Report Number: WUCS-79-11

1979-11-01

On the Nonquivalence of Shadow Prices and Dual Variables

Authors: D. C. Aucamp and D. I, Steinberg

The purpose of this paper is to demonstrate that in the event of degeneracy is present in an optimal basic solution to a linear programming problem, the optimal values of the dual variables doe no necessarily correspond to shadow prices. In such instances it will be shown how the actual values of the shadow prices may be determined and the nature of the relationship between shadow prices and dual variables.

ON THE NONEQUIVALENCE OF SHADOW PRICES AND DUAL VARIABLES

D. C. Aucamp and D. I. Steinberg

WUCS-79-11

Department of Computer Science

Washington University

St. Louis, Missouri 63130

November 1978

ABSTRACT

The purpose of this paper is to demonstrate that in the event degeneracy is present in an optimal basic solution to a linear programming problem, the optimal values of the dual variables do not necessarily correspond to shadow prices. In such instances it will be shown how the actual values of the shadow prices may be determined and the nature of the relationship between shadow prices and dual variables.

1. Introduction

Consider the linear programming problem:

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad 1 \leq i \leq m$$

$$x_j \geq 0 \; , \quad 1 \leq j \leq n$$

Appendix

A. Proof of Theorem 1

Consider the optimal tableau to the linear programming problem (1), and label the objective function coefficient in the i^{th} slack column, y_i*. Then it is well known that (see, e.g., [1], [5], [6])

$$\text{(i)} \qquad z* = \sum_{1}^{m} y_i*b_i \ .$$

Now increase b_r by ε, where $\varepsilon \geq 0$. The new basic solution using the current basis is $X_B* + \varepsilon q_r$, where q_r is an m-dimensional vector. It is easy to show that $B(X_B + \varepsilon q_r) = b + \varepsilon e_r$, so that

$$\text{(ii)} \qquad q_r = B^{-1} e_r$$

where B is the current optimal basis matrix and e_r is the r^{th} column of an m^{th} order identity matrix.

Now parameterize the current tableau in the above fashion. Since only the right-hand side coefficients change with ε, we still have $y*(\varepsilon) = y*$ for all ε. However the solution may not be feasible for $\varepsilon > 0$ (it won't be if $X_{Bi} = 0$ and $q_{ir} < 0$ for any $1 \leq i \leq m$). Define the objective function with this basis as $z(\varepsilon)$ and note that $z*(\varepsilon) \leq z(\varepsilon)$, since the current basic solution may not be feasible for $\varepsilon > 0$ (perturbing the current basic solution to make it feasible can only reduce z). From (i) and these deliberations, we conclude that

$$\text{(iii)} \qquad z*(\varepsilon) \leq z(\varepsilon) = z* + \varepsilon y_r* \ .$$

In order to facilitate the discussion, we assume that any linear programming problem with other types of constraints has been converted to the form given by (1). Therefore, the resource vector, b, need not be positive.

The dual of (1) is given by

$$(2) \qquad \left\{ \begin{array}{l} \text{Minimize } z = \sum_{i=1}^{m} b_i y_i \\ \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \ 1 \leq j \leq n \\ y_i \geq 0, \ 1 \leq i \leq m \end{array} \right\}$$

It is well-known (see, e.g., [1]-[9]) that the optimal values of the dual variables in (2) correspond to the objective function coefficients in the optimal simplex solution to (1); specifically, if $y_i{*}$ denotes the optimal objective function coefficient in the i^{th} primal slack variable, x_{si}, then $y_i{*}$ is also the optimal value of the i^{th} dual variable. This relationship is independent of the algebraic sign of b_i. Unfortunately, there is no universal agreement as to the definition and meaning of the term, "<u>shadow price</u>," p_i, of the i^{th} resource b_i; p_i is usually interpreted to mean the rate of increase of the objective function per unit increase in b_i. However, in many texts which actually define shadow prices in this manner, it is also indicated that $p_i = y_i{*}$, which is not correct in all cases. In some texts the shadow price is defined as the rate of increase of the objective function per unit increase in b_i <u>provided the optimal set of basic variables remains the same</u>; the latter proviso, however, is inadequate in cases when the optimal basis must change to maintain feasibility. In still other texts the shadow price is simply defined as $y_i{*}$. Again, this definition is not entirely satisfactory since the usual interpretation of a shadow price (as stated above) may not be consistent with this definition under certain circumstances.

In summary, none of the definitions or interpretations found in [1]-[9] is entirely satisfactory either because they are in error or because they do not cover all cases. We assert that for a shadow price to be defined in a physically meaningful way, p_i should be defined as the achievable rate of increase in the objective function per unit increase in resource i, which may be formally stated as follows:

(3) $$p_i = \frac{\partial z^*}{\partial b_i}, \quad 1 \leq i \leq m$$

(where z* denotes the optimal value of the objective function) provided only increases in b_i are allowed (i.e., p_i is a right-side partial derivative).

Dorfman, Samuelson, and Solow note that a separate shadow price may be defined for decreases in b_i (i.e., a left-side partial derivative) and that the two will be unequal when "the resource is just on the borderline of being redundant." [4, p. 166, Footnote] However, this observation is not developed in any further detail.

If $b_i<0$ in (1), then we have in effect an original surplus constraint that has been altered to a slack constraint. One realistic definition of the shadow price for the original surplus requirement is the rate of change in z* per unit of decrease in this requirement. Therefore, after the surplus constraint has been altered to the slack format in (1), with a negative b_i, the definition given by (3) is precisely what is desired. We conclude that the right-sided derivative given by (3) is a useful definition for shadow prices, whether the originating constraint is of the surplus or slack form. Similar reasoning can also be used in the event the original constraint is an equality; in this case we end up with two shadow prices (one where we increase, and the other where we decrease, the value of the right-hand side coefficient).

It is the purpose of this paper to fully explore the relationships between shadow prices as defined by (3) and dual variables. We shall demonstrate that $p_i = y_i*$ when the optimal primal basic solution is nondegenerate. However, when this solution is degenerate, there may then be multiple dual optimal solutions, and shadow prices are no longer necessarily equal to dual variables. In fact, $p \neq y$ (p is the shadow price vector) in general when multiple dual optimal solutions exist.

Although we shall confine our discussion to an investigation of the effects of marginal increases in a resource, a similar analysis applies to marginal decreases in a resource, in which case the derivative in (3) is viewed as a left-side derivative. If primal degeneracy exists, the left- and right-side derivatives of z with respect to a resource may not be equal; if the optimal primal solution is nondegenerate, however, these two values will be the same.

2. Relationships Between Dual Variables and Shadow Prices

In this section we develop some relationships between shadow prices and dual variables. Assume that $p = (p_1, \cdots, p_m)$ is the shadow price vector and that $y*^{(k)} = (y_1*^{(k)}, y_2*^{(k)}, \cdots, y_m*^{(k)})$ is the k^{th} optimal extreme point solution of the dual problem. Several of the relationships between p and $y*^{(k)}$ are cited in Theorem 1 and Corollary 1 below. If there is only one optimal dual solution, we drop the k superscript.

THEOREM 1: If there are K optimal extreme point solutions for the dual to System (1), then

$$p_i = \min_k \{y_i*^{(k)}\} \tag{4}$$

and if $K > 1$ and $b > 0$, then further

$$p \neq y*^{(k)}, \; 1 \leq k \leq K \tag{5}$$

Proof. See the Appendix.

Corollary 1: If there are K optimal extreme point solutions for the dual of (1), then

(6) $$p = y^* \quad \text{if } K = 1$$

(7) $$p \leq y^{*(k)}, \ 1 \leq k \leq K.$$

Proof. The proof is immediate from (4) and (5).

We now provide several examples to clarify the above theorem and corollary. Consider problem P1 below, which is the standard case treated by Corollary 1.

(P1)
$$\begin{aligned} \text{Maximize } z = 3x_1 + x_2 & \\ x_1 + x_2 \leq 1 & \quad \text{①} \\ 2.5x_1 + x_2 \leq 2 & \quad \text{②} \\ x_1, x_2 \geq 0 & \end{aligned}$$

The optimal solution, as shown in Tableau 1 and illustrated in Figure 1, is $x^* = (.8,0)$, with $z^* = 2.4$. From the tableau it is seen that $y_2^* = 1.2$, and it is clear that the rate at which the objective function increases per unit of increase in b_2 is also 1.2, since, for small perturbations in this nondegenerate case, an increase in b_2 by an amount α will result in the same Δz^* as a decrease in x_{s2}^* from zero to $-\alpha$. However, there is a limit to how much b_2 can be increased with the current basis. Specifically, if b_2 were to increase beyond 2.5, then x_{s1} would become negative. Thus, the shadow price of the second resource is 1.2 only within the range, $0 \leq b_2 \leq 2.5$.

Now consider the following linear programming problem (illustrated in Figure 2):

$$\text{(P2)} \qquad \begin{aligned} \text{Maximize } z = 3x_1 + x_2 & \\ x_1 + x_2 \le 1 & \quad \textcircled{1} \\ 2x_1 + x_2 \le 2 & \quad \textcircled{2} \\ x_1, x_2 \ge 0 & \quad . \end{aligned}$$

An optimal simplex tableau is given in Tableau 2. Now, however, although $y_1* = 3$, the shadow price for resource 1 is 0, since increasing b_1 (by any amount) will cause the basic solution to become infeasible, as x_{s2} will become negative. This observation is apparent in Figure 2, and also deducible from the optimal simplex tableau. The -2 coefficient in the row corresponding to the degenerate basic variable, x_{s2}, indicates that an increase in b_1 by α will result in a <u>decrease</u> in x_{s2} by 2α. Consequently, the 3α marginal profit predicted by the dual variable ($y_1* = 3$) cannot physically be achieved.

A second optimal tableau for this example is given by Tableau 3. The corresponding optimal dual solution is $y* = (0,3/2)$. Observe that $y_2* = 3/2$. However, $p_2 = 0$, since an increase in resource 2 will not increase the objective function. (Note that the coefficient -1/2 in the x_{s1} row predicts that x_{s1} will decrease from its current level of zero if b_2 is increased). Moreover, if we now label the two dual solutions as $y*^{(1)} = (y_1*^{(1)}, y_2*^{(1)})$ and $y*^{(2)} = (y_1*^{(2)}, y_2*^{(2)})$, then we see that

$$p_1 = \min_k\{y_1*^{(k)}\} = \text{Min}\{3,0\} = 0$$

$$p_2 = \text{Min}_k\{y_2*^{(k)}\} = \text{Min}\{0,3/2\} = 0 \ .$$

Further, it is clear that $p \le y*^{(k)}$ and $p \ne y*^{(k)}$ for $k=1,2$. We caution the reader that $p \ne y*^{(k)}$ is guaranteed only in cases when $b > 0$, as is the case in (P2). It is easy to construct examples[1] when $p = y^*$ in the situation when $b \not> 0$.

The above example _seems_ to demonstrate that the shadow price of resource i is 0 if, in the $i^{\underline{th}}$ slack variable column of the tableau, there is a negative number in a row corresponding to a degenerate basic variable. However, this is not necessarily true, as the next example illustrates.

Consider the linear programming problem

$$\text{Maximize } z = x_2 + 2x_3$$

(P3)
$$x_3 \leq 1 \qquad (1)$$
$$x_1 + x_2 + x_3 \leq 1 \qquad (2)$$
$$x_1, x_2, x_3 \geq 0 \quad .$$

There are two optimal tableaux for this problem, given by Tableaux 4 and 5. The two optimal dual solutions are (1,1) and (0,2), and we claim, therefore, that

$$p_1 = \text{Min}\{1,0\} = 0$$
$$p_2 = \text{Min}\{1,2\} = 1$$

Therefore, p_2 is not zero in spite of the negative coefficient in Tableau 4.

3. Shadow Price Information from Simplex Tableaux

In order to glean shadow price information from simplex tableaux, we introduce the following notation: Let x_{Bi} denote the value of the $i^{\underline{th}}$ basic variable, β_{ij} the coefficient in row i, column j of the tableau (where β_{oj} is

---

[1]One such example is the problem: maximize $z = -x_2$, subject to the two constraints $x_1+x_2 \leq 2$ and $-x_1-2x_2 \leq -2$ and the non-negativity restrictions. There are two optimal dual extreme point solutions, (0,0) and (1,1), and $p = (0,0)$.

the $j^{\underline{th}}$ coefficient in the objective function row). The following definition will be important in our analysis:

DEFINITION: Row r of an optimal simplex tableau is called a "valid degenerate pivot row" (VDPR) if and only if

(a) $x_{Br} = 0$; and

(b) $\beta_{rj} < 0 \Rightarrow \beta_{oj} > 0, \quad i \leq j \leq m+n.$

If row r of an optimal simplex tableau is a VDPR, then an alternate optimal dual solution can be obtained in one iteration by replacing (in the basic solution) the degenerate basic variable x_{Br} with nonbasic variable x_k, where

$$|\beta_{ok}/\beta_{rk}| = \underset{1 \leq j \leq m+n}{\text{Minimum}} |\beta_{oj}/\beta_{rj}| \tag{8}$$

This fact follows directly from the derivation of the dual simplex method (e.g., see [1], [2], [5]) and leads to:

THEOREM 2:

(a) If x* is nondegenerate, then y* is unique (and p = y*).

(b) If x* is degenerate with a VDPR, then there exists at least one alternate optimal extreme point for the dual; if there is no VDPR, there may or may not exist alternate dual optima.[2]

Proof. See the Appendix.

COROLLARY 2: If the optimal primal is degenerate and the corresponding dual is not, then there exists an alternate dual optimal solution.

Proof. If y* is not degenerate, then a VDPR exists. The corollary then follows from Theorem 2.

[2]Theorem 3 apparently contradicts the implication of Problem 8-12, page 268, of Hadley [5] (that primal degeneracy implies dual alternate optima) unless the latter is merely referring to alternate representations of the same optimal dual solution.

For example, in Tableau 2 for problem (P2) the basic variable x_{s2} is degenerate. The x_{s2} row is clearly a VDPR. Thus, by Theorem 2, there is at least one alternate optimal extreme point for the dual. As discussed earlier, there are in fact two alternate dual optima, given in Tableaux 2 and 3 as (3,0) and (0,3/2), respectively. Moreover, Theorem 1 tells us that p = (0,0), in this case.

In problem (P3) we have another example demonstrating the results of Theorem 2. Tableau 4 shows that the x_2 row is a VDPR, and the x_{s1} row in Tableau 5 is likewise a VDPR.

If, as stated in Theorem 2, the primal solution is degenerate but there is no VDPR, then there may or may not be an alternate dual solution. In problem (P4) below we demonstrate a case where there is no alternate dual.

(P4)

$$\text{maximize } z = 2x_2 + x_3$$

$$x_3 \leq 1 \qquad (1)$$

$$x_1 + x_2 + x_3 \leq 1 \qquad (2)$$

$$x_1, x_2, x_3 \geq 0$$

Tableau 6 provides the optimal primal solution, and the alternate representation is shown in Tableau 7. The latter tableau corresponds to a different basis, but the solutions represent the same point. Tableau 6 is degenerate, but there is no VDPR. The same is true of Tableau 7. Thus, an alternate optimal dual solution is not guaranteed and, in this case, does not exist.

Problem (P5) below demonstrates that a VDPR is not a necessary condition for the existence of alternate dual optima.

$$
(P5)\quad \left\{
\begin{array}{r}
\text{maximize } z = 2x_5 \\
x_1 + 2x_2 - x_3 + x_4 + x_5 \leq 2 \\
-x_1 - x_2 + x_3 - x_4 + x_5 \leq 2 \\
x_1 + x_2 + x_3 + x_4 \qquad \leq 1
\end{array}
\right\}
$$

While optimal Tableau 8 does not have a VDPR, there are alternate tableaux (9 and 10), and Tableau 10 demonstrates the existence of a second dual solution.

We assert without proof that, if an alternate optimal dual exists, at least one of the primal tableaux will have a VDPR.

4. Commercial Software Packages

It is of interest to know what information regarding shadow prices is provided by commercial linear programming software packages. In order to answer this question, problem (P3) was solved using IBM's Mathematical Programming System (MPS). The resulting optimal solution corresponded to that given in Tableau 4, along with the dual variables $y_1{*} = y_2{*} = 1$. No indication is given that $y_1{*} = 1$ is not the actual shadow price "marginal cost" of resource 1. However, MPS contains an option called RANGE, which provides information about the range of values of each objective function coefficient (c_j) and each resource value (b_i) for which the current basic variables remain optimal. When RANGE is employed on (P3), the MPS output does indicate, if interpreted correctly, that the shadow price (called the "unit cost" by MPS) for resource 1, is not 1, because increasing that resource does not produce any change in the solution. However, the output does not provide any information on the actual value of the shadow price of resource 1. On the other hand, the MPS output does provide sufficient information to conclude that the shadow price for resource 2 is actually $y_2^* = 1$.

(P3) was solved a second time using MPS with RANGE, this time with the two constraints interchanged.[3] The resulting optimal solution corresponds to that given in Tableau 5. Again, the output, if interpreted correctly, indicates that $y_2^* = 2$ is not the shadow price for resource 2, but gives no indication of its actual value (i.e., $p_2 = 1$). In order to ascertain the correct shadow prices using MPS, the parametric analysis option (PARARHS) must be employed for each constraint for which RANGE has indicated the given shadow price may not be valid.[4] Hence, the problem must be run twice, first using RANGE and then using PARARHS.

5. Summary and Conclusions

In this paper we have discussed the relationship of shadow prices to dual variables. We have demonstrated that shadow prices are not necessarily equal to dual variables except in the case when the primal is nondegenerate. In the case when the primal is degenerate, there is automatically at least one alternate optimal dual solution when there is a valid degenerate pivot row (VDPR), and there may be dual alternate optima even if there isn't such a row. If there is a VDPR, we have shown which variables should enter and leave the basis in order to arrive at an alternate dual. In all cases the i^{th} shadow price always equals the smallest value of the i^{th} dual variable in

---

[3] In the ensuing discussion, the resource labels and dual variable subscripts refer to the original problem.

[4] It should be noted that the standard commercial linear programming packages, such as MPS, do not provide for a parametric analysis of upper bound constraints (e.g., those of the form $x \leq 3$) if those are given as bounds on the variable rather than explicitly input as constraints.

the set of the optimal dual extreme point solutions. Further, in the case when the resources are positive and multiple dual optima exist, the shadow price vector never equals any of the dual solution vectors.

The widespread assumption that shadow prices and dual variables are identical may lead practitioners to the erroneous conclusion in some cases that increasing the value of a particular resource would be profitable. This situation is made worse by the fact that the commercial software packages use the same false relation (that dual variables and shadow prices are synonymous). It is to be hoped that this situation will be remedied by the developers of these packages.

The equality holds when $x^* + \varepsilon q_r$ is feasible. From (iii) and (3) we have

(iv) $$p_r = \lim_{\varepsilon \to 0^+} \left(\frac{z^*(\varepsilon) - z^*}{\varepsilon}\right) \leq y_r^*$$

Therefore, for any optimal basis, we must have

(v) $$p \leq y^*$$

We note that, for $\varepsilon \geq 0$, an optimal basic feasible solution always exists (since we are relaxing the constraint). It can be shown that we can find a basis such that $X_B^* + \varepsilon q_r$ is feasible for $0 \leq \varepsilon \leq \delta$ for some $\delta > 0$. If we choose this basis, then $z^*(\varepsilon) = z^* + \varepsilon y_r^*$; therefore $dz/d\varepsilon = y_r^*$. Since the shadow price must be at least as great as this particular feasible method of increasing z^*, we know that $p_r \geq y_r^*$, where y_r^* is determined only from this particular basis matrix having the properties described above. Equation (4) is therefore immediate from these results.

Now suppose that (5) is not true, and assume that $p = y^*(1)$. From (4) we require $y^{*(1)} \leq y^{*(2)}$. However, this is impossible in the case when $b > 0$ since, from (i), we would then have $z^{*(1)} < z^{*(2)}$. Therefore, (5) is valid. Q.E.D.

B. Proof of Theorem 2.

If x^* is nondegenerate, then from the results of Section A of the Appendix we know that $x^* + \varepsilon q \geq 0$ for any optimal basis, provided ε is sufficiently small. The results of that section then require that $p_r = y_r^*$, so that $p = y^*$. Since p is unique (from its definition), we conclude that y^* is unique, and Theorem 2, part (a), is proved.

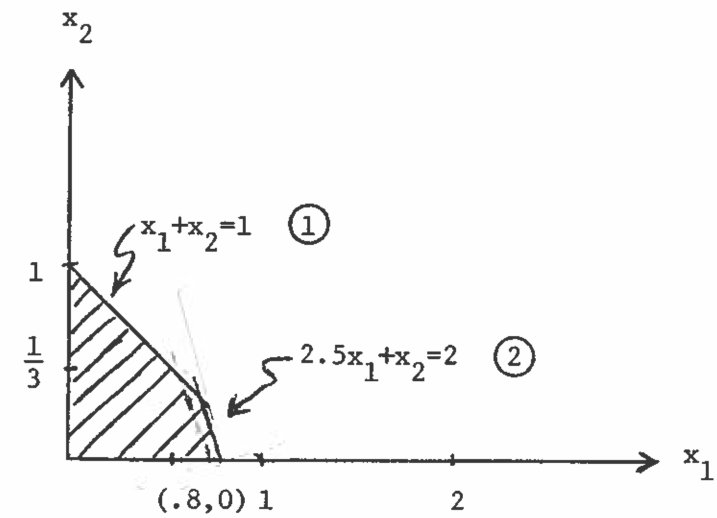

Figure 1. Feasible Region of (P1)

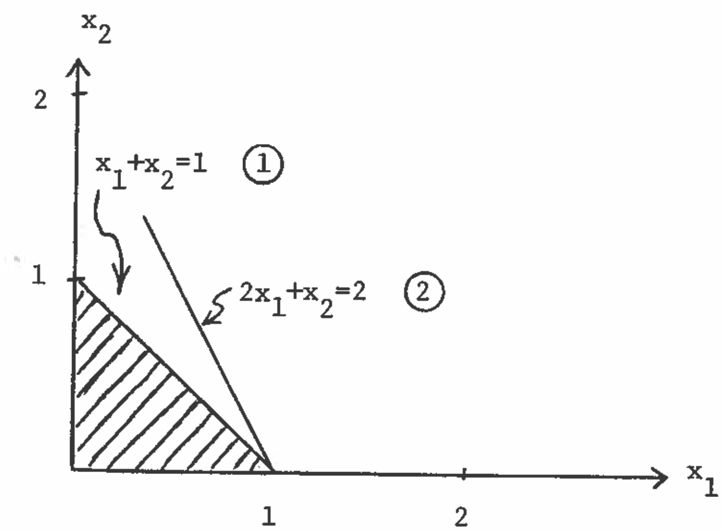

Figure 2. Feasible Region of (P2)

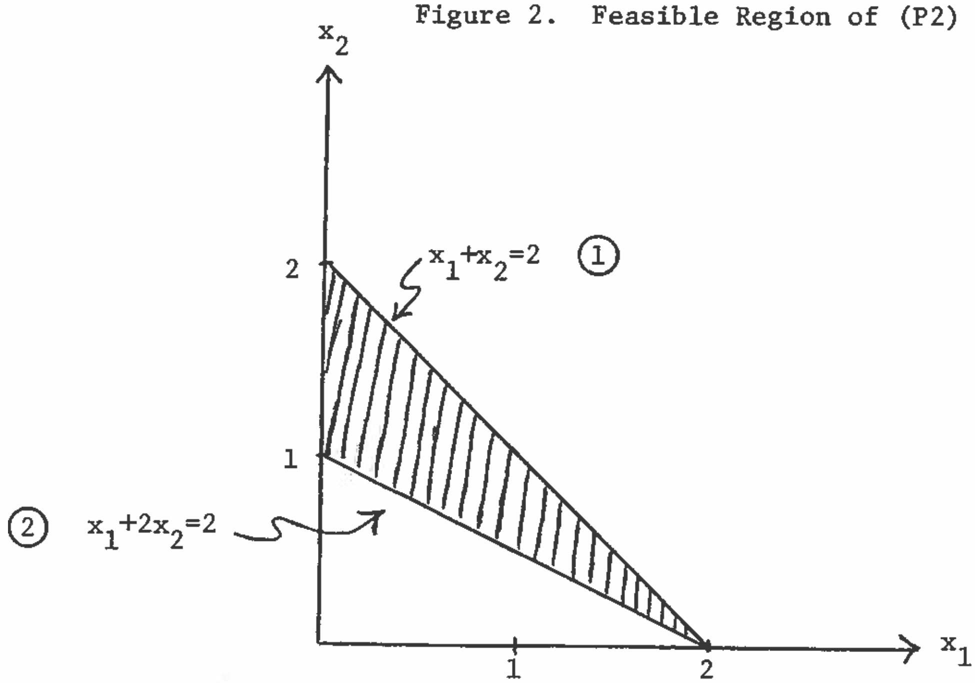

Figure 3. Feasible Region for (P4)

Basic Variables	z	x_1	x_2	x_{s1}	x_{s2}	RHS
z	1	0	.2	0	1.2	2.4
x_{s1}	0	0	.6	1	-.4	.2
x_1	0	1	.4	0	.4	.8

Tableau 1. Optimal Primal Solution for (P1)

Basic Variables	z	x_1	x_2	x_{s1}	x_{s2}	RHS	
z	1	0	2	3	0	3	
x_1	0	1	1	1	0	1	
x_{s2}	0	0	-1	-2	1	0	← VDPR

Tableau 2. Optimal Primal Solution for (P2)

Basic Variables	z	x_1	x_2	x_{s1}	x_{s2}	RHS	
z	1	0	½	0	3/2	3	
x_1	0	1	½	0	½	1	
x_{s1}	0	0	½	1	-½	0	← VDPR

Tableau 3. Alternate Representation of Optimal Primal Solution of (P2)

Basic Variables	z	x_1	x_2	x_3	x_{s1}	x_{s2}	RHS	
z	1	1	0	0	1	1	2	
x_3	0	0	0	1	1	0	1	
x_2	0	1	1	0	-1	1	0	← VDPR

Tableau 4. Optimal Primal Solution for (P3).

Basic Variables	z	x_1	x_2	x_3	x_{s1}	x_{s2}	RHS	
z	1	2	1	0	0	2	2	
x_3	0	1	1	1	0	1	1	
x_{s1}	0	-1	-1	0	1	-1	0	← VDPR

Tableau 5. Alternate Representation of Optimal Primal Solution for (P3)

Basic Variables	z	x_1	x_2	x_3	x_{s1}	x_{s2}	RHS	
z	1	2	0	0	0	2	2	
x_3	0	0	0	1	1	0	1	
x_2	0	1	1	0	-1	1	0	← not VDPR

Tableau 6. Optimal Primal Solution (P4)

Basic Variables	z	x_1	x_2	x_3	x_{s1}	x_{s2}	RHS	
z	1	2	0	0	0	2	2	
x_3	0	1	1	1	0	1	1	
x_{s1}	0	-1	-1	0	1	-1	0	← not VDPR

Tableau 7. Alternate Representation of Optimal Primal Solution for (P4)

Basic Variable	z	x_1	x_2	x_3	x_4	x_5	x_{s1}	x_{s2}	x_{s3}	x_{24}	RHS	
z	1	0	0	0	0	0	0	0	0	2	4	
x_{s1}	0	1	2	-1	1	0	1	0	0	-1	0	← not VDPR
x_{s2}	0	-1	-1	1	-1	0	0	1	0	-1	0	← not VDPR
x_{s3}	0	1	1	1	1	0	0	0	1	0	1	
x_5	0	0	0	0	0	1	0	0	0	1	2	

Tableau 8. Optimal Primal Solution for (P5)

Basic Variables	z	x_1	x_2	x_3	x_4	x_5	x_{s1}	x_{s2}	x_{s3}	x_{s4}	RHS	
z	1	0	0	0	0	0	0	0	0	2	4	
x_1	0	1	2	-1	1	0	1	0	0	-1	0	← not VDPR
x_{s2}	0	0	1	0	0	0	1	1	0	-2	0	← VDPR
x_{s3}	0	0	-1	2	0	0	-1	0	1	1	1	
x_5	0	0	0	0	0	1	0	0	0	1	2	

Tableau 9. Alternate Representation of Optimal Primal Solution for (P5)

Basic Variables	z	x_1	x_2	x_3	x_4	x_5	x_{s1}	x_{s2}	x_{s3}	x_{s4}	RHS	
z	1	0	1	0	0	0	1	1	0	0	4	
x_1	0	1	$\frac{3}{2}$	-1	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	← not VDPR
x_{s4}	0	0	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	1	0	← VDPR
x_{s3}	0	0	$-\frac{1}{2}$	2	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	1	
x_5	0	0	$\frac{1}{2}$	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	2	

Tableau 10. Alternate Representation of Optimal Primal Solution for (P5)

Now assume that X* is degenerate, and that row r is a VDPR. If X_k replaces X_{Br} in the basis, where k is defined from (8), it is straightforward to show the new basis will also be optimal and that at least one of the objective function coefficients will have been changed. Thus, y* is not unique when there is a VDPR. On the other hand, if there is no VDPR, Problems (P4) and (P5) demonstrate that there may or may not exist an alternate y*. Q.E.D.

References

1. L. Cooper and D. I. Steinberg, Methods and Applications of Linear Programming, W. B. Saunders Co., Philadelphia (1974).

2. G. B. Dantzig, Linear Programming and Extensions, Princeton University Press, Princeton, N. J. (1963).

3. S. I. Gass, Linear Programming, Fourth Edition, McGraw-Hill Book Co., New York (1975).

4. R. Dorfman, P. A. Samuelson, and R. M. Solow, Linear Programming and Economic Analysis, McGraw-Hill Book Co., New York (1958).

5. G. Hadley, Linear Programming, Addison Wesley, Reading, Mass. (1962).

6. F. S. Hillier and G. J. Lieberman, Introduction to Operations Research, Second Edition, Holden-Day, San Francisco (1974).

7. D. M. Simmons, Linear Programming for Operations Research, Holden-Day, San Francisco (1972).

8. H. A. Taha, Operations Research: An Introduction, Second Edition, The Macmillan Co., New York (1976).

9. H. M. Wagner, Principles of Operations Research, Second Edition, Prentice-Hall, Englewood Cliffs, N. J. (1975).