} = \mathbf{u}_2 - \mathbf{u}_1$, we correspondingly define $\delta\mathbf{w}$, $\delta\mathbf{v}$, $\delta\varpi$, and $\delta\mathbf{f}$. Then by subtracting the equation for $\mathbf{u}_2$ from the equation for $\mathbf{u}_1$, we find that $\delta\mathbf{u}$ satisfies:

$$\frac{\partial}{\partial t}\delta\mathbf{v} + \operatorname{curl}\mathbf{v}_2 \times \mathbf{u}_2 - \operatorname{curl}\mathbf{v}_1 \times \mathbf{u}_1 + \operatorname{grad}\delta\varpi = Re^{-1}\Delta\delta\mathbf{w} + \delta\mathbf{f} \tag{2.5}$$

Since $\operatorname{curl}\delta\mathbf{v} \times \mathbf{u}_2 + \operatorname{curl}\mathbf{v}_1 \times \delta\mathbf{u} = \operatorname{curl}\mathbf{v}_2 \times \mathbf{u}_2 - \operatorname{curl}\mathbf{v}_1 \times \mathbf{u}_1$ we may rewrite the previous equation as

$$\frac{\partial}{\partial t}\delta\mathbf{v} + \operatorname{curl}\delta\mathbf{v} \times \mathbf{u}_2 + \operatorname{curl}\mathbf{v}_1 \times \delta\mathbf{u} + \operatorname{grad}\delta\varpi = Re^{-1}\Delta\delta\mathbf{w} + \delta\mathbf{f} \tag{2.6}$$

We project onto divergence free fields.

$$\frac{\partial}{\partial t}\delta\mathbf{v} + Re^{-1}A\delta\mathbf{w} = \widetilde{B}(\mathbf{u}_2, \delta\mathbf{v}) + \widetilde{B}(\delta\mathbf{u}, \mathbf{v}_1) + \delta\mathbf{f} \tag{2.7}$$

Then, taking the L^2 inner-product with $A\delta\mathbf{u}$ we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^{1/2}\delta\mathbf{u}\|_\epsilon^2 + Re^{-1}\left\langle A\delta\mathbf{w}, A\delta\mathbf{u}\right\rangle &= \\ \langle\widetilde{B}(\mathbf{u}_2, \delta\mathbf{v}), A\delta\mathbf{u}\rangle + \langle\widetilde{B}(\delta\mathbf{u}, \mathbf{v}_1), A\delta\mathbf{u}\rangle + \langle\delta\mathbf{f}, A\delta\mathbf{u}\rangle\end{aligned} \tag{2.8}$$

We use Lemma 2.3 on the viscous term and Lemma 2.7(ii) and (iv)(with antisymmetry) on the non-linear terms to obtain the following inequality:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|A^{1/2}\delta\mathbf{u}\|_\epsilon^2 + Re^{-1}\gamma\|A\delta\mathbf{u}\|_\epsilon^2 &\le \\ c_3|A\mathbf{u}_2|\ \|\delta\mathbf{v}\|\ |A\delta\mathbf{u}| + c_3|A\delta\mathbf{u}|\ |\mathbf{v}_1|\ |A^{3/2}\delta\mathbf{u}| + \langle\delta\mathbf{f}, A\delta\mathbf{u}\rangle\end{aligned} \tag{2.9}$$

Applying Cauchy-Schwarz, Young, Poincaré, and Lemma 2.4 we obtain the following inequalities:

$$\begin{aligned}
\langle \delta\mathbf{f}, A\delta\mathbf{u}\rangle &\le \frac{Re^{-1}\gamma}{2}|A\delta\mathbf{u}|^2 + \frac{1}{2Re^{-1}\gamma}|\delta\mathbf{f}|^2 \\
c_3|A\mathbf{u}_2|\ \|\delta\mathbf{v}\|\ |A\delta\mathbf{u}| &\le c_3(1+\epsilon)|A\mathbf{u}_2|\ |A^{3/2}\delta\mathbf{u}|\ |A\delta\mathbf{u}| \\
&\le \frac{Re^{-1}\gamma\epsilon}{4}|A^{3/2}\delta\mathbf{u}|^2 + \frac{c_3^2(1+\epsilon)^2}{Re^{-1}\gamma\epsilon}|A\mathbf{u}_2|^2|A\delta\mathbf{u}|^2 \\
c_3|A\delta\mathbf{u}|\ |\mathbf{v}_1|\ |A^{3/2}\delta\mathbf{u}| &\le c_3(1+\epsilon)|A\delta\mathbf{u}|\ |A\mathbf{u}_1|\ |A^{3/2}\delta\mathbf{u}| \\
&\le \frac{Re^{-1}\gamma\epsilon}{4}|A^{3/2}\delta\mathbf{u}|^2 + \frac{c_3^2(1+\epsilon)^2}{Re^{-1}\gamma\epsilon}|A\mathbf{u}_1|^2|A\delta\mathbf{u}|^2
\end{aligned}$$

Applying these inequalities to (2.9) we find

$$\frac{d}{dt}\|A^{1/2}\delta\mathbf{u}\|_\epsilon^2 + Re^{-1}\gamma\|A\delta\mathbf{u}\|_\epsilon^2 \le \frac{2c_3^2(1+\epsilon)^2}{Re^{-1}\gamma\epsilon}(|A\mathbf{u}_1|^2 + |A\mathbf{u}_2|^2)|A\delta\mathbf{u}|^2 + \frac{1}{Re^{-1}\gamma}|\delta\mathbf{f}|^2.$$

To simplify the above equation we set

$$F = \frac{2c_3^2(1+\epsilon)^2}{Re^{-1}\gamma\epsilon^2}(|A\mathbf{u}_1|^2 + |A\mathbf{u}_2|^2)$$

and write

$$\frac{d}{dt}\|A^{1/2}\delta\mathbf{u}\|_\epsilon^2 + Re^{-1}\gamma\|A\delta\mathbf{u}\|_\epsilon^2 \le \epsilon F|A\delta\mathbf{u}|^2 + \frac{1}{Re^{-1}\gamma}|\delta\mathbf{f}|^2. \tag{2.10}$$

For the next step we want to reduce the order of A in the norm of $|A\delta\mathbf{u}|_\epsilon^2$. Instead of using Poincaré we use Lemma 2.2 on both terms to obtain the following inequality.

$$Re^{-1}\gamma|A\delta\mathbf{u}|_\epsilon^2 \ge cd_{\mathcal{E}}^{-1/2}Re^{-1}\gamma|A^{1/2}\delta\mathbf{u}|_\epsilon^2 - d_{\mathcal{E}}^{-1}Re^{-1}\gamma(\eta(\delta\mathbf{u})^2 + \epsilon\eta(A^{1/2}\delta\mathbf{u})^2)$$

We apply this to equation (2.10) and obtain

$$\frac{d}{dt}\|A^{1/2}\delta\mathbf{u}\|_\epsilon^2 + cd_{\mathcal{E}}^{-1/2}Re^{-1}\gamma\|A^{1/2}\delta\mathbf{u}\|_\epsilon^2 - \epsilon F|A\delta\mathbf{u}|^2 \le$$
$$d_{\mathcal{E}}^{-1}Re^{-1}\gamma(\eta(\delta\mathbf{u})^2 + \epsilon\eta(A^{1/2}\delta\mathbf{u})^2) + \frac{1}{Re^{-1}\gamma}|\delta\mathbf{f}|^2.$$

The above equation is almost in a form where we can apply the generalized Gronwall inequality. To get a usable form we subtract $F|A^{1/2}\mathbf{u}|^2$ from the left-hand side.

$$\frac{d}{dt}\|A^{1/2}\delta\mathbf{u}\|_\epsilon^2 + (cd_{\mathcal{E}}^{-1/2}Re^{-1}\gamma - \epsilon F)\|A^{1/2}\delta\mathbf{u}\|_\epsilon^2 \le$$
$$d_{\mathcal{E}}^{-1}Re^{-1}\gamma(\eta(\delta\mathbf{u})^2 + \epsilon\eta(A^{1/2}\delta\mathbf{u})^2) + \frac{1}{Re^{-1}\gamma}|\delta\mathbf{f}|^2 \quad (2.11)$$

The equation is now in a form to use the generalized Gronwall inequality. We make the following identifications

$$\xi = \|A^{1/2}\delta\mathbf{v}\|_\epsilon^2$$
$$a = cd_{\mathcal{E}}^{-1/2}Re^{-1}\gamma - \epsilon F$$
$$b = d_{\mathcal{E}}^{-1}Re^{-1}\gamma(\eta(\delta\mathbf{u})^2 + \epsilon\eta(A^{1/2}\delta\mathbf{u})^2) + \frac{1}{Re^{-1}\gamma}|\delta\mathbf{f}|^2$$

By the assumptions we have that

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T} b^+(\tau)\,d\tau = 0.$$

Due to our S_1^2 estimate (2.13), we conclude the following.

$$\limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T} a^-(\tau)d\tau \le \limsup_{t\to\infty}\frac{1}{T}\int_t^{t+T}\epsilon F(\tau)d\tau$$
$$\le \frac{4c_3^2(1+\epsilon)^2}{Re^{-1}\gamma\epsilon}S_1^2 < \infty.$$

All that remains is to verify condition (2.11). As in section 2.3 we set $T = Re\gamma^{-1}$.

We can make this choice of T, since the generalized Gronwall Lemma only needs the conditions to hold for some T.

$$\begin{aligned}
\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} a(\tau)\ d\tau &= \liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \Big(c d_{\mathcal{E}}^{-1/2} Re^{-1}\gamma - \epsilon F(\tau) \Big)\, d\tau \\
&= c d_{\mathcal{E}}^{-1/2} Re^{-1}\gamma - \liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \epsilon F(\tau) d\tau \\
&\geq c d_{\mathcal{E}}^{-1/2} Re^{-1}\gamma - \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \epsilon F(\tau) d\tau \\
&\geq c d_{\mathcal{E}}^{-1/2} Re^{-1}\gamma - \frac{4c_3^2(1+\epsilon)^2}{Re^{-1}\gamma\epsilon} S_1^2.
\end{aligned}$$

Thus in order for

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} a(\tau) d\tau > 0$$

we must have

$$c d_{\mathcal{E}}^{-1/2} Re^{-1}\gamma - \frac{4c_3^2(1+\epsilon)^2}{Re^{-1}\gamma\epsilon} S_1^2 > 0. \tag{2.12}$$

Solving for $d_{\mathcal{E}}$ and substituting in our bound for S_1^2 from (2.13) we find

$$d_{\mathcal{E}} < \frac{c\gamma^6\epsilon^4}{(1+\epsilon)^4 Re^8 Gr^4}. \tag{2.13}$$

where c is some constant that does not depend on any of present parameters. □

2.6 Discussion of the Nodal Distance

$$d_{\mathcal{E}} < \frac{c\gamma^6\epsilon^4}{(1+\epsilon)^4 Re^8 Gr^4} \tag{2.1}$$

It is very important to note that our estimate for the nodal distance is not sharp. Thus the above estimate merely serves as an lower bound for largest possible $d_{\mathcal{E}}$. The other important comment we make before proceeding is that these results do not exist for 3d NS. The same difficulties with well-posedness also prevent determination of a modal number.

We now discuss the functional behavior of our bound for the nodal distance. Since it is an upper bound, in this discussion we assume that we choose the largest possible distance given by the inequality. Thus when we say that $d_{\mathcal{E}}$ increases or decreases we are talking about the choice of largest possible distance based on our bound. We investigate the role of the Reynolds number first. As $Re \to \infty$, $d_{\mathcal{E}} \to 0$ and as $Re \to 0^+$, $d_{\mathcal{E}} \to \infty$. It is not surprising that one would need better resolution for a more turbulent flow. As far as the increase in nodal distance as Re decreases, this is most likely due to a highly viscous flow being over damped.

Similar behavior is show with the Grashof number. We see that as the Grashof decreases the nodal distance increases. This is not surprising since a Grashof number of zero indicates no forcing and thus due to dissipation any solution will decay to zero. As the Grashof number is increased the nodal distance decreases. This seems reasonable as in increase in forcing would compete with the viscous damping. Although since the bound is not sharp this may be an artifact.

The parameter γ represents the ratio of the filtering on the dissipative term to the filtering on the convective (non-linear) terms. Recall that we assume $0 < \gamma \leq 1$. In that case of $\gamma = 1$ we recover the LANS-α equations on the periodic domain. In [20], Korn derived a nodal distance for the LANS-α equations. Their result (note this result is for equations that have not been non-dimensionalized) is

$$d_{\mathcal{E}} = \frac{\alpha^6}{3c\nu(\lambda_1^{-1} + \alpha^2)^2(\alpha^2 + 1)\max\{1, \lambda_1^{-1/2}\}} \left[(2\lambda_1 + 4)Gr^2\lambda_1 + \frac{c(\lambda_1 + \alpha^2)^4 Gr^6}{\alpha^1 2\lambda_1^2}\right]^{-1}$$

First we believe that the $[]^{-1}$ is a typo and that it should be to the -2 power. Plus, their nodal distance is inversely proportional to the viscosity, which seems non-physical. Besides small errors in the derivation we wee that our result is a nice improvement. Back to (2.1) we see for $\epsilon \to 0^+$ that $d_{\mathcal{E}} \to 0$. While not surprising, the result is disappointing. Had we found $d_{\mathcal{E}} > 0$, we may have had a way of giving a bound for the nodal distance for the of 3d NS. Essentially we are bounded by similar difficulties that arise with the 3d Navier–Stokes equations. We also see that as ϵ increase the nodal distance increases, though we should be careful not to interpret the result as $\epsilon \to \infty$ since we are on a bounded domain.

Interestingly as $\gamma \to 0^+$, $d_{\mathcal{E}} \to 0$. In terms of the NS-$\alpha\beta$ equations having $\gamma \to 0^+$ reduces the filtering on the forcing term and thus decreases regularizing effect of dissipation. In the situation of $\gamma = 0$, determining a finite modal number would actually be more difficult than the case of 3d Navier–Stokes! This is due to the presence of higher order derivatives in the non-linear term with no corresponding higher derivatives on the dissipative term.

2.7 Determining Modes

Let $\{\mathbf{w}_j\}_1^\infty$ be the eigenfunctions of the stokes operator A and let P_m be the projection operator onto the first m eigenfunctions. Correspondingly we define the projection $Q_m = I - P_m$.

Since the smallest eigenvalue of $Q_m\mathbf{u}$ is λ_{m+1} we have the following version of Poincaré.

$$|A^s Q_m \mathbf{u}| \leq \lambda_{m+1}^{-1/2} |A^{s+1/2} Q_m \mathbf{u}| \tag{2.1}$$

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions to NS-$\alpha\beta$ with forcing functions $\mathbf{f}_1$ and $\mathbf{f}_2$ respectively. We assume that the two forcing functions have the same asymptotic behavior.

$$\lim_{t\to\infty}\int_\Omega |\mathbf{f}_1 - \mathbf{f}_2|^2\, d\mathbf{x} = 0$$

We say that m is the number of determining modes or the modal value if

$$\lim_{t\to\infty}\int_\Omega |\mathrm{P}_m\mathbf{v}_1 - \mathrm{P}_m\mathbf{v}_2|^2\, d\mathbf{x} = 0$$

implies

$$\lim_{t\to\infty}\int_\Omega |\mathbf{v}_1 - \mathbf{v}_2|^2\, d\mathbf{x} = 0.$$

Theorem 2.1. *(Brady)(Determining Modes for the NS-$\alpha\beta$ equations) Let $\gamma \le 1$ for the NS-$\alpha\beta$ equations. Let m be the least integer that satisfies*

$$\lambda_{m+1} \ge \frac{c(2+\epsilon)^2 Re^4 Gr^2}{\gamma^3\epsilon}, \tag{2.2}$$

then m is an upper bound for the number of determining modes. Note that c is some constant that does do not depend on any of the present parameters.

Before we proceed with a proof we want to comment on finding the number m. From [8] it can be shown that

$$\lambda_m \le \left[(4\pi)^{-1/3}\left(\frac{m}{2}+1\right)^{1/3} + \frac{\sqrt{3}}{2}\right]^2.$$

Thus

$$\left[(4\pi)^{-1/3}\left(\frac{m+1}{2}+1\right)^{1/3} + \frac{\sqrt{3}}{2}\right]^2 \ge \frac{c(2+\epsilon)^2 Re^4 Gr^2}{\gamma^3\epsilon}. \tag{2.3}$$

If fact as discussed in [8] this relationship for m holds asymptotically. Therefore we have

$$m \sim \frac{(2+\epsilon)^3 Re^6 Gr^3}{\gamma^{9/4}\epsilon^{3/2}} \tag{2.4}$$

Proof. We begin as we did in the case of finding an estimate of the nodal distance. Let $\delta\mathbf{u} = \mathbf{u}_2 - \mathbf{u}_1$, we correspondingly define $\delta\mathbf{w}$, $\delta\mathbf{v}$, $\delta\varpi$, and δf. Then by subtracting the equation for $\mathbf{u}_2$ from the equation for $\mathbf{u}_1$ and projecting onto divergence free vector fields, we obtain the following equation.

$$\frac{\partial}{\partial t}\delta\mathbf{v} + Re^{-1}A\delta\mathbf{w} = \widetilde{B}(\mathbf{u}_2, \delta\mathbf{v}) + \widetilde{B}(\delta\mathbf{u}, \mathbf{v}_1) + \delta\mathbf{f} \tag{2.5}$$

We take the L^2 inner product of the above equation with $Q_m\delta\mathbf{v}$.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|Q_m\delta\mathbf{v}|^2 + Re^{-1}\langle A\delta\mathbf{w}, Q_m\delta\mathbf{v}\rangle = \\ \langle\widetilde{B}(\mathbf{u}_2,\delta\mathbf{v}), Q_m\delta\mathbf{v}\rangle + \langle\widetilde{B}(\delta\mathbf{u},\mathbf{v}_1), Q_m\delta\mathbf{v}\rangle + \langle\delta\mathbf{f}, Q_m\delta\mathbf{v}\rangle\end{aligned} \tag{2.6}$$

We apply Lemma 2.3 on the viscous term to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|Q_m\delta\mathbf{v}|^2 + Re^{-1}\gamma\|Q_m\delta\mathbf{v}\|^2 \le \\ \langle\widetilde{B}(\mathbf{u}_2,\delta\mathbf{v}), Q_m\delta\mathbf{v}\rangle + \langle\widetilde{B}(\delta\mathbf{u},\mathbf{v}_1), Q_m\delta\mathbf{v}\rangle + \langle\delta\mathbf{f}, Q_m\delta\mathbf{v}\rangle\end{aligned} \tag{2.7}$$

Since $I = Q_m + P_m$, we expand the tri-linear terms as follows.

$$\langle\widetilde{B}(\mathbf{u}_2,\delta\mathbf{v}), Q_m\delta\mathbf{v}\rangle = \langle\widetilde{B}(\mathbf{u}_2, Q_m\delta\mathbf{v}), Q_m\delta\mathbf{v}\rangle + \langle\widetilde{B}(\mathbf{u}_2, P_m\delta\mathbf{v}), Q_m\delta\mathbf{v}\rangle$$

$$\langle\widetilde{B}(\delta\mathbf{u},\mathbf{v}_1), Q_m\delta\mathbf{v}\rangle = \langle\widetilde{B}(Q_m\delta\mathbf{u}, \mathbf{v}_1), Q_m\delta\mathbf{v}\rangle + \langle\widetilde{B}(P_m\delta\mathbf{u}, \mathbf{v}_1), Q_m\delta\mathbf{v}\rangle$$

We bound these terms using Lemma's 2.4 and 2.7, Young's inequality, and Poincaré.

$$
\begin{aligned}
\langle \widetilde{B}(\mathbf{u}_2, Q_m\delta\mathbf{v}), Q_m\delta\mathbf{v}\rangle &\le c_3|A\mathbf{u}_2|\ \|Q_m\delta\mathbf{v}\|\ |Q_m\delta\mathbf{v}| \\
&\le \frac{16c_3^2}{Re^{-1}\gamma}|A\mathbf{u}_2|^2|Q_m\delta\mathbf{v}|^2 + \frac{Re^{-1}\gamma}{32}\|Q_m\delta\mathbf{v}\|^2 \\
\langle \widetilde{B}(\mathbf{u}_2, P_m\delta\mathbf{v}), Q_m\delta\mathbf{v}\rangle &\le c_3|A\mathbf{u}_2|\ |P_m\delta\mathbf{v}|\ \|Q_m\delta\mathbf{v}\| \\
&\le \frac{16c_3^2}{Re^{-1}\gamma}|A\mathbf{u}_2|^2|P_m\delta\mathbf{v}|^2 + \frac{Re^{-1}\gamma}{32}\|Q_m\delta\mathbf{v}\|^2 \\
\langle \widetilde{B}(Q_m\delta\mathbf{u}, \mathbf{v}_1), Q_m\delta\mathbf{v}\rangle &\le c_3|AQ_m\delta\mathbf{u}|\ |\mathbf{v}_1|\ |Q_m\delta\mathbf{v}| \\
&\le \frac{16c_3^2}{Re^{-1}\gamma}|AQ_m\mathbf{u}_2|^2|\mathbf{v}_1|^2 + \frac{Re^{-1}\gamma}{32}\|Q_m\delta\mathbf{v}\|^2 \\
&\le \frac{16c_3^2(1+\epsilon)^2}{Re^{-1}\gamma}|AQ_m\mathbf{u}_2|^2|A\mathbf{u}_1|^2 + \frac{Re^{-1}\gamma}{32}\|Q_m\delta\mathbf{v}\|^2 \\
\langle \widetilde{B}(P_m\delta\mathbf{u}, \mathbf{v}_1), Q_m\delta\mathbf{v}\rangle &\le c_3|AP_m\delta\mathbf{u}|\ |\mathbf{v}_1|\ \|Q_m\delta\mathbf{v}\| \\
&\le \frac{16c_3^2}{Re^{-1}\gamma}|AP_m\mathbf{u}_2|^2|\mathbf{v}_1|^2 + \frac{Re^{-1}\gamma}{32}\|Q_m\delta\mathbf{v}\|^2
\end{aligned}
$$

We bound the term with the forcing function as follows.

$$
\langle \delta\mathbf{f}, Q_m\delta\mathbf{v}\rangle \le \frac{Re^{-1}\gamma}{32}\|Q_m\delta\mathbf{v}\| + \frac{16}{Re^{-1}\gamma}|\delta\mathbf{f}|^2
$$

We make the following identifications.

$$
b = \frac{32c_3^2}{Re^{-1}\gamma}|A\mathbf{u}_2|^2|P_m\delta\mathbf{v}|^2 + \frac{32c_3^2}{Re^{-1}\gamma}|AP_m\mathbf{u}_2|^2|\mathbf{v}_1|^2 + \frac{32}{Re^{-1}\gamma}|\delta\mathbf{f}|^2 \tag{2.8}
$$

$$
F = \frac{32c_3^2}{Re^{-1}\gamma}|A\mathbf{u}_2|^2 + \frac{32c_3^2(1+\epsilon)^2}{Re^{-1}\gamma}|A\mathbf{u}_1|^2 \tag{2.9}
$$

Thus applying the bounds with the above identifications we obtain the following

$$
\frac{d}{dt}|Q_m\delta\mathbf{v}|^2 + Re^{-1}\gamma\|Q_m\delta\mathbf{v}\|^2 - F|Q_m\delta\mathbf{v}|^2 \le b \tag{2.10}
$$

Now we use (2.1) on the viscous term to obtain.

$$\frac{d}{dt}|Q_m \delta \mathbf{v}|^2 + \left(\lambda_{m+1} Re^{-1}\gamma - F\right)|Q_m \delta \mathbf{v}|^2 \le b \tag{2.11}$$

We now have the equation in the form where we can apply generalized Gronwall. We identify $a = (\lambda_{m+1} Re^{-1}\gamma - F)$. By the assumptions give we have that

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} b^+(\tau)d\tau = 0.$$

We now check the conditions for a, and as before we set $T = Re\gamma^{-1}$.

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} a^-(\tau)\ d\tau \le \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} F(\tau)d\tau = \\ \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \left(\frac{32c_3^2}{Re^{-1}\gamma}|A\mathbf{u}_2(\tau)|^2 + \frac{32c_3^2(1+\epsilon)^2}{Re^{-1}\gamma}|A\mathbf{u}_1|^2\right)d\tau \\ \le \frac{32c_3^2(2+\epsilon)^2}{Re^{-1}\gamma\epsilon}S_1^2 < \infty\end{aligned}$$

Where S_1^2 is given by (2.13). We now verify the final condition and in doing so we shall obtain an estimate for the number of determining modes.

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} a(\tau)d\tau = \lambda_{m+1} Re^{-1}\gamma - \liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} F(\tau)d\tau \\ \ge \lambda_{m+1} Re^{-1}\gamma - \limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \frac{c_6}{Re^{-1}\gamma}\left(\frac{32c_3^2}{Re^{-1}\gamma}|A\mathbf{u}_2(\tau)|^2 + \frac{32c_3^2(1+\epsilon)^2}{Re^{-1}\gamma}|A\mathbf{u}_1|^2\right)d\tau \\ \ge \lambda_{m+1} Re^{-1}\gamma - \frac{32c_3^2(2+\epsilon)^2}{Re^{-1}\gamma\epsilon}S_1^2\end{aligned}$$

Thus in order to have

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} a(\tau)d\tau > 0,$$

it is sufficient for

$$\lambda_{m+1} \ge \frac{32c_3^2(2+\epsilon)^2}{Re^{-2}\gamma^2\epsilon}S_1^2.$$

Substituting in S_1^2 using (2.13) we obtain

$$\lambda_{m+1} \geq \frac{c(2+\epsilon)^2 Gr^2}{Re^{-4}\gamma^3\epsilon}. \tag{2.12}$$

where c is some constant that does do not depend on any of the present parameters.

□

2.8 Discussion of the Modal Number

We found an upper bound for the smallest possible modal number to be

$$\lambda_{m+1} \geq \frac{c(2+\epsilon)^2 Re^4 Gr^2}{\gamma^3\epsilon}. \tag{2.1}$$

It is very important to note that our estimate for the modal number is not sharp. Thus the above estimate merely serves as an upper bound and the functional dependence of the modal number could be different. The other important comment we make before proceeding is that these results do not exist for 3d Navier–Stokes. The same difficulties with well-posedness also prevent determination of a modal number.

We now discuss the functional behavior of our lower bound. In this discussion we assume that we choose the smallest possible λ_m for the inequality, thus when we say that λ_m may decrease we are talking about the choice of smallest possible value. For the Reynolds number we see that as $Re \to \infty$ $\lambda_m \to \infty$ and as $Re \to 0^+$ $\lambda_m \to 0$. It is not surprising that one would need more modes to describe a more turbulent flow. As far as the decrease of modes as Re decreases, this is most likely due to a highly viscous flow being over damped.

Similar behavior is show with the Grashof number. We see that as the Grashof decreases the number of modes needed decreases. This should not be surprising a Grashof number of zero indicates no forcing and thus due to dissipation any solution will decay to zero. As the Grashof number is increased the number of modes is suggested to increase. This seems reasonable as in increase in forcing would compete with the viscous damping.

Before discussing ϵ we investigate the influence of γ. Note that γ ranges over $0 < \gamma \leq 1$. In that case of $\gamma = 1$ we recover the LANS-α equations on the periodic domain. Korn [20] had given an estimate for the modal number of the LANS-α equations, but we believe that there is a serious flaw in their proof. Interestingly as $\gamma \to 0^+$, λ_m increases. In terms of the NS-$\alpha\beta$ equations having $\gamma \to 0^+$ reduces the filtering on the forcing term and thus decreases regularizing effect of dissipation. In the situation of $\gamma = 0$ determining a finite modal number would actually be more difficult than the case of 3d Navier–Stokes! This is due to the presence of higher order derivatives in the non-linear term with no corresponding higher derivatives on the dissipative term.

For $\epsilon \to 0^+$ we have $\lambda_m \to \infty$. While not surprising, the result is disappointing. Had this result been finite we may have had a way of giving a bound for the modal number of 3d Navier–Stokes, but due to the difficulties of 3d Navier–Stokes the result was not unexpected.

Chapter 3

A Kármán–Howarth type equation for the Navier–Stokes-$\alpha\beta$ equations

3.1 Introduction

In this chapter we develop a Kármán–Howarth type equation for the NS-$\alpha\beta$ equations, which we refer to as the KH-$\alpha\beta$ equation. In addition to developing the KH-$\alpha\beta$ our main result is that in the limit as α and β go to zero we recover the original Kármán–Howarth equation for the Navier–Stokes equations (NS). Sadly we discover that the new KH-$\alpha\beta$ equations has an even greater closure problem as compared to the original Kármán–Howarth equations. Thus, much like with the NS equations, we are able to say little about the nature of homogeneous isotropic flow for the NS-$\alpha\beta$ equations. In attempting to derive the KH-$\alpha\beta$ equation we discovered that the classical approach of using a probability density function to average was insufficient for the higher order NS-$\alpha\beta$ equations. Thus before proceeding with a derivation of the KH-$\alpha\beta$ equation we first have to discuss a new method of averaging in the context of homogeneous isotropic turbulence.

We develop a method of averaging for use in turbulence modeling. The key idea of the method is to average over the space of admissible solutions. While the method is not novel in probability theory, it has seen very little use when investigating turbulence models. The small amount of use has been in the study of statistical solutions by Vishik and Fursikov [30] and Foias et al. [8]. Also, the idea of averaging over the space of solutions is briefly mentioned by Androulakis and Dostoglou [1], but not pursued. The way to connect their work with ours is that if well-posedness is established for the model, then a probability measure on the space of initial conditions is equivalent to a probability measure over a space of solutions. But, most importantly even in that case our method of application differs from [30] and [8]. We compare our method with the traditional probability density function (pdf) method of averaging and we show that the pdf method of averaging is a consequence of averaging on the space of solutions. We also discuss the implications of homogeneous isotropic turbulence under this approach.

Traditionally in turbulent flows we imagine that we can perform an experiment over and over with almost identical conditions. But the conditions may vary in ways that we are unable to measure/control. Thus, we obtain different outcomes each time we perform our experiment. These differences may be due to variability in the initial conditions, the boundary conditions, temperature distribution, and so on. We want to be able to predict the flow, but due to this variability we cannot talk about the flow having a specific value at each point in space and time. But we can talk about the probability of the velocity field being in a certain range of values. One assumption that is often made is that the flow satisfies some set of equations, such as the Navier–Stokes equations for incompressible fluid flow. Even though we cannot predict the flow we do assume that the flow satisfies some (hopefully deterministic) equations.

Since we cannot say much about the particular values of the flow we take a probabilistic point of view. Often we are interested in the average values of the velocity field or the fluctuating portion of the velocity field. An endemic problem in the literature is that the type of averaging is often not mentioned. Or if it is mentioned, it is not clear what assumptions are placed on the method of averaging or even how the averaging is applied. One exception is the work by Vishik and Fursikov [30]. They consider a probability measure on the space of initial conditions of the Navier–Stokes equations to discuss statistical solutions. While this approach is very rigorous, it is concerned more with well-posedness questions and can be cumbersome in a general setting. For practical purposes we propose a compromise between the analytic work of Vishik and Fursikov and the traditional approaches. We want a general way to look at the probabilistic nature of turbulence that is still rigorous and makes clear which assumptions are made. We believe that our viewpoint, which is presented in section 3.3, is superior to many of the classical techniques because our assumptions are clear and the implications are not ambiguous. Plus, using our formulation it can actually be easier to derive equations such as the Kármán–Howarth equation. First we review the classical probability density function method of averaging for fluid flows. The reason for presenting this viewpoint is twofold. We want to show how our method relates to the probability density function method of averaging and we want to justify why a new perspective is needed.

3.2 Probability density function approach to averaging

Let $\mathbf{u}(\mathbf{x}, t)$ be the velocity field of the flow in some domain Ω. We follow Pope [24], although our notation differs slightly. We let $f(\mathbf{v}, \mathbf{x}, t)$ represent the probability density function for the random variable $\mathbf{u}(\mathbf{x}, t)$, where $\mathbf{v}$ represents all the possible vector values that $\mathbf{u}(\mathbf{x}, t)$ can take on. Since the flow is assumed to be related at nearby points and times it is assumed that the pdf has a certain amount of smoothness in $\mathbf{x}$ and t. Let $k : \mathcal{V} \to \mathbb{R}$ be a random function, where $\mathcal{V}$ is the space of translations associated to the euclidean space $\mathcal{E}$. The expected value of $k(\mathbf{u}(\mathbf{x}, t))$ is given by

$$\langle k(\mathbf{u}(\mathbf{x}, t))\rangle := \int_{\mathcal{V}} k(\mathbf{v}) f(\mathbf{v}, \mathbf{x}, t)\, d\mathbf{v}.$$

The idea being that we integrate over all possible values of $\mathbf{u}$, represented by all $\mathbf{v} \in \mathcal{V}$.

We can easily take an average at more than one value of $(\mathbf{x}, t)$ by using the joint pdf. i.e., $f_N(\mathbf{v}_1, \mathbf{x}_1, t_1; ...; \mathbf{v}_N, \mathbf{x}_N, t_N)$. Letting $h : \mathcal{V}^N \to \mathbb{R}$ be a random function of $\mathbf{u} \times \cdots \times \mathbf{u}$($n$-times), we have that its average of h at $\mathbf{u}(\mathbf{x}_1, t_1), ..., \mathbf{u}(\mathbf{x}_N, t_N)$ is defined as

$$\langle h(\mathbf{u}(\mathbf{x}_1, t_1), ..., \mathbf{u}(\mathbf{x}_N, t_N))\rangle$$

$$:= \int_{\mathcal{V}^N} h(\mathbf{v}_1, ..., \mathbf{v}_N) f_N(\mathbf{v}_1, \mathbf{x}_1, t_1; ...; \mathbf{v}_N, \mathbf{x}_N, t_N)\, d\mathbf{v}_1 \cdots d\mathbf{v}_N.$$

This method is very useful and been used to arrive at many important results in turbulence theory.

3.2.1 Homogeneous and isotropic flows

The idea of a homogeneous (turbulent) flow is a turbulent flow whose probabilities are invariant under translations of the coordinate system. An isotropic[1] (turbulent) flow is a homogeneous flow whose probabilities are invariant under arbitrary rotations and reflections of the coordinate system. Since the pdf encodes the statistical information about the flow we define homogeneity and isotropy in terms of the pdf.

Definition 3.1. A turbulent flow is homogeneous if every n-joint pdf is invariant under translations. *i.e.,*

$$f_N(\mathbf{v}_1, \mathbf{x}_1, t_1; ...; \mathbf{v}_N, \mathbf{x}_N, t_N) = f_N(\mathbf{v}_1, \mathbf{x}_1 + \mathbf{r}, t_1; ...; \mathbf{v}_N, \mathbf{x}_N + \mathbf{r}, t_N)$$

for all $r \in \mathcal{V}$.

Definition 3.2. A turbulent flow is isotropic if it is homogeneous and every n-joint pdf is invariant under Orth, the full orthogonal group. *i.e.,*

$$f_N(\mathbf{v}_1, \mathbf{x}_1, t_1; ...; \mathbf{v}_N, \mathbf{x}_N, t_N) = f_N(\mathbf{Q}\mathbf{v}_1, \mathbf{Q}\mathbf{x}_1, t_1; ...; \mathbf{Q}\mathbf{v}_N, \mathbf{Q}\mathbf{x}_N, t_N)$$

for all $\mathbf{Q} \in \text{Orth}$.

These definitions are an example as to the power of the pdf method. One is able to clearly define what it means for a flow to be isotropic.

3.2.2 Issues with the pdf perspective

Our first issue is with understanding what it means to average a derivative of the velocity field. We write $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in component form relative to coordinates. In Pope [24] it is claimed that averages and derivatives commute and

[1] Often in the literature isotropic refers to invariance under proper rotations, but as is standard in turbulence we use isotropic to refer to invariance under the full orthogonal group.

that the following identity holds.

$$\left\langle \frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial x_i} \right\rangle = \frac{\partial}{\partial x_i} \langle \mathbf{u}(\mathbf{x},t) \rangle . \tag{3.1}$$

How are we to make sense of the left hand side of (3.1)? Intuitively it makes sense to average a spatial derivative of the velocity field, but with our definition the pdf $f(\mathbf{v},\mathbf{x},t)$ is with respect to the random variable $\mathbf{u}$ as opposed to $(\partial/\partial x_i)\mathbf{u}$. One could define more random variables to be represented by the pdf, but then one would have the issue of how to relate pdf's of different random variables. The only way to carry on is to assume that

$$\left\langle \frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial x_i} \right\rangle$$

does exist. Under this assumption we now justify equation (3.1). We let $\Delta_{h\mathbf{e}_i}\mathbf{u}(\mathbf{x},t)$ represent the standard difference quotient. *i.e.,*

$$\Delta_{h\mathbf{e}_i}\mathbf{u}(\mathbf{x},t) = \frac{\mathbf{u}(\mathbf{x}+h\mathbf{e}_i,t) - \mathbf{u}(\mathbf{x},t)}{h}.$$

We now proceed with a standard $\epsilon - \delta$ argument. From the definition of the derivative we know that for every $\epsilon > 0$, there exists a $\delta > 0$, such that for all $|h| < \delta$, we have that

$$\left\| \frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial x_i} - \Delta_{h\mathbf{e}_i}\mathbf{u}(\mathbf{x},t) \right\| < \epsilon$$

or equivalently for each component u_j of $\mathbf{u}$

$$\Delta_{h\mathbf{e}_i} u_j(\mathbf{x},t) - \epsilon < \frac{\partial u_j(\mathbf{x},t)}{\partial x_i} < \Delta_{h\mathbf{e}_i} u_j(\mathbf{x},t) + \epsilon.$$

Averaging the above inequality we find that,

$$\langle \Delta_{h\mathbf{e}_i} u_j(\mathbf{x},t) - \epsilon \rangle < \left\langle \frac{\partial u_j(\mathbf{x},t)}{\partial x_i} \right\rangle < \langle \Delta_{h\mathbf{e}_i} u_j(\mathbf{x},t) + \epsilon \rangle .$$

Note that for each $h \neq 0$, $\langle \Delta_{h\mathbf{e}_i} \mathbf{u}(\mathbf{x}, t) \rangle$ is well defined. Summations and averages commute, and the average of a constant in the constant, thus the above inequality becomes

$$\langle \Delta_{h\mathbf{e}_i} u_j(\mathbf{x}, t) \rangle - \epsilon < \left\langle \frac{\partial u_j(\mathbf{x}, t)}{\partial x_i} \right\rangle < \langle \Delta_{h\mathbf{e}_i} u_j(\mathbf{x}, t) \rangle + \epsilon$$

and so

$$-\epsilon < \left\langle \frac{\partial u_j(\mathbf{x}, t)}{\partial x_i} \right\rangle - \langle \Delta_{h\mathbf{e}_i} u_j(\mathbf{x}, t) \rangle < \epsilon. \tag{3.2}$$

Therefore, as $\epsilon \to 0$, (3.2) implies that

$$\frac{\partial}{\partial x_i} \langle \mathbf{u}(\mathbf{x}, t) \rangle = \left\langle \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial x_i} \right\rangle.$$

Keep in mind that we had to assume $\langle (\partial/\partial x_i)\mathbf{u} \rangle$ existed. Certainly from a physical viewpoint it is reasonable that the average of $(\partial/\partial x_i)\mathbf{u}$ should exist, but we did not clearly define what $\langle (\partial/\partial x_i)\mathbf{u} \rangle$ meant mathematically. There are ways to resolve this issue, but we belabor it because it foreshadows other problems that we will see later.

Now that we have justified equation (3.1), we can see what the average of the derivative is in terms of the pdf for $\mathbf{u}(\mathbf{x}, t)$.

$$\frac{\partial}{\partial x_i} \langle \mathbf{u}(\mathbf{x}, t) \rangle = \lim_{h \to 0} \frac{1}{h} \left[\langle \mathbf{u}(\mathbf{x} + h\mathbf{e}_i, t) \rangle - \langle \mathbf{u}(\mathbf{x}, t) \rangle \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\int_{\mathcal{V}} \mathbf{v} f(\mathbf{v}, \mathbf{x} + h\mathbf{e}_i, t)\, d\mathbf{v} - \int_{\mathcal{V}} \mathbf{v} f(\mathbf{v}, \mathbf{x}, t)\, d\mathbf{v} \right]$$

$$= \lim_{h \to 0} \int_{\mathcal{V}} \mathbf{v} \left[\frac{f(\mathbf{v}, \mathbf{x} + h\mathbf{e}_i, t) - f(\mathbf{v}, \mathbf{x}, t)}{h} \right] d\mathbf{v}$$

Assuming that the pdf is smooth enough we can interchange the limit and the integral to obtain,

$$= \int_{\mathcal{V}} \mathbf{v} \lim_{h \to 0} \left[\frac{f(\mathbf{v}, \mathbf{x} + h\mathbf{e}_i, t) - f(\mathbf{v}, \mathbf{x}, t)}{h} \right] d\mathbf{v} = \int_{\mathcal{V}} \mathbf{v} \frac{\partial f}{\partial x_i}(\mathbf{v}, \mathbf{x}, t)\, d\mathbf{v}.$$

Thus, we can take the average of a derivative by the following formula.

$$\left\langle \frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial x_i} \right\rangle = \int_{\mathcal{V}} \mathbf{v} \frac{\partial f}{\partial x_i}(\mathbf{v},\mathbf{x},t)\, d\mathbf{v}$$

A similar analysis shows that for any differentiable function $k : \mathcal{V} \to \mathbb{R}$, we have

$$\left\langle \frac{\partial}{\partial x_i} k(\mathbf{u}(\mathbf{x},t)) \right\rangle = \int_{\mathcal{V}} k(\mathbf{v}) \frac{\partial f}{\partial x_i}(\mathbf{v},\mathbf{x},t)\, d\mathbf{v}.$$

For example,

$$\left\langle u_j(\mathbf{x},t)) \frac{\partial}{\partial x_i} u_j(\mathbf{x},t)) \right\rangle = \left\langle \frac{\partial}{\partial x_i} (u_j(\mathbf{x},t)))^2 \right\rangle = \int_{\mathcal{V}} v_i^2 \frac{\partial f}{\partial x_i}(\mathbf{v},\mathbf{x},t)\, d\mathbf{v}.$$

It seems like we are making progress, but there is a monster lurking the background. How do we compute the following?

$$\left\langle k\left(\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial x_i}\right) \right\rangle = \left\langle k\left(\lim_{h\to 0} \frac{\mathbf{u}(\mathbf{x}+h\mathbf{e}_i,t) - \mathbf{u}(\mathbf{x},t)}{h}\right) \right\rangle$$

If k is continuous we can factor the limit out of the function k, but in general we cannot factor out $1/h$. If we try to factor the derivative out naively, we obtain false results. All we can potentially write is,

$$\left\langle k\left(\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial x_i}\right) \right\rangle = \lim_{h\to 0} \int_{\mathcal{V}^2} k\left(\frac{\mathbf{v}-\mathbf{w}}{h}\right) f_2(\mathbf{v},\mathbf{x}+h\mathbf{e}_i,t,\mathbf{w},\mathbf{x},t)\, d\mathbf{v}\, d\mathbf{w}.$$

While this does give us a formula, it not useful. One way around this issue is to introduce more random variables and more general joint pdfs. We tried this approach and noticed that we were having to make more assumptions and solve more problems that were all similar in nature. This was when we realized that there may be a more general starting point, one that may give a more complete picture.

3.3 Averaging on the space of solutions

First we review the basic ideas of probability theory. A probability space is a triple $(\mathcal{X}, \mathcal{F}, \Pi)$, where $\mathcal{X}$ is the underlying space, $\mathcal{F}$ is a σ-algebra of measurable subsets of $\mathcal{X}$ according to the probability measure Π. A random variable is a measurable function from Ω to $\mathbb{R}$ (or $\mathbb{R}^n$), where we use Lebesgue measure on $\mathbb{R}$.

Suppose we have two probability spaces $(\mathcal{X}_1, \mathcal{F}_1, \Pi_1)$ and $(\mathcal{X}_2, \mathcal{F}_2, \Pi_2)$ and a measurable mapping $\varphi : \mathcal{X}_1 \to \mathcal{X}_2$. The push-forward of the measure Π_1 under φ, denoted by $\varphi_*\Pi$, gives a probability measure on $\mathcal{X}_2$ by the following rule: Let $A \in \mathcal{F}_2$ then

$$\varphi_*\Pi(A) = \Pi(\varphi^{-1}(A)).$$

Given a random variable $u : \mathcal{X} \to \mathbb{R}$ we can push-forward the measure Π to give the measure $u_*\Pi$ on $\mathbb{R}$. Suppose that this measure is absolutely continuous with respect to Lebesgue measure, λ, on $\mathbb{R}$. Then we can can take the Radon-Nikodym derivative of $u_*\Pi$ with respect to λ to obtain a measurable function $f : \mathbb{R} \to [0, \infty)$ with the property that given a measurable subset of B of $\mathbb{R}$ we have

$$u_*\Pi(B) = \int_{\mathbb{R}} f(x)\, d\lambda(x).$$

The function f is often referred to as the continuous distribution or the probability density function of $u_*\Pi$ with respect to Lebesgue measure. For more information on these issues see Billingsley [3].

I was inspired to take this approach after reading different work by Vishik and Fursikov [30] dealing with nature of what they call statistical solutions. In their approach they generally start with a space of initial conditions with a probability measure and show the existence of time dependent measures that satisfy certain properties. This work is very interesting, but we want a general averaging procedure that is not focused on a specific problem.

It is assumed that there is a space $\mathcal{S}$ of solutions to a given set of partial differential equations and that there is a measure Π on some σ-algebra of subsets, say $\mathcal{F}$. In practice the $\mathcal{F}$ will have to do with borel sets generated by a topology on $\mathcal{S}$. With the above terminology we have the triple $(\mathcal{S}, \mathcal{F}, \Pi)$. For turbulence we generally use the Navier–Stokes equations and that will be the prototype we have in mind for this discussion. So $\mathcal{S}$ consists of all pairs $(\mathbf{u}, p)$ that satisfy the Navier-Stokes equations. We assume that there is a source of randomness, in that if we were to repeat the experiment over and over we would get different solutions. Yet for each experiment we have a solution $(\mathbf{u}, p)$ to the Navier–Stokes equations. The probability measure Π encodes the likelihood of a solution occurring and hence encodes the nature of the turbulence.

For simplicity we often suppress the pressure when we are only dealing with the velocity field. So we write $\mathbf{u} \in \mathcal{S}$ to mean the solution pair $(\mathbf{u}, p) \in \mathcal{S}$. From this perspective the function $\mathbf{u}(\mathbf{x}, t)$ is a random variable for each $(\mathbf{x}, t)$. *i.e.,* $\mathbf{u}(\mathbf{x}, t) : \mathcal{S} \to \mathcal{V}$ is a measurable function that evaluates each velocity field at $(\mathbf{x}, t)$. Since $\mathbf{u}(\mathbf{x}, t)$ is a random variable we can find its average in the natural way. Thus, we define the average of the velocity field at $(\mathbf{x}, t)$ by

$$\langle \mathbf{u}(\mathbf{x}, t) \rangle = \int_{\mathcal{S}} \mathbf{v}(\mathbf{x}, t) \Pi(\, d\mathbf{v}).$$

Here we are integrating over all possible solutions $\mathbf{v}$. The notation $\Pi(\, d\mathbf{v})$ means that $\mathbf{v}$ is being varied, but that the integral is with respect the measure Π on $\mathcal{S}$. We also consider the pressure field and the various derivatives of the velocity and pressure fields as random variables. For example, if we want to find the average of the gradient of the velocity field we treat $\operatorname{grad} \mathbf{u}(\mathbf{x}, t) : \mathcal{S} \to \mathcal{V} \otimes \mathcal{V}$ as another random variable and take the average.

$$\langle \operatorname{grad} \mathbf{u}(\mathbf{x}, t) \rangle = \int_{\mathbf{v} \in \mathcal{S}} \operatorname{grad} \mathbf{v}(\mathbf{x}, t) \Pi(\, d\mathbf{v})$$

In fact for any measurable function $k : \mathcal{V} \otimes \mathcal{V} \to \mathbb{R}$ we have

$$\langle k\left(\operatorname{grad} \mathbf{u}(\mathbf{x}, t)\right) \rangle = \int_{\mathbf{v} \in \mathcal{S}} k\left(\operatorname{grad} \mathbf{v}(\mathbf{x}, t)\right) \Pi(\, d\mathbf{v}).$$

From this perspective there is no ambiguity on defining the average of derivatives. We now justify that derivatives commute with averages. As before we write $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in components.

$$\begin{aligned}
\frac{\partial}{\partial x_i} \langle \mathbf{u}(\mathbf{x}, t) \rangle &= \lim_{h \to 0} \frac{1}{h} \left[\langle \mathbf{u}(\mathbf{x} + h e_i, t) \rangle - \langle \mathbf{u}(\mathbf{x}, t) \rangle \right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[\int_{\mathcal{S}} \mathbf{v}(\mathbf{x} + h e_i, t) \Pi(\, d\mathbf{v}) - \int_{\mathcal{S}} \mathbf{v}(\mathbf{x}, t) \Pi(\, d\mathbf{v}) \right] \\
&= \lim_{h \to 0} \int_{\mathcal{S}} \frac{\mathbf{v}(\mathbf{x} + h e_i, t) - \mathbf{v}(\mathbf{x}, t)}{h} \Pi(\, d\mathbf{v}) \\
&= \int_{\mathcal{S}} \lim_{h \to 0} \frac{\mathbf{v}(\mathbf{x} + h e_i, t) - \mathbf{v}(\mathbf{x}, t)}{h} \Pi(\, d\mathbf{v}) \\
&= \int_{\mathcal{S}} \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial x_i} \Pi(\, d\mathbf{v}) \\
&= \left\langle \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial x_i} \right\rangle
\end{aligned}$$

To move the limit inside the integral we assume the solutions (elements of $\mathcal{S}$) have enough regularity to appeal to the Lebesgue convergence theorem. We can perform similar operations for derivatives in t and higher order derivatives as long as the velocity fields are regular enough. Of course the same results hold for the pressure field.

3.3.1 Homogeneous and isotropic measures

These definitions are inspired by the work of Androulakis and Dostoglou [1]. In fact they mention the idea of having a measure on the space of solutions, but shift their focus to measures on the space of initial conditions.

First we define a translation operator on $\mathcal{S}$. Let $\mathbf{r} \in \mathcal{V}$, then $T_{\mathbf{r}} : \mathcal{S} \to \mathcal{S}$ is the operator defined by $T_{\mathbf{r}}(\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t)) = (\mathbf{u}(\mathbf{x} - \mathbf{r}, t), p(\mathbf{x} - \mathbf{r}))$. If our space Ω had a boundary then we would have trouble defining the translation operator near the boundary. To get around this we assume that our ambient space, Ω, is either $\mathbb{R}^3$ or a periodic box of side length L. We use this operator to pushforward the measure Π on $\mathcal{S}$ to get a new measure $T_{\mathbf{r}}\Pi$ on $\mathcal{S}$. Note that we drop the standard "$_*$" on the pushforward for notational convenience.

Definition 3.1. A measure Π is said to be **homogeneous** if $T_{\mathbf{r}}\Pi = \Pi$ for all $\mathbf{r} \in \mathcal{V}$

Remember that the probability measure Π contains all the information about the turbulence. So for the probabilities to be invariant under translation, we should have that the measure is invariant under translations. Thus, we say that a turbulent flow is homogeneous if the probability measure Π is homogeneous.

Traditionally isotropic turbulent flows are homogeneous flows whose probabilities are invariant under rotations and reflections[2] of the coordinate system. We do not want a preferred point of rotation so we will consider rotations about any point $\mathbf{o} \in \mathcal{E}$. Then by homogeneity the probabilities of flow are invariant under rotations and reflections

[2] Some authors use isotropic to describe functions that are invariant under the group of rotations. As is common in turbulence theory we take isotropy to represent invariance under the full orthogonal group.

about any point. Thus, we use $\mathbf{o}$ to be our generic point in the spirit of the origin.

Suppose we take a vector field $\mathbf{v}(\mathbf{x})$ and we rotate the coordinate system about a point $\mathbf{o}$ by an orthogonal tensor $\mathbf{Q} \in \mathrm{Orth}$ giving the new vector field $\hat{\mathbf{v}}(\hat{\mathbf{x}})$. The new vector field is related to the original vector field by the following formula.

$$\hat{\mathbf{v}}(\hat{\mathbf{x}}) = \mathbf{Q}\mathbf{v}(\mathbf{Q}^T(\hat{\mathbf{x}} - \mathbf{o}) + \mathbf{o})$$

Similarly for a scalar field $p(\mathbf{x})$ we have $\hat{p}(\hat{\mathbf{x}}) = p(\mathbf{Q}^T(\hat{\mathbf{x}} - \mathbf{o}) + \mathbf{o})$. Based on these observations we define the operator $R_{\mathbf{Q}} : \mathcal{S} \to \mathcal{S}$ by $R_{\mathbf{Q}}(\mathbf{u}(\mathbf{x}, t), p(\mathbf{x})) = (\mathbf{Q}\mathbf{u}(\mathbf{Q}^T(\hat{\mathbf{x}} - \mathbf{o}) + \mathbf{o}, t), p(\mathbf{Q}^T(\hat{\mathbf{x}} - \mathbf{o}) + \mathbf{o}))$. We suppress the point $\mathbf{o}$ since it's use will be apparent from the context and by homogeneity we can always translate to any other point. Once again we can use this operator to push-forward the measure Π from $\mathcal{S}$ to $\mathcal{S}$.

Definition 3.2. We say the measure Π is **isotropic** if Π is a homogeneous measure and $R_{\mathbf{Q}}\Pi = \Pi$, for all $\mathbf{Q} \in \mathrm{Orth}$ (and for all $\mathbf{o} \in \mathcal{E}$).

An isotropic flow is a homogeneous flow whose probability measure Π is also isotropic. Thus a homogeneous isotropic turbulent flow is a probability space of solutions with an isotropic measure Π.

Traditionally in the literature (see [2], [19], or [25]), spatial averaging is also used when dealing with isotropic flows. I.e.,

$$\langle f \rangle_{\mathcal{L}} = \lim_{L \to \infty} \frac{1}{L^3} \int_{L^3} f(\boldsymbol{\xi})\, dv.$$

Our problem with spatial averaging is that the flow has to be homogeneous in order for it to be well defined. In addition, desired properties seem to be forced upon the spatial average as opposed to being consequence of the basic assumptions of homogeneity and isotropy. Under suitable hypotheses an ergodic theorem[3] gives that spatial averaging is equivalent to the probability density function approach to averaging used in many sources including Pope [24]. The beauty of averaging on the space of solutions as opposed to spacial averaging as in Batchelor [2] is that it is valid for all turbulent flows and not just homogeneous flows.

3.3.2 Discussion of random variables and distributions

As mentioned earlier we can treat $\mathbf{u}(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$ and all their derivatives as random variables. We can use any random variable to push-forward the measure. In particular for the random variable $\mathbf{u}(\mathbf{x}, t) : \mathcal{S} \to \mathcal{V}$, we can use it to push-forward the probability measure Π to the probability measure $\mathbf{u}(\mathbf{x}, t)_* \Pi$ on $\mathcal{V}$. This probability measure $\mathbf{u}(\mathbf{x}, t)_* \Pi$ is often referred to as the distribution for $\mathbf{u}(\mathbf{x}, t)$. Then given a measurable function $k : \mathcal{V} \to \mathbb{R}$ we have

$$\langle \mathbf{u}(\mathbf{x}, t) \rangle = \int_{\mathcal{S}} k(\mathbf{v}(\mathbf{x}, t)) \Pi(\, d\mathbf{v}) = \int_{\mathcal{V}} k(\mathbf{v}) \; \mathbf{u}(\mathbf{x}, t)_* \Pi(\, d\mathbf{v}). \tag{3.1}$$

Note that in the right hand term we are now integrating over all vectors $\mathbf{v} \in \mathcal{V}$ as opposed to vector fields $\mathbf{v}(\mathbf{x}, t) \in \mathcal{S}$. Let us assume that the measure $\mathbf{u}(\mathbf{x}, t)_* \Pi$ is absolutely continuous with respect to Lebesgue measure, λ, on $\mathcal{V}$. Then we can take the Radon-Nikodym derivative of $\mathbf{u}(\mathbf{x}, t)_* \Pi$ with respect to λ to get

$$f(\mathbf{v}, \mathbf{x}, t) = \frac{d\mathbf{u}(\mathbf{x}, t)_* \Pi}{d\lambda}(\mathbf{v}).$$

The fiction $f : \mathcal{V} \times \Omega \times \mathbb{R} \to \mathbb{R}^+$ is referred to as the continuous distribution or the probability density function for the random variable $\mathbf{u}(\mathbf{x}, t)$. Using this continuous

[3] See Androulakis, Dostoglou; Space Averages and Homogeneous Fluid Flows for more information.

distribution we can now write equation (3.1) as

$$\langle \mathbf{u}(\mathbf{x},t)\rangle = \int_{\mathcal{V}} k(\mathbf{v}) f(\mathbf{v},\mathbf{x},t)\, d\mathbf{v}$$

where we have written $d\mathbf{v}$ for $\lambda(d\mathbf{v})$. We will show that this distribution has the same properties as the pdf from section (3.2), but first we give evidence for our assumption that the measure $\mathbf{u}(\mathbf{x},t)_*\Pi$ is absolutely continuous with respect to Lebesgue measure.

We have turbulent flows in mind, so let us consider the opposite situation where we have a laminar flow, say $(\mathbf{u}_1(\mathbf{x},t), p_1(\mathbf{x}))$. This flow is not subject to random behavior, there are no fluctuations. Since this is the only solution that occurs we must have that measure Π is completely supported on the solution $(\mathbf{u}_1(\mathbf{x},t), p_1(\mathbf{x}))$. Thus, Π is a singular measure. In contrast, a turbulent flow is random, but more than that we have for any possible solution $(\mathbf{u}(\mathbf{x},t), p(\mathbf{x},t))$ of $\mathcal{S}$ that all the other "nearby" solutions are also possible. To be more technical we would say that the support of the measure Π is open. So for any solution, $(\mathbf{u}(\mathbf{x},t), p(\mathbf{x},t))$, in the support of Π there is a neighborhood of $(\mathbf{u}(\mathbf{x},t), p(\mathbf{x},t))$ that is also in the support of Π. Thus pushing forward Π by $\mathbf{u}(\mathbf{x},t)$ and $p(\mathbf{x},t)$ will give a measure that is absolutely continuous with respect to Lebesgue measure.

As an example let us considering the velocity field at N different points of $\Omega \times \mathbb{R}^+$, $\mathbf{u}(\mathbf{x}_1,t_1), \mathbf{u}(\mathbf{x}_2,t_2), \ldots, \mathbf{u}(\mathbf{x}_N,t_N)$. Then

$$\mathbf{u}(\mathbf{x}_1,t_1) \times \mathbf{u}(\mathbf{x}_2,t_2) \times \cdots \times \mathbf{u}(\mathbf{x}_N,t_N) : \underbrace{\mathcal{S} \times \cdots \times \mathcal{S}}_{N-\text{times}} \to \underbrace{\mathcal{V} \times \cdots \times \mathcal{V}}_{N-\text{times}}$$

is a random variable. As per our assumption we assume that the pushforward of Π by this random variable gives rise to a continuous distribution $f_N(\mathbf{v}_1,\mathbf{x}_1,t_1;...;\mathbf{v}_N,\mathbf{x}_N,t_N)$ on $\mathcal{V} \times \cdots \times \mathcal{V}$.

Theorem 3.3. *The distribution $f_N(\mathbf{v}_1, \mathbf{x}_1, t_1; ...; \mathbf{v}_N, \mathbf{x}_N, t_N)$ satisfies definitions (3.1) and(3.2).*

Proof. To simplify notation let us look at the $N = 1$ case, then general case follows exactly as this case does. For the $N = 1$ case our distribution is written as $f(\mathbf{v}, \mathbf{x}, t)$.

(homogeneity) As before let $k : \mathcal{V} \to \mathbb{R}$ be an arbitrary measurable function. Let $\mathbf{r} \in \mathcal{V}$ be arbitrary, then

$$\begin{aligned}
\int_{\mathcal{V}} k(\mathbf{v}) f(\mathbf{v}, \mathbf{x}, t)\, d\mathbf{v} &= \int_{\mathcal{S}} k(\mathbf{v}(\mathbf{x}, t)) \Pi(\, d\mathbf{v}) \\
&= \int_{\mathcal{S}} k(T_{\mathbf{r}}^{-1} \mathbf{w}(\mathbf{x}, t)) \Pi(\, dT_{\mathbf{r}}^{-1} \mathbf{w}) \qquad \text{substituting } \mathbf{v} = T_{\mathbf{r}}^{-1}\mathbf{w} \\
&= \int_{\mathcal{S}} k(T_{\mathbf{r}}^{-1} \mathbf{w}(\mathbf{x}, t)) \Pi(\, d\mathbf{w}) \qquad \text{homogeneous measure} \\
&= \int_{\mathcal{S}} k(\mathbf{w}(\mathbf{x} + \mathbf{r}, t)) \Pi(\, d\mathbf{w}) \\
&= \int_{\mathcal{V}} k(\mathbf{w}) f(\mathbf{w}, \mathbf{x} + \mathbf{r}, t)\, d\mathbf{w}
\end{aligned}$$

Since $\mathbf{v}$ and $\mathbf{w}$ are just place holders for integration we have that

$$\int_{\mathcal{V}} k(\mathbf{v}) f(\mathbf{v}, \mathbf{x}, t)\, d\mathbf{v} = \int_{\mathcal{V}} k(\mathbf{v}) f(\mathbf{v}, \mathbf{x} + \mathbf{r}, t)\, d\mathbf{v}$$

for all measurable k. Thus, we must have $f(\mathbf{v}, \mathbf{x}, t) = f(\mathbf{v}, \mathbf{x} + \mathbf{r}, t)$ for all $\mathbf{r} \in \mathcal{V}$.

(isotropy) Let $\mathbf{Q} \in \mathrm{Orth}$ and $\mathbf{o} \in \mathcal{E}$

$$\begin{aligned}
\int_{\mathcal{V}} k(\mathbf{v}) f(\mathbf{v}, \mathbf{x}, t)\, d\mathbf{v} &= \int_{\mathcal{S}} k(\mathbf{v}(\mathbf{x}, t)) \Pi(\, d\mathbf{v}) \\
&= \int_{\mathcal{S}} k(R_{\mathbf{Q}}^{-1}\mathbf{w}) \Pi(\, dR_{\mathbf{Q}}^{-1}\mathbf{w}) && \mathbf{v} = R_{\mathbf{Q}}^{-1}\mathbf{w} \\
&= \int_{\mathcal{S}} k(R_{\mathbf{Q}}^{-1}\mathbf{w}) \Pi(\, d\mathbf{w}) && \text{isotropic measure} \\
&= \int_{\mathcal{S}} k(\mathbf{Q}^T \mathbf{w}(\mathbf{Q}(\mathbf{x} - \mathbf{o}) + \mathbf{o}, t)) \Pi(\, d\mathbf{w}) \\
&= \int_{\mathcal{V}} k(\mathbf{Q}^T \mathbf{w}) f(\mathbf{w}, \mathbf{Q}(\mathbf{x} - \mathbf{o}) + \mathbf{o}, t)\, d\mathbf{w} \\
&= \int_{\mathcal{V}} k(\mathbf{v}) f(\mathbf{Q}\mathbf{v}, \mathbf{Q}(\mathbf{x} - \mathbf{o}) + \mathbf{o}, t)\, d\mathbf{w} && \text{let } \mathbf{v} = \mathbf{Q}^T\mathbf{w} \\
&= \int_{\mathcal{V}} k(\mathbf{v}) f(\mathbf{Q}\mathbf{v}, \mathbf{Q}(\mathbf{x} - \mathbf{o}) + \mathbf{o}, t)\, d\mathbf{v} && \text{change of variables}
\end{aligned}$$

In the last step where we used change of variables. Note that since $\mathbf{Q} \in \mathrm{Orth}$, the Jacobian determinant is 1. As before, since k is an arbitrary measurable function we must have that

$$f(\mathbf{v}, \mathbf{x}, t) = f(\mathbf{Q}\mathbf{v}, \mathbf{Q}(\mathbf{x} - \mathbf{o}) + \mathbf{o}, t)$$

for all $\mathbf{Q} \in \mathrm{Orth}$. Note that in the classical treatment the point of rotation $\mathbf{o}$ is implicitly assumed to be 0. Thus, the above reduces the the classical pdf averaging.

□

We have shown that we can recover the pdf averaging approach for homogeneous isotropic turbulence. But, we can do so much more! As we stated earlier taking averages of derivatives is well defined without extra assumptions (other than smoothness). Next we show what the implications of isotropy are when applied to the average of $\mathrm{grad}\, \mathbf{u}(\mathbf{x}, t)$.

3.3.3 Averaging the gradient

Let $k : \mathcal{V} \otimes \mathcal{V} \to \mathbb{R}$, then

$$
\begin{aligned}
\langle k(\operatorname{grad} \mathbf{u}(\mathbf{x}, t)) \rangle &= \int_{\mathcal{S}} k(\operatorname{grad} \mathbf{v}(\mathbf{x}, t)) \Pi(\,d\mathbf{v}) \\
&= \int_{\mathcal{S}} k(\operatorname{grad}[\mathbf{Q}^T \mathbf{w}(\mathbf{Q}(\mathbf{x} - \mathbf{o}) + \mathbf{o}, t)]) \Pi(\,dR_{\mathbf{Q}}^{-1} \mathbf{w}) \quad \text{Let } \mathbf{v} = R_{\mathbf{Q}}^{-1} \mathbf{w} \\
&= \int_{\mathcal{S}} k(\operatorname{grad}[\mathbf{Q}^T \mathbf{w}(\mathbf{Q}(\mathbf{x} - \mathbf{o}) + \mathbf{o}, t)]) \Pi(\,d\mathbf{w}) \qquad \text{invariance of } \mu \\
&= \int_{\mathcal{S}} k(\mathbf{Q}^T [\operatorname{grad} \mathbf{w}(\mathbf{Q}(\mathbf{x} - \mathbf{o}) + \mathbf{o}, t)] \mathbf{Q}) \Pi(\,d\mathbf{w}) \qquad \text{chain rule } \mathbf{w}(\mathbf{x}, t) \\
&= \left\langle k(\mathbf{Q}^T [\operatorname{grad} \mathbf{u}(\mathbf{Q}(\mathbf{x} - \mathbf{o}) + \mathbf{o}, t)] \mathbf{Q}) \right\rangle
\end{aligned}
$$

Our result is that

$$
\langle k(\operatorname{grad} \mathbf{u}(\mathbf{x}, t)) \rangle = \left\langle k(\mathbf{Q}^T [\operatorname{grad} \mathbf{u}(\mathbf{Q}(\mathbf{x} - \mathbf{o}) + \mathbf{o}, t)] \mathbf{Q}) \right\rangle . \tag{3.2}
$$

Thus, inside the average $\operatorname{grad} \mathbf{u}(\mathbf{x}, t)$ behaves as an isotropic tensor field!

Let us investigate the distribution of $\operatorname{grad} \mathbf{u}(\mathbf{x}, t)$. We use the random variable $\operatorname{grad} \mathbf{u}(\mathbf{x}, t)$ to pushforward the measure Π to $\mathcal{V} \otimes \mathcal{V}$. Let $f(\mathbf{A}, \mathbf{x}, t)$ represent the continuous distribution on $\mathcal{V} \otimes \mathcal{V}$, where $\mathbf{A}$ is a tensor that represents all the possible values of $\operatorname{grad} \mathbf{u}(\mathbf{x}, t)$. Then in terms of the continuous distribution, equation (3.2) becomes

$$
\begin{aligned}
\int_{\mathcal{V} \otimes \mathcal{V}} k(\mathbf{A}) f(\mathbf{A}, \mathbf{x}.t) \, d\mathbf{A} &= \int_{\mathcal{V} \otimes \mathcal{V}} k(\mathbf{Q}^T \mathbf{A} \mathbf{Q}) f(\mathbf{A}, \mathbf{Q}(\mathbf{x} - \mathbf{o}) + \mathbf{o}, t) \, d\mathbf{A} \\
&= \int_{\mathcal{V} \otimes \mathcal{V}} k(\mathbf{B}) f(\mathbf{Q} \mathbf{B} \mathbf{Q}^T, \mathbf{Q}(\mathbf{x} - \mathbf{o}) + \mathbf{o}, t) \, d\mathbf{A} \quad \mathbf{B} = \mathbf{Q}^T \mathbf{A} \mathbf{Q} \\
&= \int_{\mathcal{V} \otimes \mathcal{V}} k(\mathbf{B}) f(\mathbf{Q} \mathbf{B} \mathbf{Q}^T, \mathbf{Q}(\mathbf{x} - \mathbf{o}) + \mathbf{o}, t) \, d\mathbf{B} \quad d\mathbf{B} = \, d\mathbf{A} \\
&= \int_{\mathcal{V} \otimes \mathcal{V}} k(\mathbf{A}) f(\mathbf{Q} \mathbf{A} \mathbf{Q}^T, \mathbf{Q}(\mathbf{x} - \mathbf{o}) + \mathbf{o}, t) \, d\mathbf{A}
\end{aligned}
$$

Where the last equality is obtained by renaming the variable of integration. Since k is arbitrary we must have

$$f(\mathbf{Q}\mathbf{A}\mathbf{Q}^T, \mathbf{Q}(\mathbf{x}-\mathbf{o})+\mathbf{o}, t) = f(\mathbf{A}, \mathbf{x}, t)$$

for all $\mathbf{Q} \in \mathrm{Orth}$. Similar formulas follow for higher order derivatives of $\mathbf{u}(\mathbf{x}, t)$.

3.3.4 Averages of the pressure term

In the literature the pressure term is usually neglected. For example in [29], [6], and [11] a weak formulation is investigated, and so the pressure term naturally drops out. In the classical sources such as [19] and [25] it is not made clear what is being assumed about the pressure field. We think the assumption is made that the probabilities of the pressure field are invariant under translations and rotations and reflections, but this is usually ambiguous. Since the pressure field can be solved for by inverting the laplacian it is sometimes assumed that the pressure field is isotropic in the average when the velocity field is isotropic. Using our method of averaging we show that this is in fact true. Using green's function for the laplacian one can solve for the pressure field in terms of the velocity field and obtain

$$p(\mathbf{x}, t) = -\int_\Omega \frac{[\operatorname{grad}\mathbf{u}(\mathbf{y}, t)]^T : \operatorname{grad}\mathbf{u}(\mathbf{y}, t)}{|\mathbf{x}-\mathbf{y}|}\, d\mathbf{y}.$$

Taking the average of $p(\mathbf{Q}(\mathbf{x}-\mathbf{o})+\mathbf{o})$ we obtain,

$$\begin{aligned}
\langle p(\mathbf{Q}(\mathbf{x}-\mathbf{o})+\mathbf{o},t)\rangle &= \left\langle -\int_\Omega \frac{[\operatorname{grad}\mathbf{u}(\mathbf{y},t)]^T : \operatorname{grad}\mathbf{u}(\mathbf{y},t)}{|\mathbf{Q}(\mathbf{x}-\mathbf{o})-\mathbf{y}|}\,d\mathbf{y}\right\rangle \\
&= \left\langle -\int_\Omega \frac{[\operatorname{grad}\mathbf{u}(\mathbf{Qz},t)]^T : \operatorname{grad}\mathbf{u}(\mathbf{Qz},t)}{|\mathbf{Q}(\mathbf{x}-\mathbf{o})-\mathbf{Qz}|}\,d\mathbf{z}\right\rangle && \text{let } \mathbf{y}=\mathbf{Qz} \\
&= \left\langle -\int_\Omega \frac{[\mathbf{Q}\operatorname{grad}\mathbf{u}(\mathbf{z},t)\mathbf{Q}^T]^T : \mathbf{Q}\operatorname{grad}\mathbf{u}(\mathbf{z},t)\mathbf{Q}^T}{|\mathbf{x}-\mathbf{z}|}\,d\mathbf{z}\right\rangle && \text{by (3.2)} \\
&= \left\langle -\int_\Omega \frac{\mathbf{Q}[\operatorname{grad}\mathbf{u}(\mathbf{z},t)]^T\mathbf{Q}^T : \mathbf{Q}\operatorname{grad}\mathbf{u}(\mathbf{z},t)\mathbf{Q}^T}{|\mathbf{x}-\mathbf{z}|}\,d\mathbf{z}\right\rangle \\
&= \left\langle -\int_\Omega \frac{[\operatorname{grad}\mathbf{u}(\mathbf{z},t)]^T : \operatorname{grad}\mathbf{u}(\mathbf{z},t)}{|\mathbf{x}-\mathbf{z}|}\,d\mathbf{z}\right\rangle \\
&= \langle p(\mathbf{x},t)\rangle
\end{aligned}$$

Homogeneity of $\langle p(\mathbf{x},t)\rangle$ follows similarly. Showing how derivatives of the pressure field transform can be cumbersome, which is why we prefer to declare the pressure field as part of the solution space that is invariant under translations and rotations.

3.4 Discussion of functions that are invariant under rotations and reflections

Functions that are invariant in the above sense are very important when investigating isotropic turbulence. For purposes of clarity we review some of the basic ideas here. In the literature functions of this form are sometimes referred to as isotropic functions, see Noll and Truesdell [28] as an example and for more details. In the following discussion V is a n-dimensional (real) inner product space and $\mathrm{Orth}(n)$ is the space of orthogonal transformations of V. The precise mathematical formulation for a function that is invariant under $\mathrm{Orth}(n)$ depends on the type of quantity being considered. First, we consider a scalar-valued function of vectors. I.e., $\phi : \mathrm{V} \to \mathbb{R}$. We say that ϕ

is invariant under $\mathrm{Orth}(n)$, if $\phi(\mathrm{Q}\mathbf{v}) = \phi(\mathbf{v})$ for all $\mathbf{Q} \in \mathrm{Orth}(n)$. We can extend this idea to situations where we have a scalar function of more then one argument. For example suppose $\psi(\mathbf{v}_1, ..., \mathbf{v}_m)$ is a scalar-valued function with each $\mathbf{v}_i \in \mathrm{V}$. Then we say that ψ is invariant under $\mathrm{Orth}(n)$ if $\psi(\mathbf{Q}\mathbf{v}_1, ..., \mathbf{Q}\mathbf{v}_m) = \psi(\mathbf{v}_1, ..., \mathbf{v}_m)$ for all $\mathbf{Q} \in \mathrm{Orth}(n)$.

Next we consider vector-valued functions on V such as $\mathbf{f} : \mathrm{V} \to \mathrm{V}$. Let $\mathbf{a} \in \mathrm{V}$ be a fixed vector and consider the scalar product $\mathbf{f}(\mathbf{v}) \cdot \mathbf{a}$. Then, the function $\phi(\mathbf{v}, \mathbf{a}) = \mathbf{f}(\mathbf{v}) \cdot \mathbf{a}$ is a scalar-valued function on $\mathrm{V} \times \mathrm{V}$. Suppose that ϕ is invariant under $\mathrm{Orth}(n)$, then what condition must $\mathbf{f}(\mathbf{v})$ satisfy? We have that $\phi(\mathbf{Q}\mathbf{v}, \mathbf{Q}\mathbf{a}) = \phi(\mathbf{v}, \mathbf{a})$ for all $\mathbf{Q} \in \mathrm{Orth}$ Thus,

$$\mathbf{f}(\mathbf{Q}\mathbf{v}) \cdot \mathbf{Q}\mathbf{a} = \mathbf{f}(\mathbf{v}) \cdot \mathbf{a},$$

and so

$$\mathbf{Q}^T \mathbf{f}(\mathbf{Q}\mathbf{v}) \cdot \mathbf{a} = \mathbf{f}(\mathbf{v}) \cdot \mathbf{a}.$$

Since this is true for any $\mathbf{a} \in \mathrm{V}$, we must have that $\mathbf{Q}^T \mathbf{f}(\mathbf{Q}\mathbf{v}) = \mathbf{f}(\mathbf{v})$. Thus, we say that a vector-valued function f on V is invariant under $\mathrm{Orth}(n)$ if $\mathbf{f}(\mathbf{Q}\mathbf{v}) = \mathbf{Q}\mathbf{f}(\mathbf{v})$ for all $\mathbf{Q} \in \mathrm{Orth}(n)$.

Next we extend this idea to tensor-valued functions. Let $\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{v}_1, ..., \mathbf{v}_n)$ be a (2nd order) tensor field with vector arguments. Suppose $\phi(\mathbf{v}_1, ..., \mathbf{v}_n, \mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \mathbf{\Phi}(\mathbf{v}_1, ..., \mathbf{v}_n)\mathbf{b}$ is invariant under $\mathrm{Orth}(n)$. Then we as before we have,

$$(\mathbf{Q}\mathbf{a}) \cdot \mathbf{\Phi}(\mathbf{Q}\mathbf{v}_1, ..., \mathbf{Q}\mathbf{v}_n)(\mathbf{Q}\mathbf{b}) = \mathbf{a} \cdot \mathbf{\Phi}(\mathbf{v}_1, ..., \mathbf{v}_n)\mathbf{b}.$$

We also have that

$$(\mathbf{Q}\mathbf{a}) \cdot \mathbf{\Phi}(\mathbf{Q}\mathbf{v}_1, ..., \mathbf{Q}\mathbf{v}_n)(\mathbf{Q}\mathbf{b}) = \mathbf{a}(\mathbf{Q}^T \cdot \mathbf{\Phi}(\mathbf{Q}\mathbf{v}_1, ..., \mathbf{Q}\mathbf{v}_n)\mathbf{Q})\mathbf{b}$$

Finally, since this is true for all $\mathbf{a}$ and $\mathbf{b}$, we have

$$\mathbf{\Phi}(\mathbf{Q}\mathbf{v}_1, ..., \mathbf{Q}\mathbf{v}_n) = \mathbf{Q} \cdot \mathbf{\Phi}(\mathbf{v}_1, ..., \mathbf{v}_n)\mathbf{Q}^T.$$

Thus, we say that a tensor-valued function is invariant under Orth(n), we mean that the above identity is satisfied for all Q $\in$ Orth(n). One can easily generalize these definitions to higher order tensor valued functions whose arguments are vectors.

An important result concerning scalar-valued functions that are invariant under Orth(n) is Cauchy's theorem. Given scalar-valued function $\phi(\mathbf{v}_1, ..., \mathbf{v}_n)$ of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ that is is invariant under Orth(n). Cauchy's theorem [28] says that the function can only depend on the scalar products of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

3.5 Double velocity correlations

In the following our definitions mirror Robertson [25] as opposed to Kármán and Howarth [19]. Also, in our discussion all quantities will depend on a single time value, hence for simplicity we suppress the explicit dependence on the time variable t. Our goal is to understand the relationship of the dynamics at two nearby points. Let $\mathbf{x}$ and $\mathbf{x}' = \mathbf{x} + \mathbf{r}$ be two points in $\mathcal{E}$ (point space), with $\mathbf{r} \in \mathcal{V}$. We consider the following average

$$\langle \mathbf{u}(\mathbf{x}) \otimes \mathbf{u}(\mathbf{x} + \mathbf{r}) \rangle. \tag{3.1}$$

The above is a tensor that represents the correlation between the velocity field at the two points $\mathbf{x}$ and $\mathbf{x} + \mathbf{r}$. First we show that due to homogeneity the above does not depend on the base-point $\mathbf{x}$. By definition,

$$\langle \mathbf{u}(\mathbf{x}) \otimes \mathbf{u}(\mathbf{x} + \mathbf{r}) \rangle = \int_{\mathcal{S}} \mathbf{v}(\mathbf{x}) \otimes \mathbf{v}(\mathbf{x} + \mathbf{r}) \, \Pi(\mathrm{d}\mathbf{v})$$

Let $\mathbf{y}$ be an arbitrary point in Ω. Since the measure Π is invariant with respect to the translation operator we can replace $\mathbf{v}$ with $T_{\mathbf{y}-\mathbf{x}}\mathbf{v}$ inside the measure Π.

$$\int_{\mathcal{S}} \mathbf{v}(\mathbf{x}) \otimes \mathbf{v}(\mathbf{x}+\mathbf{r})\, \Pi(\mathrm{d}T_{\mathbf{y}-\mathbf{x}}\mathbf{v})$$

Then setting $\mathbf{w} = T_{\mathbf{y}-\mathbf{x}}\mathbf{v}$ we find that $\mathbf{v}(\mathbf{z}) = T_{\mathbf{x}-\mathbf{y}}\mathbf{w}(\mathbf{z}) = \mathbf{w}(\mathbf{z} - (\mathbf{x} - \mathbf{y}))$. After these substitutions the above integral becomes

$$\int_{\mathcal{S}} \mathbf{w}(\mathbf{y}) \otimes \mathbf{w}(\mathbf{y}+\mathbf{r})\, \Pi(\mathrm{d}\mathbf{w}) = \langle \mathbf{u}(\mathbf{y}) \otimes \mathbf{u}(\mathbf{y}+\mathbf{r})\rangle.$$

Thus, $\langle \mathbf{u}(\mathbf{x}) \otimes \mathbf{u}(\mathbf{x}+\mathbf{r})\rangle = \langle \mathbf{u}(\mathbf{y}) \otimes \mathbf{u}(\mathbf{y}+\mathbf{r})\rangle$ for all $\mathbf{y}$. Since the above is independent of the base point $\mathbf{x}$ we define the double velocity correlation tensor or just double correlation tensor as

$$\mathbf{R}(\mathbf{r}) = \langle \mathbf{u}(\mathbf{x}) \otimes \mathbf{u}(\mathbf{x}+\mathbf{r})\rangle. \tag{3.2}$$

The above gives the correlation tensor for the velocity fields at two points whose relative configuration is given by the vector $\mathbf{r}$. Moving forward let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors. We define the double velocity correlation as

$$R(\mathbf{r}, \mathbf{a}, \mathbf{b}) := \mathbf{a} \cdot \mathbf{R}(\mathbf{r})\mathbf{b} = \langle (\mathbf{u}(\mathbf{x}) \cdot \mathbf{a})(\mathbf{u}(\mathbf{x}+\mathbf{r}) \cdot \mathbf{b})\rangle. \tag{3.3}$$

The above is a correlation of the velocity fields at two points, with $\mathbf{a}$ representing the direction along which $\mathbf{u}(\mathbf{x})$ is projected and $\mathbf{b}$ representing the direction along which $\mathbf{u}(\mathbf{x}+\mathbf{r})$ is projected. Classically a positively oriented orthonormal frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is introduced. Then, $\mathbf{a}$ and $\mathbf{b}$ are taken to be $\mathbf{e}_i$ and $\mathbf{e}_j$ respectively. In this case the double velocity correlation is written as $R_{ij}(\mathbf{r}) = \langle u_i(\mathbf{x}) u_j(\mathbf{x}')\rangle$ and is traditionally referred to as the double correlation tensor, where it is understood that the components relative to the orthonormal basis, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, make up a tensor.

Claim 1. $\mathbf{R}(\mathbf{r})$ is invariant under Orth.

Proof. Let $\mathbf{Q} \in$ Orth. The key to this proof is the invariance of the measure Π under the operator $R_{\mathbf{Q}}$. Then,

$$\mathbf{R}(\mathbf{Q}\mathbf{r}) = \langle \mathbf{u}(\mathbf{x}) \otimes \mathbf{u}(\mathbf{x} + \mathbf{Q}\mathbf{r}) \rangle$$

$$= \int_{\mathcal{S}} \mathbf{v}(\mathbf{x}) \otimes \mathbf{v}(\mathbf{x} + \mathbf{Q}\mathbf{r}) \, \Pi(\mathrm{d}\mathbf{v})$$

By spatial invariance we can translate $\mathbf{x}$ to $\mathbf{o}$ to obtain,

$$\int_{\mathcal{S}} \mathbf{v}(\mathbf{o}) \otimes \mathbf{v}(\mathbf{o} + \mathbf{Q}\mathbf{r}) \, \Pi(\mathrm{d}\mathbf{v}).$$

Since Π is invariant with respect to $R_{\mathbf{Q}}$ we replace $\mathbf{v}$ with $R_{\mathbf{Q}}^{-1}\mathbf{v}$ inside the measure Π.

$$\int_{\mathcal{S}} \mathbf{v}(\mathbf{o}) \otimes \mathbf{v}(\mathbf{o} + \mathbf{Q}\mathbf{r}) \, \Pi(\mathrm{d}R_{\mathbf{Q}}^{-1}\mathbf{v})$$

We set $\mathbf{w} = R_{\mathbf{Q}}^{-1}\mathbf{v}$ and so we have $\mathbf{v}(\mathbf{z}) = R_{\mathbf{Q}}\mathbf{w}(\mathbf{z}) = \mathbf{Q}\mathbf{w}(\mathbf{Q}^T(\mathbf{z} - \mathbf{o}) + \mathbf{o})$. Plugging these substitutions in the above integral we obtain

$$\int_{\mathcal{S}} \mathbf{Q}\mathbf{w}(\mathbf{o}) \otimes \mathbf{Q}\mathbf{w}(\mathbf{o} + \mathbf{r}) \, \Pi(\mathrm{d}\mathbf{w}).$$

$$= \langle \mathbf{Q}\mathbf{u}(\mathbf{o}) \otimes \mathbf{Q}\mathbf{u}(\mathbf{o} + \mathbf{r}) \rangle$$

$$= \mathbf{Q} \langle \mathbf{u}(\mathbf{o}) \otimes \mathbf{u}(\mathbf{o} + \mathbf{r}) \rangle \mathbf{Q}^T$$

$$= \mathbf{Q}\mathbf{R}(\mathbf{r})\mathbf{Q}^T$$

$\square$

As a consequence we also have that $R(\mathbf{r}, \mathbf{a}, \mathbf{b})$ is an invariant under Orth as a scalar-valued function. Let $Q \in$ Orth, then $R(\mathbf{Q}\mathbf{r}, \mathbf{Q}\mathbf{a}, \mathbf{Q}\mathbf{b}) = (\mathbf{Q}\mathbf{a}) \cdot \mathbf{Q}\mathbf{R}(\mathbf{r})\mathbf{Q}^T(\mathbf{Q}\mathbf{b}) =$

$R(\mathbf{r}, \mathbf{a}, \mathbf{b})$

Claim 2. $\mathbf{R}(\mathbf{r})$ is a symmetric tensor.

Proof. By definition $\mathbf{R}(\mathbf{r})$ is symmetric if and only if $\mathbf{a} \cdot \mathbf{R}(\mathbf{r})\mathbf{b} = \mathbf{b} \cdot \mathbf{R}(\mathbf{r})\mathbf{a}$ for all vectors $\mathbf{a}$ and $\mathbf{b}$. Since $R(\mathbf{r}, \mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \mathbf{R}\mathbf{b}$ we see that $\mathbf{R}(\mathbf{r})$ is symmetric if and only if $R(\mathbf{r}, \mathbf{a}, \mathbf{b}) = R(\mathbf{r}, \mathbf{b}, \mathbf{a})$ for all $\mathbf{a}$, $\mathbf{b} \in \mathcal{V}$.

First we show that $R(\mathbf{r}, \mathbf{a}, \mathbf{b}) = R(-\mathbf{r}, \mathbf{b}, \mathbf{a})$,

$$
\begin{aligned}
R(-\mathbf{r}, \mathbf{b}, \mathbf{a}) &= \langle (\mathbf{u}(\mathbf{x}) \cdot \mathbf{b})(\mathbf{u}(\mathbf{x} - \mathbf{r}) \cdot \mathbf{a}) \rangle \\
&= \langle (\mathbf{u}(\mathbf{x} + \mathbf{r}) \cdot \mathbf{b})(\mathbf{u}(\mathbf{x}) \cdot \mathbf{a} \rangle && \text{translating by } \mathbf{r} \\
&= \langle (\mathbf{u}(\mathbf{x}) \cdot \mathbf{a}) \cdot (\mathbf{u}(\mathbf{x} + \mathbf{r}) \cdot \mathbf{b}) \rangle \\
&= R(\mathbf{r}, \mathbf{a}, \mathbf{b})
\end{aligned}
$$

Next, we note that $\mathbf{R}(-\mathbf{r}) = \mathbf{R}(\mathbf{r})$, since $\mathbf{R}(\mathbf{r})$ is invariant under reflections. Thus, $R(-\mathbf{r}, \mathbf{a}, \mathbf{b}) = R(\mathbf{r}, \mathbf{a}, \mathbf{b})$. On combining these two identities, we obtain $R(\mathbf{r}, \mathbf{a}, \mathbf{b}) = R(-\mathbf{r}, \mathbf{a}, \mathbf{b}) = R(\mathbf{r}, \mathbf{b}, \mathbf{a})$. □

Note that the above proof only needed $\mathbf{R}(\mathbf{r})$ to be spatially invariant and invariant under basic reflections and not under rotations.

We now discuss the representation of the double–velocity correlation tensor. Let $\mathbf{r}$ be a nonzero vector. Define the unit vector $\mathbf{e}$ aligned with $\mathbf{r}$ by

$$\mathbf{e} = \frac{1}{r}\mathbf{r}, \qquad r = |\mathbf{r}|. \tag{3.4}$$

Further, define the projectors $\mathbf{P}(\mathbf{e})$ onto $\mathbf{e}$ and $\mathbf{P}^{\perp}(\mathbf{e})$ onto a plane with unit normal to $\mathbf{e}$ by

$$\mathbf{P}(\mathbf{e}) = \mathbf{e} \otimes \mathbf{e} \qquad \text{and} \qquad \mathbf{P}^{\perp}(\mathbf{e}) = I - \mathbf{e} \otimes \mathbf{e}. \tag{3.5}$$

Trivially, but importantly, $\mathbf{P}(\mathbf{e})$ and $\mathbf{P}^{\perp}(\mathbf{e})$ are orthogonal. Since $\mathbf{R}$ is a symmetric isotropic tensor-valued mapping depending on $\mathbf{r}$, standard results from representation theory yield

$$\mathbf{R}(\mathbf{r}) = A(r)\mathbf{P}(\mathbf{e}) + B(r)\mathbf{P}^{\perp}(\mathbf{e}), \tag{3.6}$$

where A and B are arbitrary scalar-valued mappings. Note that, due to the orthogonality of $\mathbf{P}(\mathbf{e})$ and $\mathbf{P}^{\perp}(\mathbf{e})$, the two terms entering the representation (3.6) are independent. Consequently $R(\mathbf{r}, \mathbf{a}, \mathbf{b})$ can be written as

$$R(\mathbf{r}, \mathbf{a}, \mathbf{b}) = A(r)(\mathbf{a} \cdot \mathbf{e})(\mathbf{b} \cdot \mathbf{e}) + B(r)\big(\mathbf{a} \cdot \mathbf{b} - (\mathbf{a} \cdot \mathbf{e})(\mathbf{b} \cdot \mathbf{e})\big) \tag{3.7}$$

3.5.1 Derivatives of averaged quantities

We investigate taking derivatives of averaged quantities that depend on the separation $\mathbf{r}$. e.g., $\frac{\partial}{\partial \mathbf{r}_k}\langle \mathbf{u}(\mathbf{x}) \otimes \mathbf{u}(\mathbf{x}')\rangle$. Specifically we are interested on how the rates of change of the functions inside the average relate to derivatives of the average. For notational simplicity we restrict to one spatial dimension, higher dimensional analogs are exactly the same. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be differentiable functions and let x and $x' = x + r$ be two points. We investigate $\frac{\partial}{\partial r}\langle f(x)g(x')\rangle$. Following the previous discussion we assume that the average is independent of the base-point x. Then,

$$\frac{\partial}{\partial r}\langle f(x)g(x+r)\rangle = \langle \frac{\partial}{\partial r}\big(f(x)g(x+r)\big)\rangle = \langle f(x)\frac{\partial}{\partial r}g(x+r)\rangle = \langle f(x)\frac{\partial}{\partial x'}g(x')\rangle.$$

Since the average is independent of the base-point x, we translate x by $-r$ and take the partial derivative.

$$\frac{\partial}{\partial r}\langle f(x-r)g(x)\rangle = \langle \frac{\partial}{\partial r}\big(f(x-r)g(x)\big)\rangle = \langle \frac{\partial}{\partial r}\big(f(x-r)\big)g(x)\rangle$$
$$= \langle -\frac{\partial}{\partial (x-r)}\big(f(x-r)\big)g(x)\rangle$$

Finally translating x by r we obtain

$$\langle -\frac{\partial}{\partial x} f(x)g(x+r)\rangle$$

Thus written succinctly we have the following rule,

$$\frac{\partial}{\partial r}\langle f(x)g(x')\rangle = \langle f(x)\frac{\partial}{\partial x'}g(x')\rangle = \langle -\frac{\partial}{\partial x}\big(f(x)\big)g(x')\rangle \tag{3.8}$$

The second equality above can be interpreted as an integration by parts type property. In fact if one uses a spatial average over a periodic region such as in Batchelor [2], then this identity follows from integrations by parts in x. Note, that if we replace x with $\mathbf{x}$, x' with $\mathbf{x}'$, and r with $\mathbf{r}_i$, then all of the above formulas hold with the appropriate replacements.

3.5.2 Divergence free tensors with application to the double correlation tensor

Let $\mathbf{T}(\mathbf{x})$ be a (second-order) tensor–valued function, the divergence of this tensor is defined as follows.

$$\mathbf{a}\cdot \operatorname{div}\mathbf{T} = \operatorname{div}(\mathbf{T}(\mathbf{x})^T\mathbf{a})$$

for all $\mathbf{a} \in \mathcal{V}$.

Definition 3.1. We say that a tensor–valued function $\mathbf{T}(\mathbf{x})$ of $\mathbb{R}^n$ is divergence free if $\operatorname{div} \mathbf{T}(\mathbf{x}) = \mathbf{0}$.

This definition generalizes the idea of a divergence free vector field. Note that for a symmetric tensor field $\mathbf{R}$, $\mathbf{R}$ is divergence free if and only if $\mathbf{R}^T$ is divergence free. As we now show the double velocity correlation tensor defined in section 3.5 is an example of a divergence free tensor. Since we are dealing with incompressible fluids we have that $\operatorname{div} \mathbf{u}(\mathbf{x}) = \mathbf{0}$. Next we compute $\operatorname{div} \mathbf{R}(\mathbf{r})$. Let $\mathbf{a} \in \mathcal{V}$, then

$$\operatorname{div}_{\mathbf{r}} \langle (\mathbf{u} \otimes \mathbf{u}')^T \mathbf{a} \rangle = \operatorname{div}_{\mathbf{r}} \langle (\mathbf{u} \cdot \mathbf{a}) \mathbf{u}' \rangle \tag{3.9}$$

$$= \langle (\mathbf{u} \cdot \mathbf{a}) \operatorname{div}_{\mathbf{x}'} \mathbf{u}' \rangle \tag{3.10}$$

$$= \mathbf{0}. \tag{3.11}$$

Since $\mathbf{R}$ is divergence free, certain restrictions are imposed on the scalar quantities A and B. On computing $\operatorname{div} \mathbf{R}$, we find that

$$\begin{aligned}
\operatorname{div} \mathbf{R}\big(\mathbf{r}) &= \operatorname{div}(A(r)\mathbf{e} \otimes \mathbf{e} + B(r)(I - \mathbf{e} \otimes \mathbf{e})\big) \\
&= \mathbf{e} \otimes \mathbf{e} \operatorname{grad} A(r) + A(r) \operatorname{div}(\mathbf{e} \otimes \mathbf{e}) \\
&\quad + (I - \mathbf{e} \otimes \mathbf{e}) \operatorname{grad} B(r) - B(r) \operatorname{div}(\mathbf{e} \otimes \mathbf{e}) \\
&= \mathbf{e} \otimes \mathbf{e}(A'(r)\mathbf{e}) + A(r)(\frac{2}{r}\mathbf{e}) + (I - \mathbf{e} \otimes \mathbf{e})(B'(r)\mathbf{e}) - B(r)(\frac{2}{r}\mathbf{e}) \\
&= A'(r)\mathbf{e} + \frac{2}{r}A(r)\mathbf{e} + B'(r)\mathbf{e} - B'(r)\mathbf{e} - \frac{2}{r}B(r)\mathbf{e} \\
&= [A'(r) + \frac{2}{r}A(r) - \frac{2}{r}B(r)]\mathbf{e} = 0.
\end{aligned}$$

Since this must hold for all $\mathbf{e} = \mathbf{r}/r$, $A(\mathbf{r})$ and $B(\mathbf{r})$ must satisfy

$$A'(r) + \frac{2}{r}A(r) - \frac{2}{r}B(r) = 0$$

Using this equation we can solve for $B(r)$ in terms or $A(r)$ to give the following equation for $\mathbf{R}(\mathbf{r})$,

$$\mathbf{R}(\mathbf{r}) = A(r)\mathbf{e}\otimes\mathbf{e} + \left(\frac{r}{2}A'(r) + A(r)\right)(I - \mathbf{e}\otimes\mathbf{e}). \tag{3.12}$$

The double velocity correlation tensor is now written in terms of a single scalar function. In these situations we will refer to $A(r)$ as the scalar defining function for $\mathbf{R}(\mathbf{r})$.

3.5.3 Divergence free vector-valued functions that are invariant under Orth

Divergence free, orthogonally invariant vectors often arise when investigating correlations. Let $\mathbf{u}(\mathbf{r})$ be an vector–valued mapping dependent on $\mathbf{r}$, that is divergence free and invariant under Orth. By the theory of invariants, $\mathbf{u}$ must have the form $\mathbf{u}(\mathbf{r}) = l(r)\mathbf{r}$, for some scalar-valued function l of r. Since $\mathbf{u}(\mathbf{r})$ is divergence free, $\operatorname{div}\mathbf{u}(\mathbf{r}) = 0$. Thus,

$$\operatorname{div}\big(l(r)\mathbf{r}\big) = \big(\operatorname{grad} l(r)\big)\cdot\mathbf{r} + l(r)\operatorname{div}\mathbf{r} = l'(r)\frac{\mathbf{r}}{r}\cdot\mathbf{r} + 3l(r) = rl'(r) + 3l(r) = 0.$$

The ODE, $rl'(r) + 3l(r) = 0$, has the solution $l(r) = C/r^3$ where C is an arbitrary constant. Since l is bounded near $r = 0$, we must have $C = 0$. Thus, $l(r) \equiv 0$. An important example of such a vector field is $\langle p(\mathbf{x})\mathbf{u}(\mathbf{x}+\mathbf{r})\rangle$, where p is the pressure field. We quickly check that $\langle p(\mathbf{x})\mathbf{u}(\mathbf{x}+\mathbf{r})\rangle$ is in fact divergence free and orthogonally invariant. We use the notation $\operatorname{div}_{\mathbf{r}}$ to emphasize that the divergence is taking place with respect to $\mathbf{r}$ and not $\mathbf{x}$.

$$\operatorname{div}_{\mathbf{r}}\langle p(\mathbf{x})\mathbf{u}(\mathbf{x}+\mathbf{r})\rangle = \langle p(\mathbf{x})\operatorname{div}_{\mathbf{r}}\mathbf{u}(\mathbf{x}+\mathbf{r})\rangle = 0$$

and for $\mathbf{Q} \in Orth$

$$\begin{aligned}
\langle p(\mathbf{x})\mathbf{u}(\mathbf{x}+\mathbf{Q}\mathbf{r})\rangle &= \int_{\mathcal{S}} q(\mathbf{x})\mathbf{v}(\mathbf{x}+\mathbf{Q}\mathbf{r})\,\Pi\big(\mathrm{d}(\mathbf{v},q)\big) \\
&= \int_{\mathcal{S}} q(\mathbf{o})\mathbf{v}(\mathbf{o}+\mathbf{Q}\mathbf{r})\,\Pi\big(\mathrm{d}(\mathbf{v},q)\big) && \text{translate } \mathbf{x} \text{ to } \mathbf{o} \\
& && \text{by homogeneity of } \Pi \\
&= \int_{\mathcal{S}} q_1(\mathbf{o})\mathbf{Q}\mathbf{w}(\mathbf{o}+\mathbf{r})\,\Pi\big(\mathrm{d}(\mathbf{w},q_1)\big) && \text{set } (\mathbf{v},q) = R_{\mathbf{Q}}(\mathbf{w},q_1) \\
& && \text{and invariance of } \Pi \text{ under } R_{\mathbf{Q}} \\
&= \int_{\mathcal{S}} q_1(\mathbf{x})\mathbf{Q}\mathbf{w}(\mathbf{x}+\mathbf{r})\,\Pi\big(\mathrm{d}(\mathbf{w},q_1)\big) && \text{translate back by } \mathbf{x} \\
&= \langle p(\mathbf{x})\mathbf{Q}\mathbf{u}(\mathbf{x}+\mathbf{r})\rangle \\
&= \mathbf{Q}\langle p(\mathbf{x})\mathbf{u}(\mathbf{x}+\mathbf{r})\rangle
\end{aligned}$$

Thus by the preceding remarks,

$$\langle p(\mathbf{x})\mathbf{u}(\mathbf{x}+\mathbf{r})\rangle = 0.$$

3.6 The Kármán–Howarth equation for the Navier–Stokes-$\alpha\beta$ equations

We take the NS-$\alpha\beta$ equations in the following form

$$\begin{gathered}
\tfrac{\partial \mathbf{v}}{\partial t} + (\operatorname{grad}\mathbf{v})\mathbf{u} + (\operatorname{grad}\mathbf{u})^T\mathbf{v} = -\operatorname{grad}p + \nu\Delta\mathbf{w}, \\
\operatorname{div}\mathbf{u} = 0,
\end{gathered} \tag{3.1}$$

where $\mathbf{v} = (1-\alpha^2\Delta)\mathbf{u}$ and $\mathbf{w} = (1-\beta^2\Delta)\mathbf{u}$. Of course we continue to assume that we are dealing with a homogeneous isotropic flow with an isotropic measure.

3.6.1 Applying the dynamics of the NS-$\alpha\beta$ to the velocity correlation tensor

Since we view $\mathbf{u}$ as the evolving filtered velocity field, we seek to determine how the the correlation $\langle \mathbf{u}(\mathbf{x}) \otimes \mathbf{u}(\mathbf{x})' \rangle$ evolves in time. To simplify the following notation we write $\mathbf{u}$ for $\mathbf{u}(\mathbf{x})$ and $\mathbf{u}'$ for $\mathbf{u}(\mathbf{x}')$. As issue is that $\partial \mathbf{v}/\partial t$ appears in (3.1) instead of $\partial \mathbf{u}/\partial t$. We could invert $(1-\alpha^2\Delta)$, but this would introduce other complications. Instead, we consider

$$\langle \frac{\partial \mathbf{v}}{\partial t} \otimes \mathbf{u}' \rangle + \langle \mathbf{u} \otimes \frac{\partial \mathbf{v}'}{\partial t} \rangle \tag{3.2}$$

Using the ideas from section 3.5.1 we switch taking the laplacian with respect to $\mathbf{x}$ and $\mathbf{x}'$ in $\mathbf{v} = (1-\alpha^2\Delta)\mathbf{u}$ and $\mathbf{v}' = (1-\alpha^2\Delta')\mathbf{u}'$ respectively, to $\mathbf{r}$. After changing the derivative we can factor the operator $(1-\alpha^2\Delta)$ out of both averages to obtain,

$$(1-\alpha^2\Delta)\left(\langle \frac{\partial \mathbf{u}}{\partial t} \otimes \mathbf{u}' \rangle + \langle \mathbf{u} \otimes \frac{\partial \mathbf{u}'}{\partial t} \rangle \right)$$

$$= (1-\alpha^2\Delta)\left(\frac{\partial}{\partial t} \langle \mathbf{u} \otimes \mathbf{u}' \rangle \right), \tag{3.3}$$

with the understanding that the laplacian Δ is now with respect to the separation vector $\mathbf{r}$. The above gives us the time rate change of the double correlation tensor multiplied by the operator $(1-\alpha^2\Delta)$. Looking back at (3.2), we may now substitute in for $\partial \mathbf{v}/\partial t$ using the NS-$\alpha\beta$ equations. The following derivation is done using coordinates to avoid cumbersome expressions, but can be done coordinate-free as long as one is careful when dealing with the triple tensor products. In terms of components, (3.2) reads

$$\langle \frac{\partial v_i}{\partial t} u_j' \rangle + \langle u_i \frac{\partial v_j'}{\partial t} \rangle.$$

Isolating the first term and substituting for $\partial \mathbf{v}/\partial t$ we obtain,

$$\langle \frac{\partial v_i}{\partial t} u_j' \rangle = \langle \big[-\frac{\partial v_i}{\partial x_k} u_k - \frac{\partial u_k}{\partial x_i} v_k - \frac{\partial p}{\partial x_k} \delta_{ik} + \nu \Delta (1 - \beta^2 \Delta) u_i \big] u_j' \rangle$$

$$= \langle - \frac{\partial v_i}{\partial x_k} u_k u_j' - \frac{\partial u_k}{\partial x_i} v_k u_j' - \frac{\partial p}{\partial x_k} u_j' \delta_{ik} + \nu \big(\Delta (1 - \beta^2 \Delta) u_i \big) u_j' \rangle. \tag{3.4}$$

Now our goal is to express all the terms as double and triple correlations tensors in $\mathbf{u}$. To this end, we will use the property that $\mathbf{u}$ and that $\mathbf{v} = (1 - \alpha^2 \Delta)\mathbf{u}$ are divergence free. Thus,

$$\frac{\partial u_k}{\partial x_k} = 0$$

and so the first term in (3.4) becomes

$$\langle - \frac{\partial v_i}{\partial x_k} u_k u_j' \rangle = \langle - \frac{\partial}{\partial x_k} (v_i u_k) u_j' \rangle = \frac{\partial}{\partial r_k} \langle v_i u_k u_j' \rangle$$

Then writing $v_i = (1 - \alpha^2 \Delta) u_i$ the previous equation becomes

$$\frac{\partial}{\partial r_k} \langle \big((1 - \alpha^2 \Delta) u_i \big) u_k u_j' \rangle = \frac{\partial}{\partial r_k} \langle u_i u_k u_j' \rangle - \alpha^2 \frac{\partial}{\partial r_k} \langle (\Delta u_i) u_k u_j' \rangle.$$

As we did to obtain equation (3.3) we take the laplacian with respect to $\mathbf{r}$ and factor the laplacian out from the average. To do so we use the standard product rule to yield

$$(1 - \alpha^2 \Delta) \frac{\partial}{\partial r_k} \langle u_i u_k u_j' \rangle + \alpha^2 \frac{\partial}{\partial r_k} \langle u_i (\Delta u_k) u_j' \rangle + 2\alpha^2 \frac{\partial}{\partial r_k} \langle \frac{\partial u_i}{\partial x_l} \frac{\partial u_k}{\partial x_l} u_j' \rangle. \tag{3.5}$$

The last step might seem counter productive, but is motivated by the desire to group terms in a way that preserves the operator $(1 - \alpha^2 \Delta)$. Next, considering the second term in (3.4) we obtain

$$\langle -\frac{\partial u_k}{\partial x_i} v_k u_j' \rangle = \langle -\frac{\partial u_k}{\partial x_i} u_k u_j' \rangle - \alpha^2 \langle -\frac{\partial u_k}{\partial x_i} (\Delta u_k) u_j' \rangle$$

$$= \frac{1}{2} \frac{\partial}{\partial r_i} \langle u_k u_k u_j' \rangle + \alpha^2 \langle \frac{\partial u_k}{\partial x_i} (\Delta u_k) u_j' \rangle.$$

Note that the components of the average in the first term, $\langle u_k u_k u_j' \rangle = \langle \mathbf{u} \cdot \mathbf{u} u_j' \rangle$, form a linear orthogonally invariant tensor and so it is zero! Thus, the above expression simplifies to,

$$\alpha^2 \langle \frac{\partial u_k}{\partial x_i} (\Delta u_k) u_j' \rangle. \tag{3.6}$$

The third term in (3.4) is $-\langle (\partial p / \partial x_k) u_j' \delta_{ik} \rangle$ and this is zero since $\langle p \mathbf{u}' \rangle$ is an orthogonally invariant divergence free vector. As before we have $\langle p \mathbf{u}' \rangle = \mathbf{0}$ and so $-\langle \frac{\partial p}{\partial x_k} u_j' \delta_{ik} \rangle = \frac{\partial}{\partial r_k} \langle p u_j' \rangle \delta_{ik} = 0$. Finally, the last term in (3.4) can be written as

$$\nu \Delta (1 - \beta^2 \Delta) \langle u_i u_j' \rangle. \tag{3.7}$$

Next, (3.4) equals

$$\begin{aligned} &(1 - \alpha^2 \Delta) \tfrac{\partial}{\partial r_k} \langle u_i u_k u_j' \rangle + \alpha^2 \tfrac{\partial}{\partial r_k} \langle u_i (\Delta u_k) u_j' \rangle \\ &+ 2\alpha^2 \tfrac{\partial}{\partial r_k} \langle \tfrac{\partial u_i}{\partial x_l} \tfrac{\partial u_k}{\partial x_l} u_j' \rangle + \alpha^2 \langle \tfrac{\partial u_k}{\partial x_i} (\Delta u_k) u_j' \rangle + \nu \Delta (1 - \beta^2 \Delta) \langle u_i u_j' \rangle \end{aligned} \tag{3.8}$$

Now, using the same ideas we find that the second term in (3.2) is equal to

$$\begin{aligned} &-(1 - \alpha^2 \Delta) \tfrac{\partial}{\partial r_k} \langle u_i u_k' u_j' \rangle - \alpha^2 \tfrac{\partial}{\partial r_k} \langle u_i (\Delta' u_k') u_j' \rangle \\ &- 2\alpha^2 \tfrac{\partial}{\partial r_k} \langle u_i \tfrac{\partial u_k'}{\partial x_l'} \tfrac{\partial u_j'}{\partial x_l'} \rangle + \alpha^2 \langle u_i \tfrac{\partial u_k'}{\partial x_j'} (\Delta' u_k') \rangle + \nu \Delta (1 - \beta^2 \Delta) \langle u_i u_j' \rangle \end{aligned} \tag{3.9}$$

Using invariance of the measure Π under translation and reflection, one can show the following identities for the second, third and fourth terms above.

$$\begin{aligned}
\langle u_i(\Delta' u'_k)u'_j\rangle &= -\langle u'_i(\Delta u_k)u_j\rangle \\
\langle u_i \frac{\partial u'_k}{\partial x'_l}\frac{\partial u'_j}{\partial x'_l}\rangle &= -\langle u'_i \frac{\partial u_k}{\partial x_l}\frac{\partial u_j}{\partial x_l}\rangle \\
\langle u_i \frac{\partial u'_k}{\partial x'_j}(\Delta' u'_k)\rangle &= \langle u'_i \frac{\partial u_k}{\partial x_j}(\Delta u_k)\rangle.
\end{aligned}$$

Thus, (3.9) equals

$$\begin{aligned}
(1-\alpha^2\Delta)\tfrac{\partial}{\partial r_k}\langle u'_i u_k u_j\rangle + \alpha^2\tfrac{\partial}{\partial r_k}\langle u'_i(\Delta u_k)u_j\rangle + 2\alpha^2\tfrac{\partial}{\partial r_k}\langle u'_i \tfrac{\partial u_i}{\partial x_l}\tfrac{\partial u_j}{\partial x_l}\rangle \\
-\alpha^2\langle u'_i \tfrac{\partial u_k}{\partial x_j}(\Delta u_k)\rangle + \nu\Delta(1-\beta^2\Delta)\langle u_i u'_j\rangle.
\end{aligned} \tag{3.10}$$

We make the following definitions:

$$R_{ij}(\mathbf{r}) = \langle u_i u'_j\rangle \tag{3.11}$$

$$T_{ij}(\mathbf{r}) = \frac{\partial}{\partial r_k}\langle u_i u_k u'_j\rangle + \frac{\partial}{\partial r_k}\langle u'_i u_k u_j\rangle \tag{3.12}$$

$$\begin{aligned}
S_{ij}(\mathbf{r}) = {} & \frac{\partial}{\partial r_k}\langle u_i(\Delta u_k)u'_j\rangle \\
& + \frac{\partial}{\partial r_k}\langle u'_i(\Delta u_k)u_j\rangle + 2\frac{\partial}{\partial r_k}\langle \frac{\partial u_i}{\partial x_l}\frac{\partial u_k}{\partial x_l}u'_j\rangle + 2\frac{\partial}{\partial r_k}\langle u'_i\frac{\partial u_k}{\partial x_l}\frac{\partial u_j}{\partial x_l}\rangle
\end{aligned} \tag{3.13}$$

$$F_{ij}(\mathbf{r}) = \langle \frac{\partial u_k}{\partial x_i}(\Delta u_k)u'_j\rangle - \langle u'_i\frac{\partial u_k}{\partial x_j}(\Delta u_k)\rangle \tag{3.14}$$

Let $\mathbf{R}(\mathbf{r})$, $\mathbf{T}(\mathbf{r})$, $\mathbf{S}(\mathbf{r})$, and $\mathbf{F}(\mathbf{r})$ be the tensors whose components are given by $R_{ij}(\mathbf{r})$, $T_{ij}(\mathbf{r})$, and $S_{ij}(\mathbf{r})$, and $F_{ij}(\mathbf{r})$ respectively. Now we add equations (3.8) and (3.10) as in (3.2) and use definitions (3.11)-(3.14) to yield

$$(1-\alpha^2\Delta)R_{ij}(\mathbf{r}) = 2\nu\Delta(1-\beta^2\Delta)R_{ij}(\mathbf{r}) + (1-\alpha^2\Delta)T_{ij}(\mathbf{r}) + \alpha^2 S_{ij}(\mathbf{r}) + \alpha^2 F_{ij}(\mathbf{r}) \tag{3.15}$$

and in terms of the tensor fields

$$(1-\alpha^2\Delta)\mathbf{R}(\mathbf{r}) = 2\nu\Delta(1-\beta^2\Delta)\mathbf{R}(\mathbf{r}) + (1-\alpha^2\Delta)\mathbf{T}(\mathbf{r}) + \alpha^2\mathbf{S}(\mathbf{r}) + \alpha^2\mathbf{F}(\mathbf{r}). \quad (3.16)$$

By inspection of their component forms we see that $\mathbf{R}(\mathbf{r})$, $\mathbf{T}(\mathbf{r})$, and $\mathbf{S}(\mathbf{r})$ are all symmetric. On the other hand equation (3.14) suggests that $\mathbf{F}(\mathbf{r})$ is skew-symmetric, but by equation (3.16) we must have that $\mathbf{F}(\mathbf{r})$ is symmetric. Thus, we must have $\mathbf{F}(\mathbf{r}) = \mathbf{0}$ and so (3.16) becomes

$$(1-\alpha^2\Delta)\mathbf{R}(\mathbf{r}) = 2\nu\Delta(1-\beta^2\Delta)\mathbf{R}(\mathbf{r}) + (1-\alpha^2\Delta)\mathbf{T}(\mathbf{r}) + \alpha^2\mathbf{S}(\mathbf{r}). \quad (3.17)$$

Now we show that $\mathbf{T}(\mathbf{r})$ is divergence free.

$$\begin{aligned}
\frac{\partial}{\partial r_j}T_{ij}(\mathbf{r}) &= \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_k}[\langle u_i u_k u_j'\rangle + \langle u_i' u_k u_j\rangle] \\
&= \frac{\partial}{\partial r_k}\langle u_i u_k \frac{\partial u_j'}{\partial x_j'}\rangle - \frac{\partial}{\partial r_k}\langle u_i' \frac{\partial u_k}{\partial x_j} u_j)\rangle \\
&= 0 + \langle u_i' \frac{\partial u_k}{\partial x_j}\frac{\partial u_j}{\partial x_k}\rangle
\end{aligned}$$

but the components of this last term form an orthogonally invariant vector and so as in Section 3.5.3 it must be zero! Therefore, $\mathbf{T}(\mathbf{r})$ is divergence free. From Section 3.5, $\mathbf{R}(\mathbf{r})$ is divergence free, and from equation (3.17) we conclude that $\mathbf{S}(\mathbf{r})$ is divergence free.

3.6.2 Deriving the Kármán–Howarth type equation for the NS-$\alpha\beta$ equations

As discussed in Section 3.5.3 an orthogonally invariant, symmetric, divergence free tensor can be written in terms of a what we refer to as the scalar defining function. Let $Q(r)$, $T(r)$, and $S(r)$ be the scalar defining functions for $\mathbf{R}(\mathbf{r})$, $\mathbf{T}(\mathbf{r})$, and $\mathbf{S}(\mathbf{r})$ respectively. We want to write (3.17) as an equation involving only the defining scalar functions. To do so, we need to compute the laplacian of the tensor fields in terms of their defining scalar functions. Let $\mathbf{F}(\mathbf{r})$ be an orthogonally invariant divergence free tensor with defining scalar $f(r)$.

We let D be a differential operator defined on scalar valued function of r by

$$D(f(r)) = \frac{1}{r^4}\frac{\partial}{\partial r} r^4 \frac{\partial}{\partial r} f(r) = f''(r) + \frac{4}{r} f'(r).$$

Our goal is to show that if $f(r)$ is a scalar defining function for $\mathbf{F}(\mathbf{r})$, then $D(f(r))$ is the corresponding scalar defining function for $\Delta\mathbf{F}(\mathbf{r})$. So we compute:

$$\Delta\mathbf{F}(\mathbf{r}) = \Delta\Big(f(r)\mathbf{e}\otimes\mathbf{e} + \big(\frac{r}{2}f'(r) + f(r)\big)\big(I - \mathbf{e}\otimes\mathbf{e}\big)\Big) \tag{3.18}$$

$$= \Delta\Big[-\frac{1}{2} r f'(r)\mathbf{e}\otimes\mathbf{e} + \Big(\frac{1}{2} r f'(r) + f(r)\Big) I\Big] \tag{3.19}$$

$$= \Delta\Big[-\frac{1}{2r} f'(r)\mathbf{r}\otimes\mathbf{r} + \Big(\frac{1}{2} r f'(r) + f(r)\Big) I\Big] \tag{3.20}$$

To simplify the following calculation we let

$$g(r) = -\frac{1}{2r} f'(r)$$

and

$$h(r) = \frac{1}{2} r f'(r) + f(r).$$

Remembering that Δ is taken with respect to $\mathbf{r}$, we have the identity $\Delta g(r) = g''(r) + (2/r)g'(r)$. We need the following identities, which are easily derived in the appendix using tensor diagrams:

$$\operatorname{grad}(\mathbf{r}\otimes\mathbf{r})\mathbf{r} = 2\mathbf{r}\otimes\mathbf{r}$$

$$\Delta(\mathbf{r}\otimes\mathbf{r}) = 2I$$

We carry on the calculation from equation (3.20) with $g(r)$ and $h(r)$

$$\begin{aligned}
\Delta[g(r)\mathbf{r}\otimes\mathbf{r} + h(r)I] &= \Delta[g(r)\mathbf{r}\otimes\mathbf{r}] + \Delta h(r)I \\
&= \Delta g(r)\mathbf{r}\otimes\mathbf{r} + 2\operatorname{grad}(\mathbf{r}\otimes\mathbf{r})\operatorname{grad} g(r) + g(r)\Delta\mathbf{r}\otimes\mathbf{r} + \Delta h(r)I \\
&= \Delta g(r)\mathbf{r}\otimes\mathbf{r} + 2\operatorname{grad}(\mathbf{r}\otimes\mathbf{r})\Big(\frac{g'(r)}{r}\mathbf{r}\Big) + g(r)(2I) + \Delta h(r)I \\
&= \Delta g(r)\mathbf{r}\otimes\mathbf{r} + \frac{2g'(r)}{r}\operatorname{grad}(\mathbf{r}\otimes\mathbf{r})\mathbf{r} + 2g(r)I + \Delta h(r)I \\
&= \Delta g(r)\mathbf{r}\otimes\mathbf{r} + \frac{4g'(r)}{r}(\mathbf{r}\otimes\mathbf{r}) + 2g(r)I + \Delta h(r)I \\
&= \Big(\Delta g(r) + \frac{4g'(r)}{r}\Big)\mathbf{r}\otimes\mathbf{r} + \big(2g(r) + \Delta h(r)\big)I
\end{aligned}$$

Then in terms of the function $f(r)$ the above equation becomes,

$$\Delta\mathbf{F}(\mathbf{r}) = \Big(\frac{2}{r^3}f'(r) + \frac{2}{r^2}f''(r) - \frac{1}{2r}f'''(r)\Big)\mathbf{r}\otimes\mathbf{r} + \Big(\frac{2}{r}f'(r) + 3f''(r) + \frac{r}{2}f'''(r)\Big)I \quad (3.21)$$

Therefore, to verify that $D(f(\mathbf{r}))$ is the scalar defining function of $\Delta\mathbf{F}(\mathbf{r})$ we must show that

$$D(f(r))\mathbf{e}\otimes\mathbf{e} + \big(\frac{r}{2}\frac{\partial}{\partial r}D(f(r)) + D(f(r))\big)\big(I - \mathbf{e}\otimes\mathbf{e}\big)\Big) \quad (3.22)$$

is equal to the right hand side of equation (3.21). We can rewrite the above equation as

$$-\frac{1}{2r}\frac{\partial}{\partial r}D(f(r))\mathbf{r}\otimes\mathbf{r}+\Big(\frac{1}{2}r\frac{\partial}{\partial r}D(f(r))+D(f(r))\Big)I \tag{3.23}$$

$$\tag{3.24}$$

Expanding the operator D we obtain,

$$-\frac{1}{2r}\frac{\partial}{\partial r}\big[f''(r)+\frac{4}{r}f'(r)\big]\mathbf{r}\otimes\mathbf{r}+\big(\frac{1}{2}r\frac{\partial}{\partial r}\big[f''(r)+\frac{4}{r}f'(r)\big]+f''(r)+\frac{4}{r}f'(r)\big)I$$

$$=\Big(\frac{2}{r^3}f'(r)-\frac{2}{r^2}f''(r)-\frac{1}{2r}f'''(r)\Big)\mathbf{r}\otimes\mathbf{r}+\Big(\frac{2}{r}f'(r)+3f''(r)+\frac{r}{2}f'''(r)\Big)I=\mathbf{\Delta F}(\mathbf{r})$$

Thus, the defining scalar function for $\mathbf{\Delta F}$ is $D(f(\mathbf{r}))$.

Now we return to equation (3.17). Using the fact that applying the laplacian to one of the tensors in (3.17) is the same as applying the operator D to the defining scalar function, we obtain an equivalent equation relating the defining scalar functions.

$$(1-\alpha^2D)\frac{\partial Q}{\partial t}=2\nu D(1-\beta^2D)Q+(1-\alpha^2D)T+\alpha^2S. \tag{3.25}$$

This is one form of a Kármán–Howarth type equation for the NS-$\alpha\beta$ equations. Henceforth, we with refer to this as the KH-$\alpha\beta$ equation.

3.6.3 Discussion of the KH-$\alpha\beta$ equation

At this stage a few comments are in order. First if we set $\beta=\alpha$, we arrive at the LANS-α equations as a special case of the NS-$\alpha\beta$ equations. So in the above equation setting $\beta = \alpha$ we obtain,

$$(1-\alpha^2 D)\frac{\partial Q}{\partial t} = 2\nu D(1-\alpha^2 D)Q + (1-\alpha^2 D)T + \alpha^2 S. \tag{3.26}$$

A Kármán–Howarth type equation for the LANS-α equations. Note how this equation differs from the equation Holm derived in [16],

$$\frac{\partial Q}{\partial t} = \left(r\frac{\partial}{\partial r} + 5\right)(T - \alpha^2 S) + 2\nu D(Q). \tag{3.27}$$

There are two main reasons for the difference in these two equations. One is that Holm investigated the velocity correlation tensor $\langle \mathbf{v} \otimes \mathbf{u}' \rangle$, so in his equation Q is the scalar defining function for $\mathbf{Q}(\mathbf{r}) = \langle \mathbf{v} \otimes \mathbf{u}' \rangle$. The other difference is that our T is the defining scalar function for the same triple correlation tensor $\langle \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u}' \rangle$ that appears in the papers by Robertson [25] and in Kármám and Howarth [19], whereas Holm's T arises from the triple correlation tensor $\langle (\mathbf{v} \otimes \mathbf{u}' + \mathbf{u}' \otimes \mathbf{v} + \mathbf{v}' \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v}') \otimes \mathbf{u} \rangle$. Also the scalar functions S arise from different tensors, but this is expected based on the afore mentioned differences.

The next observation is that with $\beta = \alpha$ if we take the limit as α goes to zero, we recover the Navier Stokes equations from the NS-$\alpha\beta$ equations. So in equation (3.26) taking the limit as α goes to zero we arrive at the Kármán–Howarth equation that Roberson [25] obtained for the Navier Stokes equations.

$$\frac{\partial Q}{\partial t} = 2\nu D(Q) + T \tag{3.28}$$

We now return to equation (3.25). We can rewrite the equation in a slightly different form as,

$$(1-\alpha^2 D)\frac{\partial Q}{\partial t} = 2\nu D(1-\alpha^2 D)(Q) + (1-\alpha^2 D)T + \alpha^2 S + 2\nu\alpha^2(1-\frac{\beta^2}{\alpha^2})\mathrm{D}^2(Q). \quad (3.29)$$

In the NS-$\alpha\beta$ equation, equation (3.1), the terms involving α appear on the the left hand side and these terms represent the dispersive properties of the equation. On the other hand the term with β appears on the right hand side and corresponds to the diffusive properties of the equation. Then as α and β increase, the filtering of the dispersive and diffusive effects increase respectively. Thus β^2/α^2 relates to the ratio of diffusion filtering and dispersion filtering. Writing the Kármán–Howarth equation for the NS-$\alpha\beta$ as we have in equation (3.29) allows us to better see the influence of β^2/α^2. First we consider $\beta^2/\alpha^2 \ll 1$, then equation (3.29) becomes

$$(1-\alpha^2 D)\frac{\partial Q}{\partial t} = 2\nu D(Q) + (1-\alpha^2 D)T + \alpha^2 S. \quad (3.30)$$

Which is to be expected if the dispersive filtering is much greater than the diffusive filtering. On the other hand if $\beta/\alpha = O(1)$ then one cannot neglect neglect the influence of either type of filtering.

There has been a major issue with the Kármán–Howarth equation for Navier–Stokes. Looking at equation equation (3.28) we see that there are two unknown functions in the partial differential equation. Many hypothesis have been proposed to link Q and T, but they all fail severely over a certain range. Finding a way to relate Q and T has been termed the closure problem of the Kármán–Howarth and it remains one of the most outstanding problems in turbulence.

Our hope was that with the extra filtering parameters we may have had a way to control the terms and somehow solve the closure problem for the NS-$\alpha\beta$ equations. But looking at equation (3.25) we see that there are three unknowns in the KH-$\alpha\beta$ equation. Thus, the closure problem is even greater than the closure problem for the original Kármán–Howarth equation. One could attempt to apply the various closure hypotheses used for the Kármán–Howarth equation for $\mathbf{R}(\mathbf{r})$ and $\mathbf{T}(\mathbf{r})$, but $\mathbf{S}(\mathbf{r})$ is new and has not been treated before. In fact the tensor $\mathbf{S}(\mathbf{r})$ is very complicated compared to $\mathbf{R}(\mathbf{r})$ or $\mathbf{T}(\mathbf{r})$. And so since any physically inspired closure hypothesis for the Kármán–Howarth equation will almost certainly fail for the KH-$\alpha\beta$ equation.

Chapter 4

Investigations of an anisotropic generalization of the Navier–Stokes-$\alpha\beta$ equations.

4.1 Introduction

In this chapter we derive energy type inequalities for an anisotropic generalization of the Navier–Stokes-$\alpha\beta$ equations derived by Capriz and Fried in [4]. Specifically we are interested in the following generalization.

$$\begin{aligned} \rho\left(\dot{u} - \tfrac{1}{2}\operatorname{div}(Y\dot{W} + \dot{W}Y)\right) &= -\operatorname{grad} p + \mu\Delta w + \rho f \\ \dot{Y} &= YW - WY \\ w &= (1-\beta^2\Delta)u \\ \operatorname{div} u &= 0. \end{aligned} \tag{4.1}$$

Where u is the velocity of the flow, $W = \operatorname{skw}\operatorname{grad} u$, ρ is the mass density, μ is the viscosity, ρf is the force per volume. The tensor Y is the most notable difference from NS. This tensor is a moment of inertia tensor for the ephemeral continua. The for each (x, t) the tensor is positive-definite. In addition we assume that we are working on the periodic domain of side length $2\pi\ell$. Our reason is two-fold. One is that the boundary conditions have not been decided for the anisotropic equations, and second much like in chapter 2 the assumption greatly simplifies the functional analysis. We also make the comment that previously in the dissertation we used bold symbols for points in $\mathcal{E}$, vectors in $\mathcal{V}$, and tensors in $\mathcal{V} \otimes \mathcal{V}$, but in keeping with the literature on ephemeral continua we will not use boldface type for these symbols. Our functional setting is similar to the setting used in chapter 2, but due to some difference and for completeness we present the functional setting in the section.

4.2 Functional setting for the anisotropic equations

Most of the following is developed in Constantin and Foias [5]. We define the following functional spaces:

(i) $\mathcal{V} = \{\phi \in C^\infty(\Omega)^3 : \operatorname{div} \phi = 0 \quad \text{and} \quad \int_\Omega \phi(x) dx = 0\}$

(ii) H is the closure of $\mathcal{V}$ in $L^2(\Omega)^3$

(iii) V is the closure of $\mathcal{V}$ in $H^1(\Omega)^3$

The reason we define $\mathcal{V}$ to contain functions whose average is zero is given as follows. If we integrate the first equation in (4.1) over Ω, we find using integration by parts that

$$\frac{d}{dt} \int_\Omega u(x, t)\, dx = \int_\Omega f(x, t)\, dx. \tag{4.1}$$

Thus if we assume that

$$\int_\Omega u(x, 0)\, dx = 0 \qquad \text{and} \qquad \int_\Omega f(x, t)\, dx = 0,$$

then we can conclude that

$$\int_\Omega u(x,t)\,dx = 0 \qquad \text{for all } t.$$

Our reason for making this assumption is in the simplification of the analysis while keeping the original nature of the problem. This assumption is commonly made for the Navier–Stokes equations, see [8] or [5]. Specifically it allows us to use a simpler form of Poincaré's inequality.

We let P denote Helmholtz-Leray projection onto divergence free vector fields, *I.e.,* $P : L^2(\Omega)^3 \to H$. We define the stokes operator by $A = -P\Delta$. The domain of A is given by $D(A) = V \cap H^2(\Omega)^3$. The inverse of stokes operator is a self-adjoint compact operator, therefore there exists an orthonormal basis of eigenfunctions $\{\phi_j\}$ and eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ such that $A\phi_j = \lambda_j \phi_j$.

We make the following observations about the stokes operator from [5]. First is that on the periodic domain (and $\mathbb{R}^n$) the projection operator commutes with the laplacian. Thus we could have defined the stokes operator as $-\Delta$, but in keeping with tradition we define the stokes operator as $A = -P\Delta$. Using the eigenvalues of A we may define A^s for any $s \in \mathbb{R}$. For $s > 0$ the domain of A^s may be identified as $D(A^s) = V \cap H^{2s}(\Omega)^3$. For $s < 0$ we identify the duals spaces as $D(A^s) = D(A^{-s})'$. We use the shorthand notation $V^s = D(A^s)$. Then, for $0 \leq r \leq s$ the embedding $V^s \hookrightarrow V^r$ is compact. For $r \leq s$, i.e., r and/or s could be negative, we have that $V^s \hookrightarrow V^r$ is a continuous embedding.

Let u be a time dependent function on a Banach space X. Then we write

$$\|u\|_{L^p([0,T];X)} = \left(\int_0^T \|u\|_X^p \, dt \right)^{1/p}$$

and

$$\|u\|_{L^\infty([0,T];X)} = \operatorname{ess\,sup}_{t\in[0,T]} \|u\|_X$$

In addition due to the presence of the tensor Y, we must define norms on tensors fields. We have that $Y : \Omega \times \mathbb{R} \to \mathcal{V} \otimes \mathcal{V}$, thus for each t we write $\|Y\|_{L^2}$ to be the L^2 norm of Y over Ω using the Euclidean product on tensors. We similarly define $\|Y\|_{H^s}$ for higher order Sobolev norms. Then as above we write

$$\|Y\|_{L^p([0,T];X)} = \left(\int_0^T \|Y\|_X^p \, dt \right)^{1/p}$$

and

$$\|Y\|_{L^\infty([0,T];X)} = \operatorname{ess\,sup}_{t\in[0,T]} \|Y\|_X$$

We make frequent use of the following inequalities.

1. (Poincaré's inequality) Let $u \in V$ and

   $$\int_\Omega u \, dx = 0.$$

   Then

   $$\|u\|_{L^2} \leq \lambda_1^{-1/2} \|A^{1/2}u\|_{L^2},$$

   where λ_1 is the first eigenvalue of the stokes operator and for the periodic box of side length L, $\lambda_1 = 4\pi^2/L^2$.

2. (Young's Inequality) Given $a, b, p, q > 0$ with $1/p + 1/q = 1$ we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

3. We will mostly use the following version of Young's Inequality where $p = q = 2$. Let $a, b, \epsilon > 0$, then

$$ab < \frac{1}{2\epsilon} a^2 + \frac{\epsilon}{2} b^2$$

Let $\langle \cdot, \cdot \rangle$ represent the L^2 inner product and $|\cdot| = \|\cdot\|_{L^2}$ the L^2 norm. We also define $\|\cdot\| = |A^{1/2} \cdot|$, it is important to note that on V, we have $|A^{1/2} \cdot| = |\operatorname{grad} \cdot|$. Using Poincaré's inequality and the periodic boundary conditions it can be shown that the H^s norm is equivalent to the norm associated to $V^s = D(A^{s/2})$. I.e., for a given s there are constants c, c' such that $c|A^{s/2}u| \leq \|u\|_{H^s(\Omega)^3} \leq c'|A^{s/2}u|$. Thus we will use $|A^{s/2} \cdot|$ for the $H^s(\Omega)^3$ norm on V^s. Before proceeding with the analysis of anisotropic equations we investigate the properties that Y must satisfy.

4.3 The tensor-field Y

Recall that the principal invariants $I_1(A)$, $I_2(A)$, and $I_3(A)$ of a second-order tensor A can be expressed in terms of $\operatorname{tr}(A)$, $\operatorname{tr}(A^2)$, and $\operatorname{tr}(A^3)$ as

$$\begin{aligned} I_1(A) &= \operatorname{tr} A, \\ I_2(A) &= \frac{1}{2} \operatorname{tr}^2(A) - \frac{1}{2} \operatorname{tr}(A^2), \\ I_3(A) &= \frac{1}{6} \operatorname{tr}^3(A) - \frac{1}{2} \operatorname{tr}(A) \operatorname{tr}(A^2) + \frac{1}{3} \operatorname{tr}(A^3). \end{aligned} \tag{4.1}$$

Theorem 4.1. *Let Y be a non-singular differentiable second-order tensor field that evolves according to the evolution equation*

$$\dot{Y} = YW - WY \tag{4.2}$$

where the superimposed dot represents the material time derivative and W is some given second-order tensor field. Then the principal invariants are invariant under the flow. I.e.,

$$\overset{\cdot}{\overline{I_1(Y)}} = \overset{\cdot}{\overline{I_2(Y)}} = \overset{\cdot}{\overline{I_3(Y)}} = 0.$$

Proof. Since the derivative commutes with the trace we have

$$\begin{aligned}\overset{\cdot}{\overline{I_1(Y)}} = \overset{\cdot}{\overline{\operatorname{tr}(Y)}} &= \operatorname{tr}(\dot{Y}) = \operatorname{tr}(YW - WY)\\ &= \operatorname{tr}(YW) - \operatorname{tr}(WY) = \operatorname{tr}(YW) - \operatorname{tr}(YW) = 0.\end{aligned}$$

To show that $I_2(Y)$ is invariant under the flow it is sufficient to show that the trace of Y^2 is invariant under the flow.

$$\begin{aligned}\overset{\cdot}{\overline{\operatorname{tr}(Y^2)}} &= \operatorname{tr}(\dot{Y}Y + Y\dot{Y}) = 2\operatorname{tr}(\dot{Y}Y) = 2\operatorname{tr}(YWY - WY^2)\\ &= \operatorname{tr}(YWY) - \operatorname{tr}(WY^2) = \operatorname{tr}(WY^2) - \operatorname{tr}(WY^2) = 0\end{aligned}$$

Similarly to show that $I_3(Y)$ it is sufficient to show $\overset{\cdot}{\overline{\operatorname{tr} Y^3}} = 0$.

$$\overset{\cdot}{\overline{\operatorname{tr} Y^3}} = 3\operatorname{tr}(Y^2\dot{Y}) = 3\operatorname{tr}(Y^2(YW - WY)) = 0$$

$\square$

Suppose in addition that Y is symmetric. Let η_1, η_2, and η_3 be the (not necessarily distinct) eigenvalues of Y. Then, the principal invariants of Y take the following form.

$$\begin{aligned}I_1(Y) &= \eta_1 + \eta_2 + \eta_3\\ I_2(Y) &= \eta_1\eta_2 + \eta_1\eta_3 + \eta_2\eta_3\\ I_3(Y) &= \eta_1\eta_2\eta_3\end{aligned}$$

Theorem 4.2. *Suppose that Y is a symmetric positive definite tensor field with $Y \in C^1(\Omega \times \mathbb{R})$. We assume that Y satisfies the evolution equation $\dot{Y} = YW - WY$ for some tensor field W. Then, the eigenvalues are invariant under the flow. I.e., $\dot{\eta}_1 = \dot{\eta}_2 = \dot{\eta}_3 = 0$. In particular, if we are given an initial tensor field $Y(x, t_o)$, then we know the eigenvalue distribution for each later time by following the distribution along the flow.*

Proof. Recall that for a symmetric positive definite tensor, that all of the eigenvalues are positive. We begin by assuming that all three eigenvalues are distinct. Since $\dot{\overline{I_1(Y)}} = \dot{\overline{I_2(Y)}} = \dot{\overline{I_3(Y)}} = 0$ we obtain the following system of equations.

$$\begin{aligned} \dot{\eta}_1 + \dot{\eta}_2 + \dot{\eta}_3 &= 0 \\ (\eta_2 + \eta_3)\dot{\eta}_1 + (\eta_1 + \eta_3)\dot{\eta}_2 + (\eta_1 + \eta_2)\dot{\eta}_3 &= 0 \\ \eta_2\eta_3\dot{\eta}_1 + \eta_1\eta_3\dot{\eta}_2 + \eta_1\eta_2\dot{\eta}_3 &= 0 \end{aligned} \tag{4.3}$$

After some algebraic manipulations we obtain the following equivalent system.

$$\begin{aligned} \dot{\eta}_1 + \dot{\eta}_2 + \dot{\eta}_3 &= 0, \\ (\eta_1 - \eta_2)(\eta_1 - \eta_3)\dot{\eta}_1 &= 0, \\ (\eta_1 - \eta_2)(\eta_2 - \eta_3)\dot{\eta}_2 &= 0. \end{aligned}$$

Since all of the eigenvalues are distinct we conclude that $\dot{\eta}_1 = \dot{\eta}_2 = \dot{\eta}_3 = 0$. Next we investigate the degenerate case where two of the eigenvalues are equal at a specific x and t_o. Thus, the following are evaluated at (x, t_o). Without loss of generality we assume that $\eta_2 = \eta_3$ and so we obtain the following matrix for the system.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2\eta_2 & \eta_1 + \eta_2 & \eta_1 + \eta_2 & 0 \\ \eta_2^2 & \eta_1\eta_2 & \eta_1\eta_2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus we conclude that $\dot{\eta}_1 = 0$ and that $\dot{\eta}_2 = -\dot{\eta}_3$. Certainly $\dot{\eta}_2 = \dot{\eta}_3 = 0$ satisfies this condition, but we must preclude the possibility of $\dot{\eta}_2 \neq \dot{\eta}_3$. Suppose towards a contradiction that $\eta_2 = \eta_3$ and $\dot{\eta}_2 = -\dot{\eta}_3 \neq 0$ at time t_o and position $x(t_o)$. Without loss of generality we assume that $\dot{\eta}_2 > 0$, and so we have $\dot{\eta}_3 < 0$. Then at time $t_o + T$ and position $x(t_o + T)$ we have $\eta_2 > \eta_3$. By the non-degenerate case above we have $\dot{\eta}_2 = \dot{\eta}_3 = 0$ for all $T > 0$. Since $Y \in C^1(\eta \times \mathbb{R}$ we have that the eigenvalues of Y depend continuously on Y. This is a consequence of the fact that roots of polynomials depend continuously on their coefficients (see [23]) and by the Cayley-Hamilton equation. Thus,

$$\eta_2(t+T, x(t+T)) > \eta_3(t+T, x(t+T))$$

and

$$\eta_2((t+T, x(t+T)) \quad \text{and} \quad \eta_3(t+T, x(t+T)) \quad \text{are constant for all } T > 0$$

imply that

$$\eta_2(t, x(t)) = \lim_{T \to 0} \eta_2(t+T, x(t+T)) > \lim_{T \to 0} \eta_3(t+T, x(t+T)) = \eta_3(t, x(t)).$$

Thus we conclude that $\eta_2 \neq \eta_3$, which contradicts our assumption. Thus, we find that $\dot{\eta}_1 = \dot{\eta}_2 = \dot{\eta}_3 = 0$. Finally we consider a case where all of the eigenvalues are equal at a specific (x, t_o). I.e., $\eta = \eta_1 = \eta_2 = \eta_3$. We obtain the system.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2\eta & 2\eta & 2\eta & 0 \\ \eta^2 & \eta^2 & \eta^2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Suppose that there is a non-trivial solution, we assume without loss of generality that $\dot{\eta}_2 > 0$ and $\dot{\eta}_3 < 0$ with no restriction on $\dot{\eta}_1$. Thus at time $t_o + T$ the second eigenvalue will have increased and the third will have decreased. Depending on the behavior of the first eigenvalue we will either be in the first or second case. Thus as shown above we have that $\dot{\eta}_1(t_o + T) = \dot{\eta}_2(t_o + T) = \dot{\eta}_3(t_o + T) = 0$ for all $T > 0$. And by the continuity of the eigenvalues as discussed in case 2 we may conclude that $\dot{\eta}_1(t_o) = \dot{\eta}_2(t_o) = \dot{\eta}_3(t_o) = 0$. □

Next we find a bound on the action of Y in terms of its eigenvalues. Let W, L be arbitrary time dependent tensor fields on the domain η. For each time t we denote the L^2 inner product over η as

$$\langle W, L \rangle = \int_\Omega W \cdot L \, dx$$

where $W \cdot L$ is the standard euclidean product on tensors. We donate the L^2 norm as

$$|W|^2 = \langle W, W \rangle.$$

As before let Y be a symmetric positive definite time dependent tensor field over η, that evolves according to the evolution equation $\dot{Y} = YW - WY$. Then for each x and t we can find a orthonormal basis of eigenvectors for the translation space $\mathcal{V}$. We let e_i and η_i for $i = 1, 2, 4$ denote the eigenvectors and eigenvalues respectively. It is important to note that these eigenvectors and eigenvalues depend on x and t. If the domain η is compact then by the continuity of Y we have that the eigenvalues are bounded from above and from below away from 0. Let $\eta_{\max}$ and η_{min} denote the maximum and minimum eigenvalues respectively. If the domain is unbounded or open then we will assume that maximum and minimum eigenvalues are bounded from above and from below away from 0. Then by Theorem 4.2, since the eigenvalues are invariant under the flow, we have that these bounds for the eigenvalues hold for all t.

Now we state one of key theorems for showing boundedness of solutions.

Theorem 4.3. *Let T be any tensor field in L^2 and let Y be given as above. In addition to Y, we consider all positive powers of Y. I.e., Y^s, where s is any positive number. Let η_{max} and η_{min} be the maximum and minimum eigenvalues of Y over η as discussed above. Then we have the following inequality.*

$$\eta_{min}^{2s}|T|^2 \leq |Y^s T|^2 \leq \eta_{max}^{2s}|T|^2 \tag{4.4}$$

Where η_{max} and η_{min} are independent of t.

Proof. In terms of the eigenvectors e_i we may write T as

$$T = \sum_{i,j=1}^{n} a_{ij} e_i(x) e_j(x).$$

Thus,

$$Y^s T = \sum_{i,j=1}^{n} Y^s(a_{ij} e_i(x) e_j(x)) = \sum_{i,j=1}^{n} a_{ij} Y^s(e_i(x)) e_j(x) = \sum_{i,j=1}^{n} a_{ij} \eta_i^s e_i(x) e_j(x).$$

Calculating the L^2 norm we find

$$|Y^s T|^2 = \int_\eta \sum_{i,j=1}^{n} a_{ij}^2 \eta_i^{2s} dx$$

and so

$$\eta_{min}^{2s}|T|^2 \leq |YT|^{2s} \leq \eta_{max}^{2s}|T|^2.$$

In theorem 4.2 we had shown that the eigenvalues are invariant under the flow. By the chain rule the same holds true for η_i^s. Thus, the bounds obtained above are independent of time. $\square$

4.4 Energy estimate for the anisotropic equations

The anisotropic equations are given as follows.

$$\begin{aligned} \rho\left(\dot{u} - \tfrac{1}{2}\operatorname{div}(Y\dot{W} + \dot{W}Y)\right) &= -\operatorname{grad} p + \mu\Delta w + \rho f \\ \dot{Y} &= YW - WY \\ w &= (1-\beta^2\Delta)u \\ \operatorname{div} u &= 0 \end{aligned} \tag{4.1}$$

Where $W = \operatorname{skw}(\operatorname{grad} u)$. The second-order tensor field Y is assumed to be symmetric positive definite. As discussed in the previous section the eigenvalues of Y are invariant under the flow. We shall refer to the eigenvalues of Y as η_1, η_2, and η_3. Next we divide by sides by ρ and set $\varpi = p/\rho$ to obtain the following form of the anisotropic equations.

$$\begin{aligned} \dot{u} - \operatorname{div}(Y\dot{W} + \dot{W}Y) &= -\operatorname{grad}\varpi + \nu\Delta w + f \\ \dot{Y} &= YW - WY \\ w &= (1-\beta^2\Delta)u \\ \operatorname{div} u &= 0 \end{aligned} \tag{4.2}$$

Next we apply the Helmholtz-Leray projection P onto the first equation above to obtain

$$\begin{aligned} \dot{u} + \nu Aw &= \tfrac{1}{2}P\operatorname{div}(Y\dot{W} + \dot{W}Y) + f, \\ \dot{Y} &= YW - WY, \\ w &= (1+\beta^2 A)u, \end{aligned} \tag{4.3}$$

where A is the Stokes operator. Recall some of the assumptions from section 4.2. We assume that the domain Ω is periodic of side length $2\pi\ell$. And that for simplicity of the analysis we assume that the average of of the initial velocity field is zero and that the average of the forcing function is zero for all time. This allows us to assume that the average of the velocity filed is zero for all t. We proceed with finding bounds on solutions u and Y. We begin by taking the inner product of the first equation in (4.5) with u.

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{1}{2}\langle Y\dot{W} + \dot{W}Y, \operatorname{grad} u\rangle + \nu\left(|A^{1/2}u| + \beta^2|Au|^2\right) = \langle f, u\rangle \tag{4.4}$$

To simplify the second expression on the right hand side we first note that $Y\dot{W}+\dot{W}Y$ is skew symmetric, thus we have that

$$\begin{aligned}
\langle Y\dot{W} + \dot{W}Y, \operatorname{grad} u\rangle &= \langle Y\dot{W} + \dot{W}Y, W\rangle \\
&= -\langle Y, W\dot{W} + \dot{W}W\rangle \\
&= -\langle Y, \dot{\overline{W^2}}\rangle \\
&= -\frac{d}{dt}\langle Y, W^2\rangle + \langle \dot{Y}, W^2\rangle \\
&= \frac{d}{dt}\langle YW, W\rangle + \langle YW - WY, W^2\rangle \\
&= \frac{d}{dt}\langle Y^{1/2}W, Y^{1/2}W\rangle - \langle Y, W^3\rangle + \langle Y, W^3\rangle \\
&= \frac{d}{dt}|Y^{1/2}W|^2
\end{aligned}$$

We obtain

$$\frac{1}{2}\frac{d}{dt}\left(|u|^2 + |Y^{1/2}W|^2\right) + \nu\left(|A^{1/2}u|^2 + \beta^2|Au|^2\right) = \langle f, u\rangle. \tag{4.5}$$

We bound the forcing term using Young and Poincaré.

$$\langle f, u\rangle = \langle A^{-1/2}f, A^{1/2}u\rangle \le \frac{1}{2\nu}|A^{-1/2}f|^2 + \frac{\nu}{2}|A^{1/2}|^2 \le \frac{1}{2\lambda_1\nu}|f|^2 + \frac{\nu}{2}|A^{1/2}u|^2.$$

Applying the above bound in equation (4.5) we find

$$\frac{d}{dt}\left(|u|^2 + |Y^{1/2}W|^2\right) + \nu\left(|A^{1/2}u|^2 + \beta^2|Au|^2\right) \leq \frac{1}{\lambda_1\nu}|f|^2 \tag{4.6}$$

We use Poincaré's inequality on the viscous term

$$|A^{1/2}u|^2 + \beta^2|Au|^2 \geq \lambda_1\left(|u|^2 + \beta^2|A^{1/2}u,|^2\right)$$

where λ_1 is the first eigenvalue of the stokes operator A. For the forcing term we use Young's inequality.

$$\langle f, u\rangle \leq \frac{1}{2\lambda_1\nu}|f|^2 + \frac{\lambda_1\nu}{2}|u|^2$$

Combing these identities with equation (4.5) we obtain the following inequality.

$$\frac{d}{dt}\left(|u|^2 + |Y^{1/2}W|^2\right) + \lambda_1\nu\left(|u|^2 + \beta^2|A^{1/2}u|^2\right) \leq \frac{1}{\lambda_1\nu}|f|^2 \tag{4.7}$$

Since $\operatorname{div} u = 0$ we have

$$|A^{1/2}u|^2 = |\operatorname{grad} u|^2 = 2|W|^2.$$

Thus we may write:

$$\frac{d}{dt}\left(|u|^2 + |Y^{1/2}W|^2\right) + \lambda_1\nu\left(|u|^2 + 2\beta^2|W|^2\right) \leq \frac{1}{\lambda_1\nu}|f|^2. \tag{4.8}$$

Our next step is to compare the sizes of $|u|^2 + |Y^{1/2}W|^2$ and $|u|^2 + 2\beta^2|W|^2$. From Theorem 4.3 we have that $|Y^{1/2}W|^2 \leq \eta_{max}|W|^2$. We first compare $|u|^2 + \eta_{max}|W|^2$ and $|u|^2 + 2\beta^2|W|^2$. There are two cases to consider.

Case 1. ($2\beta^2 > \eta_{max}$) For this case we easily obtain

$$\begin{aligned}|u|^2 + 2\beta^2|W|^2 &\geq |u|^2 + \eta_{max}|W|^2 \\ &\geq |u|^2 + |Y^{1/2}W|^2\end{aligned}$$

Case 2. ($2\beta^2 \leq \eta_{max}$) Then,

$$\begin{aligned}|u|^2 + 2\beta^2|W|^2 &= \frac{2\beta^2}{\eta_{max}}(\frac{\eta_{max}}{2\beta^2}|u|^2 + \eta_{max}|W|^2) \\ &\geq \frac{2\beta^2}{\eta_{max}}(|u|^2 + \eta_{max}|W|^2) \\ &\geq \frac{2\beta^2}{\eta_{max}}(|u|^2 + |Y^{1/2}W|^2)\end{aligned}$$

We define

$$\gamma = \min\left\{\frac{2\beta^2}{\eta_{max}}, 1\right\}$$

and obtain

$$|u|^2 + 2\beta^2|W|^2 \geq \gamma(|u|^2 + |Y^{1/2}W|^2).$$

Applying the above inequality in (4.8), we find

$$\frac{d}{dt}\left(|u|^2 + |Y^{1/2}W|^2\right) + \lambda_1\nu\gamma(|u|^2 + |Y^{1/2}W|^2) \leq \frac{1}{\lambda_1\nu}|f|^2. \tag{4.9}$$

Now we may apply a classical Gronwall inequality to obtain the following estimate.

$$\begin{aligned}&|u(t)|^2 + |Y^{1/2}(t)W(t)|^2 \leq \\ &\qquad e^{-\lambda_1\nu\gamma t}\left(|u(0)|^2 + |Y^{1/2}(0)W(0)|^2\right) + e^{-\lambda_1\nu\gamma t}\int_0^t \frac{1}{\lambda_1\nu}e^{\lambda_1\nu\gamma s}|f(s)|^2 ds\end{aligned} \tag{4.10}$$

If we apply the result $\eta_{min}|W|^2 \leq |Y^{1/2}W|^2$ and use $|\operatorname{grad} u|^2 = 2|W|^2$, then we obtain

$$
\begin{aligned}
&|u(t)|^2 + \frac{1}{2}\eta_{min}|\operatorname{grad} u(t)|^2 \leq \\
&\qquad e^{-\lambda_1\nu\gamma t}\left(|u(0)|^2 + |Y^{1/2}(0)W(0)|^2\right) + e^{-\lambda_1\nu\gamma t}\int_0^t \frac{1}{\lambda_1\nu}e^{\lambda_1\nu\gamma s}|f(s)|^2 ds
\end{aligned} \tag{4.11}
$$

Next we integrate equation (4.9) over $[s, t]$.

$$
\begin{aligned}
&|u(t)|^2 + |Y^{1/2}W(t)|^2 + \int_s^t \lambda_1\nu\gamma(|u(\tau)|^2 + |Y^{1/2}(\tau)W(\tau)|^2)\ d\tau \leq \\
&\qquad |u(s)|^2 + |Y^{1/2}W(s)|^2 + \int_s^t \frac{1}{\lambda_1\nu}|f(\tau)|^2\ d\tau
\end{aligned} \tag{4.12}
$$

We neglect the first term on the left hand side, bound the first two term on the right hand side using (4.10). We also use $\eta_{min}|\operatorname{grad} u|^2 \leq 2|Y^{1/2}W|^2$ for the term inside the integral.

$$
\begin{aligned}
&\int_s^t \lambda_1\nu\gamma\left(|u(\tau)|^2 + \frac{1}{2}\eta_{min}|\operatorname{grad} u(\tau)|^2\right)\ d\tau \leq \\
&e^{-\lambda_1\nu\gamma s}\left(|u(0)|^2 + |Y^{1/2}(0)W(0)|^2\right) + e^{-\lambda_1\nu\gamma s}\int_0^s \frac{1}{\lambda_1\nu}e^{\lambda_1\nu\gamma\tau}|f(\tau)|^2\ d\tau + \int_s^t \frac{1}{\lambda_1\nu}|f(\tau)|^2\ d\tau
\end{aligned}
$$

In particular for $s = 0$ we find

$$
\begin{aligned}
&\int_0^t \lambda_1\nu\gamma\left(|u(\tau)|^2 + \frac{1}{2}\eta_{min}|\operatorname{grad} u(\tau)|^2\right)\ d\tau \leq \\
&\qquad |u(0)|^2 + |Y^{1/2}(0)W(0)|^2 + \int_0^t \frac{1}{\lambda_1\nu}|f(\tau)|^2\ d\tau.
\end{aligned} \tag{4.13}
$$

We repeat the above procedure by integrating (4.6) over $[s,t]$. We find

$$\int_s^t \nu \left(|A^{1/2}u(\tau)|^2 + \beta^2|Au(\tau)|^2\right)\, d\tau \leq$$

$$e^{-\lambda_1\nu\gamma s}\left(|u(0)|^2 + |Y^{1/2}(0)W(0)|^2\right) + e^{-\lambda_1\nu\gamma s}\int_0^s \frac{1}{\lambda_1\nu} e^{\lambda_1\nu\gamma\tau}|f(\tau)|^2\, d\tau + \int_s^t \frac{1}{\lambda_1\nu}|f(\tau)|^2\, d\tau$$

and with $s=0$

$$\int_0^t \nu \left(|A^{1/2}u(\tau)|^2 + \beta^2|Au(\tau)|^2\right)\, d\tau \leq$$

$$|u(0)|^2 + |Y^{1/2}(0)W(0)|^2 + \int_0^t \frac{1}{\lambda_1\nu}|f(\tau)|^2\, d\tau. \quad (4.14)$$

Let $T > 0$ be fixed, we consider the interval of time $[0,T]$. We assume that $f \in L^2([0,T];H) \cap L^\infty([0,\infty);H)$. Equations (4.13) and (4.14) shows that $u \in L^2([0,T];V^s)$ for $s = 0,1,2$ and equation (4.11) implies that $u \in L^\infty([0,T];V^s)$ for $s = 0,1$. Next we investigate bounds on Y.

We take the L^2 inner product of $\dot{Y}$ with Y and we obtain

$$\langle \dot{Y}, Y\rangle = \langle YW - WY, Y\rangle = 0.$$

Thus,

$$\frac{d}{dt}|Y|^2 = 0.$$

As mentioned in section 4.2 we write $\|Y\|_{L^2}$ be the L^2 space for tensor fields over the domain Ω. Thus we have $Y \in L^2([0,T];L^2) \cap L^\infty([0,\infty);L^2)$. Knowing that Y is bounded we subsequently obtain the following $L^2([0,T];L^2)$ bound for $\dot{Y}$.

$$\begin{aligned}\|\dot{Y}\|_{L^2([0,T];L^2)} &\leq 2\|YW\|_{L^2([0,T];L^2)} \\ &\leq \|Y\|_{L^2([0,T];L^2)}\|u\|_{L^2([0,T];V)} < \infty.\end{aligned}$$

and

$$\begin{aligned}\|\dot{Y}\|_{L^\infty([0,\infty);L^2)} &\leq 2\|YW\|_{L^\infty([0,\infty);L^2)} \\ &\leq \|Y\|_{L^\infty([0,\infty);L^2)}\|u\|_{L^\infty([0,\infty);V)} < \infty.\end{aligned}$$

We summarize these results as

Theorem 4.1. *Let u and Y be a solution to the anisotropic equations* (4.5)*. Let $T > 0$ be fixed and suppose that $f \in L^2([0,T];H) \cap L^\infty([0,\infty);H)$, then*

$$\begin{aligned}&u \in L^2([0,T];V^2) \cap L^\infty([0,\infty);V), \\ &Y \in L^2([0,T];L^2) \cap L^\infty([0,\infty);L^2), \\ &\dot{Y} \in L^2([0,T];L^2) \cap L^\infty([0,\infty);L^2).\end{aligned} \tag{4.15}$$

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