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Billiard Markov Operators and Second-Order Differential Operators

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Billiard Markov Operators and Second-Order Differential Operators

by

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To my **Los Angeles family** (Wendy, Jack, Scarlet, and Ron)
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Contents

Acknowledgements iii

1 Introduction to Billiards 1

1.1 Random Billiards with Microstructure 1

1.2 Motivation and the Markov Operator P 3

1.3 Main Results 7

1.3.1 Results About Eigenvalues and Eigenfunctions 8

1.3.2 Billiards in Higher Dimensions 9

1.4 A Look Ahead 14

2 A detailed discussion on the billiard map and P 15

2.1 The Billiard Map 15

2.2 The Markov Operator 17

2.3 The Markov chains determined by P 19

2.4 Other Properties of P 19

3 The Billard and Legendre Operators in One Dimension 21

3.1 Approximating $\mathcal{L}_h$ by $\mathcal{L}$ 21

3.2 Billard Operators From Families With Symmetric Cells 24

3.3 Relating the spectra of P_h and $\mathcal{L}$ 32

4 The Billard and Legendre Operators in N Dimensions 38

4.1 Billiard Families in N Dimensions 38

4.2 The Convergence of $\mathcal{L}_h$ to $\mathcal{L}$ 44

4.3 The Eigenfunctions of $\mathcal{L}$ 49

Chapter 1

Introduction to Billiards

This chapter presents a brief introduction to random billiards with microstructure and explains the motivation behind the thesiswork. It also includes an overview of the main results and some examples to illustrate them.

1.1 Random Billiards with Microstructure

Billiards are dynamical systems that have been well-studied by mathematicians during the past half-century [9], [1]. A typical billiard describes the path that a point particle travels inside a domain in $\mathbb{R}^2$ with piecewise smooth boundary, alternating between straight-line motion and specular reflection (i.e. the angle of incidence equals the angle of reflection) from a boundary, which we call *the billiard table*. Therefore, a billiard is usually a deterministic dynamical system.

However, in this thesis, we are concerned with *random billiards*, which are probabilistic dynamical systems akin to ordinary billiards, except the point particle's elastic (i.e., no loss of speed) reflection with the billiard table may not be specular. Instead, the particle's post-collision angle is specified by a stochastic (scattering) operator that depends, in general, on the pre-collision angle.

In particular, my work focuses on *billiards with microstructure*, which are random billiards that are derived in the following way. Consider the collisions of a billiard particle with the macroscopically flat walls of a channel (i.e. billiard table). Suppose that the walls have a *microgeometry*, i.e., a periodic microstructure that is specified by a deterministic billiard system associated with a second billiard table, Q. We call Q the *billiard cell*, and it takes the shape of one interation in the pattern of the microstructure. See Figure 1.1.

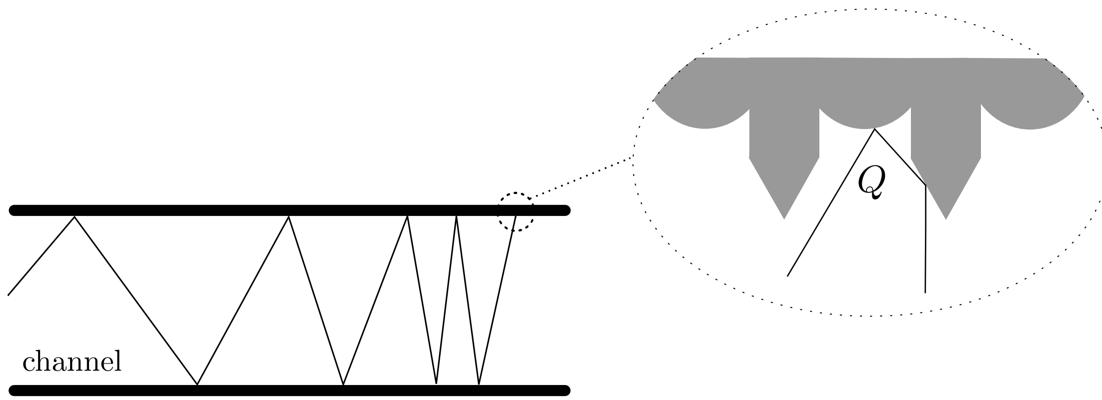

Figure 1.1: A macroscopically flat channel with its periodic microstructure.

The particle's angle of incidence, $\theta \in (0, \pi)$, is fixed, but the precise position, r, that it enters one of the billiard cells, Q, of the microstructure, is random. Once inside Q, the particle undergoes ordinary billiard reflection until it exits the cell at angle $\Theta = \Psi_\theta(r)$. See Figure 1.2.

Since r is generally assumed to be uniformly random on $[0, 1]$, the post-collision angle, Θ, is a random function, and the transition probabilities operator, P, that describes the scattering process depends on the *microgeometry* of the billiard table.

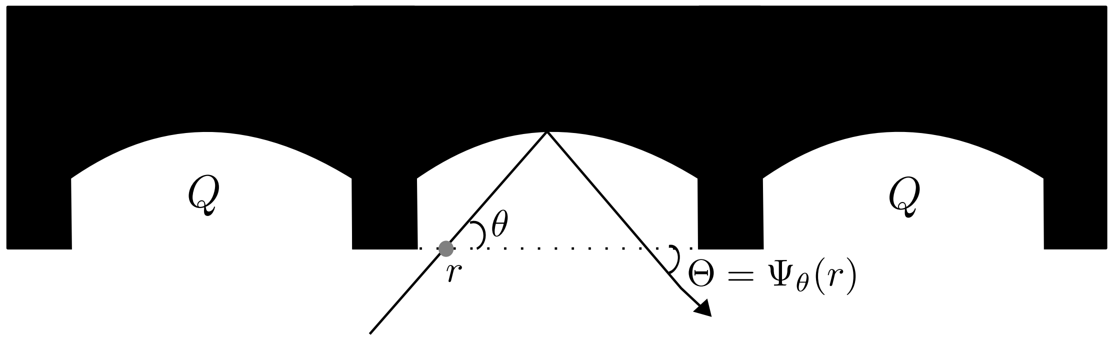

Figure 1.2: A particle enters a billiard cell at position r and angle θ and exits at angle $\Theta = \Psi_\theta(r)$.

1.2 Motivation and the Markov Operator P

The transition probabilities operator, P, gives rise to a class of Markov chains with continous state space. Hence, P is a Markov operator.

Remark. Recall that a Markov chain is a sequence of random variables $X_1, X_2, X_3, ...$ with the Markov property, that is, given the present state, the future and past states are independent.

One of the main questions of Markov chains is to determine the stationary distribution(s) and the rate of convergence to stationarity. It turns out that the measure ν, which is given by $d\nu = \frac{1}{2}\sin\theta d\theta$ and called the (Knudsen) cosine law [8], is the (often unique) stationary measure, ν, for the types of billiards that we will study.

The Markov operator P can be defined on $L^2([0,\pi],\nu)$. Then P takes the form

$$(Pf)(\theta) = \int_0^1 f(\psi_\theta(r))\, dr \tag{1.1}$$

and, under reasonably general conditions, turns out to be a bounded, self-adjoint operator [4].

A natural question for the study of billiards with microstructure is to determine how the spectrum and microgeometry of P are related. It has been shown that the rate of convergence to stationarity is related to the spectral properties of P [2], [6]. Of particular interest in connection with the spectrum is the *spectral gap* of P, i.e., the difference between 1 and the eigenvalue with the next highest modulus. (The operators we are interested in are often compact so it makes sense to talk about their discrete spectrum of eigenvalues.)

Another quantity, which seems at first to be very different from the spectral gap and has not, as far as we know, been considered before in such a setting, has to do with what we have termed the **second moment of scattering**.

Definition 1.1. Define the jth *moment of scattering*, $j = 0, 1, ...,$ by

$$\mathcal{E}_j(\theta) := E[(\Theta - \theta)^j] = \int_0^1 (\Psi_\theta(r) - \theta)^j dr,$$

where E denotes expectation.

It has been observed through my numerical experiments (see Figures 1.3 and 1.4) and by analysis of some very specific examples, e.g., [6], that these two quantities are very closely related to each other in many cases.

The significance of this relationship lies in the observation that the second moment has a more intuitive geometric interpretation than the spectral gap and can, in certain cases, be computed by elementary geometric calculations.

There are a few cases, however, for which $\frac{\mathcal{E}_2^h(\theta)}{\gamma(h)} \xrightarrow{h \to 0} 1$ does not hold true. For example, consider Figure 1.5.

A natural next step is to formulate a question concerning the asymptotic equality of the spectral gap and the second moment for certain parametric families of random billards (e.g., the one described in Figure 1.6). We denote the Markov operators associated with these parametric families by $\{P_h\}$, and require that as the parameter h tends to 0, the second moment $\mathcal{E}_2^h(\theta)$ also tends to 0, i.e., the

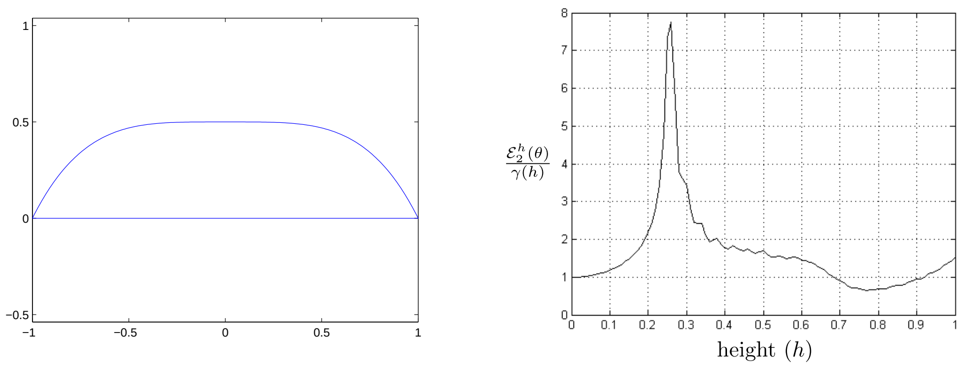

Figure 1.3: A random billiard with microstructure that has its cell shape defined by the graph (on the left) of $h(1-x^4)$ for $x \in [-1, 1]$ is examined. On the right, the ratio (y-axis) of the second moment ($\mathcal{E}_2^h(\theta)$) and the spectral gap ($\gamma(h)$) of the associated Markov operator, P_h, tends to 1 as h tends to 0.

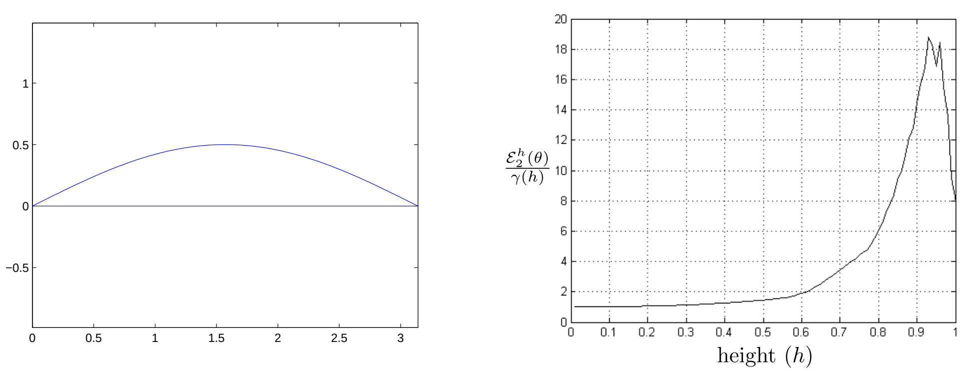

Figure 1.4: A random billiard with microstructure that has its cell shape defined by the graph (on the left) of $h \sin x$ for $x \in [0, \pi]$ is examined. On the right, the ratio (y-axis) of the second moment ($\mathcal{E}_2^h(\theta)$) and the spectral gap ($\gamma(h)$) of the associated Markov operator, P_h, tends to 1 as h tends to 0.

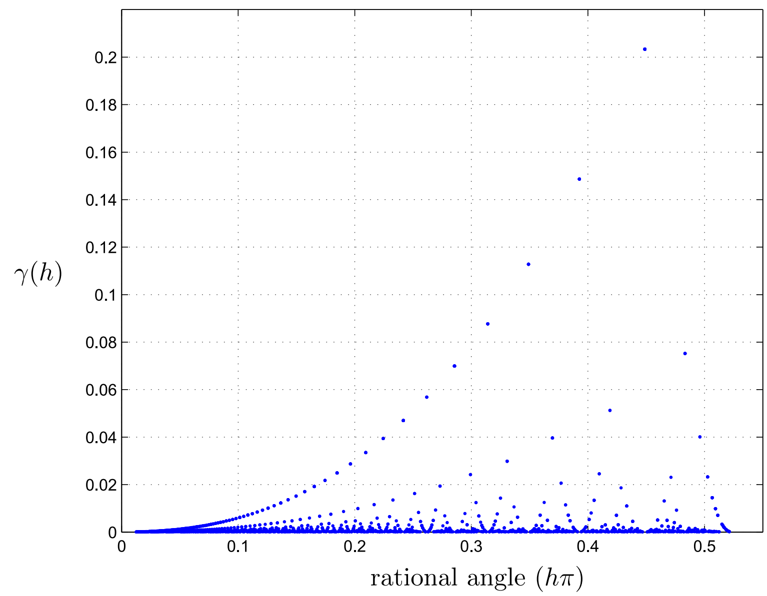

Figure 1.5: A Markov operator approximating the P_h associated with a microstructure cell shape that is defined by the graph of an isosceles triangle with a rational base angle $h\pi$ is examined. The wild fluctuation in the spectral gap $\gamma(h)$ implies that its ratio with the second moment will not tend towards 1 as $h \to 0$.

reflection becomes more specular. In addition, the associated Markov chains are assumed to be geometrically ergodic. In particular, for any delta measure δ_θ (not concentrated on 0 or π), $\delta_\theta P_h^n \xrightarrow{n\to 0} \nu$ in terms of total variation. Then we can ask the following question for parametric families (e.g., the one in Figure 1.6) that possess the aforementioned properties.

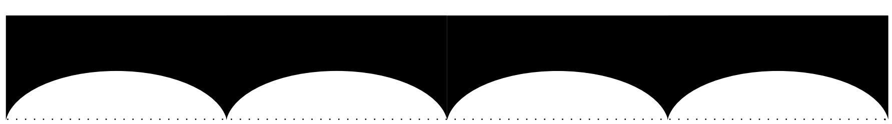

Figure 1.6: A random billiard with microstructure that has its cell shape defined by the graph of $h(1-x^2)$ for $x \in [-1,1]$.

Question 1. Denote the spectral gap and second moment of P_h by $\gamma(h)$ and $\mathcal{E}_2^h(\theta)$, respectively. Let θ be bounded away from 0 and π. Under what conditions on a parametric billiard family will $\dfrac{\mathcal{E}_2^h(\theta)}{\gamma(h)} \longrightarrow 1$ as $h \to 0$?

Most of my current work has consisted of obtaining information, both numerical and analytical, to help answer this question.

1.3 Main Results

The conjecture was originally suggested by a connection between Sturm -Liouville operators and certain random billiards with microstructure.

Let P_h be the Markov operator associated with a parametric family of random billiards with the properties described above, and let θ and Θ denote the incoming and outgoing angles, respectively. We also define the absolute moments by $\overline{\mathcal{E}}_j(\theta) := E[|\Theta - \theta|^j]$. The main results of the thesis are rooted in a comparison between the billiard Laplacian $P_h - I$ and Legendre's equation

$$(\mathcal{L}\Phi)(\theta) = \frac{1}{2\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Phi}{d\theta}\right)$$

on the interval $(0, \pi)$. For $h > 0$, we define a normalized billiard Laplacian, $\mathcal{L}_h = \frac{P_h - I}{l(h)}$, on $L^2([0,\pi],\nu)$, where $l(h)$ is dependent on the geometric properties of the parametric family. In addition, we write the moments and absolute moments of scattering as $\mathcal{E}_j^h(\theta)$ and $\overline{\mathcal{E}}_j^h(\theta)$, respectively, to show their dependence on the parameter h.

Definition 1.2. *We say that the billiard scattering family with perturbation parameter h satisfies the **L-weak scattering condition** if there is a $c > 0$ and, for each h, a $\theta_h > 0$ so that*

1. $\mathcal{E}_1^h(\theta) = \frac{ch^2}{2}\cot\theta + o(h^2)$ *and* $\mathcal{E}_2^h(\theta) = ch^2 + o(h^2)$ *for all* θ *in* $[\theta_h, \pi - \theta_h]$,

2. $\overline{\mathcal{E}}_1^h(\theta) = \mathcal{O}(h)$, $\overline{\mathcal{E}}_2^h(\theta) = \mathcal{O}(h^2)$, *and* $\overline{\mathcal{E}}_3^h(\theta) = o(h^2)$ *for all* θ *in* $[0, \pi]$,

3. $\theta_h = o(h^{\frac{1}{2}})$

We now consider parametric billiard families that satisfy the L-weak scattering condition, so we may write the normalized billiard Laplacian as

$$\mathcal{L}_h := \frac{P_h - I}{ch^2}\; c > 0.$$

Remark. It is important to observe that if a parametric billiard family satisfies the L-weak scattering condition, its first and second moments approximate the first and second coefficients of Legendre's equation for small values of h.

With this in mind, it is reasonable that the following proposition holds.

Proposition 1.1. *Assume that the L-weak scattering condition is satisfied by a billiard scattering system with perturbation parameter h. Let $\Phi \in C^3([0,\pi])$ be such that $\Phi'(\theta) = \mathcal{O}(\sin\theta)$, so Φ' vanishes to first order at 0 and π. Then $\lim_{h\to 0} ||\mathcal{L}_h\Phi - \mathcal{L}\Phi||_2 = 0$.*

Now, a geometric study of the moments leads to the following main theorem.

Theorem 1.1. *Consider a parametric billiard family whose microstructure consist of cells with a piecewise smooth boundary given by a symmetric function $f_h(x)$. Without loss of generality, we assume that the cell entrance has normalized length 1 and lies on the x=axis with $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Define the parameter by $h = \sup_{x\in\left[-\frac{1}{2},\frac{1}{2}\right]} |f_h'(x)|$. Furthermore, suppose that $\lim_{h\to 0} \frac{8}{h^2}\int_0^{\frac{1}{2}} (f_h'(x))^2\,dx$ exists. Then $\lim_{h\to 0} ||\mathcal{L}_h\Phi - \mathcal{L}\Phi||_2 = 0$.*

1.3.1 Results About Eigenvalues and Eigenfunctions

Let $\Phi_n(\theta) = P_n(\cos\theta)$, where $P_n(\cos\theta)$ are the Legendre polynomials. It is well-known in spherical harmonics that the $P_n(x)$ form an orthogonal basis for

$L^2((0,\pi),\nu)$ and are eigenfunctions for $\mathcal{L}$ with eigenvalues $-\frac{1}{2}n(n+1)$. (Note that our $\mathcal{L}$ is the standard Legendre operator multiplied by a factor of $\frac{1}{2}$.)

It is natural to ask whether the eigenvalues of $\mathcal{L}_h$ will converge to those of the limit operator $\mathcal{L}$. The following theorem provides an answer to this question.

Theorem 1.2. *Let σ denote the spectrum of $\mathcal{L}$. The spectrum of $\mathcal{L}_h$ converges to a subset of $\sigma \cup \{-\infty\}$ as h goes to 0.*

1.3.2 Billiards in Higher Dimensions

So far, we have assumed that our billiard surface is in $\mathbb{R}^2$ and the associated Markov operator is one-dimensional. One can ask if there are analgous results about the convergence of $\mathcal{L}_h$ to $\mathcal{L}$ and the approximation of $\mathcal{L}_h$'s spectrum in higher dimensions. It turns out that we can prove a similar convergence theorem in n-dimensions.

Consider point particle collisions on a parametric billiard family whose microstructures are give by the graph of functions $f_h : \mathbb{R}^n \to \mathbb{R}$ (see Figure 1.7) with the following two properties:

1. **periodicity:** $f_h(x + \sum_i m_i a_i e_i) = f(x),\ a_i > 0,\ m_i \in \mathbb{Z}$
2. **symmetry** $f_h(-x) = f_h(x)$.

Let $n(x)$ and $\overline{n}(x)$ denote the unit normal vector to the graph of f_h at $x \in \mathbb{R}^n$ and its orthogonal projection to $\mathbb{R}^n$, respectively. We define the parameter h by $h = \sup_{x\in\mathbb{T}^n} ||\overline{n}(x)||$.

Let

$$A(h) := \int_{\mathbb{T}^n} \overline{n}(x)^* \otimes \overline{n}(x)\, dx,$$

where the $\overline{n}(x)$ arises from the graph of f_h. Also, let $\Phi \in L^2(\mathbb{D}^n, dx)$, where $\mathbb{D}^n$ is

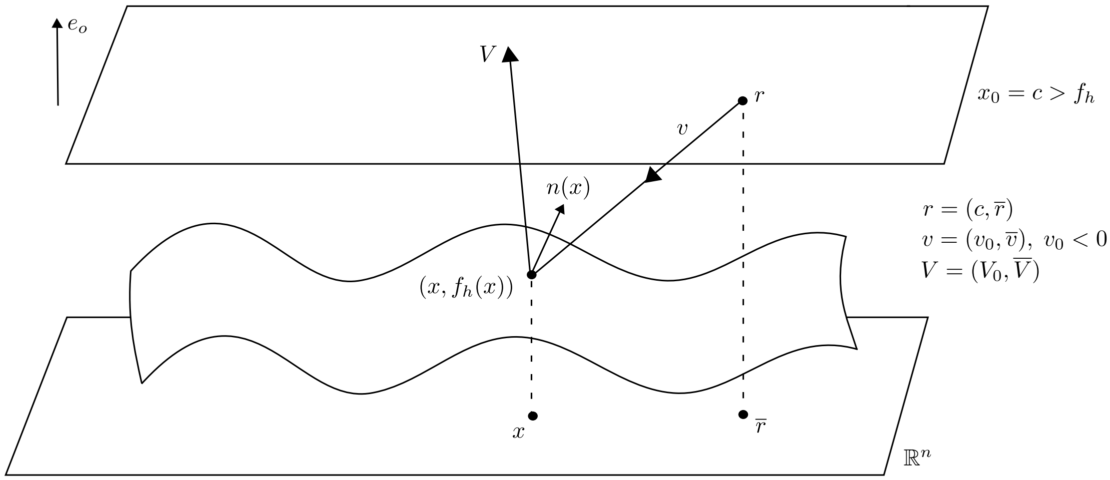

Figure 1.7: A collision of a point particle with a n+1-dimensional billiard surface. $\overline{r}, \overline{v}$, and $\overline{V}$ are the orthogonal projections of the r, v, and V. We take $\overline{v}$ to be a vector in the open unit disc $\mathbb{D}^n$.

the n-dimensional unit disc and dx is normalized Lebesgue measure. Define

$$\mathcal{L}_h = \frac{P_h - I}{2h^2}$$

and

$$\mathcal{L}\Phi = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i}(1 - ||\overline{v}||^2)\frac{\partial \Phi}{\partial x_i},$$

where $c_i = \lim_{h \to 0} \dfrac{\lambda_i(h)}{h^2}$ and the λ_i's are the eigenvalues of the operator A(h) associated to the particular f_h we are examining. Note that this limit will exist in most of the surfaces we consider.

The newly defined $\mathcal{L}$ reduces to the well-studied Legendre operator in dimension one. For this reason, $\mathcal{L}$ can be viewed as a higher-dimensions generalization of the Legendre operator. As far as we know, this natural extension has not been considered in other literature. Therefore, we will include a brief examination of its spectral information later. For now, we will familiarize ourselves with $\mathcal{L}$ with an example.

Example 1.3. Consider a 2-dimensional random parametric billiard family whose microstructure cells have the shape of a pyramid with a rectangular base (see Figure 1.8). For simplicity, instead of $l(h)$, we will denote the height of the pyramid by l. In order to find $\mathcal{L}$ in this case, we only need to determine the values of c_1 and c_2.

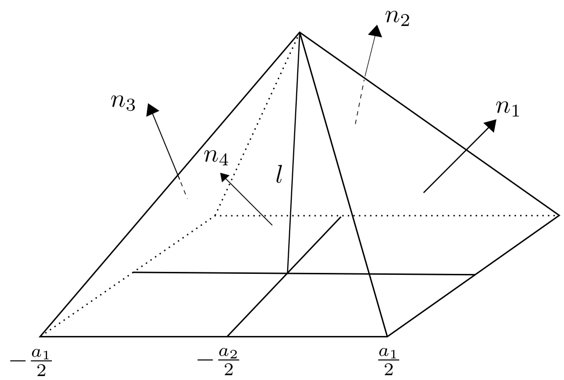

Figure 1.8: A random billiard with microstructure that has its cell shape defined by a 3-dimensional rectangular-base pyramid. The height of the pyramid is denoted by l and the outward-pointing normal vectors to each face are represented by n_i, $i = 1, 2, 3, 4$.

A geometric calculation based on Figure 1.8 shows that

$$n_1 = \frac{1}{\sqrt{l^2 + \frac{a_1^2}{4}}}\left(l, 0, \frac{a_1}{2}\right) \qquad n_2 = \frac{1}{\sqrt{l^2 + \frac{a_2^2}{4}}}\left(0, l, \frac{a_2}{2}\right)$$
$$n_3 = \frac{1}{\sqrt{l^2 + \frac{a_1^2}{4}}}\left(-l, 0, -\frac{a_1}{2}\right) \qquad n_4 = \frac{1}{\sqrt{l^2 + \frac{a_2^2}{4}}}\left(0, -l, \frac{a_2}{2}\right).$$

Normalizing and projecting the normal vectors to $\mathbb{R}^2$, we obtain

$$\overline{n}_1 = \frac{1}{\sqrt{l^2 + \frac{a_1^2}{4}}}(l, 0) \qquad \overline{n}_2 = \frac{1}{\sqrt{l^2 + \frac{a_2^2}{4}}}(0, l)$$
$$\overline{n}_3 = \frac{1}{\sqrt{l^2 + \frac{a_1^2}{4}}}(-l, 0) \qquad \overline{n}_4 = \frac{1}{\sqrt{l^2 + \frac{a_2^2}{4}}}(0, -l).$$

The parameter h is given by $\max\left\{\frac{1}{\sqrt{l^2+\frac{a_1^2}{4}}}, \frac{1}{\sqrt{l^2+\frac{a_2^2}{4}}}\right\}$. Assuming that $a_2 \geq a_1$,

$$h = \frac{1}{\sqrt{l^2 + \dfrac{a_1^2}{4}}}.$$

Observe that

$$\begin{aligned}
\overline{n}_1^* \otimes \overline{n}_1 &= \frac{l^2}{l^2 + \frac{a_1^2}{4}} (1,0)^* \otimes (1,0) = \overline{n}_3^* \otimes \overline{n}_3 \\
\overline{n}_2^* \otimes \overline{n}_2 &= \frac{l^2}{l^2 + \frac{a_2^2}{4}} (0,1)^* \otimes (0,1) = \overline{n}_4^* \otimes \overline{n}_4.
\end{aligned}$$

Since the area of the base is $a_1 a_2$, it follows that

$$\begin{aligned}
A(h) =& \frac{1}{a_1 a_2}\left[\frac{a_1 a_2}{2} \frac{l^2}{l^2 + \frac{a_1^2}{4}} (1,0)^* \otimes (1,0) + \frac{a_1 a_2}{2} \frac{l^2}{l^2 + \frac{a_2^2}{4}} (0,1)^* \otimes (0,1)\right] \\
=& \frac{1}{2}\left[\frac{l^2}{l^2 + \frac{a_1^2}{4}} e_1^* \otimes e_1 + \frac{l^2}{l^2 + \frac{a_2^2}{4}} e_2^* \otimes e_2\right]
\end{aligned}$$

So

$$A(h) = \begin{pmatrix} \dfrac{l^2}{l^2 + \dfrac{a_1^2}{4}} & 0 \\ 0 & \dfrac{l^2}{l^2 + \dfrac{a_2^2}{4}} \end{pmatrix}.$$

It is immediate that the eigenvalues of $A(h)$ are

$$\lambda_1(h) = \frac{l^2}{l^2 + \dfrac{a_1^2}{4}}, \text{ and } \lambda_2(h) = \frac{l^2}{l^2 + \dfrac{a_2^2}{4}}.$$

We can now see that

$$c_1 = \lim_{h\to 0} \frac{\lambda_1(h)}{h^2} = \frac{1}{2}$$
$$c_2 = \lim_{h\to 0} \frac{\lambda_2(h)}{h^2} = \frac{1}{2}\frac{4l^2 + a_1^2}{4l^2 + a_2^2}.$$

Assuming that there are at most a finite number of collisions before each billiard trajectory exits the cell, we have the following result concerning the convergence of $\mathcal{L}_h$ to $\mathcal{L}$ in higher dimensions.

Theorem 1.4. *Assume that a billiard scattering system with perturbation parameter h has microstructure given by the graph of a symmetric and periodic function $f_h(x)$. Let $h = \sup_{x\in\mathbb{T}^n} ||\overline{n}(x)||$, where $\overline{n}(x)$ is the orthogonal projection to $\mathbb{R}^n$ of the unit normal to the graph of $f_h(x)$. Moreover, we assume that each billiard particle collides with the walls of a cell surface at most k times before exiting for some $k > 0$. Let $\Phi \in C^3(\mathbb{D}^n)$ and denote the eigenvalues of the self-adjoint operator $A(h)$ by $\lambda_i(h)$. Suppose $c_i = \lim_{h\to 0} \frac{\lambda_i(h)}{h^2}$ exists for $1 \leq i \leq n$. Then $\lim_{h\to 0} ||\mathcal{L}_h\Phi - \mathcal{L}\Phi||_2 = 0$.*

We can also say a little more about the eigenfunctions of $\mathcal{L}$. Notice that we can view $\mathcal{L}$ as a symmetric operator on the smooth functions on $\mathbb{D}^n \subset [-1,1]^n$. Let P_k denote the k^{th} Legendre polynomial on $[-1,1]^n$ and let $\Pi_i : [-1,1]^n \longrightarrow [-1,1]$ be the i^{th} coordinate projection map. Then for a multi-index $m = (m_1, ..., m_n) \in \mathbb{N}^n$, we define Φ_m as $\Phi_m := \prod_{i=1}^{n} P_{m_i} \circ \Pi_i|_{\mathbb{D}^n}$.

Theorem 1.5. $\Big\{\Phi_m\Big\}_{m\in\mathbb{N}^n}$ *is a complete family of eigenfunctions for $\mathcal{L}$ and forms a basis for $L^2(\mathbb{D}^n, dx)$.*

1.4 A Look Ahead

In Chapter 2, we become more familiar with the billiard map and the associated Markov operator by delving into a discussion on the background material that is necessary to present the main results. Next, we examine random billiard systems in one dimension and explore the L^2 convergence of $\mathcal{L}_h$ to $\mathcal{L}$ and the comparison of the two operators' spectra in Chapter 3. Then we consider billiard families in higher dimensions in Chapter 4, where we generalize the Legendre operator to n dimensions, study its spectral information, and prove a similar convergence theorem for $\mathcal{L}_h$ and $\mathcal{L}$.

Chapter 2

A detailed discussion on the billiard map and P

We will now elaborate on the properties of the billiard map and the operator P for random billiards with microstructure which possess the properties discussed in the introduction. We will always assume that the boundary segment Γ_0 (where the particle enters the billiard cell Q) is flat and of normalized length 1.

2.1 The Billiard Map

Denote by $\Gamma = \partial Q = \cup_i \tilde{\Gamma}_i$ the decomposition of the boundary of the billiard cell Q into its smooth component curves, or walls. We call $\mathbf{n}$ the unit normal field that points inward on each smooth component and orient the boundary of the cell by requiring that Q is left of each $\tilde{\Gamma}_i$. Moreover, let

$$\mathcal{M}_i = \{(q, v) \in \tilde{\Gamma}_i : \langle v, \mathbf{n}(q) \rangle \geq 0\}$$

and define the *collision space* $\mathcal{M}$ as $\mathcal{M} = \cup_i \mathcal{M}_i$. Excusing the abuse of notation, let the union of the boundaries of the $\mathcal{M}_i$'s be called $\partial\mathcal{M}$ and $\mathcal{M}^\circ = \mathcal{M} \backslash \partial\mathcal{M}$. Then let $\mathcal{F} : \mathcal{M} \longrightarrow \mathcal{M}$ denote the *billiard map.* (The detailed definition of $\mathcal{F}$ can be found in [1].)

Consider $x = (q, v) \in \mathcal{M}^\circ$. Let q' be the first intersection of the ray $q+tv$, $t > 0$ with ∂Q and v' denote the orthogonal reflection of v on the tangent space to ∂Q at q'. If q' is not a corner point (i.e. an endpoint to a wall) and v is not a tangent vector to ∂Q at q', then $\mathcal{F}(x) = (q', v')$.

Other items of interest include $\mathcal{F}^{\pm m}$, the iterates of $\mathcal{F}$, and their associated singular sets, $\mathcal{S}_{\pm m}$, which are defined using induction. Let $\mathcal{S}_0 = \partial\mathcal{M}$, $\mathcal{S}_{\pm 1} = \mathcal{S}_0 \cup \{x \in \mathcal{M}^\circ : \mathcal{F}^{\pm 1}(x) \notin \mathcal{M}^\circ\}$, and continuing this way,

$$\mathcal{S}_{\pm(m+1)} = \mathcal{S}_{\pm m} \cup \mathcal{F}^{\mp m}(\mathcal{S}_{\pm m}).$$

Also, let $\tilde{\mathcal{M}} := \mathcal{M} \backslash \cup_{i=-\infty}^{\infty} \mathcal{S}_j$. It can be shown that $\mathcal{M} \backslash \mathcal{S}_{\pm 1}$ are open sets and $\mathcal{F} : \mathcal{M} \backslash \mathcal{S}_1 \longrightarrow \mathcal{M} \backslash \mathcal{S}_{-1}$ is a smooth diffeomorphism. In addition, $\mathcal{F}$ has an associated invariant measure on $\mathcal{M}$, and $\mathcal{F}$'s iterates are smooth on $\tilde{\mathcal{M}}$, which is a dense G_δ-subset of $\mathcal{M}$ of full Lebesgue measure.

Define $M = \mathcal{M}_0$ as the subset of $\mathcal{M}$ that consists of pairs (q, v) such that $q \in \Gamma_0$. Let μ be the measure obtained by restricting and normalizing the invariant measure on $\mathcal{M}$ to M. Note that μ is a probability measure. Then Poincaré recurrence implies that there exists $E_0 \subset M \cap \tilde{\mathcal{M}}$ such that E_0 has full μ-measure, and billiard orbits that begin in E_0 will return to M and are non-singular. The number of steps that the orbits take to return to M will be finite, so each $x \in M$ has a neighborhood in M on which the return map is smooth, and all the points in this open set return to M in the same number of steps as the orbit of x. It follows that E_0 is open, has full measure, and the first return map to M is well-defined and smooth on it. Moreover, we can extend E_0 to a set E that also contains singular orbits that eventually still return to M. Then the first return map $T : E \longrightarrow M$ is defined for all vectors that enter the cell at angles close enough to 0 or π, and all vectors that are based at the two endpoints of Γ_0 are included in E_0.

Let S_1 and S_{-1} denote the singular sets of T and T^{-1}, respectively. Note that S_1 and S_{-1} are compact subsets of M. Then $T : M \backslash S_1 \longrightarrow M \backslash S_{-1}$ is a smooth diffeomorphism. We often consider T as a map from M to itself for simplicity and call T the *reduced billiard map* on M. Observe that μ is T-invariant.

Defining I and V as $I = [0, 1]$ and $V = [0, \pi]$, we can parametrize M by writing $M = I$ x V. Denote by $r \in I$ the position at which the particle enters Γ_0 and by $\theta \in V$ the angle the initial vector makes with the positive tangent unit vector to the boundary. Note that this definition of θ differs from the usual one where it is measure from the normal vector. As a consequence, $\mu = \lambda \otimes \nu$, where λ is the Lebesgue measure on I and $d\nu(\theta) = \frac{1}{2} \sin(\theta) d\theta$.

2.2 The Markov Operator

Given a random billiard with microstructure whose cells have a symmetric shape, we can define an associated *Markov operator*, P. Also called the *collision operator*, P is the transition probabilities operator for certain Markov chains that will be discussed soon. Using the notations for the billiard map and letting $\pi_2 : M \longrightarrow V$ be the coordinate projection map, we define the operator P on $L^\infty(V, \nu)$ by

$$(Pf)(\theta) = \int_I f(\pi_2 \circ T(s, \theta))\, ds.$$

Then for each measurable $A \subset V$, the conditional probability that the post-collision angle lies in A is given by

$$P(A|\theta) := (P\mathbb{1}_A)(\theta_0).$$

In addition, if η is a signed measure on V and $f \in L^\infty(V,\nu)$, then P acts naturally on η via the relation

$$(\eta P)(f) = \eta(Pf),$$

where $\eta(f)$ is the integral of f with respect to η. In particular, when η is the probability distribution of pre-collision angles, then ηP will be the distribution of post-collision angles. It is simple to check that another representation for ηP is

$$\eta P = (\pi_2 \circ T)_* \lambda \otimes \eta, \tag{2.1}$$

where the underscore asterisk denotes the push-forward of the measure. Using Equation 2.1 and the T-invariance of μ, we can show that ν is the *stationary probability* of P on V (i.e. ν is P-invariant) since

$$\nu P = (\pi_2)_* T_* \mu = (\pi_2)_* \mu = \nu.$$

Let $L^2(V,\nu)$ be the Hilbert space with the integral inner product $< f, g >= \int_V fg\, d\nu$. Note that we usually deal with real functions, so the conjugation that appears in the standard inner product is not necessary. We can combine the P-invariance ν and an appeal to Jensen's inequality to argue that P is a bounded operator on $L^2(V,\nu)$ of norm 1.

2.3 The Markov chains determined by P

Consider the Markov chains that arise from the angles associated with the collisions of billiard trajectories. Let $R_0, R_1, ...$ be independent, uniformly distributed random variables with values in I. In addition, let Θ_0 be a random variable with distribution η on V. (Θ_0 can be thought of as the initial angle of the billiard trajectory.) For $n = 0, 1, ...$, we define the random angle Θ_{n+1} by

$$\Theta_{n+1} = \pi_2(T(R_n, \Theta_n)).$$

Then $\Theta_1, \Theta_2, ...$ is the Markov chain associated to P with initial probability distribution η. It is easy to see that the transition probabilities are obtained using the equation

$$\text{Prob}(\Theta_{n+1} \in A | \Theta_n = \theta) = P(A|\theta) := (P\mathbb{1}_A)(\theta).$$

Moreover, $\text{Prob}(\Theta_n \in A) = (\eta P^n)(\mathbb{1}_A)$. As shown in [4], under very general assumptions, these Markov chains are irreducible and aperiodic. So ηP^n will converge to the stationary distribution ν as $n \to \infty$ for any starting distribution η.

2.4 Other Properties of P

The L^2 dual of P is given by the following:

$$(P^* f)(\theta) = \int_I f(\pi_2 \circ T^{-1}(s, \theta)) ds.$$

Since the billiard cell Q has a symmetric shape, it turns out that P is self-adjoint [5]. Therefore, its spectrum is contained in [-1,1].

Moreover, let $M_1 \subset M$ be the open subset consisting of the pairs (r, θ) such that $\Psi'_\theta(r) \neq 0$, where $\Psi_\theta(r) := \pi_2 T(r, \theta)$. Consider the partition of M into M_1 and $M_2 = M \backslash M_1$. Since T is smooth on the set $E_0 \subset M$, which has full measure, we will disregard $M \backslash E_0$ for simplicity and assume that T is also smooth on M_2. Then we can write the Lebesgue decomposition of P as

$$P = \alpha_1 P_1 + \alpha_2 P_2,$$

where P_1 is absolutely continuous with respect to ν and P_2 is the singular part. It can be shown that P_1 is an integral operator on $L^2(V, \nu)$. That is,

$$(P_1 f)(\theta) = \int_V \omega_1(\theta, \phi) f(\phi)\, d\nu_1(\phi).$$

The proof and a detailed discussion of the kernel ω_1 is provided in [5]. Two items to note are that $\omega_1(\theta, \phi) = \omega_1(\phi, \theta)$ because Q is symmetric and time reversibility of the Markov chains implies $\omega_1(\theta, \phi) = \omega(\pi - \phi, \pi - \theta)$.

Chapter 3

The Billard and Legendre Operators in One Dimension

3.1 Approximating $\mathcal{L}_h$ by $\mathcal{L}$

The importance of Proposition 1.1 is that it relates the operator $\mathcal{L}_h$ to the Legendre operator, which, in turn, allows us to approximate the spectrum of $\mathcal{L}_h$ using the eigenvalues of $\mathcal{L}$.

Remark. It is well-known [3] that when the limit operator $\mathcal{L}$ has the form of the Legendre operator, there exists a domain in which $\mathcal{L}$ will be a self-adjoint operator whose eigenfunctions and eigenvalues are precisely the standard Legendre functions and polynomials.

Proposition 1.1. *Assume that the L-weak scattering condition is satisfied by a billiard scattering system with perturbation parameter h. Let* $\Phi \in C^3([0, \pi])$ *be such that* $\Phi'(\theta) = \mathcal{O}(\sin\theta)$*, so* Φ' *vanishes to first order at 0 and* π*. Then* $\lim_{h\to 0} ||\mathcal{L}_h\Phi - \mathcal{L}\Phi||_2 = 0$*.*

Proof. Let Φ be a C^3 differentiable function with bounded first, second, and third derivatives. Consider the second-order Taylor expansion of $\Phi(\Theta)$ around θ,

$$\Phi(\Theta) = \Phi(\theta) + \Phi'(\theta)(\Theta - \theta) + \frac{1}{2}\Phi''(\theta)(\Theta - \theta)^2 + R_2(\theta, \Theta),$$

where

$$R_2(\theta, \Theta) = \frac{1}{2}\int_{I(\theta,\Theta)} |\Phi'''(t)|(\Theta - t)^2\, dt$$

is the remainder term and $I(\theta, \Theta)$ is the interval with endpoints θ and Θ. Recalling that $(P_h\Phi)(\theta) = E_\theta^h[\Phi(\Theta)]$, we take expectations and obtain

$$\begin{aligned}
&(P_h\Phi)(\theta) - \Phi(\theta) \\
=& E_\theta^h(\Phi(\Theta)) - \Phi(\theta) \\
=& \mathcal{E}_1^h(\theta)\Phi'(\theta) + \frac{1}{2}\mathcal{E}_2^h(\theta)\Phi''(\theta) + E_\theta^h(R_2(\theta, \Theta)).
\end{aligned}$$

Since $|R_2(\theta, \Theta)| \le \frac{1}{2}||\Phi'''||_\infty|\Theta - \theta|^3$, we obtain

$$\left|((P_h\Phi)(\theta) - \Phi(\theta)) - \left(\mathcal{E}_1^h(\theta)\Phi'(\theta) + \frac{1}{2}\mathcal{E}_2^h(\theta)\Phi''(\theta)\right)\right| \le \frac{1}{2}||\Phi||_\infty\overline{\mathcal{E}}_3^h(\theta) = o(h^2).$$

Dividing through by ch^2 and taking limits results in

$$\lim_{h\to 0}\left|(\mathcal{L}_h\Phi)(\theta) - \frac{1}{ch^2}\left(\mathcal{E}_1^h(\theta)\Phi'(\theta) + \frac{1}{2}\mathcal{E}_2^h(\theta)\Phi''(\theta)\right)\right| = 0$$

for all $\theta \in [0, \pi]$. For convenience, define

$$(\overline{\mathcal{L}}_h\Phi)(\theta) := \frac{1}{ch^2}\left(\mathcal{E}_1^h(\theta)\Phi'(\theta) + \frac{1}{2}\mathcal{E}_2^h(\theta)\Phi''(\theta)\right).$$

We have just shown that $|\mathcal{L}_h\Phi(\theta) - \overline{\mathcal{L}}_h\Phi(\theta)|$ is bounded and converges to 0 pointwise as h goes to 0, which implies that

$$\lim_{h\to 0} ||\mathcal{L}_h\Phi(\theta) - \overline{\mathcal{L}}_h\Phi(\theta)||_2 = 0.$$

We are done if we show that $||\overline{\mathcal{L}}_h\Phi(\theta) - \mathcal{L}\Phi(\theta)||_2$ goes to 0 for each $\Phi \in C^3([0,\pi])$. In order to accomplish this, we write

$$||\overline{\mathcal{L}}_h\Phi(\theta) - \mathcal{L}\Phi(\theta)||_2^2 = \mathcal{J}_1 + \mathcal{J}_2,$$

where we define $\mathcal{J}_1$ and $\mathcal{J}_2$ by

$$\mathcal{J}_1 := \int_{[\theta_h,\pi-\theta_h]} |\overline{\mathcal{L}}_h\Phi(\theta) - \mathcal{L}\Phi(\theta)|^2 \, d\nu(\theta),$$
$$\mathcal{J}_2 := \int_{[\theta_h,\pi-\theta_h]^c} |\overline{\mathcal{L}}_h\Phi(\theta) - \mathcal{L}\Phi(\theta)|^2 \, d\nu(\theta).$$

It is immediate from parts 1 and 3 of the L-weak scattering condition that $\mathcal{J}_1 \to 0$ as $h \to 0$. For $\mathcal{J}_2$, we first consider the interval $[0,\theta_h]$. Recalling that $d\nu(\theta) = \frac{1}{2}\sin\theta\, d\theta$ and using the definition of $\overline{\mathcal{L}}_h\Phi(\theta)$, we have

$$\int_0^{\theta_h} |\overline{\mathcal{L}}_h\Phi|^2 \, d\nu(\theta) \le \frac{1}{2c^2}\int_0^{\theta_h} \left(\frac{\overline{\mathcal{E}}_1^h(\theta)}{h^2}|\Phi'(\theta)| + \frac{\overline{\mathcal{E}}_2^h(\theta)}{2h^2}|\Phi''(\theta)| \right)^2 \sin\theta\, d\theta.$$

Consider the two terms under the square of the integrand on the right-hand side. By the L-weak scattering condition, $\overline{\mathcal{E}}_1^h(\theta) = \mathcal{O}(h)$ and $\overline{\mathcal{E}}_2^h(\theta) = \mathcal{O}(h^2)$. Then using the assumption that $\Phi'(\theta)$ vanishes to the first order at 0, we observe that the first term is of order $\mathcal{O}(\frac{\theta}{h})$ and the second term is bounded. Thus, we have

$$\frac{1}{2c^2}\int_0^{\theta_h} \left(\frac{\overline{\mathcal{E}}_1^h(\theta)}{h^2}|\Phi'(\theta)| + \frac{\overline{\mathcal{E}}_2^h(\theta)}{2h^2}|\Phi''(\theta)| \right)^2 \sin\theta\, d\theta = \mathcal{O}\left(\frac{\theta_h^4}{h^2}\right)$$

so the integral goes to 0 since $\theta_h = o(h^{\frac{1}{2}})$. A similar argument can be employed for the interval $[\pi-\theta_h,\pi]$, and it follows that

$$\lim_{h\to 0}\int_{[\theta_h,\pi-\theta_h]^c}|\overline{\mathcal{L}}_h\Phi|^2\,d\nu(\theta)=0.$$

Since Φ is a C^3 function with bounded derivatives, we have that $|\mathcal{L}\Phi|^2\sin\theta$ is bounded, i.e., $\exists\, M>0$ such that $|\mathcal{L}\Phi|^2\sin\theta\le M$. Then

$$\begin{aligned}&\lim_{h\to 0}\int_{[\theta_h,\pi-\theta_h]^c}|\mathcal{L}\Phi|^2\,d\nu(\theta)\\ &=\lim_{h\to 0}\frac{1}{2}\int_{[\theta_h,\pi-\theta_h]^c}|\mathcal{L}\Phi|^2\sin\theta\,d\theta\\ &\le\frac{1}{2}M\lim_{h\to 0}\int_{[\theta_h,\pi-\theta_h]^c}1\,d\theta\\ &=0.\end{aligned}$$

Both $\lim_{h\to 0}\int_{[\theta_h,\pi-\theta_h]^c}|\overline{\mathcal{L}}_h\Phi|^2\,d\nu(\theta)=0$ and $\lim_{h\to 0}\int_{[\theta_h,\pi-\theta_h]^c}|\mathcal{L}\Phi|^2\,d\nu(\theta)=0$, so we conclude that $\mathcal{J}_2$ also goes to zero, thereby completing the proof. $\square$

3.2 Billard Operators From Families With Symmetric Cells

One of the main assumptions that we make in order to obtain our spectral comparison between $\mathcal{L}_h$ and $\mathcal{L}$ is that the billiard system satisfies the L-weak scattering condition. We will now discuss various billiard families that fulfill this assumption. Thus, Proposition 1.1 implies the $\mathcal{L}_h$ converges to $\mathcal{L}$ for these families.

Consider a billiard family consisting of microstructure cells with a piecewise smooth boundary given by a symmetric function $f_h(x)$ (see Figure 3.1). Without loss of generality, we assume that $f_h(x)$ is symmetric about the y-axis and the normalized entrance segment is of length one and lies on the x-axis on the interval

$\left[-\frac{1}{2}, \frac{1}{2}\right]$. The parameter is defined by $h = \sup_{x\in\left[-\frac{1}{2},\frac{1}{2}\right]} |f_h'(x)|$.

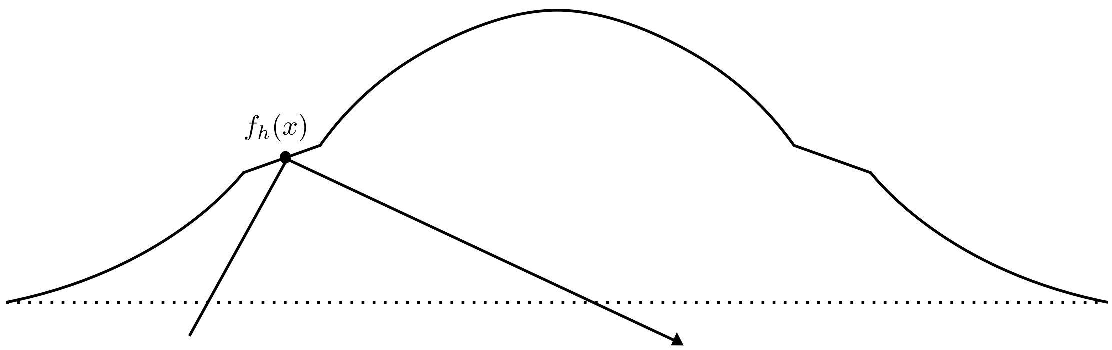

Figure 3.1: A microstructure cell consisting of a piecewise smooth boundary given by a symmetric function $f_h(x)$.

Note that for any bilaterally symmetric billiard cell shape, the identity

$$\pi - \Psi_\theta(r) = \Psi_{\pi-\theta}(1-r)$$

holds. Therefore, we will perform our analyses assuming $\theta \leq \frac{\pi}{2}$ because the results for $\theta > \frac{\pi}{2}$ can be obtained using the aforementioned identity.

We will show that the billiard system with the aforementioned properties satisfies the L-weak scattering condition. To achieve this, we start by finding θ_h. We want θ_h to have the property that if $\sin\theta \geq \sin\theta_h$, the billiard trajectory will only collide with the walls of the cell once.

Let $x_0 \in [-\frac{1}{2}, \frac{1}{2}]$ be such that $f_h'(x_0) \geq f_h'(x)$ for any $x \in [-\frac{1}{2}, \frac{1}{2}]$. Define $\frac{\beta}{2}$ as the angle between the normal and downward vertical vectors at $f_h(x)$. (see Figures 3.2 and 3.3).

Suppose a trajectory only collides once with the walls of the cell. A geometric calculation based on Figure 3.3 shows that $\theta = \Theta - 2\alpha$, and note that $\alpha, \Theta \leq \frac{\beta}{2}$.

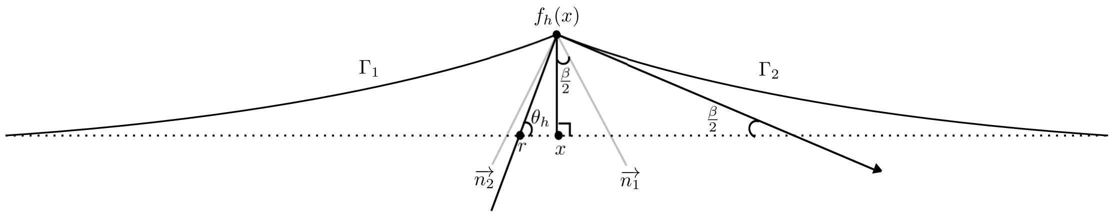

Figure 3.2: An initial angle of θ_h results in a billiard trajectory hitting the corner point, reflecting off Γ_1, and exiting at a grazing angle to Γ_2.

It follows that $\theta_h \leq \frac{\beta}{2} + 2\left(\frac{\beta}{2}\right)$, so

$$\theta_h \leq \frac{3\beta}{2}.$$

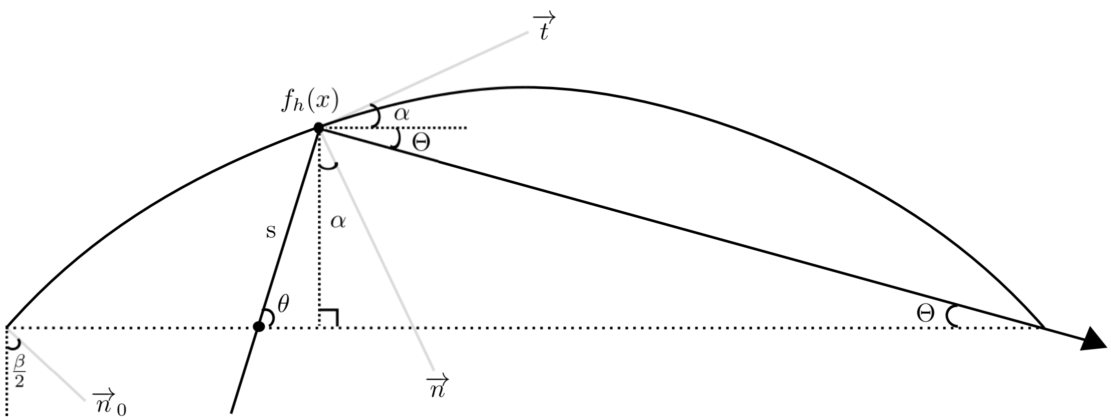

Figure 3.3: A focusing cell displaying the definition of β and a billiard trajectory that exits at $x = \frac{1}{2}$. The normal and tangent vectors to the cell are shown in gray.

Now assume that θ is such that $\sin\theta \geq \sin\theta_h$.

Geometric calculations based on Figure 3.4, which hold true for all symmetric cell shapes, show that for $x \in \left[-\frac{1}{2}, 0\right]$,

$$\Theta = \theta - 2\alpha, \tag{3.1}$$

$$\tan\alpha = f_h'(x). \tag{3.2}$$

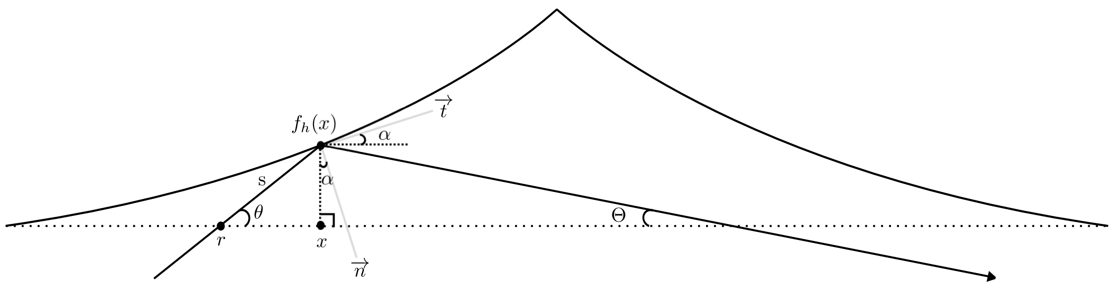

Figure 3.4: A billiard from a dispersing family with parameter h. The entrance segment (dotted line) lies on the x-axis on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. The normal and tangent vectors to the curve at the collision point are shown in gray and α is the angle between the normal and downward vertical vectors at the collision point.

It follows from Equations 3.1 and 3.2 that

$$\Theta = \theta - 2\tan^{-1}(f_h'(x)). \tag{3.3}$$

Another look at Figure 3.4 gives the relation $(x, f_h(x)) = (r, 0) + s(\cos\theta, \sin\theta)$, which implies that

$$r = x - f_h(x)\cot\theta. \tag{3.4}$$

By a similar argument Equations 3.3 and 3.4 are also true for $x \in \left[0, \frac{1}{2}\right]$. Differentiating gives us

$$\frac{d\Theta}{dx} = -2\frac{d}{dx}\tan^{-1}(f_h'(x)) = -2\frac{f_h''(x)}{1 + f_h'(x)^2}$$
$$\frac{dr}{dx} = 1 - f_h'(x)\cot\theta$$

for $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Using this, we can write the operator P_h as an integral over the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Keeping Equation 3.3 in mind, for $\Phi \in L^2([0,\pi],\nu)$, we have

$$
\begin{aligned}
&(P_h\Phi)(\theta)\\
=&\int_0^1 \Phi\left(\Psi_\theta(r)\right)\, dr\\
=&\int_{-\frac{1}{2}}^{\frac{1}{2}} \Phi\left(\Psi_\theta\left(r\left(x\right)\right)\right)\, \frac{dr}{dx}dx\\
=&\int_{-\frac{1}{2}}^{\frac{1}{2}} \Phi(\theta - 2\tan^{-1}(f_h'(x))(1 - f_h'(x)\cot\theta)\, dx.
\end{aligned}
$$

Recall that $\mathcal{E}_j^h(\theta) := E^h\left[(\Theta-\theta)^j\right]$ and the symmetry of $f_h(x)$ implies $f_h'(-x) = -f_h'(x)$. Then employing a similar strategy as in the calculation for P_h, we obtain

$$
\begin{aligned}
&\mathcal{E}_j^h(\theta)\\
=&E^h\left[(\Theta-\theta)^j\right]\\
=&\int_0^1 (\Psi_\theta(r)-\theta)^j\, dr\\
=&\int_{-\frac{1}{2}}^{\frac{1}{2}} (\Psi_\theta(r(x))-\theta)^j\, \frac{dr}{dx}dx\\
=&\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[2\tan^{-1}(f_h'(x)\right]^j (1 - f_h'(x)\cot\theta)\, dx\\
=&\int_0^{\frac{1}{2}} \left[2\tan^{-1}(f_h'(x)\right]^j (1 + f_h'(x)\cot\theta) + \left[-2\tan^{-1}(f_h'(x)\right]^j (1 - f_h'(x)\cot\theta)\, dx\\
=&\int_0^{\frac{1}{2}} 2^j \left[\tan^{-1}(f_h'(x)\right]^j \left[(1 + f_h'(x)\cot\theta) + (-1)^j(1 - f_h'(x)\cot\theta)\right]\, dx\\
=&2^j \int_0^{\frac{1}{2}} \left[\tan^{-1}(f_h'(x)\right]^j \left[(1+(-1)^j) + (1-(-1)^j)f_h'(x)\cot\theta\right]\, dx.
\end{aligned}
$$

Thus,

$$\mathcal{E}_j^h(\theta) = \begin{cases} 2^{j+1} \displaystyle\int_0^{\frac{1}{2}} [\tan^{-1}(f_h'(x)]^j \, dx & \text{if } j \text{ is even} \\ 2^{j+1} \left(\displaystyle\int_0^{\frac{1}{2}} [\tan^{-1}(f_h'(x)]^j f_h'(x) \right) \cot\theta \, dx & \text{if } j \text{ is even} \end{cases} \tag{3.5}$$

It follows that

$$\mathcal{E}_1^h(\theta) = 4 \left(\int_0^{\frac{1}{2}} \tan^{-1}(f_h'(x) f_h'(x) \right) \cot\theta \, dx$$

$$\text{and } \mathcal{E}_2^h(\theta) = 8 \int_0^{\frac{1}{2}} [\tan^{-1}(f_h'(x)]^2 \, dx.$$

With one extra assumption, we can show that these billiard systems satisfy the L-weak scattering condition.

Theorem 1.1. *Consider a parametric billiard family whose microstructure consist of cells with a piecewise smooth boundary given by a symmetric function $f_h(x)$. Without loss of generality, we assume that the cell entrance has normalized length 1 and lies on the x=axis with $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Define the parameter by $h = \sup_{x \in \left[-\frac{1}{2}, \frac{1}{2}\right]} |f_h'(x)|$. Furthermore, suppose that $\lim_{h\to 0} \frac{8}{h^2} \int_0^{\frac{1}{2}} (f_h'(x))^2 \, dx$ exists. Then $\lim_{h\to 0} ||\mathcal{L}_h\Phi - \mathcal{L}\Phi||_2 = 0$.*

Proof. Recall that $\frac{\beta}{2}$ is the angle between the normal and downward vertical vectors at the point of the cell boundary that has the largest slope (after taking absolute value). (See Figure 3.3). This implies that

$$\tan(\frac{\beta}{2}) = \sup_{x \in \left[-\frac{1}{2}, \frac{1}{2}\right]} |f_h'(x)| = h,$$

so

$$\frac{\beta}{2} = \tan^{-1}(h).$$

Then $\theta_h \leq \frac{3\beta}{2} = 3\tan^{-1}(h)$. Expanding $\tan^{-1}(h)$ as a Taylor series gives $\theta_h = o(h^{\frac{1}{2}})$ and proves part 3 of the condition.

Let $\theta \in [\theta_h, \pi - \theta_h]$. To conclude that $\mathcal{E}_2^h(\theta) = ch^2 + o(h^2)$, it suffices to show that

$$\lim_{h\to 0} \frac{8\int_0^{\frac{1}{2}} [\tan^{-1}(f_h'(x))]^2\, dx - ch^2}{h^2} = 0 \tag{3.6}$$

for some $c > 0$. Solving for c, we see that the limit is zero if and only if $c = \lim_{h\to 0} \frac{8}{h^2} \int_0^{\frac{1}{2}} [\tan^{-1}(f_h'(x))]^2\, dx$. Since the difference between $\tan^{-1}(f_h'(x))$ and $f_h'(x)$ is of order $o(h^2)$, $c = \lim_{h\to 0} \frac{8}{h^2} \int_0^{\frac{1}{2}} (f_h'(x))^2\, dx$. Then by our assumption, Equation 3.6 is true. Similarly, to see that $\mathcal{E}_1^h(\theta) = \frac{ch^2}{2}\cot\theta + o(h^2)$, it suffices to show that

$$\lim_{h\to 0} \frac{\left(4\int_0^{\frac{1}{2}} \tan^{-1}(f_h'(x))(f_h'(x)\, dx\right)\cot(\theta) - \frac{ch^2}{2}\cot\theta}{h^2} = 0 \tag{3.7}$$

Solving for c, we have that the equality holds if and only if

$$c = \lim_{h\to 0} \frac{8}{h^2} \int_0^{\frac{1}{2}} \tan^{-1}(f_h'(x)) f_h'(x)\, dx.$$

Since $\tan^{-1}(f_h'(x)) - f_h'(x) = o(h^2)$, it follows that $c = \lim_{h\to 0} \frac{8}{h^2} \int_0^{\frac{1}{2}} (f_h'(x))^2\, dx$. Appealing once more to our assumption on c, we may conclude that part 1 of the condition is satisfied.

As for part 2, let θ in $[0,\pi]$. Let α_i be the angle between the normal and downward vertical vectors at the i^{th} collision point. Recall that $\alpha_i \leq \frac{\beta}{2}$. If there is only one collision, the relation $|\Theta - \theta| = 2|\alpha|$ implies that $|\Theta - \theta| \leq 2(\frac{\beta}{2}) = 2\tan^{-1}(h)$.

If there are multiple collisions, we only have to consider the relationships between the first two and the last two (see Figure 3.5). Note that $\theta_1 = \theta_{last}$ if the

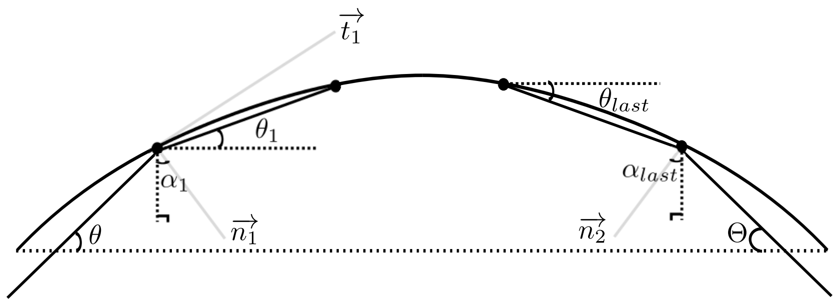

Figure 3.5: A trajectory with multiple collisions. The normal and tangent vectors to the points of collision are in gray.

trajectory collides twice. By straightforward calculations based on Figure 3.5 we obtain

$$|\theta| \leq |\theta_1| + 2|\alpha_1|$$
$$|\Theta| \leq |\theta_{last}| + 2|\alpha_{last}|.$$

Observe that $|\theta_1|, |\theta_{last}| \leq \tan^{-1}(h)$ and $|\alpha_1|, |\alpha_{last}| \leq \frac{\beta}{2} \leq \tan^{-1}(h)$. It follows that $|\Theta - \theta| \leq |\Theta| + |\theta| \leq 6\tan^{-1}(h)$.

Thus, for $\theta \in [0, \pi]$,

$$|\Theta - \theta| \leq k\tan^{-1}(h)$$

where $k = 2$ if there is only one collision and $k = 6$ if there are multiple collisions. It follows from the Taylor expansion $\tan^{-1}(h) = [h - \frac{1}{3}h^3 + ...]$ that

$$\overline{\mathcal{E}}_1^h(\theta) = E^h[|\Theta - \theta|] \leq k\tan^{-1}(h) = k[h - \frac{1}{3}h^3 + ...] = \mathcal{O}(h).$$

Similar calculations give $\overline{\mathcal{E}}_2^h(\theta) = \mathcal{O}(h^2)$ and $\overline{\mathcal{E}}_3^h(\theta) = o(h^2)$. Thus, our proof is finished. □

Recall that the Legendre polynomials, $\Phi_n(\theta) = P_n(\cos\theta)$, are eigenfunctions for $\mathcal{L}$ with eigenvalues $-\frac{1}{2}n(n+1)$. Moreover, the P_n form an orthogonal basis for $L^2((0,\pi),\nu)$. An important observation to make about the $\Phi_n(\theta)$ is that the derivative $\Phi'(\theta) = -P_n'(\cos\theta)\sin\theta$ vanishes at 0 and π, which satisfies the last assumption in Proposition 1.1.

3.3 Relating the spectra of P_h and $\mathcal{L}$

By assuming the L-weak-scattering condition and applying Proposition 1.1, we will show that the eigenvalues and eigenfunctions of the normalized billiard Laplacian $\mathcal{L}_h = \frac{1}{ch^2}(P_h - I)$ can be approximated by those of the Legendre operator $\mathcal{L}$. We start by letting Π_h denote the spectral family of the (bounded, self-adjoint) operator $\mathcal{L}_h$. Then define the spectral measure $\mu_h^{\psi,\phi}$ by

$$\mu_h^{\psi,\phi} := \langle \Pi_h(\cdot)\psi, \phi\rangle \,,$$

where ψ and ϕ are any two functions in $L^2([0,\pi],\nu)$. In particular, if ψ is a unit vector, $\mu_h^{\psi} := \mu_h^{\psi,\psi}$ is a probability measure over the spectrum of $\mathcal{L}_h$.

Proposition 3.1. *Let σ denote the spectrum of $\mathcal{L}$ and assume that the L-weak scattering condition is satisfied. Then for any two functions ψ and ϕ in $L^2([0,\pi],\nu)$, the support of $\mu_h^{\psi,\phi}$ limits to a subset of σ in the folowing sense: For every open neighborhood U of σ and every compact subset K of $\mathbb{R}$,*

$$\lim_{h\to 0} \mu_h^{\psi,\phi}(U^c \cap K) = 0.$$

In addition, for each n, $\langle \Pi_h(K)\psi, \Phi_n\rangle$ goes to 0 with h for any compact set K that does not contain $\lambda_n = -\frac{1}{2}n(n+1)$.

Proof. Let U be an open subset of $\mathbb{R}$ that contains $\left\{\lambda_n := -\frac{1}{2}n(n+1) : n = 0, 1, ...\right\}$ and I be a component interval of U^c. For each n, choose $\epsilon > 0$ such that

$|z-\lambda_n| \geq \epsilon$ for all $z \in I$. Let $I^+_{h,n}, I^-_{h,n}$ be a measurable partition of I such that $A \mapsto \pm \left\langle \Pi_h(I^{\pm}_{h,n} \cap A)\psi, \Phi_n \right\rangle$ are non-negative measures. Then

$$
\begin{aligned}
&\left|\left\langle \Pi_h(I^{\pm}_{h,n})\psi, (\mathcal{L}_h - \mathcal{L})\Phi_n \right\rangle\right| &(3.8)\\
&= \int_{I^{\pm}_{h,n}} |\langle d\Pi_h(z)\psi, (\mathcal{L}_h - \lambda_n)\Phi_n\rangle| \\
&= \int_{I^{\pm}_{h,n}} |\langle (\mathcal{L}_h - \lambda_n) d\Pi_h(z)\psi, \Phi_n\rangle| \\
&= \int_{I^{\pm}_{h,n}} |\langle (z - \lambda_n) d\Pi_h(z)\psi, \Phi_n\rangle| \\
&\geq \epsilon \left|\left\langle \Pi_h(I^{\pm}_{h,n})\psi, \Phi_n \right\rangle\right|. &(3.9)
\end{aligned}
$$

By Proposition 1.1, Expression (3.8), and hence Expression (3.9), goes to 0 with h. It follows that $\left\langle \Pi_h(I^{\pm}_{h,n})\psi, \Phi_n \right\rangle \to 0$ as $h \to 0$, so the same is true for $\langle \Pi_h(I)\psi, \Phi_n\rangle$ (thus proving the last claim in the proposition). Now given $\epsilon > 0$, let N be a positive integer such that $\sum_{m=N+1}^{\infty} |\langle \Phi_m, \phi\rangle|^2 \leq \epsilon^2$. Observe that

$$
\begin{aligned}
&\left|\mu_h^{\psi,\phi}(I) - \sum_{m=0}^{N} \langle \Pi_h(I)\psi, \Phi_m\rangle \langle \Phi_m, \phi\rangle\right| &(3.10)\\
=&\left|\langle \Pi_h(I)\psi, \phi\rangle - \sum_{m=0}^{N} \langle \Pi_h(I)\psi, \Phi_m\rangle \langle \Phi_m, \phi\rangle\right| \\
=&\left|\sum_{m=N+1}^{\infty} \langle \Pi_h(I)\psi, \Phi_m\rangle \langle \Phi_m, \phi\rangle\right| \\
\leq&\left(\sum_{m=N+1}^{\infty} |\langle \Pi_h(I)\psi, \Phi_m\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{m=N+1}^{\infty} |\langle \Phi_m, \phi\rangle|^2\right)^{\frac{1}{2}} \\
\leq&\epsilon||\psi||_2
\end{aligned}
$$

where we used the Cauchy-Schwarz inequality to obtain the fourth line and the fact that $\left(\left|\sum_{m=N+1}^{\infty}\langle\Pi_h(I)\psi,\Phi_m\rangle\right|\right)^{\frac{1}{2}} \leq \left(\left|\sum_{m=0}^{\infty}\langle\psi,\Phi_m\rangle\right|\right)^{\frac{1}{2}} = ||\psi||_2$ to establish the last inequality. Since ϵ is arbitrary, we have that Expression (3.10) goes to 0 with h. We showed above that $\langle\Pi_h(I))\psi,\Phi_m\rangle \to 0$ as $h \to 0$ for each $m = 0, 1, ..., N$, so the summation in Expression (3.10) also goes to 0 with h. By the last two statements, we have that $\lim_{h\to 0}\mu_h^{\psi,\phi}(I) = 0$. As there are finitely many such intervals I which intersect K, the first claim of the proposition follows. □

The following result is an immediate consequence of Proposition 3.1 when we recall that $\mu_h^{\psi,\psi}$ is a probability measure over the spectrum of $\mathcal{L}_h$ for any unit vector $\psi \in L^2([0,\pi],\nu)$.

Theorem 1.2. *Let σ denote the spectrum of $\mathcal{L}$. The spectrum of $\mathcal{L}_h$ converges to a subset of $\sigma \cup \{-\infty\}$ as h goes to 0.*

Theorem 1.2 also gives us information about the spectrum of P_h. Let $\sigma^{\mathcal{L}_h}$ and σ^{P_h} denote the spectra of $\mathcal{L}_h$ and P_h, respectively. Since $P_h = I + ch^2\mathcal{L}_h$, we can obtain a correspondence between $\sigma^{\mathcal{L}_h}$ and σ^{P_h} via the map

$$\begin{aligned}\sigma^{\mathcal{L}_h} &\longrightarrow \sigma^{P_h}\\ z &\longmapsto 1 + ch^2 z\end{aligned}$$

Besides the L-weak scattering condition, we need to assume one more property about $\mathcal{L}_h$ in order to further approximate its spectrum with that of $\mathcal{L}$.

Definition 3.1. *The family $\mathcal{L}_h$ has **non-dissipating spectrum** if 0 is a simple eigenvalue and for every ψ and ϕ in $L^2([0,\pi],\nu) \subset \mathbb{C}^{\perp}$*

$$\lim_{j\to\infty}\limsup_{h\to 0}\left|\mu_h^{\psi,\phi}(K_j^c)\right| = 0$$

where K_j is some increasing sequence of compact sets exhausting $(-\infty, 0)$.

We now prove three properties about open neighborhoods in $\mathbb{R}$ that only contain one eigenvalue of the Legendre operator.

Lemma 3.1. *Let the family $\mathcal{L}_h$ satisfy the L-weak scattering condition and have non-dissipating spectrum. Given an open interval neighborhood I of $-\frac{1}{2}l(l+1)$ that does not contain any other eigenvalues λ_m, the follow statements are true:*

1. $\lim\limits_{h\to 0} \langle \Pi_h(I)\psi, \phi_l\rangle = \langle \psi, \Phi_l\rangle$
2. $\lim\limits_{h\to 0} \langle \Pi_h(I^c)\psi, \phi_l\rangle = 0$
3. $\lim\limits_{h\to 0} \mu_h^{\psi}(I) = |\langle \psi, \Phi_l\rangle|^2$.

Proof. To prove the second statement, write $I^c = K \cup K^c$, where K is a large compact set. By the final claim of Proposition 3.1 and the assumption of a non-dissipative spectrum, $\langle \Pi_h(K)\psi, \phi_l\rangle$ and $\langle \Pi_h(K)\psi, \phi_l\rangle$ go to 0 with h, respectively. It follows that $\lim\limits_{h\to 0} \langle \Pi_h(I^c)\psi, \phi_l\rangle = 0$. Then the conclusion of the first statement is immediate since $\Pi_h(I) + \Pi_h(I^c)$ is the identity operator. For the third limit, given $\epsilon > 0$, choose N such that $N > l$ and $\left(\sum\limits_{m=N+1}^{\infty} |\langle \Pi_h(I)\psi, \Phi_m\rangle|^2\right)^{\frac{1}{2}} \leq \epsilon^2$. In addition, we can write

$$\langle \Pi_h(I)\psi, \psi\rangle = \sum_{m=0}^{\infty} |\langle \Pi_h(I)\psi, \Phi_m\rangle|^2 \tag{3.11}$$

due to the following calculation:

$$\begin{aligned}
&\langle \Pi_h(I)\psi, \psi\rangle \\
&= \langle \Pi_h(I)\psi, [\Pi_h(I) + \Pi_h(I^c)]\,\psi\rangle \\
&= \langle \Pi_h(I)\psi, \Pi_h(I)\psi\rangle \\
&= \langle \langle \Pi_h(I)\psi, \Phi_m\rangle\, \Phi_m, \langle \Pi_h(I)\psi, \Phi_m\rangle\, \Phi_m\rangle \\
&= \sum_{m=0}^{\infty} |\langle \Pi_h(I)\psi, \Phi_m\rangle|^2 .
\end{aligned}$$

Now using Equation (3.11) and the Cauchy-Schwarz Inequality, we obtain

$$\begin{aligned}
&\left|\mu_h^\psi(I) - \sum_{m=0}^{N} |\langle \Pi_h(I)\psi, \Phi_m\rangle|^2\right| \\
=&\left|\langle \Pi_h(I)\psi, \psi\rangle - \sum_{m=0}^{N} |\langle \Pi_h(I)\psi, \Phi_m\rangle|^2\right| \\
=&\left|\sum_{m=N+1}^{\infty} \langle \Pi_h(I)\psi, \Phi_m\rangle \left\langle \Pi_h(I)\psi, \Phi_m\right\rangle\right| \\
\leq&\left(\sum_{m=N+1}^{\infty} |\langle \Pi_h(I)\psi, \Phi_m\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{m=N+1}^{\infty} |\langle \Pi_h(I)\psi, \Phi_m\rangle|^2\right)^{\frac{1}{2}} \\
\leq&\epsilon||\psi||_2
\end{aligned}$$

where the the last inequality is established by bounding one of the square roots with our assumption and the other with $\left(\left|\sum_{m=0}^{\infty} \langle \psi, \Phi_m\rangle\right|\right)^{\frac{1}{2}} = ||\psi||_2$. Therefore,

$$\left|\mu_h^\psi(I) - |\langle \Pi_h(I)\psi, \Phi_l\rangle|^2\right| \leq \epsilon||\psi||_2 + \sum_{m\neq l, m\leq N} |\langle \Pi_h(I)\psi, \Phi_m\rangle|^2 .$$

By part 2 of the lemma, the summation on the right-hand side goes to 0 with h, implying that the left-hand side can be made arbitarily small as $h \to 0$. Combining this information with the first statement of the lemma, we may conclude that $\lim_{h\to 0} \mu_h^\psi(I) = |\langle \psi, \Phi_l\rangle|^2$. □

We are now ready to compare the spectral information of $\mathcal{L}_h$ and $\mathcal{L}$.

Proposition 3.2. *Suppose that the L-weak scattering and non-dissipating conditions are satisfied. Let* $\psi \in L^2([0,\pi],\nu)$ *and* δ_{λ_m} *be the delta measure at* $\lambda_m = -\frac{1}{2}m(m+1)$. *Then*

$$\lim_{h\to 0} \mu_h^\psi = \sum_{m=0}^{\infty} |\langle \psi, \Phi_m\rangle|^2 \delta_{\lambda_m}.$$

Moreover, let $\epsilon > 0$ *be small enought so that* $I_l^\epsilon := (\lambda_l - \epsilon, \lambda_l + \epsilon)$ *does not contain* *any eigenvalue of $\mathcal{L}$ except λ_l. Then*

$$\lim_{h\to 0} \left|\left|\Pi_h\left(I_l^\epsilon\right)\psi - \left\langle \psi, \Phi_l\right\rangle \Phi_l\right|\right|_2 = 0.$$

Proof. The first statement follows immediately from Lemma 3.1. To prove the second statement, we write

$$\begin{aligned} &\left|\left|\Pi_h\left(I_l^\epsilon\right)\psi - \left\langle \psi, \Phi_l\right\rangle \Phi_l\right|\right|_2^2 \\ = &\left(\mu_h^\psi\left(I_l^\epsilon\right) - \left|\left\langle \psi, \Phi_l\right\rangle\right|^2\right) + 2\left(\left|\left\langle \psi, \Phi_l\right\rangle\right|^2 - \left\langle \Pi_h\left(I_l^\epsilon\right)\psi, \Phi_l\right\rangle \left\langle \psi, \Phi_l\right\rangle\right), \end{aligned}$$

where the two differences in the second line go to 0 with h due to part 3 and part 1 of Lemma 3.1, respectively. The final limit in the proposition is then a direct consequence. $\square$

Chapter 4

The Billard and Legendre Operators in N Dimensions

4.1 Billiard Families in N Dimensions

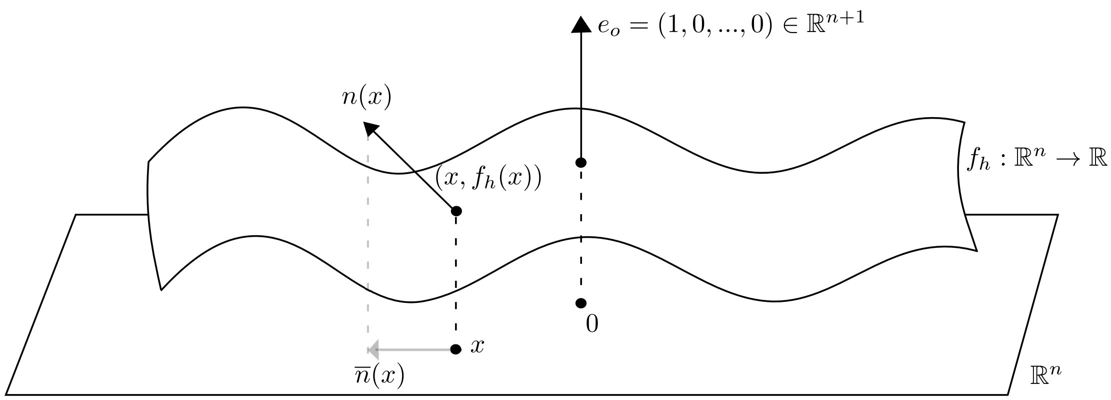

Figure 4.1: An n+1-dimensional billiard surface. Let $n(x)$ and $\overline{n}(x)$ denote the unit normal vector to the graph of f at $x \in \mathbb{R}^n$ and its orthogonal projection to $\mathbb{R}^n$, respectively.

Let $\{e_i\}_{i=0}^{n}$ be the standard basis for $\mathbb{R}^n$. Consider a parametric billiard family consisting of microstucture cells whose shapes are given by functions $f_h : \mathbb{R}^n \to \mathbb{R}$ (see Figure 4.1) with the following two properties:

1. **periodicity:** $f_h(x + \sum_i m_i a_i e_i) = f(x),\ a_i > 0,\ m_i \in \mathbb{Z}$

2. **symmetry** $f_h(-x) = f_h(x)$.

We define the parameter h by $h = \sup_{x \in \mathbb{T}^n} ||\overline{n}(x)||$. Observe that we may write

$$n(x) = \overline{n}(x) + n_0(x)e_0 = \frac{e_0 - grad_x f_h}{\sqrt{1 + ||grad_x f_h||^2}}. \tag{4.1}$$

It follows from Equation 4.1 that $n_0 = \frac{1}{\sqrt{1+||grad_x f_h||^2}} = \sqrt{1 + ||\overline{n}||^2}$. Now fix a constant $c > \sup_{x \in \mathbb{T}^n} ||f_h(x)||$ and consider intial velocities v for trajectories that intersect the horizontal plane at height c in $\mathbb{R}^{n+1}$ at point r and collide with the billiard surface (i.e. graph of $f_h(x)$). We denote the after-collision velocities by V (see Figure 4.2).

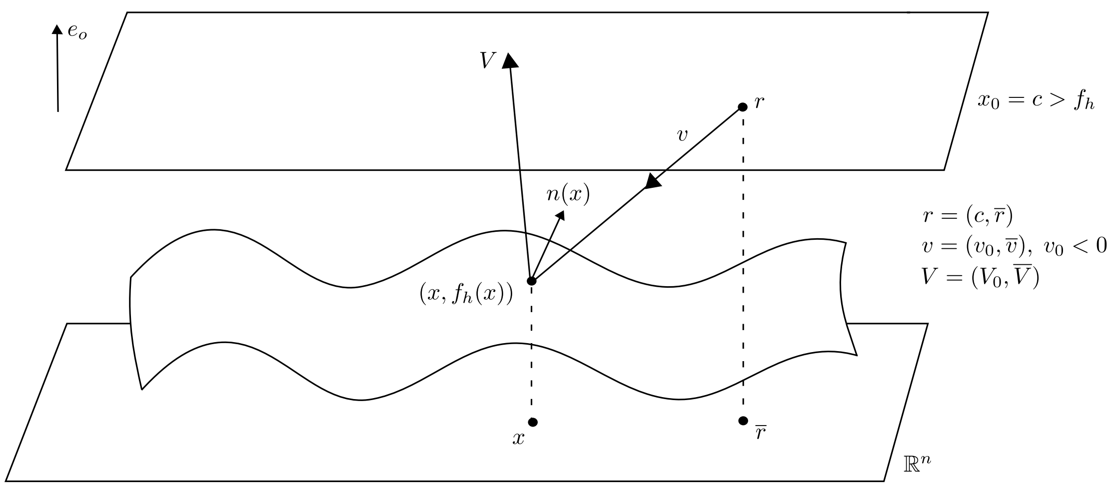

Figure 4.2: A collision of a point particle with a n+1-dimensional billiard surface. $\overline{r}, \overline{v}$, and $\overline{V}$ are the orthogonal projections of r, v, and V. We take $\overline{v}$ to be a vector in the open unit disc $\mathbb{D}^n$.

A calculation based on Figure 4.2 shows that for single collisions, $\frac{\overline{r}-x}{f_h(x)-c} = -\frac{\overline{v}}{v_0}$. This implies $\overline{r} = x - (f_h(x) - c)\frac{\overline{v}}{v_0}$, and it follows that

$$d\overline{r}_x = I - (df_h)_x \otimes \frac{\overline{v}}{v_0}.$$

Using this equation and the matrix determinant formula [7], we have that

$$det(d\overline{r}_x) = 1 - (df_h)_x \left(\frac{\overline{v}}{v_0}\right).$$

From Equation 4.1 one can obtain the relation $grad_x f_h = -\frac{\overline{n}(x)}{n_0(x)}$, so $(df_h)_x \left(\frac{\overline{v}}{v_0}\right) = \left\langle -\frac{\overline{n}(x)}{n_0(x)}, \frac{\overline{v}}{v_0} \right\rangle$. Then

$$det(d\overline{r}_x) = 1 + \left\langle \frac{\overline{n}(x)}{n_0(x)}, \frac{\overline{v}}{v_0} \right\rangle. \tag{4.2}$$

We are now ready to find an equation for the billiard operator P_h. Let $\Phi \in L^2(\mathbb{D}^n, dx)$, where dx is the normalized Lebesgue measure. Since the reflection law states that the post-collision velocity $V(r,v)$ is given by $V = v - 2 < n, v > n$, the following also holds:

$$\overline{V}(\overline{r}, \overline{v}) = \overline{v} - 2 < n, v > \overline{n}. \tag{4.3}$$

For $\overline{v}$ that result in only one collision, we can use Equations 4.3 and 4.2 to show that the billiard operator is given by

$$\begin{aligned}
&(P_h\Phi)(\overline{v}) \\
&= \int_{\mathbb{T}^n} \Phi(\overline{V}(\overline{r}, \overline{v}))\, d\overline{r} \\
&= \int_{\mathbb{T}^n} \Phi(\overline{v} - 2 < n, v > \overline{n})\, d\overline{r} \\
&= \int_{\mathbb{T}^n} \Phi(\overline{v} - 2(< \overline{n}(x), \overline{v} > + n_0(x)v_0)\overline{n}(x)) \left(1 + \left\langle \frac{\overline{n}(x)}{n_0(x)}, \frac{\overline{v}}{v_0} \right\rangle\right) dx.
\end{aligned}$$

The symmetry of $f_h(x)$ means $\overline{n}(-x) = -\overline{n}(x)$ and $n_0(-x) = n_0(x)$. Using these two properties, we can expand the integral formula for P_h as

$$
\begin{aligned}
&(P_h\Phi)(\overline{v})\\
&= \frac{1}{2}\int_{\mathbb{T}^n}\Bigg[\Phi\Bigg(\overline{v}-2(<\overline{n},\overline{v}>+n_0v_0)\overline{n}\Bigg)\left(1+\left\langle\frac{\overline{n}(x)}{n_0(x)},\frac{\overline{v}}{v_0}\right\rangle\right)\\
&\qquad\qquad\qquad\qquad +\\
&\qquad \Phi\Bigg(\overline{v}+2(-<\overline{n},\overline{v}>+n_0v_0)\overline{n}\Bigg)\left(1-\left\langle\frac{\overline{n}(x)}{n_0(x)},\frac{\overline{v}}{v_0}\right\rangle\right)\Bigg]\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{T}^n}\Bigg[\frac{\Phi(\overline{v}-2<\overline{n},\overline{v}>\overline{n}-2n_0v_0\overline{n})+\Phi(\overline{v}-2<\overline{n},\overline{v}>\overline{n}+2n_0v_0\overline{n})}{2} \qquad (4.4)\\
&\qquad\qquad\qquad\qquad +\\
&\frac{\Phi(\overline{v}-2<\overline{n},\overline{v}>\overline{n}-2n_0v_0\overline{n})-\Phi(\overline{v}-2<\overline{n},\overline{v}>\overline{n}+2n_0v_0\overline{n})}{2}\left\langle\frac{\overline{n}(x)}{n_0(x)},\frac{\overline{v}}{v_0}\right\rangle\Bigg]\,dx.
\end{aligned}
$$

Consider the second-order Taylor approximation of $\Phi(\overline{V})$ around $\overline{v}$:

$$\Phi(\overline{V}) = \Phi(\overline{v}) + d\Phi_{\overline{v}}(\overline{V}-\overline{v}) + \frac{1}{2}d^2\Phi_{\overline{v}}(\overline{V}-\overline{v}) + R_2(\overline{v},\overline{V}),$$

where

$$R_2(\overline{v},\overline{V}) = \frac{1}{2}\int_{U_{\overline{v},\overline{V}}} d^3\Phi_{\overline{v}}(\overline{v}-t)(\overline{V}-t)^2\,dt$$

is the remainder term and $U_{\overline{v},\overline{V}}$ is open set of $\mathbb{D}^n$ containing the set of vectors whose length is between $||\overline{v}||$ and $||\overline{V}||$. Recalling that $(P_h\Phi)(\overline{v}) = \int_{\mathbb{T}^n}\Phi(\overline{V}(\overline{r},\overline{v}))\,d\overline{r}$, we obtain

$$(P_h\Phi)(\overline{v}) - \Phi(\overline{v}) = \int_{\mathbb{T}^n} d\Phi_{\overline{v}}(\overline{V}-\overline{v})\,d\overline{r}_x + \int_{\mathbb{T}^n}\frac{1}{2}d^2\Phi_{\overline{v}}(\overline{V}-\overline{v})\,d\overline{r}_x + \int_{\mathbb{T}^n} R_2(\overline{v},\overline{V})\,d\overline{r}_x.$$

Equation 4.4 can be used to find $\int_{\mathbb{T}^n} d\Phi_{\overline{v}}(\overline{V} - \overline{v})\, d\overline{r}_x$ and $\int_{\mathbb{T}^n} \frac{1}{2} d^2\Phi_{\overline{v}}(\overline{V} - \overline{v})\, d\overline{r}_x$.

$$
\begin{aligned}
& d\Phi_{\overline{v}}(\overline{V} - \overline{v}) \\
&= \frac{1}{2}\Bigg[d\Phi_{\overline{v}}(-2 < \overline{n}, \overline{v} > \overline{n} - 2n_0 v_0 \overline{n}) + d\Phi_{\overline{v}}(-2 < \overline{n}, \overline{v} > \overline{n} + 2n_0 v_0 \overline{n}) \\
&\qquad + \\
&\Big(d\Phi_{\overline{v}}(-2 < \overline{n}, \overline{v} > \overline{n} - 2n_0 v_0 \overline{n}) - d\Phi_{\overline{v}}(-2 < \overline{n}, \overline{v} > \overline{n} + 2n_0 v_0 \overline{n}) \Big) \left\langle \frac{\overline{n}(x)}{n_0(x)}, \frac{\overline{v}}{v_0} \right\rangle \Bigg] \\
&= d\Phi_{\overline{v}}\Bigg[-2 < \overline{n}, \overline{v} > \overline{n} - 2n_0 v_0 \overline{n} \left\langle \frac{\overline{n}(x)}{n_0(x)}, \frac{\overline{v}}{v_0} \right\rangle \Bigg] \\
&= -4 < \overline{n}, \overline{v} > d\Phi_{\overline{v}} \\
&= -4 d\Phi_{\overline{v}} \circ (\overline{n}(x)^* \otimes \overline{n}(x))\, (\overline{v})
\end{aligned}
$$

Therefore,

$$
\int_{\mathbb{T}^n} d\Phi_{\overline{v}}(\overline{V} - \overline{v})\, d\overline{r}_x = -4 d\Phi_{\overline{v}} \left[\int_{\mathbb{T}^n} \overline{n}(x)^* \otimes \overline{n}(x)\, dx \right] \overline{v}. \tag{4.5}
$$

Furthermore,

$$
\begin{aligned}
&\frac{1}{2}d^2\Phi_{\overline{v}}(\overline{V}-\overline{v})\\
&= d^2\Phi_{\overline{v}}(\overline{n},\overline{n})(-2<\overline{n},\overline{v}>-2n_0v_0)^2\\
&\quad + d^2\Phi_{\overline{v}}(\overline{n},\overline{n})(-2<\overline{n},\overline{v}>+2n_0v_0)^2\\
&\quad + d^2\Phi_{\overline{v}}(\overline{n},\overline{n})(-2<\overline{n},\overline{v}>-2n_0v_0)^2\left\langle \frac{\overline{n}}{n_0},\frac{\overline{v}}{v_0}\right\rangle\\
&\quad + d^2\Phi_{\overline{v}}(\overline{n},\overline{n})(-2<\overline{n},\overline{v}>+2n_0v_0)^2\left\langle \frac{\overline{n}}{n_0},\frac{\overline{v}}{v_0}\right\rangle\\
&= d^2\Phi_{\overline{v}}(\overline{n},\overline{n})\left[2<\overline{n},\overline{v}>^2+2n_0^2v_0^2+\left\langle \frac{\overline{n}}{n_0},\frac{\overline{v}}{v_0}\right\rangle\left(4<\overline{n},\overline{v}>n_0v_0\right)\right]\\
&= d^2\Phi_{\overline{v}}(\overline{n},\overline{n})\left[6<\overline{n},\overline{v}>^2+2(1-||\overline{n}||^2)(1-||\overline{v}||^2)\right]\\
&= 2(1-||\overline{v}||^2)d^2\Phi_{\overline{v}}(\overline{n},\overline{n})+\left(6<\overline{n},\overline{v}>^2-2||\overline{n}||^2(1-||\overline{v}||^2)\right)d^2\Phi_{\overline{v}}(\overline{n},\overline{n})\\
&= 2(1-||\overline{v}||^2)d^2\Phi_{\overline{v}}(\overline{n},\overline{n})+\mathcal{O}(||\overline{n}||^4).
\end{aligned}
$$

Therefore,

$$
\int_{\mathbb{T}^n}\frac{1}{2}d^2\Phi_{\overline{v}}(\overline{V}-\overline{v}) = 2(1-||\overline{v}||^2)d^2\Phi_{\overline{v}}\left[\int_{\mathbb{T}^n}\overline{n}(x)\otimes\overline{n}(x)\,dx\right]+\mathcal{O}(h^4). \tag{4.6}
$$

We can write

$$
\begin{aligned}
A &:= \int_{\mathbb{T}^n}\overline{n}(x)^*\otimes\overline{n}(x)\,dx = \sum_{i,j}a_{i,j}e_i^*\otimes e_j\\
B &:= \int_{\mathbb{T}^n}\overline{n}(x)\otimes\overline{n}(x)\,dx = \sum_{i,j}a_{i,j}e_i\otimes e_j,
\end{aligned}
$$

and observe that A is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ which is self-adjoint and non-negative definite. Note that A and B depend on h, so we can denote them as $A(h)$ and $B(h)$. However, to simplify notation, we will continue to refer to them as A and B. Then

$$d\Phi_{\overline{v}}(A\overline{v}) = \sum_{i,j} a_{i,j} < \overline{v}, e_i > d\Phi_{\overline{v}}(e_j)$$
$$d^2\Phi_{\overline{v}}(B) = \sum_{i,j} a_{i,j} d^2\Phi_{\overline{v}}(e_i, e_j),$$

so if $x_1, ..., x_n$ is the orthonormal eigenbasis of R^n associated with A, then $A = \sum_{i=1}^{n} \lambda_i u_i^* \otimes u_i$, where the $\lambda_i' s$ denote the eigenvalues of A. Thus, for intial velocities $\overline{v} \in \mathbb{D}^n$ that cause only one collision,

$$\begin{aligned}
&\int_{\mathbb{T}^n} d\Phi_{\overline{v}}(\overline{V} - \overline{v})\, d\overline{r}_x + \int_{\mathbb{T}^n} \frac{1}{2} d^2\Phi_{\overline{v}}(\overline{V} - \overline{v})\, d\overline{r}_x \\
=& -4d\Phi_{\overline{v}}(A\overline{v}) + 2(1 - ||\overline{v}||^2) d^2\Phi_{\overline{v}}(B) + \mathcal{O}(h^4) \\
=& -4\sum_{i=1}^{n} \lambda_i < \overline{v}, x_i > d\Phi_{\overline{v}}(x_i) + 2(1 - ||\overline{v}||^2) \sum_{i=1}^{n} \lambda_i d^2\Phi_{\overline{v}}(x_i, x_i) + \mathcal{O}(h^4) \\
=& 2\sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial x_i}(1 - ||\overline{v}||^2)\frac{\partial \Phi}{\partial x_i} + \mathcal{O}(h^4),
\end{aligned} \tag{4.7}$$

where the second equality is due to Equations 4.5 and 4.6.

4.2 The Convergence of $\mathcal{L}_h$ to $\mathcal{L}$

The convergence of $\mathcal{L}_h$ to $\mathcal{L}$ in the one-dimensional case holds true in higher dimensions given that the maximum number of collisions before a particle exits the billiard surface is finite. For the n-dimensional case, we define $\mathcal{L}_h$ as

$$\mathcal{L}_h = \frac{P_h - I}{2h^2}$$

and take inspiration from Equation 4.7 to define $\mathcal{L}$ as

$$\mathcal{L}\Phi = \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i}(1 - ||\overline{v}||^2)\frac{\partial \Phi}{\partial x_i},$$

where $c_i = \lim_{h \to 0} \dfrac{\lambda_i(h)}{h^2}$ and the λ_i's are the eigenvalues of the operator A associated to the particular f_h we are examining. Note that this limit will exist in most of the surfaces we consider.

Before we are ready to prove the convergence of $\mathcal{L}_h$ to $\mathcal{L}$, we need the following lemma.

Lemma 4.1. *Suppose the microstructure of a billiard table is given by the graph of f_h, which has the properties of symmetry and periodicity as described at the beginning of the chapter. Let $\overline{v} \in \mathbb{D}^n$ be an intial velocity of a trajectory that has at most k collisions before exiting the billiard cell. Then $\exists\, M > 0$ such that*

$$||\overline{V} - \overline{v}|| \leq 2kMh^2.$$

Proof. Let $\overline{v}^i$ be the velocity vectors after the i^{th} collision. Note that by Equation 4.1, $grad_x f_h = -\frac{\overline{n}(x)}{n_0(x)}$, so

$$\left|\frac{v_0^i}{\overline{v}^i}\right| \leq ||grad f_h|| \leq \frac{||\overline{n}(x_i)||}{|n_0(x_i)|}.$$

Then

$$|v_0^i| \leq \frac{||\overline{n}(x_i)||}{|n_0(x_i)|}||\overline{v}^i|| \leq \frac{||\overline{n}(x_i)||}{|n_0(x_i)|} \leq m_i||\overline{n}(x_i)||$$

for some constant $m_i > 0$. This implies that $\exists\, M > 0$ such that for any i,

$$\begin{aligned}
&\left\langle v^i, n(x_i)\right\rangle \\
=&\left\langle \overline{v}^i, \overline{n}(x_i)\right\rangle + \left\langle v_0^i, n_0(x_i)\right\rangle \\
\leq& M \sup_{x\in\mathbb{T}^n} ||\overline{n}(x)|| \\
=& Mh.
\end{aligned}$$

Consider the relation

$$\overline{v}^i = \overline{v}^{i-1} - 2 < v^i, n(x_i) > \overline{n}(x_i).$$

Noting that $||\overline{n}(x_i)|| \leq h$ for any i, it follows from the relation that $||\overline{V} - \overline{v}|| \leq 2kMh^2$. ☐

Theorem 1.4. *Assume that a billiard scattering system with perturbation parameter h has microstructure given by the graph of a symmetric and periodic function $f_h(x)$. Let $h = \sup_{x\in\mathbb{T}^n} ||\overline{n}(x)||$, where $\overline{n}(x)$ is the orthogonal projection to $\mathbb{R}^n$ of the unit normal to the graph of $f_h(x)$. Moreover, we assume that each billiard particle collides with the walls of a cell surface at most k times before exiting for some $k > 0$. Let $\Phi \in C^3(\mathbb{D}^n)$ and denote the eigenvalues of the self-adjoint operator $A(h)$ by $\lambda_i(h)$. Suppose $c_i = \lim_{h\to 0} \dfrac{\lambda_i(h)}{h^2}$ exists for $1 \leq i \leq n$. Then $\lim_{h\to 0} ||\mathcal{L}_h\Phi - \mathcal{L}\Phi||_2 = 0$.*

Proof. Let Φ be a C^3 function with bounded derivatives up to the third order. Consider the second-order Taylor approximation of $\Phi(\overline{V})$ around $\overline{v}$:

$$\Phi(\overline{V}) = \Phi(\overline{v}) + d\Phi_{\overline{v}}(\overline{V} - \overline{v}) + \frac{1}{2} d^2\Phi_{\overline{v}}(\overline{V} - \overline{v}) + R_2(\overline{v}, \overline{V}),$$

where

$$R_2(\overline{v},\overline{V}) = \frac{1}{2}\int_{U_{\overline{v},\overline{V}}} d^3\Phi_{\overline{v}}(\overline{v}-t)(\overline{V}-t)^2\,dt$$

is the remainder term and $U_{\overline{v},\overline{V}}$ is open set of $\mathbb{D}^n$ containing the set of vectors whose length is between $||\overline{v}||$ and $||\overline{V}||$. Recalling that $(P_h\Phi)(\overline{v}) = \int_{\mathbb{T}^n}\Phi(\overline{V}(\overline{r},\overline{v}))\,d\overline{r}$, we obtain

$$(P_h\Phi)(\overline{v}) - \Phi(\overline{v}) = \int_{\mathbb{T}^n} d\Phi_{\overline{v}}(\overline{V}-\overline{v})\,d\overline{r}_x + \frac{1}{2}\int_{\mathbb{T}^n} d^2\Phi_{\overline{v}}(\overline{V}-\overline{v})\,d\overline{r}_x + \int_{\mathbb{T}^n} R_2(\overline{v},\overline{V})\,d\overline{r}_x.$$

By Lemma 4.1, $\left|R_2(\overline{v},\overline{V})\right| \le \frac{1}{2}||d^3\Phi_{\overline{v}}||||\overline{V}-\overline{v}||^3 = o(h^2)$, so

$$\left|((P_h\Phi)(\overline{v}) - \Phi(\overline{v})) - \left(\int_{\mathbb{T}^n} d\Phi_{\overline{v}}(\overline{V}-\overline{v})\,d\overline{r}_x + \frac{1}{2}\int_{\mathbb{T}^n} d^2\Phi_{\overline{v}}(\overline{V}-\overline{v})\,d\overline{r}_x\right)\right| \le o(h^2).$$

Dividing through by $2h^2$ and taking limits results in

$$\lim_{h\to 0}\left|(\mathcal{L}_h\Phi)(\overline{v}) - \frac{1}{2h^2}\left(\int_{\mathbb{T}^n} d\Phi_{\overline{v}}(\overline{V}-\overline{v})\,d\overline{r}_x + \frac{1}{2}\int_{\mathbb{T}^n} d^2\Phi_{\overline{v}}(\overline{V}-\overline{v})\,d\overline{r}_x\right)\right| = 0$$

for all $\overline{v}\in\mathbb{D}^n$. For convenience, define

$$(\overline{\mathcal{L}}_h\Phi)(\overline{v}) := \frac{1}{2h^2}\left(\int_{\mathbb{T}^n} d\Phi_{\overline{v}}(\overline{V}-\overline{v})\,d\overline{r}_x + \frac{1}{2}\int_{\mathbb{T}^n} d^2\Phi_{\overline{v}}(\overline{V}-\overline{v})\,d\overline{r}_x\right).$$

We have just shown that $|\mathcal{L}_h\Phi(\overline{v}) - \overline{\mathcal{L}}_h\Phi(\overline{v})|$ is bounded and converges to 0 pointwise as h goes to 0, which implies that

$$\lim_{h\to 0}||\mathcal{L}_h\Phi(\overline{v}) - \overline{\mathcal{L}}_h\Phi(\overline{v})||_2 = 0.$$

We are done if we show that $||\overline{\mathcal{L}}_h\Phi(\overline{v}) - \mathcal{L}\Phi(\overline{v})||_2$ goes to 0 for each $\Phi\in C^3(\mathbb{D}^n)$.

In order to accomplish this, we write

$$||\overline{\mathcal{L}}_h\Phi(\overline{v}) - \mathcal{L}\Phi(\overline{v})||_2^2 = \mathcal{J}_1 + \mathcal{J}_2,$$

where we define $\mathcal{J}_1$ and $\mathcal{J}_2$ by

$$\mathcal{J}_1 := \int_{U_h} |\overline{\mathcal{L}}_h\Phi(\overline{v}) - \mathcal{L}\Phi(\overline{v})|^2 \, d\overline{r}_x,$$
$$\mathcal{J}_2 := \int_{U_h^c} |\overline{\mathcal{L}}_h\Phi(\overline{v}) - \mathcal{L}\Phi(\overline{v})|^2 \, d\overline{r}_x.$$

Here, $U_h \subset \mathbb{D}^n$ is the set of initial velocities that result in only 1 collision. Note the U_h is getting larger as h goes to zero and the billiard surface becomes flatter. In particular, $U_h = \mathbb{D}^n$ when $h = 0$. It is immediate from Equation 4.7 that $\mathcal{J}_1 \to 0$ as $h \to 0$. For $\mathcal{J}_2$, using the definition of $\overline{\mathcal{L}}_h\Phi(\overline{v})$, we have

$$\int_{U_h^c} |\overline{\mathcal{L}}_h\Phi|^2 \, d\overline{r}_x \leq \frac{1}{4}\int_{U_h^c} \left(\frac{\int_{\mathbb{T}^n} d\Phi_{\overline{v}}(\overline{V} - \overline{v}) \, d\overline{r}_x}{h^2} + \frac{\int_{\mathbb{T}^n} d^2\Phi_{\overline{v}}(\overline{V} - \overline{v}) \, d\overline{r}_x}{2h^2} \right)^2 d\overline{r}_x.$$

Consider the two terms under the square of the integrand on the right-hand side. By Lemma 4.1,

$$\int_{\mathbb{T}^n} d\Phi_{\overline{v}}(\overline{V} - \overline{v}) \, d\overline{r}_x \leq ||d\Phi_{\overline{v}}|| \, ||\overline{V} - \overline{v}|| = \mathcal{O}(h^2)$$
$$\text{and} \int_{\mathbb{T}^n} d^2\Phi_{\overline{v}}(\overline{V} - \overline{v}) \, d\overline{r}_x \leq ||d^2\Phi_{\overline{v}}|| \, ||\overline{V} - \overline{v}||^2 = \mathcal{O}(h^4).$$

It follows that the first term under the square is bounded and the second is of order $\mathcal{O}(h^2)$. Since U_h^c shrinks to the empty set as h goes to zero, it follows that

$$\lim_{h\to 0} \int_{U_h^c} |\overline{\mathcal{L}}_h\Phi|^2 \, d\overline{r}_x = 0. \tag{4.8}$$

Now consider $\int_{U_h^c} |\mathcal{L}\Phi|^2 \, d\overline{r}_x$. Since the coefficients of $\mathcal{L}$ are bounded over the disc, $\mathcal{L}\Phi$ is bounded over the disc since Φ belongs in $C^3(\mathbb{D}^n)$ and has bounded derivatives. Combined with the fact that U_h^c's volume shrinks to zero as h goes to zero, we obtain

$$\lim_{h \to 0} \int_{U_h^c} |\mathcal{L}_h \Phi|^2 \, d\overline{r}_x = 0. \tag{4.9}$$

By Equations 4.8 and 4.9, we conclude that $\mathcal{J}_2$ also goes to zero, thereby completing the proof. $\square$

4.3 The Eigenfunctions of $\mathcal{L}$

Recall that the Legendre polynomials $\{P_k\}_{k=0,1,...}$ are the eigenfunctions (say, with associated eigenvalues $\{\lambda_k\}_{k=0,1,...}$) of the Legendre operator. We can use this information to find the eigenvalues and eigenfunctions of the n-dimensional $\mathcal{L}$. Moreover, we can show that the eigenfunctions that we find will form a basis for $L^2(\mathbb{D}^n, dx)$.

Let $\Pi_i : [-1,1]^n \longrightarrow [-1,1]$ be the i^{th} coordinate projection map. Then for a multi-index $m = (m_1, ..., m_n) \in \mathbb{N}^n$, we define Φ_m as $\Phi_m := \prod_{i=1}^{n} P_{m_i} \circ \Pi_i|_{\mathbb{D}^n}$.

Theorem 1.5. $\left\{\Phi_m\right\}_{m \in \mathbb{N}^n}$ *is a complete family of eigenfunctions for $\mathcal{L}$ and forms a basis for $L^2(\mathbb{D}^n, dx)$.*

Proof. With the definition of $\mathcal{L}$ in mind, let $\rho = 1 - ||\overline{v}||^2$ and $\mathcal{L}_i\phi = \frac{\partial}{\partial x_i}(1 - ||\overline{v}||^2)\frac{\partial \phi}{\partial x_i}$. Consider ϕ and ψ, two smooth functions on $\mathbb{D}^n \subset [-1,1]^n$ such that

$\phi_i\psi_i = 0$ for $i = 1, ..., n$. Then

$$\begin{aligned}
\mathcal{L}(\phi\psi) =& \sum_{i=1}^{n} c_i \Big[\rho_i(\phi\psi)_i + \rho(\phi\psi)_{ii} \Big] \\
=& \sum_{i=1}^{n} \Big[(\rho_i\phi_i\psi + \rho\phi_{ii}\psi) + (\rho_i\psi_i\phi + \rho\psi_{ii}\phi) + (2\rho\phi_i\psi_i) \Big] \\
=& \sum_{i=1}^{n} c_i(\mathcal{L}_i\phi)\psi + \sum_{i=1}^{n} c_i\phi(\mathcal{L}_i\psi) + 0 \\
=& (\mathcal{L}\phi)\psi + \phi(\mathcal{L}\psi).
\end{aligned}$$

This implies that if we let $\phi_k^i := P_k \circ \Pi_i|_{\mathbb{D}^n}$ and $\Phi_m := \prod_{i=1}^{n} \phi_{m_i}^i$ (where $m = (m_1, ..., m_n) \in \mathbb{N}^n$), then

$$\mathcal{L}\Phi_m = \sum_{i=1}^{n} \phi_{m_1}^1 \cdots \mathcal{L}\phi_{m_i}^i \cdots \phi_{m_n}^n = \left(\sum_{i=1}^{n} c_i\lambda_{m_i} \right) \Phi_m.$$

This shows that $\Big\{ \Phi_m \Big\}_{m\in\mathbb{N}^n}$ is a complete set of eigenfunctions for $\mathcal{L}$ with associated eigenvalues $\Big\{ \sum_{i=1}^{n} c_i\lambda_{m_i} \Big\}$. Therefore, we have proved the first claim. For the second part, we need to show that *span* $\Big\{ \Phi_m : m \in \mathbb{N}^n \Big\}$ is dense in $L^2(\mathbb{D}^n, dx)$. Let Ψ be a function in $L^2(\mathbb{D}^n, dx)$. It can be approximated by $\tilde{\Psi}|_{\mathbb{D}^n}$, where $\tilde{\Psi}$ is a continuous function on $[-1, 1]^n$. (Recall that the continuous functions are dense in $[-1, 1]^n$.) By the Weierstrass Approximation Theorem, $\tilde{\Psi}$ can be uniformly approximated by a polynomial $\sum_{|m|\leq N} a_m x^m$, where $x^m = \prod_{i=1}^{n} x_i^{m_i}$. Now each $x_i^{m_i}$ can be written as a linear combination of Legendre polynomials, i.e. $x_i^{m_i} = \sum_{k=0}^{N_i} b_k P_k(x_i)$, because $\Big\{ P_k \Big\}_{k=0,1,...}$ is basis in the 1-dimensional L^2 space. It follows that *span* $\Big\{ \Phi_m : m \in \mathbb{N}^n \Big\}$ is dense in $L^2(\mathbb{D}^n, dx)$.

□

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