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# Coordinates Arising from Affine Fibrations

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# Chapter 1

# Introduction

Let $R$ (along with all other rings throughout) be a commutative Noetherian ring. We will denote the set of units in $R$ by $R^*$, and use $R^{[n]}$ to denote the polynomial ring in $n$ variables over $R$. By $GA_n(R)$, we denote the general automorphism group $\mathrm{Aut}_{\mathrm{Spec}\, R}(\mathrm{Spec}\, R^{[n]})$. This group is naturally anti-isomorphic to the group $\mathrm{Aut}_R\, R^{[n]}$. We will abuse this correspondence somewhat; given $\phi \in GA_n(R)$, and $f \in R^{[n]}$, we will write $\phi(f) \in R^{[n]}$. We will use $\circ$ for multiplication in $GA_n(R)$, so given $\phi_1 = (F_1, \ldots, F_n), \phi_2 = (G_1, \ldots, G_n) \in GA_n(R)$, we have the natural composition $\phi_1 \circ \phi_2 = (F_1(G_1, \ldots, G_n), \ldots, F_n(G_1, \ldots, G_n))$.

$GA_n(R)$ contains many important subgroups. Among those of interest to us are

- $GL_n(R)$, the group of all *linear* automorphisms (each component is homogeneous of degree one)
- $Af_n(R)$, the group of all *affine* automorphisms (each component is of degree one)
- $GA_n^k(R)$ ($k \in \mathbb{N}$), the *tangent preserving* automorphism groups. This group consists of all automorphisms of the form $\phi = (F_1, \ldots F_n)$ where $F_i = X_i + P_i$, and $P_i$ has order[1] at least $k+1$.

- $EA_n(R)$, the group generated by *elementary* automorphisms (automorphisms that fix $n-1$ variables)

- $EA_n^k(R) = EA_n(R) \cap GA_n^k(R)$ ($k \in \mathbb{N}$), the elementary tangent preserving groups

- $TA_n(R)$, the group of *tame* automorphisms, generated by $GL_n(R)$ and $EA_n(R)$. Equivalently, it is generated by $Af_n(R)$ and $EA_n^1(R)$.

We are also sometimes interested in $MA_n(R) := \mathrm{End}_{\mathrm{Spec}\, R}(\mathrm{Spec}\, R^{[n]}) \cong (R^{[n]})^n$, the monoid of all algebraic endomorphisms of $\mathrm{Spec}\, R^{[n]}$. We of course then have $GA_n(R) \subset MA_n(R)$.

The structure of $GA_n(R)$ is a question where surprisingly little is known. The question of how to identify and construct automorphisms has been the subject of much study over the last half century, but not much is known beyond $n = 2$.

**Definition 1.1.** A set of polynomials $f_1, \ldots, f_m \in R^{[n]}$ is called a *partial system of coordinates* if there exist $g_{m+1}, \ldots, g_n \in R^{[n]}$ such that $(f_1, \ldots, f_m, g_{m+1}, \ldots, g_n) \in GA_n(R)$. When $m = 1$, $f_1$ is simply called a *coordinate*.

We can now state precisely the overarching question motivating much of the work in the field of affine algebraic geometry.

**Question 1.1.** *Given $f_1, \ldots, f_m \in R^{[n]}$, what conditions are sufficient to guarantee that $f_1, \ldots, f_m$ is a partial system of coordinates?*

At one extreme, $m = n$, lies the famous Jacobian Conjecture, first posed by Keller (for $n = 2$ and $R = \mathbb{Z}$) in 1939:

**Jacobian Conjecture.** *Let $f_1, \ldots, f_n \in R^{[n]}$. $(f_1, \ldots, f_n) \in GA_n(R)$ if and only if the Jacobian determinant is a unit, i.e.* $\det(\frac{\partial f_i}{\partial x_j}) \in R^*$.

[1]The *order* of a polynomial $f$ is the minimum of the degrees of the monomials with nonzero coefficients appearing in $f$. We will denote this by $\mathrm{ord}\, f$.

This is false in positive characteristic; in characteristic zero it is known only for $n = 1$. When $n = 2$, although the Jacobian Conjecture is unknown, we do have the following:

**Theorem 1.1** (Jung-van der Kulk)**.** *Let $k$ be a field. Then $TA_2(k) = GA_2(k)$.*

Elements of $TA_2(k)$ are algorithmically recognizable (this can be seen from the Abhyankar-Moh-Suzuki epimorphism theorem), so this gives a very nice structure on $GA_2(k)$. However, in 2001 Shestakov and Umirbaev [17] showed that while elements of $TA_3(\mathbb{C})$ are algorithmically recognizable, $TA_3(\mathbb{C}) \neq GA_3(\mathbb{C})$.

At the other extreme, $m = 1$, an important idea is the notion of hyperplanes In this case, a single polynomial $f \in R^{[n]}$ is called a *hyperplane* if $R^{[n]}/f \cong R^{[n-1]}$.

**Embedding Conjecture** (Abhyankar-Sathaye)**.** *Hyperplanes are coordinates*

This also fails in positive characteristic; it was proven over a field of characteristic zero with $n = 2$ by Abhyankar and Moh [1], and Suzuki [19] independently. This was later generalized for $R$ a polynomial ring in a single variable over a field of characteristic zero (with $n = 2$) by Russell and Sathaye [14]. It remains open for $n \geq 3$.

For arbitrary $m$, we require the notion of *affine fibrations*.

**Definition 1.2.** An inclusion of $R$-algebras $A \hookrightarrow B$ is called an *affine fibration* or an $\mathbb{A}^r$*-fibration* if

1. B is flat over A
2. B is finitely generated over $A$
3. $B \otimes_A \kappa(\mathfrak{p}) \cong \kappa(\mathfrak{p})^{[r]}$ for each $\mathfrak{p} \in \operatorname{Spec} A$ (where $\kappa(\mathfrak{p})$ denotes the residue field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$)

*Remark* 1.1. If $A, B$ are both polynomial rings over a field, then one only needs to check the third condition to see that $A \hookrightarrow B$ is an affine fibration (c.f. [6] Lemma 2.1).

The simplest examples of affine fibrations are polynomial rings $A \hookrightarrow A^{[r]}$. Dolgachev and Weisfeiler, working in the context of group schemes [22], were led to conjecture

**Dolgachev-Weisfeiler Conjecture.** *Let $k$ be a field and $B = k^{[t]}$. Then $A \hookrightarrow B$ is an $\mathbb{A}^r$-fibration if and only if $B = A^{[r]}$.*

Asanuma [2] produced counterexamples in positive characteristic (for $r = 2$). He also showed

**Theorem 1.2** (Asanuma)**.** *Suppose $A \hookrightarrow B$ is an $\mathbb{A}^r$-fibration. Then $\Omega_{B/A}$ is a projective $B$-module. If $\Omega_{B/A}$ is free (as a $B$-module), then there exists $s \in \mathbb{N}$ such that $A^{[r+s]} = B^{[s]}$.*

The Quillen-Suslin theorem provides the following corollary:

**Corollary 1.3.** *Let $k$ be a field, $B = k^{[t]}$, and let $A \hookrightarrow B$ be an $\mathbb{A}^r$-fibration. Then there exists $s \in \mathbb{N}$ such that $A^{[r+s]} = B^{[s]}$.*

To see how this relates to identifying partial coordinate systems, we define

**Definition 1.3.** If $R[f_1, \ldots, f_m] \hookrightarrow R^{[n]}$ is an $\mathbb{A}^{n-m}$-fibration, we say $f_1, \ldots, f_m$ are a *partial system of residual coordinates*; in the case $m = 1$, we call $f_1$ a *residual coordinate.*

We will frequently be concerned with the case when $R$ is a $\mathbb{C}[x]$-algebra. In this case, we have the notion of *strongly residual coordinates*:

**Definition 1.4.** Let $R$ be a $\mathbb{C}[x]$-algebra, and $f_1, \ldots, f_m \in R^{[n]}$. Then $f_1, \ldots, f_m$ is called a *partial system of strongly $x$-residual coordinates* if

1. $R^{[n]}$ is flat over $R[f_1, \ldots, f_m]$

2. $\bar{f}_1, \ldots, \bar{f}_m$, the images modulo $x$, form a partial coordinate system over $\bar{R}$

3. $f_1, \ldots, f_m$ is a partial system of coordinates over the localization $R_x = R \otimes_{\mathbb{C}[x]} \mathbb{C}[x, x^{-1}]$

*Remark* 1.2. This definition depends on the choice of the variable $\mathbb{C}[x] = \mathbb{C}^{[1]}$; if the choice of variable is clear, we may simply say *strongly residual.*

*Remark* 1.3. If $R$ is a polynomial ring over a field, then the flatness condition follows from the latter two conditions. Most of our examples will be in this situation.

*Remark* 1.4. We use $\mathbb{C}$ merely for convenience; $\mathbb{C}$ can be replaced by any field of characteristic zero.

The term *strongly $x$-residual* is motivated by the following.

**Proposition 1.4.** *Let $R$ be a $\mathbb{C}[x]$-algebra, and $f_1, \ldots, f_m \in R^{[n]}$. If $f_1, \ldots, f_m$ is a partial system of strongly $x$-residual coordinates, then it is a partial system of residual coordinates.*

We now formulate a special case of the Dolgachev-Weisfeiler conjecture

**Dolgachev-Weisfeiler Coordinate Conjecture** (DWC($R$,$m$,$n$))**.** *Let $R$ be a polynomial ring over a field of characteristic zero, and let $f_1, \ldots, f_m \in R^{[n]}$. This forms a partial system of coordinates if and only if it is a partial system of residual coordinates.*

Once again, this fails in positive characteristic; we summarize what is known in the following:

**Theorem 1.5.** *Let $k$ be a field of characteristic zero. The following cases of the Dolgachev-Weisfeiler coordinate conjecture are known:*

1. *DWC($k^{[r]}$,$m$,$m$) is true for all $r \geq 0$, $m \geq 1$*

2. *DWC($k^{[r]}$,$m$,$m+1$) is true for all $r \geq 0$, $m \geq 1$.*

3. *DWC(k,1,3) is true.*

*Proof.* The first statement can be shown directly using the Formal Inverse Function Theorem. The arguments for the latter two are laid out in [6]. To show (2), suppose $f_1, \ldots, f_m \in k^{[r+m+1]}$ is a partial system of residual coordinates over $k^{[r]}$. Then $B = k^{[r+m+1]}$ is an $\mathbb{A}^1$-fibration over $A = k^{[r]}[f_1, \ldots, f_m]$. By Corollary 1.3, $A^{[s+1]} = B^{[s]}$ for some $s \in \mathbb{N}$; a theorem of Hamann [10] then implies $A^{[1]} = B$.

The third part follows from Sathaye's theorem [16].

**Theorem 1.6** (Sathaye)**.** *Let $A$ be a DVR of characteristic zero, and $A \hookrightarrow B$ an $\mathbb{A}^2$-fibration. Then $B = A^{[2]}$.*

Assume $k[f_1] \hookrightarrow k^{[3]}$ is an $\mathbb{A}^2$-fibration. Since $k[f_1]$ is a PID, Sathaye's theorem yields that $f_1$ is locally a coordinate in $B = k^{[3]}$; thus by the Bass-Connell-Wright theorem [3], $B$ is a symmetric $k[f_1]$-algebra of a projective rank 2 $B$-module. But since $k[f_1]$ is a PID, this is actually a free module, hence $B = k[f_1]^{[2]}$.

□

*Remark* 1.5. For $m = 1$ and $R = k$ a field, the hypothesis "$k[f_1] \hookrightarrow k^{[n]}$ is an affine fibration" implies that $f_1 - \lambda$ is a hyperplane for all $\lambda \in k$ ($f_1$ is then sometimes called a *general hyperplane* or a *hyperplane fibration*). In this case, DWC($k$,1,$n$) is implied by the Sathaye Conjecture [15], which is in turn slightly weaker than the Embedding Conjecture. For $n \leq 3$ and $\mathbb{Q} \subset k$, Kaliman (for $k = \mathbb{C}$ in [11], and generalized to arbitrary $k \supset \mathbb{Q}$ with Daigle in [7]) showed something in between the latter two: If $f - \lambda$ is a hyperplane for all but finitely many $\lambda \in k$, then it is a coordinate.

*Remark* 1.6. It is easy to check that DWC($k^{[r]}$,$m$,$n$) implies DWC($k^{[r+1]}$,$m-1$,$n-1$).

We are generally interested in constructing coordinate-like polynomials. The motivating example is the Nagata automorphism

$$\sigma := (x, y + x(xz - y^2), z + 2y(xz - y^2) + x(xz - y^2)^2) \in GA_3(\mathbb{C}) \qquad (1.1)$$

This map was written down by Nagata in 1972 [13] as an example of an automorphism which was not known to be tame. While Smith [18] and Wright (unpublished) gave elementary proofs that it is stably tame, it was not shown to be wild until the work of Shestakov and Umirbaev [17] some 30 years later. Nagata constructed this map by observing the following (an easy application of the Formal Inverse Function Theorem):

**Theorem 1.7.** $GA_n(\mathbb{C}[x]) = \{\phi \in GA_n(\mathbb{C}[x,x^{-1}]) \cap MA_n(\mathbb{C}[x]) \mid J\phi \in \mathbb{C}\}$

*Remark* 1.7. This is a special case of the overring principle ([20] Prop. 1.1.7)

Perhaps the simplest thing one can then do is choose $\alpha, \beta \in EA_n(\mathbb{C}[x,x^{-1}])$ with $J\alpha, J\beta \in \mathbb{C}^*$ such that $\alpha^{-1} \circ \beta \circ \alpha \in MA_n(\mathbb{C}[x])$. If $\alpha \in GL_n(\mathbb{C}[x,x^{-1}])$ or $n = 1$, it is easy to see that the composition is always tame over $\mathbb{C}[x]$; so really the simplest interesting thing is Nagata's map, which can be written

$$\sigma = (x, y, z + \frac{y^2}{x}) \circ (x, y + x^2 z, z) \circ (x, y, z - \frac{y^2}{x})$$

We are mainly interested in studying residual coordinates that arise from this kind of construction. For $(f_1, \ldots, f_n) \in GA_n(\mathbb{C}[x,x^{-1}][y])$, consider

$$\phi = (y + x^m Q, z_1, \ldots, z_n) \circ (y, f_1, \ldots, f_n) \in GA_{n+1}(\mathbb{C}[x,x^{-1}])$$

If $x^m Q(f_1, \ldots, f_n) \in x\mathbb{C}[x,y]^{[n]}$, then $\phi(y) = y + x^m Q(f_1, \ldots, f_n)$ is a strongly $x$-residual coordinate over $\mathbb{C}[x]$. From Theorem 1.5, we see that these must be coordinates if $n = 1$, but the question of whether such things are coordinates is open for $n \geq 2$ (the $n = 1$ case gives $y + x(xz - y^2)$, the $y$ component of the Nagata automorphism (1.1)). With minimal effort, we can see that if $m$ is sufficiently large, you do in fact have a coordinate.

**Theorem 1.8.** *Let* $\psi = (y, f_1, \ldots, f_n) \in GA_{n+1}(\mathbb{C}[x, x^{-1}])$ *and* $Q \in \mathbb{C}[x]^{[n]}$. *Then for all* $m >> 0$, $y + x^m Q(f_1, \ldots, f_n)$ *is a* $\mathbb{C}[x]$*-coordinate. To be precise, write* $\psi^{-1} = (y, g_1, \ldots, g_n)$ *and define, for* $k > 0$ *and* $1 \leq j \leq n$,

$$q_{j,k} = \min\{q | x^{qk} \frac{\partial^k g_j}{\partial y^k}(y, f_1, \ldots, f_n) \in \mathbb{C}[x, y]^{[n]}\}$$

*Also define*

$$m_1 := \max_{1 \leq j \leq n;\ 0 < k} \{q_{j,k}\} \tag{1.2}$$

$$m_2 := \min\{m \in \mathbb{N} | x^m Q(f_1, \ldots, f_n) \in \mathbb{C}[x, y]^{[n]}\} \tag{1.3}$$

*Then* $y + x^m Q(f_1, \ldots, f_n)$ *is a coordinate if* $m \geq m_1 + m_2$.

*Proof.* This is just an application of Taylor's formula. Let

$$\phi = \psi^{-1} \circ (y + x^m Q, z_1, \ldots, z_n) \circ \psi$$

and compute

$$\begin{aligned}
\phi(z_j) &= g_j(y + x^m Q(f_1, \ldots, f_m), f_1, \ldots, f_n) \\
&= \sum_{k=0} \frac{1}{k!} \frac{\partial^k g_j}{\partial y^k}(y, f_1, \ldots, f_n)(x^m Q(f_1, \ldots, f_n))^k \\
&= z_j + \sum_{k=1} \frac{1}{k!} x^{k(m - m_2 - m_1)} \left( x^{km_1} \frac{\partial^k g_j}{\partial y^k}(y, f_1, \ldots, f_n) \right) (x^{m_2} Q(f_1, \ldots, f_n))^k
\end{aligned}$$

It is now immediate from (1.2) and (1.3) that $\phi(z_j) \in \mathbb{C}[x, y]^{[n]}$. One quickly checks that $J\phi = 1$, so by Theorem 1.7 $\phi \in GA_n(\mathbb{C}[x])$. □

The most well known example of these kinds of strongly residual coordinates are called the *Vénéreau polynomials*, first written down in 2001 by Vénéreau ([23], [12]) and Berson ([4]) independently. They are given by

$$b_m := y + x^m(xz + y(yu + z^2)) \in \mathbb{C}[x,y]^{[2]} \tag{1.4}$$

Vénéreau noted that the $b_m$ are hyperplanes as well as residual coordinates for each $m \geq 1$. Furthermore, doing essentially the same thing as Theorem 1.8, he showed that for $m \geq 3$, $b_m$ is a coordinate. Freudenburg [9] was able to show that $b_1$ and $b_2$ are 1-stable coordinates; however, the question of whether $b_1$ and $b_2$ are coordinates remained open.

In chapter 2, we investigate the Vénéreau polynomials. We show that $b_2$ is in fact a coordinate. We introduce a related class of strongly residual coordinates, which we term *Vénéreau-type polynomials*. We show that all of these are also hyperplanes; we show that many of them are coordinates, and the remainder are 1-stable coordinates. This gives many more potential counterexamples to DWC($\mathbb{C}$,1,4), DWC($\mathbb{C}^{[1]}$,1,3), and the Embedding Conjecture.

One of the obstructions in dealing with the Vénéreau polynomials is that they arise from a wild automorphism; indeed, they appear as $\phi(y)$ where $\phi = (y + x(xz), z, u) \circ (y, f_1, f_2)$ where $(y, f_1, f_2) \in GA_2(\mathbb{C}[x, x^{-1}, y])$ is wild. In chapter 3, we thus restrict our attention to considering tame strongly residual coordinates; that is, strongly residual coordinates that arise as the $y$-component of an automorphism of the form $(y+xQ, z_1, \ldots, z_n) \circ (y, f_1, \ldots, f_n)$ where $(y, f_1, \ldots, f_n) \in TA_n(\mathbb{C}[x, x^{-1}, y])$. Our main result is an improvement in the required $m$ via a modification of the conjugation approach of Theorem 1.8. In addition, we show that in the case of $GA_2(\mathbb{C}[x, x^{-1}, y])$, the Vénéreau polynomials can be characterized as the simplest strongly residual coordinates which cannot be shown to be coordinates.

# Chapter 2

# Vénéreau and Vénéreau-type Polynomials

## 2.1 The Vénéreau Polynomials

Throughout this chapter, we will let $R = \mathbb{C}[x]$ and $S = \mathbb{C}[x, x^{-1}]$. In addition, when working with $\mathbb{C}^{[3]}$, $R^{[3]}$, or $S^{[3]}$, we will use the variables $\mathbb{C}^{[3]} = \mathbb{C}[y, z, u]$. Define a derivation on $\mathbb{C}[y, z, u]$ by

$$D := y\frac{\partial}{\partial z} - 2z\frac{\partial}{\partial u} \tag{2.1}$$

$D$ is triangular and thus locally nilpotent. We also define

$$p := yu + z^2 \qquad v := xz + yp \qquad w := x^2u - 2xzp - yp^2 \tag{2.2}$$

Note that $\ker D = \mathbb{C}[y, p]$, and $\exp(pD) \in GA_3(\mathbb{C})$ is essentially the Nagata map (1.1). By defining $Dx = 0$, $D$ naturally extends to a derivation on $R^{[3]}$ and thus $S^{[3]}$ with kernel $S[y, p]$ (c.f. [8]). Thus we may consider $\psi := \exp(\frac{p}{x}D) \in GA_3(S)$. One quickly computes

$$\psi = \left(y, \frac{v}{x}, \frac{w}{x^2}\right) \tag{2.3}$$

We note that since $\psi$ fixes $p$, we quickly obtain the relation

$$yw + v^2 = x^2 p \tag{2.4}$$

From (2.3), we see the Vénéreau polynomials naturally as $S$-coordinates; indeed, recall from the introduction (1.4) that (with $m \geq 1$), they are defined by

$$b_m = y + x^m v \tag{2.5}$$

Thus we have $(y + x^{m+1}z, z, u) \circ \psi = (b_m, \frac{v}{x}, \frac{w}{x^2}) \in GA_3(S)$. Noting that $b_m \equiv y \pmod{x}$, we see that each $b_m$ is a strongly $x$-residual (and hence residual) coordinate. Vénéreau [23] also showed that each $b_m$ is a $\mathbb{C}[x]$-hyperplane; thus they satisfy the hypotheses of both the Dolgachev-Weisfeiler and Embedding Conjectures. Vénéreau also showed that for $m \geq 3$, $b_m$ is a coordinate. Similar to Theorem 1.8, he constructed $\phi_m \in GA_3(S)$ by

$$\phi_m = \psi^{-1} \circ (y + x^{m+1}z, z, u) \circ \psi \tag{2.6}$$

As in the proof of Theorem 1.8, one can check that $\phi_m \in GA_3(R)$ for $m \geq 3$ and $\phi_m(y) = b_m$. However, Vénéreau was unable to resolve $b_1$ and $b_2$. We first show that $b_2$ is a coordinate.

**Theorem 2.1.** *The second Vénéreau polynomial* $b_2 = y + x^2(xz + y(yu + z^2))$ *is a coordinate.*

*Proof.* We make use of Theorem 1.7. Define $\varphi_m \in GA_3(S)$ by

$$\varphi_m = \psi^{-1} \circ (y, z - \frac{1}{2}x^{m+1}u, u) \circ (y + x^{m+1}z, z, u) \circ \psi \tag{2.7}$$

One quickly checks that $\varphi_m(y) = b_m$, and $J\varphi_m = 1$. So it now suffices to check that $\varphi_m \in MA_3(R)$ for $m \geq 2$. We show this by direct computation. We first compute $\varphi_m(v)$, $\varphi_m(w)$, and $\varphi_m(p)$, and use these to derive that $\varphi_m(z), \varphi_m(u) \in R^{[3]}$. Set

$\alpha = (y + x^{m+1}z, z - \frac{1}{2}x^{m+1}u, u)$ (so $\varphi_m = \psi^{-1} \circ \alpha \circ \psi$).

$$\begin{aligned} \varphi_m(v) &= (\psi^{-1} \circ \alpha \circ \psi)(v) & \varphi_m(w) &= (\psi^{-1} \circ \alpha \circ \psi)(w) \\ &= (\alpha \circ \psi)(xz) & &= (\alpha \circ \psi)(x^2u) \\ &= \psi\left(xz - \frac{1}{2}x^{m+2}u\right) & &= \psi(x^2u) \\ &= v - \frac{1}{2}x^m w & &= w \end{aligned}$$

Now, using (2.4),

$$\begin{aligned} \varphi_m(p) &= \frac{1}{x^2}\varphi_m(yw + v^2) \\ &= \frac{1}{x^2}\left((y + x^m v)w + \left(v - \frac{1}{2}x^m w\right)^2\right) \\ &= p + \frac{1}{4}x^{2m-2}w^2 \\ \varphi_m(z) &= \frac{1}{x}\varphi_m(v - yp) \\ &= \frac{1}{x}\left(\left((v - \frac{1}{2}x^m w\right) - (y + x^m v)\left(p + \frac{1}{4}x^{2m-2}w^2\right)\right) \\ &= z - x^{m-1}\left(\frac{1}{2}w + vp\right) - \frac{1}{4}x^{2m-3}w^2 b_m \\ \varphi_m(u) &= \frac{1}{x^2}\varphi_m(w + 2vp - yp^2) \\ &= \frac{1}{x^2}\left(w + 2\left(v - \frac{1}{2}x^m w\right)\left(p + \frac{1}{4}x^{2m-2}w^2\right) - (y + x^m v)\left(p + \frac{1}{4}x^{2m-2}w^2\right)^2\right) \\ &= u - x^{m-2}p(w + pv) + \frac{1}{2}x^{2m-4}w^2(v - b_m p) - \frac{1}{4}x^{3m-4}w^3 - \frac{1}{16}x^{4m-6}w^4 b_m \end{aligned}$$

Clearly if $m \geq 2$, then $\varphi_m(z), \varphi_m(u) \in R^{[3]}$. $\square$

To see how $b_1$ is different, we need the following notation.

**Notation 2.1.** Let $\alpha \in GA_n(S)$. Define

$$\alpha^*(GA_n(R)) = \{\alpha^{-1} \circ \phi \circ \alpha \mid \phi \in GA_n(R)\} \subset GA_n(S)$$

In this notation, we see immediately from (2.6) and (2.7) that $\phi_m, \varphi_m \in \psi^*(GA_n(R))$. In fact, we have something stronger: let $\tilde{\psi} \in GA_3(S)$ be given by

$$\tilde{\psi} = (y, xz, x^2u) \circ \psi = (y, v, w) \tag{2.8}$$

Then $\varphi_m, \phi_m \in \tilde{\psi}^*(GA_n(R))$; indeed, one can check that $\phi_m = \tilde{\psi}^{-1} \circ (y + x^m z, z, u) \circ \tilde{\psi}$ and $\varphi_m = \tilde{\psi}^{-1} \circ (y + x^m z, z - \frac{1}{2}x^{m-1}u, u) \circ \tilde{\psi}$. So for $m \geq 2$, $\varphi_m \in GA_3(R) \cap \tilde{\psi}^*(GA_3(R))$. It turns out that this is too much to ask for $b_1$, however.

**Theorem 2.2.** *There is no automorphism $\phi \in GA_3(R) \cap \tilde{\psi}^*(GA_3(R))$ with $\phi(y) = b_1$.*

This follows from the following technical result, whose proof we defer:

**Theorem 2.3.** *Let $\alpha \in GA_3(R)$.*

1. *If $\tilde{\psi}^{-1} \circ \alpha \circ \tilde{\psi} \in GA_3(R)$, then $\alpha(p) \in (p, x^2)$.*
2. *Suppose that $\alpha(y) \equiv y \pmod{x}$. Then $\tilde{\psi}^{-1} \circ \alpha \circ \tilde{\psi} \in GA_3(R)$ if and only if $\alpha(p) = p + x^2F + xpA + x^3B$ for some $A, B \in R^{[3]}$ and $F \in \mathbb{C}[y, p]$ such that $\alpha \equiv \exp(FD) \pmod{x}$ (recall $D = y\frac{\partial}{\partial z} - 2z\frac{\partial}{\partial u}$ from (2.1)).*
3. *Suppose that $\alpha(y) \equiv y \pmod{x}$, $\phi = \tilde{\psi}^{-1} \circ \alpha \circ \tilde{\psi} \in GA_3(R)$, and $\phi(p) \equiv p \pmod{x}$. If $\alpha$ is stably tame over $R$, then $\phi$ is stably tame over $R$ as well.*

*Proof of Theorem 2.2.* Suppose $\phi \in GA_3(R) \cap \tilde{\psi}^*(GA_3(R))$ and $\phi(y) = b_1$. Then $\phi = \tilde{\psi}^{-1} \circ \alpha \circ \tilde{\psi}$ for some $\alpha \in GA_3(R)$. In particular, since $\phi(y) = b_1$, we may write

$$\alpha = \left(y + xz, \sum_{i=0} x^i G_i, \sum_{i=0} x^i H_i\right) \tag{2.9}$$

for some $G_i, H_i \in \mathbb{C}^{[3]}$. By Theorem 2.3 (2), we have $\alpha \equiv \exp(FD) \pmod{x}$ for some $F \in \mathbb{C}[y, p]$. Thus

$$G_0 = z + yF \qquad H_0 = u - 2zF - yF^2 \tag{2.10}$$

Write $\alpha(p) = \sum_{i=0} x^i P_i$ for some $P_i \in \mathbb{C}^{[3]}$. On the one hand, we can compute directly from (2.9)

$$P_0 = yH_0 + G_0^2$$

$$P_1 = yH_1 + zH_0 + 2G_0G_1 \tag{2.11}$$

$$P_2 = yH_2 + zH_1 + 2G_0G_2 + G_1^2 \tag{2.12}$$

On the other hand, from Theorem 2.3 (2), we must have $P_0 = p$, $P_1 \in (p)$, and $P_2 \equiv F \pmod{p}$. We will use these to derive a contradiction.

**Claim 2.4.** $G_1 \equiv -\frac{1}{2}u \pmod{y, z}$.

*Proof.* Since $P_1 \equiv 0 \pmod{p}$, we apply (2.11) and compute

$$\begin{aligned} 0 &\equiv yH_1 + zH_0 + 2G_0G_1 && \pmod{p} \\ &\equiv yH_1 + z(u - 2zF - yF^2) + 2(z - yF)G_1 && \pmod{p} \end{aligned}$$

with the second line following from (2.10). Noting that $p = yu + z^2 \in (y, z^2)$, we may go modulo $(y, z^2)$ and obtain $0 \equiv z(u + 2G_1) \pmod{y, z^2}$, hence $G_1 \equiv -\frac{1}{2}u \pmod{y, z}$. □

Now, since $P_2 \equiv F \pmod{p}$, we apply (2.12) and compute

$$F \equiv yH_2 + zH_1 + 2(z + yF)G_2 + G_1^2 \qquad \pmod{p}$$

Similar to above, since $p \in (y, z)$, we see $F \equiv G_1^2 \pmod{y, z}$. Applying Claim 2.4, we then have

$$F \equiv \frac{1}{4}u^2 \pmod{y, z}$$

However, $F \in \mathbb{C}[y,p]$, so $F \equiv a \pmod{y,z}$ for some $a \in \mathbb{C}$, a contradiction. □

So while $b_1$ cannot be a coordinate of an automorphism of $GA_3(R) \cap \tilde{\psi}^*(GA_3(R))$, we pose the following:

**Conjecture 2.1.** *$b_1$ is a coordinate if and only if it is a coordinate of an automorphism $\phi \in GA_3(R) \cap \psi^*(GA_3(R))$.*

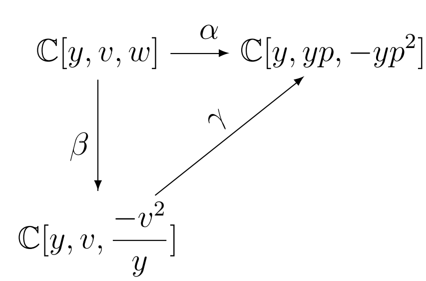
We devote the rest of this section to proving Theorem 2.3. First, we require a couple lemmas.

**Lemma 2.5.** $\mathbb{C}[y,v,w] \cap (x)R[y,z,u] = \mathbb{C}[y,v,w] \cap (x^2)R[y,z,u] = (yw+v^2)\mathbb{C}[y,v,w]$

*Proof.* Clearly the relation $yw + v^2 = x^2p$ (from (2.4)) guarantees

$$\mathbb{C}[y,v,w] \cap xR[y,z,u] \supset \mathbb{C}[y,v,w] \cap x^2R[y,z,u] \supset (yw+v^2)\mathbb{C}[y,v,w]$$

So we simply need to see $\mathbb{C}[y,v,w] \cap xR[y,z,u] \subset (yw+v^2)\mathbb{C}[y,v,w]$. Note that we have a map $\alpha : \mathbb{C}[y,v,w] \to \mathbb{C}[y, yp, -yp^2]$ obtained from going $\bmod x$. Clearly $yw + v^2$ is in the kernel of this map, so it descends to the quotient. Observing that $\mathbb{C}[y,v,w]/(yw+v^2) \cong \mathbb{C}[y, v, \frac{-v^2}{y}]$, we have the following commutative diagram:




Note that $\gamma$ is in fact an isomorphism; hence $\ker\beta = \ker\alpha$. But $\beta$ is the quotient map, so $\ker\beta = (yw+v^2)$, and $\ker\alpha = \mathbb{C}[y,v,w] \cap (x)R[y,z,u]$.

□

**Corollary 2.6.** $\tilde{\psi}^{-1}(x^2R[y,z,u]) \cap R[y,z,u] = (x^2,p)R[y,z,u]$

*Proof.* Applying $\tilde{\psi}$ throughout, this is equivalent to showing

$$x^2R[y,z,u] \cap R[y,v,w] = (x^2, yw+v^2)R[y,v,w]$$

That the right side is contained in the left is immediate (recall from (2.4) that $yw+v^2 \in x^2R[y,z,u]$). For the opposite containment, suppose $A \in x^2R[y,z,u] \cap R[y,v,w]$ and write $A = \sum_{i=0} x^iA_i$ for some $A_i \in \mathbb{C}[y,v,w]$. It suffices to assume that $A = A_0 + xA_1$. Since $A \in x^2R[y,z,u] \subset xR[y,z,u]$, and (trivially) $xA_1 \in xR[y,z,u]$, we must also have $A_0 \in xR[y,z,u]$. So $A_0 \in xR[y,z,u] \cap \mathbb{C}[y,v,w] = (yw+v^2)\mathbb{C}[y,v,w]$ by Lemma 2.5. Thus, we may now assume $A = xA_1$. Since $A \in x^2R[y,z,u]$, we see $A_1 \in xR[y,z,u]$, and again applying Lemma 2.5 yields $A_1 \in (yw+v^2)\mathbb{C}[y,v,w]$. $\square$

**Lemma 2.7.** *Suppose $\phi \in GA_3(\mathbb{C})$ with $\phi(y) = y$ and $\phi(p) \in (p)$. Then $\phi = (y, cz, c^2u) \circ \exp(PD)$ for some $c \in \mathbb{C}^*$, $P \in \mathbb{C}[y,p]$.*

*Proof.* Write $\phi = (y, F_1, F_2)$ for some $F_1, F_2 \in \mathbb{C}^{[3]}$. Since $\phi(p) \in (p)$ and $p$ is irreducible, we see $yF_2 + F_1 = rp$ for some $r \in \mathbb{C}^*$. Set $c = \frac{1}{r}J\phi \in \mathbb{C}^*$. Now we compute

$$cy = \frac{1}{r}yJ(y,F_1,F_2) = \frac{1}{r}J(y,F_1,yF_2+F_1^2) = J(y,F_1,p)$$

Observe that $J(y,\cdot,p) = D$ (recall from (2.1) $D = y\frac{\partial}{\partial z} - 2z\frac{\partial}{\partial u}$), so we may rewrite this as $D(F_1) = cy \in \ker D$; thus $F_1 = cz + \tilde{P}$ for some $\tilde{P} \in \ker D = \mathbb{C}[y,p]$. We now recompute

$$rp = \phi(p) = yF_2 + (cz+\tilde{P})^2$$

Comparing the $z^2$ terms on each side, we deduce $r = c^2$. We also must have $y|\tilde{P}$. Set $P = \frac{\tilde{P}}{cy} \in \mathbb{C}[y,p]$. Plugging this back in to the relation $\phi(p) = rp = c^2p$, we obtain

$$c^2yu = yF_2 + 2c^2yzP + c^2y^2P^2$$

Thus $F_2 = c^2(u - 2zP - yP^2)$, and we have $\phi = (y, c(z+yP), c^2(u-2zP-yP^2)) =$

$(y, cz, c^2u) \circ \exp(PD)$. □

We now have the required tools, and may proceed with the proof of Theorem 2.3.

*Proof of Theorem 2.3.* Define $\phi = \tilde{\psi}^{-1} \circ \alpha \circ \tilde{\psi}$. For (1), suppose $\phi \in GA_3(R)$; in particular, $\phi(p) \in R^{[3]}$. We thus compute, noting that (2.8) and (2.4) imply $\tilde{\psi}^{-1}(p) = \frac{p}{x^2}$,

$$\begin{aligned}(\tilde{\psi}^{-1} \circ \alpha \circ \tilde{\psi})(p) &\in R[y, z, u]\\ (\alpha \circ \tilde{\psi})(\frac{p}{x^2}) &\in R[y, z, u]\\ \alpha(\frac{p}{x^2}) &\in \tilde{\psi}^{-1}(R[y, z, u])\\ \alpha(p) &\in \tilde{\psi}^{-1}(x^2 R[y, z, u])\end{aligned}$$

Since $\alpha \in GA_3(R)$, we thus have $\alpha(p) \in \tilde{\psi}^{-1}(x^2 R[y, z, u]) \cap R[y, z, u]$. Applying Corollary 2.6, we thus have $\alpha(p) \in (x^2, p)R[y, z, u]$, establishing assertion (1) of the theorem.

We now assume for the remainder that $\alpha(y) \equiv y \pmod{x}$. Write

$$\alpha = (y + xQ, \sum_{i=0} x^i G_i, \sum_{i=0} x^i H_i)$$

for some $Q \in R^{[3]}$,$G_i, H_i \in \mathbb{C}^{[3]}$. Let

$$Q = \sum_{i=0} x^i Q_i \qquad\qquad \alpha(p) = \sum_{i=0} x^i P_i \tag{2.13}$$

for some $Q_i, P_i \in \mathbb{C}^{[3]}$. Direct computation shows

$$P_0 = yH_0 + G_0^2 \qquad\qquad P_1 = yH_1 + Q_0 H_0 + 2G_0 G_1 \tag{2.14}$$

For one direction of (2), assume $\phi \in GA_3(R)$. In particular, $\phi(p) \in R^{[3]}$. We then compute, using (2.13),

$$\phi(x^2p) = (\tilde{\psi}^{-1} \circ \alpha \circ \tilde{\psi})(x^2p) = (\alpha \circ \tilde{\psi})(p) = \tilde{\psi}\left(\sum_{i=0} x^i P_i\right)$$

Since $\phi(x^2p) \in x^2R^{[3]}$, we thus see $P_0 + xP_1 \in \tilde{\psi}^{-1}(x^2R^{[3]}) \cap R^{[3]} = (x^2, p)R^{[3]}$ by Corollary 2.6. Since $P_0, P_1 \in \mathbb{C}^{[3]}$, we then see $P_0 + xP_1 \in pR^{[3]}$, whence $P_0, P_1 \in (p)\mathbb{C}^{[3]}$.

Consider $\bar{\alpha} = (y, G_0, H_0) \in GA_3(\mathbb{C})$, the image modulo $x$; Since $p$ is irreducible and $\bar{\alpha}(p) = P_0 \in (p)$, by Lemma 2.7 $\bar{\alpha} = (y, cz, c^2u) \circ \exp(FD)$ for some $c \in \mathbb{C}^*$, $F \in \mathbb{C}[y, p]$. Thus we have

$$G_0 = c(z + yF) \qquad\qquad H_0 = c^2(u - 2zF - yF^2)$$

In particular, we see from (2.14) that we must have $P_0 = c^2p$. Since $\alpha(p) \in (p, x^2)$ by part (1) and $P_1 \in (p)C^{[3]}$, we can write $\alpha(p) = c^2p + xpA' + x^2B'$ for some $A', B' \in R^{[3]}$. Now,

$$\begin{aligned}
\phi(xz) &= \phi(v - yp) \\
&= (\tilde{\psi}^{-1} \circ \alpha \circ \tilde{\psi})(v - yp) \\
&= (\alpha \circ \tilde{\psi})(z - y\frac{p}{x^2}) \\
&= \tilde{\psi}\left(\sum_{i=0} x^i G_i - (y + xQ)(\frac{p}{x^2}(c^2 + xA') + B')\right) \\
&= \sum_{i=0} x^i \tilde{\psi}(G_i) - (y + x\tilde{\psi}(Q))\left(p(c^2 + x\tilde{\psi}(A')) + \tilde{\psi}(B')\right)
\end{aligned}$$

with the first equality arising from (2.2). Since $\phi(z) \in R^{[3]}$, we must have $\phi(xz) \in$

$xR^{[3]}$, and thus

$$
\begin{aligned}
0 &\equiv \tilde{\psi}(G_0) - y(pc^2) - y\tilde{\psi}(B') && \pmod{x} \\
0 &\equiv \tilde{\psi}(c(z + yF)) - y(pc^2) - y\tilde{\psi}(B') && \pmod{x} \\
0 &\equiv c(v - ypc) + y\tilde{\psi}(cF - B') && \pmod{x} \\
0 &\equiv cyp(1 - c) + y\tilde{\psi}(cF - B') && \pmod{x}
\end{aligned}
$$

Observing that $\tilde{\psi}(R^{[3]}) \subset (x, y)R^{[3]}$ (coming from the fact that $y, v, w \in (x, y)R^{[3]}$), we must have $c = 1$. We also must have $F - B' \in R^{[3]} \cap \tilde{\psi}^{-1}(xR^{[3]}) = (x, p)R^{[3]}$ (from Corollary 2.6). Thus we write $B' = F + xB + pA''$ for some $B \in R^{[3]}$. Setting $A = A' + xA''$ gives $\alpha(p) = p + x^2F + xpA + x^3B$ as required.

For the converse, assume $\alpha(p) = p + x^2F + xpA + x^3B$ and $\bar{\alpha} \equiv \exp(FD) \pmod{x}$. In particular, we have

$$G_0 = z + yF \qquad\qquad H_0 = u - 2zF - yF^2 \tag{2.15}$$

Since $J\phi = J\alpha = J\bar{\alpha} = 1$, it suffices to check that $\phi \in MA_3(R)$; compute

$$\phi(y) = (\tilde{\psi}^{-1} \circ \alpha \circ \tilde{\psi})(y) = (\alpha \circ \tilde{\psi})(y) = y + xQ(y, v, w) \in R^{[3]}$$

Next, we show $\phi(xz) \in xR^{[3]}$. Again, using (2.2),

$$
\begin{aligned}
\phi(xz) &= (\tilde{\psi}^{-1} \circ \alpha \circ \tilde{\psi})(v - yp) \\
&= (\alpha \circ \tilde{\psi}) \left( z - y\frac{p}{x^2} \right) \\
&= \tilde{\psi} \left( \sum_{i=0} x^i G_i - (y + xQ) \left( \frac{1}{x^2} \right) \left( p + x^2F + xpA + x^3B \right) \right) \\
&= \sum_{i=0} x^i \tilde{\psi}(G_i) - \left( y + x\tilde{\psi}(Q) \right) \left( p + \tilde{\psi}(F) + x(p\tilde{\psi}(A) + x\tilde{\psi}(B)) \right)
\end{aligned}
$$

So $\phi(xz) \in R^{[3]}$. Going modulo $x$, and recalling from (2.15) that $G_0 = z + yF$ (and thus $\tilde{\psi}(G_0) = v + y\tilde{\psi}(F)$), we obtain

$$\phi(xz) \equiv (v + y\tilde{\psi}(F)) - y(p + \tilde{\psi}(F)) \equiv 0 \pmod{x}$$

So we have $\phi(xz) \in xR^{[3]}$ and thus $\phi(z) \in R^{[3]}$. Since $\phi(y) \notin xR^{[3]}$, it suffices to check that $\phi(p) \in R^{[3]}$ as well (as then $\phi(u) \in R^{[3]}$).

$$\begin{aligned}\phi(p) &= (\tilde{\psi}^{-1} \circ \alpha \circ \tilde{\psi})(p) \\ &= (\alpha \circ \tilde{\psi})(\frac{p}{x^2}) \\ &= \tilde{\psi}\left(\frac{1}{x^2}(p + x^2F + xpA + x^3B)\right) \\ &= p + \tilde{\psi}(F) + x(p\tilde{\psi}(A) + \tilde{\psi}(B))\end{aligned}$$

So $\phi(p) \in R^{[3]}$, and thus $\phi \in GA_3(S) \cap MA_3(R)$. Since $J\phi = J\alpha = 1$, by Theorem 1.7 $\phi \in GA_3(R)$.

For the final part, we appeal to recent results of Berson, van den Essen, and Wright [5]; the following is Theorem 4.5 of that paper applied to $R = \mathbb{C}[x]$.

**Theorem 2.8** (Berson, van den Essen, and Wright)**.** *Let $\phi \in GA_n(R)$ with $J\phi = 1$. If $\phi \in TA_n(S)$, and $\bar{\phi} \in EA_n(\mathbb{C})$ (where $\bar{\phi}$ denotes the image modulo $x$), then $\phi$ is stably tame.*

We also use the following well known result of Smith [18]:

**Smith's formula.** *Let $A$ be a $\mathbb{Q}$-algebra and $D$ be a triangular derivation of $A^{[n]}$. Then for any $P \in \ker D$, $\exp(PD)$ is 1-stably tame.*

Suppose $\phi$, $\alpha$ are as in (3). While $\phi \notin TA_3(S)$, Smith's formula shows that $\tilde{\psi}$ is 1-stably tame (over $S$), hence $\phi$ is stably tame over $S$. One also quickly checks that $J\phi = J\alpha \in \mathbb{C}^*$. So we only need to see that $\bar{\phi}$ is stably a composition of elementaries.

By Lemma 2.7, since $\bar{\phi}(y) = y$ and $\bar{\phi}(p) = p$, a composition with a diagonal map allows us to assume $\bar{\phi}(p) = \exp(PD)$ for some $P \in \mathbb{C}[y,p]$ (and thus $J\phi = J\bar{\phi} = 1$). Again appealing to Smith's formula, we thus have $\bar{\phi}(p) \in EA_4(\mathbb{C})$. Thus by Theorem 2.8, $\phi$ is stably tame. □

## 2.2 Exponentials

It is often useful to express a given automorphism as an exponential when possible. We thus give an expression of $\varphi_n$ as an exponential in Proposition 2.10 below. In [9], Freudenburg showed that the automorphism $\exp(x^{n-3}vd)$ contained $b_m$ as a coordinate ($m \geq 3$), where $d = J(v, w, \cdot) \in \mathrm{LND}_R\, R[y, z, u]$. We observe that this is the same as our $\phi_m$:

**Proposition 2.9.** $\phi_m = \exp(x^{m-3}vd)$

*Proof.* Let $D' = x^{m+1}z\frac{\partial}{\partial y} = x^{m+1}zJ(z, u, \cdot)$. Note that since $J\psi = 1$, we have

$$\psi D'\psi^{-1} = x^m v(\psi J(z, u, \cdot)\psi^{-1}) = x^{m-3}vJ(v, w, \cdot)\left(\frac{1}{J\psi}\right) = x^{m-3}vd$$

Thus $\phi_m = \psi^{-1} \circ \exp(D') \circ \psi = \exp(\psi D'\psi^{-1}) = \exp(x^{m-3}vd)$. □

Similarly for $\varphi_m$, let $e = J(p + \frac{1}{2}x^{m-2}vw, w, \cdot)$.

**Proposition 2.10.** $\varphi_m = \exp(\frac{1}{2}x^{m-1}e)$

*Proof.* Let $E' = x^{m+1}\left((z + \frac{1}{4}x^{m+1}u)\frac{\partial}{\partial y} - \frac{1}{2}u\frac{\partial}{\partial z}\right) = \frac{1}{2}x^{m+1}J(p + \frac{1}{2}x^{m+1}zu, u, \cdot)$. As above, observe that

$$\begin{aligned}\psi E'\psi^{-1} &= \frac{1}{2}x^{m+1}\left(\psi J\left(p + \frac{1}{2}x^{m+1}zu, u, \cdot\right)\psi^{-1}\right) \\ &= \frac{1}{2}x^{m-1}J\left(p + \frac{1}{2}x^{m-2}vw, w, \cdot\right)\left(\frac{1}{J\psi}\right) \\ &= \frac{1}{2}x^{m-1}e\end{aligned}$$

So we see $\varphi_m = \psi^{-1} \circ \exp(E') \circ \psi = \exp(\psi E' \psi^{-1}) = \exp(\frac{1}{2}x^{m-1}e)$. $\square$

## 2.3 Vénéreau Complements

A naïve approach to showing $b_1$ is a coordinate is to simply attempt to compose an element of $GA_3(S)$ with $\varphi_1$ that knocks off the terms with negative $x$-degree; if we could do this with an automorphism that fixed $y$, we would have $b_1$ is a coordinate. This appears to be quite difficult. However, if we drop our insistence on fixing $y$, we can obtain an element of $GA_3(R)$ quite easily, at the expense of no longer having $\phi(y) = b_m$. Indeed, by Theorem 2.3, we seek an $\alpha \in GA_3(R)$ with $\alpha(p) \in (x^2, p)$. Perhaps the closest[1] one to that used in constructing $\varphi_m$ is

$$\alpha_m = (y + x^m z - \frac{1}{4}x^{2m}u, z - \frac{1}{2}x^m u, u)$$

Define

$$\theta_m := \tilde{\psi}^{-1} \circ \alpha_m \circ \tilde{\psi} \tag{2.16}$$

One easily checks that $\alpha_m(p) = p$ for all $m \geq 1$; thus, by Theorem 2.3 (2), $\theta_m \in GA_3(R)$ for all $m \geq 1$, and we have $\theta_m(y) = y + x^m v - \frac{1}{4}x^{2m}w$.

**Definition 2.2.** The polynomials $c_m := y + x^m w$ $(m \geq 1)$ are called *Vénéreau complements*

This definition is motivated by the following:

**Theorem 2.11.** *The Vénéreau polynomial $b_m$ is a coordinate if and only if the Vénéreau complement $c_{2m}$ is.*

*Proof.* Since $\theta_m \in GA_3(R)$ and $\theta_m(y) = b_m - \frac{1}{4}x^{2m}w$, it follows that $b_m$ is a coordinate if and only if $y + \frac{1}{4}x^{2m}w$ is. Consider the automorphism $\beta_m \in GA_4(\mathbb{C})$ given by

[1] The careful reader will note that this $\alpha_m$ essentially drops out of the proof of Theorem 2.2

$\beta_m = (\lambda x, y, \lambda z, \lambda^2 u)$ where $\lambda^{2m+4} = 4$. One easily checks that $\beta_m(w) = \lambda^4 w$ and thus $\beta_m(y + \frac{1}{4}x^{2m}w) = c_{2m}$. Thus conjugation by $\beta_m$ shows that $c_{2m}$ is a coordinate if and only if $y + \frac{1}{4}x^{2m}w$ is as well. □

This motivates the following:

**Question 2.2.** *Are $c_m = y + x^m w$ coordinates?*

It turns out the answer is "yes" if $m \geq 3$. So this slightly stronger fact gives an alternate proof of the fact that $b_m$ is a coordinate for all $m \geq 2$. The proof of this is given in more generality below in Theorem 2.12.

*Remark* 2.1. Theorem 2.11 can be generalized slightly: let $P(w) \in R[w]$. Then $y + xvP(w)$ is a coordinate if and only if $y + x^2wP(w)^2$ is. The proof is almost identical.

It turns out that like the Vénéreau polynomials $b_m$, the complements $c_m$ are very coordinate like: they are hyperplanes; strongly $x$-residual coordinates; and stably tame, 1-stable coordinates.[2] This is shown in more generality in the subsequent section. We mention these in particular here because of their special relationship to the Vénéreau polynomials via Theorem 2.11.

## 2.4 Vénéreau-type Polynomials

Instead of only considering the Vénéreau polynomials and the complements defined in the preceding section, one may generalize these slightly and still retain all the coordinate-like properties.

**Definition 2.3.** A *Vénéreau-type polynomial* is a polynomial of the form $y + xQ$ for some $Q \in R[v, w]$.

[2]To be precise, there exist $\phi'_m \in GA_4(R)$ with $\phi'_m(y) = c_m$, and $\phi_m$ is stably tame.

Note that $Q = x^{m-1}v$ gives the Vénéreau polynomials $b_m$, while $Q = x^{m-1}w$ gives the complements $c_m$. We first give a sufficient condition for a Vénéreau-type polynomial to be a coordinate.

**Theorem 2.12.** *If $Q = x^2Q_1 + xvQ_2$ for some $Q_1 \in R[v,w]$ and $Q_2 \in \mathbb{C}[v^2,w]$, then the Vénéreau-type polynomial $y + xQ$ is a coordinate.*

*Proof.* Write $Q_2 = \sum \alpha_{a,b} v^{2a} w^b$ for some $\alpha_{a,b} \in \mathbb{C}$. Define $\alpha \in GA_3(R)$ by

$$\alpha = \left(y + xQ(z,u), z - \frac{1}{2}x^2 \sum \alpha_{a,b}(-1)^a y^a u^{a+b+1}, u\right)$$

Direct computation shows

$$\begin{aligned}
\alpha(p) &\equiv (y + xQ(z,u))\,u + \left(z - \frac{1}{2}x^2 \sum \alpha_{a,b}(-1)^a y^a u^{a+b+1}\right)^2 && \pmod{x^3} \\
&\equiv (yu + z^2) + x^2 z \left(uQ_2(z,u) - \sum \alpha_{a,b}(-1)^a y^a u^{a+b+1}\right) && \pmod{x^3} \\
&\equiv p + x^2 z \left(\sum \alpha_{a,b} u^{b+1}(z^{2a} - (-yu)^a)\right) && \pmod{x^3}
\end{aligned}$$

Noting that $z^{2a} - (-yu)^a \in (z^2 + yu) = (p)$, we thus have that $\alpha(p) \equiv p \pmod{(xp, x^3)}$. Thus, by Theorem 2.3 (2), $\phi := \tilde{\psi}^{-1} \circ \alpha \circ \tilde{\psi} \in GA_3(R)$, and one quickly checks $\phi(y) = y + xQ$. □

It is interesting to note that the above automorphisms are stably tame; moreover, any automorphism of the above type, with a Vénéreau-type polynomial as a coordinate, must be stably tame.

**Theorem 2.13.** *If $Q = x^2Q_1 + xvQ_2$ for some $Q_1 \in R[v,w]$ and $Q_2 \in \mathbb{C}[v^2,w]$, any $\varphi \in GA_3(R)$ with $\varphi(y) = y + xQ$ is stably tame.*

*Proof.* First, note that the $\phi$ constructed in the proof of Theorem 2.12 is stably tame; indeed, the fact that $\alpha(p) \equiv p \pmod{(xp, x^3)}$ guarantees $\phi(p) \equiv p \pmod{x}$, thus $\phi$ is stably tame by part 3 of Theorem 2.3. Now, consider arbitrary $\varphi \in GA_3(R)$

with $\varphi(y) = y + xQ$. Then $(\varphi \circ \phi^{-1})(y) = (\phi \circ \phi^{-1})(y) = y$. In other words, $\varphi \circ \phi^{-1} \in GA_2(R[y])$. The main result of [5] states that any automorphism in two variables over a regular ring is stably tame; since $R$ is regular, so is $R[y]$, thus the composition $\varphi \circ \phi^{-1}$ is stably tame. Since $\phi$ is also stably tame, we thus see that $\varphi$ is stably tame. □

The remainder of this section is devoted to showing that all Vénéreau-type polynomials satisfy a variety of coordinate-like properties that are known to hold for the Vénéreau polynomial. In particular, we show that they are strongly $x$-residual coordinates (Theorem 2.14), hyperplanes (Theorem 2.17), hyperplane fibrations (Theorem 2.19), and stably tame 1-stable coordinates (Theorem 2.27).

**Theorem 2.14.** *Let $f = y + xQ$, $Q \in R[v, w]$ be a Vénéreau-type polynomial. Then $f$ is a strongly $x$-residual coordinate.*

*Proof.* Since $Q \in R[v, w]$, $\phi = \tilde{\psi}^{-1} \circ (y + xQ(z, u), z, u) \circ \tilde{\psi} \in GA_3(S)$ and $\phi(y) = y + xQ$, so $y + xQ$ is an $S$-coordinate. Clearly $\bar{f} \equiv y \pmod{x}$, so $f$ is a strongly $x$-residual coordinate. □

To see that Vénéreau-type polynomials are hyperplanes, we use the following fact (pointed out to me by Arno van den Essen), which also appears (with $b = 0$) in [12]. A special case of this was used in [23] to show that $f_1$ is a hyperplane.

**Lemma 2.15.** *Let $A$ be a commutative ring, $a, b \in A$, and $g \in A[y]$. Then $A[y]/(y + ag(y) - b) \cong A[y]/(y + g(ay + b))$.*

*Proof.* We compute below, where the first isomorphism is given by sending $y$ to $ay+b$; we then identify $t = -g(ay + b)$, and use this to rewrite the relation in terms of $t$

only.

$$\begin{aligned} A[y]/(y+ag(y)-b) &\cong A[ay+b]/\left(a(y+g(ay+b))\right) \\ &\cong A[ay+b,t]/\left(a(y+g(ay+b)),t+g(ay+b)\right) \\ &\cong A[ay+b,t]/\left(a(y-t),t+g(ay+b)\right) \\ &\cong A[t]/\left(t+g(at+b)\right) \end{aligned}$$

□

**Corollary 2.16.** *Let $A$ be a commutative ring, $a, b \in A^{[n]}$ and $g \in A^{[n]}[y]$. Then $y + ag(y) - b$ is a hyperplane (over $A$) if and only if $y + g(ay + b)$ is as well.*

**Theorem 2.17.** *Let $f = y + xQ$, $Q \in R[v, w]$, be a Vénéreau-type polynomial. Then $f$ is an $R$-hyperplane of $R^{[3]}$; that is, $R^{[3]}/(f) \cong_R R^{[2]}$.*

*Proof.* Similar to (2.2), define

$$p_0 = xyu + z^2 \qquad v_0 = z + yp_0 \qquad w_0 = xu - 2v_0p_0 + yp_0^2 \tag{2.17}$$

**Claim 2.18.** *It suffices to check that $y + xQ_0$ is a coordinate for any $Q_0 \in R[v_0, w_0]$.*

*Proof.* Applying Corollary 2.16 to $f$ (with $a = x$ and $b = 0$) yields that $f$ is a hyperplane if and only if $f_0 = y+Q(xv_0, xw_0)$ is. We can write $Q(xv_0, xw_0) = xQ_0+\lambda$ for some $Q_0 \in R[v_0, w_0]$ and $\lambda \in \mathbb{C}$; hence $f_0 = y + xQ_0 + \lambda$. Thus, if $y + xQ_0$ is a coordinate, so is $f_0$, hence $f_0$ is a hyperplane and (by Corollary 2.16) so is $f$. □

We have thus reduced the theorem to showing that $y+xQ_0$ is a coordinate for any $Q_0 \in R[v_0, w_0]$. The proof of this is quite analagous to that of Theorem 2.1. First, define $D_0 = xy\frac{\partial}{\partial z} - 2z\frac{\partial}{\partial u}$, and observe that $\ker D_0 = R[y, p_0]$. Also define

$$\psi_0 = \exp\left(\frac{p_0}{x}D_0\right) = \left(y, v_0, \frac{w_0}{x}\right)$$

In particular, since $\psi_0$ fixes $p_0$, we see

$$yw_0 + v_0^2 = p_0 \tag{2.18}$$

Also define

$$\alpha_0 := (y + xQ_0(z,u), z, u) \qquad \phi_0 := \psi_0^{-1} \circ \alpha_0 \circ \psi_0$$

One immediately sees that $\phi_0 \in GA_3(S)$, $J\phi_0 = 1$, and $\phi_0(y) = y + xQ_0$. So by Theorem 1.7, we need only see that $\phi_0 \in MA_3(R)$. We first compute $\phi_0(v_0)$, $\phi_0(w_0)$, and $\phi_0(p_0)$, and use those to compute $\phi_0(z)$, $\phi_0(u)$.

$$\begin{aligned}\phi_0(v_0) &= (\psi_0^{-1} \circ \alpha_0 \circ \psi_0)(v_0) \\ &= (\alpha_0 \circ \psi_0)(z) \\ &= (\psi_0)(z) \\ &= v_0\end{aligned} \qquad \begin{aligned}\phi_0(w_0) &= (\psi_0^{-1} \circ \alpha_0 \circ \psi_0)(w_0) \\ &= (\alpha_0 \circ \psi_0)(xu) \\ &= (\psi_0)(xu) \\ &= w_0\end{aligned}$$

Now from (2.18) we see

$$\phi_0(p_0) = \phi_0(yw_0 + v_0^2) = (y + xQ_0)w_0 + v_0^2 = p_0 + xQ_0w_0$$

Finally, we compute, using (2.17)

$$\begin{aligned}\phi_0(z) &= \phi_0(v_0 - yp_0) \\ &= v_0 - (y + xQ_0)(p_0 + xQ_0w_0)\end{aligned}$$

and

$$\begin{aligned}
\phi_0(u) &= \frac{1}{x}\phi_0(w_0 + 2v_0p_0 - yp_0^2) \\
&= \frac{1}{x}\Big(w_0 + 2v_0(p_0 + xQ_0w_0) - (y + xQ_0)(p_0 + xQ_0w_0)^2\Big) \\
&= \frac{1}{x}\Big(w_0 + 2v_0p_0 - yp_0^2\Big) + Q_0(2w_0v_0 - 2w_0yp_0 - p_0^2) + \\
&\quad - xQ_0^2(yw_0^2 - 2w_0p_0) - x^2Q_0^3w_0^2 \\
&= u + Q_0(2w_0v_0 - 2w_0yp_0 - p_0^2) - xQ_0^2(yw_0^2 - 2w_0p_0) - x^2Q_0^3w_0^2
\end{aligned}$$

Thus $\phi_0 \in MA_3(R)$ and hence $\phi_0 \in GA_3(R)$. □

We even have something stronger, namely that all Vénéreau-type polynomials define hyperplane fibrations. To show this for $b_1$, Vénéreau made use of the fact that $b_1 = y + x(xz + y(yu + z^2))$ is linear in $u$; this does not hold for Vénéreau-type polynomials in general, so we have to do a bit more work.

**Theorem 2.19.** *Let $f = y + xQ$, $Q \in R[v, w]$, be a Vénéreau-type polynomial. Then for each $c \in \mathbb{C}$, $f - c$ is an $R$-hyperplane of $R^{[3]}$.*

*Proof.* By Theorem 2.17, we assume $c \neq 0$. As in the proof of that theorem, we would like to apply Corollary 2.16. However, now we must do so with $b = c$ (still $a = x$). Define

$$p_1 = (xy + c)u + z^2 \quad v_1 = xz + (xy + c)p_1 \quad w_1 = x^2u - 2v_1p_1 + (xy + c)p_1^2 \qquad (2.19)$$

Then Corollary 2.16 yields that we need only show $y + Q(v_1, w_1)$ is a hyperplane (in fact, we show it is a coordinate). The general idea is as follows: construct an automorphism $(y + Q(v_1, w_1), *, *) \in GA_3(S)$, then compose on the left with automorphisms fixing $y$ until it is also in $MA_3(S)$. Then we apply Theorem 1.7. We start by defining a derivation over $S$ by

$$D_1 = (xy+c)\frac{\partial}{\partial z} - 2z\frac{\partial}{\partial u}$$

$D_1$ is triangular, and hence locally nilpotent. One easily checks that $\ker D_1 = R[y, p_1]$. Define

$$\phi_0 = \exp(\frac{p_1}{x}D_1) = (y, \frac{v_1}{x}, \frac{w_1}{x^2})$$

Since $x^2 p_1 \in \ker D_1$, we must have $\phi_0(x^2 p_1) = x^2 p_1$ and thus obtain

$$(xy+c)w_1 + v_1^2 = x^2 p_1 \tag{2.20}$$

Our first step is to define

$$\phi_1 = (y + Q(xz, x^2u), z, u) \circ \phi_0 = (y + Q(v_1, w_1), \frac{v_1}{x}, \frac{w_1}{x^2})$$

Next, we set

$$\begin{aligned}\phi_2 &= (y, z, u + c^{-1}z^2) \circ \phi_1 \\ &= \left(y + Q(v_1, w_1), \frac{v_1}{x}, \frac{cw_1 + v_1^2}{cx^2}\right) \\ &= \left(y + Q(v_1, w_1), \frac{v_1}{x}, c^{-1}p_1 - \frac{yw_1}{cx}\right)\end{aligned} \tag{2.21}$$

with the last equality following from (2.20). For the next step, we require the following:

**Claim 2.20.** *For any $G \in R^{[2]}$, $G(v_1, -c^{-1}v_1^2) \equiv G(v_1, w_1) + xyG'(v_1, w_1) \pmod{x^2}$ for some $G' \in R^{[2]}$.*

*Proof.* This is a straightforward computation, appealing to (2.20). Indeed, note that

$$-c^{-1}v_1^2 = w_1 + c^{-1}x(yw_1 - xp_1)$$

We thus compute, applying Taylor's formula at the second step,

$$\begin{aligned} G(v_1, -c^{-1}v_1^2) &\equiv G(v_1, w_1 + c^{-1}xyw_1) && \pmod{x^2} \\ &\equiv G(v_1, w_1) + c^{-1}xyw_1 \frac{\partial G}{\partial w_1}(v_1, w_1) && \pmod{x^2} \end{aligned}$$

Setting $G' = c^{-1}w_1 \frac{\partial G}{\partial w_1}(v_1, w_1)$ yields the claim. $\square$

**Claim 2.21.** *For any $H \in R^{[3]}, G \in R^{[2]}$, there exists $H' \in R^{[3]}$ such that*

$$H(y+G(v_1, w_1)-G(v_1, -c^{-1}v_1^2), v_1, -c^{-1}v_1^2) \equiv H(y, v_1, w_1)+xH'(y, v_1, w_1) \pmod{x^2} \tag{2.22}$$

*Moreover, if $x|H$, then $x|H'$.*

*Proof.* Let

$$L := H(y + G(v_1, w_1) - G(v_1, -c^{-1}v_1^2), v_1, -c^{-1}v_1^2)$$

First, apply Claim 2.20 to $G$, obtaining $L = H(y + xyG'(v_1, w_1) + x^2T_1, v_1, -c^{-1}v_1^2)$ for some $T_1 \in R^{[3]}$. Next, apply Taylor's formula in the first component, obtaining for some $H' \in R[y, v_1, w_1]$ and $T_2 \in R^{[3]}$,

$$\begin{aligned} L &= H(y, v_1, -c^{-1}v_1^2) + xH'(y, v_1, w_1) + x^2T_2 \\ &= H(y, v_1, w_1 + xyw_1 - x^2p_1) + xH'(y, v_1, w_1) + x^2T_2 \end{aligned}$$

with the second equality following from (2.20). We now apply Taylor's formula in the third component, obtaining

$$L \equiv H(y, v_1, w_1) + xH''(y, v_1, w_1) \pmod{x^2}$$

This is precisely the desired claim. □

**Claim 2.22.** *Suppose $\phi \in GA_3(S)$ is of the form*

$$\phi = \left(y + F(v_1, w_1), \frac{v_1}{x}, A + x^r B +^{r-2} G(y, v_1, w_1)\right)$$

*for some $r \in \mathbb{Z}$, $A, B \in R^{[3]}$, $F \in R[v_1, w_1]$, and $G \in R[y, v_1, w_1]$. Then there exists $\tau \in GA_3(S)$ such that $\tau \circ \phi = (y + F(v_1, w_1), \frac{v_1}{x}, A + x^r B')$ for some $B' \in R^{[3]}$.*

*Proof.* Define

$$\tau = \left(y, z, u - x^{r-2} G\left(y - F(xz, -c^{-1}(xz)^2), xz, -c^{-1}(xz)^2\right)\right)$$

Now compute

$$\tau \circ \phi = \Big(y + F(v_1, w_1), \frac{v_1}{x}, A + x^r B + \\ x^{r-2}\left(G(y, v_1, w_1) - G\left(y + F(v_1, w_1) - F(v_1, -c^{-1}v_1^2), v_1, -c^{-1}v_1^2\right)\right)\Big)$$

Applying Claim 2.21, we obtain

$$\tau \circ \phi = \left(y + F(v_1, w_1), \frac{v_1}{x}, A + x^r B' + x^{r-1} G'(y, v_1, w_1)\right)$$

Note that, by Claim 2.21, if $x|G$, then $x|G'$.

Now we can apply an induction step (replacing $B$ with $B'$ and $G$ with $xG'$) to obtain the desired result. □

Now, apply the preceding claim to $\phi_2$ (2.21) to obtain

$$\phi_3 := \Big(y + Q(v_1, w_1), \frac{v_1}{x}, c^{-1}p_1 + xB\Big)$$

for some $B \in R^{[3]}$. Finally, set

$$\begin{aligned}\phi_4 &= (y, z - \frac{c^2 u}{x}, u) \circ \phi_3 \\ &= \Big(y + Q(v_1, w_1), \frac{v_1}{x} - \frac{c}{x}(p_1 + xcB), c^{-1}p_1 + xB\Big) \\ &= \Big(y + Q(v_1, w_1), z + yp_1 - c^2 B, c^{-1}p_1 + xB\Big)\end{aligned}$$

with the last equality coming from (2.19). Thus we have $\phi_4 \in MA_3(R) \cap GA_3(S)$, hence $\phi_4 \in GA_3(R)$ and $\phi_4(y) = y + Q(v_1, w_1)$, i.e. $y + Q(v_1, w_1)$ is a coordinate as required. □

Before proceeding, we remark that Corollary 2.16 raises an interesting question:

**Conjecture 2.3.** *Let $A$ be a $\mathbb{Q}$ algebra, $a, b \in A^{[n]}$, and $g \in A^{[n]}[y]$. Then $y+g(ay+b)$ is a coordinate, if and only if $y + ag(y) - b$ is.*

This is weaker than the Embedding Conjecture, but a proof of this conjecture would make every Vénéreau-type polynomial a coordinate.

The next thing we prove is a generalization of a fact about $f_1$ from [21]. The motivation is the following well known lemma:

**Lemma 2.23.** *Let $A$ be a ring, and $f \in A^{[n]}$. Then the following are equivalent*

1. *$f$ is a coordinate*
2. *$A^{[n]} \cong_{A[f]} A[f]^{[n-1]}$*
3. *$A[c]^{[n]}/(f - c) \cong_{A[c]} A[c]^{[n-1]}$, where $c$ is an additional indeterminate*

In light of this, we may observe that

**Corollary 2.24.** *Let* $P(x,c) \in \mathbb{C}[x,c] = R[c]$*, and set* $\tilde{R} = R[c]/(P)$*. Let* $f$ *be a Vénéreau-type polynomial. If* $f$ *is a coordinate of* $R[y,z,u]$*, then* $f - c$ *is a* $\tilde{R}$*-hyperplane; i.e.* $\tilde{R}[y,z,u]/(f-c) \cong_{\tilde{R}} \tilde{R}^{[2]}$*.*

So if we found one such $P$ for which a Vénéreau-type polynomial $f$ is not a $\tilde{R}$-hyperplane, then it could not be a coordinate.

**Theorem 2.25.** *For any of the following polynomials* $P_i \in R[c]$ *and any Vénéreau-type polynomial* $f = y + xQ$*,* $f - c$ *is a* $\tilde{R}_i$*-hyperplane (where* $\tilde{R}_i := R[c]/(P_i)$*).*

1. $P_1 := x - x_0$ *(where* $x_0 \in \mathbb{C}$*)*

2. $P_2 := c - c_0$ *(where* $c_0 \in \mathbb{C}$*)*

3. $P_3 := x^2 - c^3$

*Proof.* For (1), first suppose $x_0 = 0$. Then $f - c \equiv y - c \pmod{x}$ is obviously a coordinate, hence a hyperplane. If $x_0 \in \mathbb{C}^*$, then since $f - c$ is a $S$ coordinate, $\bar{f} - c$ is a coordinate when we go mod $x - x_0$, and hence is a hyperplane. For part (2), note that $R^{[3]}[c]/(c - c_0, f - c) \cong R^{[3]}/(f - c_0)$. By Theorem 2.19, we thus have $R^{[3]}/(f - c_0) \cong R^{[2]} \cong (R[c]/(c - c_0))^{[2]}$ as required.

For (3), we follow the approach sketched in [21] for the Vénéreau polynomials; this requires Corollary 2.16 and the following (Corollary 1.31 from [12]), which is an application of the Abhyankar-Moh-Suzuki theorem:

**Lemma 2.26.** *Let* $\alpha(y,z,u) \in \mathbb{C}[t][y,z,u]$ *be a* $\mathbb{C}[t]$*-coordinate, and let* $\beta \in \mathbb{C}[t]$*. If* $\alpha(y,z,\beta u)$ *is a* $\mathbb{C}[t]$*-residual coordinate, then it is also a* $\mathbb{C}[t]$*-coordinate, and hence a* $\mathbb{C}[t]$*-hyperplane.*

The key idea from [21] is that $\tilde{R}_3 \cong \mathbb{C}[t^2, t^3]$, and using the main theorem from that paper, it suffices to show the image of $f - c$ in $\mathbb{C}[t^2, t^3]$ is a $\mathbb{C}[t]$-hyperplane. Note that the image of $f-c$ is

$$f_0 = y + t^3Q(t^3z + yp, t^6u - 2t^3zp - yp^2) - t^2$$

As in the proof of Theorem 2.17, we apply Corollary 2.16 with $a = b = t^2$; hence, it suffices to show

$$f_1 = y + t^3Q(tz + (y+1)p_1, t^4u - 2tzp_1 - (y+1)p_1^2)$$

is a coordinate over $\mathbb{C}[t]$, where $p_1 = t^2(y+1)u + z^2$. In order to apply Lemma 2.26, we first establish that $f_1$ is a strongly $t$-residual coordinate, and hence a residual coordinate over $\mathbb{C}[t]$. Clearly it is a coordinate modulo $t$; to see it is a coordinate over $\mathbb{C}[t, t^{-1}]$, define $D_1 = t^2(y+1)\frac{\partial}{\partial z} - 2z\frac{\partial}{\partial u}$, and set $\phi_1 = (y + t^3Q(tz, t^4u), z, u) \circ \exp(\frac{p_1}{t^3}D_1)$. One may quickly check that $\phi_1(y) = f_1$.

Now, taking $\beta = t^2$ in Lemma 2.26, it suffices to show that $y + t^3Q(tz + (y+1)p_2, t^2u - 2tzp_2 - yp_2^2)$ is a coordinate, where $p_2 = (y+1)u + z^2$. But aside from replacing $t$ by $x$, we recognize this as $y + x^3Q(v, w)$ conjugated by $(y-1, z, u)$, so we are done since $y + x^3Q(v, w)$ is a coordinate by Theorem 2.12. □

Contained in the work of Freudenburg and Daigle [6] is the fact that all Vénéreau-type polynomials are stable coordinates (we can see this using Theorem 2.14 and Corollary 1.3). However, no bound was given on the number of additional variables needed. It was previously shown by Freudenburg [9] that for $f_1$ and $f_2$, only one additional variable is needed. It turns out one is sufficent for all Vénéreau-type polynomials.

**Theorem 2.27.** *Let $f = y + xQ$, $Q \in R[v, w]$ be a Vénéreau-type polynomial. Then there exists $\phi \in GA_4(R)$ with $\phi(y) = f$ and $\phi$ stably tame.*

This follows from Theorem 3.14 in Chapter 3. The proof will be given there.

# Chapter 3

# Tame Strongly Residual Coordinates

Throughout this chapter, $R$ will denote a $\mathbb{C}[x]$-algebra, and $S$ will be the localization $S = R \otimes_{\mathbb{C}[x]} \mathbb{C}[x, x^{-1}]$. The aim of this chapter is to show that a class of residual coordinates are in fact coordinates. In particular, we will require that they be *tame strongly $x$-residual coordinates.*

**Definition 3.1.** Let $f \in R^{[n]} = R[y, z_1, \ldots, z_{n-1}]$ be a strongly $x$-residual coordinate. Then there exist $\phi \in GA_n(\mathbb{C})$ with $\phi(y) \equiv \bar{f} \pmod{x}$, and $\psi \in GA_n(S)$ with $\psi(y) = f$. We call $f$ a *tame strongly $x$-residual coordinate* if $\phi^{-1} \circ \psi \in TA_n(S)$.

*Remark* 3.1. If $f$ is a tame strongly residual coordinate, then $(\phi^{-1} \circ \psi)(y) \equiv y \pmod{x}$. For this reason, we restrict our attention to the case $f = y + xQ$, for some $Q \in R^{[n]}$, noting that this simply reflects a choice of coordinates of $\mathbb{C}^{[n]}$.

The chapter is organized as follows: Section 1 contains the necessary preliminary computations; section 2 contains the main results; and section 3 is a discussion of some interesting examples. The reader may wish to proceed immediately to section 2, and refer back to section 1 as necessary.

## 3.1 Preliminaries

Throughout this section, we will use the variables $R^{[n]} = R[z_1, \ldots, z_n]$ (similarly for $S^{[n]}$). Given $\psi \in EA_n(S)$, we can write $\psi = \prod_{i=1}^{q} \Phi_i$ where for each $1 \le i \le q$, $\Phi_i$ is elementary (fixing $n-1$ variables). For convenience, we will write

$$\psi_{a,b} := \Phi_a \circ \cdots \circ \Phi_b$$

Given such a factorization, define, for each $0 \le a \le q$, $1 \le k \le n$

$$\begin{aligned} t_{a,k} &:= \min\{t \mid \psi_{a+1,q}(x^t z_k) \in R^{[n]}\} \\ A_a &:= R[x^{t_{a,1}} z_1, \ldots, x^{t_{a,n}} z_n] \\ A_a[\hat{z}_k] &:= A_a \cap R^{[n]}[\hat{z}_k] \end{aligned}$$

By convention, $\psi_{q+1,q} = id$, whence $t_{q,k} = 0$ for each $k$ and $A_q = R^{[n]}$. If $\Phi_a$ is elementary in $z_k$, we define

$$\begin{aligned} \delta_a &:= t_{a-1,k} - t_{a,k} \\ \epsilon_a &:= \max\{\epsilon \in \mathbb{Z} | \Phi_a(x^{t_{a-1,k}} z_k) - x^{t_{a-1,k}} z_k \in x^{\epsilon} A_a\} \end{aligned}$$

We can write each $\Phi_a$ in a canonical form; suppose it is elementary in $z_k$. Then we write

$$\Phi_a = \Big(z_1, \ldots, z_{k-1}, z_k + x^{-t_{a-1,1}+\epsilon_a} P_a(\hat{z}_k), z_{k+1}, \ldots, z_n\Big) \tag{3.1}$$

for some $P_a(\hat{z}_k) \in A_a[\hat{z}_k]$. Moreover, by the definition of $\epsilon_a$, $P_a \notin x A_a[\hat{z}_k]$.

**Definition 3.2.** Let $\psi \in EA_n(S)$, and write $\psi = \prod_{i=1}^{q} \Phi_i$. We call this a *reducing factorization* if, for each $1 \le a \le q$,

1. $\delta_a \ge 0$

2. $\psi_{a+1,q}(P_a) \in A_q \setminus xA_q$

**Lemma 3.1.** *Let $\psi = \prod_{i=1}^{q} \Phi_i \in EA_n(S)$ be a reducing factorization. Let $1 \le a \le q$. Then*

1. $\epsilon_a \ge 0$

2. $\Phi_a(A_{a-1}) = A_{a-1} \subset A_a$.

*Proof.* Suppose that $\Phi_a$ is elementary in $z_k$. Observe that

$$\psi_{a,q}(x^{t_{a-1,k}} z_k) = x^{\delta_a} \psi_{a+1,q}(x^{t_{a,k}} z_k) + x^{\epsilon_a} \psi_{a+1,q}(P_a) \in A_q \setminus xA_q \tag{3.2}$$

Since $\delta_a \ge 0$ and $\psi_{a+1,q}(x^{t_{a,k}} z_k), \psi_{a+1,q}(P_a) \in A_q \setminus xA_q$, we must have $\epsilon_a \ge 0$. Part (2) follows immediately from (3.1), noting that $P_a(\hat{z}_k) \in A_a[\hat{z}_k] = A_{a-1}[\hat{z}_k]$. □

The following is a straightforward application of Taylor's formula.

**Lemma 3.2.** *Let $1 \le a \le q$ and $P(\hat{z}_1) \in A_a[\hat{z}_1]$, and set $w_k = z_k + xz_kG_k + xH_k$ for some $G_k, H_k \in A_a$ ($2 \le k \le n$). Then there exist $Q_0 \in A_a[\hat{z}_1]$ and $Q_1 \in A_a$ such that*

$$P\left(x^{t_{a,2}} w_2, \ldots, x^{t_{a,n}} w_n\right) = P(\hat{z}_1) + x\left(Q_0(\hat{z}_1) + (x^{t_{a,1}} z_1)Q_1\right)$$

It will be convenient to have the following definition. Note that this depends on a given $A_a$.

**Definition 3.3.** Given $A_a$, we define the subgroup $IA_n^a(R) \subset GA_n(R)$ to be the set of all automorphisms of the form

$$\alpha = (z_1 + xz_1G_1 + xH_1, \ldots, z_n + xz_nG_n + xH_n)$$

where $G_1, H_1, \ldots, G_n, H_n \in A_a$.

A priori, $IA_n^a(R)$ is simply a subset, but one quickly checks that is indeed multiplicatively closed and closed under inversion. The following properties of any $\alpha \in IA_n^a(R)$ are immediate:

1. $\alpha \equiv id \pmod{x}$
2. $\alpha(A_a) = A_a$

*Remark* 3.2. Given any $\tau_a \in \mathbb{N}^n$, one can write $\tau_a = (t_{a,1}, \ldots, t_{a,n})$ and, abusing our notation, define $A_a = R[x^{t_{a,1}}z_1, \ldots, x^{t_{a,n}}z_n]$. The definition of $IA_n^a(R)$ depends only on the choice of $\tau_a$. All statements from Corollary 3.3 through Lemma 3.8 continue to hold in this context. We will make use of this in the proof of Theorem 3.17 below, particularly in the context of (3.14).

With this definition in hand, Lemma 3.2 has a few useful corollaries. The first is immediate.

**Corollary 3.3.** *Let* $\alpha \in IA_n^a(R)$ *and* $P(\hat{z}_1) \in A_a[\hat{z}_1]$. *Then* $\alpha(P(\hat{z}_1)) - P(\hat{z}_1) \in xA_a$.

**Corollary 3.4.** *Let* $\alpha \in IA_n^a(R)$ *and* $\Phi = (z_1 + x^{-s}P_a(\hat{z}_1), z_2, \ldots, z_n) \in EA_n(S)$ *for some* $P_a(\hat{z}_1) \in A_a[\hat{z}_1]$. *If* $s \leq t_{a,1}$, *then there exists* $\alpha' \in IA_n^a(R)$ *and* $\Phi' = (z_1 + x^{-s+1}P'_a(\hat{z}_1), z_2, \ldots, z_n) \in EA_n(S)$ *such that* $P'_a(\hat{z}_1) \in A_a[\hat{z}_1]$ *and* $\Phi^{-1} \circ \alpha \circ \Phi = \alpha' \circ \Phi'$. *Moreover, if* $\Phi \in EA_n^1(S)$, *then* $\Phi' \in EA_n^1(S)$.

*Proof.* This is a straightforward computation, the key hypothesis being that $s \leq t_{a,1}$ gives $\Phi(A_a) = A_a$. Write, for some $F_i, G_i \in A_a$,

$$\alpha = (z_1 + xz_1G_1 + xH_1, \ldots, z_n + xz_nG_n + xH_n)$$

For $2 \leq k \leq n$, we have

$$(\Phi^{-1} \circ \alpha \circ \Phi)(z_k) = (\alpha \circ \Phi)(z_k) = \Phi(z_k + xz_kG_k + xH_k) = z_k + xz_kG'_k + xH'_k \quad (3.3)$$

where $G'_k = \Phi(G_k) \in A_a$, $H'_k = \Phi(H_k) \in A_a$. Next, from Corollary 3.3, we write

$$\alpha(P_a(\hat{z}_1)) = P_a(\hat{z}_1) + xQ$$

for some $Q \in A_a$. Letting $\Phi(G_1) = \tilde{P}_0(\hat{z}_1) + x^{t_{a,1}} z_1 \tilde{G}_1$ and $\Phi(Q) = \tilde{Q}_0(\hat{z}_1) + x^{t_{a,1}} z_1 \tilde{Q}$ for some $\tilde{P}_0(\hat{z}_1), \tilde{Q}_0(\hat{z}_1) \in A_a[\hat{z}_1]$ and $\tilde{G}_1, \tilde{Q} \in A_a$, we compute

$$\begin{aligned}
(\Phi^{-1} \circ \alpha \circ \Phi)(z_1) &= (\alpha \circ \Phi)(z_1 - x^{-s} P_a(\hat{z}_1)) \\
&= \Phi\left(z_1 + xz_1G_1 + xH_1 - x^{-s}\alpha(P_a(\hat{z}_1))\right) \\
&= \Phi\left(z_1 + xz_1G_1 + xH_1 - x^{-s}\left(P_a(\hat{z}_1) + xQ\right)\right) \\
&= \left(z_1 + x^{-s}P_a(\hat{z}_1)\right) + x\left(z_1 + x^{-s}P_a(\hat{z}_1)\right)\Phi(G_1) + \\
&\quad\; x\Phi(H_1) - x^{-s}\left(P_a(\hat{z}_1) + x\Phi(Q)\right) \\
&= z_1 + xz_1\left(\Phi(G_1) + x^{t_{a,1-s}}(P_a(\hat{z}_1)\tilde{G}_1 + \tilde{Q})\right) + \\
&\quad\; x\Phi(H_1) + x^{-s+1}T(\hat{z}_1)
\end{aligned} \tag{3.4}$$

where $T(\hat{z}_1) = P_a(\hat{z}_1)\tilde{P}_0(\hat{z}_1) + \tilde{Q}_0(\hat{z}_1) \in A_a[\hat{z}_1]$. Note that $\operatorname{ord} \tilde{Q}_0(\hat{z}_1) \geq \operatorname{ord} P_a(\hat{z}_1)$, and thus $\operatorname{ord} T(\hat{z}_1) \geq \operatorname{ord} P_a(\hat{z}_1)$. We require a brief lemma:

**Lemma 3.5.** *Let $G, H \in A_a$ and $P(\hat{z}_1) \in A_a[\hat{z}_1]$. If $s \leq t_{a,1}$, then there exist $G', H' \in A_a$, $P'(\hat{z}_1) \in A_a[\hat{z}_1]$ such that*

$$z_1 + xz_1G + xH + x^{-s}P(\hat{z}_1) = (z_1 + x^{-s}P'(\hat{z}_1)) + x(z_1 + x^{-s}P'(\hat{z}_1))G' + xH'$$

*Moreover,* $\operatorname{ord} P'(\hat{z}_1) \geq \operatorname{ord} P(\hat{z}_1)$.

Before proving this, let us first note that this completes the proof of the corollary: applying the lemma to (3.4), we may then write

$$(\Phi^{-1} \circ \alpha \circ \Phi)(z_1) = \left(z_1 + x^{-s+1}P'(\hat{z}_1)\right) + x\left(z_1 + x^{-s+1}P'(\hat{z}_1)\right)G'_1 + xH'_1 \tag{3.5}$$

for some $G_1', H_1' \in A_a$ and $P'(\hat{z}_1) \in A_a[\hat{z}_1]$. Setting

$$\Phi' = (z_1 + x^{-s+1}P'(\hat{z}_1), z_2, \ldots, z_n)$$
$$\alpha' = (z_1 + xz_1(\Phi')^{-1}(G_1') + x(\Phi')^{-1}(H_1'), \ldots, z_n + xz_n(\Phi')^{-1}(G_n') + x(\Phi')^{-1}(H_n'))$$

we easily see from (3.5) and (3.3) that $\Phi^{-1} \circ \alpha \circ \Phi = \alpha' \circ \Phi'$. We note that $\alpha' \in IA_n^a(R)$ since $\Phi'(A_a) = A_a$.

To see the moreover statement, note that Lemma 3.5 gives us $\operatorname{ord}(P'(\hat{z}_1)) \geq \operatorname{ord}(T) \geq \operatorname{ord}(P_a(\hat{z}_1))$. Thus $\Phi_a \in EA_n^1(S)$ implies $\Phi' \in EA_n^1(S)$. □

*Proof of Lemma 3.5.* We proceed by downward induction on $s$. First, define

$$\beta := (z_1 + x^{-s}P(\hat{z}_1), z_2, \ldots, z_n) \tag{3.6}$$

Since $s \leq t_{a,1}$, we have $\beta(A_a) = A_a$. In particular, $\beta^{-1}(G) \in A_a$, so we may write

$$\beta^{-1}(G) = Q_0(\hat{z}_1) + x^{t_{a,1}} z_1 G_1 \tag{3.7}$$

for some $Q_0(\hat{z}_1) \in A_a[\hat{z}_1]$, $G_1 \in A_a$. Then, setting $G'' = \beta(G_1) \in A_a$, we have

$$G = Q_0(\hat{z}_1) + x^{t_{a,1}} \left(z_1 + x^{-s}P(\hat{z}_1)\right) G''$$

To see the base case of the induction, suppose $s \leq 0$. Then we have (letting $w_1 := z_1 + x^{-s}P(\hat{z}_1)$)

$$\begin{aligned} z_1 + xz_1G + xH + x^{-s}P(\hat{z}_1) &= w_1 + x(w_1 - x^{-s}P(\hat{z}_1))G + xH \\ &= w_1 + xw_1G + x(H - P(\hat{z}_1)G) \\ &= w_1 + xw_1\left(G - x^{t_{a,1}-s}P(\hat{z}_1)G''\right) + \\ &\quad x\left(H - x^{-s}P(\hat{z}_1)Q_0(\hat{z}_1)\right) \end{aligned}$$

Since $s \leq 0$, we can set $G' = G - x^{t_{a,1}-s}P(\hat{z}_1)G'' \in A_a$ and $H' = H - x^{-s}P(\hat{z}_1)Q(\hat{z}_1) \in A_a$ to achieve the desired result.

Now suppose $s > 0$. Define $G_0, H_0 \in A_a$ and $P_0 \in A_a[\hat{z}_1]$ by

$$\begin{aligned} G_0 &= \beta^{-1}(G) - x^{t_{a,1}-s}P(\hat{z}_1)G_1 \\ H_0 &= \beta^{-1}(H) \\ P_0(\hat{z}_1) &= -P(\hat{z}_1)Q_0(\hat{z}_1) \end{aligned}$$

Observe $\operatorname{ord} P_0(\hat{z}_1) \geq \operatorname{ord} P(\hat{z}_1)$. Note that these definitions combined, with (3.6) and (3.7), yield

$$\beta^{-1}\left(z_1 + xz_1G + xH + x^{-s}P(\hat{z}_1)\right) = z_1 + xz_1G_0 + xH_0 + x^{-s+1}P_0(\hat{z}_1) \tag{3.8}$$

By the induction hypothesis, there exists $\tilde{G}, \tilde{H} \in A_a$ and $\tilde{P}(\hat{z}_1) \in A_a[\hat{z}_1]$ (with $\operatorname{ord} \tilde{P}(\hat{z}_1) \geq \operatorname{ord} P_0(\hat{z}_1)$) such that (letting $\tilde{w} = z_1 + x^{-s+1}\tilde{P}(\hat{z}_1)$)

$$z_1 + xz_1G_0 + xH_0 + x^{-s+1}P_0(\hat{z}_1) = \tilde{w} + x\tilde{w}\tilde{G} + x\tilde{H}$$

Since from (3.8), the left hand side is equal to $\beta^{-1}(z_1 + xz_1G + xH + x^{-s}P(\hat{z}_1))$, it suffices to show that applying $\beta$ to the right hand side produces something in the desired form: indeed, if we define $P' := P + x\tilde{P}$, $G' := \beta(\tilde{G})$,and $H' := \beta(\tilde{H})$, we need only note that $\beta(\tilde{w}) = z_1 + x^{-s}P'(\hat{z}_1)$. Since $\beta(A_a) = A_a$, we have $G', H' \in A_a$ as required. It is also clear that $\operatorname{ord} P'(\hat{z}_1) \geq \operatorname{ord} P(\hat{z}_1)$. □

The following will be the crucial ingredient in the proof of Theorem 3.14.

**Corollary 3.6.** *Let $\alpha \in IA_n^a(R) \cap EA_n(S)$ and $\Phi = (z_1 + x^{-s}P_a(\hat{z}_1), z_2, \ldots, z_n) \in EA_n(S)$ for some $P_a(\hat{z}_1) \in A_a[\hat{z}_1]$. If $s \leq t_{a,1}$, then there exists $\Phi' \in EA_n(S)$ such that $\Phi' \circ \alpha \circ \Phi \in IA_n^a(R) \cap EA_n(S)$. Moreover, if $\alpha, \Phi \in EA_n^1(S)$, then $\Phi' \circ \alpha \circ \Phi \in EA_n^1(S)$*

*as well.*

*Proof.* The proof is by induction downward on $s$. If $s < 0$ then $\Phi \in IA_n^a(R)$ and the claim is immediate. If $s \geq 0$, then by Corollary 3.4, $\Phi^{-1} \circ \alpha \circ \Phi = \tilde{\alpha} \circ \tilde{\Phi}$ for some $\tilde{\alpha} \in IA_n^a(R) \cap EA_n(S)$ and $\tilde{\Phi} = (z_1 + x^{-s+1}\tilde{P}_a(\hat{z}_1), z_2, \ldots, z_n)$. By the induction hypothesis, there exits $\tilde{\Phi}'$ such that $\tilde{\Phi}' \circ \tilde{\alpha} \circ \tilde{\Phi} \in IA_n^a(R) \cap EA_n(S)$. Then setting $\Phi' = \tilde{\Phi}' \circ \Phi^{-1}$, we have

$$\Phi' \circ \alpha \circ \Phi = \tilde{\Phi}' \circ \Phi^{-1} \circ \alpha \circ \Phi = \tilde{\Phi}' \circ \tilde{\alpha} \circ \tilde{\Phi} \in IA_n^a(R) \cap EA_n(S)$$

giving the claim. For the moreover statement, note that by Corollary 3.4, if $\Phi \in EA_n^1(S)$ then so is $\tilde{\Phi}$; if $\alpha \in EA_n^1(S)$ as well, then so is $\tilde{\alpha}$, so by induction $\tilde{\Phi}' \circ \tilde{\alpha} \circ \tilde{\Phi} \in EA_n^1(S)$ giving the desired statement (as $\Phi' \circ \alpha \circ \Phi = \tilde{\Phi}' \circ \tilde{\alpha} \circ \tilde{\Phi}$). □

We now rephrase this slightly for use in the proof of Theorem 3.12 below.

**Corollary 3.7.** *Let $\Phi_a = (z_1 + x^{-t_{a,1}}P_a(\hat{z}_1), z_2, \ldots, z_n) \in EA_n(S)$ for some $P_a \in A_a[\hat{z}_1]$ and $\alpha \in IA_n^a(R)$. Suppose that $A_{a-1} \subset A_a$. Then there exists $\alpha' \in IA_n^{a-1}(R)$, and $\Phi_a' = (z_1 + x^{-t_{a,1}}P_a'(\hat{z}_1), z_2, \ldots, z_n)$ (where $P_a' \in A_a[\hat{z}_1]$) such that $\Phi_a \circ \alpha = \alpha' \circ \Phi_a'$.*

*Proof.* By Corollary 3.4, there exists $\alpha' \in IA_n^a(R)$ with $\Phi_a \circ \alpha \circ \Phi_a^{-1} = \alpha' \circ \Phi'$. Since we assume $A_a \subset A_{a-1}$, we see $IA_n^a(R) \subset IA_n^{a-1}(R)$ and thus $\alpha' \in IA_n^{a-1}(R)$. Letting $\Phi_a' = \Phi' \circ \Phi_a$, we thus have $\Phi_a \circ \alpha = \alpha \circ \Phi_a'$. □

The remainder of this section is devoted to results that will help us in the $n = 2$ case.

**Lemma 3.8.** *Suppose $\delta_{a+1} = 0$ and let $\Phi_1 = (z_1 + x^{-t_{a,1}}P(\hat{z}_1), z_2) \in EA_2^1(S)$ and $\Phi_2 = (z_1 + x^{-t_{a,1}+1}F_1, z_2 + x^{-t_{a,2}+1}F_2) \in GA_2(S)$ for some $P \in A_a[\hat{z}_1]$, $F_1, F_2 \in A_a = A_{a+1}$. Then there exists $\Phi_1' = (z_1 + x^{-t_{a,1}}P'(\hat{z}_1), z_2) \in EA_2^1(S)$ (with $P'(\hat{z}_1) \in A_a[\hat{z}_1]$) , $\Phi_2' = (z_1, z_2 + x^{-t_{a,2}+1}Q'(\hat{z}_2)) \in EA_2^1(S)$ (with $Q'(\hat{z}_2) \in A_{a+1}[\hat{z}_2]$), and $\alpha \in IA_2^a(R)$ such that $\Phi_1 \circ \Phi_2 = \alpha \circ \Phi_2' \circ \Phi_1'$.*

*Proof.* First, note that $\Phi_1(A_a) = A_a = \Phi_2(A_a)$. This follows from Lemma 3.5 and Taylor's formula. First, we compute

$$\begin{aligned}\Phi_1 \circ \Phi_2 &= \left(z_1 + x^{-t_{a,1}} \left(xF_1 + P(x^{t_{a,2}} z_2 + xF_2)\right), z_2 + x^{-t_{a,2}+1} F_2\right) \\ &= \left(z_1 + xz_1G_1 + x^{-t_{a,1}} P_1(x^{t_{a,2}} z_2), z_2 + xz_2G_2 + x^{-t_{a,2}+1} Q(x^{t_{a,1}} z_1)\right)\end{aligned}$$

for some $G_1, G_2 \in A_a$,$P_1 \in A_a[\hat{z}_1]$, $Q \in A_a[\hat{z}_2]$ by Taylor expansion. Next, we apply Lemma 3.5 in the $z_1$ coordinate to obtain $G_1', H_1' \in A_a$ and $\Phi_1' = (z_1 + x^{-t_{a,1}} P'(\hat{z}_1), z_2) \in EA_2^1(S)$ such that

$$\Phi_1 \circ \Phi_2 = \left(\Phi_1'(z_1) + x\Phi'(z_1)G_1' + xH_1', z_2 + xz_2G_2 + x^{-t_{a,2}+1} Q(x^{t_{a,1}} z_1)\right)$$

We thus compute

$$\begin{aligned}\Phi_1 \circ \Phi_2 &= \left(\Phi_1'(z_1) + x\Phi'(z_1)G_1' + xH_1', z_2 + xz_2G_2 + x^{-t_{a,2}+1} Q(x^{t_{a,1}} z_1)\right) \circ (\Phi_1')^{-1} \circ \Phi_1' \\ &= \left(z_1 + xz_1G_1'' + xH_1'', z_2 + xz_2G_2' + x^{-t_{a,2}+1} Q(x^{t_{a,1}} z_1 - P'(\hat{z}_1))\right) \circ \Phi_1'\end{aligned}$$

where $G_1'' = (\Phi_1')^{-1}(G_1')$, $H_1'' = (\Phi_1')^{-1}(H_1')$, and $G_2' = (\Phi_1')^{-1}(G_2)$. Note since $\Phi_1'(A_a) = A_a$, we have $G_1'', H_1'', G_2' \in A_a$. Now applying Taylor's formula to $Q(x^{t_{a,1}} z_1 - P'(x^{t_{a,2}} z_2))$, we obtain

$$Q(x^{t_{a,1}} z_1 - P'(x^{t_{a,2}} z_2)) = Q(x^{t_{a,1}} z_1) + x^{t_{a,2}} z_2 G_2''$$

for some $G_2'' \in A_a$. Thus, we have

$$\Phi_1 \circ \Phi_2 = \left(z_1 + xz_1G_1'' + xH_1'', z_2 + xz_2G_2'' + x^{-t_{a,2}+1} Q(x^{t_{a,1}} z_1)\right) \circ \Phi_1'$$

We now apply Lemma 3.5, this time in the second component, to obtain $G_2''', H_2''' \in$

$A_a$ and $\Phi_2' = (z_1, z_2 + x^{-t_{a,2}+1}Q'(\hat{z}_2)) \in EA_2^1(S)$ such that

$$\begin{aligned}\Phi_1 \circ \Phi_2 &= (z_1 + xz_1G_1'' + xH_1'', \Phi_2'(z_2) + x\Phi_2'(z_2)G_2''' + xH_2''') \circ (\Phi_2')^{-1} \circ \Phi_2' \circ \Phi_1' \\ &= (z_1 + xz_1G_1''' + xH_1''', z_2 + xz_2G_2'''' + xH_2'''') \circ \Phi_2' \circ \Phi_1'\end{aligned}$$

where $G_1''' = (\Phi_2')^{-1}(G_1'')$, $H_1''' = (\Phi_2')^{-1}(H_1'')$, $G_2'''' = (\Phi_2')^{-1}(G_2''')$, and $H_2'''' = (\Phi_2')^{-1}(H_2''')$. Now we simply set $\alpha = (z_1 + xz_1G_1''' + xH_1''', z_2 + xz_2G_2'''' + xH_2'''')$; noting that $\Phi_2'(A_a) = A_a$ yields $\alpha \in IA_2^a(R)$ and thus the desired result. □

We would like to generalize $EA_n(S)$ slightly. The idea is that elements of $IA_n^a(R)$ can be inserted at the $a$-th step and essentially nothing changes. This will prove to be of great utility in showing tame strongly residual coordinates to be coordinates.

**Definition 3.4.** $\psi \in GA_n(S)$ is said to have a *generalized elementary factorization* if there exist $q \in \mathbb{N}$, and for each $1 \leq a \leq q$, $\tau_a = (t_{a,1}, \ldots, t_{a,n}) \in \mathbb{N}^n$, $\alpha_a \in IA_n^{a-1}(R)$, and $\Phi_a \in EA_n(S)$ such that

1. $\tau_{a-1} - \tau_a = (0, \ldots, 0, \delta_a, 0, \ldots, 0)$ for some $\delta_a = t_{a-1,k} - t_{a,k} \in \mathbb{Z}$
2. $\Phi_a = (z_1, \ldots, z_{k-1}, z_k + x^{-s_a}P_a(\hat{z}_k), z_{k+1}, \ldots, z_k)$ for some $P_a(\hat{z}_k) \in A_a[\hat{z}_k]$, where $s_a = \max\{t_{a-1,k}, t_{a,k}\}$.
3. $\psi = \prod_{i=1}^{q} \alpha_i \circ \Phi_i$
4. $\psi_{a,q} := \prod_{i=a}^{q} \alpha_i \circ \Phi_i$ satisfies $\psi_{a,q}(x^{t_{a-1,j}}z_j) \in R^{[n]} \setminus xR^{[n]}$ for each $1 \leq a \leq q$, $1 \leq j \leq n$.

If in addition, $\delta_a \geq 0$ for each $1 \leq a \leq q$, we call this a *generalized reducing factorization.*

Observe that if $\psi \in EA_n(S)$ can be written $\psi = \prod_{i=1}^{q} \Phi_i$, then our previous notion of $t_{i,j}$ satisfies this definition (with each $\alpha_i = id$), justifying our conflation of notations.

**Lemma 3.9.** *Let $\psi \in GA_2(S)$ have a generalized elementary factorization $\psi = \prod_{i=1}^{q} \alpha_i \circ \Phi_i$. Let $1 \leq a \leq q$, and without loss of generality suppose $\Phi_a$ is elementary in $z_1$.*

1. *If $\delta_a > 0$, then $\Phi_a = (z_1 + x^{-t_{a-1,1}} P_a(\hat{z}_1), z_2)$ for some $P_a \in A_a[\hat{z}_1] \setminus xA_a[\hat{z}_1]$*

2. *If $\delta_a < 0$, then $\Phi_a = (z_1 + x^{-t_{a,1}} P_a(\hat{z}_1), z_2)$ for some $P_a \in A_a[\hat{z}_1] \setminus xA_a[\hat{z}_1]$. Moreover, we may assume $\Phi_{a+1}$ is elementary in $z_2$; and if $\Phi_a \in EA_2^1(S)$, then $\delta_{a+1} \leq 0$.*

*Proof.* First, note that $\psi_{a,q}(x^{t_{a-1,j}} z_j) \in R^{[2]} \setminus xR^{[2]}$ implies $(\Phi_a \circ \psi_{a+1,q})(x^{t_{a-1,j}} z_j) \in R^{[2]} \setminus xR^{[2]}$ (since $\alpha_a \equiv id \pmod{x}$ and $\alpha_a(A_{a-1}) = A_{a-1}$). Without loss of generality, assume $\Phi_a$ is elementary in $z_1$. Similar to (3.2), we thus compute

$$(\Phi_a \circ \psi_{a+1,q})(x^{t_{a-1,1}} z_1) = x^{\delta_a} \psi_{a+1,q}(x^{t_{a,1}} z_1) + x^{t_{a-1,1} - s_a} \psi_{a+1,q}(P_a(\hat{z}_1)) \tag{3.9}$$

Since $P_a \in R[x^{t_{a,2}} z_2]$, we must have $\psi_{a+1,q}(P_a) \in R^{[2]}$, with $\psi_{a+1,q}(P_a) \in xR^{[2]}$ only if $P_a \in xR[x^{t_{a,2}} z_2]$. Note also that $\psi_{a+1,q}(x^{t_{a,1}} z_1) \in R^{[2]}$. Thus, if $\delta_a > 0$, $t_{a-1,1} - s_a = 0$, and we must have $P_a \notin xA_a[\hat{z}_1]$ in order to have $(\Phi_a \circ \psi_{a+1,q})(x^{t_{a-1,1}} z_1) \notin xR^{[2]}$ (from (3.9)). If instead $\delta_a < 0$ , we must have $t_{a-1,1} - s_a = \delta_a$ (whence $s_a = t_{a,1}$). Thus as before, we must have $\psi_{a+1,q}(P_a) \notin xR^{[2]}$ and thus $P_a \notin xA_a[\hat{z}_1]$.

For the moreover statement of part (2), suppose $\delta_a < 0$ and $\Phi_a \in EA_2^1(S)$. We assume for contradiction that $\delta_{a+1} > 0$. Since $\delta_a < 0$, we may assume $\alpha_{a+1} = id$ (by Corollary 3.7, we have $\alpha_a \circ \Phi_a \circ \alpha_{a+1} = (\alpha_a \circ \alpha'_{a+1}) \circ \Phi'_a$ with $\alpha_a \circ \alpha'_{a+1} \in IA_n^{a-1}(R)$). Thus, if $\Phi_{a+1}$ is elementary in $z_1$, we can replace $\Phi_a$ by $\Phi_a \circ \Phi_{a+1}$; so we may assume $\Phi_{a+1}$ is elementary in $z_2$ and write

$$\Phi_{a+1} = \Big(z_1, z_2 + x^{-t_{a,2}} Q(x^{t_{a+1,1}} z_1)\Big)$$
$$\psi_{a+2,q} = \Big(z_1 + x^{-t_{a+1,1}} F_1, z_2 + x^{-t_{a+1,2}} F_2\Big)$$

where $Q \in R^{[1]} \backslash xR^{[1]}$ (by part (1)) and $F_1, F_2 \in R^{[2]} \backslash xR^{[2]}$ (from part (4) of Definition 3.4). We then compute

$$\begin{aligned}(\Phi_a \circ \psi_{a+1,q})(x^{t_{a-1,1}} z_1) &= (\Phi_a \circ \Phi_{a+1} \circ \psi_{a+2,q})(x^{t_{a-1,1}} z_1) \\ &= (\Phi_{a+1} \circ \psi_{a+2,q}) \left( x^{t_{a-1,1}} z_1 + x^{\delta_a} P_a(x^{t_{a,2}} z_2) \right) \\ &= \psi_{a+2,q} \left( x^{t_{a-1,1}} z_1 + x^{\delta_a} P_a(x^{t_{a,2}} z_2 + Q(x^{t_{a+1,1}} z_1)) \right) \\ &= x^{t_{a-1,1}} z_1 + x^{\delta_a} \left( F_1 + P_a \left( x^{t_{a,2}} z_2 + x^{\delta_{a+1}} F_2 + Q(x^{t_{a+1,1}} z_1 + F_1) \right) \right)\end{aligned}$$

Since $\psi_{a,q}(x^{t_{a-1,1}} z_1) \in R^{[2]} \setminus xR^{[2]}$, we must also have $(\Phi_a \circ \psi_{a+1,q})(x^{t_{a-1,1}} z_1) \in R^{[2]} \setminus xR^{[2]}$, and thus have

$$F_1 + P_a \left( x^{t_{a,2}} z_2 + x^{\delta_{a+1}} F_2 + Q(x^{t_{a+1,1}} z_1 + F_1) \right) \in x^{-\delta_a} A_q \tag{3.10}$$

Since $\Phi_{a+1}$ is elementary in $z_2$ and $\delta_a < 0$, we must have $t_{a+1,1} = t_{a,1} > 0$. Since we assumed $\delta_{a+1} > 0$, we also have $t_{a,2} > t_{a+1,2} \geq 0$. Thus, taking (3.10) modulo $x$ , we have

$$F_1 + P_a(Q(F_1)) \equiv 0 \pmod{xA_q} \tag{3.11}$$

Note that since $\psi_{a+2,q}(x^{t_{a+1,1}} z_1) \in R^{[2]} \setminus xR^{[2]}$, $\overline{F_1} \neq 0$ ($\overline{F_1}$ denotes the image modulo $x$). We also have $Q, P_a \notin xR^{[1]}$; but since $\Phi_a \in EA_2^1(S)$, $\deg \overline{P_a(Q(F_1))} > \deg \overline{F_1}$, contradicting (3.11). □

**Lemma 3.10.** *Let $\psi \in GA_n(S)$ have a generalized reducing factorization $\psi = \prod_{i=1}^{q} \alpha_i \circ \Phi_i$. Then for $1 \leq a \leq b \leq q$, there exists $F_1, G_1, H_1, \ldots, F_n, G_n, H_n \in A_b$ such that*

$$\psi_{a,b} = \left( z_1 + xz_1 G_1 + xH_1 + x^{-t_{a-1,1}} F_1, \ldots, z_n + xz_n G_n + xH_n + x^{-t_{a-1,n}} F_n \right)$$

*Proof.* We induct on $b-a$. Assume

$$\psi_{a,b-1} = \left(z_1 + xz_1\tilde{G}_1 + x\tilde{H}_1 + x^{-t_{a-1,1}}\tilde{F}_1, \ldots, z_n + xz_n\tilde{G}_n + x\tilde{H}_n + x^{-t_{a-1,n}}\tilde{F}_n\right)$$

for some $\tilde{G}_j, \tilde{H}_j, \tilde{F}_j \in A_{b-1}$. Since $\alpha_b \in IA_n^{b-1}(R)$, we have $\alpha_b(A_{b-1}) = A_{b-1}$. It is thus easy to see that $\psi_{a,b-1} \circ \alpha_b$ is in the same form as $\psi_{a,b-1}$, so without loss of generality we assume $\alpha_b = id$. Without loss of generality, assume $\Phi_b$ is elementary in $z_1$, and by Lemma 3.9, write $\Phi_b = (z_1 + x^{-t_{b-1,1}}P_b(\hat{z}_1), z_2, \ldots, z_n)$. For $2 \leq j \leq n$, set $G_j = \Phi_b(\tilde{G}_j)$, $H_j = \Phi_b(\tilde{H}_j)$, and $F_j = \Phi_b(\tilde{F}_j)$. Then $G_j, H_j, F_j \in A_b$ and $\psi_{a,b}(z_j) = z_j + xz_jG_j + xH_j + x^{-t_{a-1,j}}F_j$. We now just need to compute $\psi_{a,b}(z_1)$. Note that $\Phi_b(\tilde{G}_1) \in A_{b-1}$, so we may write $\Phi_b(\tilde{G}_1) = Q_b(\hat{z}_1) + (x^{t_{b-1,1}}z_1)G_1'$ for some $Q_b \in A_{b-1}[\hat{z}_1] = A_b[\hat{z}_1]$ and $G_1' \in A_{b-1} \subset A_b$. Thus we compute

$$\begin{aligned}
\psi_{a,b}(z_1) &= (\psi_{a,b-1} \circ \Phi_b)(z_1) \\
&= (z_1 + x^{-t_{b-1,1}}P_b(\hat{z}_1)) + x(z_1 + x^{-t_{b-1,1}}P_b(\hat{z}_1))\Phi_b(\tilde{G}_1) + \\
&\quad x\Phi_b(\tilde{H}_1) + x^{-t_{a-1,1}}\Phi_b(\tilde{F}_1) \\
&= z_1 + xz_1\left(\Phi_b(\tilde{G}_1) + P_b(\hat{z}_1)G_1'\right) + x\Phi_b(\tilde{H}_1) + \\
&\quad x^{-t_{a-1,1}}\left(\Phi_b(\tilde{F}_1) + x^{t_{a-1,1}-t_{b-1,1}}P_b(\hat{z}_1)(1 + xQ_b(\hat{z}_1))\right)
\end{aligned}$$

Since it is a generalized reducing factorization and $a-1 \leq b-1$, we thus have $t_{a-1,1} \geq t_{b-1,1}$. Hence $\psi_{a,b}$ is in the desired form, with $G_1 = \Phi_b(\tilde{G}_1) + P_b(\hat{z}_1)G_1' \in A_b$, $H_1 = \Phi_b(\tilde{H}_1) \in A_b$, and $F_1 = \Phi_b(\tilde{F}_1) + x^{t_{a-1,1}-t_{b-1,1}}P_b(\hat{z}_1)(1 + xQ_b(\hat{z}_1)) \in A_b$. □

**Lemma 3.11.** *Let $\psi_{a,b} = \prod_{i=a}^{b}\Phi_i \in EA_2^1(S)$, and suppose $\delta_i = 0$ and $\epsilon_i = 0$ for $a < i \leq b$. Suppose also that $\Phi_a$ is elementary in $z_2$. Then $\psi_{a,b} = (z_1 + x^{-t_{a-1,1}}F_1, z_2 + x^{-t_{a-1,2}}F_2)$ for some $F_1, F_2 \in A_b$. Moreover, letting $\bar{F}_k$ denote the image modulo $xA_b$, we have*

1. *If* $\deg \bar{F}_2 > 0$*, then* $\deg \bar{F}_2 > \deg \bar{F}_1$

*2. If* $\deg \bar{F}_2 \leq 0$*, then* $\deg \bar{F}_1 \leq 0$

*Proof.* We induct on $b - a$. If $b = a$, $\psi_{a,b} = \Phi_a$ and the claim follows from (3.1). So assume $b > a$. Then by the induction hypothesis, we have

$$\psi_{a+1,b} = (z_1 + x^{-t_{a,1}} F_1, z_2 + x^{-t_{a,2}} F_2')$$

for some $F_1, F_2' \in A_b$. From (3.1), we can write $\Phi_a = (z_1, z_2 + x^{-t_{a-1}} P_a(\hat{z}_2))$ for some $P_a \in A_a[\hat{z}_2] \setminus x A_a[\hat{z}_2]$ (since $\epsilon_a = 0$ by assumption). Set $F_2 = F_2' + P_a(x^{t_{a-1,1}} z_1 + F_1) \in A_b$. We compute, noting $t_{a-1,j} = t_{a,j}$,

$$\begin{aligned}\psi_{a,b} &= \Phi_a \circ \psi_{a+1,b} \\ &= \left(z_1 + x^{-t_{a-1,1}} F_1, z_2 + x^{-t_{a-1,2}} P_a(x^{t_{a-1,1}} z_1 + F_1) + x^{-t_{a-1,2}} F_2'\right) \\ &= \left(z_1 + x^{-t_{a-1,1}} F_1, z_2 + x^{-t_{a-1,2}} F_2\right)\end{aligned}$$

thus giving the first claim. For the remainder, we note that (since $t_{a-1,1} = t_{b,1}$)

$$\bar{F}_2 = \bar{F}_2' + \bar{P}_a(x^{t_{b,1}} z_1 + \bar{F}_1) \tag{3.12}$$

First, we note that if $\deg \bar{F}_1 \leq 0$, (1) and (2) are both immediate. So assume $\deg \bar{F}_1 > 0$. Note that since each $\Phi_i \in EA_2^1(S)$, we have

$$\bar{F}_1 = \sum_{i+j>1} \alpha_{i,j} (x^{t_{b,1}} z_1)^i (x^{t_{b,2}} z_2)^j$$

for some $\alpha_{i,j} \in \mathbb{C}$. Thus we must have $\deg(x^{t_{b,1}} z_1 + \bar{F}_1) \geq \deg \bar{F}_1$. Observing that $\Phi_a \in EA_2^1(S)$, we see

$$\deg(\bar{P}_a(x^{t_{b,1}} z_1 + \bar{F}_1)) \geq 2 \deg(x^{t_{b,1}} z_1 + \bar{F}_1) \geq 2 \deg \bar{F}_1 > \deg \bar{F}_1 \tag{3.13}$$

By the induction hypothesis, we must have $\deg \bar{F}_1 > \deg \bar{F}_2'$. Thus, from (3.12) and (3.13)

$$\deg \bar{F}_2 = \deg(\bar{P}_a(x^{t_{b,1}} z_1 + \bar{F}_1) > \deg \bar{F}_1$$

Thus $\deg \bar{F}_2 > 0$ and $\deg \bar{F}_2 > \deg \bar{F}_1$ as desired. □

The remainder of this section is the proof of the following proposition, which provides the main ingredient in the proof of Theorem 3.17

**Theorem 3.12.** *Let $\psi = \prod_{i=1}^{q} \Phi_i \in EA_2^1(S)$. Then $\psi$ admits a generalized reducing factorization $\psi = \prod_{i=1}^{q'} \alpha_i \circ \tilde{\Phi}_i$.*

*Proof.* Let $a \leq q$ be minimal such that $\delta_a < 0$. In particular, we must have $a \leq q-2$. We induct on $q - a$, with the base case coming when $\psi$ has a reducing factorization. By the induction hypothesis, we can write $\psi_{a+1,q} = \prod_{i=a+1}^{\tilde{q}} \alpha_i \circ \tilde{\Phi}_i$. We will use $\tilde{t}_{i,j}$, $\tilde{\delta}_i$, etc. to distinguish the respective quantities for this generalized reducing factorization from those of the original $\psi = \prod_{i=1}^{q} \Phi_i$. The induction hypothesis guarantees $\tilde{\delta}_i \geq 0$ for each $a + 1 \leq i \leq \tilde{q}$.

Without loss of generality, assume that $\Phi_a$ is elementary in $z_1$. By Lemma 3.9, we have for some $P_a \in A_a[\hat{z}_1]$,

$$\Phi_a = \left(z_1 + x^{-t_{a,1}} P_a(\hat{z}_1), z_2\right)$$

Let $b > a$ be minimal such that $\tilde{t}_{b,1} < t_{a,1}$. By part (2) of Definition 3.4, for some $Q_b \in \tilde{A}_b[\hat{z}_1] \setminus x\tilde{A}_b[\hat{z}_1]$, we have

$$\tilde{\Phi}_b = \left(z_1 + x^{-t_{a,1}} Q_b(\hat{z}_1), z_2\right)$$

**Claim 3.13.** $\tilde{\delta}_i = 0$ *for each $a + 1 \leq i \leq b - 1$.*

*Proof.* Suppose not, so $t_{a,2} > t_{b,2}$. By Lemma 3.10 we can write

$$\tilde{\psi}_{a+1,b-1} = \left(z_1 + xz_1G_1 + xH_1 + x^{-t_{a,1}}F_1, z_2 + xz_2G_2 + xH_2 + x^{-t_{a,2}}F_2\right)$$

where $F_1, G_1, H_1, F_2, G_2, H_2 \in \tilde{A}_{b-1}$. Since $t_{a,1} > t_{b,1}$, we must have $\tilde{\psi}_{a,b}(x^{t_{a,1}}z_1) \in x\tilde{A}_b$. Compute

$$\begin{aligned}\tilde{\psi}_{a,b}(x^{t_{a,1}}z_1) &= (\Phi_a \circ \tilde{\psi}_{a+1,b-1} \circ \tilde{\Phi}_b)(x^{t_{a,1}}z_1)\\ &= (\tilde{\psi}_{a+1,b-1} \circ \tilde{\Phi}_b)(x^{t_{a,1}}z_1 + P_a(\hat{z}_1))\\ &= \tilde{\Phi}_b\left(x^{t_{a,1}}z_1 + F_1 + x(x^{t_{a,1}}z_1G_1 + H_1) + P_a\left(x^{t_{a,2}}z_2 + F_2 + x(x^{t_{a,2}}z_2G_2 + H_2)\right)\right)\end{aligned}$$

Applying $\tilde{\Phi}_b$ and going modulo $x\tilde{A}_b$,we obtain

$$\begin{aligned}\tilde{\psi}_{a,b}(x^{t_{a,1}}z_1) &\equiv Q_b + F_1(Q_b, x^{t_{a,2}}z_2) + P_a(x^{t_{a,2}}z_2 + F_2(Q_b, x^{t_{a,2}}z_2)) \pmod{x}\\ &\equiv Q_b + F_1(Q_b, 0) + P_a(F_2(Q_b, 0)) \pmod{x}\end{aligned}$$

with the second line following since $t_{a,2} > t_{b,2}$. But note that $F_1(t, 0)$ and $P_a(F_2(t, 0))$ must have $t$-order at least 2, since $\psi \in EA_2^1(S)$, thus giving $\tilde{\psi}_{a,b}(x^{t_{a,1}}z_1 \notin x\tilde{A}_b$, a contradiction. □

Now let $r > a$ be minimal such that $\tilde{\delta}_r > 0$. By Claim 3.13, we must have $\tilde{\Phi}_r$ is elementary in $z_1$, so we have, for some $Q_r \in \tilde{A}_r[\hat{z}_1]$,

$$\tilde{\Phi}_r = (z_1 + x^{-t_{a,1}}Q_r(\hat{z}_1), z_2)$$

First, we observe that we may assume $\alpha_i = id$ for $a + 1 \le i \le r$; since $\tilde{\delta}_i \le 0$ for $a \le i \le r$ we can push each $\alpha_i$ to the left of each preceding $\tilde{\Phi}_j$ by Lemma 3.7. We may also assume that $\tilde{\Phi}_{a+1}$ is elementary in $z_2$. Note also that if $\tilde{\epsilon}_i > 0$ for some $a < i < r$, then by Lemma 3.8, we can push $\tilde{\Phi}_i$ past all preceding $\tilde{\Phi}_j$; so we may assume $\tilde{\epsilon}_i = 0$ for $a \leq i < r$. Then by Lemma 3.11, we may write

$$\tilde{\psi}_{a+1,r-1} = (z_1 + x^{-t_{a,1}} F_1, z_2 + x^{-t_{a,2}} F_2)$$

for some $F_1, F_2 \in \tilde{A}_{r-1}$ with $\deg \bar{F}_2 > \deg \bar{F}_1$. Note that we must have $\tilde{\psi}_{a,r}(x^{t_{a,1}} z_1) \in x\tilde{A}_r$. Compute

$$\begin{aligned}
\tilde{\psi}_{a,r}(x^{t_{a,1}} z_1) &= (\Phi_a \circ \tilde{\psi}_{a+1,r-1} \circ \tilde{\Phi}_r)(x^{t_{a,1}} z_1) \\
&= (\tilde{\psi}_{a+1,r-1} \circ \tilde{\Phi}_r) \left(x^{t_{a,1}} z_1 + P_a(\hat{z}_1)\right) \\
&= \tilde{\Phi}_r \left(x^{t_{a,1}} z_1 + F_1 + P_a(x^{t_{a,2}} z_2 + F_2)\right)
\end{aligned}$$

Applying $\tilde{\Phi}_r$ and going modulo $x\tilde{A}_b$, we obtain

$$\tilde{\psi}_{a,b}(x^{t_{a,1}} z_1) \equiv Q_r + F_1(Q_r, x^{t_{a,2}} z_2) + P_a(x^{t_{a,2}} z_2 + F_2(Q_r, x^{t_{a,2}} z_2)) \pmod{x}$$

Note that the condition $\deg \bar{F}_2 > \deg \bar{F}_1$ implies that $\bar{F}_2 \in I$ where $I = (x^{t_{a,1}} z_1 - Q_r(x^{t_{a,2}} z_2))\, \tilde{A}_r$. However, this forces $\bar{F}_2 = 0$ (and hence $\bar{F}_1 = 0$): For if not, from computing the Jacobian of $\tilde{\psi}_{a+1,r-1}$, one can obtain that $\bar{F}_1$, $\bar{F}_2$ are algebraically dependent; thus $\bar{F}_1 \in I$ as well. But note that we must also have, as in Lemma 3.11

$$\tilde{\psi}_{a+2,r-1} = (z_1 + x^{-t_{a,1}} F_1, z_2 + x^{-t_{a,2}} F_2')$$

for some $F_2' \in \tilde{A}_{r-1}$. Then $\bar{F}_1$ and $\bar{F}_2{}'$ are also algebraically dependent, so $\bar{F}_2{}' \in I$; but recall from (3.12) that $\bar{F}_2 = \bar{F}_2{}' + \tilde{P}_{a+1}(x^{t_{a+1,1}} z_1 + \bar{F}_1$, which implies $P_{a+1}(x^{t_{a+1,1}}) \in I$, a contradiction.

□

## 3.2 Main results

**Theorem 3.14.** *Let $\psi = \prod_{i=1}^{q} \Phi_i \in EA_n(S)$ have a reducing factorization (see Definition 3.2). Suppose that for $0 \leq i \leq q$, $\epsilon_i > 0$ for each $\Phi_i$ that is elementary in any of $z_{k_1}, \ldots, z_{k_r}$. Then there exists $\theta \in GA_n(R)$ with the property*

1. *$\theta(z_{k_j}) = \psi(z_{k_j})$ for each $1 \leq j \leq r$.*

2. *$\theta$ is stably tame.*

*Proof.* We first note that as a consequence of $\psi = \prod_{i=1}^{q} \Phi_i$ being a reducing factorization, $\delta_{a+1} \geq 0$ and thus $IA_n^a(R) \subset IA_n^{a+1}(R)$ for each $0 \leq a < q$. The theorem follows immediately from the following claim, which we prove by induction.

**Claim 3.15.** *For each $a = 0, \ldots, q$, there exists $\phi_a \in IA_n^a(R) \cap EA_n(S)$ such that $\phi_a(z_{k_j}) = (\prod_{i=1}^{a} \Phi_i)(z_{k_j})$ for each $1 \leq j \leq r$. Moreover, $\phi_a$ is stably tame.*

The $a = 0$ case is trivial (set $\phi_0 := id$). So suppose $\phi_a$ is in the prescribed form. Without loss of generality, let $\Phi_{a+1}$ be elementary in $z_1$. Since we assume $\psi$ has a reducing factorization, we may write $\Phi_{a+1}(z_1) = z_1 + x^{-t_{a,1}+\epsilon_{a+1}} P_{a+1}(\hat{z}_1)$ with $P_{a+1} \in A_{a+1}[\hat{z}_1] = A_a[\hat{z}_1]$. If $z_1 \in \{z_{k_1}, \ldots, z_{k_r}\}$, then we must have $t_{a,1} = 0$ and $\epsilon_{a+1} > 0$, in which case $\Phi_{a+1} \in IA_n^a(R) = IA_n^{a+1}(R)$. Then we may set $\phi_{a+1} = \phi_a \circ \Phi_{a+1} \in IA_n^a(R) = IA_n^{a+1}(R)$. Otherwise, we may note that since $\epsilon_{a+1} \geq 0$, $t_{a,1} - \epsilon_a \leq t_{a,1}$, and apply Corollary 3.6 to obtain $\phi_{a+1} := \Phi'_{a+1} \circ \phi_a \circ \Phi_{a+1} \in IA_n^a(R) \subset IA_n^{a+1}(R) \cap EA_n(S)$. Since $\Phi'_{a+1}$ is elementary in $z_1$, for $2 \leq j \leq n$, $\phi_{a+1}(z_j) = \phi_a \circ \Phi_{a+1}$. Since by the induction hypothesis, $\phi_a(z_{k_j}) = (\prod_{i=1}^{a} \Phi_i)(z_{k_j})$, in both cases we have $\phi_{a+1}(z_{k_j}) = (\prod_{i=1}^{a+1} \Phi_i)(z_{k_j})$ for each $1 \leq j \leq r$. The stable tameness claim follows immediately from Theorem 2.8. ◻

One useful application is the following corollary:

**Corollary 3.16.** *Let $\psi = \prod_{i=1}^{q} \Phi_i \in EA_{n-1}(S[y])$ be a reducing factorization, and let $F \in A_0$. Then $y + x\psi(F)$ is an $R$-coordinate of a stably tame automorphism.*

This furnishes the deferred proof of Theorem 2.27, restated here for convenience.

**Theorem 2.21.** *Let $f = y + xQ$, $Q \in R[v,w]$ be a Vénéreau-type polynomial. Then there exists $\phi \in GA_4(R)$ with $\phi(y) = f$ and $\phi$ stably tame.*

*Proof.* Define

$$\begin{aligned}
\Phi_1 &= (y, z + yt, u, t) \\
\Phi_2 &= (y, z, u - 2zt - yt^2, t) \\
\Phi_3 &= (y, z, u, t + \frac{yu + z^2}{x}) \\
\Phi_4 &= (y, z - yt, u, t) \\
\Phi_5 &= (y, z, u + 2zt - yt^2, t)
\end{aligned}$$

One easily checks that $\psi = \Phi_1 \circ \cdots \circ \Phi_5$ is a reducing factorization and $Q(xz, x^2u) \in A_0$, with $\psi(Q(xz, x^2u)) = Q(v, w)$. Then the previous corollary gives the theorem. $\square$

For the remainder of this section, let $B = R[y_1, \dots, y_n]$, $B_x = B \otimes_R S \cong B \otimes_{\mathbb{C}[x]} \mathbb{C}[x, x^{-1}]$, and $B^{[2]} = B[z_1, z_2]$. Consider $\psi \in \langle EA_2^1(B_x), GA_n(R[z_1, z_2])\rangle \subset GA_{n+2}(S)$. Write $\psi = \prod_{i=1}^{q} \Psi_i \circ \Phi_i$ where $\Psi_i \in GA_n(R[z_1, z_2])$ and $\Phi_i \in EA_2^1(B_x)$. For $0 \le a \le q$, $1 \le j \le 2$, define

$$\begin{aligned}
\tilde{t}_{a,j} &= \min\{t \mid (\prod_{i=a+1}^{q} \Psi_i \circ \Phi_i)(x^t z_j) \in B^{[2]}\} \\
\tilde{A}_a &= B[x^{t_{a,1}} z_1, x^{t_{a,2}} z_2]
\end{aligned}$$

We will also write $\widetilde{IA^a_{n+2}}(R)$ for the subgroup of all automorphisms of the form

$$\alpha = (y_1 + xF_1, \dots, y_n + xF_n, z_1 + xz_1G_1 + xH_1, z_2 + xz_2G_2 + xH_2)$$

where $F_j, G_1, H_1, G_2, H_2 \in \tilde{A}_a$.

**Theorem 3.17.** *Let $B = R[y_1, \ldots, y_n]$ and $B_x = B \otimes_R S$. Let $\psi = \prod_{i=1}^{q} \Psi_i \circ \Phi_i \in EA_{n+2}(S)$ where $\Phi_i \in EA_2^1(B_x)$ and $\Psi_i \in \widetilde{IA_{n+2}^{a-1}}(R) \cap GA_n(R^{[2]})$. Then $\psi$ is elementarily equivalent to an automorphism $\theta \in GA_{n+2}(R)$ and $\theta(y_j) = \psi(y_j)$ for $1 \leq j \leq n$. Moreover, if each $\Psi_i \in EA_n(S^{[2]})$, then $\theta$ is stably tame.*

*Proof.* Similar to the proof of Theorem 3.14, the theorem follows from

**Claim 3.18.** *For each $1 \leq a \leq q$, there exists $\Theta_a \in \widetilde{IA_{n+2}^{a}}(R)$ with $\Theta_a(y_j) = (\prod_{i=1}^{a} \Psi_i \circ \Phi_i)(y_j)$ for each $1 \leq j \leq n$. Moreover, if $\Psi_i \in EA_{n+2}(S)$ for $1 \leq i \leq a$, then $\Theta_a$ is stably tame.*

We induct on $a$, with $a = 0$ being trivial. First set $\Theta'_a = \Theta_a \circ \Psi_{a+1}$. By the induction hypothesis, $\Theta_a \in \widetilde{IA_{n+2}^{a}}(R)$, so $\Theta'_a$ is as well. So we simply replace $\Theta_a$ by $\Theta'_a$ and assume $\Psi_{a+1} = id$. Since $\Phi_{a+1} \in EA_2(B_x)$, we may write

$$\Phi_{a+1} = \prod_{k=1}^{q_{a+1}} \varphi_k$$

and define for $0 \leq b \leq q_{a+1}$, $1 \leq j \leq 2$

$$t_{a+1,b,j} = \min\{t \in \mathbb{N} | (\prod_{k=b+1}^{q_{a+1}} \varphi_k \prod_{r=a+2}^{q} \Psi_r \circ \Phi_r)(x^t z_j) \in B^{[2]}\}$$

$$A_{a+1,b} = B[x^{t_{a+1,b,1}} z_1, x^{t_{a+1,b,2}} z_2] \tag{3.14}$$

This is simply a reindexing of our usual definition of $t_{i,j}$. Now by Theorem 3.12, we may assume $\Phi_{a+1}$ has a generalized reducing factorization.

$$\Phi_{a+1} = \prod_{b=1}^{\tilde{q}_{a+1}} \alpha_b \tilde{\varphi}_b$$

where $\alpha_b \in IA_2^{b-1}(B)$ and $\tilde{\varphi}_b \in EA_2^1(B_x)$. Thus, by Corollary 3.6, there is some $\gamma \in EA_2(B_x)$ with $\Theta_{a+1} := \gamma \circ \Theta_a \circ \Phi_{a+1} \in \widetilde{IA_{n+2}^{a+1}}(R)$. Since $y_j \in B$ and $\gamma \in EA_2(B_x)$, we have $\Theta_{a+1}(y_j) = (\Theta_a \circ \Phi_{a+1})(y)$. The stable tameness claim follows immediately from Theorem 2.8. $\square$

This has the following nice corollary:

**Corollary 3.19.** *Let* $\psi = \prod_{i=1}^{q} \Phi_i \in EA_2^1(S[y])$ *and* $F \in A_0 \cap R^{[2]}$*. Then* $y + x\psi(F)$ *is a coordinate of a stably tame automorphism.*

This has the following consequence: When considering DW($\mathbb{C}^{[1]}$,1,3), one essentially must either use a wild $\psi$ or an $F \in \psi^{-1}(R^{[2]}) \setminus A_0$ to construct a strongly residual coordinate if there is any hope of it not being a coordinate as well. Since Vénéreau-type polynomials arise from perhaps the simplest wild automorphism, they can thus be considered to be the simplest strongly residual coordinates that are not known to be coordinates.

## 3.3 Examples

The Vénéreau polynomial (and Vénéreau-type polynomials) provide examples of strongly residual coordinates that are not known to be coordinates. However, they are wild over $\mathbb{C}[x, x^{-1}]$. Here we provide examples of tame strongly residual coordinates that are not known to be coordinates.

*Example* 1. Define $\phi = \Phi_0 \circ \cdots \Phi_6$, where

$$\Phi_0 = (y + x(z_3(xz_1)), z_1, z_2, z_3)$$

$$\Phi_1 = (y, z_1, z_2, z_3 - \frac{yz_2 + z_1^2}{x}) \qquad \Phi_4 = (y, z_1, z_2, z_3 + \frac{yz_2 + z_1^2}{x})$$

$$\Phi_2 = (y, z_1 + yz_3, z_2, z_3) \qquad \Phi_5 = (y, z_1 - yz_3, z_2, z_3)$$

$$\Phi_3 = (y, z_1, z_2 - 2z_1z_3 - yz_3^2, z_3) \qquad \Phi_6 = (y, z_1, z_2 + 2z_1z_3 - yz_3^2, z_3)$$

Then $\phi(y) = y + xz_3(xz_1 + y(yz_2 + z_1^2))$ is a tame strongly residual coordinate.

However, the composition $\phi$ is not a reducing factorization. It is unkown whether $\phi(y)$ is a coordinate (although it is a 1-stable coordinate). Note that this is quite similar to the Vénéreau polynomial $b_1$.

*Example* 2. Define $\phi = \Phi_0 \circ \cdots \Phi_6$, where

$$\Phi_0 = (y + x(xz_3(xz_1)), z_1, z_2, z_3)$$

$$\Phi_1 = (y, z_1, z_2, z_3 + \frac{(yz_2 + z_1^2)^2}{x}) \qquad \Phi_4 = (y, z_1, z_2, z_3 + \frac{yz_2 + z_1^2}{x})$$

$$\Phi_2 = (y, z_1 + yz_3, z_2, z_3) \qquad \Phi_5 = (y, z_1 - yz_3, z_2, z_3)$$

$$\Phi_3 = (y, z_1, z_2 - 2z_1z_3 - yz_3^2, z_3) \qquad \Phi_6 = (y, z_1, z_2 + 2z_1z_3 - yz_3^2, z_3)$$

Then $\phi(y) = y+x(xz_3+(yz_2+z_1^2)+(yz_2+z_1^2)^2)(xz_1+y(yz_2+z_1^2))$ is a tame strongly residual coordinate. However, the composition $\phi$ is not a reducing factorization. While the previous example failed both conditions of Definition 3.2, this does satisfy that all $\delta_i$ are nonnegative. It is unkown whether $\phi(y)$ is a coordinate.

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