Report Number: WUCS-89-10

1989-02-24

Asynchronous Algorithms for Optimal Flow Control of BCMP Networks

Authors: Andreas D. Bovopoulos and Aurel A. Lazar

The decentralized flow control problem for an open multiclass BCMP network is studied. The power based optimization criterion is employed for the derivation of the optimal flow control for each of the network's users. It is shown that that optimal arrival rates correspond to the unique Nash equilibrium point of a noncooperative game problem. Asynchronous algorithms are presented for the computation of the Nash equilibrium point of the network. Among them, the nonlinear Gauss-Seidel algorithms is distinguished for its robustness and speed of convergence.

ASYNCHRONOUS ALGORITHMS FOR OPTIMAL FLOW CONTROL OF BCMP NETWORKS

Andreas D. Bovopoulos
Aurel A. Lazar

WUCS-89-10

February 1989

Department of Computer Science
Washington University
Campus Box 1045
One Brookings Drive
Saint Louis, MO 63130-4899

Asynchronous Algorithms for Optimal Flow Control of BCMP Networks

Andreas D. Bovopoulos
Department of Computer Science
Washington University, St. Louis, MO 63130-4899
e-mail: andreas@patti.wustl.edu

and

Aurel A. Lazar
Department of Electrical Engineering
and
Center for Telecommunications Research
Columbia University, New York, NY 10027-6699
e-mail: aurel@ctr.columbia.edu

ABSTRACT

The decentralized flow control problem for an open multiclass BCMP network is studied. The power based optimization criterion is employed for the derivation of the optimal flow control for each of the network's users. It is shown that the optimal arrival rates correspond to the unique Nash equilibrium point of a noncooperative game problem. Asynchronous algorithms are presented for the computation of the Nash equilibriun point of the network. Among them, the nonlinear Gauss-Seidel algorithm is distinguished for its robustness and speed of convergence.

Index Terms: Asynchronous computation, computer networks, distributed algorithms, flow control, game theory, optimization.

February 24, 1989

The research reported here was supported in part by the National Science Foundation under Grant # CDR-84-21402.

Asynchronous Algorithms for Optimal Flow Control of BCMP Networks

1. Introduction

For computer communication networks supporting different classes of packets with (often) conflicting objectives, a game theoretical approach has been investigated for achieving efficient resource allocation. Nash, Pareto and Stackelberg criteria [2] have been studied [7], [9], [11], [12]. In this paper the decentralized flow control problem of a multiclass model of a computer communications network is analyzed. Each network user operates with an arrival rate which maximizes its own objective function, the power criterion.

In the remainder of this section, previous work related to the class of problems above is reviewed, and the content of the paper is outlined.

Barath-Kummar and Jaffe in [1], [14] first introduced a heuristic algorithm which was termed "greedy" and was found to possess good convergence properties in the computation of an efficient flow control of a multiclass queueing system. Cansever [9] and Douligeris and Mazumdar [11] showed the convergence of the "greedy" algorithm for specific networks. Specifically, in [9] the convergence of the "greedy" algorithm to the Nash equilibrium was proven for a simple network shared by two users. In [11] the existence of a Nash equilibrium point was proven for the case in which a number of users shared the resources of an exponential processor. In addition, the "greedy" algorithm was proven to converge to the Nash equilibrium when *the exponential processor was shared by two classes of packets.* The methodology used for the proof was direct algebraic manipulation.

Note that the methodology used in [9] *cannot be extended* to an arbitrary network. The proof corresponding to the particular example analyzed in [9] is rather tedious. Furthermore, the methodology used in the derivation cannot be used for the proof of the existence and uniqueness of the Nash equilibrium point of an arbitrary BCMP network [4].

The technical disadvantage of the methodology used in [11] is that even though the network analyzed is the simplest possible, the proof of the convergence of the "greedy" algorithm to a Nash equilibrium point cannot be extended to cover the case in which the exponential processor is shared by more than two users. Furthermore, in [11] the possibility of the existence of multiple Nash equilibrium points is not considered, and consequently no explicit study of the uniqueness of the Nash equilibrium point was attempted.

The apparent limitations of the approach followed in [9] and [11] were addressed both technically and conceptually in [7] using a more systematic formulation of the problem. The existence and uniqueness of the Nash equilibrium point was proven. Furthermore, it was pointed out that *the "greedy" algorithm was simply the Gauss-Seidel iterative procedure* for the solution of a system of equations.

The set of equations that a Nash equilibrium point satisfies is a set of linear equations. In [7] a class of asynchronous algorithms was introduced for the computation of the Nash equilibrium point. In that framework each user updated its flow control policy asynchronously, that is, a particular user could change its flow control policy even if it had not obtained an updated version of the policies of the other users. This paradigm was along the lines of previous work on decentralized computation [10], [2], [5], [18].

In [8] the existence and uniqueness of the Nash equilibrium point was proven in the context of decentralized flow control of an arbitrary BCMP network with K

users.

This paper is organized as follows. In Section 2, the statement of the problem is presented. In Section 3, monotonicity properties that hold in a BCMP network are derived. The existence and uniqueness of the Nash equilibrium point for a BCMP network is proven in Section 4. To make the analysis more transparent, we first present asynchronous algorithms for the computation of the Nash equilibrium point for a network consisting of an exponential server with rate μ. We then apply the analysis to a general BCMP network. Asynchronous algorithms that converge to the Nash equilibrium point corresponding to the solution of the decentralized flow control problem of a BCMP network are given in Sections 5. Finally, in Section 6, the results corresponding to the derivation of the Nash equilibrium point are applied to specific examples.

2. The Statement of the Problem

A BCMP network with I exponential servers is shared by K different classes of packets that are processed according to the FIFO policy. Let μ^i be the service rate of the i^{th} server, and M be the $1 \times I$ matrix $[\mu^1 \ \cdots \ \mu^I]$. Let $\mathbf{R}^k = [r^{kij}]$ be the $I \times I$ routing matrix of the k^{th} class of packets ($1 \leq k \leq K,\ 1 \leq i \leq I,\ 1 \leq j \leq I$). In this notation, class k packets are routed from node i to node j with probability r^{kij}, and enter into the network at node j with probability $r^{k \cdot j}$. $\mathbf{\Lambda} = [\lambda^1 \ \cdots \ \lambda^K]$ represents the set of arrival rates of the K packet classes. Thus, user k operates with a state independent Poisson arrival rate λ^k. With respect to the power criterion, users operate at a Nash equilibrium point; that is, given the strategies of the other users, none of them has an incentive to unilaterally change its operating policy [2].

The power-based user optimization criterion requires that each user k maximize the ratio of the weighted average throughput over the expected time delay:

$$P^k \stackrel{def}{=} \frac{(E\gamma^k)^{\beta_k}}{E\tau^k} \quad ,$$

for all k, $k = 1, 2, \cdots, K$. The parameter β_k (which is assumed to be greater than or equal to zero) may be adjusted to achieve different trade offs between average throughput and expected time delay.

In order to achieve the Nash equilibrium, two classes of asynchronous distributed flow control algorithms are investigated. In the first class, users attempt to reach the Nash operating point using a Gauss-Seidel type iterative procedure. The algorithm requires only one user to change its flow control policy at a time. After adjusting its policy to maximize its power, the user broadcasts the value of its new policy to all other users. Using this step-by-step iterative procedure, the optimal operating point for the network is reached.

The second class of decentralized algorithms investigated is based on a paradigm borrowed from the field of distributed computation [10]. In this case, each user recomputes its flow control policy asynchronously. The computation of a new flow control policy takes place even if the latest changes in the control strategies of the other users are not available. However, the policies of other users are guaranteed to be available within a bounded time.

3. Monotonicity Properties of BCMP Networks

Let the $1 \times I$ matrix $\Theta^k = [\theta^{k1} \ \cdots \ \theta^{kI}]$ be the solution of the traffic flow equations for the k^{th} class of packets of the BCMP network defined in the previous section, *i.e.*,

$$\Theta^k = \Lambda^k + (\Theta^k \wedge \mathbf{M})\mathbf{R}^k \quad . \tag{3.1}$$

Here Λ^k denotes the load vector of the input traffic flows of the k^{th} traffic class

$$\Lambda^k = \lambda^k [r^{k\cdot 1} \ \cdots \ r^{k\cdot I}] \quad ,$$

for all k, $k = 1, 2, \cdots, K$. If the BCMP network is stable, then

$$\Theta^k = \Lambda^k + \Theta^k \mathbf{R}^k \quad ,$$

or

$$\Theta^k = \Lambda^k (\mathbf{I} - \mathbf{R}^k)^{-1} \quad .$$

With

$$\alpha^k = [\alpha^{k1} \ \cdots \ \alpha^{kI}] \stackrel{\text{def}}{=} [r^{k\cdot 1} \ \cdots \ r^{k\cdot I}](\mathbf{I} - \mathbf{R}^k)^{-1} \quad ,$$

we obtain

$$\theta^{kj} = \alpha^{kj}\lambda^k \quad ,$$

for all j, $j = 1, 2, \cdots, I$.

Proposition 3.1. *The expected time delay $E\tau^k$ of the k^{th} class packets of a BCMP network is a nondecreasing function of λ^l for all l, $l = 1, 2, \cdots, K$, and a convex increasing function of λ^l for all l, $l = 1, 2, \cdots, K$, if and only if $\alpha^{kj}\alpha^{lj} > 0$ for some j, $j = 1, 2, \cdots, I$.*

Proof : The throughput of a particular class of packets in a stable multiclass BCMP network equals its arrival rate. Therefore,

$$E\gamma^k = \lambda^k \quad . \tag{3.2}$$

The time delay amounts to

$$E\tau^k = \frac{EQ^k}{E\gamma^k} = \frac{EQ^{k1} + \cdots + EQ^{kI}}{\lambda^k} \quad ,$$

and

$$EQ^{kj} = \frac{\theta^{kj}}{\mu^j - \sum_{l=1}^{K} \theta^{lj}} = \lambda^k \frac{\alpha^{kj}}{\mu^j - \sum_{l=1}^{K} \alpha^{lj}\lambda^l} \quad .$$

Thus,

$$EQ^k = \lambda^k \sum_{j=1}^{I} \frac{\alpha^{kj}}{\mu^j - \sum_{l=1}^{K} \alpha^{lj}\lambda^l} \quad , \tag{3.3}$$

and

$$E\tau^k = \sum_{j=1}^{I} \frac{\alpha^{kj}}{\mu^j - \sum_{l=1}^{K} \alpha^{lj}\lambda^l} \quad . \tag{3.4}$$

Using equation (3.4), one can easily prove Proposition 3.1.

4. Decentralized Flow Control: Existence and Uniqueness of the Nash Equilibrium Point

In this section, we prove the existence and uniqueness of a Nash equilibrium point for a BCMP network which consists of one queue with an exponential server of rate μ shared by K users. The existence and uniqueness of a Nash equilibrium point for arbitrary BCMP networks is derived based upon these results.

The power of user k is given by

$$P^k \stackrel{def}{=} (\lambda^k)^{\beta_k}(\mu - \sum_{l=1}^{K} \lambda^l) \quad . \tag{4.1}$$

In this section two approaches are used in order to prove the existence and uniqueness of the Nash equilibrium point. One of these approaches will be further used for the proof that a general BCMP network has a unique Nash equilibrium point as well.

If z_1^k is the value of the arrival rate λ^k for which the first derivative of the k^{th} user's power is zero, then

$$z_1^k = \frac{\beta_k}{1+\beta_k}(\mu - \sum_{l \neq k} \lambda^l) \quad . \tag{4.2}$$

Let

$$z_2^k \stackrel{\text{def}}{=} max\left\{\frac{\beta_k - 1}{\beta_k + 1}(\mu - \sum_{l \neq k} \lambda^l) \ , \ 0\right\} \quad , \tag{4.3}$$

for all k, $k = 1, 2, \cdots, K$.

Observe that $z_1^k \geq z_2^k$ and that the power of the k^{th} user is a convex increasing function in the interval $[0,\ z_2^k]$ and a concave function of λ^k in the interval $[z_2^k,\ \mu - \sum_{l \neq k} \lambda^l]$. Furthermore, $\mu - (\sum_{l \neq k} \lambda^l + z_1^k) = \frac{1}{1+\beta_k}(\mu - \sum_{l \neq k} \lambda^l) \neq 0$. Observe also that if $\beta_k > 0$, then $z_1^k \neq 0$.

Let

$$\mathcal{A} \stackrel{\text{def}}{=} \left\{(\lambda^1, \cdots, \lambda^K) : \lambda^k \geq z_1^k \textit{for every } k,\ 1 \leq k \leq K, \textit{and } \sum_{l=1}^{K} \lambda^l \leq \mu \right\} \quad ,$$

which is closed, compact, and convex. Furthermore the qualifications constraints for the Kuhn-Tucker conditions hold because the space $\mathcal{A}$ is specified by a set of linear equations. Note that the users of the network *never operate on the border of the space* $\mathcal{A}$.

From [ROS65] we know that *an equilibrium point exists for every concave K-person game with concave rewards and a bounded, closed, and convex space* $\mathcal{A}$. Furthermore, $\sum_{k=1}^{K} P^k$ is a diagonally strictly concave function [ROS65] in $\mathcal{A}$. Thus,

Proposition 4.1: *An M/M/1 queueing system shared by a number of users, each operating under a power criterion, has a unique Nash equilibrium point.*

Since the optimal arrival rate of the k^{th} user is zero if $\beta_k = 0$, we can assume for the rest of the section that $\beta_k \neq 0$ for every k. Furthermore the fact that $\beta_k \neq 0$ implies that $z_1^k > 0$. A different proof of Proposition 4.1 is given in the sequel [BOV88a].

Let H^k be the function

$$H^k \stackrel{\text{def}}{=} \frac{1}{P^k} \quad , \tag{4.4}$$

for all k, $k = 1, 2, \cdots, K$. Then,

$$H^k = \frac{1}{(\lambda^k)^{\beta_k}(\mu \ - \ \sum_{l=1}^{K} \lambda^l)} \quad , \tag{4.5}$$

for all k, $k = 1, 2, \cdots, K$. The first derivative of H^k with respect to λ^k is negative for $\lambda^k < z_1^k \neq 0$ and positive for $\lambda^k > z_1^k$. Furthermore, the second derivative of H^k is always positive. Therefore, *H^k is a convex function for every value of β_k, and $-H^k$ is a concave function with respect to λ^k, for all k, $k = 1, 2, \cdots, K$.* Observe that if $\lambda^k = 0$, then $H^k \ = \ \infty$ and if $\mu \ - \ \sum_{l=1}^{K} \lambda^l = 0$, then $H^k \ = \ \infty$. Therefore, there always exists a small $\epsilon > 0$ such that the optimal arrival rate of each user λ^k, for all k, $k = 1, 2, \cdots, K$, belongs to the set $\mathcal{A}$, defined by the set

$$\mathcal{A} \stackrel{\text{def}}{=} \left\{(\lambda^1, \cdots, \lambda^K) : \lambda^k \geq \frac{z_1^k}{2}, \textit{for every } k,\ 1 \leq k \leq K, \textit{and } \sum_{l=1}^{K} \lambda^l \leq \mu - \epsilon \right\} \quad ,$$

for an appropriate small positive value of ϵ. $\mathcal{A}$ is a closed, compact, and convex set. The inequality constraints validate the qualification conditions of the Kuhn-Tucker necessary conditions for optimality of the nonlinear optimization problem. Furthermore, $\sum_{k=1}^{K} H^k$ is a diagonally strictly convex function in $\mathcal{A}$. Thus, there exists a unique Nash equilibrium point for every value of the parameters β_k, $1 \le k \le K$ [ROS65].

We would like to extend the previous analysis to the case of a BCMP network. In Section 3, we found that the expected time delay of the k^{th} class of packets is given by the equation

$$E\tau^k \; = \; \sum_{j=1}^{M} \frac{\alpha^{kj}}{\mu^j \; - \; \sum_{l=1}^{K} \alpha^{lj} \lambda^l} \quad . \tag{3.4}$$

Thus, for a BCMP network

$$H^k \; = \; (\lambda^k)^{-\beta_k} \sum_{j=1}^{M} \frac{\alpha^{kj}}{\mu^j \; - \; \sum_{l=1}^{K} \alpha^{lj} \lambda^l} \quad , \tag{4.6}$$

is a convex function with respect to λ^k.

Let z_1^k be the value of the arrival rate λ^k which minimizes the function H^k (or, equivalently, which maximizes the power function P^k). It can be easily proven that if $\beta_k \neq 0$, then $z_1^k > 0$. In the same way as before, we define the space $\mathcal{A}$ as the following:

$$\mathcal{A} = \left\{ (\lambda^1, \cdots, \lambda^K) : \lambda^k \ge \epsilon^1, 1 \le k \le K; \; \sum_{l=1}^{K} \alpha^{lj} \lambda^l \le \mu^j - \epsilon^2, \; 1 \le j \le I, \right\}$$

for appropriate small positive ϵ^1 and ϵ^2. $\mathcal{A}$ is a closed, compact, and convex set. The constraints fulfill the Kuhn-Tucker qualification constraints, and $\sum_{k=1}^{K} H^k$ *is a diagonally strictly convex function in* $\mathcal{A}$. Therefore there exists a *unique* Nash equilibrium point [ROS65]. This implies that

Theorem 4.2: *A BCMP queueing system shared by K users, each operating under a power criterion, has a unique Nash equilibrium point.*

5. Asynchronous Algorithms for Decentralized Flow Control

5.1 A Simple Network with an Exponential Server

In the sequel, the K user BCMP network is assumed to consist of one queue with an exponential server of rate μ. The power of user k is given by

$$P^k = (\lambda^k)^{\beta_k}(\mu - \sum_{l=1}^{K} \lambda^l) \quad , \tag{5.1}$$

for all k, $k = 1, 2, \cdots, K$. The Nash operating point is the solution of the set of equations [7]

$$(1 + \beta_k)\lambda^k + \beta_k \sum_{l \neq k} \lambda^l = \beta_k \mu \quad , \tag{5.2}$$

for all k, $k = 1, 2, \cdots, K$. Note that the above system of K linear equations has K unknowns.

Let

$$\delta_k \stackrel{\text{def}}{=} \frac{\beta_k}{1 + \beta_k} \quad , \tag{5.3}$$

for all k, $k = 1, 2, \cdots, K$.
Also, x^T denotes the vector $[\lambda^1 \cdots \lambda^K]$ and A the $K \times K$ matrix $[a_{kl}]$, with $a_{kk} = 1$ and $a_{kl} = \delta_k$ for all l, $l \neq k$. b^T denotes the vector $[b_1 \cdots b_K]$, where $b_k = \delta_k \mu$. Thus, the derivation of the Nash equilibrium of this network requires the solution of the system of linear equations

$$Ax = b \quad . \tag{5.4}$$

Since the diagonal elements of the matrix A are equal to 1, $D = diag(A) = I$, the $K \times K$ matrix $B = [b_{kl}]$ defined by

$$B = I - D^{-1}A = I - A$$

amounts to

$$B = \begin{pmatrix} 0 & -\delta_1 & \dots & -\delta_1 \\ -\delta_2 & 0 & \dots & -\delta_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\delta_K & -\delta_K & \dots & 0 \end{pmatrix} \quad .$$

Therefore Equation (5.4) has been reduced to the fixed point equation

$$x \;=\; Bx \;+\; b \quad . \tag{5.5}$$

In [10], Chazan and Miranker consider chaotic relaxation methods for solving linear systems of equations. In the sequel these results are briefly presented and subsequently applied to the problem above.

The computations are done in steps called iterations. These iterations are indexed by $t = 0, 1, 2, \cdots$ where t can be viewed as a discrete time variable for the system. At time $t+1$, a set $U(t+1)$ of coordinates of the vector $x(t+1)$ is updated as follows:

$$\begin{aligned} x_k(t+1) \;&=\; \sum_{l=1}^{K} b_{kl} x_l(t - \tau(t,k,l)) \;+\; b_k \;, \quad if\ k \in U(t) \quad , \\ x_k(t+1) \;&=\; x_k(t) \;, \quad if\ k \;\notin\; U(t) \quad , \end{aligned} \tag{5.6}$$

where $\{\tau(\cdot,\cdot,\cdot)\}$ are delay terms. The initial vector is $x(0)$ with $x_k(t) \;=\; x_k(0)$ for all $t < 0$. Let Γ denote a choice for $\{U(t), \tau(t,k,l),\ t \geq 0,\ 1 \leq k, l \leq K\}$. Then $x(0)$ and Γ completely define the evolution of $x(t),\ t > 0$ (the sample path of $x(t),\ t > 0$) given by Equation (5.6).

Let $\Gamma_{d,s}$ and Γ_d correspond to the case in which the (random) delays are bounded by d (*i.e.*, $\tau(t,k,l) \leq d$) and each component $x_k,\ k = 1, \cdots, K$, of the vector x is updated during the iterative procedure either at least every s steps or infinitely often respectively. Then the following holds:

Theorem 5.1: *The sequence of iterates $x(t)$, $t = 0, 1, 2, \cdots$ converges for all choices of $\Gamma_{d,s}$ or Γ_d iff $\rho(|B|) < 1$.*

Proof : For a proof and extensions, see [10].

∎

Lemma 5.2: *If $\delta_l > 0$ for all l, $1 \leq l \leq K$, the dominant nonnegative eigenvalue of the associated nonnegative matrix $|B|$*

$$|B| \;=\; \begin{pmatrix} 0 & \delta_1 & \dots & \delta_1 \\ \delta_2 & 0 & \dots & \delta_2 \\ \vdots & \vdots & \ddots & \vdots \\ \delta_K & \delta_K & \dots & 0 \end{pmatrix}$$

is bounded by

$$\sum_{k=1}^{K} \delta_k \; - \; \max_{l} \delta_l \leq \rho(|B|) \leq \sum_{k=1}^{K} \delta_k \; - \; \min_{l} \delta_l$$

and

$$\sum_{i=1}^{K} \delta_i - \frac{\sum_{i=1}^{K} \delta_i^2 \; - \; \min_l \delta_l^2}{\sum_{i=1}^{K} \delta_i \; - \; \min_l \delta_l} \leq \rho(|B|) \; \leq \; \sum_{i=1}^{K} \delta_i - \frac{\sum_{i=1}^{K} \delta_i^2 \; - \; \max_l \delta_l^2}{\sum_{i=1}^{K} \delta_i \; - \; \max_l \delta_l} \quad , \; (5.7)$$

with equality on either side of the previous two inequalities implying equality throughout.

Proof: Let A be the $K \times K$ nonnegative matrix $[a_{ij}]$, with row sums, $r_1, r_2, \cdots, r_K$. If $\rho(A)$ were the maximum eiguenvalue of matrix A, then ([15])

$$\min_{i} \{ \frac{1}{r_i} \sum_{l=1}^{K} a_{il} r_l \} \; \leq \; \rho(A) \; \leq \; \max_{i} \{ \frac{1}{r_i} \sum_{l=1}^{K} a_{il} r_l \} \quad .$$

Recall that $\rho(A) = \rho(A^T)$. The inequalities (5.7) follow by applying the above inequality to the matrices $|B|$ and $|B|^T$. The bounds given by the second inequality are better than the bounds given by the first one.

∎

Proposition 5.3 : *For an $M/M/1$ queueing system with a power-based user criterion, the chaotic relaxation algorithm converges for all choices of $\Gamma_{d,s}$ and Γ_d if the upper bound of $\rho(|B|)$ (given in Equation (5.7)) is smaller than one, and does not converge if the lower bound of $\rho(|B|)$ (given in Equation (5.7)) is greater or equal to one.*

Proof: Proposition 5.3, is a direct conclusion of Theorem 5.1 and Lemma 5.2.

∎

5.2 The General BCMP Network

As proven in Proposition 4.2, every BCMP network has a unique Nash equilibrium point. This particular point corresponds to arrival rates which are the unique solution of the following set of nonlinear equations involving the first derivatives of the power expression:

$$\frac{\partial P^k}{\partial \lambda^k} = 0 \quad , \tag{5.8}$$

for all k, $k = 1, \cdots, K$.

The nonlinear Gauss-Seidel method is known to have very good convergence properties [16], [5] and can be used for the computation of the Nash equilibrium point. With this algorithm, the users optimize their arrival rates one user at a time. Furthermore, user's k optimal arrival rate is the solution of the $\frac{\partial P^k}{\partial \lambda^k} = 0$, with respect to λ^k. The nonlinear Gauss-Seidel algorithm is appealing because of the simplicity of its implementation, its robustness and its excellent convergence rate, properties we observed in numerous examples. A comparison fo the nonlinear Gauss-Seidel method with a number of different methods for the computation of the Nash equilibrium point is presented in Section 6.

6. Applications

In Section 5 we described the Gauss-Seidel iterative procedure for the computation of the Nash equilibrium point of a BCMP network. Let

$$\mathbf{P}(\mathbf{\Lambda}) \stackrel{\text{def}}{=} [\frac{\partial P^1}{\partial \lambda^1} \cdots \frac{\partial P^K}{\partial \lambda^K}]^T$$

and

$$\mathbf{J}(\mathbf{\Lambda}) \stackrel{\text{def}}{=} [\frac{\partial^2 P^k}{\partial \lambda^k \partial \lambda^l}] \quad ,$$

for all k, l, $1 \leq k \leq K$, $1 \leq l \leq K$.

The Nash equilibrium point might be computed by deriving the least-square solution of the equation $\mathbf{P}(\mathbf{\Lambda}) = \mathbf{0}$, or equivalently, by deriving the unique global minimizer $\mathbf{\Lambda}^*$ of the function $\frac{1}{2}(\mathbf{P}(\mathbf{\Lambda})^T)(\mathbf{P}(\mathbf{\Lambda}))$. It is known [16] that $\mathbf{\Lambda}^*$ is the unique global minimizer of $\frac{1}{2}(\mathbf{P}(\mathbf{\Lambda})^T)(\mathbf{P}(\mathbf{\Lambda}))$ if and only if $\mathbf{P}(\mathbf{\Lambda}^*) = \mathbf{0}$.

A good optimization technique for the derivation of the least-square solution of a set of nonlinear equations is *the modified Gauss-Newton iteration procedure* which calls for the following iteration:

$$\mathbf{\Lambda}_{n+1} = \mathbf{\Lambda}_n - \omega_n[(\mathbf{J}(\mathbf{\Lambda}_n))^{\mathrm{T}}(\mathbf{J}(\mathbf{\Lambda}_n)) + \epsilon I]^{-1}(\mathbf{J}(\mathbf{\Lambda}_n))^{\mathrm{T}}\mathbf{P}(\mathbf{\Lambda}_n) \tag{6.1}$$

In the previous expression the inverse matrix always exists provided only that $\epsilon > 0$. The parameter ω_n may be chosen [16] to ensure that

$$\frac{1}{2}(\mathbf{P}(\mathbf{\Lambda}_{n+1})^T)(\mathbf{P}(\mathbf{\Lambda}_{n+1})) \leq \frac{1}{2}(\mathbf{P}(\mathbf{\Lambda}_n)^T)(\mathbf{P}(\mathbf{\Lambda}_n)) \quad .$$

In [16] this method is further explained. Unfortunately the modified Gauss-Newton iterative procedure is inefficient for large networks. Furthermore the algorithm could converge to a local minimum. We applied the modified Gauss-Newton iterative procedure to a number of examples. We noticed that it converges to global minimum only in the case that the power coefficients $\beta_1 = \beta_2 = \cdots = \beta_I$.

In this section the previous results are applied to specific examples. In Fig. 6.1 a two stages switch is modeled as a multiclass BCMP network. μ^1, μ^2, μ^3 are the processors of the first stage and μ^4, μ^5, μ^6 are the processors of the second stage. $\mu^1 = \cdots = \mu^6 = 2$ *packets/sec* and, $\beta^1 = \beta^2 = \beta^3 = 1.0$. The switch is shared by three classes of packets, each operating with a state independent arrival rate. Each packet, upon completion of its service in a processor of the first stage is routed to a processor of the second stage with probabilistic routing which is given in the Fig. 6.1 .

The optimal decentralized flow control of the switch, *i.e.*, the Nash equilibrium point is obtained below using different algorithms.

In 6.2 the Nash equilibrium point is computed using the Gauss-Seidel procedure.

In 6.3 the Nash equilibrium point is computed using the Newton procedure [16].

In 6.4 the Nash equilibrium point is computed using the Gauss-Newton iterative procedure described above. Observe that the Gauss-Seidel algorithm converges in the least number of iterations to the unique Nash equilibrium point.

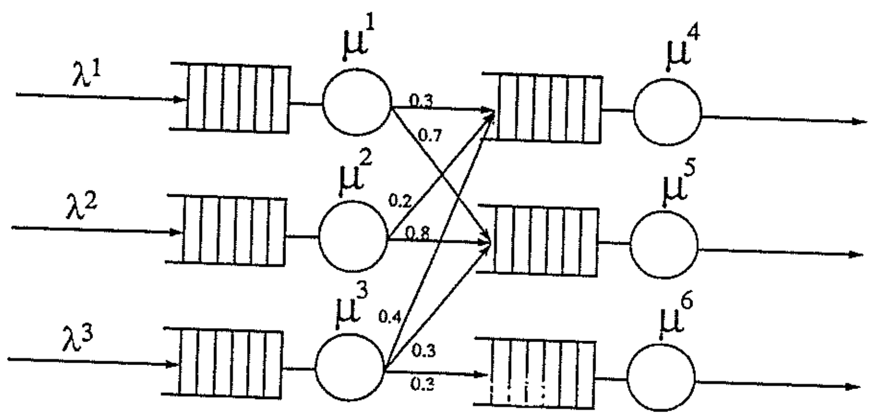

FIGURE 6.1. A two stages switch shared by three classes of customers.

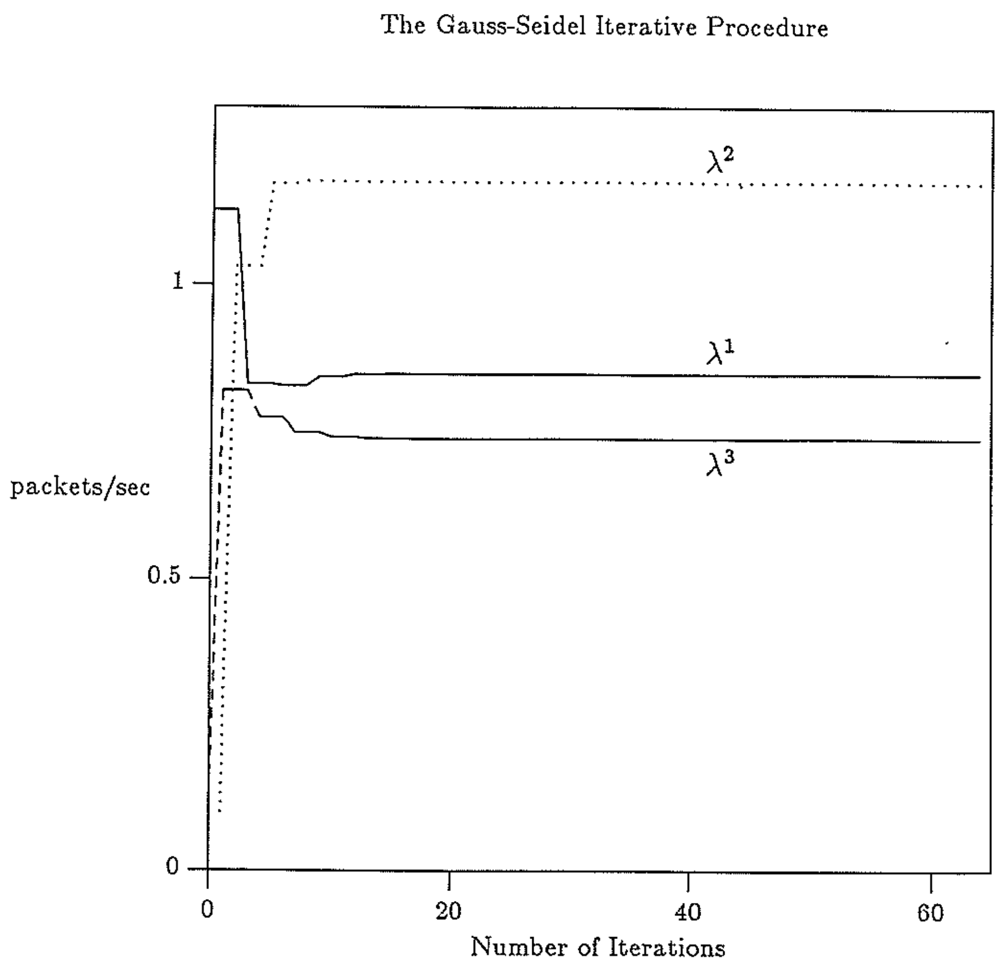

FIGURE 6.2. The computation of the Nash equilibrium point using the Gauss-Seidel procedure for $\beta^1 = \beta^2 = \beta^3 = 1.0$.

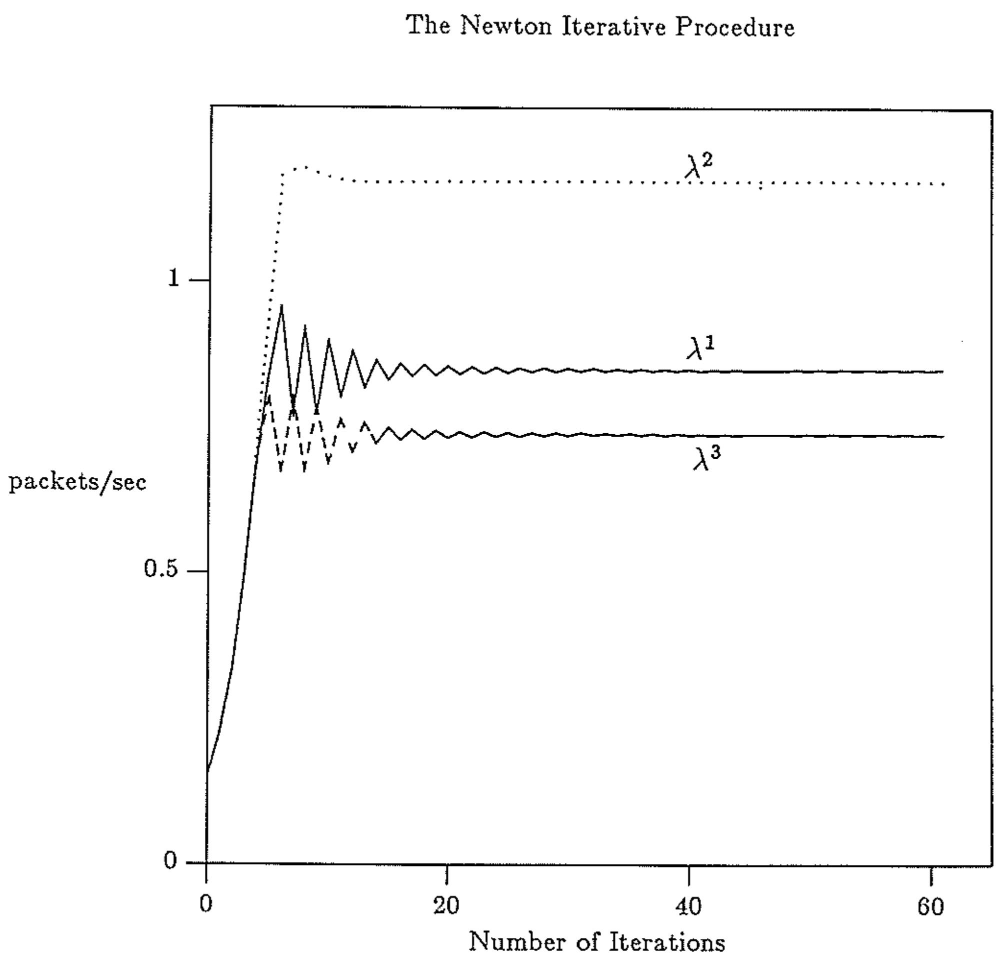

FIGURE 6.3. The computation of the Nash equilibrium point using the Newton procedure for $\beta^1 = \beta^2 = \beta^3 = 1.0$.

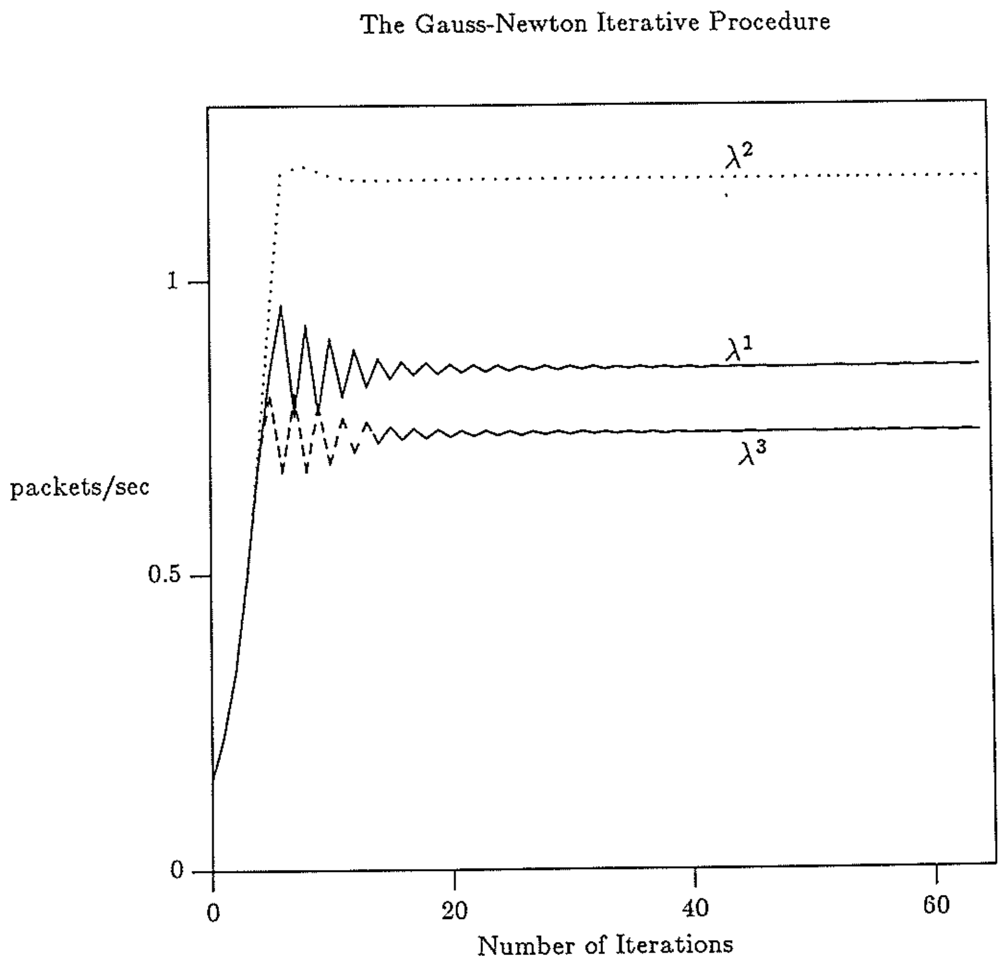

FIGURE 6.4. The computation of the Nash equilibrium point using the Gauss-Newton procedure for $\beta^1 = \beta^2 = \beta^3 = 1.0$.

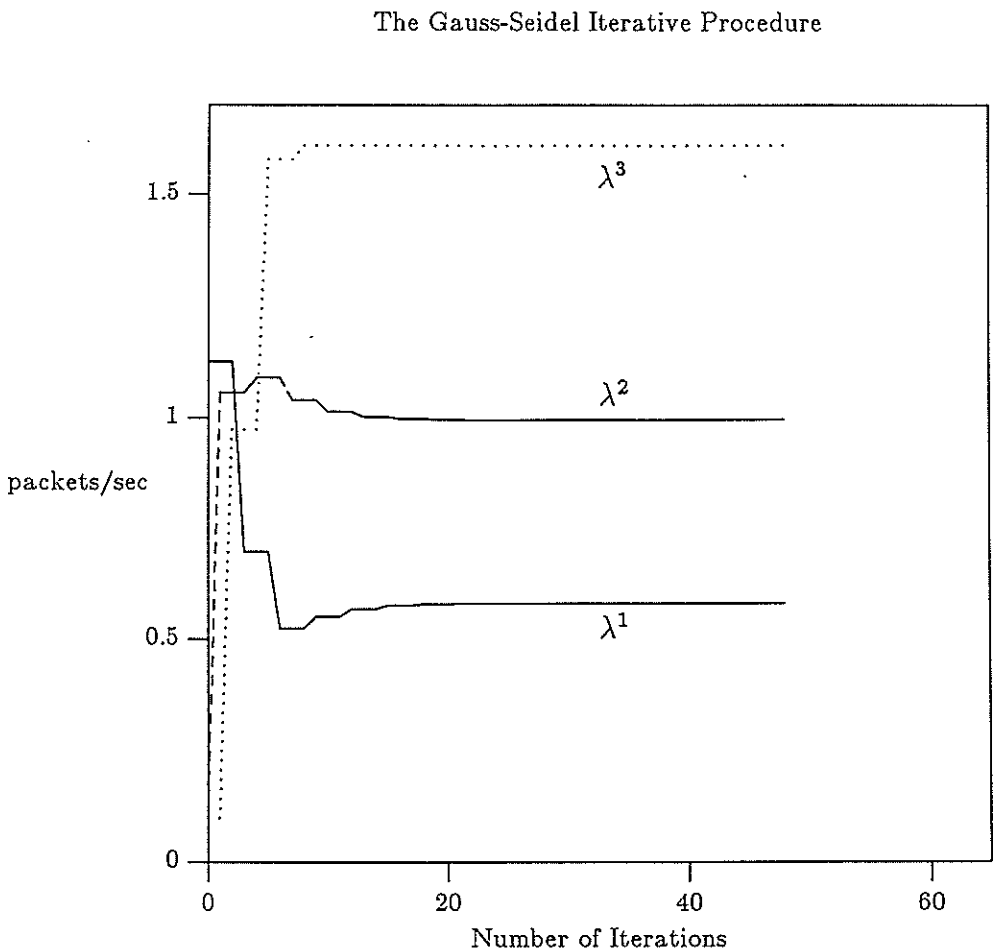

FIGURE 6.5. The computation of the Nash equilibrium point using the Gauss-Seidel procedure, for $\beta^1 = 1.0$, $\beta^2 = 2.0$, and $\beta^3 = 3.0$.

In Fig. 6.5 we compute the Nash equilibrium point of the switch presented in Fig. 6.2 for $\beta^1 = 1.0$, $\beta^2 = 2.0$, and $\beta^3 = 3.0$.

7. Conclusions

The decentralized resource allocation problem was studied from the users point of view, in a multiclass BCMP environment. Such a problem can be seen as a noncooperative game problem. The power-based user optimization criterion was employed in the derivation of the optimal flow control for each network user. The existence of a unique Nash equilibrium point was proven. Synchronous and asynchronous algorithms were presented for the computation of the Nash equilibrium point of the network. Among them, the nonlinear Gauss-Seidel algorithm was distinguished for its robustness and speed of convergence. These results are of prime importance in any resource sharing environment.

References

[1] K. Barath-Kummar and J. Jaffe, "A New Approach to Performance Oriented Flow Control," *IEEE Transactions on Communications,* vol. COM-29, April 1981, pp. 427-435.

[2] T. Basar and G.J. Olsder, *Dynamic Noncooperative Game Theory,* Academic Press, New York, 1982.

[3] G. M. Baudet, "Asynchronous Iterative Methods for Multiprocessors," *Journal of the Association for Computing Machinery,* vol. 25, no. 2, April 1978, pp. 226-244.

[4] F. Basket and M.M. Chandy and R.R. Muntz and F.G. Palacios, "Open, Closed, and Mixed Networks of Queues with Different Classes of Customers," *Journal of the ACM,*vol. 22, no. 2, April 1975, pp.248-260.

[5] D.P. Bertsekas, "Distributed Asynchronous Computation of Fixed Points," *Mathematical Programming,* vol. 27, 1983, pp. 107-120.

[6] D.P. Bertsekas and R. Gallager, *Data Networks,* Prentice-Hall, New Jersey, 1987.

[7] A.D. Bovopoulos and A.A. Lazar, "Decentralized Algorithms for Optimal Flow Control," Proceedings of the Twenty-Fifth Annual Allerton Conference on Communication, Control and Computing, University of Illinois, Urbana-Champaign, Illinois, September 30-October 2, 1987, pp. 979-988.

[8] A.D. Bovopoulos and A.A. Lazar, "Asynchronous Iterative Algorithms for Optimal Load Balancing," Proceedings of the Twenty-Second Annual Conference on Information Sciences and Systems, Princeton University, Princeton, New Jersey, March 16-18, 1988, pp. 1051-1057.

[9] D.H. Cansever, "Decentralized Algorithms for Flow Control in Networks," *Proceedings of the 25th Conference in Decision and Control,* Athens, Greece, December 1986, pp. 2107-2112.

[10] D. Chazan and W. Miranker, "Chaotic Relaxation," *Linear Algebra and its Applications,* vol. 2, 1969, pp. 199-222.

[11] C. Douligeris and R. Mazumdar, "An Approach to Flow Control in an Integrated Environment", *CTR Technical Report, CU-CTR-TR-50,* Columbia University, 1987.

[12] M.T. Hsiao and A.A. Lazar, "A Game Theoretic Approach to Optimal Decentralized Flow Control of Markovian Queueing Networks with Multiple Controllers," *Proceedings of Performance87,* Brussels, Belgium, December 9-11, 1987.

[13] J.R. Jackson, "Networks of Waiting Lines," *Operat. Res.,* vol. 5, 1957, pp. 518-521.

[14] J.M. Jaffe, "Flow Control Power is Nondecentralizable," *IEEE Transactions on Communications,* vol. COM-29, September 1981, pp. 1301-1306.

[15] H. Minc, *Nonnegative Matrices,* J. Wiley & Sons, New York, 1988.

[16] J.M. Ortega and W.C. Rheinboldt, *Iterative Solutions of Nonlinear Equations in Several Variables,* Academic Press, 1970.

[17] J.B. Rosen, " Existence and Uniqueness of Equilibrium Points for Concave N-Person Games," *Econometrica,* vol. 33, no. 3, July 1965, pp. 520-534.

[18] J.N. Tsitsiklis and D.P. Bertsekas, "Distributed Asynchronous Optimal Routing in Data Networks," *IEEE Transactions on Automatic Control,* vol. AC-31, no. 4, July 1986, pp. 325-332.