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1. Introduction

We investigate the dynamical behavior of *a decentralized network* that is shared by users, each trying to achieve *its own objectives.*

The objectives, or demands, of each of the network's users are explicitly expressed in the form of a criterion.

Different criteria encapsulate different performance objectives and result in different user behavior.

Optimization Criteria

$$P^k \stackrel{def}{=} \frac{(E\gamma^k)^{\beta_k}}{E\tau^k} \quad .$$

$$\min EQ^k \quad .$$

Talk given at the review of the Advanced Communications Systems Project, March 20-23, 1989.

$$\max_{E\tau^k \leq T^k \ and \ E\gamma^k \geq \Gamma^k} E\gamma^k \quad .$$

$$\min_{E\gamma^k \geq \Gamma^k \ and \ E\tau^k \leq T^k} E\tau^k \quad .$$

2. Existence and Uniqueness of Equilibrium Points for Concave N-Person Games

Nash Equilibrium Point: A number of rational *noncooperative players* operate at a Nash equilibrium point, if none of the players can improve its reward by altering its strategy *unilaterally.*

N-person Constraint Game: We consider an $n-$person constraint game in which the constraints of each of the players, as well as a player's payoff function, may depend on the strategy of every other player. Let x_i describe the strategy of the i^{th} player, and let

$$x \stackrel{def}{=} (x_1, \cdots, x_n) \quad .$$

Let $\phi_i(x)$ be the payoff function of the i^{th} player.

Assume:

(i) $\phi_i(x)$ is continuous in x and concave in x_i.

(ii) The state space R defined by the product space of the individual strategy spaces, lies in a convex, closed, and bounded region.

Then by Kakutani's fixed point theorem an equilibrium point *exists* for every concave n-person game. Let

$$\sigma(x, r) \stackrel{def}{=} \sum_{i=1}^{n} r_i \phi_i(x) \quad , \tag{2.1}$$

for $r_i \geq 0$ for all i, $1 \leq i \leq n$.

$$g(x,r) \stackrel{def}{=} \begin{pmatrix} r_1 \nabla_1 \phi_1(x) \\ r_2 \nabla_2 \phi_2(x) \\ \vdots \\ r_n \nabla_n \phi_n(x) \end{pmatrix} \quad , \tag{2.2}$$

where $r = (r_1, \cdots, r_n)$ and $r_i \geq 0$, for all i, $1 \leq i \leq n$.

$\sigma(x,r)$ is *diagonally strictly concave* for $x \in R$ and fixed $r \geq 0$ if for every $x^0, x^1 \in R$

$$(x^1 - x^0) g(x^0, r) - (x^1 - x^0) g(x^1, r) \ > \ 0 \quad . \tag{2.3}$$

If $\sigma(x,r)$ is diagonally strictly concave for some $r = \bar{r} > 0$, then, if there exists an equilibrium point for the noncooperative game, that point is *unique*.

3. Decentralized Optimal Routing

Let $\mathcal{P}_k$ be the set of all directed paths connecting the origin and destination nodes of user k, $1 \leq k \leq K$.

Let y^{ki} be the flow on path i, $i \in \mathcal{P}_k$, of the k class of packets.

Let

$$y^k \stackrel{def}{=} [y^{k1} \cdots y^{k|\mathcal{P}_k|}] \quad , \tag{3.1}$$

be the corresponding vector of path flows of the k class of packets. Then,

$$\lambda^k = \sum_{i \in \mathcal{P}_k} y^{ki} \quad . \tag{3.2}$$

Let

$$F^{ki} \stackrel{def}{=} \sum_{\substack{\textit{all paths } p \\ \textit{belonging to class } k \\ \textit{and utilizing server } i}} y^{kp} \tag{3.3}$$

be the total load at node i that belongs to class k, and

$$F^i \stackrel{def}{=} \sum_{k=1}^{K} F^{ki} \tag{3.4}$$

be the total load at node i. Then

$$E\tau = \frac{1}{\sum_{k=1}^{K} \sum_{i \in \mathcal{P}_k} y^{ki}} \sum_{j=1}^{I} \frac{F^j}{\mu^j - F^j} \tag{3.5}$$

and

$$EQ = \sum_{j=1}^{I} \frac{F^j}{\mu^j - F^j} \quad . \tag{3.6}$$

It is proven that in a multiclass environment the minimization of EQ gives a unique set of F^j for $j = 1, \cdots, I$. But such a set could correspond to multiple solutions with respect to the path flows of the different classes of customers.

In order to obtain a unique solution we instead minimize the quantity

$$E\bar{Q} \stackrel{def}{=} EQ + \alpha^2 \left(\sum_{k=1}^{K} \sum_{i \in \mathcal{P}_k} (y^{ki})^{\alpha^1} \right) \tag{3.7}$$

where $\alpha^1 > 1$, and α^2 is a very small positive number.

The Game Approach:

$$E\tau^k = \frac{1}{\sum_{i \in \mathcal{P}_k} y^{ki}} \sum_{j=1}^{I} \frac{F^{kj}}{\mu^j - F^j} \tag{3.8}$$

and

$$EQ^k = \sum_{j=1}^{I} \frac{F^{kj}}{\mu^j - F^j} \quad . \tag{3.9}$$

Let

$$\mathbf{y} \stackrel{\text{def}}{=} (y^1, \cdots y^K) \quad . \tag{3.10}$$

The state space

$$\mathcal{A} = \left\{ \mathbf{y} : \sum_{i \in \mathcal{P}_k} y^{ki} = \lambda^k, \; y^{ki} \geq 0, and \; F^j \leq \mu^j - \epsilon \right\}$$

for every k, $1 \leq k \leq K$, and every j, $1 \leq j \leq I$.
The state space $\mathcal{A}$ is a closed, compact, and convex set. The objective function of each of the network's users, is an increasing function with respect to each of its path flows. Therefore it is easy to see that even if we only require that

$$\sum_{i \in \mathcal{P}_k} y^{ki} \geq \lambda^k \tag{3.11}$$

user k will achieve its objective with path flows validating

$$\sum_{i \in \mathcal{P}_k} y^{ki} = \lambda^k \quad . \tag{3.12}$$

Let

$$\mathcal{A}^* = \left\{ \mathbf{y} : \sum_{i \in \mathcal{P}_k} y^{ki} \geq \lambda^k, \ y^{ki} \geq 0, and\ F^j \leq \mu^j - \epsilon \right\}$$

for every k, $1 \leq k \leq K$, and every j, $1 \leq j \leq I$.

The state space $\mathcal{A}^*$ is also a closed, compact, and convex set. EQ^k is continuous in $\mathbf{y}$ and convex in y^k in both $\mathcal{A}$ and $\mathcal{A}^*$. Furthermore $\sum_{l=1}^{K} EQ^l$ is diagonally strictly convex function in both $\mathcal{A}$ and $\mathcal{A}^*$.

Proposition 3.1. *If there exist a set of feasible path flows* $\mathbf{y}$ *in* $\mathcal{A}$ *or* $\mathcal{A}^*$*, then the decentralized routing problem as formulated in this section has a unique Nash equilibrium point.*

4. Decentralized Flow Control

Each user operates under the following criterion:

$$\max_{EQ^k - T^k E\gamma^k \leq 0 \ and \ E\gamma^k \geq \Gamma^k} (E\gamma^k)^\alpha \quad , \tag{4.1}$$

or equivalently,

$$\max_{EQ^k \ - \ T^k \lambda^k \leq 0 \ and \lambda^k \geq \Gamma^k} (\lambda^k)^\alpha$$

for some α, $0 < \alpha < 1$.

Observe that the time delay constraint is convex with respect to the path rates, and is active only when the arrival rate $\lambda^k \neq 0$.

The second optimization criterion can be written as follows:

$$\min_{\lambda^k \geq \Gamma^k \ and \ EQ^k - T^k E\gamma^k \leq 0} E\tau^k \quad , \tag{4.2}$$

for $k = 1, 2, \cdots, K$.
The state space can be defined as before.
Proposition 4.1. : *A set of active users may have multiple Nash equilibrium points. In general, a network may have more than one set of active users.*

5. Decentralized Resource Allocation

Optimal Routing and Flow Control under Power Criterion:

$$P^k \stackrel{def}{=} \frac{(E\gamma^k)^{\beta_k}}{E\tau^k} \quad . \tag{5.1}$$

Let H^k be the function

$$H^k \stackrel{\text{def}}{=} \frac{1}{P^k} \quad . \tag{5.2}$$

For a BCMP network

$$H^k = (\lambda^k)^{-2\beta_k} \sum_{j=1}^{I} \frac{F^{kj}}{\mu^j - F^j} \quad . \tag{5.3}$$

Let

$$x^k \stackrel{\text{def}}{=} [\lambda^k, y^{k1} \cdots y^{k|\mathcal{P}_k|}] \quad ,$$

and

$$\mathbf{x} \stackrel{\text{def}}{=} [x^1 \cdots x^K] \quad .$$

Let

$$\mathcal{A} = \left\{ \mathbf{x} : \lambda^k \geq \epsilon^1, \ F^j \leq \mu^j - \epsilon^2, \sum_{i \in \mathcal{P}_k} y^{ki} = \lambda^k \right\}$$

for every k, $1 \leq k \leq K$, and every j, $1 \leq j \leq I$.

$\mathcal{A}$ is a convex, closed, and compact space with respect to $\mathbf{x}$.

H^k is a continuous function in $\mathbf{x}$ and convex in x^k. Furthermore $\sum_{l=1}^{K} H^l$ is diagonally strictly convex function in $\mathcal{A}$.

Proposition 5.1. : *A BCMP queueing system which is shared by a number of noncooperative users each maximizing its own power, has a unique Nash equilibrium point.*

Notice that users that operate under a power criterion *always* send packets inside a network.

6. A Packet Switched Network Operating under Preemptive Priorities

The objective function of the k user is

$$\max_{EQ^k - T^k E\gamma^k \leq 0 \ and \ E\gamma^k \geq \Gamma^k} (E\gamma^k)^\alpha \quad , \tag{6.1}$$

for some α, $0 < \alpha < 1$.

We assume that the traffic of class k has lower service priority than the traffic of class $1, \cdots, k-1$, and higher priority than the traffic of class $k+1, \cdots$.

The total load at node j amounts to

$$F^j = \sum_{k=1}^{K} \sum_{\substack{all \ paths \ p \\ belonging \ to \ class \ k \\ and \ utilizing \ server \ j}} y^{kp} \quad . \tag{6.2}$$

Let

$$F^{l,j} = \sum_{k=1}^{l} \sum_{\substack{all \ paths \ p \\ belonging \ to \ class \ k \\ and \ utilizing \ server \ j}} y^{kp} \quad . \tag{6.3}$$

Let EQ^l be the expected number of packets that belong to class l, inside the network, and $EQ^{1,k}$ be the expected number of packets that belong to classes $1, \cdots, k$, inside the network. The expected number of packets that belong to class 1, inside the network is given by

$$EQ^1 = \sum_{j=1}^{I} \frac{F^{1,j}}{\mu^j - F^{1,j}} \quad . \tag{6.4}$$

$$EQ^{1,2} = EQ^1 + EQ^2 \quad .$$

But

$$EQ^{1,2} = \sum_{j=1}^{I} \frac{F^{2,j}}{\mu^j - F^{2,j}} \quad . \tag{6.5}$$

Therefore

$$EQ^2 = \sum_{j=1}^{I} \frac{F^{2,j}}{\mu^j - F^{2,j}} - \sum_{j=1}^{I} \frac{F^{1,j}}{\mu^j - F^{1,j}} \quad . \tag{6.6}$$

Similarly we can prove that

$$EQ^l = \sum_{j=1}^{I} \frac{F^{l,j}}{\mu^j - F^{l,j}} - \sum_{j=1}^{I} \frac{F^{l-1,j}}{\mu^j - F^{l-1,j}} \tag{6.7}$$

for all l, $l = 2, \cdots, K$.

The state space of the network is a closed, compact, and convex set. It is easy to prove that EQ^k for all k, $k = 1, \cdots, K$ is a continuous function in $\mathbf{y}$ and convex in y^k. Furthermore $\sum_{l=1}^{K} EQ^l$ is diagonally strictly convex function in the state space. Therefore reasoning as before, we can prove that

Proposition 6.1. : *A packet switched network that is shared by a number of noncooperative users, each of which operates under a constraint optimization criterion and shares the network resources based on preemptive priorities, has always a unique Nash equilibrium point.*

7. Conclusions

- A number of resource allocation problems have been investigated as Nash games.
- Such a formulation is advantageous in a decentralized environment.
- Efficient algorithms are currently under development for the solution of this class of problems. Current experience shows that Gauss-Seidel type algorithms are appropriate for this class of problems.

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