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**OPTIMAL RESOURCE ALLOCATION FOR
MARKOVIAN QUEUEING NETWORKS: THE
COMPLETE INFORMATION CASE**

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ABSTRACT

The problem of finding the optimal routing and flow control of a single-class Markovian network under a suitable optimization criterion is analyzed. It is proven that, if complete information about the state of the network is made available to the network controller, the optimal state dependent routing is essentially deterministic, and the optimal flow control is of a generalized window type. An iterative linear programming algorithm is given for the derivation of the optimal routing and flow control policy.

Index Terms: complete information, flow control, linear programming, optimization, queueing networks, resource allocation, routing.

1. Introduction

This paper addresses the problem of finding the optimal routing and flow control of a Markovian queueing network under a suitable optimization criterion. Our aim is to introduce a general methodology of handling the combined routing and flow control problem of systems that can be modeled as state dependent Markovian queueing networks. *The general approach is based on finding a transformation that reduces a class of non-linear optimization problems to linear optimization problems.* Using this transformation, the powerful theory of linear programming can be applied.

The class of problems analyzed in this paper is reduced to a *centralized* optimization problem. The network controller, has *complete information* about the state of the network. Because the routing is state dependent the queueing networks under analysis are not of product form [22]. For this class of networks, the solution to the problem posed here leads to an upper bound on the achievable performance under any control policy (based on any information pattern). In [3], [4] and [5], the same problem was investigated but with a controller using different information patterns. Because of the difference in the available information, different algorithms were needed there for obtaining the optimal routing and flow control.

The paper is organized as follows. In Section 2, specific examples that can be solved using the methodology introduced here are presented. These examples serve as a motivation for the general framework investigated in the sequel. In Section 3, the queueing network model and the optimization criterion are introduced. In addition the basic notation is defined. In Section 4, the Markovian queueing model is analyzed, and the optimal solution is derived. In Section 5, the optimization methodology introduced is applied to specific examples.

2. Motivation

In this section some specific problems that can be solved using the methodology introduced in this paper are presented.

A substantial amount of work in the literature has investigated *the optimal resource allocation in a network of parallel processors* under different assumptions

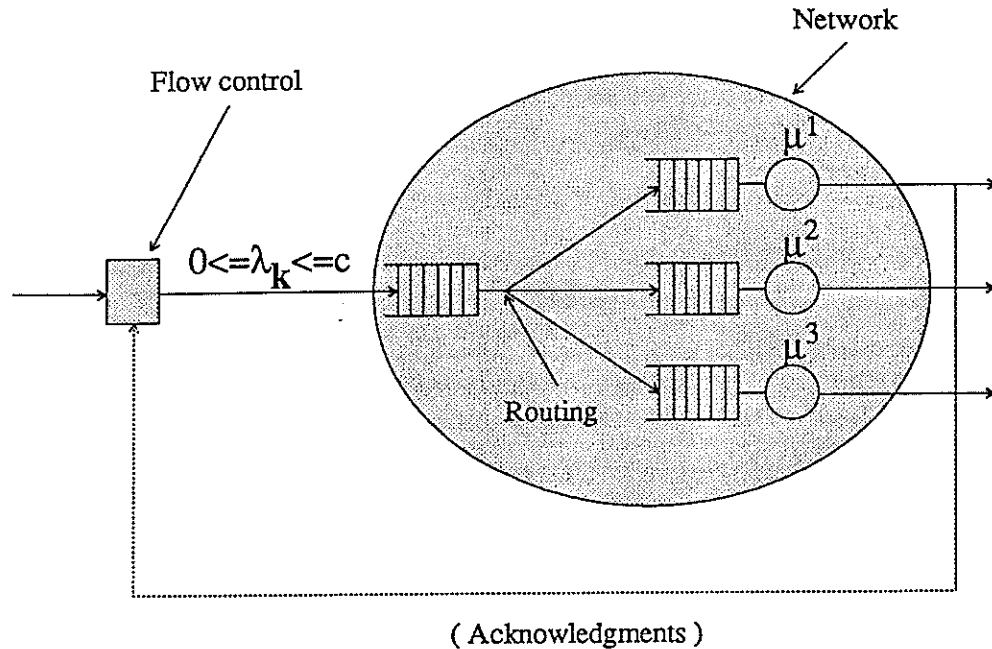


Figure 2.1 A network of parallel processors.

and using various optimization criteria ([24], [23], [7], [18], [11], [12], [1], [19], [15]). A precise description of a network of parallel processors that can be analyzed using the methodology introduced in this paper follows.

Packets arrive at a common buffer to be routed to an arbitrary number of parallel processors, each having its own buffer (see Fig. 2.1). Every processor can serve only one packet at a time, and a processor is accessible if its buffer is not full. At each arrival or departure instant, all currently accessible processors are offered to the packet occupying the first position of the common buffer. The packet can choose either to join one of these processors or to decline them all and continue to wait in the common buffer. All remaining accessible processors are offered next to the packet in the second position, then to the packet in the third position and so on until all the packets in the buffer have been considered. If the common buffer is of size zero, packets are directly routed to one of the available parallel processors. The analysis of this network is undertaken in Section 5.

The methodology introduced in this paper can also be used in the analysis of *multi-echelon repairable inventory models*. In these models the trade-off between the level of spares in the system and a service level objective, such as the expected number of operational units, is investigated (see Fig. 2.2).

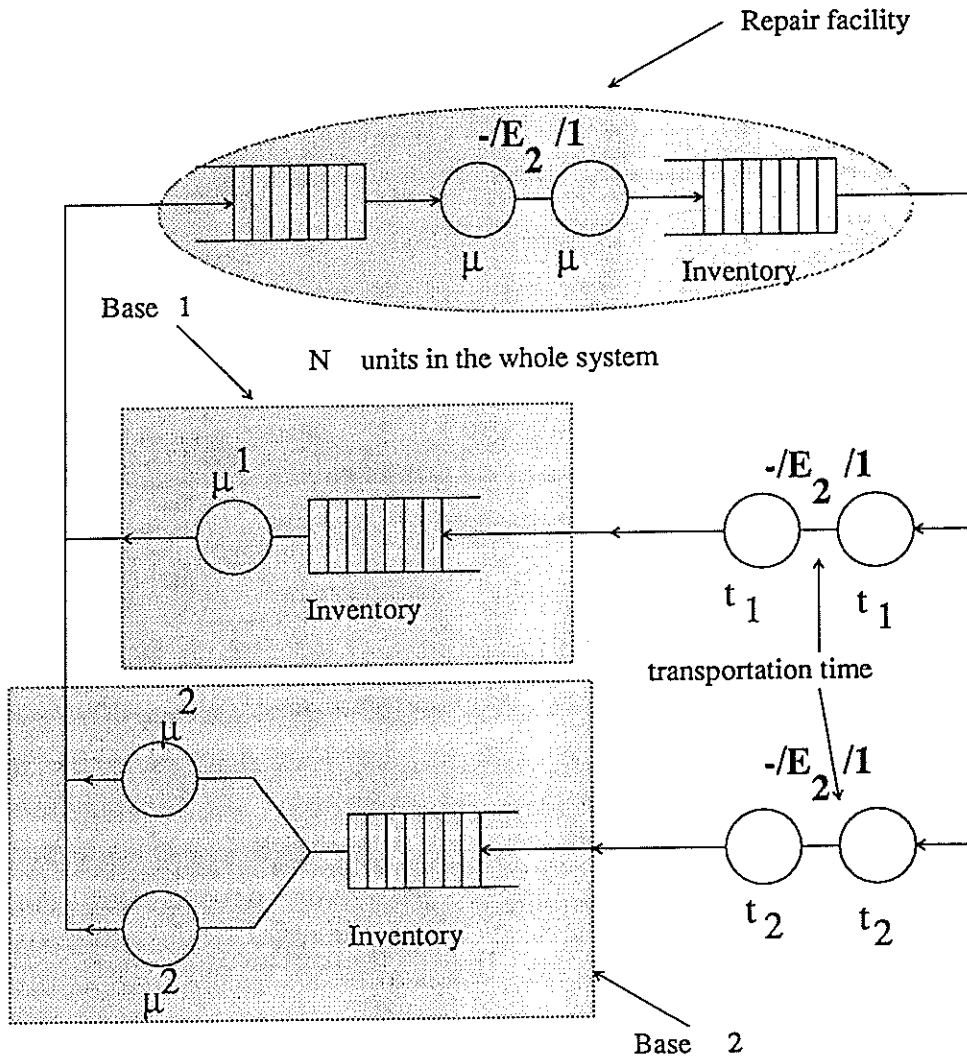


Figure 2.2 Multi-echelon repairable inventory model.

Units in operation are subject to failure. If a unit fails, it is sent to the repair facility and replaced with a spare from the on-hand inventory of spares in the location of the failure. In the event of a stock-out at that location (the base), the number of units in operation is reduced until a spare can be shipped from the repair facility. If we assume that the repair facility and the transportation activities in the system can be modeled as multi-server queues with Erlang service times and that the failure process at some location is Poisson with rate proportional to the number of operational units there, the entire multi-echelon problem can be modeled as an optimization problem over a queueing network.

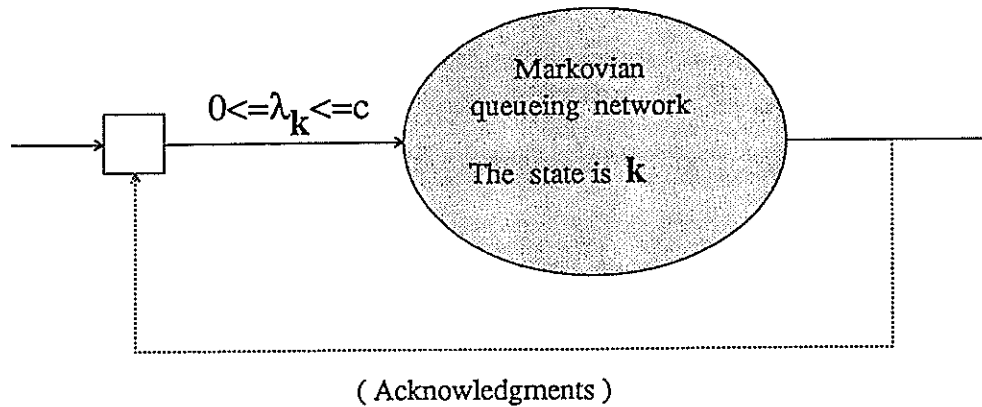


Figure 3.1. The model of a general Markovian queueing network under state dependent flow control

The analysis of the repairable inventory models just described is simplified if it is assumed that the failure rate at some location is independent of the number of units in operation there and that the routing is determined by the order in which repair requests are placed. These two assumptions are common to almost all repairable inventory problems ([16], [17], [20], [21]). For models with exponentially distributed failure times and Erlang repair and transportation times, both of the above assumptions can be relaxed.

3. Problem Definition

Packets processed by a control node arrive at a Markovian queueing network (Fig. 3.1) with exponentially distributed service times and finite or infinite buffers. The service rates of the queueing network are given and are state dependent. The rate at which packets are sent into the network is unknown and remains to be determined. Similarly, the routing parameters in the network are to be derived.

The evolution of the queueing network is described by the stochastic vector $Q_t = (Q_t^1, \dots, Q_t^{I-1}, Q_t^I)$, where Q_t^i refers to the number of packets in processor i , $1 \leq i \leq I$.

Let k_i be the number of packets at processor i , for all i , $i = 1, 2, \dots, I$. The state space of the system is

$$E = \{k = (k_1, \dots, k_I) \mid 0 \leq k_i \leq N_i, \quad i = 1, 2, \dots, I\} \quad ,$$

N_i is either a finite positive integer or infinity, for $i = 1, 2, \dots, I$.

For simplicity of notation, the operators T_i , T_i and T_{ij} for all $i, j, 1 \leq i, j \leq I$, defined by

$$T_i k = (k_1, \dots, k_i + 1, \dots, k_I) \quad ,$$

for $k_i \geq 0$, for all $i, 1 \leq i \leq I$,

$$T_i k = (k_1, \dots, k_i - 1, \dots, k_I) \quad ,$$

and

$$T_{ij} k = (k_1, \dots, k_i - 1, \dots, k_j + 1, \dots, k_I) \quad ,$$

for $k_i \geq 1$, for all $i, 1 \leq i \leq I$, are introduced.

In what follows the unknown flow control and routing parameters will be formally introduced. We will first start with the *flow control* parameter.

Definition 1. $\lambda = (\lambda_k), k \in E$, denotes the flow control: that is, λ_k denotes the dependence of the flow control parameter on the state vector k .

Definition 2. The class of controls $\lambda = (\lambda_k), k \in E$, that satisfies the peak constraint

$$0 \leq \lambda_k \leq c \quad (3.1)$$

is called *admissible*.

Here c represents the maximum number of packets that the control node can send into the network. It models the maximum capacity of the communication link that connects the controller with the network [13].

In the sequel we need the following lemma:

Lemma 3.1. *A queueing system with service rate λ_k , $0 \leq \lambda_k \leq c$, is first order equivalent to a queueing system with service rate c that feeds back to its input a percentage of its output stream equal to*

$$r_k \stackrel{\text{def}}{=} 1 - \frac{\lambda_k}{c} \quad .$$

Proof : In order to show that the two systems are first order equivalent, we have to show that they have equal equilibrium probabilities of being in certain state. This follows directly from Jackson's theorem. ■

Note that, the service rate of the control queue as perceived by the network can be written as:

$$\lambda_k = c(1 - r_k^{\cdot}) \quad . \quad (3.2)$$

Therefore, r_k^{\cdot} represents the probability that a packet leaving the control server with rate c joins the control queue.

The following *routing parameters* at times of arrival from outside the Markovian queueing network are defined:

r_k^i is the probability that a packet leaving the control server joins processor i (*i.e.*, that a transition from state k to state $T_i:k$ takes place).

Note that,

$$r_k^{\cdot} + \sum_{j=1}^I r_k^j = 1 \quad . \quad (3.3)$$

In addition, the following *routing parameters* at times of departure from a certain processor of the Markovian queueing network are also introduced:

r_k^{ij} is the probability that a packet leaving processor i joins processor j (*i.e.*, that a transition from state k to state $T_{i;j:k}$ takes place when $k_i > 0$).

r_k^i is the probability that a packet leaves processor i to join the control queue (*i.e.*, that a transition from state k to state $T_i:k$ takes place when $k_i > 0$).

Note also that, for all i , $1 \leq i \leq I$,

$$\sum_{j=1}^I r_k^{ij} + r_k^i = 1 \quad . \quad (3.4)$$

In what follows, r_k denotes the set of routing probabilities:

$$r_k = (r_k^i, r_k^{\cdot}, r_k^{ij}) \quad ,$$

for all $i, j, 1 \leq i, j \leq I$, and $k \in E$. Note that $r_k^{\cdot\cdot}$ is not included in the r_k 's since it can be derived from the flow control parameter λ_k using Equation 3.2.

We are now in the position to define the transition rates $q = (q(k, j)), (k, j) \in E \times E$, of the Markov process Q_t . These are:

$$q(k, k) = c r_k^{\cdot\cdot} \quad , \quad (3.5)$$

$$q(k, T_i k) = c r_k^i \quad , \quad (3.6)$$

$$q(k, T_i k) = \mu^i r_k^i \quad , \quad (3.7)$$

$$, \quad q(k, T_{ij} k) = \mu^i r_k^{ij} \quad , \quad (3.8)$$

for all $i, j, 1 \leq i, j \leq I$.

Let $E\gamma$ and $E\tau$ denote, respectively, the average throughput and the average time delay of the network of Fig. 3.1.

Definition 3. The set of routing and flow control parameters $(r_k, \lambda_k), k \in E$, is the *action space* of the optimization problem.

Definition 4. The set of routing and flow control parameters $(r_k, \lambda_k), k \in E$, are said to be optimum over the class of admissible controls for a given upper bound of the average time delay of the network $T, T \in R_+$, if the maximum average throughput

$$F(T) \stackrel{\text{def}}{=} \max_{E\tau \leq T} E\gamma \quad (3.9)$$

is achieved.

Note that by introducing the routing probability $r_k^{\cdot\cdot}$, the problem of optimal routing and flow control is reduced to a routing problem [8].

4. The Optimal Routing and Flow Control

The solution of the previous optimization problem is accomplished with the use of the "prime methods" from the theory of nonlinear programming. A prime

algorithm that finds the optimal routing and flow control that maximizes the average network throughput under an average time delay constraint consists of the following three steps.

In the first step it is shown in Lemma 4.1 that the question of finding the optimal routing and flow control parameters under the constraint that at any given moment there are at most N packets in the network, can be cast as a nonlinear optimization problem. The latter is reduced to an iterative algorithm as described in Proposition 4.2. The iterative algorithm is based on a linear program. By exploiting the structure of the linear program structural results of the solution are derived and given in Proposition 4.3, 4.4 and 4.5. It is demonstrated that the optimal routing is essentially deterministic, and the flow control mechanism is of a generalized window type.

The state space of the problem under the additional condition that at any given moment there are at most N packets in the system is given by

$$E_N = \left\{ \mathbf{k} = (k_1, \dots, k_I) \mid 0 \leq k_i \leq N_i, \quad i = 1, 2, \dots, I; \quad \text{and} \quad \sum_{j=1}^I k_j \leq N \right\},$$

where N_i is either a finite positive integer or infinite. Note that the bounds N_i above are due to the finite buffers of the queueing network. In an effort to make the equations as transparent as possible, the following notational rule will be used for the rest of the paper: *terms that express transitions from (or to) a state outside E_N , have the value zero.* Let $p(\mathbf{k})$ represent the probability that the network is in state \mathbf{k} . We have the following

Lemma 4.1. $(\lambda_{\mathbf{k}}, r_{\mathbf{k}})$, $\mathbf{k} \in E_N$, are the optimal routing and flow control parameters if the expected throughput

$$\sum_{\mathbf{k} \in E_N} p(\mathbf{k}) \lambda_{\mathbf{k}} \tag{4.1}$$

is maximized subject to:

(i) the global balance equations

$$p(\mathbf{k}) \left\{ \sum_{i=1}^I c r_{\mathbf{k}}^i + \sum_{i=1}^I \mu^i r_{\mathbf{k}}^i + \sum_{j=1}^I \mu^i r_{\mathbf{k}}^{ij} \right\} =$$

$$\sum_{i=1}^I p(T_i; \mathbf{k}) c r_{T_i; \mathbf{k}}^i + \sum_{i=1}^I p(T_i; \mathbf{k}) \mu^i r_{T_i; \mathbf{k}}^i + \sum_{1 \leq i, j \leq I} p(T_{ji}; \mathbf{k}) \mu^i r_{T_{ji}; \mathbf{k}}^{ij}, \quad (4.2)$$

for all $\mathbf{k} \in E_N$,

$$\sum_{\mathbf{k} \in E_N} p(\mathbf{k}) = 1, \quad (4.3)$$

(ii) the time delay constraint

$$\sum_{\mathbf{k} \in E_N} p(\mathbf{k}) \sum_{i=1}^I k_i - T \sum_{\mathbf{k} \in E_N} p(\mathbf{k}) \lambda_{\mathbf{k}} \leq 0, \quad (4.4)$$

(iii) the capacity constraint $0 \leq \lambda_{\mathbf{k}} \leq c$ and routing probabilities $r_{\mathbf{k}}^i \geq 0$, $r_{\mathbf{k}}^i \geq 0$ and $r_{\mathbf{k}}^{ij} \geq 0$, for all $\mathbf{k} \in E_N$, $1 \leq i, j \leq I$, with

$$r_{\mathbf{k}}^i + \sum_{j=1}^I r_{\mathbf{k}}^{ij} = 1, \quad (4.5)$$

$$\sum_{j=1}^I r_{\mathbf{k}}^{ij} + r_{\mathbf{k}}^i = 1. \quad (4.6)$$

Proof : Using the transition rates given by the equations 3.5-3.8, the global balance equations

$$p(\mathbf{k}) \sum_{i=1}^I \left\{ q(\mathbf{k}, T_i; \mathbf{k}) + q(\mathbf{k}, T_i; \mathbf{k}) + \sum_{j=1}^I q(\mathbf{k}, T_{ij}; \mathbf{k}) \right\} =$$

$$\sum_{i=1}^I \left\{ p(T_i; \mathbf{k}) q(T_i; \mathbf{k}, \mathbf{k}) + p(T_i; \mathbf{k}) q(T_i; \mathbf{k}, \mathbf{k}) + \sum_{j=1}^I p(T_{ji}; \mathbf{k}) q(T_{ji}; \mathbf{k}, \mathbf{k}) \right\}$$

can be easily reduced to equation 4.2.

Under the additional constraint that there are at most N packets in the network, the throughput of the network depicted in Fig. 3.1 is

$$E\gamma_N = \sum_{\mathbf{k} \in E_N} p(\mathbf{k}) \lambda_{\mathbf{k}}.$$

Since the average time delay constraint amounts to

$$E\tau_N = \frac{\sum_{\mathbf{k} \in E_N} p(\mathbf{k}) \sum_{i=1}^I k_i}{E\gamma_N} ,$$

the condition $E\tau_N \leq T$ is equivalent to

$$\sum_{\mathbf{k} \in E_N} p(\mathbf{k}) \sum_{i=1}^I k_i - T \sum_{\mathbf{k} \in E_N} p(\mathbf{k}) \lambda_{\mathbf{k}} \leq 0 . \quad \blacksquare$$

One observes that in the equations 4.1-4.4 of Lemma 4.1, the unknowns are the equilibrium probabilities and the routing probabilities. In order to linearize the optimization problem of Lemma 4.1, a new set of variables is introduced. With each point \mathbf{k} , $\mathbf{k} \in E_N$, are associated the variables $x = (x(\mathbf{k}, \mathbf{j}))$, $(\mathbf{k}, \mathbf{j}) \in E_N \times E_N$, defined by:

$$x(\mathbf{k}, \mathbf{k}) \stackrel{\text{def}}{=} p(\mathbf{k}) r_{\mathbf{k}}^{\cdot\cdot} , \quad (4.7)$$

$$x(T_i \cdot \mathbf{k}, \mathbf{k}) \stackrel{\text{def}}{=} p(T_i \cdot \mathbf{k}) r_{T_i \cdot \mathbf{k}}^{i\cdot} , \quad (4.8)$$

$$x(\mathbf{k}, T_i \cdot \mathbf{k}) \stackrel{\text{def}}{=} p(\mathbf{k}) r_{\mathbf{k}}^{\cdot i} , \quad (4.9)$$

$$x(\mathbf{k}, T_{ij} \mathbf{k}) \stackrel{\text{def}}{=} p(\mathbf{k}) r_{\mathbf{k}}^{ij} , \quad (4.10)$$

$$x(\mathbf{k}, T_i \cdot \mathbf{k}) \stackrel{\text{def}}{=} p(\mathbf{k}) r_{\mathbf{k}}^{i\cdot} , \quad (4.11)$$

for all i, j , $1 \leq i, j \leq I$.

Proposition 4.2. *The optimal routing and flow control parameters $(\lambda_{\mathbf{k}}, r_{\mathbf{k}})$, $\mathbf{k} \in E$, are given by the equations:*

$$r_{\mathbf{k}}^{\cdot i} = \frac{x(\mathbf{k}, T_i \cdot \mathbf{k})}{x(\mathbf{k}, \mathbf{k}) + \sum_{j=1}^I x(\mathbf{k}, T_j \cdot \mathbf{k})} , \quad (4.12)$$

$$r_{\mathbf{k}}^{i\cdot} = \frac{x(\mathbf{k}, T_i \cdot \mathbf{k})}{\sum_{j=1}^I x(\mathbf{k}, T_{ij} \mathbf{k}) + x(\mathbf{k}, T_i \cdot \mathbf{k})} , \quad (4.13)$$

for all i , $1 \leq i \leq I$,

$$r_{\mathbf{k}}^{ij} = \frac{x(\mathbf{k}, T_{ij} \mathbf{k})}{\sum_{j=1}^I x(\mathbf{k}, T_{ij} \mathbf{k}) + x(\mathbf{k}, T_i \cdot \mathbf{k})} , \quad (4.14)$$

for all $i, j, 1 \leq i, j \leq I$, and

$$\lambda_k = c \left(1 - \frac{x(k, k)}{x(k, k) + \sum_{j=1}^I x(k, T_j k)} \right), \quad (4.15)$$

where $x = (x(k, j)), (k, j) \in E \times E$, is the solution of the following iterative algorithm:

Algorithm:

Step 0 : $N=1$,

Step 1 : For the current value of N , solve the following linear program:

$$\max c \sum_{k \in E_N} \sum_{i=1}^I x(k, T_i k), \quad (4.16)$$

subject to the linear constraints:

$$\sum_{i=1}^I x(k, T_i k) + \sum_{i=1}^I x(k, T_i k) \frac{\mu^i}{c} + \sum_{i=1}^I \sum_{j=1}^I x(k, T_{ij} k) \frac{\mu^i}{c} = \quad (4.17)$$

$$\sum_{i=1}^I x(T_i k, k) + \sum_{i=1}^I x(T_i k, k) \frac{\mu^i}{c} + \sum_{i=1}^I \sum_{j=1}^I x(T_{ji} k, k) \frac{\mu^i}{c},$$

for every $k \in E_N$,

$$\sum_{k \in E_N} \left\{ x(k, k) + \sum_{j=1}^I x(k, T_j k) \right\} 1(\sum_{l=1}^I k_l < N) + \quad (4.18)$$

$$\sum_{k \in E_N} \left\{ \sum_{j=1}^I x(k, T_{ij} k) + x(k, T_i k) \right\} 1(\sum_{l=1}^I k_l = N) = 1$$

for one i , such that $k_i > 0, 1 \leq i \leq I$,

$$\sum_{k \in E_N} \sum_{n=1}^I \left\{ x(k, k) + \sum_{j=1}^I x(k, T_j k) \right\} \frac{k_n}{c} 1(\sum_{l=1}^I k_l < N) + \quad (4.19)$$

$$\sum_{k \in E_N} \left\{ \sum_{j=1}^I x(k, T_{ij} k) + x(k, T_i k) \right\} \frac{N}{c} 1(\sum_{l=1}^I k_l = N)$$

$$-Tc \sum_{\mathbf{k} \in E_N} \sum_{j=1}^I x(\mathbf{k}, T_j \mathbf{k}) \leq 0$$

for one i , such that $k_i > 0$, $1 \leq i \leq I$,

$$\sum_{j=1}^I x(\mathbf{k}, T_{ij} \mathbf{k}) + x(\mathbf{k}, T_i \mathbf{k}) = \sum_{j=1}^I x(\mathbf{k}, T_{lj} \mathbf{k}) + x(\mathbf{k}, T_l \mathbf{k}) \quad (4.20)$$

for every i and l such that $k_i > 0$ and $k_l > 0$, $1 \leq i, l \leq I$ and $0 < \sum_{j=1}^I k_j \leq N$,

and

$$x(\mathbf{k}, \mathbf{k}) + \sum_{j=1}^I x(\mathbf{k}, T_j \mathbf{k}) = \sum_{j=1}^I x(\mathbf{k}, T_{ij} \mathbf{k}) + x(\mathbf{k}, T_i \mathbf{k}) \quad (4.21)$$

for every i , such that $k_i > 0$, $1 \leq i \leq I$ and $0 < \sum_{j=1}^I k_j \leq N$,

Step 2 : If $E\gamma_N = E\gamma_{N-1}$, stop; The computed routing parameters as well as the flow control parameters are optimal and are given by the equations 4.12-4.15 for $\mathbf{k} \in E_N$, and the arrival rates $\lambda_{\mathbf{k}} = 0$, for $\mathbf{k} \in E - E_N$. Else, $N := N + 1$, and return to Step 1, using the optimal solution of the linear program as the initial feasible point of the next iteration.

Proof : By summing equations 4.7, 4.9 and using equation 3.3 we obtain

$$p(\mathbf{k}) = x(\mathbf{k}, \mathbf{k}) + \sum_{j=1}^I x(\mathbf{k}, T_j \mathbf{k}), \quad \text{for } \sum_{j=1}^I k_j \leq N \quad (4.22)$$

and by summing equations 4.10, 4.11 and using equation 3.4 we obtain

$$p(\mathbf{k}) = \sum_{j=1}^I x(\mathbf{k}, T_{ij} \mathbf{k}) + x(\mathbf{k}, T_i \mathbf{k}) \quad (4.23)$$

for $1 \leq i \leq I$ and $k_i > 0$.

Therefore,

$$\sum_{j=1}^I x(\mathbf{k}, T_{i,j}\mathbf{k}) + x(\mathbf{k}, T_i\mathbf{k}) = \sum_{j=1}^I x(\mathbf{k}, T_{l,j}\mathbf{k}) + x(\mathbf{k}, T_l\mathbf{k})$$

for every i and l such that $k_i > 0$ and $k_l > 0$, $1 \leq i, l \leq I$ and $0 < \sum_{j=1}^I k_j \leq N$, and

$$x(\mathbf{k}, \mathbf{k}) + \sum_{j=1}^I x(\mathbf{k}, T_j\mathbf{k}) = \sum_{j=1}^I x(\mathbf{k}, T_{i,j}\mathbf{k}) + x(\mathbf{k}, T_i\mathbf{k})$$

for all i , $1 \leq i \leq I$, such that $k_i > 0$ and $0 < \sum_{j=1}^I k_j \leq N$. This proves equations 4.20, and 4.21.

Noting that, by definition (4.8), (4.10) and the fact that $\mathbf{k} = T_{i,j}(T_{j,i}\mathbf{k})$

$$p(T_i\mathbf{k})r_{T_i\mathbf{k}}^i = x(T_i\mathbf{k}, \mathbf{k}) ,$$

$$p(\mathbf{k})r_{\mathbf{k}}^{ij} = x(\mathbf{k}, T_{i,j}\mathbf{k}) ,$$

$$p(T_{j,i}\mathbf{k})r_{T_{j,i}\mathbf{k}}^{ij} = x(T_{j,i}\mathbf{k}, \mathbf{k}) ,$$

and finally from (4.11) and the fact that $\mathbf{k} = T_i(T_i\mathbf{k})$:

$$p(T_i\mathbf{k})r_{T_i\mathbf{k}}^i = x(T_i\mathbf{k}, \mathbf{k}) ,$$

the global balance equations 4.2 become:

$$\begin{aligned} \sum_{i=1}^I x(\mathbf{k}, T_i\mathbf{k}) + \sum_{i=1}^I x(\mathbf{k}, T_i\mathbf{k}) \frac{\mu^i}{c} + \sum_{i=1}^I \sum_{j=1}^I x(\mathbf{k}, T_{i,j}\mathbf{k}) \frac{\mu^i}{c} = \\ \sum_{i=1}^I x(T_i\mathbf{k}, \mathbf{k}) + \sum_{i=1}^I x(T_i\mathbf{k}, \mathbf{k}) \frac{\mu^i}{c} + \sum_{i=1}^I \sum_{j=1}^I x(T_{j,i}\mathbf{k}, \mathbf{k}) \frac{\mu^i}{c} . \end{aligned}$$

Since the sum of the probabilities over the state space is equal to one, we have

$$\sum_{\mathbf{k} \in E_N} \left\{ x(\mathbf{k}, \mathbf{k}) + \sum_{j=1}^I x(\mathbf{k}, T_j\mathbf{k}) \right\} 1\left(\sum_{l=1}^I k_l < N\right) +$$

$$\sum_{\mathbf{k} \in E_N} \left\{ \sum_{j=1}^I x(\mathbf{k}, T_{ij}\mathbf{k}) + x(\mathbf{k}, T_i.\mathbf{k}) \right\} 1\left(\sum_{l=1}^I k_l = N\right) = 1$$

for one i , such that $k_i > 0$, $1 \leq i \leq I$.

The throughput of the network is given by the equation

$$E\gamma_N = \sum_{\mathbf{k} \in E_N} p(\mathbf{k}) \lambda_{\mathbf{k}} 1\left(\sum_{i=1}^I k_i < N\right),$$

or equivalently,

$$E\gamma_N = c \sum_{\mathbf{k} \in E_N} \sum_{i=1}^I x(\mathbf{k}, T_i.\mathbf{k}).$$

Finally, the time delay constraint can be written as:

$$\sum_{\mathbf{k} \in E_N} p(\mathbf{k}) \sum_{i=1}^I k_i - TE\gamma_N \leq 0,$$

or, equivalently,

$$\begin{aligned} & \sum_{\mathbf{k} \in E_N} \sum_{n=1}^I \left\{ x(\mathbf{k}, \mathbf{k}) + \sum_{j=1}^I x(\mathbf{k}, T_{.j}\mathbf{k}) \right\} \frac{k_n}{c} 1\left(\sum_{l=1}^I k_l < N\right) + \\ & \sum_{\mathbf{k} \in E_N} \left\{ \sum_{j=1}^I x(\mathbf{k}, T_{ij}\mathbf{k}) + x(\mathbf{k}, T_i.\mathbf{k}) \right\} \frac{N}{c} 1\left(\sum_{l=1}^I k_l = N\right) \\ & - Tc \sum_{\mathbf{k} \in E_N} \sum_{j=1}^I x(\mathbf{k}, T_{.j}\mathbf{k}) \leq 0 \end{aligned}$$

for one i , such that $k_i > 0$, $1 \leq i \leq I$.

The solution of the linear program gives as output parameters the values of $x(\mathbf{k}, \mathbf{j})$, $(\mathbf{k}, \mathbf{j}) \in E_N \times E_N$ for the current value of N . The solution of the iterative algorithm gives as output parameters the values of $x(\mathbf{k}, \mathbf{j})$, $(\mathbf{k}, \mathbf{j}) \in E \times E$. From equations 4.7-4.11 one can easily see that the optimal routing and flow control parameters are given by the equations 4.12-4.15. ■

Proposition 4.2 gives us a constructive way to compute the optimal routing and flow control policy. In order to better understand the meaning of this result the structure of the linear program is analyzed below. Let S be the subset of the state space E that is accessible under the optimal policy obtained in Proposition 4.2. Then,

$$S = \{ k \in E \mid p(k) > 0 \} .$$

Let S_N be the subset of the state space E_N that is accessible under the solution of the linear program (in Step 2 of the Iterative Algorithm) for the current value of N . Then,

$$S_N = \{ k \in E_N \mid p(k) > 0 \} .$$

Proposition 4.3. *The optimal routing is deterministic for all but at most one state in S . The optimal flow control is of a generalized window type, by which it is meant that packets enter the network with arrival rate c in all but at most one point of S and with rate 0 elsewhere (bang-bang control).*

Proof : Associated with every state k , $k \in S_N$, are the (global balance) equations 4.17, and the set of equations 4.20 and 4.21. In addition equation 4.19 represents a global inequality constraint for the entire network. If $k \in S_N$, we conclude from equations 4.22 and 4.23 that

$$x(k, k) + \sum_{j=1}^I x(k, T_j k) > 0 ,$$

for $\sum_{j=1}^I k_j \leq N$, and

$$\sum_{j=1}^I x(k, T_{ij} k) + x(k, T_i k) > 0 ,$$

for all i , $1 \leq i \leq I$ such that $k_i > 0$. Therefore, *at least one* of the elements in the sets $(x(k, k), x(k, T_i k))$, $1 \leq i \leq I$, and $(x(k, T_{ij} k), x(k, T_i k))$, $1 \leq i \leq I$, is nonzero for each k , $k \in S_N$. These nonzero elements represent basic variables of an optimal basic feasible solution to the linear program 4.16-4.21 and their number is equal to the

number of constraints given by the equations 4.17, 4.20 and 4.21. The time delay constraint 4.19 allows one more variable of the optimal basic feasible solution to be nonzero. The existence of an optimal basic feasible solution to the linear program 4.16-4.21 is guaranteed by the existence of an optimal feasible solution [14]. This proves, then, that the optimal routing probabilities specified by the equations 4.12-4.14 take the values zero or one *except* for at most one state in S_N . For this state one more of the routing probabilities in r_k , or r_k^* is nonzero. Thus, there exists an optimal policy with no more than one random point in the whole action space (r_k, λ_k) , $k \in E$. ■

The original optimization problem defined in equation 3.9 exhibited one constraint. The proof of Proposition 4.3 above showed that there is at most one random point in the action space due to the time delay constraint. Therefore, the following extension to Proposition 4.3 is straightforward.

Proposition 4.4. *If the number of linear constraints is P , there exists an optimal policy with no more than P random points in the whole action space (r_k, λ_k) , $k \in E$.*

The random point in the action space specifies the probability of routing from a given state to the next state. Thus, the practical implementation of the control policy at a random point in the action space is based on probabilistic routing. We also note the following structural result.

Proposition 4.5. *If the time delay constraint is not present or it is present but the optimal value of the dual linear program variable corresponding to the time delay constraint is zero, the optimal routing probabilities only depend on the ratios $\frac{\mu^i}{c}$ for $i = 1, 2, \dots, I$.*

Proof : All the linear programming constraints, with the exception of the time delay constraint, depend on only the ratios $\frac{\mu^i}{c}$, for $i = 1, 2, \dots, I$. The time delay constraint 4.19 depends on the value of c as well as the value of μ^i , for $i = 1, 2, \dots, I$. When the optimal value of the variable corresponding to the time delay constraint in the dual linear program is zero, the linear program 4.16-4.21 obtained by ignoring the time delay constraint has the same solution. These observations prove the proposition. ■

Another optimization criterion (see also [4]) that can be applied to the analysis of queueing problems considered in this paper is the following:

$$\min_{E\gamma \geq \Gamma} E\tau$$

for $0 \leq \lambda_k \leq c$, and for $0 < \Gamma$. Via the transformation introduced in Section 4 of [4], (or a similar transformation introduced in [6]) this problem can also be transformed into a linear optimization problem, and the above analysis and conclusions hold for this criterion as well.

5. Applications

In this section we apply the optimization methodology introduced above to some specific examples. A program written in the C programming language, automatically creates the linear program from a diagram that shows all possible transitions out of a state. The outputs of the program are the optimal flow control and the optimal routing parameters. From these values the optimal equilibrium probabilities and other quantities of interest (such as the average throughput and the time delay) can be derived.

(i) *Optimal Routing and Flow Control of a Network of Parallel Processors.*

The network of this example (Fig. 2.1.) consists of three processors in parallel with service rates $\mu^1 = 2.0 \text{ packets/sec}$, $\mu^2 = 1.0 \text{ packets/sec}$, and $\mu^3 = .5 \text{ packets/sec}$, respectively. Each of the processors has a buffer size equal to 10. Packets arrive into the network with state dependent arrival rate λ_k where $0 \text{ packets/sec} \leq \lambda_k \leq 4 \text{ packets/sec}$. We want to find the optimal state dependent routing and the optimal flow control that maximize the throughput of the network for different upper bounds T of the time delay constraint. In Fig. 5.1. the maximum throughput of the network for different upper bounds of the time delay constraint is given. In Fig. 5.2. the optimal routing parameters out of a given state for $T = 1.97 \text{ sec}$ is depicted. An arrow out of state (Q_t^1, Q_t^2, Q_t^3) points towards the state that the system will enter as a result of an arrival under the optimal routing policy. All routing probabilities are arrows pointing towards the switching surface (Fig. 5.2).

(ii) *Optimal Flow Control of a Tree-like Network.*

The network of this example (Fig. 5.3.) is a tree-like network with service rates $\mu^1 = 4.0 \text{ packets/sec}$, $\mu^2 = 3.0 \text{ packets/sec}$, $\mu^3 = 2.0 \text{ packets/sec}$, $\mu^4 = 3.0$

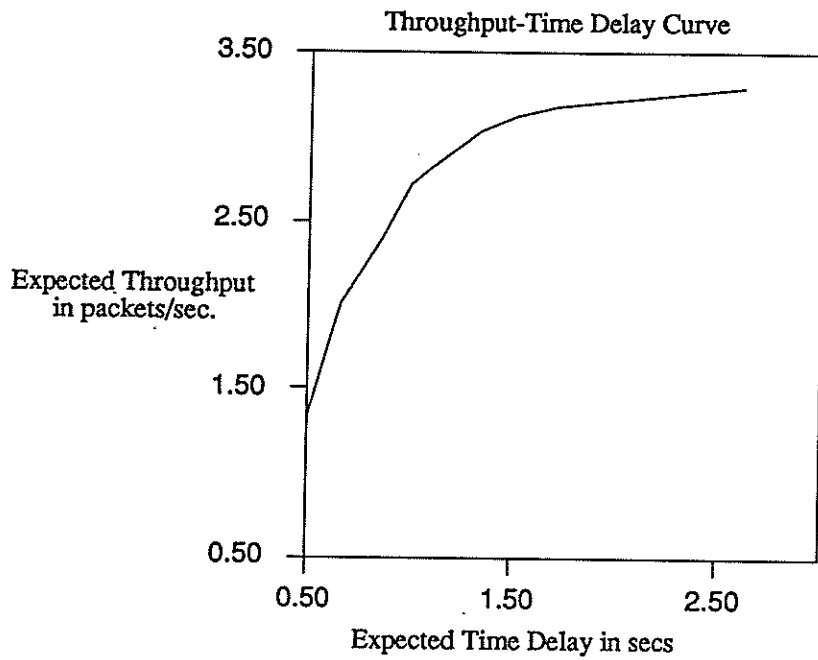


Figure 5.1. The optimal throughput under different time delay requirements for a network of three parallel processors.

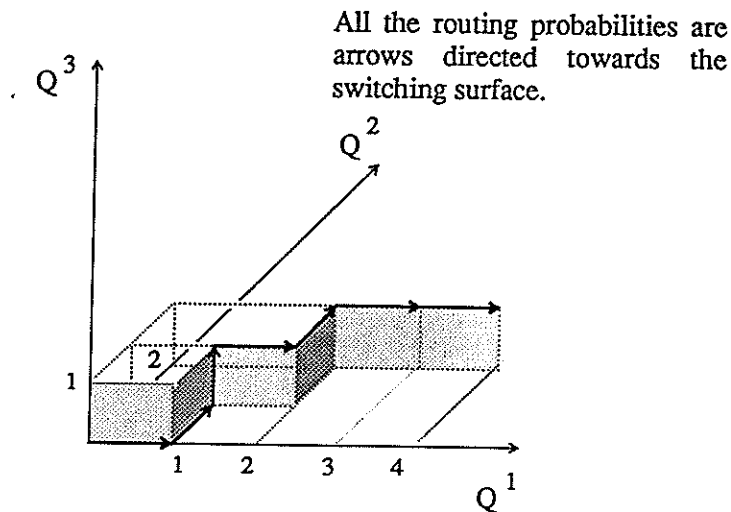


Figure 5.2. Optimal routing probabilities for a network of three parallel processors.

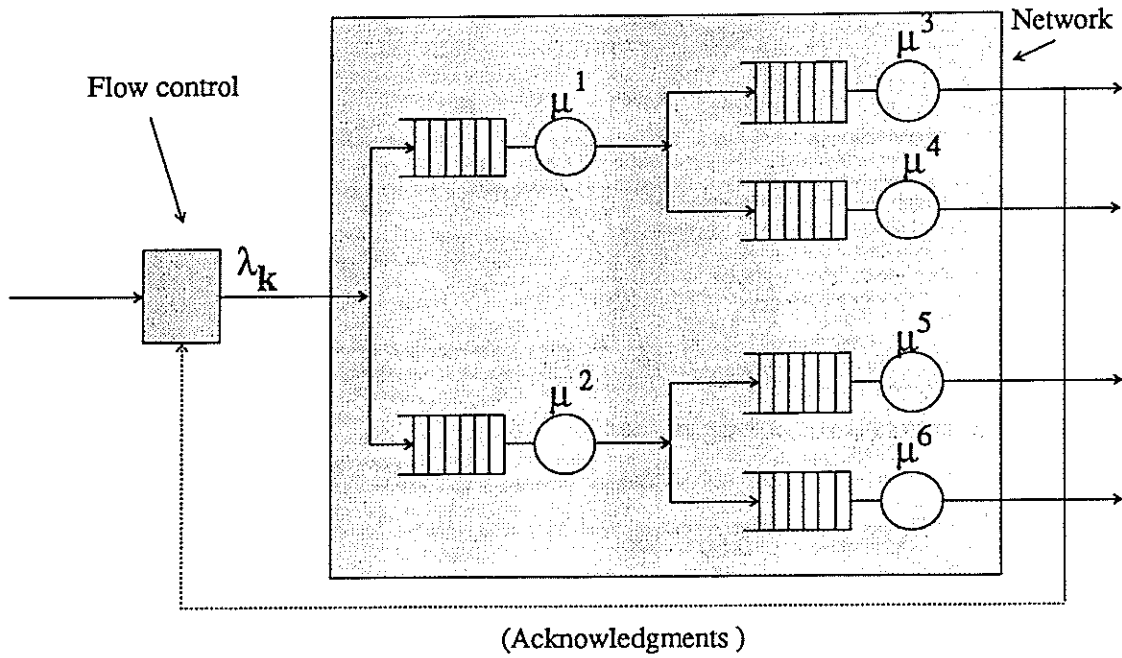


Figure 5.3. A tree-like network subject to state dependent routing and flow control.

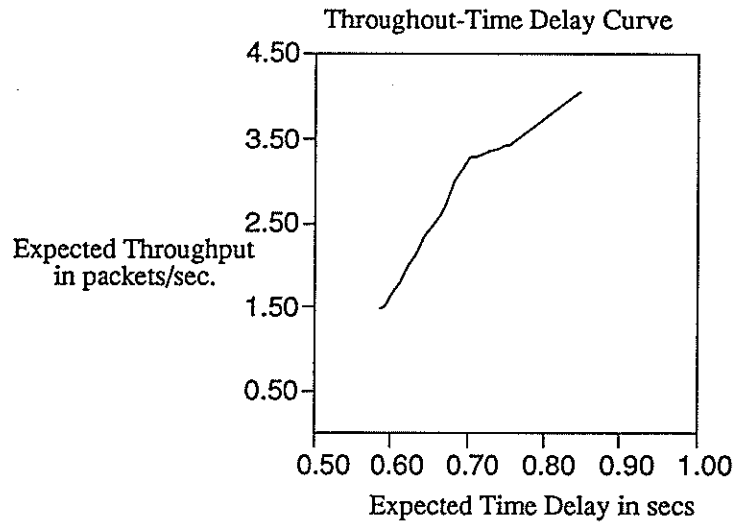


Figure 5.4. The optimal throughput under different time delay requirements for a tree-like network.

packets/sec, $\mu^5 = 1.0$ packets/sec, and $\mu^6 = 2.0$ packets/sec, respectively. Each of the processors has a buffer size equal to 10. Packets arrive into the network with state dependent arrival rate λ_k where 0 packets/sec $\leq \lambda_k \leq 10$ packets/sec. In Fig. 5.4. the throughput of the network for different upper bounds of the time delay constraint is given.

Consider now the example depicted in Fig. 2.2. There exist two bases. Base 1 has one operating unit and base 2 has two. The repair facility is an $-/E_2/1$ queue with rate per phase μ . The transportation time from the repair facility to base i is Erlang with two stages and rate per stage t_i . For simplicity of exposition, we assume that transportation from the bases to the repair facility is instantaneous. The total inventory cost per unit time is defined by:

$$aM - \sum_{i=1}^2 b^i \left(K^i - \frac{E\gamma^i}{\mu^i} \right)$$

where:

a : is a constant,

M : is the number of spares in the multi-echelon system,

b^i : is a penalty cost per non-operational unit per unit time, at base i , $i = 1, 2$.

K^i : is the maximum number of units in operation at base i .

μ^i : is the failure rate for an operating unit, at base i .

$E\gamma$: is the throughput measured at the repair facility.

$E\gamma^i$: is the throughput measured at base i . Note that $K^i - \frac{E\gamma^i}{\mu^i}$, is the expected number of non-operational units at base i . The objective is to find the optimal number of spares M in the system, that minimizes the total inventory cost. We can easily solve this problem using the optimization methodology introduced in this paper.

Finally note that in [25] our methodology was applied to the study of flow control of Metropolitan Area Networks.

6. Conclusions

An optimization methodology that can be applied to a variety of Markovian queueing problems was introduced. This methodology was applied to the derivation

of the optimal state dependent routing and flow control of a Markovian queueing network.

The optimization problem investigated calls for maximizing the average network throughput subject to an upper bound on the average network time delay. It was demonstrated that the latter can be reduced to an iterative algorithm whose steps require the solution of a linear program. Structural properties of the optimal solution were also given.

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