5-24-2012

Comparison Theorems in Elliptic Partial Differential Equations with Neumann Boundary Conditions

Jeffrey Langford
Washington University in St. Louis

WASHINGTON UNIVERSITY IN ST. LOUIS

Department of Mathematics

Dissertation Examination Committee:
Albert Baernstein II, Co-Chair
John McCarthy, Co-Chair
Richard Laugesen
Richard Rochberg
James Schilling
Guido Weiss

Comparison Theorems in Elliptic Partial Differential Equations with Neumann Boundary Conditions

by

Jeffrey Joseph Langford

A dissertation presented to the
Graduate School of Arts and Sciences
of Washington University in
partial fulfillment of the
requirements for the degree
of Doctor of Philosophy

May 2012

Saint Louis, Missouri

Acknowledgements

I am truly grateful to have had two phenomenal advisors in Albert Baernstein II and Richard Laugesen. I am thankful to Al for introducing me to the beauty of symmetrization and for all of your time and patience over the years. I thank Rick for the countless Skype chats and all of the hospitality you showed me during my visits to Illinois. Without the help of my advisors, this work would not have been possible.

I would like to thank Washington University and the Mathematics Department for financial support over the years from 2006–2012. I especially thank professors John McCarthy and Richard Rochberg for their grant support as well.

I am grateful to my mom, dad, sister, and grandparents for their constant encouragement and support. I am also thankful for the friends who have made my time at Washington University the most enjoyable period of my life.

Last I would like to thank Kelly, who makes life so much more fun.

Contents

List of Figures

Chapter 0: A Preview

The history of comparison theorems in elliptic partial differential equations dates to the mid 1970's, when G. Talenti proved his now famous result known as Talenti's Theorem [**T**]. Talenti compared the solutions of two partial differential equations (PDEs) that impose homogeneous Dirichlet boundary conditions. To be precise, let $0 \le f \in L^2(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and consider the solution u to the Poisson PDE

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where Δ is the standard Laplacian operator. Talenti then considered a second PDE defined on a ball $\Omega^\#$ with the same volume as Ω. In this second PDE, the input data is obtained by "rearranging" f's values into a radial function $f^\#$ that decreases as the radial variable r increases. After solving the PDE

$$\begin{aligned} -\Delta v &= f^\# \quad \text{in} \quad \Omega^\#, \\ v &= 0 \quad \text{on} \quad \partial\Omega^\#. \end{aligned}$$

Talenti found that the two solutions u and v are comparable through their decreasing rearrangements, a finding with consequences about L^p norms: $\|u\|_{L^p(\Omega)} \le \|v\|_{L^p(\Omega^\#)}$, $1 \le p \le \infty$, and oscillation: $\operatorname*{osc}_{\Omega} u = \max_{\Omega} u \le \max_{\Omega^\#} v = \operatorname*{osc}_{\Omega^\#} v$.

Talenti's Theorem has the following physical interpretation. Suppose we are standing inside of an arbitrarily shaped room where the walls are held at temperature zero. At each spot inside the room heat is being generated, and some spots generate more heat than others. Now consider a circular room with the same size as the original room and with a heat source that is hottest at the center of the room and coolest near the walls. Additionally, the spots inside the first and second room that generate a given amount of heat occupy the same area. Talenti's Theorem implies that the maximum temperature of the first room is no larger than the maximum temperature of the second room.

The process of rearranging a function f's values into a function $f^{\#}$ that is radially decreasing is known as the Schwarz rearrangement. Since there are different ways to rearrange a function and other types of boundary conditions, the work of Talenti sparked a study of comparison theorems using different rearrangements and/or different boundary conditions. Alvino, Lions, and Trombetti [**ALT**] compared the solutions of two PDEs with homogeneous Dirichlet boundary conditions, one with initial data f, and the other with data $f^{\#}$ obtained from f by performing a Steiner symmetrization. Under Steiner symmetrization, $f^{\#}$ is obtained by performing a Schwarz rearrangement on slice functions of f. They do not reach as strong a conclusion as Talenti's, but they still deduce the same L^p and oscillation inequalities as in Talenti's Theorem.

Theorems also appear in the literature comparing the solution of an initial PDE imposing homogeneous Neumann boundary conditions to a coupled system of PDEs taking various forms: in [**MS**], the coupled system of PDEs imposes homogeneous Dirichlet boundary conditions; in [**AMT**], the coupled system of PDEs imposes inhomogeneous Neumann and Dirichlet boundary conditions; and in [**FM**], the coupled system of PDEs imposes mixed boundary conditions. But what appears missing from the literature are comparison theorems imposing Neumann boundary conditions on the first and second PDE, following in the true spirit of Talenti's Theorem. In this thesis, we prove several such results.

In general, we will begin with a PDE of the form

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where Ω will either be a ball, an annulus, a sphere, or a hemisphere. Our rearranged PDE will be defined on the same space, with

$$\begin{aligned} -\Delta v &= f^{\#} \quad \text{in} \quad \Omega, \\ \frac{\partial v}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $f^{\#}$ is some rearrangement of f. We identify spaces and rearrangements that yield comparison theorems with the same L^p norm consequences as Talenti's Theorem as well as oscillation inequalities. Namely, that $\|u\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega)}$, $1 \leq p \leq \infty$, and $\operatorname*{osc}_{\Omega} u \leq \operatorname*{osc}_{\Omega} v$.

Chapter 1 presents a dictionary of the various rearrangements, star functions, and subharmonicity results that will be used throughout the thesis. We begin Chapter 2 with a conjecture of B. Kawohl from his 1985 book [**Ka**], and its solution by A. Baernstein in 1986. Kawohl conjectured that the oscillation of a solution to Poisson's equation in a rectangle increases when the slice functions of the input data f are rearranged in a monotone decreasing manner. Baernstein proved Kawohl's conjecture by first proving a comparison theorem on an annulus involving cap symmetrization. By a conformal mapping, a comparison theorem on a rectangle is deduced which consequently gives Kawohl's conjecture. The ideas in Chapter 2 are Baernstein's, but the proof of the annular comparison result (Theorem 2.2) is different than the original sent to Kawohl, in light of the development of the theory of the "star function." Also, the deduction of the rectangular comparison result (Theorem 2.4) from the annular comparison result did not appear in the original correspondence, and is due to the author. The proof of Kawohl's conjecture (Corollary 2.6) from the rectangular comparison result differs from the one sent by Baernstein to Kawohl and is also by the author.

The heart of this thesis, and of the original work by the author, begins in Chapter 3, where our main results appear as Theorem 3.1 and Corollary 3.3. These results generalize the two-dimensional annular comparison result (Theorem 2.2) of Baernstein from Chapter 2. It should be mentioned at the outset that generalizing Theorem 2.2 to higher dimensions is not immediate, but requires a domain approximation argument that is not necessary in the two-dimensional case. Here is the main result from Chapter 3, which appears as Theorem 3.1.

Theorem (Comparison Theorem in Spherical Shells)**.** *Let $A = \{x \in \mathbb{R}^n : a < |x| < b\}$ be a spherical shell and let $f \in L^2(A)$ with $\int_A f \, \mathrm{d}x = 0$. Assume u and v are weak solutions to*

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad A, & -\Delta v &= f^{\#} \quad \text{in} \quad A, \\ \tfrac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial A, & \tfrac{\partial v}{\partial n} &= 0 \quad \text{on} \quad \partial A, \\ \textstyle\int_A u \, \mathrm{d}x &= 0, & \textstyle\int_A v \, \mathrm{d}x &= 0, \end{aligned}$$

where $f^{\#}$ is the $(n-1, n)$ cap symmetrization of f (explained in Chapter 1).

Then for almost every $r \in (a, b)$ and each convex function $\phi : \mathbb{R} \to \mathbb{R}$ we have

$$\int_{\mathbb{S}^{n-1}} \phi(u(r\xi)) \, \mathrm{d}\sigma_{n-1}(\xi) \;\leq\; \int_{\mathbb{S}^{n-1}} \phi(v(r\xi)) \, \mathrm{d}\sigma_{n-1}(\xi).$$

Consequently,

$$\|u\|_{L^p(A)} \leq \|v\|_{L^p(A)} \quad 1 \leq p \leq \infty,$$

and

$$\operatorname*{osc}_A u \leq \operatorname*{osc}_A v.$$

The remainder of Chapters 3 and Chapter 4 are devoted to consequences of the main results. The second section of Chapter 3 discusses comparison results on spheres (Corollary 3.5) and hemispheres (Corollary 3.7) that follow from Theorem 3.1. In Chapter 4, we project the hemispherical comparison result (Corollary 3.7) into the unit ball with stereographic projection, to obtain a weighted comparison result in the unit ball (Theorem 4.4). We use this result to obtain oscillation estimates in terms of the input data for solutions to a weighted Poisson equation in the unit disk (Corollary 4.6). Consequently, we obtain oscillation estimates for the standard (unweighted) Poisson equation as well (Corollary 4.7). Corollaries 4.6 and 4.7 are among the most appealing results of the thesis, because their statements can be read without any knowledge of rearrangements.

The results of Chapter 5 are independent of the rest of the thesis. The first main result (Theorem 5.5) can be viewed as a one-dimensional analogue of Talenti's Theorem for an interval, except instead of imposing homogeneous Dirichlet boundary conditions as Talenti did, we impose homogeneous Neumann boundary conditions. In Talenti's situation, the input data should be symmetric decreasing in order to maximize L^p norms and oscillation. In contrast, when imposing homogeneous Neumann boundary conditions, Theorem 5.5 says to make the input data monotone decreasing to maximize L^p norms and oscillation. We next prove a comparison result in the disk using cap symmetrization (Theorem 5.7) that uses the Neumann Green's function for the unit disk discussed in Appendix A. We end Chapter 5 by showing that no (reasonable) comparison theorem exists for the Schwarz rearrangement under Neumann boundary conditions. Specifically, Example 5.9 shows the following on the unit disk $\mathbb{D}$:

Example. There exists a function $f \in L^2(\mathbb{D})$ with $\int_{\mathbb{D}} f \, dx = 0$ such that when u and v are weak solutions to

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \mathbb{D}, \qquad & -\Delta v &= f^{\#} \quad \text{in} \quad \mathbb{D}, \\ \tfrac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}, \qquad & \tfrac{\partial v}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}, \end{aligned}$$

and $f^{\#}$ is the Schwarz rearrangement of f, we have $\operatorname*{osc}_{\mathbb{D}} u > \operatorname*{osc}_{\mathbb{D}} v$.

The example above comes by taking f equal to 1 on the right half of the unit disk and -1 on the left half of the unit disk.

We end the thesis by discussing some open problems in Chapter 6, one of which was originally motivated by the above example. Which function f defined in the unit disk and taking the values 1 and -1, each on half of $\mathbb{D}$, generates a solution (to Poisson's equation with homogeneous Neumann boundary conditions) having the greatest oscillation? This problem can be interpreted physically, as the problem below describes.

Problem. You are standing in a perfectly insulated circular room. In half of the locations in the room, heat is generated at unit rate. In the remainder of the room, heat is absorbed, also at unit rate. If you are allowed to choose where heat is generated and where heat is absorbed, which arrangement will produce the greatest difference in temperature across the room?

Mathematically, we solve a Poisson equation in the unit disk $\mathbb{D} \subset \mathbb{R}^2$ with Neumann boundary conditions:

$$\begin{aligned} -\Delta u &= \mathbf{1}_E - \mathbf{1}_{\mathbb{D}\setminus E} \quad \text{in} \quad \mathbb{D}, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}, \end{aligned}$$

where the sets E and $\mathbb{D} \setminus E$ each have area $\frac{\pi}{2}$. We can think of u as solving a steady state heat equation, with u independent of time t; then u represents the equilibrium temperature. The problem asks to find the set E that maximizes the temperature gap $\operatorname*{osc}_{\mathbb{D}} u = \max_{\mathbb{D}} u - \min_{\mathbb{D}} u$.

We finally discuss a rearrangement originally studied by Leckband [**Le**] to prove Moser's inequality on the unit ball of $\mathbb{R}^n$. We conjecture that this rearrangement plays the role for PDEs with Neumann boundary conditions that the Schwarz rearrangement did for PDEs with Dirichlet boundary conditions.

CHAPTER 1

Background: Rearrangements and PDEs

In this chapter, we lay out the basic concepts used throughout the thesis. We begin with rearrangements, and give meaning to the concept of two functions having the "same size." We discuss several canonical rearrangements and the associated star functions. For functions that satisfy a partial differential equation (PDE), we have a differential inequality involving "star functions." We call these differential inequalities "Subharmonicity Results" and they are described below as well.

We next touch briefly on existence and estimates of solutions to Poisson's equation with homogeneous Neumann boundary conditions. The main theorems of this thesis assume the existence of solutions to Poisson's equation. The results of Chapter 1 show that these solutions really do exist.

1.1. A catalogue of rearrangements and star functions

This thesis studies how the behavior of solutions to PDEs changes when the data are rearranged. We therefore begin by defining several canonical rearrangements that will be used throughout the thesis. To each rearrangement corresponds the notion of a star function, and associated commutativity and subharmonicity results. We will see that star functions allow us to compare the solution of a PDE to the solution of its corresponding "rearranged" PDE.

1.1.1. Rearrangements. Our first definition makes precise the notion of two functions having the same size. We measure the size of functions by measuring the size of their "upper" level sets.

Definition 1.1 (Rearrangements)**.** Given functions $f \in L^1(X)$ and $g \in L^1(Y)$ defined on measure spaces (X, μ) and (Y, ν), we say f and g are *rearrangements* of each other if

$$\mu(\{x \in X : t < f(x)\}) \quad = \quad \nu(\{y \in Y : t < g(y)\})$$

for every $t \in \mathbb{R}$.

A good starting point for analysts who want to learn about rearrangement methods is the book by Lieb and Loss [**LL**], which tackles a number of standard and not so standard results in analysis using rearrangement methods.

1.1.2. The decreasing rearrangement. Throughout this subsection, (X, μ) denotes a fixed finite measure space. Given a function $f : X \to \mathbb{R}$, we can construct a fundamental rearrangement of f, called the decreasing rearrangement and denoted by f^*, that is defined on the (possibly infinite) interval $[0, \mu(X)]$. The decreasing rearrangement f^* is a decreasing right continuous rearrangement of f defined on an interval.

Definition 1.2 (Decreasing Rearrangement)**.** Let $f \in L^1(X)$ and define $f^* : [0, \mu(X)] \to \mathbb{R}$ by the formula

$$f^*(t) = \begin{cases} \operatorname*{ess\,sup}_X f & \text{if } t = 0 \\ \inf\{s : \mu(\{x : s < f(x)\}) \leq t\} & \text{if } t \in (0, \mu(X)) \\ \operatorname*{ess\,inf}_X f & \text{if } t = \mu(X). \end{cases}$$

We call f^* the *decreasing rearrangement* of f.

Lemma 1.3. *f^* defines a rearrangement of f.*

Proof. Write $\lambda_f(t) = \mu(\{x \in X : t < f(x)\})$ and similarly $\lambda_{f^*}(t) = |\{x \in [0, \mu(X)] : t < f^*(x)\}|$, where we have written absolute value for one-dimensional Lebesgue measure. We show $\lambda_f = \lambda_{f^*}$. First fix $t \in (\operatorname{ess\,inf}_X f, \operatorname{ess\,sup}_X f)$. The equality $\lambda_f(t) = \lambda_{f^*}(t)$ follows if we can show

$$\big(0, \lambda_f(t)\big) \quad = \quad \{x \in (0, \mu(X)) : t < f^*(x)\},$$

which is equivalent to proving

$$x < \lambda_f(t) \quad \text{if and only if} \quad t < f^*(x).$$

To establish the above equivalence, we write $f^*(x) = \inf\{s : \lambda_f(s) \leq x\}$. If $\lambda_f(t) \leq x$, then by definition, $f^*(x)$ is the smallest s where $\lambda_f(s) \leq x$. Since $\lambda_f(t) \leq x$, it follows that $f^*(x) \leq t$. Conversely, assume $\inf\{s : \lambda_f(s) \leq x\} = f^*(x) \leq t$. Since λ_f is a decreasing function, it follows that $\lambda_f(s) \leq x$ for any $f^*(x) \leq s$. Taking $s = t$ gives $\lambda_f(t) \leq x$.

When $t = \operatorname{ess\,inf}_X f$, we use the right continuity of λ_f and λ_{f^*}, to conclude $\lambda_f(\operatorname{ess\,inf}_X f) = \lambda_{f^*}(\operatorname{ess\,inf}_X f)$. When $t < \operatorname{ess\,inf}_X f$, both λ_f and λ_{f^*} equal $\mu(X)$. Finally, when $t \geq \operatorname{ess\,sup}_X f$, both λ_f and λ_{f^*} equal zero. Hence, f^* is a rearrangement of f. $\square$

The following result says that the decreasing rearrangement is a contraction in the L^p distance. It appears as Proposition 1.2.1 of [**Ke**].

Theorem 1.4 (Decreasing Rearrangement Contracts L^p Distance)**.** *Let $f, g \in L^1(X)$. Then for each $1 \leq p < \infty$, we have*

$$\int_0^{\mu(X)} |f^* - g^*|^p \ \mathrm{d}t \ \ \leq \ \ \int_X |f - g|^p \ \mathrm{d}\mu.$$

The first two pictures in Figure 1.1 on page 18 show a function f together with its decreasing rearrangement f^*.

We now define the star function for a general measure space. Proposition 1.8 establishes the connection between the star function and the decreasing rearrangement.

Definition 1.5 (Star Function for a General Measure Space)**.** Let $f \in L^1(X)$. The star function of f will be denoted by $f^\bigstar$ and is defined on the interval $[0, \mu(X)]$ by the formula

$$f^\bigstar(t) \ \ = \ \ \sup_{\mu(E)=t} \int_E f \ \mathrm{d}\mu,$$

where the sup is taken over all measurable subsets $E \subseteq X$ with $\mu(E) = t$.

Before proceeding, we need the following definition.

Definition 1.6. Assume (X, μ) is a measure space and $B \subseteq X$ with $0 < \mu(B)$. We say B is an *atom* if for every subset $A \subseteq B$, either $\mu(A) = \mu(B)$ or $\mu(A) = 0$. The measure space (X, μ) is called *non-atomic* if it contains no atoms.

A subset of $\mathbb{R}^n$ with Lebesgue measure, for example, is a non-atomic measure space.

A result of W. Sierpiński says that a non-atomic measure space assumes a continuum of values. Precisely, given a subset $B \subseteq X$ with $0 < \mu(B)$, for any $a \leq \mu(B)$ there exists a subset of $A \subseteq B$ with $\mu(A) = a$. See Theorem 13 of [**Fry**].

It is a result of Baernstein that for any t value, there exists a subset $E \subseteq X$ for which the sup defining $f^\bigstar(t)$ is achieved (Proposition 1 of [**Ba2**]). We prove this result below. When we define star functions later on, we are thus justified using max rather than sup.

Proposition 1.7. *Assume $f \in L^1(X)$ with (X, μ) a finite non-atomic measure space. Given $t \in [0, \mu(X)]$, there exists a subset $E \subseteq X$ such that*

$$f^\bigstar(t) \quad = \quad \int_E f \; \mathrm{d}\mu.$$

Thus, the sup *defining $f^\bigstar$ is really a* max.

PROOF. Equality holds when $t = 0$ or $t = \mu(X)$ by taking $E = \emptyset$ or $E = X$, respectively. Assume $t \in (0, \mu(X))$. As a function of s,

$$\mu(\{x : s < f(x)\})$$

is decreasing and right continuous. Hence there exists an s where

$$\mu(\{x : s < f(x)\}) \quad \leq \quad t \quad \leq \quad \mu(\{x : s \leq f(x)\}). \tag{1.1}$$

Since (X, μ) is non-atomic, we use Sierpiński's result (mentioned after Definition 1.6) to pick a subset $E \subseteq X$ with $\mu(E) = t$ and where

$$\{x : s < f(x)\} \quad \subseteq \quad E \quad \subseteq \quad \{x : s \leq f(x)\}.$$

If F is any subset of X with $\mu(F) = t$, we have

$$\begin{aligned}\int_F f \; \mathrm{d}\mu &= \int_F (f - s) \; \mathrm{d}\mu + st \\ &\leq \int_X (f - s)^+ \; \mathrm{d}\mu + st \\ &= \int_E (f - s) \; \mathrm{d}\mu + st \\ &= \int_E f \; \mathrm{d}\mu,\end{aligned}$$

which gives the proposition. □

The set of length t on which the decreasing rearrangement f^* is biggest is the interval $[0, t]$. Since f^* is a rearrangement of f, it seems plausible that $f^\star(t) = \int_0^t f^*(x) \; \mathrm{d}x$. Our next proposition verifies this equality.

Proposition 1.8. *Assume $f \in L^1(X)$ with (X, μ) a finite non-atomic measure space. Then for each $t \in [0, \mu(X)]$,*

$$f^\star(t) = \int_0^t f^*(x) \; \mathrm{d}x.$$

PROOF. When $t = 0$, equality obviously holds. When $t = \mu(X)$, equality holds because $\int_0^{\mu(X)} f^* \; \mathrm{d}x = \int_X f \; \mathrm{d}\mu$. Let $t \in (0, \mu(X))$. Let s and E be as in the proof of Proposition 1.7. Then,

$$\begin{aligned}f^\star(t) &= \int_E f \; \mathrm{d}\mu \\ &= \int_X (f - s)^+ \; \mathrm{d}\mu + st \\ &= \int_0^{\mu(X)} \big(f^*(x) - s\big)^+ \; \mathrm{d}x + st,\end{aligned}$$

where the last equality holds because $(f-s)^+$ and $(f^*-s)^+$ are rearrangements of each other. It follows from equation (1.1) that

$$\{x : s < f^*(x)\} \;\subseteq\; [0,t) \;\subseteq\; \{x : s \leq f^*(x)\}.$$

Thus,

$$\begin{aligned} \int_0^{\mu(X)} \big(f^*(x)-s\big)^+ \,\mathrm{d}x + st &= \int_0^t \big(f^*(x)-s\big)^+ \,\mathrm{d}x + st \\ &= \int_0^t f^*(x)\,\mathrm{d}x. \end{aligned}$$

□

Star functions first appeared as a tool to prove Edrei's "Spread Conjecture" [**Ba1**] about growth of meromorphic functions in the plane, and have since been used to solve other extremal problems involving various norms of Schlicht functions and Green's functions [**Ba2, Ba3**].

Star function inequalities define a type of "majorization." Our next proposition says that star function inequalities can be rephrased in terms of convex mean inequalities and appears as Proposition 3 of [**Ba2**]. The proof of Proposition 3 in [**Ba2**] is for functions defined on an interval. By passing to decreasing rearrangements, the result also holds for functions defined on a general measure space.

Proposition 1.9 (Majorization)**.** *Let $u, v \in L^1(X)$. Then*

$$u^\star \;\leq\; v^\star$$

on $[0, \mu(X)]$ *if and only if the inequality*

$$\int_X \phi(u)\ \mathrm{d}\mu \ \ \leq \ \ \int_X \phi(v)\ \mathrm{d}\mu$$

holds for every increasing convex function $\phi : \mathbb{R} \to \mathbb{R}$.

Moreover, if $\int_X u\ \mathrm{d}\mu = \int_X v\ \mathrm{d}\mu$, *then the word "increasing" may be removed from the previous statement.*

WARNING: The above proposition does not assert that the phi integrals are finite.

We need a definition before the next corollary.

Definition 1.10 (Oscillation)**.** If $u : X \to \mathbb{R}$ is measurable, we define the *oscillation* by

$$\operatorname*{osc}_X u \ \ = \ \ \operatorname*{ess\,sup}_X u - \operatorname*{ess\,inf}_X u.$$

The next corollary gives two important consequences of majorization in the sense of star functions.

Corollary 1.11. *Let* $u, v \in L^1(X)$ *where* $\int_X u\ \mathrm{d}\mu = \int_X v\ \mathrm{d}\mu$ *and assume* (X, μ) *is a finite measure space. If* $u^\star \leq v^\star$ *on* $[0, \mu(X)]$, *then*

$$\|u\|_{L^p(X, \mathrm{d}\mu)} \ \ \leq \ \ \|v\|_{L^p(X, \mathrm{d}\mu)}, \quad 1 \leq p \leq \infty.$$

Moreover,

$$\begin{aligned}
\operatorname*{ess\,sup}_X u &\le \operatorname*{ess\,sup}_X v, \\
\operatorname*{ess\,inf}_X u &\ge \operatorname*{ess\,inf}_X v, \\
\operatorname*{osc}_X u &\le \operatorname*{osc}_X v.
\end{aligned}$$

WARNING: It is not assumed that the L^p norms, ess inf, ess sup, and osc above are finite. Rather, if the L^p norm of v is finite, then so is the L^p norm of u. Likewise, if the L^p norm of u is infinite, then so is the L^p norm of v. Similar considerations apply to the other inequalities.

Proof. By Proposition 1.9, the inequality

$$\int_X \phi(u)\, \mathrm{d}\mu \;\le\; \int_X \phi(v)\, \mathrm{d}\mu$$

holds for each convex function $\phi : \mathbb{R} \to \mathbb{R}$. Taking $\phi(x) = |x|^p$ establishes the L^p norm inequality for $1 \le p < \infty$. Letting $p \to \infty$ gives the case when $p = \infty$.

To establish the ess sup inequality, we rewrite $u^\star \le v^\star$ using Proposition 1.8 as

$$\int_0^t u^*(s)\, \mathrm{d}s \;\le\; \int_0^t v^*(s)\, \mathrm{d}s$$

for every $0 \le t \le \mu(X)$. Multiplying the inequality above by $\frac{1}{t}$ and taking the limit as $t \to 0$, we obtain

$$\begin{aligned}
\operatorname*{ess\,sup}_X u &= \operatorname*{ess\,sup}_{[0,\mu(X)]} u^* \\
&= \lim_{t\to 0} \frac{1}{t}\int_0^t u^*(s)\ \mathrm{d}s \\
&\le \lim_{t\to 0} \frac{1}{t}\int_0^t v^*(s)\ \mathrm{d}s \\
&= \operatorname*{ess\,sup}_{[0,\mu(X)]} v^* \\
&= \operatorname*{ess\,sup}_X v.
\end{aligned}$$

Since $\int_X u\ \mathrm{d}\mu = \int_X v\ \mathrm{d}\mu$, we also have $(-v)^\bigstar \le (-u)^\bigstar$. Hence the argument above implies that

$$\operatorname*{ess\,sup}_X\ -v \ \le\ \operatorname*{ess\,sup}_X\ -u$$

and consequently

$$\operatorname*{ess\,inf}_X v \ \le\ \operatorname*{ess\,inf}_X u.$$

The osc inequality now follows by combining the ess sup and ess inf inequalities. □

1.1.3. The Schwarz rearrangement. Throughout this subsection, $\Omega \subseteq \mathbb{R}^n$ denotes a non-empty subset and $\Omega^\# \subseteq \mathbb{R}^n$ denotes the open ball centered at the origin with the same Lebesgue measure as Ω. Write $B(0,R) = \{x \in \mathbb{R}^n : |x| < R\}$ for this ball. That is,

$$\Omega^\# \ =\ B(0,R).$$

When Ω has infinite measure, $R = \infty$ and consequently $\Omega^\#$ will be all of $\mathbb{R}^n$. Given a function $f : \Omega \to \mathbb{R}$, we can construct a radially decreasing rearrangement $f^\# : \Omega^\# \to \mathbb{R}$ called the Schwarz rearrangement of f.

Definition 1.12 (Schwarz Rearrangement)**.** If $f \in L^1(\Omega)$, define $f^\# : \Omega^\# \to \mathbb{R}$ by the formula

$$f^\#(x) \quad = \quad f^*(\alpha_n |x|^n),$$

where α_n is the volume of unit ball in $\mathbb{R}^n$ and f^* is the decreasing rearrangement of f. We call $f^\#$ the *Schwarz rearrangement* of f. The Schwarz rearrangement is also sometimes called the *symmetric decreasing rearrangement*, or *s.d.r.* for short.

Figure 1.1 below shows the graph of a function f in one dimension, together with its decreasing rearrangement f^* and its Schwarz rearrangement $f^\#$.

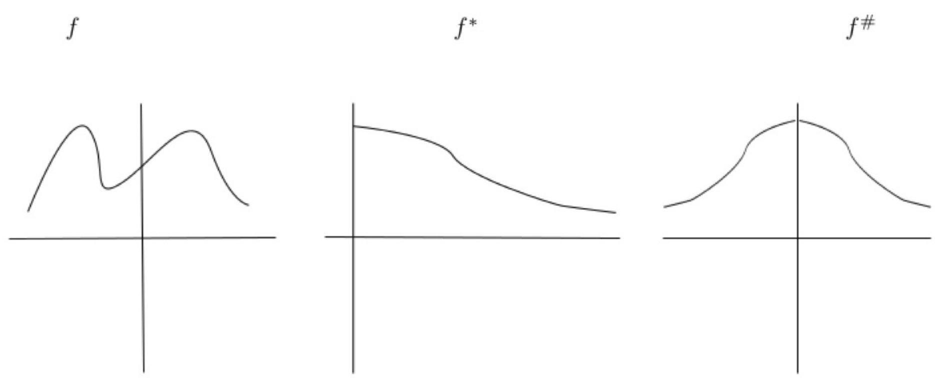

FIGURE 1.1. A function f together with the decreasing rearrangement f^* and Schwarz rearrangement $f^\#$.

The following result says that the Schwarz rearrangement is a contraction in the L^p distance. It appears as a special case of Theorem 3 in [**Ba5**].

Theorem 1.13 (Schwarz Rearrangement Contracts L^p Distance)**.** *Let* $f, g \in L^1(\Omega)$*. Then for each* $1 \le p < \infty$ *we have*

$$\int_{\Omega^\#} \left|f^\# - g^\#\right|^p \, \mathrm{d}x \quad \le \quad \int_\Omega \left|f - g\right|^p \, \mathrm{d}x.$$

The star function corresponding to the Schwarz rearrangement is a function of one variable, defined on the interval

$$\Omega^\star \quad = \quad (0, R).$$

Definition 1.14 (Star Function for Schwarz Rearrangement)**.** Let $f \in L^1(\Omega)$. The star function of f associated with the Schwarz rearrangement is defined on the interval $\Omega^\star = (0, R)$ by the formula

$$f^\star(r) \quad = \quad \max_{|E|=|B(0,r)|} \int_E f \, \mathrm{d}x,$$

where absolute value $|\cdot|$ denotes Lebesgue measure and the max is taken over all measurable subsets $E \subseteq \Omega$ with the same Lebesgue measure as the ball centered at the origin of radius r.

Just as in Proposition 1.7, the max defining $f^\star$ is achieved for some subset E, which explains our use of max instead of sup.

The set of size $\left|B(0, r)\right|$ on which $f^\#$ is biggest is the ball $B(0, r)$, so it follows just as in Proposition 1.8 that

$$f^\star(r) \quad = \quad \int_{B(0,r)} f^\# \, \mathrm{d}x.$$

Now we state two important results that require new notation.

Let $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ denote the standard Laplacian operator in $\mathbb{R}^n$ and define operators $\Delta^\star$ and $\Delta^{\star t}$ acting on $G \in C^2(0,R)$ by

$$\begin{aligned} \Delta^\star G(r) &= G''(r) - \frac{n-1}{r} G'(r), \\ \Delta^{\star t} G(r) &= G''(r) + (\frac{n-1}{r} G)'(r) \end{aligned}$$

for $0 < r < R$.

Next, define an operator J that takes a function $u \in L^1(\Omega^\#)$ to a function Ju defined on $(0,R)$ by the equation

$$Ju(r) = \int_{B(0,r)} u \, \mathrm{d}x$$

for $r \in (0,R)$.

Theorem 1.15 below appears as formula (5.9) in [**Ba5**] and Theorem 1.16 appears as Theorem 5 in [**Ba5**].

Theorem 1.15 (Commutativity Relation for Schwarz Rearrangement)**.** *For each $u \in C^2(\Omega^\#)$ the following relation holds*

$$J\Delta u = \Delta^\star Ju.$$

PROOF. For $0 < r < R$ we compute

$$\begin{aligned} (J\Delta u)(r) &= \int_{B(0,r)} \Delta u \, \mathrm{d}x \\ &= \int_{\partial B(0,r)} \frac{\partial u}{\partial n} \, \mathrm{d}S \end{aligned}$$

by Green's Theorem. On the other hand,

$$\begin{aligned}(\Delta^\star Ju)(r) &= r^{n-1}\frac{d}{dr}\Big(\frac{1}{r^{n-1}}\frac{d}{dr}\int_0^r\int_{\mathbb{S}^{n-1}} u(s\xi)\ \mathrm{d}\sigma_{n-1}(\xi)\ s^{n-1}\mathrm{d}s\Big) \\ &= \int_{\mathbb{S}^{n-1}} u_r(r\xi)\ r^{n-1}\mathrm{d}\sigma_{n-1}(\xi) \\ &= \int_{\partial B(0,r)} \frac{\partial u}{\partial n}\ \mathrm{d}S,\end{aligned}$$

which establishes the result. □

The name of the next theorem comes from the special case where u is a harmonic function: Theorem 1.16 then says that $u^\star$ is $\Delta^\star$ subharmonic, meaning that $\Delta^\star u^\star \geq 0$ in an appropriate sense.

Theorem 1.16 (Subharmonicity for Schwarz Rearrangement)**.** *Suppose* $u \geq 0$ *and* $\lim_{x\to x_0} u(x) = 0$ *for every* $x_0 \in \partial\Omega$*.*

If $-\Delta u = f$ *in* Ω*, then*

$$-\Delta^\star u^\star \ \leq\ f^\star$$

in the weak sense, meaning that for each nonnegative $G \in C_c^2(\Omega^\star)$*,*

$$-\int_0^R u^\star \Delta^{\star t} G\ \mathrm{d}r \ \leq\ \int_0^R f^\star G\ \mathrm{d}r.$$

1.1.4. The spherical rearrangement. The spherical rearrangement is an analogue of the Schwarz rearrangement for functions defined on spheres. We write $\mathbb{S}^n = \{(\xi_1, \xi_2, \ldots, \xi_{n+1}) \in \mathbb{R}^{n+1} : \xi_1^2 + \xi_2^2 + \ldots + \xi_{n+1}^2 = 1\}$ for the unit $n-$sphere in $\mathbb{R}^{n+1}$, and σ_n for surface measure on $\mathbb{S}^n$. So, for example, $\sigma_1(\mathbb{S}^1) = 2\pi$. Let d denote the standard distance on $\mathbb{S}^n$ whereby the distance between any two points is calculated by computing the length of the shorter arc of the great circle that joins them. We write

$$K(\theta) = \{\xi \in \mathbb{S}^n : d(\xi, e_1) < \theta\}$$

for the open polar cap centered at the "east pole" $e_1 = (1, 0, \ldots, 0)$ and of radius θ (in the spherical distance). For example, when $n = 1$, $K(\theta) = \{e^{i\phi} : -\theta < \phi < \theta\}$. A function defined on the sphere can be rearranged into one that is constant on boundaries of caps centered at the east pole, and that decreases on these cap boundaries as they sweep out the sphere from e_1 to $-e_1$. The spherical rearrangement thus provides an analogue of the Schwarz rearrangement for the sphere.

Definition 1.17 (Spherical Rearrangement)**.** Given $F \in L^1(\mathbb{S}^n)$, we define $F^\# : \mathbb{S}^n \to \mathbb{R}$ by the formula

$$F^\#(\xi) = F^*(\sigma_n(K(\theta))),$$

where θ is the spherical distance between the point ξ and e_1, and F^* is the decreasing rearrangement of F. We call $F^\#$ the *spherical rearrangement* of F.

What does the spherical rearrangement look like? Figure 1.2 below graphs several level sets $F^{\#-1}(t)$ for a spherically rearranged function $F^\#$. These level sets are circles centered at the pole e_1 and $F^\#$ decreases on these circles as they sweep out the sphere from e_1 to its antipode $-e_1$.[1]

[1]The image in Figure 1.2 was modified from an image on R. Harwood's website http://facweb.bhc.edu/academics/science/harwoodr/geog101/study/LongLat.htm and has been used with his permission.

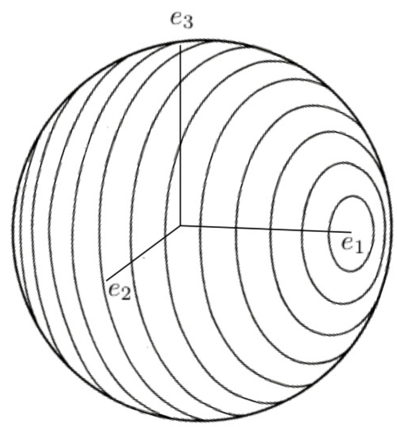

FIGURE 1.2. The level sets of a spherically rearranged $F^{\#}$.

The following result says that the spherical rearrangement is a contraction in the L^p distance and is again a special case of Theorem 3 in [**Ba5**].

Theorem 1.18 (Spherical Rearrangement Contracts L^p Distance)**.** *Let $F, G \in L^1(\mathbb{S}^n)$. Then for each $1 \leq p < \infty$ we have*

$$\int_{\mathbb{S}^n} \left|F^{\#} - G^{\#}\right|^p \, \mathrm{d}\sigma_n \;\; \leq \;\; \int_{\mathbb{S}^n} \left|F - G\right|^p \, \mathrm{d}\sigma_n. \tag{1.2}$$

The following star function definition is a direct generalization of the star function for the Schwarz rearrangement.

Definition 1.19 (Star Function for Spherical Rearrangement)**.** Given $F \in L^1(\mathbb{S}^n)$, we define $F^\star : (0, \pi) \to \mathbb{R}$ by the formula

$$F^\star(\theta) \;\; = \max_{\sigma_n(E) = \sigma_n(K(\theta))} \int_E F \, \mathrm{d}\sigma_n,$$

where the max is taken over all measurable subsets E of $\mathbb{S}^n$ with the same surface measure as the open cap $K(\theta)$.

Just as in Proposition 1.7, the max defining $F^\bigstar$ is achieved for some subset E, which explains our use of max instead of sup.

The set of surface measure $\sigma_n(K(\theta))$ on which $F^\#$ is biggest is the polar cap $K(\theta)$. Since F and $F^\#$ are rearrangements, it follows just as in Proposition 1.8 that

$$F^\bigstar(\theta) \quad = \quad \int_{K(\theta)} F^\# \, \mathrm{d}\sigma_n.$$

Thus, when $n = 1$, we have

$$F^\bigstar(\theta) \quad = \quad \int_{-\theta}^{\theta} F^\#(e^{i\phi}) \, \mathrm{d}\phi.$$

There are versions of Theorems 1.15 and 1.16 for the spherical rearrangement. They can be viewed as restrictions of analogous results for cap symmetrization on spherical shells, as we proceed to explain.

1.1.5. Cap symmetrization. Throughout this subsection, $A \subset \mathbb{R}^n$ denotes a spherical shell $A = A(a, b) = \{x \in \mathbb{R}^n : a < |x| < b\}$ for real numbers $0 < a < b < \infty$. Given a function $f : A \to \mathbb{R}$, we can spherically rearrange f on each concentric $(n-1)$-sphere. Doing so gives the $(n-1, n)$ cap symmetrization.

Definition 1.20 ($(n-1, n)$ Cap Symmetrization)**.** Given $f \in L^1(A)$, we define $f^\# : A \to \mathbb{R}$ in the following manner. If $r \in (a, b)$ and $\xi \in \mathbb{S}^{n-1}$, then

$$f^\#(r\xi) \quad = \quad (f^r)^\#(\xi),$$

where $(f^r)^\#$ denotes the spherical rearrangement of the slice function $f^r : \mathbb{S}^{n-1} \to \mathbb{R}$ defined by $f^r(\xi) = f(r\xi)$. We call $f^\#$ the $(n-1, n)$ *cap symmetrization* of f.

Star functions corresponding to cap symmetrization will be defined in a polar rectangle. Write $A^\bigstar = \{(r,\theta) \in \mathbb{R}^2 : a < r < b \text{ and } 0 < \theta < \pi\}$. The idea is to take a spherical star function on each slice function f^r. Since cap symmetrization is a partial symmetrization, meaning rearrangement takes place inside subsets of codimension one, the corresponding star function will be of two variables, r and θ.

Definition 1.21 (Star Function for $(n-1,n)$ Cap Symmetrization)**.** If $f \in L^1(A)$, define $f^\bigstar : A^\bigstar \to \mathbb{R}$ by the formula

$$f^\bigstar(r,\theta) \quad = \max_{\sigma_{n-1}(E)=\sigma_{n-1}(K(\theta))} \int_E f(r\xi)\, \mathrm{d}\sigma_{n-1}(\xi) \quad = \quad \int_{K(\theta)} f^\#(r\xi)\, \mathrm{d}\sigma_{n-1}(\xi),$$

where the max is taken over all measurable subsets E of $\mathbb{S}^{n-1}$ with the same surface measure as $K(\theta)$ and $f^\#$ denotes the $(n-1,n)$ cap symmetrization of f.

Just as in Proposition 1.7, the max defining $f^\bigstar$ is achieved for some subset E, which explains our use of max rather than sup.

If we multiply inequality (1.2) in the $\mathbb{S}^{n-1}$ version of Theorem 1.18 by r^{n-1} and integrate from $r = a$ to $r = b$ we obtain the following theorem.

Theorem 1.22 ($(n-1,n)$ Cap Symmetrization Contracts L^p Distance)**.** *Let $f, g \in L^1(A)$. Then for each $1 \le p < \infty$ we have*

$$\int_A \left|f^\# - g^\#\right|^p \,\mathrm{d}x \;\; \le \;\; \int_A |f-g|^p \,\mathrm{d}x.$$

The following result will come in handy later. It says that if a sequence of functions converges in L^1, then by passing to a subsequence we have almost everywhere pointwise convergence for the star functions involved.

Theorem 1.23 (Convergence of Star Functions)**.** *Assume $u, u_k \in L^1(A)$ and $u_k \to u$ in $L^1(A)$. Then for some subsequence and for almost every $r \in (a,b)$, we have*

$$\int_{\mathbb{S}^{n-1}} |u_{k_j}(r\xi) - u(r\xi)| \; \mathrm{d}\sigma_{n-1}(\xi) \quad \to \quad 0$$

and

$$u^{\bigstar}_{k_j}(r,\theta) \quad \to \quad u^{\bigstar}(r,\theta)$$

for every $\theta \in (0,\pi)$. In particular, $u^{\bigstar}_{k_j} \to u^{\bigstar}$ a.e. in $A^{\bigstar}$.

Proof. Define $\Psi_k : (a,b) \to \mathbb{R}$ by the formula

$$\Psi_k(r) \quad = \quad \int_{\mathbb{S}^{n-1}} \left|u_k(r\xi) - u(r\xi)\right| \; \mathrm{d}\sigma_{n-1}(\xi).$$

By assumption, $u_k \to u$ in $L^1(A)$. That is,

$$\int_a^b \int_{\mathbb{S}^{n-1}} \left|u_k(r\xi) - u(r\xi)\right| \; \mathrm{d}\sigma_{n-1}(\xi) \; r^{n-1}\mathrm{d}r \quad \to \quad 0,$$

which implies that $\Psi_k \to 0$ in $L^1((a,b), r^{n-1}\mathrm{d}r)$. Thus, we can pass to a subsequence where $\Psi_{k_j} \to 0$ a.e. in (a,b). By the very definition of Ψ_{k_j}, this implies

$$\int_{\mathbb{S}^{n-1}} \left|u_{k_j}(r\xi) - u(r\xi)\right| \; \mathrm{d}\sigma_{n-1}(\xi) \quad \to \quad 0$$

for almost every $r \in (a,b)$, which gives the first conclusion. Fix an r so that convergence holds above. Then for any $\theta \in (0,\pi)$ we have

$$\begin{aligned}
|u^{\star}_{k_j}(r,\theta) - u^{\star}(r,\theta)| &= \Big| \int_{K(\theta)} u^{\#}_{k_j}(r\xi) - u^{\#}(r\xi) \ \mathrm{d}\sigma_{n-1}(\xi) \Big| \\
&\leq \int_{\mathbb{S}^{n-1}} |u^{\#}_{k_j}(r\xi) - u^{\#}(r\xi)| \ \mathrm{d}\sigma_{n-1}(\xi) \\
&\leq \int_{\mathbb{S}^{n-1}} |u_{k_j}(r\xi) - u(r\xi)| \ \mathrm{d}\sigma_{n-1}(\xi),
\end{aligned}$$

where the last inequality holds by Theorem 1.18. Letting $j \to \infty$, we conclude

$$u^{\star}_{k_j}(r,\theta) \ \ \to \ \ u^{\star}(r,\theta).$$

$\square$

Just like for the Schwarz rearrangement, we will state commutativity and subharmonicity results for cap symmetrization. They require some notation. Given $u \in L^1(A)$, we define $Ju : A^{\star} \to \mathbb{R}$ by

$$Ju(r,\theta) = \int_{K(\theta)} u(r\xi) \ \mathrm{d}\sigma_{n-1}(\xi).$$

With this notation, we have

$$u^{\star}(r,\theta) = \int_{K(\theta)} u^{\#}(r\xi) \ \mathrm{d}\sigma_{n-1}(\xi) = Ju^{\#}(r,\theta),$$

where $u^{\#}$ denotes the $(n-1,n)$ cap symmetrization of u.

We let Δ denote the standard Laplacian operator in $\mathbb{R}^n$ expressed in polar coordinates

$$\Delta F \ = \ \partial_{rr}F + \frac{n-1}{r}\partial_r F + r^{-2}[\partial_{\theta\theta}F + (n-2)(\cot\theta)\partial_\theta F],$$

and define new operators $\Delta^\bigstar$ and $\Delta^{\bigstar t}$ which act on $C^2(A^\bigstar)$ as follows:

$$\begin{aligned}
\Delta^\bigstar F &= \partial_{rr}F + \frac{n-1}{r}\partial_r F + r^{-2}[\partial_{\theta\theta}F - (n-2)(\cot\theta)\partial_\theta F], \\
\Delta^{\bigstar t} F &= \partial_{rr}F + \frac{n-1}{r}\partial_r F + r^{-2}[\partial_{\theta\theta}F + (n-2)\partial_\theta((\cot\theta)F)].
\end{aligned} \tag{1.3}$$

Theorem 1.24 and 1.25 appear as equation (5.9) and Theorem 5 in [**Ba5**], respectively.

Theorem 1.24 (Commutativity Relation for $(n-1,n)$ Cap Symmetrization)**.** *If $u \in C^2(A)$, then*

$$J\Delta u = \Delta^\bigstar Ju$$

on $A^\bigstar$.

Theorem 1.25 (Subharmonicity for $(n-1,n)$ Cap Symmetrization)**.** *Suppose $u \in C^2(A)$ satisfies $-\Delta u = f$.*

Then

$$-\Delta^\bigstar u^\bigstar \leq f^\bigstar$$

in the weak sense, meaning that for all $g \in C_c^2(A^\bigstar)$ nonnegative,

$$-\int_{A^\bigstar} u^\bigstar \Delta^{\bigstar t} g \; r^{n-1} \mathrm{d}r \; \mathrm{d}\theta \leq \int_{A^\bigstar} f^\bigstar g \; r^{n-1} \mathrm{d}r \; \mathrm{d}\theta.$$

The two-dimensional case. In dimension $n = 2$ it is often helpful to use complex notation. In this case, $A = A(a,b) = \{z \in \mathbb{C} : a < |z| < b\}$ and we will write $A^\bigstar = \{z \in A : \mathrm{Im}(z) > 0\}$. So if $u \in L^1(A)$, for each $a < r < b$ we write

$$Ju(re^{i\theta}) = \int_{-\theta}^{\theta} u(re^{i\phi}) \; \mathrm{d}\phi,$$

and for the star function $u^\bigstar : A^\bigstar \to \mathbb{R}$ we write

$$u^\bigstar(re^{i\theta}) = \max_{|E|=2\theta} \int_E u(re^{i\phi})\ \mathrm{d}\phi.$$

Then $u^\bigstar = Ju^\#$ where $u^\#$ is the $(1,2)$ cap symmetrization of u. The following commutativity result, which appears as Proposition 3.1 in **[Ba6]**, is a special case of Theorem 1.24. Note that in dimension $n = 2$, the star operator $\Delta^\bigstar$ equals the standard planar Laplacian Δ expressed in polar coordinates.

Theorem 1.26 (Commutativity Relation for (1, 2) Cap Symmetrization)**.** *Let $u \in C^2(A)$. Then*

$$\Delta Ju \;=\; J\Delta u$$

on $A^\bigstar$.

PROOF. For $re^{i\theta} \in A^\bigstar$ we compute

$$\begin{aligned}
\Delta(Ju)(re^{i\theta}) &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) \int_{-\theta}^{\theta} u(re^{i\phi})\ \mathrm{d}\phi \\
&= \int_{-\theta}^{\theta} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) u(re^{i\phi})\ \mathrm{d}\phi + \frac{1}{r^2}\left(u_\theta(re^{i\theta}) - u_\theta(re^{-i\theta})\right) \\
&= \int_{-\theta}^{\theta} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right) u(re^{i\phi})\ \mathrm{d}\phi \\
&= (J\Delta u)(re^{i\theta}).
\end{aligned}$$

□

The following subharmonicity result is a special case of Theorem 1.25.

Theorem 1.27 (Subharmonicity for $(1,2)$ Cap Symmetrization)**.** *Suppose $u \in C^2(A)$ with $-\Delta u = f$ in A. Then*

$$-\Delta u^\bigstar \leq f^\bigstar$$

in the weak sense, meaning that for each $g \in C_c^2(A^\bigstar)$ nonnegative,

$$-\int_{A^\bigstar} u^\bigstar \Delta g \; r\mathrm{d}r \; \mathrm{d}\theta \leq \int_{A^\bigstar} f^\bigstar g \; r\mathrm{d}r \; \mathrm{d}\theta.$$

Theorems 1.26 and 1.27 explain the reason for the labels "Commutativity" and "Subharmonicity." When the rearrangement under consideration is $(1,2)$ cap symmetrization, the star function operator $\Delta^\bigstar$ equals the ordinary Laplacian Δ. In this case, the J operator commutes with the Laplacian. Additionally, if u is harmonic in an annulus, then the star function $u^\bigstar$ is subharmonic in the upper annulus $A^\bigstar$.

1.2. Solutions to PDEs with Neumann boundary conditions

The results in this thesis assume the existence of solutions to various PDEs. This section establishes the existence of such solutions.

We begin with preliminary notation. For $\Omega \subseteq \mathbb{R}^n$ an open subset, we write $W^{1,2}(\Omega)$ for the Sobolev space of functions with weak partial derivatives up to first order living in $L^2(\Omega)$. For more information on Sobolev spaces, we direct the reader to [**Ev**].

We have the following existence result which bounds the solution of Poisson's equation with homogeneous Neumann boundary conditions in terms of the data. The result is essentially a consequence of the Poincaré Inequality and Riesz Representation Theorem.

Theorem 1.28 (Existence and Bounds on Solutions). *Let Ω be a bounded Lipschitz domain and $f \in L^2(\Omega)$ with $\int_\Omega f \, \mathrm{d}x = 0$. Then there exists a unique weak solution u to the problem*

$$\begin{aligned} -\Delta u &= f \quad \textit{in} \quad \Omega, \\ \tfrac{\partial u}{\partial n} &= 0 \quad \textit{on} \quad \partial\Omega, \end{aligned}$$

where $\int_\Omega u \, \mathrm{d}x = 0$. Moreover,

$$\|u\|_{W^{1,2}(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

for some constant C depending only on the domain Ω.

Proof. Let H be the space

$$H = \{u \in W^{1,2}(\Omega) : \int_\Omega u \, \mathrm{d}x = 0\}.$$

It is easy to check that H is closed in $W^{1,2}(\Omega)$ and so H is a Hilbert space with the norm inherited from $W^{1,2}(\Omega)$. Define an inner product $\langle \cdot, \cdot \rangle_H$ by

$$\langle u, v \rangle_H = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x,$$

noting that the norm generated by this inner product is $\|\nabla u\|_{L^2(\Omega)}$, which is equivalent to the standard Sobolev norm by the Poincaré inequality (Theorem 7.16 in [**Sa**]):

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}.$$

Now fix $f \in L^2(\Omega)$ with mean zero and define a linear functional $T : H \to H$ by the formula

$$T(v) \;=\; \int_\Omega fv \, \mathrm{d}x.$$

We check

$$\begin{aligned} |T(v)| &= \Big| \int_\Omega fv \, \mathrm{d}x \Big| \\ &\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq C\|f\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &= C\|f\|_{L^2(\Omega)} \|v\|_H, \end{aligned}$$

where the first inequality follows from Cauchy-Schwarz and the second inequality holds for a domain-dependent constant C by the Poincare Inequality (Theorem 7.16 in [**Sa**]). Hence, T is a bounded linear functional on H and by the Riesz Representation Theorem, it follows that there exists a unique $u \in H$ such that

$$\langle u, v \rangle_H \;=\; T(v)$$

for all $v \in H$. Using the definitions of $\langle \cdot, \cdot \rangle_H$ and T, the above equality becomes

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \;=\; \int_\Omega fv \, \mathrm{d}x \tag{1.4}$$

for every $v \in W^{1,2}(\Omega)$ with $\int_\Omega v \, \mathrm{d}x = 0$. Because $\int_\Omega f \, \mathrm{d}x = 0$, it follows that the above equation holds for all $v \in W^{1,2}(\Omega)$, which is precisely what it means to solve the PDE weakly.

To establish the theorem's second conclusion, we compute

$$\begin{aligned}
\|u\|^2_{W^{1,2}(\Omega)} &= \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \\
&\leq C^2\|\nabla u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \\
&= (1+C^2)\|\nabla u\|^2_{L^2(\Omega)} \\
&\leq (1+C^2)\|u\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)} \\
&\leq (1+C^2)\|u\|_{W^{1,2}(\Omega)}\|f\|_{L^2(\Omega)},
\end{aligned}$$

where the first inequality holds by the Poincare Inequality and the second inequality holds by equation (1.4) with $u = v$. Dividing through by $\|u\|_{W^{1,2}(\Omega)}$ completes the proof. □

The following corollary will be used repeatedly in the following chapters.

Corollary 1.29 (Convergence of Solutions by Approximation of Data)**.** *Let Ω be a bounded Lipschitz domain and $f, f_k \in L^2(\Omega)$ with $\int_\Omega f \; \mathrm{d}x = \int_\Omega f_k \; \mathrm{d}x = 0$. Let u, u_k be weak solutions of*

$$\begin{aligned}
-\Delta u &= f \quad \text{in} \quad \Omega, & \qquad -\Delta u_k &= f_k \quad \text{in} \quad \Omega, \\
\tfrac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega, & \qquad \tfrac{\partial u_k}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega,
\end{aligned}$$

where $\int_\Omega u \; \mathrm{d}x = \int_\Omega u_k \; \mathrm{d}x = 0$.

If $f_k \to f$ in $L^2(\Omega)$ then $u_k \to u$ in $W^{1,2}(\Omega)$. In particular, $u_k \to u$ in $L^2(\Omega)$.

PROOF. The function $u - u_k$ solves

$$\begin{aligned}
-\Delta(u-u_k) &= f - f_k \quad \text{in} \quad \Omega, \\
\tfrac{\partial(u-u_k)}{\partial n} &= 0 \qquad\quad\;\; \text{on} \quad \partial\Omega,
\end{aligned}$$

and since $\int_\Omega (u - u_k)\,\mathrm{d}x = 0$, Theorem 1.28 gives $\|u - u_k\|_{W^{1,2}(\Omega)} \le C\|f - f_k\|_{L^2(\Omega)}$. Letting $k \to \infty$ gives the result. $\square$

CHAPTER 2

The Baernstein–Kawohl Correspondence

This chapter discusses a correspondence between Kawohl and Baernstein from the mid 1980's. The main result of this thesis grows out of this correspondence.

The solution sent to Kawohl by Baernstein has three components: 1) an annular comparison result, 2) a rectangular comparison result, and 3) a rephrasing of the rectangular comparison result in terms of convex means. The third component is then used to prove Kawohl's conjecture. We will follow this structure, but provide different (and simpler) proofs for each one.

We begin Section 1 with a precise statement of Kawohl's conjecture from his text [**Ka**] and its physical meaning. Section 2 presents an annular comparison result. The theory of the star function has been further developed since the correspondence, and so the proof we present differs from the one originally sent by Baernstein. In Section 3, we show how to obtain a rectangular comparison result from the annular comparison result. Baernstein indicated to Kawohl that this rectangular comparison result was the key to the conjecture. However, Baernstein did not provide the details of how this rectangular comparison result is obtained, and so we provide them in Section 3, before closing with a proof of Kawohl's conjecture in Section 4 that differs from the one originally sent by Baernstein. Our proof of Kawohl's conjecture does not rely on rephrasing the rectangular comparison result in terms of convex means. Instead, we use the rectangular comparison result directly.

2.1. Kawohl's conjecture

Hot spots: Kawohl's motivation from heat flow. Why was Kawohl interested in Neumann boundary value problems? He was thinking about Jeffrey Rauch's "hot spots" problem, which claims that for heat flow in a convex perfectly insulated domain, the hottest (and coldest) spot will approach the boundary as time goes to infinity. The hot spots conjecture is intimately connected with Neumann eigenfunctions, as we proceed to explain.

Consider the heat equation

$$\begin{aligned} u_t &= \Delta u \quad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where Ω is some bounded domain in $\mathbb{R}^n$ and $u = u(x,t)$ for $x \in \Omega$ and $t \geq 0$.

To obtain a general solution u to the above PDE, let $\phi_j(x)$ denote the j^{th} eigenfunction of the Neumann Laplacian on the domain Ω so that

$$\begin{aligned} -\Delta\phi_j &= \mu_j\phi_j \quad \text{in} \quad \Omega, \\ \frac{\partial \phi_j}{\partial n} &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where μ_j is the j^{th} Neumann eigenvalue of the Laplacian for $j \geq 1$. When $j = 1$, we take $\mu_1 = 0$ and the associated eigenfunction ϕ_1 equals a constant. The general solution of the heat equation above is given by

$$u(x,t) = \sum_{j=1}^{\infty} c_j e^{-\mu_j t}\phi_j(x),$$

where the c_j are determined by some initial condition. It is straightforward to formally verify that u defined above satisfies the heat equation; it also has vanishing outer normal derivative because each Neumann eigenfunction does.

For the sake of simplicity, assume that the eigenvalues are all *simple* (of multiplicity 1) written as

$$0 = \mu_1 < \mu_2 < \mu_3 < \dots$$

Then

$$u(x,t) \quad = \quad \text{constant} + c_2 e^{-\mu_2 t}\phi_2(x) + \text{higher order terms.}$$

We see that as $t \to \infty$, the behavior of u is governed by the behavior of the term $c_2 e^{-\mu_2 t}\phi_2(x)$. As $t \to \infty$, where will the "hottest spot" be? In other words, where will u be biggest (or smallest) as $t \to \infty$? This is equivalent to asking: where is ϕ_2 biggest and smallest?

Thus the hot spots problem asks: does the first non-constant Neumann eigenfunction of the Laplacian assume its maximum and minimum values on the boundary? The hot spots problem is difficult. It has been resolved in the affirmative for obtuse triangles [**BB**] and remains open for acute triangles. There are counterexamples for doubly-connected domains [**BW**]. For more results and references see [**BPP**].

Kawohl's conjecture below does not involve the heat equation or eigenfunctions, but it does involve the Poisson equation, which is a steady state heat equation. Moreover, the conjecture's conclusion involves making the (oscillation of the) solution biggest, and so its goals are similar to those of the hot spots problem.

In 1985, Kawohl raised the following conjecture on p.61 of [**Ka**].

Conjecture 2.1 (Kawohl's Conjecture)**.** *Let R be the unit square $(0,1)\times(0,1)$ in $\mathbb{R}^2$ and $f : R \to \mathbb{R}$ a sufficiently smooth function with mean value zero. Consider the problems:*

$$\begin{aligned} -\Delta u &= f \quad in \quad R, & -\Delta v &= f^{\#} \quad in \quad R,\\ \tfrac{\partial u}{\partial n} &= 0 \quad on \quad \partial R, & \tfrac{\partial v}{\partial n} &= 0 \quad on \quad \partial R, \end{aligned}$$

where $f^{\#}$ is the monotone decreasing rearrangement of f in the direction y.

Then the oscillation of u over $\overline{R}$ should be dominated by the oscillation of v.

To be precise, the monotone decreasing rearrangement of f in the direction y is defined in the following manner. Fix $x \in (0,1)$ and let $f^x : (0,1) \to \mathbb{R}$ denote the slice function $f^x(y) = f(x,y)$. Then $f^{\#}(x,y) = (f^x)^*(y)$ where $(f^x)^*$ is the decreasing rearrangement of the slice function f^x.

When thought of physically, Kawohl's conjecture seems quite plausible. We illustrate the discrete case. Imagine yourself inside of a perfectly insulated square room with the floor covered in square tiles. On each tile, heat is either being uniformly generated or absorbed. If, on each row, all of the tiles with heat sources are moved next to each other and all of the tiles with heat sinks are moved next to each other, with heat sources near one wall and heat sinks near the opposite wall, it seems intuitive that the temperature gap across the entire room should increase.

2.2. Two-dimensional annular comparison result

In 1986 Baernstein wrote Kawohl outlining a solution (unpublished). In the correspondence, Baernstein proves a comparison result on a planar annulus involving $(1,2)$ cap symmetrization, using the star function method. We present a complete proof.

The proof is cleaner than in the original correspondence, making use of the systematic development of properties of the star function.

In this section, we use the notation from subsection 1.1.5 of Chapter 1, dealing with cap symmetrizations. The reader might find it useful to review the definitions and theorems within that section before continuing. The result below compares solutions of two PDEs, one with given data and the other with cap symmetrized data. The conclusion states that the solution to the PDE with cap symmetrized data has larger star function.

Theorem 2.2 (Two-Dimensional Annular Comparison Theorem)**.** *Let* $f \in L^2(A)$ *with* $\int_A f \; \mathrm{d}x = 0$*. Assume* u *and* v *are weak solutions of*

$$\begin{aligned} -\Delta u &= f \quad in \quad A, & \qquad -\Delta v &= f^{\#} \quad in \quad A, \\ \tfrac{\partial u}{\partial n} &= 0 \quad on \quad \partial A, & \qquad \tfrac{\partial v}{\partial n} &= 0 \quad on \quad \partial A, \end{aligned}$$

where $f^{\#}$ *denotes the* $(1,2)$ *cap symmetrization of* f*.*

If u *and* v *are additively normalized so that* $\int_A u \; \mathrm{d}x = \int_A v \; \mathrm{d}x = 0$*, then for almost every* $r \in (a,b)$*,*

$$\int_{-\pi}^{\pi} u(re^{i\theta}) \; \mathrm{d}\theta \;\; = \;\; \int_{-\pi}^{\pi} v(re^{i\theta}) \; \mathrm{d}\theta$$

and the inequality

$$u^{\star}(re^{i\theta}) \;\; \leq \;\; v^{\star}(re^{i\theta})$$

holds for every $\theta \in (0,\pi)$*. In particular,* $u^{\star} \leq v^{\star}$ *a.e. in* $A^{\star}$*.*

Proof. **Step 1: Reduce by maximum principle to boundary estimate.**

We first assume f is Lipschitz continuous on $\overline{A}$. Since cap symmetrization decreases the modulus of continuity (this follows from Corollary 3 of [**Ba5**]), it follows that $f^{\#}$ is also Lipschitz continuous on $\overline{A}$. Consequently, the solutions u and v above belong to $C^2(\overline{A})$ by Theorem 3.2 of [**LU**]. Let Q solve

$$\begin{aligned} \Delta Q &= 0 \quad \text{in} \quad A, \\ \frac{\partial Q}{\partial n}(re^{i\theta}) &= \sin\theta \quad \text{for } r = a, b \text{ and } \theta \in [-\pi, \pi], \end{aligned}$$

normalized so $\int_A Q \, \mathrm{d}x = 0$. Theorem 3.2 of [**LU**] also implies that Q belongs to $C^2(\overline{A})$. Since the function $Q_1(z) = -Q(\bar{z})$ solves the same PDE that Q does together with the normalization assumption, it follows by uniqueness that $Q(z) = Q_1(z)$. That is, $Q(z) = -Q(\bar{z})$ so that in particular, Q vanishes along the real axis. Now define for $\epsilon > 0$

$$w(z) = u^\star(z) - Jv(z) - \epsilon Q(z), \ z \in A^\star.$$

We remind the reader that $u^\star$ and the J operator are connected by the formula

$$u^\star = Ju^{\#},$$

with $u^{\#}$ the $(1, 2)$ cap symmetrization of u.

To compute $-\Delta w$, we recall

$$-\Delta u^\star \leq f^\star$$

by Theorem 1.27. Also,

$$\begin{aligned} \Delta Jv &= J\Delta v \\ &= -Jf^{\#} \\ &= -f^\star \end{aligned}$$

by Theorem 1.26 and the definition of v. Since also $\Delta Q = 0$, we get

$$-\Delta w \quad \leq \quad 0.$$

Thus we've shown that for each non-negative $g \in C_c^2(A^\bigstar)$,

$$-\int_{A^\bigstar} w \Delta g \, \mathrm{d}x \quad \leq \quad 0.$$

This implies w is distributionally subharmonic in $A^\bigstar$. Since w is continuous on $\overline{A^\bigstar}$, we have from the maximum principle (Theorem 2.11 in [**Fra**])

$$\max_{\partial A^\bigstar} w = \max_{A^\bigstar} w. \tag{2.1}$$

We use this fact in the next step to show $w \leq 0$ in $A^\bigstar$. If we can show this, then we have

$$u^\bigstar \leq Jv \leq v^\bigstar$$

by letting $\epsilon \to 0$, which gives the theorem for Lipschitz continuous f.

Step 2: Analysis of w on $\partial A^\bigstar$.

We use equation (2.1) and split up $\partial A^\bigstar$ into 4 pieces, as the figure below shows. We will show the maximum of w on $\overline{A^\bigstar}$ cannot be attained on pieces 3 or 4, and hence by equation (2.1) must be attained on Piece 1 or Piece 2. On those pieces, we show $w = 0$. Hence $w \leq 0$ on $A^\bigstar$ as we wanted.

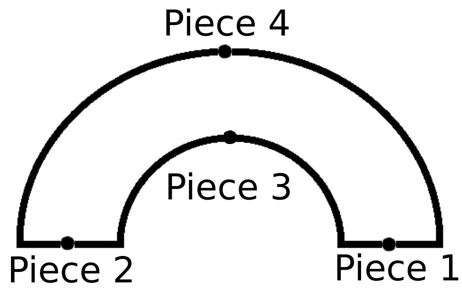

FIGURE 2.1. Subdivision of $A^\star$ into 4 pieces.

Piece 1: Consider $z = r$ for $a \le r \le b$. In this case, $w(r) = 0$ by definition of $u^\star$, Jv, and Q.

Piece 2: Consider $z = re^{i\pi}$ for $a \le r \le b$. Since Q vanishes along the real axis, we have $w(re^{i\pi}) = \int_{-\pi}^{\pi} u(re^{i\theta})\ \mathrm{d}\theta - \int_{-\pi}^{\pi} v(re^{i\theta})\ \mathrm{d}\theta$. Define a function $\Phi(r) = r\frac{d}{dr}w(re^{i\pi})$. We then compute

$$\begin{aligned}
\Phi(r) &= \int_{-\pi}^{\pi} ru_r(re^{i\theta})\ \mathrm{d}\theta - \int_{-\pi}^{\pi} rv_r(re^{i\theta})\ \mathrm{d}\theta \\
&= \int_{\partial A(a,r)} \frac{\partial u}{\partial n}\ \mathrm{d}S - \int_{\partial A(a,r)} \frac{\partial v}{\partial n}\ \mathrm{d}S \\
&= \int_{A(a,r)} \Delta u\ \mathrm{d}x - \int_{A(a,r)} \Delta v\ \mathrm{d}x \\
&= -\int_{A(a,r)} f\ \mathrm{d}x + \int_{A(a,r)} f^{\#}\ \mathrm{d}x \\
&= 0,
\end{aligned}$$

where the second equality holds because u and v have vanishing outer normals at radius a and the last equality holds by the definition of rearrangement. We have shown that $\Phi \equiv 0$, and so $w(re^{i\pi})$ is constant throughout $[a, b]$. This constant must be zero, because our assumption that u and v both have mean zero implies that $\int_a^b rw(re^{i\pi})\ \mathrm{d}r = \int_{A(a,b)} u\ \mathrm{d}x - \int_{A(a,b)} v\ \mathrm{d}x = 0$. Thus, $w(re^{i\pi}) = 0$ on $[a, b]$, so that

$w = 0$ on Piece 2 of the boundary. Hence, u and v have the same mean over each circle $\{|z| = r\}$.

Piece 3: Consider $z = ae^{i\theta}$ for $0 < \theta < \pi$. We use (2.1) to show that w cannot be maximized on Piece 3. Write $E(a, 2\theta)$ for a set of length 2θ for which $u^\star(ae^{i\theta}) = \int_{E(a,2\theta)} u(ae^{i\phi})\, \mathrm{d}\phi$ (see Proposition 1.7), and do the same for $E(a+h, 2\theta)$. If $h > 0$ we compute

$$\begin{aligned}
w((a+h)e^{i\theta}) - w(ae^{i\theta}) &= \int_{E(a+h,2\theta)} u((a+h)e^{i\phi})\, \mathrm{d}\phi - \int_{E(a,2\theta)} u(ae^{i\phi})\, \mathrm{d}\phi \\
&\quad - \int_{-\theta}^{\theta} [v((a+h)e^{i\phi}) - v(ae^{i\phi})]\, \mathrm{d}\phi \\
&\quad - \epsilon[Q((a+h)e^{i\theta}) - Q(ae^{i\theta})] \\
&\geq \int_{E(a,2\theta)} [u((a+h)e^{i\phi}) - u(ae^{i\phi})]\, \mathrm{d}\phi \\
&\quad - \int_{-\theta}^{\theta} [v((a+h)e^{i\phi}) - v(ae^{i\phi})]\, \mathrm{d}\phi \\
&\quad - \epsilon[Q((a+h)e^{i\theta}) - Q(ae^{i\theta})].
\end{aligned}$$

If we divide the above inequality by h we see

$$\liminf_{h\to 0} \frac{w((a+h)e^{i\theta}) - w(ae^{i\theta})}{h} \quad \geq \quad \epsilon \sin\theta,$$

since $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0$ at radius a. From the above inequality, we see that for $h > 0$ sufficiently small,

$$w((a+h)e^{i\theta}) - w(ae^{i\theta}) \geq \frac{\epsilon}{2} h \sin\theta > 0$$

which shows that w cannot be maximized at such a boundary point.

Piece 4: Consider $z = be^{i\theta}$ for $0 < \theta < \pi$. We proceed just as with Piece 3, replacing $a + h$ and a with $b - h$ and b.

Step 3: Approximation argument for arbitrary input data.

Now consider the general case of $f \in L^2(A)$ with $\int_A f \; \mathrm{d}x = 0$. Choose an approximating sequence of compactly supported smooth functions $f_k \in C_c^\infty(A)$ having mean zero such that $f_k \to f$ in $L^2(A)$. Let u and v be as in the statement of Theorem 2.2. Let u_k and v_k solve

$$\begin{aligned} -\Delta u_k &= f_k \quad \text{in} \quad A, & -\Delta v_k &= f_k^{\#} \quad \text{in} \quad A, \\ \tfrac{\partial u_k}{\partial n} &= 0 \quad \text{on} \quad \partial A, & \tfrac{\partial v_k}{\partial n} &= 0 \quad \text{on} \quad \partial A, \end{aligned}$$

and assume that the solutions u_k and v_k satisfy the normalization $\int_A u_k \; \mathrm{d}x = \int_A v_k \; \mathrm{d}x = 0$. Since each f_k is Lipschitz continuous on $\overline{A}$, the work of Step 1 and Step 2 give

$$u_k^\star \le v_k^\star \tag{2.2}$$

in $A^\star$ for every k. Corollary 1.29 gives that $u_k \to u$ in $L^2(A)$. By Theorem 1.22, $f_k^{\#} \to f^{\#}$ in $L^2(A)$ and consequently $v_k \to v$ in $L^2(A)$ again by Corollary 1.29. Hence by using Theorem 1.23 we can pass to a subsequence of the original f_k and assume that for almost every $r \in (a,b)$,

$$\begin{aligned} u_k^\star(re^{i\theta}) &\to u^\star(re^{i\theta}), \\ \int_{-\pi}^{\pi} u_k(re^{i\theta}) \; \mathrm{d}\theta &\to \int_{-\pi}^{\pi} u(re^{i\theta}) \; \mathrm{d}\theta, \\ v_k^\star(re^{i\theta}) &\to v^\star(re^{i\theta}), \\ \int_{-\pi}^{\pi} v_k(re^{i\theta}) \; \mathrm{d}\theta &\to \int_{-\pi}^{\pi} v(re^{i\theta}) \; \mathrm{d}\theta, \end{aligned}$$

and where the star functions converge for every $\theta \in (0,\pi)$. Our analysis of Piece 2 in Step 2 shows

$$\int_{-\pi}^{\pi} u_k(re^{i\theta}) \; \mathrm{d}\theta = \int_{-\pi}^{\pi} v_k(re^{i\theta}) \; \mathrm{d}\theta$$

for each k and hence,

$$\int_{-\pi}^{\pi} u(re^{i\theta})\ \mathrm{d}\theta \quad = \quad \int_{-\pi}^{\pi} v(re^{i\theta})\ \mathrm{d}\theta$$

for almost every r.

Finally, letting $k \to \infty$ in (2.2), we have have for almost every $r \in (a, b)$,

$$u^{\bigstar}(re^{i\theta}) \leq v^{\bigstar}(re^{i\theta})$$

holds for every $\theta \in (0, \pi)$. □

Remark 2.3. Investigating the above proof, we reach the same conclusion $u^{\bigstar} \leq v^{\bigstar}$ a.e. if we make the weaker assumption that u and v have the same mean over the annulus, rather than both having mean zero.

2.3. Mapping the annulus to the square

Baernstein indicated to Kawohl two keys to proving his conjecture. The first of these, the annular comparison result, was discussed in the last section. Now we discuss the second key, a conformal mapping, and show how it is used to obtain a rectangular comparison result. As mentioned at the outset of this chapter, some work will be involved. A Jacobian factor is introduced from the conformal change of variables. We will see that this Jacobian factor interacts well with each of the rearrangements involved.

Before we prove an analogue of Theorem 2.2 in a square, we need to define the notion of a star function using the rearrangement Kawohl considered. Let $R = (0, 1) \times (0, 1)$. For $f : R \to \mathbb{R}$, we let $f^{\#}$ denote the monotone decreasing rearrangement of f in the direction y. That is, $f^{\#}(x, y) = (f^x)^*(y)$ where $(f^x)^*$ is the decreasing rearrangement of the slice function $f^x(y) = f(x,y)$. Let $f^\star : R \to \mathbb{R}$ be the star function corresponding to this rearrangement, defined by the formula

$$f^\star(s,t) = \max_{|E|=t} \int_E f(s,\tau)\ \mathrm{d}\tau,$$

where $s \in (0,1)$ and the max is taken over all measurable subsets $E \subseteq (0,1)$ of one-dimensional Lebesgue measure $t \in (0,1)$. Just as in Proposition 1.7, the max is achieved for some subset E, which explains our use of max instead of sup.

We now state the rectangular comparison result. We compare the solutions of two PDEs, one with given data and one with data rearranged monotonically in the y direction. We see that the solution with rearranged data has a larger star function.

In the following result and in the remainder of the thesis, we use dx and dm interchangeably for two-dimensional Lebesgue measure.

Theorem 2.4 (Rectangular Comparison Theorem). *Let $f \in L^2(R)$ where $\int_R f\ \mathrm{d}m = 0$ and suppose u and v are weak solutions to*

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad R, & -\Delta v &= f^\# \quad \text{in} \quad R, \\ \tfrac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial R, & \tfrac{\partial v}{\partial n} &= 0 \quad \text{on} \quad \partial R, \end{aligned}$$

where $f^\#$ denotes the monotone decreasing rearrangement of f in the direction y.

If u and v are additively normalized so that $\int_R u\ \mathrm{d}m = \int_R v\ \mathrm{d}m = 0$, then for almost every $s \in (0,1)$,

$$\int_0^1 u(s,t)\ \mathrm{d}t \ = \ \int_0^1 v(s,t)\ \mathrm{d}t$$

and the inequality

$$u^\star(s,t) \ \leq \ v^\star(s,t)$$

holds for every $t \in (0,1)$*. In particular,* $u^\star \le v^\star$ *a.e. in* R*.*

Proof. **Step 1: Conformally convert data f and $f^\#$ on R into data g and $g^\#$ on an annulus A.**

First assume f is Lipschitz continuous in $\overline{R}$ and let $A = A(1, e^\pi) = \{z \in \mathbb{C} : 1 < |z| < e^\pi\}$. The exponential function $T(\zeta) = e^{\pi\zeta}$ maps R conformally onto $A^\star$.

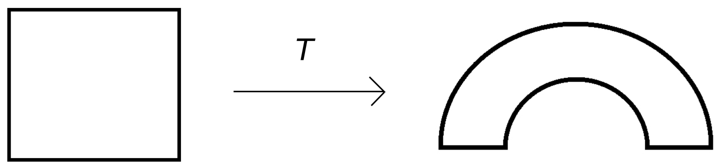

Figure 2.2. A picture of the conformal mapping T.

Define

$$\begin{aligned} g(z) &= f(T^{-1}(z))|(T^{-1})'(z)|^2 \\ &= \frac{f(T^{-1}(z))}{|\pi z|^2} \quad \text{for } z \in \overline{A^\star}, \end{aligned}$$

where we choose the branch cut of $T^{-1}(z) = \frac{1}{\pi}\log z$ to lie along the negative imaginary axis. Notice g is Lipschitz continuous on $\overline{A^\star}$. Extend g to all of $\overline{A}$ by reflection across the real axis, that is, $g(z) = g(\bar{z})$. This extended function g is Lipschitz continuous on $\overline{A}$. Since cap symmetrization is performed on circles and $|\pi z|^2$ is positive and constant on circles, it follows that

$$g^\#(z) = \frac{f^\#(T^{-1}(z))}{|\pi z|^2} \tag{2.3}$$

for $z \in A^\bigstar$. A cautionary note regarding the above equation: $g^\#$ denotes the $(1, 2)$ cap symmetrization of g whereas $f^\#$ is the monotone decreasing rearrangement of f in the y direction.

The normalization of f implies that

$$\begin{aligned} 0 &= \int_R f(\zeta)\ \mathrm{d}m(\zeta) \\ &= \int_{A^\bigstar} f(T^{-1}(z))\ |\ (T^{-1})'(z)\ |^2\ \ \mathrm{d}m(z) \end{aligned}$$

by a change of variable. Hence

$$\begin{aligned} 0 &= 2\int_{A^\bigstar} g(z)\ \mathrm{d}m(z) \\ &= \int_A g(z)\ \mathrm{d}m(z). \end{aligned}$$

This computation shows that g satisfies the admissibility condition for input data into a Poisson equation with Neumann boundary conditions. It also explains why g must include the Jacobian factor in its definition.

Let U and V solve

$$\begin{aligned} -\Delta U &= g \quad \text{in} \quad A, & \qquad -\Delta V &= g^\# \quad \text{in} \quad A, \\ \tfrac{\partial U}{\partial n} &= 0 \quad \text{on} \quad \partial A, & \qquad \tfrac{\partial V}{\partial n} &= 0 \quad \text{on} \quad \partial A, \end{aligned}$$

where U and V are normalized so that $\int_A U\ \mathrm{d}m = \int_A V\ \mathrm{d}m = 0$. By Theorem 3.2 of [**LU**], U and V belong to $C^2(\overline{A})$.

Step 2: Obtain potential solutions u and v for the original problem.

Define $u, v \in C^2(R) \cap C^1(\bar{R})$ by

$$\begin{aligned} u &= U \circ T, \\ v &= V \circ T. \end{aligned}$$

We calculate

$$\begin{aligned} -\Delta u(\zeta) &= -\Delta U(T(\zeta))|T'(\zeta)|^2 \\ &= g(T(\zeta))|T'(\zeta)|^2 \\ &= f(\zeta). \end{aligned}$$

Similarly,

$$\begin{aligned} -\Delta v(\zeta) &= -\Delta V(T(\zeta))|T'(\zeta)|^2 \\ &= g^{\#}(T(\zeta))|T'(\zeta)|^2 \\ &= f^{\#}(\zeta), \end{aligned}$$

where the last equality follows from equality (2.3).

T is conformal, hence takes arcs that are perpendicular to ∂R into arcs that are perpendicular to $\partial A^\star$. Moreover, T maps ∂R onto $\partial A^\star$. It follows that $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0$ on ∂R.

We next show that u and v have the same mean over any vertical strip through R. To see this, we fix an $s \in (0,1)$ and compute the integral

$$\begin{aligned} \int_0^1 u(s,t)\ \mathrm{d}t &= \int_0^1 U(e^{\pi s}e^{i\pi t})\ \mathrm{d}t \\ &= \frac{1}{2}\int_{-1}^1 U(e^{\pi s}e^{i\pi t})\ \mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} U(e^{\pi s}e^{it})\ \mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} V(e^{\pi s}e^{it})\ \mathrm{d}t \\ &= \int_0^1 v(s,t)\ \mathrm{d}t, \end{aligned}$$

where the second to last equality follows from the first conclusion of Theorem 2.2. Now Theorem 2.2 implies $U^\star \leq V^\star$ on $A^\star$ and writing $\zeta = (s,t)$ we then calculate

$$\begin{aligned} u^\star(\zeta) &= \int_0^t u^\#(s,\tau)\ \mathrm{d}\tau \\ &= \frac{1}{2\pi}\int_{-\pi t}^{\pi t} U^\#(e^{\pi s}e^{i\tau})\ \mathrm{d}\tau \\ &= \frac{1}{2\pi}U^\star(T(\zeta)) \\ &\leq \frac{1}{2\pi}V^\star(T(\zeta)) \\ &= v^\star(\zeta). \end{aligned}$$

One last issue to resolve is that the hypothesis of Theorem 2.4 require that u and v have mean zero. Since u and v above have the same mean over each vertical segment, they certainly have the same mean over the rectangle R. Hence by subtracting that constant from u and v we can assume that both u and v have mean zero, and the conclusion $u^\star \leq v^\star$ will still hold.

Step 3: Approximation argument for arbitrary input data f.

Now let f be a general function in $L^2(R)$ with $\int_R f \, \mathrm{d}m = 0$. Choose a sequence of compactly supported smooth functions $f_k \in C_c^\infty(R)$ with mean zero where $f_k \to f$ in $L^2(R)$. Let u_k and v_k solve

$$\begin{aligned} -\Delta u_k &= f_k \quad \text{in} \quad R, \qquad & -\Delta v_k &= f_k^\# \quad \text{in} \quad R, \\ \tfrac{\partial u_k}{\partial n} &= 0 \quad \text{on} \quad \partial R, \qquad & \tfrac{\partial v_k}{\partial n} &= 0 \quad \text{on} \quad \partial R, \end{aligned}$$

and assume that the u_k and v_k are normalized to have mean zero. Since the f_k are Lipschitz continuous on $\overline{R}$, Step 2 shows that

$$u_k^\bigstar \le v_k^\bigstar \tag{2.4}$$

in R, for each k. Step 2 also shows that for every $s \in (0,1)$,

$$\int_0^1 u_k(s,t) \, \mathrm{d}t \;=\; \int_0^1 v_k(s,t) \, \mathrm{d}t. \tag{2.5}$$

By Theorem 1.4, for each fixed s

$$\int_0^1 \left| f_k^\#(s,t) - f^\#(s,t) \right|^2 \mathrm{d}t \le \int_0^1 \left| f_k(s,t) - f(s,t) \right|^2 \mathrm{d}t,$$

and if we integrate this inequality for $s \in (0,1)$ we have

$$\int_R \left| f_k^\# - f^\# \right|^2 \mathrm{d}m \le \int_R \left| f_k - f \right|^2 \mathrm{d}m.$$

Thus, $f_k^\# \to f^\#$ in $L^2(R)$, since $f_k \to f$ in $L^2(R)$. Corollary 1.29 implies that $u_k \to u$ and $v_k \to v$ in $L^2(R)$ and hence also in $L^1(R)$.

Now we mimic the argument used to prove Theorem 1.23. Define $\Psi_k : (0,1) \to \mathbb{R}$ by the formula

$$\Psi_k(s) \;=\; \int_0^1 |u_k(s,t) - u(s,t)| \, \mathrm{d}t.$$

Since $u_k \to u$ in $L^1(R)$,

$$\int_0^1 \int_0^1 \big|u_k(s,t) - u(s,t)\big| \; \mathrm{d}t \; \mathrm{d}s \quad \to \quad 0,$$

which implies that $\Psi_k \to 0$ in $L^1((0,1), \mathrm{d}s)$. Hence some subsequence of the Ψ_k converges pointwise a.e. to 0. By passing to a subsequence of the original f_k, we may assume that $\Psi_k \to 0$ a.e. in $(0,1)$, which means that

$$\int_0^1 |u_k(s,t) - u(s,t)| \; \mathrm{d}t \quad \to \quad 0$$

for almost every $s \in (0,1)$.

Fix an s so that convergence holds above. Then for any $t \in (0,1)$ we have

$$\begin{aligned} |u_k^\star(s,t) - u^\star(s,t)| \quad &\leq \quad \int_0^t |u_k^\#(s,\tau) - u^\#(s,\tau)| \; \mathrm{d}\tau \\ &\leq \quad \int_0^1 |u_k^\#(s,\tau) - u^\#(s,\tau)| \; \mathrm{d}\tau \\ &\leq \quad \int_0^1 |u_k(s,\tau) - u(s,\tau)| \; \mathrm{d}\tau, \end{aligned}$$

the last inequality following from Theorem 1.4. Letting $k \to \infty$ we have that for almost every $s \in (0,1)$,

$$u_k^\star(s,t) \quad \to \quad u^\star(s,t)$$

for every $t \in (0,1)$. By similar considerations applied to the sequence v_k (and passing to another subsequence) we also have for almost every $s \in (0,1)$

$$\int_0^1 |v_k(s,t) - v(s,t)| \; \mathrm{d}t \quad \to \quad 0$$

and

$$v_k^\bigstar(s,t) \quad \to \quad v^\bigstar(s,t)$$

for each $t \in (0,1)$.

Letting $k \to \infty$ in equation (2.5), we conclude

$$\int_0^1 u(s,t)\ \mathrm{d}t \quad = \quad \int_0^1 v(s,t)\ \mathrm{d}t$$

for almost every $s \in (0,1)$. Letting $k \to \infty$ in inequality (2.4), we conclude that for almost every $s \in (0,1)$, the inequality

$$u^\bigstar(s,t) \quad \leq \quad v^\bigstar(s,t)$$

holds for every $t \in (0,1)$. □

Remark 2.5. We can reach the same conclusion in Theorem 2.4 if we only assume u and v have the same mean over the rectangle R, rather than assuming they both have mean zero.

2.4. Proof of Kawohl's conjecture

We now prove Kawohl's conjecture for an arbitrary input function f. The original proof by Baernstein used a characterization of the conclusion $u^\bigstar \leq v^\bigstar$ of Theorem 2.4 in terms of convex means (Proposition 1.9). The argument below avoids this characterization and is much simpler.

Corollary 2.6 (Kawohl's Conjecture)**.** *If f, u, and v are as in Theorem 2.4, then*

$$\operatorname*{osc}_R u \quad \leq \quad \operatorname*{osc}_R v.$$

Proof. By Theorem 2.4, for almost every $s \in (0,1)$, we have

$$\int_0^t u^{\#}(s,\tau)\ \mathrm{d}\tau \quad \leq \quad \int_0^t v^{\#}(s,\tau)\ \mathrm{d}\tau$$

for every $t \in (0,1)$. Additionally, u and v have the same mean over almost every vertical segment of R, hence Corollary 1.11 gives

$$\operatorname*{ess\,sup}_{t\in(0,1)} u(s,t) \quad \leq \quad \operatorname*{ess\,sup}_{t\in(0,1)} v(s,t)$$

and

$$\operatorname*{ess\,inf}_{t\in(0,1)} v(s,t) \quad \leq \quad \operatorname*{ess\,inf}_{t\in(0,1)} u(s,t).$$

Taking the ess sup over $s \in (0,1)$ in the first inequality and the ess inf over $s \in (0,1)$ in the second inequality, we have

$$\operatorname*{ess\,sup}_{R} u \quad \leq \quad \operatorname*{ess\,sup}_{R} v$$

and

$$\operatorname*{ess\,inf}_{R} v \quad \leq \quad \operatorname*{ess\,inf}_{R} u.$$

Finally, if we combine the above ess sup and ess inf inequalities, we conclude

$$\operatorname*{osc}_{R} u \quad \leq \quad \operatorname*{osc}_{R} v$$

as desired. □

CHAPTER 3

Challenges and New Results in Higher Dimensions

In this chapter, we present the main result of this thesis, Theorem 3.1. This result in a generalization of Theorem 2.2 from Chapter 2. The proof is not an immediate generalization however; we use a domain approximation argument. This approximation technique must be used because in dimension $n > 2$, the operator $\Delta^\star$ blows up near the boundary of $A^\star$, and hence the maximum principle must be used on an approximating domain where $\Delta^\star$ is better behaved. After establishing the theorem, we proceed to discuss comparison results on spheres and hemispheres.

3.1. New shell comparison results in all dimensions

The key to the proof of Theorem 2.2 is the subharmonicity property contained in Theorem 1.27. This subharmonicity result holds in all dimensions (Theorem 1.25), and so it should come as no surprise that a comparison result holds for spherical shells in higher dimensions.

The reader might find it useful to review the cap symmetrization definitions and theorems within subsection 1.1.5 before continuing. We begin this chapter with the higher dimensional version of Theorem 2.2. We compare the solutions of two PDEs, one with given initial data and one with cap symmetrized data. We see that the solution with cap symmetrized data has a larger star function. As an easy corollary, the solution with cap symmetrized data has larger L^p norms and oscillation.

Theorem 3.1 (Comparison Theorem in Spherical Shells)**.** *Let $A = A(a,b) \subset \mathbb{R}^n$ be a spherical shell and let $f \in L^2(A)$ with $\int_A f \, \mathrm{d}x = 0$. Assume u and v are weak solutions to*

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad A, & \qquad -\Delta v &= f^{\#} \quad \text{in} \quad A, \\ \tfrac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial A, & \qquad \tfrac{\partial v}{\partial n} &= 0 \quad \text{on} \quad \partial A, \end{aligned}$$

where $f^{\#}$ denotes the $(n-1,n)$ cap symmetrization of f.

If the solutions u and v are additively normalized so that $\int_A u \, \mathrm{d}x = \int_A v \, \mathrm{d}x = 0$, then for almost every $r \in (a,b)$,

$$\int_{\mathbb{S}^{n-1}} u(r\xi) \, \mathrm{d}\sigma_{n-1}(\xi) \;=\; \int_{\mathbb{S}^{n-1}} v(r\xi) \, \mathrm{d}\sigma_{n-1}(\xi)$$

and the inequality

$$u^{\star}(r,\theta) \;\le\; v^{\star}(r,\theta)$$

holds for every $\theta \in (0,\pi)$. In particular, $u^{\star} \le v^{\star}$ a.e. in $A^{\star}$.

Proof. **Step 1: Construct approximating domains.**

First suppose that f is Lipschitz continuous on $\overline{A}$. Since $(n-1,n)$ cap symmetrization decreases the modulus of continuity, it follows that $f^{\#}$ is also Lipschitz continuous on $\overline{A}$ (this follows from Corollary 3 of [**Ba5**]). Consequently, u and v belong to $C^2(\overline{A})$ by Theorem 3.2 of [**LU**].

Fix $\epsilon > 0$ and let $R_\epsilon = (a,b) \times (\epsilon, \pi - \epsilon)$. A picture of R_ϵ is given in Figure 3.1.

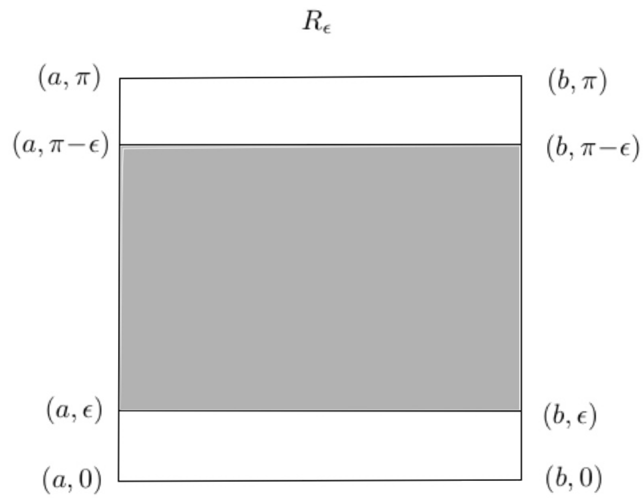

FIGURE 3.1. A picture of the domain R_ϵ.

Let S_ϵ be a C^∞ domain nested between R_ϵ and R $\big(R_\epsilon \subseteq S_\epsilon \subseteq R\big)$ satisfying $\partial S_\epsilon \cap \partial R = \partial S_\epsilon \cap \partial R_\epsilon = \big(\{a\} \times [\epsilon, \pi - \epsilon]\big) \cup \big(\{b\} \times [\epsilon, \pi - \epsilon]\big)$. A picture of such a domain S_ϵ is given in Figure 3.2.

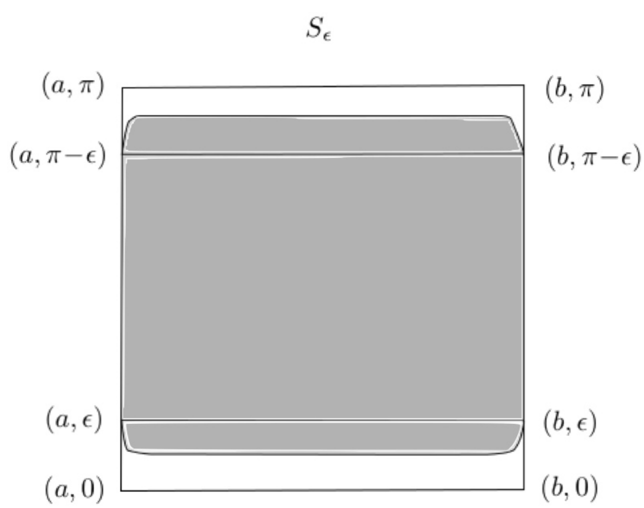

FIGURE 3.2. A picture of a possible domain S_ϵ.

Let

$$Q(r,\theta) = (r-a)(r-b) + C\theta(\pi-\theta), \quad \text{for} \quad (r,\theta) \in A^{\bigstar},$$

where C is chosen sufficiently large so that

$$\Delta^{\bigstar} Q = 1 + \frac{n-1}{r}(2r-a-b) - C\left(2 + (n-2)\cot\theta(\pi - 2\theta)\right) \leq 0.$$

Note that

$$Q_r(r,\theta) = 2\left(r - \frac{a+b}{2}\right)$$

and so it follows that

$$0 < \frac{\partial Q}{\partial n} \quad \text{on} \quad (\{a\}\times(0,\pi)) \cup (\{b\}\times(0,\pi))\,. \tag{3.1}$$

Multiplying Q by a suitable positive constant, we may assume that

$$\|Q\|_{L^\infty(A^{\bigstar})} \;\; \leq \;\; 1.$$

Define

$$w_\epsilon \;\; = \;\; u^{\bigstar} - Jv - \epsilon Q \quad \text{for } (r,\theta) \in R_\epsilon.$$

Step 2: Maximum principle on approximating domains.

In the distributional sense, we compute

$$\begin{aligned} -\Delta^{\bigstar} w_\epsilon &= -\Delta^{\bigstar} u^{\bigstar} + \Delta^{\bigstar} Jv + \epsilon\Delta^{\bigstar} Q \\ &\leq f^{\bigstar} + J\Delta v + 0 \\ &= f^{\bigstar} - f^{\bigstar} \\ &= 0, \end{aligned}$$

where the inequality follows from the Subharmonicity and Commutativity properties (Theorems 1.24 and 1.25). By the maximum principle applied to $\Delta^\bigstar$ (Theorem 3 of [**Li**]),

$$\max_{R_\epsilon} w_\epsilon \quad \leq \quad \max_{\partial R_\epsilon} w_\epsilon. \tag{3.2}$$

We claim that the max over the boundary cannot be attained at a point of $\big(\{a\} \times (\epsilon, \pi - \epsilon)\big) \cup \big(\{b\} \times (\epsilon, \pi - \epsilon)\big)$. We prove this by cases.

Case 1: Fix (a, θ_1) with $\theta_1 \in (\epsilon, \pi - \epsilon)$ and let $E(a, K(\theta_1))$ denote a subset of $\mathbb{S}^{n-1}$ with the same surface measure as $K(\theta_1)$ for which the max defining $u^\bigstar(a, \theta_1)$ is achieved. We compute for $h > 0$

$$\begin{aligned}\frac{w_\epsilon(a+h,\theta_1) - w_\epsilon(a,\theta_1)}{h} \quad &\geq \quad \int_{E(a,K(\theta_1))} \frac{u((a+h)\xi) - u(a\xi)}{h}\, \mathrm{d}\sigma_{n-1}(\xi) \\ &\quad - \int_{K(\theta_1)} \frac{[v((a+h)\xi) - v(a\xi)]}{h}\, \mathrm{d}\sigma_{n-1}(\xi) \\ &\quad - \epsilon \frac{[Q(a+h,\theta_1) - Q(a,\theta_1)]}{h}.\end{aligned}$$

Taking the $\liminf\limits_{h\to 0}$ and using that $\frac{\partial u}{\partial n}$ and $\frac{\partial v}{\partial n}$ vanish, we get

$$\liminf_{h\to 0} \frac{w_\epsilon(a+h,\theta_1) - w_\epsilon(a,\theta_1)}{h} \geq \epsilon\kappa,$$

where $\kappa = \frac{\partial Q}{\partial n}(a, \theta_1)$ is some positive number by (3.1). Hence, for all $h > 0$ sufficiently small, we have

$$\begin{aligned} w_\epsilon(a+h,\theta_1) \quad &> \quad w_\epsilon(a,\theta_1) + h\frac{\epsilon}{2}\kappa \\ &> \quad w_\epsilon(a,\theta_1). \end{aligned}$$

Thus, the maximum of w_ϵ over ∂R_ϵ does not occur at (a, θ_1), since otherwise the maximum principle (3.2) would be violated.

Case 2: Similar to Case 1, we find the maximum of w_ϵ over ∂R_ϵ does not occur at (b, θ_2) with $\theta_2 \in (\epsilon, \pi - \epsilon)$.

Our casework above shows that the portion $\big(\{a\} \times (\epsilon, \pi - \epsilon)\big) \cup \big(\{b\} \times (\epsilon, \pi - \epsilon)\big)$ of ∂R_ϵ may be removed in the max inequality (3.2). If we write $T_\epsilon = \big([a, b] \times \{\epsilon\}\big) \cup \big([a, b] \times \{\pi - \epsilon\}\big)$ then inequality (3.2) becomes

$$\max_{R_\epsilon} w_\epsilon \leq \max_{T_\epsilon} w_\epsilon,$$

from which we deduce

$$\begin{aligned} -\epsilon + \max_{R_\epsilon}(u^\star - Jv) &\leq \max_{R_\epsilon} w_\epsilon \\ &\leq \max_{T_\epsilon} w_\epsilon \\ &\leq \max_{T_\epsilon}(u^\star - Jv) + \epsilon, \end{aligned}$$

where the first and last inequalities hold because $|Q| \leq 1$. Letting $\epsilon \to 0$ we conclude

$$\max_{R}(u^\star - Jv) \leq \max_{T}(u^\star - Jv), \tag{3.3}$$

where $T = \big([a, b] \times \{0\}\big) \cup \big([a, b] \times \{\pi\}\big)$.

By definition, $u^\star - Jv = 0$ on $[a, b] \times \{0\}$, that is, when $\theta = 0$. We claim $u^\star - Jv = 0$ on $[a, b] \times \{\pi\}$ too. In other words, we claim that u and v have the same integral over each sphere of radius $r \in (a, b)$. Let $\Phi(r) = r^{n-1} \frac{\partial}{\partial r}\big(\int_{\mathbb{S}^{n-1}} \big(u(r\xi) - v(r\xi)\big) \, \mathrm{d}\sigma_{n-1}(\xi)\big) = \int_{\{|x|=r\}} (u_r - v_r) \, \mathrm{d}S$. Then since $\frac{\partial u}{\partial n}$ and $\frac{\partial v}{\partial n}$ are zero when $r = a$, we compute from

Green's Theorem that

$$\begin{aligned}\Phi(r) &= \int_{A(a,r)} (\Delta u - \Delta v)\ \mathrm{d}x \\ &= \int_{A(a,r)} (-f + f^{\#})\ \mathrm{d}x \\ &= 0.\end{aligned}$$

Thus, $\Phi \equiv 0$ on $[a,b]$. By the definition of Φ, this implies that $\int_{\mathbb{S}^{n-1}}(u(r\xi) - v(r\xi))\ \mathrm{d}\sigma_{n-1}(\xi) = c$ is constant on $[a,b]$. And since $\int_a^b \int_{\mathbb{S}^{n-1}} \big(u(r\xi) - v(r\xi)\big)\ \mathrm{d}\sigma_{n-1}(\xi)\ r^{n-1}\mathrm{d}r = \int_A (u-v)\ \mathrm{d}x = 0$ we conclude $c = 0$. Hence $\max_R [u^\star - Jv] = 0$ so that $u^\star - Jv \leq 0$ by (3.3). Thus, $u^\star \leq Jv \leq v^\star$, giving the theorem in the case where f is Lipschitz continuous on $\overline{A}$.

Step 3: Approximation argument for general f.

Now let $f \in L^2(A)$ be a general function with mean zero. Choose a sequence of compactly supported smooth functions $f_k \in C_c^\infty(A)$ each with mean zero and where $f_k \to f$ in $L^2(A)$. Assume u_k and v_k solve

$$\begin{aligned} -\Delta u_k &= f_k \quad \text{in} \quad A, & \qquad -\Delta v_k &= f_k^{\#} \quad \text{in} \quad A, \\ \tfrac{\partial u_k}{\partial n} &= 0 \quad \text{on} \quad \partial A, & \tfrac{\partial v_k}{\partial n} &= 0 \quad \text{on} \quad \partial A, \end{aligned}$$

where the u_k and v_k are normalized so that $\int_A u_k\ \mathrm{d}x = \int_A v_k\ \mathrm{d}x = 0$. Since each f_k is Lipschitz continuous on $\overline{A}$, our work above shows

$$\int_{\mathbb{S}^{n-1}} u_k(r\xi)\ \mathrm{d}\sigma_{n-1}(\xi) = \int_{\mathbb{S}^{n-1}} v_k(r\xi)\ \mathrm{d}\sigma_{n-1}(\xi) \tag{3.4}$$

for every $r \in (a,b)$ and

$$u_k^\star \leq v_k^\star \tag{3.5}$$

in $A^\bigstar$ for every k. By Theorem 1.22, $f_k^\# \to f^\#$ in $L^2(A)$ since $f_k \to f$ in $L^2(A)$. Consequently, $u_k \to u$ and $v_k \to v$ in $L^2(A)$ by Corollary 1.29. By Theorem 1.23, we can pass to a subsequence of the original f_k and assume that for almost every $r \in (a,b)$,

$$\int_{\mathbb{S}^{n-1}} u_k(r\xi)\ \mathrm{d}\sigma_{n-1}(\xi) \;\;\to\;\; \int_{\mathbb{S}^{n-1}} u(r\xi)\ \mathrm{d}\sigma_{n-1}(\xi)$$

and that

$$u_k^\bigstar(r,\theta) \;\;\to\;\; u^\bigstar(r,\theta)$$

for every $\theta \in (0,\pi)$. By another application of Theorem 1.23 and passing to yet another subsequence of the f_k, we may additionally assume that for almost every $r \in (a,b)$,

$$\int_{\mathbb{S}^{n-1}} v_k(r\xi)\ \mathrm{d}\sigma_{n-1}(\xi) \;\;\to\;\; \int_{\mathbb{S}^{n-1}} v(r\xi)\ \mathrm{d}\sigma_{n-1}(\xi)$$

and that

$$v_k^\bigstar(r,\theta) \;\;\to\;\; v^\bigstar(r,\theta)$$

for every $\theta \in (0,\pi)$. Letting $k \to \infty$ in (3.4) and (3.5), we therefore conclude that for almost every $r \in (a,b)$,

$$\int_{\mathbb{S}^{n-1}} u(r\xi)\ \mathrm{d}\sigma_{n-1}(\xi) \;\;=\;\; \int_{\mathbb{S}^{n-1}} v(r\xi)\ \mathrm{d}\sigma_{n-1}(\xi)$$

and

$$u^\bigstar(r,\theta) \;\;\leq\;\; v^\bigstar(r,\theta)$$

for every $\theta \in (0,\pi)$. $\square$

Remark 3.2. We obtain the same conclusion $u^\star \leq v^\star$ in Theorem 3.1 if we make the weaker assumption that u and v have the same mean over the shell A, not necessarily that they both have zero mean. Moreover, the slice functions of u and v still have the same mean almost everywhere under this weaker assumption.

The corollary below restates the conclusion of Theorem 3.1 in terms of convex means. This characterization goes all the way back to Hardy, Littlewood, and Polya [**HLP**]. Consequently, we obtain L^p and oscillation estimates.

Corollary 3.3. *Let f, u, and v be as in Theorem 3.1. Then for almost every $r \in (a,b)$ and each convex function $\phi : \mathbb{R} \to \mathbb{R}$ we have*

$$\int_{\mathbb{S}^{n-1}} \phi(u(r\xi)) \, \mathrm{d}\sigma_{n-1}(\xi) \;\leq\; \int_{\mathbb{S}^{n-1}} \phi(v(r\xi)) \, \mathrm{d}\sigma_{n-1}(\xi).$$

Hence,

$$\|u(r\,\cdot)\|_{L^p(\mathbb{S}^{n-1})} \;\leq\; \|v(r\,\cdot)\|_{L^p(\mathbb{S}^{n-1})}, \quad 1 \leq p \leq \infty,$$

$$\operatorname*{ess\,sup}_{|x|=r} u \;\leq\; \operatorname*{ess\,sup}_{|x|=r} v,$$

$$\operatorname*{ess\,inf}_{|x|=r} u \;\geq\; \operatorname*{ess\,inf}_{|x|=r} v,$$

$$\operatorname*{osc}_{|x|=r} u \;\leq\; \operatorname*{osc}_{|x|=r} v.$$

Consequently, for each convex function $\phi : \mathbb{R} \to \mathbb{R}$,

$$\int_A \phi(u) \, \mathrm{d}x \;\leq\; \int_A \phi(v) \, \mathrm{d}x.$$

Moreover,

$$\begin{aligned}
\|u\|_{L^p(A)} &\le \|v\|_{L^p(A)}, \quad 1 \le p \le \infty, \\
\operatorname*{ess\,sup}_A u &\le \operatorname*{ess\,sup}_A v, \\
\operatorname*{ess\,inf}_A u &\ge \operatorname*{ess\,inf}_A v, \\
\operatorname*{osc}_A u &\le \operatorname*{osc}_A v.
\end{aligned}$$

PROOF. By Theorem 3.1, for almost every $r \in (a,b)$, u and v have the same mean over the sphere $\{|x| = r\}$ and the inequality $u^\star(r,\theta) \le v^\star(r,\theta)$ holds for every $\theta \in (0,\pi)$. Hence, Proposition 1.9 gives

$$\int_{\mathbb{S}^{n-1}} \phi(u(r\xi))\, \mathrm{d}\sigma_{n-1}(\xi) \;\le\; \int_{\mathbb{S}^{n-1}} \phi(v(r\xi))\, \mathrm{d}\sigma_{n-1}(\xi)$$

for each convex function $\phi : \mathbb{R} \to \mathbb{R}$. The remaining spherical inequalities now follow from Corollary 1.11. The spherical shell inequalities follow from the spherical ones in obvious fashion. □

The next corollary tells us that the solution v to the symmetrized problem is cap symmetrized.

Corollary 3.4. *If f and v are as in Theorem 3.1, then $v = v^\#$ a.e.*

PROOF. First assume f is Lipschitz continuous on $\overline{A}$. Taking $v = u$, Step 2 in the proof of Theorem 3.1 shows $v^\star \le Jv$. Since $Jv \le v^\star$ by definition, we have $v^\star = Jv$ on $A^\star$. Fix $r \in (a,b)$. We show $v = v^\#$ by first proving that the slice function v^r is constant on $\partial K(\theta)$ for each $\theta \in (0,\pi)$.

Claim: v^r is constant on $\partial K(\theta)$ for each $\theta \in (0,\pi)$.

Assume the claim is false. Choose $\theta_0 \in (0,\pi)$ with v^r non-constant on $\partial K(\theta_0)$. Let $\xi_1, \xi_2 \in \partial K(\theta_0)$ be such that

$$\begin{aligned}\min_{\partial K(\theta_0)} v^r &= v^r(\xi_1),\\ \max_{\partial K(\theta_0)} v^r &= v^r(\xi_2).\end{aligned}$$

Let $\epsilon_1, \epsilon_2 > 0$ be small enough so that the spherical balls $B(\xi_1,\epsilon_1)$ and $B(\xi_2,\epsilon_2)$ are disjoint and

$$\sup_{B(\xi_1,\epsilon_1)} v^r < \inf_{B(\xi_2,\epsilon_2)} v^r.$$

Additionally, assume ϵ_1 and ϵ_2 are such that $\big(K(\theta_0) \cup B(\xi_2,\epsilon_2)\big)\backslash B(\xi_1,\epsilon_1)$ has the same surface measure as $K(\theta_0)$. Since $v^\star = Jv$, we have by definition

$$\int_E v^r \, \mathrm{d}\sigma_{n-1} \;\le\; v^\star(r,\theta_0) \;=\; \int_{K(\theta_0)} v \, \mathrm{d}\sigma_{n-1} \tag{3.6}$$

for all measurable subsets $E \subseteq \mathbb{S}^{n-1}$ with the same surface measure as $K(\theta_0)$. Take $E = \big(K(\theta_0) \cup B(\xi_2,\epsilon_2)\big)\backslash B(\xi_1,\epsilon_1)$. Geometrically, E is constructed from $K(\theta_0)$ by replacing the portion of $B(\xi_1,\epsilon_1)$ contained in $K(\theta_0)$ by the portion of $B(\xi_2,\epsilon_2)$ contained inside $\mathbb{S}^{n-1}\backslash K(\theta_0)$. Then $\sigma_{n-1}(E) = \sigma_{n-1}(K(\theta_0))$ and we compute

$$\begin{aligned}\int_E v^r \, \mathrm{d}\sigma_{n-1} &= \int_{K(\theta_0)} v^r \, \mathrm{d}\sigma_{n-1} + \int_{B(\xi_2,\epsilon_2)\backslash K(\theta_0)} v^r \, \mathrm{d}\sigma_{n-1} - \int_{B(\xi_1,\epsilon_1)\cap K(\theta_0)} v^r \, \mathrm{d}\sigma_{n-1}\\ &> \int_{K(\theta_0)} v^r \, \mathrm{d}\sigma_{n-1},\end{aligned}$$

which contradicts the equation (3.6). The claim is therefore proved.

By the claim, v^r is constant on $\partial K(\theta)$ for each θ. Write $v^r(\theta)$ for that value. Additionally, $\big(v^\#\big)^r$is constant on $\partial K(\theta)$ for each θ by definition, so write $\big(v^\#\big)^r(\theta)$ for that value. Since $Jv = Jv^{\#}$, we have

$$\int_{K(\theta)} v^r \, \mathrm{d}\sigma_{n-1} \;=\; \int_{K(\theta)} \left(v^{\#}\right)^r \, \mathrm{d}\sigma_{n-1}.$$

Using spherical coordinates, the above integral becomes

$$\beta_{n-2} \int_0^{\theta} v^r(\theta) \sin\theta \, \mathrm{d}\theta \;=\; \beta_{n-2} \int_0^{\theta} \left(v^{\#}\right)^r(\theta) \sin\theta \, \mathrm{d}\theta,$$

where $\beta_{n-2} = \sigma_{n-2}(\mathbb{S}^{n-2})$. Differentiating the above equation with respect to θ implies $v^r(\theta) = \left(v^{\#}\right)^r(\theta)$. That is, $v = v^{\#}$.

Now let $f \in L^2(A)$ be a general function with mean zero and let f_k and v_k be as in Step 3 in the proof of Theorem 3.1. By the above, $v_k = v_k^{\#}$ on A. By passing to a subsequence of the original f_k we may assume that $v_k \to v$ and $v_k^{\#} \to v^{\#}$ a.e. Hence $v = v^{\#}$ a.e. □

3.2. New comparison results on spheres and hemispheres

In this section we study consequences of Theorem 3.1 for comparison theorems on spheres and hemispheres. Theorem 3.1 concerns cap symmetrization, a partial symmetrization. The corollaries below, on the other hand, deal with total symmetrizations, meaning the rearrangement takes place on the whole space rather than on submanifolds.

3.2.1. Spheres. We write $\Delta_{\mathbb{S}}$ and $\nabla_{\mathbb{S}}$ for the spherical Laplacian and spherical gradient on $\mathbb{S}^n$. We write $W^{1,2}(\mathbb{S}^n)$ for the Sobolev space of functions in $L^2(\mathbb{S}^n)$ that, once expressed in spherical coordinates, have weak partial derivatives in $L^2(\mathbb{S}^n)$.

Given $F \in L^2(\mathbb{S}^n)$, we say that $U \in W^{1,2}(\mathbb{S}^n)$ is a weak solution to

$$-\Delta_{\mathbb{S}} U \;=\; F$$

provided

$$\int_{\mathbb{S}^n} \nabla_{\mathbb{S}} U \cdot \nabla_{\mathbb{S}} G \; \mathrm{d}\sigma_n \;=\; \int_{\mathbb{S}^n} FG \; \mathrm{d}\sigma_n$$

for every $G \in W^{1,2}(\mathbb{S}^n)$.

In this subsection, we use spherical rearrangements and star functions introduced in Subsection 1.1.4. The reader may find it useful to review that material before continuing. We now have the following corollary to Theorem 3.1.

Corollary 3.5 (Spherical Comparison Theorem)**.** *Let $F \in L^2(\mathbb{S}^n)$ with $\int_{\mathbb{S}^n} F \; \mathrm{d}\sigma_n = 0$. Assume U and V are weak solutions to*

$$-\Delta_{\mathbb{S}} U \;=\; F \quad in \quad \mathbb{S}^n, \qquad -\Delta_{\mathbb{S}} V \;=\; F^{\#} \quad in \quad \mathbb{S}^n,$$

where $F^{\#}$ is the spherical rearrangement of F. Additionally assume U and V are additively normalized so that $\int_{\mathbb{S}^n} U \; \mathrm{d}\sigma_n = \int_{\mathbb{S}^n} V \; \mathrm{d}\sigma_n = 0$. Then

$$U^{\star} \;\le\; V^{\star}$$

on $(0, \pi)$. Thus for every convex function $\phi : \mathbb{R} \to \mathbb{R}$,

$$\int_{\mathbb{S}^n} \phi(U) \; \mathrm{d}\sigma_n \;\le\; \int_{\mathbb{S}^n} \phi(V) \; \mathrm{d}\sigma_n.$$

Consequently,

$$
\begin{aligned}
\|U\|_{L^p(\mathbb{S}^n)} &\leq \|V\|_{L^p(\mathbb{S}^n)}, \quad 1 \leq p \leq \infty, \\
\operatorname*{ess\,sup}_{\mathbb{S}^n} U &\leq \operatorname*{ess\,sup}_{\mathbb{S}^n} V, \\
\operatorname*{ess\,inf}_{\mathbb{S}^n} U &\geq \operatorname*{ess\,inf}_{\mathbb{S}^n} V, \\
\operatorname*{osc}_{\mathbb{S}^n} U &\leq \operatorname*{osc}_{\mathbb{S}^n} V.
\end{aligned}
$$

PROOF. The idea is to extend from the sphere to a spherical shell by homogeneity. Fix any $0 < a < 1 < b < \infty$ and let $A = A(a, b)$ be the spherical shell in $\mathbb{R}^{n+1}$ with inner radius a and outer radius b. Define functions $f, u : A \to \mathbb{R}$ by the homogeneity formulas

$$
\begin{aligned}
f(r\xi) &= \frac{1}{r^2} F(\xi), \\
u(r\xi) &= U(\xi),
\end{aligned}
$$

for $r \in (a, b)$ and $\xi \in \mathbb{S}^n$.

We first observe that u solves

$$
\begin{aligned}
-\Delta u &= f \quad \text{in} \quad A \subset \mathbb{R}^{n+1}, \\
\tfrac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial A,
\end{aligned}
$$

because $\Delta = \partial_{rr} + nr^{-1}\partial_r + r^{-2}\Delta_{\mathbb{S}}$. The normalization $\int_A u \, \mathrm{d}x = 0$ follows immediately from the definition of u, since $\int_{\mathbb{S}^n} U \, \mathrm{d}\sigma_n = 0$.

Define v on A by

$$
v(r\xi) = V(\xi)
$$

and observe that the $(n+1,n)$ cap symmetrization of f is

$$f^{\#}(r\xi) \;=\; \frac{1}{r^2}F^{\#}(\xi).$$

Hence v solves

$$\begin{aligned} -\Delta v &= f^{\#} \quad \text{in} \quad A, \\ \tfrac{\partial v}{\partial n} &= 0 \quad \text{on} \quad \partial A, \end{aligned}$$

and $\int_A v \; \mathrm{d}x = 0$. Theorem 3.1 implies that for almost every $r \in (a,b)$ the inequality

$$u^{\bigstar}(r,\theta) \;\leq\; v^{\bigstar}(r,\theta)$$

holds for every $\theta \in (0,\pi)$. Pick any r so that the above inequality holds. The definitions of u and v imply

$$\begin{aligned} U^{\bigstar}(\theta) &= u^{\bigstar}(r,\theta) \\ &\leq v^{\bigstar}(r,\theta) \\ &= V^{\bigstar}(\theta) \end{aligned}$$

for every $\theta \in (0,\pi)$ which gives the main conclusion of the corollary. The conclusions about convex means and so on now follow from Proposition 1.9 and Corollary 1.11. □

The following corollary tells us that the solution V to the symmetrized problem is spherically rearranged.

Corollary 3.6. *If F and V are as in Corollary 3.5, then $V = V^{\#}$ a.e.*

PROOF. Let f and v be obtained by homogeneity as in the proof of Corollary 3.5. By Corollary 3.4, $v = v^{\#}$ a.e. which implies $V = V^{\#}$ a.e. □

3.2.2. Hemispheres. We write $\mathbb{S}^n_+ = \{(\xi_1, \xi_2, \ldots, \xi_{n+1}) \in \mathbb{S}^n : \xi_{n+1} > 0\}$ for the upper hemisphere of $\mathbb{S}^n$. For a function $F : \mathbb{S}^n_+ \to \mathbb{R}$, we extend F to $\mathbb{S}^n$ by reflection through the plane $(x_{n+1} = 0)$. That, is we define $\tilde{F} : \mathbb{S}^n \to \mathbb{R}$ a.e. by

$$\tilde{F}(\xi_1, \xi_2, \ldots, \xi_{n+1}) = \begin{cases} F(\xi_1, \xi_2, \ldots, \xi_{n+1}) & \text{if } \xi_{n+1} > 0 \\ F(\xi_1, \xi_2, \ldots, -\xi_{n+1}) & \text{if } \xi_{n+1} < 0. \end{cases} \tag{3.7}$$

We define the hemispherical rearrangement $F^\# : \mathbb{S}^n_+ \to \mathbb{R}$ of the function F by the formula

$$F^\#(\xi_1, \xi_2, \ldots, \xi_{n+1}) \;=\; \tilde{F}^\#(\xi_1, \xi_2, \ldots, \xi_{n+1}),$$

where $\tilde{F}^\#$ denotes the spherical rearrangement of $\tilde{F}$. The star function of F, denoted by $F^\bigstar$, is defined on the interval $(0, \pi)$ by the formula

$$F^\bigstar(\theta) \;=\; \max_{\sigma_n(E) = \sigma_n(K(\theta)_+)} \int_E F(\xi)\, \mathrm{d}\sigma_n(\xi),$$

where the max is taken over all subsets $E \subset \mathbb{S}^n_+$ with the same surface measure as $K(\theta)_+$; we have written $K(\theta)_+ = \{(\xi_1, \xi_2, \ldots, \xi_{n+1}) \in K(\theta) : \xi_{n+1} > 0\}$. As in Proposition 1.7, the max defining $F^\bigstar$ is achieved for some subset E, explaining our use of max rather than sup. As in Proposition 1.8, it follows that

$$F^\bigstar(\theta) \;=\; \int_{K(\theta)_+} F^\#(\xi)\, \mathrm{d}\sigma_n(\xi).$$

Before we state the next corollary, we need to discuss what it means to solve Poisson's equation on a hemisphere with Neumann boundary conditions.

We write $W^{1,2}(\mathbb{S}^n_+)$ for the Sobolev space of functions in $L^2(\mathbb{S}^n_+)$ that, once expressed in spherical coordinates, have weak partial derivatives that also belong to $L^2(\mathbb{S}^n_+)$.

Given $F \in L^2(\mathbb{S}^n_+)$ we say a function $U \in W^{1,2}(\mathbb{S}^n_+)$ is a weak solution to

$$\begin{aligned} -\Delta_{\mathbb{S}} U &= F \quad \text{in} \quad \mathbb{S}^n_+, \\ \tfrac{\partial U}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{S}^n_+, \end{aligned}$$

provided

$$\int_{\mathbb{S}^n_+} \nabla_{\mathbb{S}} U \cdot \nabla_{\mathbb{S}} G \, \mathrm{d}\sigma_n = \int_{\mathbb{S}^n_+} FG \, \mathrm{d}\sigma_n \tag{3.8}$$

for each $G \in W^{1,2}(\mathbb{S}^n_+)$.

We can now state and prove the hemispherical comparison result.

Corollary 3.7 (Hemisphere Comparison Theorem). *Let $F \in L^2(\mathbb{S}^n_+)$ with $\int_{\mathbb{S}^n_+} F \, \mathrm{d}\sigma_n = 0$. Assume U and V are weak solutions to*

$$\begin{aligned} -\Delta_{\mathbb{S}} U &= F \quad \textit{in} \quad \mathbb{S}^n_+, & -\Delta_{\mathbb{S}} V &= F^{\#} \quad \textit{in} \quad \mathbb{S}^n_+, \\ \tfrac{\partial U}{\partial n} &= 0 \quad \textit{on} \quad \partial\mathbb{S}^n_+, & \tfrac{\partial V}{\partial n} &= 0 \quad \textit{on} \quad \partial\mathbb{S}^n_+, \end{aligned}$$

Additionally assume U and V are additively normalized so that $\int_{\mathbb{S}^n_+} U \, \mathrm{d}\sigma_n = \int_{\mathbb{S}^n_+} V \, \mathrm{d}\sigma_n = 0$. Then

$$U^\star \le V^\star$$

in $(0, \pi)$. Consequently, for every convex function $\phi : \mathbb{R} \to \mathbb{R}$ we have

$$\int_{\mathbb{S}^n_+} \phi(U) \, \mathrm{d}\sigma_n \le \int_{\mathbb{S}^n_+} \phi(V) \, \mathrm{d}\sigma_n.$$

Additionally,

$$\begin{aligned}
\|U\|_{L^p(\mathbb{S}^n_+)} &\leq \|V\|_{L^p(\mathbb{S}^n_+)}, \quad 1 \leq p \leq \infty, \\
\operatorname*{ess\,sup}_{\mathbb{S}^n_+} U &\leq \operatorname*{ess\,sup}_{\mathbb{S}^n_+} V, \\
\operatorname*{ess\,inf}_{\mathbb{S}^n_+} U &\geq \operatorname*{ess\,inf}_{\mathbb{S}^n_+} V, \\
\operatorname*{osc}_{\mathbb{S}^n_+} U &\leq \operatorname*{osc}_{\mathbb{S}^n_+} V.
\end{aligned}$$

PROOF. First assume F is Lipschitz continuous on $\overline{\mathbb{S}^n_+}$. Then $F^\#$ is also Lipschitz continuous on $\mathbb{S}^n_+$ by Corollary 3 of [**Ba5**]. The solutions U and V then belong to $C^2(\overline{\mathbb{S}^n_+})$ by Theorem 3.2 of [**LU**]. Extend U, V, and F to $\mathbb{S}^n$ by reflection through the plane $(x_{n+1} = 0)$ just as we did in equation (3.7) and denote these extensions by $\tilde{U}$, $\tilde{V}$, and $\tilde{F}$ respectively. Then $\tilde{U}$ and $\tilde{V}$ belong to $W^{1,2}(\mathbb{S}^n)$ since they have classically vanishing outer normals along the equator $(x_{n+1} = 0)$.

We first claim that $\tilde{U}$ and $\tilde{V}$ solve

$$-\Delta_{\mathbb{S}}\tilde{U} = \tilde{F} \text{ in } \mathbb{S}^n, \qquad -\Delta_{\mathbb{S}}\tilde{V} = \tilde{F}^\# \text{ in } \mathbb{S}^n, \tag{3.9}$$

together with the normalizations $\int_{\mathbb{S}^n} \tilde{U}\ \mathrm{d}\sigma_n = \int_{\mathbb{S}^n} \tilde{V}\ \mathrm{d}\sigma_n = 0$.

The normalization assumption follows immediately from the definition of $\tilde{U}$ and $\tilde{V}$ and since $\int_{\mathbb{S}^n_+} U\ \mathrm{d}\sigma_n = \int_{\mathbb{S}^n_+} V\ \mathrm{d}\sigma_n = 0$.

To show (3.9) we appeal directly to the definition in equation (3.8). Let $G \in W^{1,2}(\mathbb{S}^n)$ and define $\tilde{G}(\xi_1, \ldots, \xi_{n+1}) = G(\xi_1, \ldots, -\xi_{n+1})$ for $(\xi_1, \ldots, \xi_{n+1}) \in \mathbb{S}^n$. We compute

$$\begin{aligned} \int_{\mathbb{S}^n} \nabla_{\mathbb{S}} \tilde{U} \cdot \nabla_{\mathbb{S}} G \, \mathrm{d}\sigma_n &= \int_{\mathbb{S}^n_+} \nabla_{\mathbb{S}} U \cdot \nabla_{\mathbb{S}} G \, \mathrm{d}\sigma_n + \int_{\mathbb{S}^n_+} \nabla_{\mathbb{S}} U \cdot \nabla_{\mathbb{S}} \tilde{G} \, \mathrm{d}\sigma_n \\ &= \int_{\mathbb{S}^n_+} FG \, \mathrm{d}\sigma_n + \int_{\mathbb{S}^n_+} F\tilde{G} \, \mathrm{d}\sigma_n \\ &= \int_{\mathbb{S}^n} \tilde{F} G \, \mathrm{d}\sigma_n. \end{aligned}$$

Similarly, $-\Delta_{\mathbb{S}} \tilde{V} = \tilde{F}^{\#}$ in $\mathbb{S}^n$.

By Corollary 3.5, $\tilde{U}^{\star} \leq \tilde{V}^{\star}$ on $(0, \pi)$ which immediately implies $U^{\star} \leq V^{\star}$ on $(0, \pi)$. This gives the first conclusion of the theorem in the case where F is Lipschitz continuous on $\overline{\mathbb{S}^n_+}$.

Now let $F \in L^2(\mathbb{S}^n_+)$ be a general function of mean zero and choose a sequence of test functions F_k in $C_c^{\infty}(\mathbb{S}^n_+)$ with $F_k \to F$ in $L^2(\mathbb{S}^n_+)$. Let U_k and V_k solve

$$\begin{aligned} -\Delta_{\mathbb{S}} U_k &= F_k \quad \text{in} \quad \mathbb{S}^n_+, & \qquad -\Delta_{\mathbb{S}} V_k &= F_k^{\#} \quad \text{in} \quad \mathbb{S}^n_+, \\ \tfrac{\partial U_k}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{S}^n_+, & \tfrac{\partial V_k}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{S}^n_+, \end{aligned}$$

and assume the U_k and V_k are normalized to have mean zero. The work above gives

$$U_k^{\star} \leq V_k^{\star}$$

on $(0, \pi)$ for every k.

As in Corollary 1.29, $U_k \to U$ and $V_k \to V$ in $L^2(\mathbb{S}^n_+)$. By Theorem 1.18, it follows that $U_k^{\star} \to U^{\star}$ and $V_k^{\star} \to V^{\star}$ on $(0, \pi)$. Hence, letting $k \to \infty$ in $U_k^{\star} \leq V_k^{\star}$, we obtain $U^{\star} \leq V^{\star}$. The remaining inequalities about L^p norms and so on follow from Proposition 1.9 and Corollary 1.11. □

The following corollary tells us that the solution V to the rearranged problem is hemispherically rearranged.

Corollary 3.8. *Let F and V be as in Corollary 3.7. Then $V = V^{\#}$ a.e.*

Proof. Let $\tilde{F}$ and $\tilde{V}$ be as in the proof of Corollary 3.7. By Corollary 3.6, $\tilde{V} = \tilde{V}^{\#}$ a.e. which implies $V = V^{\#}$ a.e. □

CHAPTER 4

Weighted Comparison Results in Balls

In Chapters 2 and 3 we obtained comparison results on a few canonical spaces: annuli, spherical shells, spheres, and hemispheres. But another natural space, the ball, is missing from this list. In this chapter and the next, we concern ourselves with comparison theorems in balls. The goal of this chapter is to stereographically project the hemispherical comparison result, Corollary 3.7, into the unit ball. We choose stereographic projection because it is a conformal mapping. Hence, if a function on the upper hemisphere is related to a function in the ball by composition with stereographic projection, then one function has vanishing outer normal on the boundary if and only if the other does.

We begin the chapter by projecting surface measure and the spherical Laplacian from the hemisphere into the unit ball of $\mathbb{R}^n$, to directly convert Corollary 3.7 into a comparison theorem in the unit ball with homogeneous Neumann boundary conditions. In Section 2, we discuss estimates in terms of the data. Sometimes one wants estimates comparing the solution of a PDE to its input data, rather than to the solution of a rearranged problem. We obtain such estimates for a "weighted" Poisson equation from which we deduce estimates for the standard Poisson equation.

4.1. Stereographic projection

In this section, our task is to stereographically project Corollary 3.7 into the unit ball $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$. We let

$$\Phi : \mathbb{S}^n \to \mathbb{R}^n \cup \{\infty\}$$

denote stereographic projection where the north pole $e_{n+1} = (0, 0, \ldots, 1)$ corresponds to the origin and the south pole $-e_{n+1} = (0, 0, \ldots, -1)$ corresponds to ∞. To be precise, given $(\xi_1, \ldots, \xi_{n+1}) \in \mathbb{S}^n$, define

$$\Phi(\xi_1, \ldots, \xi_{n+1}) \quad = \quad (x_1, \ldots, x_n),$$

where

$$x_i \quad = \quad \frac{\xi_i}{1 + \xi_{n+1}} \quad \text{for } 1 \leq i \leq n.$$

The figure below shows a cross-section of stereographic projection.

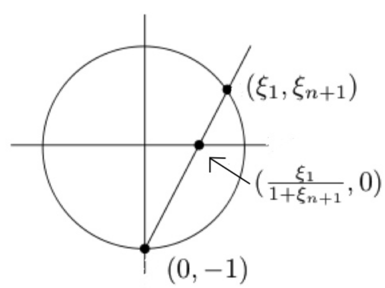

FIGURE 4.1. A cross-section of stereographic projection.

Stereographic projection of surface measure into $\mathbb{R}^n$. We first stereographically project surface measure σ_n into $\mathbb{R}^n$, to obtain a measure we denote by μ_n. Given a subset $E \subseteq \mathbb{R}^n$ we stereographically lift E onto the sphere to obtain the set $\Phi^{-1}(E)$. The surface measure of this lifted set provides the μ_n measure of E.

Definition 4.1 (Stereographic Measure on $\mathbb{R}^n$)**.** Define a measure μ_n on $\mathbb{R}^n$ by the formula

$$\mu_n(E) \quad = \quad \sigma_n(\Phi^{-1}(E)),$$

where Φ^{-1} is the inverse of stereographic projection, σ_n is surface measure on $\mathbb{S}^n$, and $E \subseteq \mathbb{R}^n$. We call μ_n *stereographic measure* on $\mathbb{R}^n$.

Explicitly,

$$\mu_n(E) \quad = \quad \int_E \rho(x)^{\frac{n}{2}}\ \mathrm{d}x,$$

where the density (Radon-Nikodym derivative) is

$$\rho(x) \quad = \quad \Big(\frac{2}{1+|x|^2}\Big)^2; \tag{4.1}$$

see [**G**].

Stereographic rearrangements in the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$. We now define stereographic rearrangements for functions defined in the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$. The idea is to lift a function onto the upper hemisphere $\mathbb{S}^n_+$, perform a hemispherical rearrangement, and then stereographically project back down to the unit ball. The end result produces a rearrangement with respect to stereographic measure μ_n in the unit ball.

Definition 4.2 (Stereographic Rearrangements in $\mathbb{B}^n$)**.** Given $f : \mathbb{B}^n \to \mathbb{R}$, stereographically lift f onto $\mathbb{S}^n_+$ to produce the function $F : \mathbb{S}^n_+ \to \mathbb{R}$ defined by

$$F \quad = \quad f \circ \Phi.$$

Taking the hemispherical rearrangement $F^\#$ and then projecting back to $\mathbb{B}^n$ gives

$$f^\# \quad = \quad F^\# \circ \Phi^{-1} \quad = \quad (f \circ \Phi)^\# \circ \Phi^{-1}.$$

We call $f^\#$ the *stereographic rearrangement* of f.

The figure below illustrates $f^\#$ in dimension $n = 2$. The level sets of $f^\#$ are circular arcs meeting the boundary of the unit disk $\mathbb{D}$ orthogonally, as the first picture shows below. Additionally, $f^\#$ decreases as these arcs sweep out the disk from $(1, 0)$ to $(-1, 0)$ as shown by the second picture; $f^\#$ has bigger values in the lighter areas and smaller values in the darker areas.

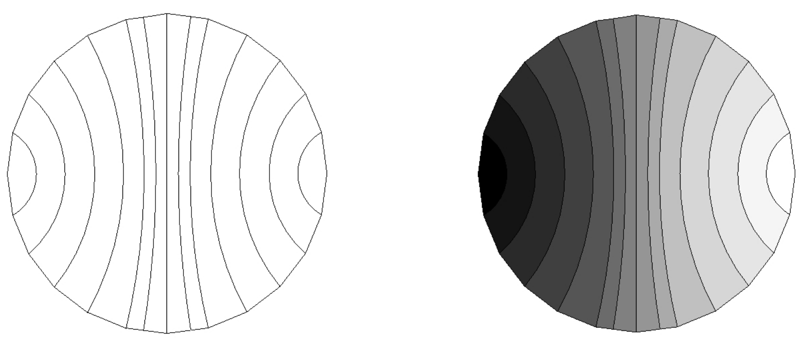

FIGURE 4.2. Level sets of a stereographically rearranged function $f^\#$; $f^\#$ is bigger in lighter areas and smaller in darker areas.

Associated with the stereographic rearrangement, we have the following star function. Given $u \in L^1(\mathbb{B}^n, \mathrm{d}\mu_n)$, define $u^\star$ on $(0, \pi)$ by the formula

$$u^\star(\theta) \quad = \max_{\mu_n(E) = \mu_n(A(\theta))} \int_E u \; \mathrm{d}\mu_n,$$

where $A(\theta) = \Phi(K(\theta)_+)$ is the image of the upper half of $K(\theta)$ under stereographic projection. As in Proposition 1.7, the max defining $u^\bigstar$ is achieved for some subset E, which explains our use of max instead of sup. As in Proposition 1.8, we have

$$u^\bigstar(\theta) \;\; = \;\; \int_{A(\theta)} u^{\#} \, \mathrm{d}\mu_n,$$

where $u^{\#}$ is the stereographic rearrangement of u. Obviously, this "stereographic star function" equals the hemispherical star function of $u \circ \Phi$.

Stereographic projection of $\Delta_{\mathbb{S}}$ from $\mathbb{S}^n$ into $\mathbb{R}^n$.

Definition 4.3 (Stereographic Operator)**.** Define an operator L acting on functions $u \in C^2(\mathbb{R}^n)$ by the formula

$$Lu(x) \;\; = \;\; \Delta_{\mathbb{S}}(u \circ \Phi)(\Phi^{-1}(x)).$$

We call L the *stereographic operator*.

If we write

$$\psi(x) \;\; = \;\; \log\big(\frac{2}{1+|x|^2}\big) \;\; = \;\; \frac{1}{2}\log \rho(x), \tag{4.2}$$

then by Theorem B.1 in Appendix B,

$$Lu \;\; = \;\; \frac{1}{\rho}\big(\Delta u + (n-2)\nabla\psi \cdot \nabla u\big). \tag{4.3}$$

Alternatively, one can obtain the above equality by using equation (0.5) of K. Richardson's paper [**R**] and the work in Section 2 of R. Graham's paper [**G**]. (Our formula above differs from Richardson's by a factor of $\frac{1}{2}$, owing to the probabilist convention of having a factor of $\frac{1}{2}$ in front of the Laplacian.)

In dimension $n = 2$, (4.3) simplifies to

$$L \;=\; \frac{1}{\rho}\Delta.$$

Before we state the main result of this section we remark that since stereographic projection is a conformal mapping, a function $u : \mathbb{B}^n \to \mathbb{R}$ satisfies $\frac{\partial u}{\partial n} = 0$ on $\partial\mathbb{B}^n$ if and only if $U : \mathbb{S}^n_+ \to \mathbb{R}$ defined by $U = u \circ \Phi$ satisfies $\frac{\partial U}{\partial n} = 0$ on $\partial\mathbb{S}^n_+$.

Corollary 4.4 (Weighted Comparison Result in $\mathbb{B}^n$)**.** *Let $f \in L^2(\mathbb{B}^n, \mathrm{d}\mu_n)$ with $\int_{\mathbb{B}^n} f \;\mathrm{d}\mu_n = 0$. Suppose u and v are weak solutions to*

$$\begin{aligned} -\tfrac{1}{\rho}(\Delta u + (n-2)\nabla\psi \cdot \nabla u) &= f \quad \textit{in} \quad \mathbb{B}^n, \\ \tfrac{\partial u}{\partial n} &= 0 \quad \textit{on} \quad \partial\mathbb{B}^n, \end{aligned}$$

$$\begin{aligned} -\tfrac{1}{\rho}(\Delta v + (n-2)\nabla\psi \cdot \nabla v) &= f^{\#} \quad \textit{in} \quad \mathbb{B}^n, \\ \tfrac{\partial v}{\partial n} &= 0 \quad \textit{on} \quad \partial\mathbb{B}^n, \end{aligned}$$

where $f^{\#}$ is the stereographic rearrangement of f in the unit ball and ρ and ψ are defined in equations (4.1) and (4.2) respectively.

If u and v are additively normalized so that $\int_{\mathbb{B}^n} u \;\mathrm{d}\mu_n = \int_{\mathbb{B}^n} v \;\mathrm{d}\mu_n = 0$, then

$$u^\star \;\leq\; v^\star$$

in $(0, \pi)$. Consequently,

$$\int_{\mathbb{B}^n} \phi(u) \;\mathrm{d}\mu_n \;\leq\; \int_{\mathbb{B}^n} \phi(v) \;\mathrm{d}\mu_n$$

for each convex function $\phi : \mathbb{R} \to \mathbb{R}$. In particular

$$\|u\|_{L^p(\mathbb{B}^n, \mathrm{d}\mu_n)} \;\leq\; \|v\|_{L^p(\mathbb{B}^n, \mathrm{d}\mu_n)}, \quad 1 \leq p \leq \infty. \tag{4.4}$$

Moreover,

$$
\begin{aligned}
\operatorname*{ess\,sup}_{\mathbb{B}^n} u &\leq \operatorname*{ess\,sup}_{\mathbb{B}^n} v, \\
\operatorname*{ess\,inf}_{\mathbb{B}^n} u &\geq \operatorname*{ess\,inf}_{\mathbb{B}^n} v, \\
\operatorname*{osc}_{\mathbb{B}^n} u &\leq \operatorname*{osc}_{\mathbb{B}^n} v.
\end{aligned}
$$

The proof of Corollary 4.4 follows by stereographically lifting to the hemisphere, applying Corollary 3.7, and then stereographically projecting back into the unit ball.

The next corollary says that the solution v with rearranged data $f^\#$ is stereographically rearranged. It is obtained by stereographically projecting Corollary 3.8 into $\mathbb{B}^n$.

Corollary 4.5. *Let f and v be as in Theorem 4.4. Then $v = v^\#$ a.e.*

In the remainder of this chapter, we focus on the two-dimensional case.

4.2. Further weighted results in dimension $n = 2$

In the remainder of this chapter, we discuss estimates in terms of the data. The results thus far compare the solutions of two PDEs. The results below, on the other hand, compare the solution u of a PDE to the input data f. The results below are consequences of the weighted comparison result, Corollary 4.4, together with the Neumann Green's function discussed in Appendix A. The reason for our restriction to dimension $n = 2$ is because we know the Neumann Green's function (for the Laplacian, and hence for L) explicitly for the unit disk $\mathbb{D}$ in two dimensions. If we knew the Neumann Green's function for L in higher dimensions, then Corollary 4.6 below would extend to all dimensions.

For convenience, in the remainder of the thesis we write μ for two-dimensional stereographic measure rather than μ_2.

Corollary 4.6 (Estimates in Terms of the Data)**.** *Let $f \in L^\infty(\mathbb{D}, \mathrm{d}\mu)$ with $\int_{\mathbb{D}} f \ \mathrm{d}\mu = 0$. Suppose u is a weak solution to*

$$\begin{aligned} -\tfrac{1}{\rho}\Delta u &= f \quad \textit{in} \quad \mathbb{D}, \\ \tfrac{\partial u}{\partial n} &= 0 \quad \textit{on} \quad \partial\mathbb{D}. \end{aligned}$$

Then

$$\operatorname*{osc}_{\mathbb{D}} u \ \leq \ (2\log 2) \operatorname*{osc}_{\mathbb{D}} f$$

with equality when f is constant on complementary half disks. Additionally, if u is additively normalized by $\int_{\mathbb{D}} u \ \mathrm{d}\mu = 0$, then

$$\|u\|_{L^q(\mathbb{D},\mathrm{d}\mu)} \ \leq \ K_2(q)\|f\|_{L^1(\mathbb{D},\mathrm{d}\mu)}, \quad 1 \leq q < \infty$$

for some universal constant $K_2(q)$ and

$$\|u\|_{L^\infty(\mathbb{D},\mathrm{d}\mu)} \ \leq \ K_1(p)\|f\|_{L^p(\mathbb{D},\mathrm{d}\mu)}, \quad 1 < p \leq \infty \tag{4.5}$$

for some universal constant $K_1(p)$.

Proof. Assume u is normalized so that $\int_{\mathbb{D}} u \ \mathrm{d}\mu = 0$ (which does not affect the oscillation). Let v solve

$$\begin{aligned} -\tfrac{1}{\rho}\Delta v &= f^\# \quad \text{in} \quad \mathbb{D}, \\ \tfrac{\partial v}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}, \end{aligned}$$

where $f^\#$ is the stereographic rearrangement of f in the unit disk and v is normalized so that $\int_{\mathbb{D}} v \ \mathrm{d}\mu = 0$. By Corollary 4.4,

$$\operatorname*{osc}_{\mathbb{D}} u \ \leq \ \operatorname*{osc}_{\mathbb{D}} v. \tag{4.6}$$

In addition, by Corollary 4.5, v is stereographically rearranged. Hence v is biggest at 1 and smallest at -1. Using the Green's representation for v as in Corollary A.2 of Appendix A, we compute

$$\begin{aligned}\operatorname*{osc}_{\mathbb{D}} v &= v(1) - v(-1)\\ &= \frac{1}{\pi}\int_{\mathbb{D}} \log\frac{|1+w|}{|1-w|} f^{\#}(w)\rho(w)\ \mathrm{d}m(w)\\ &= \frac{1}{\pi}\int_{\mathbb{D}^+} \log\frac{|1+w|}{|1-w|} f^{\#}(w)\rho(w)\ \mathrm{d}m(w)\\ &\quad -\frac{1}{\pi}\int_{\mathbb{D}^-} \log\frac{|1-w|}{|1+w|} f^{\#}(w)\rho(w)\ \mathrm{d}m(w),\end{aligned}$$

where we have written $\mathbb{D}^+ = \{z \in \mathbb{D} : \operatorname{Re} z > 0\}$ for the right half of the unit disk and $\mathbb{D}^- = \{z \in \mathbb{D} : \operatorname{Re} z < 0\}$ for the left half. Noting $\log\frac{|1+w|}{|1-w|}$ is positive on $\mathbb{D}^+$ and negative on $\mathbb{D}^-$, we have

$$\begin{aligned}\operatorname*{osc}_{\mathbb{D}} v &\leq \left(\operatorname*{ess\,sup}_{\mathbb{D}^+} f^{\#}\right)\frac{1}{\pi}\int_{\mathbb{D}^+} \log\frac{|1+w|}{|1-w|}\rho(w)\ \mathrm{d}m(w)\\ &\quad -\left(\operatorname*{ess\,inf}_{\mathbb{D}^-} f^{\#}\right)\frac{1}{\pi}\int_{\mathbb{D}^-} \log\frac{|1-w|}{|1+w|}\rho(w)\ \mathrm{d}m(w)\\ &= \left(\operatorname*{osc}_{\mathbb{D}} f^{\#}\right)\frac{1}{\pi}\int_{\mathbb{D}^+} \log\frac{|1+w|}{|1-w|}\rho(w)\ \mathrm{d}m(w)\\ &= \left(2\log 2\right)\left(\operatorname*{osc}_{\mathbb{D}} f\right),\end{aligned}$$

where we have lifted onto the sphere to compute

$$
\begin{aligned}
\frac{1}{\pi}\int_{\mathbb{D}^+} \log\frac{|1+w|}{|1-w|}\rho(w)\ \mathrm{d}m(w) &= 2\int_0^{\frac{\pi}{2}} \sin\theta \log\frac{|1+e^{i\theta}|}{|1-e^{i\theta}|}\ \mathrm{d}\theta \\
&= 2\int_0^{\frac{\pi}{2}} \sin\theta\log(\cos\frac{\theta}{2})\ \mathrm{d}\theta \\
&\quad -2\int_0^{\frac{\pi}{2}} \sin\theta\log(\sin\frac{\theta}{2})\ \mathrm{d}\theta \\
&= 8\int_{\frac{\sqrt{2}}{2}}^{1} x\log x\ \mathrm{d}x \\
&\quad -8\int_0^{\frac{\sqrt{2}}{2}} x\log x\ \mathrm{d}x \\
&= 2\log 2.
\end{aligned}
$$

We next prove inequality (4.5). Fix $1 < p \leq \infty$. Inequality (4.4) with "$p = \infty$" gives

$$
\|u\|_{L^\infty(\mathbb{D},\mathrm{d}\mu)} \leq \|v\|_{L^\infty(\mathbb{D},\mathrm{d}\mu)}.
$$

Because $v(1) = \max_{\mathbb{D}} v$, $v(-1) = \min_{\mathbb{D}} v$, and $\int_{\mathbb{D}} v\ \mathrm{d}\mu = 0$, it follows that

$$
\|v\|_{L^\infty(\mathbb{D},\mathrm{d}\mu)} = \max\{v(1), -v(-1)\}.
$$

We use the Green's representation for the solution v as in Corollary A.2 of Appendix A. Hence for any $c \in \mathbb{R}$,

$$
\begin{aligned}
v(1) &= \int_{\mathbb{D}} \big(\frac{1}{\pi}\log\frac{1}{|1-w|} + \frac{|w|^2}{4\pi} + c\big) f^{\#}(w)\rho(w)\ \mathrm{d}m(w) \\
&\leq \|f^{\#}\|_{L^p(\mathbb{D},\mathrm{d}\mu)} \min_{c\in\mathbb{R}} \Big\|\frac{1}{\pi}\log\frac{1}{|1-w|} + \frac{|w|^2}{4\pi} + c\Big\|_{L^q(\mathbb{D},\mathrm{d}\mu)} \\
&= M_1 \|f\|_{L^p(\mathbb{D},\mathrm{d}\mu)},
\end{aligned}
$$

where

$$M_1 = \min_{c\in\mathbb{R}} \Big\| \frac{1}{\pi} \log \frac{1}{|1-w|} + \frac{|w|^2}{4\pi} + c \Big\|_{L^q(\mathbb{D},\mathrm{d}\mu)},$$

and $\frac{1}{p} + \frac{1}{q} = 1$. Similarly,

$$\begin{aligned} -v(-1) &\leq \|f\|_{L^p(\mathbb{D})} \min_{c} \Big\| \frac{1}{\pi} \log \frac{1}{|1+w|} + \frac{|w|^2}{4\pi} + c \Big\|_{L^q(\mathbb{D},\mathrm{d}\mu)} \\ &= M_2 \|f\|_{L^p(\mathbb{D})}. \end{aligned}$$

Taking K_1 to be the maximum of M_1 and M_2 yields inequality (4.5). Note K_1 is finite because $1 \leq q < \infty$.

We obtain the theorem's second conclusion by a duality argument as follows. Fix $1 < p \leq \infty$. Let $g \in C_c^\infty(\mathbb{D})$ and let $g_A = \frac{1}{\mu(\mathbb{D})} \int_{\mathbb{D}} g \, \mathrm{d}\mu$ be the average of g over $\mathbb{D}$ with respect to stereographic measure μ.

Let ϕ solve

$$\begin{aligned} -\tfrac{1}{\rho}\Delta\phi &= g - g_A \quad \text{in} \quad \mathbb{D}, \\ \tfrac{\partial\phi}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}, \end{aligned}$$

and assume ϕ is normalized so that $\int_{\mathbb{D}} \phi \, \mathrm{d}\mu = 0$. Writing $\langle \cdot, \cdot \rangle_{L^2(\mathbb{D},\mathrm{d}\mu)}$ for the standard inner product on $L^2(\mathbb{D}, \mathrm{d}\mu)$ we compute

$$\begin{aligned} \langle u, g \rangle_{L^2(\mathbb{D},\mathrm{d}\mu)} &= \langle u, g - g_A \rangle_{L^2(\mathbb{D},\mathrm{d}\mu)} \\ &= \langle u, -\frac{1}{\rho}\Delta\phi \rangle_{L^2(\mathbb{D},\mathrm{d}\mu)} \\ &= \langle f, \phi \rangle_{L^2(\mathbb{D},\mathrm{d}\mu)}, \end{aligned} \tag{4.7}$$

where the first equality holds because $\int_{\mathbb{D}} u \; \mathrm{d}\mu = 0$ and the third equality holds by Green's theorem. Continuing on,

$$\begin{aligned} \langle f, \phi \rangle_{L^2(\mathbb{D},\mathrm{d}\mu)} &\leq \|f\|_{L^1(\mathbb{D},\mathrm{d}\mu)} \|\phi\|_{L^\infty(\mathbb{D},\mathrm{d}\mu)} \\ &\leq K_1(p)\|g - g_A\|_{L^p(\mathbb{D},\mathrm{d}\mu)} \|f\|_{L^1(\mathbb{D},\mathrm{d}\mu)} \\ &\leq 2K_1(p)\|g\|_{L^p(\mathbb{D},\mathrm{d}\mu)} \|f\|_{L^1(\mathbb{D},\mathrm{d}\mu)}, \end{aligned} \tag{4.8}$$

where the second inequality follows by applying inequality (4.5) to the functions ϕ and $g - g_A$. If we take $K_2(q) = 2K_1(p)$, where $\frac{1}{p} + \frac{1}{q} = 1$, and combine equality (4.7) with inequality (4.8) we have

$$\langle u, g \rangle_{L^2(\mathbb{D},\mathrm{d}\mu)} \leq K_2(q)\|g\|_{L^p(\mathbb{D},\mathrm{d}\mu)} \|f\|_{L^1(\mathbb{D},\mathrm{d}\mu)}.$$

If we take the sup over all $g \in C_c^\infty(\mathbb{D})$ with $\|g\|_{L^p(\mathbb{D},\mathrm{d}\mu)} \leq 1$ in the above inequality, we conclude

$$\|u\|_{L^q(\mathbb{D},\mathrm{d}\mu)} \leq K_2(q)\|f\|_{L^1(\mathbb{D},\mathrm{d}\mu)}.$$

□

The corollary above also gives a result for the unweighted Poisson's equation with homogeneous Neumann boundary conditions.

Corollary 4.7 (Estimates in Terms of the Data for Poisson's Equation)**.** *Let $f \in L^\infty(\mathbb{D})$ with $\int_{\mathbb{D}} f \; \mathrm{d}x = 0$. Let u be a weak solution to*

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \mathbb{D}, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}. \end{aligned}$$

Then $\operatorname*{osc}_{\mathbb{D}} u \leq (2 \log 2) \operatorname*{osc}_{\mathbb{D}} f$.

Proof. We have that

$$\begin{aligned} -\tfrac{1}{\rho}\Delta u &= \tfrac{1}{\rho}f \quad \text{in} \quad \mathbb{D}, \\ \tfrac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}, \end{aligned}$$

and because $\int_{\mathbb{D}} \frac{1}{\rho} f \, \mathrm{d}\mu = \int_{\mathbb{D}} f \, \mathrm{d}x = 0$, we can apply Corollary 4.6 to conclude

$$\begin{aligned} \operatorname*{osc}_{\mathbb{D}} u &\leq 2\log 2 \operatorname*{osc}_{\mathbb{D}} \left(\frac{1}{\rho} f\right) \\ &\leq 2\log 2 \operatorname*{osc}_{\mathbb{D}} f, \end{aligned}$$

since $\rho(x) \geq 1$ when $|x| \leq 1$. $\square$

CHAPTER 5

One-Dimensional Methods and Consequences

The results of this chapter are completely independent of Theorem 3.1, the main result thus far, or any of its consequences. In the first section of this chapter, we prove a comparison result on an interval. This result can be thought of as a one-dimensional analogue of Talenti's theorem, but with Neumann boundary conditions rather than Dirichlet boundary conditions. Talenti's Theorem in one dimension tells us that our input data should be symmetric decreasing for our solution to have maximal L^p norms and oscillation. Theorem 5.5 tells how to maximize the L^p norms and oscillation of a solution to Poisson's equation when imposing homogeneous Neumann boundary conditions: take the input data to be monotone decreasing. Section 2 contains a comparison result in the unit disk, similar to Theorem 2.2 using $(1, 2)$ cap symmetrization. The difference here is that the proof uses the Neumann Green's function discussed in Appendix A. If one knew the Neumann Green's function for a planar annulus, Theorem 2.2 would potentially have a much shorter proof. We end the chapter by presenting two examples in Section 3 that show no (reasonable) Neumann comparison theorem exists for the unit interval or disk using the Schwarz rearrangement. The Schwarz rearrangement is the correct rearrangement to use for comparison theorems imposing homogeneous Dirichlet boundary conditions, and this is precisely Talenti's Theorem. The examples of Section 3 show that the Schwarz rearrangement is not the correct rearrangement to use when imposing homogeneous Neumann boundary conditions.

5.1. One-dimensional comparison result for an interval

In this section we prove a one-dimensional comparison theorem in the flavor of Talenti's Theorem, except imposing Neumann boundary conditions. The end result, Theorem 5.5, is revealing. The Neumann comparison theorem comes by rearranging the data f in a "one-sided" decreasing manner. We begin with the following existence result. Although it holds for any interval $[a,b]$, we give a proof for the interval $[-\pi,\pi]$.

Proposition 5.1. (Poisson's Equation in One Dimension). *Let $f \in L^1[-\pi,\pi]$ with $\int_{-\pi}^{\pi} f \, \mathrm{d}x = 0$. A unique $u \in C^1[-\pi,\pi]$ exists satisfying:*

1. *u' is absolutely continuous on $[-\pi,\pi]$.*

2. *$-u'' = f$ a.e. in $[-\pi,\pi]$.*

3. *$u'(-\pi) = u'(\pi) = 0$.*

4. *$\int_{-\pi}^{\pi} u \, \mathrm{d}x = 0$.*

Proof. We first show uniqueness. Suppose u and v both satisfy all the properties listed above. Let $w = u - v$. Since w' is absolutely continuous, for each $x \in [-\pi,\pi]$ we have

$$w'(x) = w'(x) - w'(-\pi) = \int_{-\pi}^{x} (-f + f) = 0,$$

which shows that w is constant. Property 4 implies $w \equiv 0$. Thus $u \equiv v$, giving uniqueness. For existence, we simply take

$$u(x) = -\int_{-\pi}^{x} \int_{-\pi}^{t} f(s) \, \mathrm{d}s \, \mathrm{d}t + c,$$

where $c \in \mathbb{R}$ is chosen to make $\int_{-\pi}^{\pi} u \, \mathrm{d}x = 0$. □

If we are interested only in finding a function u with the property that $-u'' = f$, we can consider the function whose Fourier series is given by

$$u(x) \quad = \quad \sum_{n \neq 0} \frac{1}{n^2} \hat{f}(n) e^{inx}, \tag{5.1}$$

where $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, \mathrm{d}x$. If we formally differentiate the above equation termwise, we see

$$-u''(x) = \sum_{n \neq 0} \hat{f}(n) e^{inx} = f(x).$$

We are led to consider the function K whose Fourier coefficients are given by

$$\hat{K}(n) = \begin{cases} \frac{1}{n^2} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0. \end{cases}$$

One readily verifies that

$$K(x) \quad = \quad \frac{1}{2} x^2 - \pi |x| + \frac{1}{3} \pi^2 \quad \text{for} \quad -\pi \leq x \leq \pi.$$

We now extend K and f to all of $\mathbb{R}$ by making them 2π-periodic. Under this extension, K is Lipschitz continuous on $\mathbb{R}$ and K' exists everywhere except at even multiples of π. The function u in equation (5.1) has the property that $\hat{u}(n) = \hat{K}(n) \hat{f}(n) = \widehat{K * f}(n)$ which leads us the study the convolution

$$u(x) = (K * f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K(x - y) f(y) \, \mathrm{d}y. \tag{5.2}$$

At this point, we investigate how the u defined in (5.2) differs from the u in Proposition 5.1. We will see that u defined in equation (5.2) satisfies properties 1, 2, and 4 of Proposition 5.1. Below, u always denotes the convolution $K * f$ and we identify K and f with their 2π-periodic extensions.

Proposition 5.2. *The function $u = K * f$ satisfies the following three properties:*

1. *u is continuously differentiable on $\mathbb{R}$ with $u' = K' * f$.*

2. *u' is absolutely continuous on $[-\pi, \pi]$. Moreover for $x \in [-\pi, \pi]$*

$$u'(x) - u'(-\pi) \quad = \quad -\int_{-\pi}^{x} f(y)\ \mathrm{d}y$$

and $-u'' = f$ a.e. in $[-\pi, \pi]$.

3. *$u'(\pi) = u'(-\pi) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} x f(x)\ \mathrm{d}x$.*

4. *$\int_{-\pi}^{\pi} u\ \mathrm{d}x = 0$.*

Proof. Property 1 is a standard fact about convolutions. For property 2, we calculate

$$\begin{aligned}
(K' * f)(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K'(y) f(x-y)\ \mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} (y+\pi) f(x-y)\ \mathrm{d}y \\
&\quad + \frac{1}{2\pi}\int_{0}^{\pi} (y-\pi) f(x-y)\ \mathrm{d}y \\
&= \frac{1}{2\pi}\int_{x-\pi}^{x} (x-y-\pi) f(y)\ \mathrm{d}y \\
&\quad + \frac{1}{2\pi}\int_{x}^{x+\pi} (x-y+\pi) f(y)\ \mathrm{d}y,
\end{aligned}$$

so $(K' * f)(x)$ is absolutely continuous. Its derivative equals $-f(x)$ by direct calculation, for almost every x. Thus property 2 holds. For property 3, we use the above calculation to see

$$\begin{aligned}(K' * f)(\pi) &= -\frac{1}{2\pi}\int_0^{\pi} yf(y)\ \mathrm{d}y \\ &\quad +\frac{1}{2\pi}\int_{\pi}^{2\pi}(2\pi - y)f(y)\ \mathrm{d}y \\ &= -\frac{1}{2\pi}\int_0^{\pi} yf(y)\ \mathrm{d}y \\ &\quad -\frac{1}{2\pi}\int_{-\pi}^{0} yf(y+2\pi)\ \mathrm{d}y \\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} yf(y)\ \mathrm{d}y.\end{aligned}$$

A similar calculation holds for $(K' * f)(-\pi)$. Finally, to establish property 4 we simply observe

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)\ \mathrm{d}x = \hat{u}(0) = \hat{K}(0)\hat{f}(0) = 0.$$

□

Thus we have shown that $K * f$ satisfies properties 1,2, and 4 in Proposition 5.1. The only question remaining is about its derivative at $-\pi$ and π. Property 3 of Proposition 5.2 gives

$$(K' * f)(\pi) = (K' * f)(-\pi) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} xf(x)\ \mathrm{d}x$$

which, if zero, tells us that $K * f$ is the unique solution u as stated in Proposition 5.1. When f is an even function, for example, the above integral equals zero. We summarize our findings in the following result.

Proposition 5.3. *Let f and u be as in Proposition 5.1. If $\int_{-\pi}^{\pi} xf(x)\ \mathrm{d}x = 0$ then u is given by convolution in equation (5.2).*

Next, we prove a comparison result. For $f \in L^1[-\pi, \pi]$, let $f^{\#}$ denote the Schwarz rearrangement, also called the symmetric decreasing rearrangement, of f. Similarly, let $f^{\bigstar}$, the star function of f, be defined on the interval $[0, 2\pi]$ by

$$f^{\bigstar}(t) = \max_{|E|=t} \int_E f(x)\ \mathrm{d}x = \int_{-\frac{t}{2}}^{\frac{t}{2}} f^{\#}(x)\ \mathrm{d}x,$$

where the max is taken over all measurable subsets $E \subseteq [-\pi, \pi]$ of measure t. By Proposition 1.7, the max defined above is achieved for some subset E, which explains our use of max rather than sup.

Theorem 5.4 (Preliminary Comparison Theorem on an Interval with Neumann Boundary Conditions)**.** *Let $f \in L^1[-\pi, \pi]$ where $\int_{-\pi}^{\pi} f(x)\ \mathrm{d}x = \int_{-\pi}^{\pi} x f(x)\ \mathrm{d}x = 0$. Let u and v be the solutions as in Proposition 5.1 to*

$$-u'' = f \ \textit{in}\ (-\pi, \pi), \qquad -v'' = f^{\#}\ \textit{in}\ (-\pi, \pi),$$
$$u'(-\pi) = u'(\pi) = 0, \qquad v'(-\pi) = v'(\pi) = 0,$$

where $f^{\#}$ is the Schwarz rearrangement of f, and $\int_{-\pi}^{\pi} u(x)\ \mathrm{d}x = \int_{-\pi}^{\pi} v(x)\ \mathrm{d}x = 0$. Then

$$u^{\bigstar} \ \leq \ v^{\bigstar}$$

in $[0, 2\pi]$.

Proof. By Proposition 5.3, we have $u = K * f$ and $v = K * f^{\#}$ (because $f^{\#}$ is even and so $\int_{-\pi}^{\pi} x f^{\#}(x)\ \mathrm{d}x = 0$). Fix $t \in [0, 2\pi]$ and let E be any measurable subset $E \subseteq [-\pi, \pi]$ of measure t. Applying the Riesz-type inequality on $\mathbb{S}^1$ (Theorem 1 of **[Ba4]**) to the functions χ_E, K, and f, we get

$$\int_E u = \int_E K * f \leq \int_{E^{\#}} K * f^{\#} = \int_{E^{\#}} v.$$

Taking the max in the above inequality over all subsets E of $[-\pi,\pi]$ of measure t, we obtain the conclusion $u^\bigstar(t) \le v^\bigstar(t)$. $\square$

A reflective interlude. Let $f \in L^1[0,\pi]$ with $\int_0^\pi f(x)\ \mathrm{d}x = 0$. Extend f to $[-\pi,\pi]$ by even reflection across the origin and denote the extended function by $\tilde{f}$. Observe that $\int_{-\pi}^{\pi} x\tilde{f}(x)\ \mathrm{d}x = 0$. Clearly,

$$(\tilde{f})^\#(x) = f^*(x)\ , \quad 0 \le x \le \pi. \tag{5.3}$$

Let u correspond to f in the analogue of Proposition 5.1 over $[0,\pi]$ and similarly let v correspond to f^*. Let $\tilde{u}$ and $\tilde{v}$ correspond to $\tilde{f}$ and $(\tilde{f})^\#$ in the $[-\pi,\pi]$ version of Proposition 5.1. Theorem 5.3 gives $\tilde{u} = K * \tilde{f}$ and $\tilde{v} = K * (\tilde{f})^\#$.

We first show that $\tilde{u}$ is obtained from u by reflection. We show $\tilde{u}$ is even and that $\tilde{u}$ also satisfies the properties of Proposition 5.1 corresponding to f over the interval $[0,\pi]$. First, $\tilde{u}$ is even:

$$\begin{aligned}
\tilde{u}(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K(x-y)\tilde{f}(y)\ \mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} K(x+y)\tilde{f}(y)\ \mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} K(-x-y)\tilde{f}(y)\ \mathrm{d}y \\
&= \tilde{u}(-x),
\end{aligned}$$

where the second equality follows by a change of variables and the third since K is even. Since $\tilde{u} \in C^1[-\pi,\pi]$ and is even, we must have $\tilde{u}'(0) = 0$. Additionally, we have $\tilde{u}'(\pi) = 0$ by assumption. Again, being even implies $\int_0^\pi \tilde{u}(x)\ \mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} \tilde{u}(x)\ \mathrm{d}x = 0$. Hence by uniqueness, $\tilde{u}(x) = u(x)$ on $[0,\pi]$. Hence, $(\tilde{u})^\#(x) = u^*(x)$ on $[0,\pi]$. We similarly have $(\tilde{v})^{\#}(x) = v^*(x)$ on $[0,\pi]$. By Theorem 5.4 we have for each $0 \le t \le 2\pi$

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} (\tilde{u})^{\#}(x)\ \mathrm{d}x \le \int_{-\frac{t}{2}}^{\frac{t}{2}} (\tilde{v})^{\#}(x)\ \mathrm{d}x$$

which implies

$$\int_{0}^{\frac{t}{2}} (\tilde{u})^{\#}(x)\ \mathrm{d}x \le \int_{0}^{\frac{t}{2}} (\tilde{v})^{\#}(x)\ \mathrm{d}x$$

finally giving

$$\int_{0}^{\frac{t}{2}} u^*(x)\ \mathrm{d}x \le \int_{0}^{\frac{t}{2}} v^*(x)\ \mathrm{d}x. \tag{5.4}$$

If we define $u^\bigstar$ and $v^\bigstar$ on $[0,\pi]$ by the formulas

$$\begin{aligned} u^\bigstar(t) &= \max_{|E|=t} \int_E u(x)\ \mathrm{d}x = \int_0^t u^*(x)\ \mathrm{d}x, \\ v^\bigstar(t) &= \max_{|E|=t} \int_E v(x)\ \mathrm{d}x = \int_0^t v^*(x)\ \mathrm{d}x, \end{aligned}$$

then inequality (5.4) shows that $u^\bigstar \le v^\bigstar$ on $[0,\pi]$. These findings give the first conclusion in the following theorem.

Theorem 5.5 (Comparison Theorem on an Interval with Neumann Boundary Conditions)**.** *Let $f \in L^1[0,\pi]$ with $\int_0^\pi f(x)\ \mathrm{d}x = 0$ and assume u and v are the solutions as in Proposition 5.1 to*

$$-u'' = f \text{ in } (0,\pi), \qquad -v'' = f^* \text{ in } (0,\pi),$$

$$u'(0) = u'(\pi) = 0, \qquad v'(0) = v'(\pi) = 0,$$

where f^* is the decreasing rearrangement of f, and u and v satisfy the normalization $\int_0^\pi u(x)\ \mathrm{d}x = \int_0^\pi v(x)\ \mathrm{d}x = 0$. Then the inequality

$$u^\bigstar \le v^\bigstar$$

holds on $[0,\pi]$. *Moreover, for each convex function* $\phi : \mathbb{R} \to \mathbb{R}$ *we have*

$$\int_0^\pi \phi(u)\ \mathrm{d}x \quad \le \quad \int_0^\pi \phi(v)\ \mathrm{d}x.$$

In particular,

$$\begin{aligned} \|u\|_{L^p([0,\pi],\mathrm{d}x)} \ &\le\ \|v\|_{L^p([0,\pi],\mathrm{d}x)}, \qquad 1 \le p \le \infty, \\ \max_{[0,\pi]} u \ &\le\ \max_{[0,\pi]} v, \\ \min_{[0,\pi]} u \ &\ge\ \min_{[0,\pi]} v, \\ \operatorname*{osc}_{[0,\pi]} u \ &\le\ \operatorname*{osc}_{[0,\pi]} v. \end{aligned}$$

PROOF. The star function inequality follows from the work preceding the statement of the theorem. The inequalities about convex means and so on follow from Proposition 1.9 and Corollary 1.11. □

Remark 5.6. There is an analogue of the above theorem for any interval $[a,b]$, not just $[0,\pi]$.

5.2. Disk comparison result with $(1,2)$ cap symmetrization

In this section, we work in dimension $n = 2$ and prove a result analogous to Theorem 2.2 for the unit disk. The disk result has a simpler proof because of the Neumann Green's function from Appendix A. The Neumann Green's function allows us to forego the maximum principle used to establish Theorem 2.2. We continue to let $\mathbb{D} \subset \mathbb{C}$ denote the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and write $\mathbb{D}^\star = \{z \in \mathbb{D} : \mathrm{Im}\ z > 0\}$ for the upper half of the unit disk. For a function $u : \mathbb{D} \to \mathbb{R}$ we will write $u^\#$ for the

$(1,2)$ cap symmetrization of u and define the star function of u on $\mathbb{D}^\star$ by the formula

$$u^\star(re^{i\theta}) \;=\; \max_{|E|=2\theta} \int_E u(re^{i\phi})\ \mathrm{d}\phi,$$

where the max is taken over all subsets $E \subseteq [-\pi, \pi]$ with length $|E| = 2\theta$. We recall that the max defined above is achieved for some subset E, as in Proposition 1.7, which is why we write max instead of sup. Then

$$u^\star(re^{i\theta}) \;=\; \int_{-\theta}^{\theta} u^\#(re^{i\phi})\ \mathrm{d}\phi,$$

where $u^\#$ is the $(1,2)$ cap symmetrization of u.

Theorem 5.7. *Let $f \in L^2(\mathbb{D})$ with $\int_{\mathbb{D}} f\ \mathrm{d}m = 0$. Suppose u and v are weak solutions to*

$$\begin{aligned} -\Delta u &= f \quad in \quad \mathbb{D}, & \qquad -\Delta v &= f^\# \quad in \quad \mathbb{D},\\ \tfrac{\partial u}{\partial n} &= 0 \quad on \quad \partial\mathbb{D}, & \qquad \tfrac{\partial v}{\partial n} &= 0 \quad on \quad \partial\mathbb{D}, \end{aligned}$$

where $f^\#$ denotes the $(1,2)$ cap symmetrization of f.

If u and v are additively normalized so that $\int_{\mathbb{D}} u\ \mathrm{d}m = \int_{\mathbb{D}} v\ \mathrm{d}m = 0$, then for almost every $r \in (0,1)$

$$\int_{-\pi}^{\pi} u(re^{i\theta})\ \mathrm{d}\theta \;=\; \int_{-\pi}^{\pi} v(re^{i\theta})\ \mathrm{d}\theta$$

and the inequality

$$u^\star(re^{i\theta}) \;\le\; v^\star(re^{i\theta})$$

holds for every $\theta \in (0,\pi)$. In particular, $u^\star \le v^\star$ a.e. in $\mathbb{D}^\star$. Consequently, for each convex function $\phi : \mathbb{R} \to \mathbb{R}$ we have

$$\int_{-\pi}^{\pi} \phi(u(re^{i\theta}))\ \mathrm{d}\theta \;\le\; \int_{-\pi}^{\pi} \phi(v(re^{i\theta}))\ \mathrm{d}\theta.$$

In particular, the following inequalities hold for almost every slice function u^r and v^r:

$$\begin{aligned}
\|u^r\|_{L^p([-\pi,\pi],\mathrm{d}\theta)} &\le \|v^r\|_{L^p([-\pi,\pi],\mathrm{d}\theta)}, \quad 1 \le p \le \infty, \\
\operatorname*{ess\,sup}_{[-\pi,\pi]} u^r &\le \operatorname*{ess\,sup}_{[-\pi,\pi]} v^r, \\
\operatorname*{ess\,inf}_{[-\pi,\pi]} u^r &\ge \operatorname*{ess\,inf}_{[-\pi,\pi]} v^r, \\
\operatorname*{osc}_{[-\pi,\pi]} u^r &\le \operatorname*{osc}_{[-\pi,\pi]} v^r.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|u\|_{L^p(\mathbb{D},\mathrm{d}m)} &\le \|v\|_{L^p(\mathbb{D},\mathrm{d}m)}, \quad 1 \le p \le \infty, \\
\operatorname*{ess\,sup}_{\mathbb{D}} u &\le \operatorname*{ess\,sup}_{\mathbb{D}} v, \\
\operatorname*{ess\,inf}_{\mathbb{D}} u &\ge \operatorname*{ess\,inf}_{\mathbb{D}} v, \\
\operatorname*{osc}_{\mathbb{D}} u &\le \operatorname*{osc}_{\mathbb{D}} v.
\end{aligned}$$

PROOF. **Step 1: Star function inequality for Lipschitz input data f.**

First assume f is Lipschitz continuous on $\overline{\mathbb{D}}$. Since cap symmetrization decreases the modulus of continuity (this follows from Corollary 3 of [**Ba5**]), it follows that $f^\#$ is also Lipschitz continuous on $\overline{\mathbb{D}}$. We appeal to the Green's representation for the solutions u and v as described by Corollary A.2 in Appendix A. We have

$$\begin{aligned}
u(z) &= \int_{\mathbb{D}} G(z,w) f(w) \, \mathrm{d}m(w), \\
v(z) &= \int_{\mathbb{D}} G(z,w) f^\#(w) \, \mathrm{d}m(w),
\end{aligned}$$

where

$$G(z,w) = \frac{1}{2\pi} \log \frac{1}{|z-w|} + \frac{1}{2\pi} \log \frac{1}{|1-z\overline{w}|} + \frac{|z|^2+|w|^2}{4\pi} - \frac{3}{8\pi}.$$

Observe

$$\begin{aligned}
u(re^{i\phi}) &= \int_{-\pi}^{\pi}\int_0^1 G(re^{i\phi},\rho e^{i\psi})f(\rho e^{i\psi})\ \rho \mathrm{d}\rho\ \mathrm{d}\psi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 f(\rho e^{i\psi})\log\frac{1}{\left|r-\rho e^{i(\psi-\phi)}\right|}\ \rho \mathrm{d}\rho\ \mathrm{d}\psi \\
&\quad +\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1 f(\rho e^{i\psi})\log\frac{1}{\left|1-r\rho e^{i(\psi-\phi)}\right|}\ \rho \mathrm{d}\rho\ \mathrm{d}\psi \\
&\quad +\int_{-\pi}^{\pi}\int_0^1 (\frac{r^2+\rho^2}{4\pi}-\frac{3}{8\pi})f(\rho e^{i\psi})\ \rho \mathrm{d}\rho\ \mathrm{d}\psi
\end{aligned}$$

because $\int_{\mathbb{D}} f\ \mathrm{d}m = 0$. Fix a subset $E \subseteq [-\pi,\pi]$ of length 2θ and write $E^{\#} = [-\theta,\theta]$. The functions $\log\frac{1}{|r-\rho e^{i\phi}|}$ and $\log\frac{1}{|1-r\rho e^{i\phi}|}$ are symmetric decreasing in ϕ. Hence the Riesz-type inequality on $\mathbb{S}^1$ (Theorem 1 of [**Ba4**]) gives

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathbf{1}_E(e^{i\phi})f(\rho e^{i\psi})\log\frac{1}{\left|r-\rho e^{i(\psi-\phi)}\right|}\ \mathrm{d}\psi\ \mathrm{d}\phi \\
\leq\ &\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathbf{1}_{E^{\#}}(e^{i\phi})f^{\#}(\rho e^{i\psi})\log\frac{1}{\left|r-\rho e^{i(\psi-\phi)}\right|}\ \mathrm{d}\psi\ \mathrm{d}\phi
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathbf{1}_E(e^{i\phi})f(\rho e^{i\psi})\log\frac{1}{\left|1-r\rho e^{i(\psi-\phi)}\right|}\ \mathrm{d}\psi\ \mathrm{d}\phi \\
\leq\ &\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathbf{1}_{E^{\#}}(e^{i\phi})f^{\#}(\rho e^{i\psi})\log\frac{1}{\left|1-r\rho e^{i(\psi-\phi)}\right|}\ \mathrm{d}\psi\ \mathrm{d}\phi.
\end{aligned}$$

Since $\int_{-\pi}^{\pi}\mathbf{1}_E(e^{i\phi})\ \mathrm{d}\phi = \int_{-\pi}^{\pi}\mathbf{1}_{E^{\#}}(e^{i\phi})\ \mathrm{d}\phi$ and $\int_{-\pi}^{\pi} f(\rho e^{i\psi})\ \mathrm{d}\psi = \int_{-\pi}^{\pi} f^{\#}(\rho e^{i\psi})\ \mathrm{d}\psi$,

$$\begin{aligned}
&(\frac{r^2+\rho^2}{4\pi}-\frac{3}{8\pi})\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathbf{1}_E(e^{i\phi})f(\rho e^{i\psi})\ \mathrm{d}\psi\ \mathrm{d}\phi \\
=\ &(\frac{r^2+\rho^2}{4\pi}-\frac{3}{8\pi})\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\mathbf{1}_{E^{\#}}(e^{i\phi})f^{\#}(\rho e^{i\psi})\ \mathrm{d}\psi\ \mathrm{d}\phi.
\end{aligned}$$

Multiplying the two inequalities and the equality above through by ρ, integrating from $\rho = 0$ to $\rho = 1$, and adding all three together gives

$$\begin{aligned}\int_E u(re^{i\phi})\ \mathrm{d}\phi &= \int_E \int_{\mathbb{D}} G(re^{i\phi}, w) f(w)\ \mathrm{d}m(w)\ \mathrm{d}\phi \\ &\leq \int_{E^\#} \int_{\mathbb{D}} G(re^{i\phi}, w) f^\#(w)\ \mathrm{d}m(w)\ \mathrm{d}\phi \\ &\leq v^\bigstar(re^{i\theta}).\end{aligned}$$

If we take the max in the above inequality over all subsets $E \subseteq [-\pi, \pi]$ of length 2θ, we conclude

$$u^\bigstar(re^{i\theta}) \ \leq \ v^\bigstar(re^{i\theta}).$$

Step 2: Slice functions of u and v have same mean.

Let

$$w(r) \ = \ \int_{-\pi}^{\pi} u(re^{i\theta})\ \mathrm{d}\theta - \int_{-\pi}^{\pi} v(re^{i\theta})\ \mathrm{d}\theta$$

and compute

$$\begin{aligned} r\frac{dw}{dr} &= r\int_{-\pi}^{\pi} \big(u_r(re^{i\theta}) - v_r(re^{i\theta})\big)\ \mathrm{d}\theta \\ &= \int_{\partial(|z|<r)} \left(\frac{\partial u}{\partial n} - \frac{\partial v}{\partial n}\right)\ \mathrm{d}S \\ &= \int_{|z|<r} (\Delta u - \Delta v)\ \mathrm{d}m \\ &= \int_{|z|<r} (-f + f^\#)\ \mathrm{d}m \\ &= 0,\end{aligned}$$

so that w is constant. Now

$$\int_0^1 w(r)r \, \mathrm{d}r \quad = \quad \int_A (u-v) \, \mathrm{d}m \quad = \quad 0$$

by assumption, and hence $w \equiv 0$ throughout $[0,1]$.

Hence, we have established the star function conclusion and that the slice functions of u and v have the same mean when f is Lipschitz continuous on $\overline{\mathbb{D}}$.

Step 3: Slice function means and star function inequality for arbitrary f.

Now let $f \in L^2(\mathbb{D})$ with $\int_{\mathbb{D}} f \, \mathrm{d}m = 0$. Choose an approximating sequence of compactly supported smooth functions $f_k \in C_c^\infty(\mathbb{D})$ having mean zero where $f_k \to f$ in $L^2(\mathbb{D})$. Let u and v be as in the statement of Theorem 5.7. Let u_k and v_k solve

$$\begin{aligned} -\Delta u_k &= f_k \quad \text{in} \quad \mathbb{D}, & \qquad -\Delta v_k &= f_k^{\#} \quad \text{in} \quad \mathbb{D}, \\ \tfrac{\partial u_k}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}, & \qquad \tfrac{\partial v_k}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}, \end{aligned}$$

and assume that the solutions u_k and v_k satisfy the normalization $\int_{\mathbb{D}} u_k \, \mathrm{d}m = \int_{\mathbb{D}} v_k \, \mathrm{d}m = 0$. Then each f_k is Lipschitz continuous on $\overline{\mathbb{D}}$ and so by Step 1 and Step 2, we have for each r that

$$\int_{-\pi}^{\pi} u_k(re^{i\theta}) \, \mathrm{d}\theta \quad = \quad \int_{-\pi}^{\pi} v_k(re^{i\theta}) \, \mathrm{d}\theta \tag{5.5}$$

and

$$u_k^\star \leq v_k^\star \tag{5.6}$$

for every k on $\mathbb{D}^\star$. Corollary 1.29 gives that $u_k \to u$ in $L^2(A)$. By Theorem 1.22 $f_k^{\#} \to f^{\#}$ in $L^2(A)$ and consequently $v_k \to v$ in $L^2(A)$. Hence by using Theorem 1.23 we can pass to a subsequence of the original f_k and assume that for almost every

$r \in (0,1)$

$$\int_{-\pi}^{\pi} u_k(re^{i\theta})\ \mathrm{d}\theta \ \ \to \ \ \int_{-\pi}^{\pi} u(re^{i\theta})\ \mathrm{d}\theta$$

and

$$u_k^{\star}(re^{i\theta}) \ \ \to \ \ u^{\star}(re^{i\theta})$$

for every $\theta \in (0,\pi)$. By passing to another subsequence of the f_k, we may additionally assume that for almost every $r \in (0,1)$

$$\int_{-\pi}^{\pi} v_k(re^{i\theta})\ \mathrm{d}\theta \ \ \to \ \ \int_{-\pi}^{\pi} v(re^{i\theta})\ \mathrm{d}\theta$$

and

$$v_k^{\star}(re^{i\theta}) \ \ \to \ \ v^{\star}(re^{i\theta})$$

for every $\theta \in (0,\pi)$. Hence, letting $k \to \infty$ in (5.5) gives

$$\int_{-\pi}^{\pi} u(re^{i\theta})\ \mathrm{d}\theta \ \ = \ \ \int_{-\pi}^{\pi} v(re^{i\theta})\ \mathrm{d}\theta$$

for almost every $r \in (0,1)$. Letting $k \to \infty$ in (5.6) gives that for almost every $r \in (0,1)$

$$u^{\star}(re^{i\theta}) \ \ \leq \ \ v^{\star}(re^{i\theta})$$

holds for every $\theta \in (0,\pi)$.

Step 4: Convex mean, L^p norm, and oscillation inequalities.

The inequalities about L^p norms and so on over $[-\pi, \pi]$ follow from Proposition 1.9 and Corollary 1.11. The disk inequalities follow from the interval inequalities in obvious fashion. □

5.3. Failure of Neumann comparison for the Schwarz rearrangement

Recall that Talenti's Theorem [**T**] gives a comparison theorem with homogeneous Dirichlet boundary conditions, with the oscillation and L^p norms increasing under rearrangement of the data. In contrast, the examples below show that the Schwarz rearrangement does not give a reasonable comparison theorem with Neumann boundary conditions, because under rearrangement of data, the oscillation of the solution can decrease. In all of the comparison results proven thus far, the solution corresponding to the symmetrized data exhibits greater oscillation, and this inequality would go the opposite way.

Example 5.8 (One Dimension)**.** Consider the function $\mathbf{1}_{[-1,0]} - \mathbf{1}_{[0,1]}$ on the interval $[-1, 1]$ and its Schwarz rearrangement $(\mathbf{1}_{[-1,0]} - \mathbf{1}_{[0,1]})^{\#} = -\mathbf{1}_{[-1,-\frac{1}{2}]} + \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]} - \mathbf{1}_{[\frac{1}{2},1]}$ graphed in the figure below.

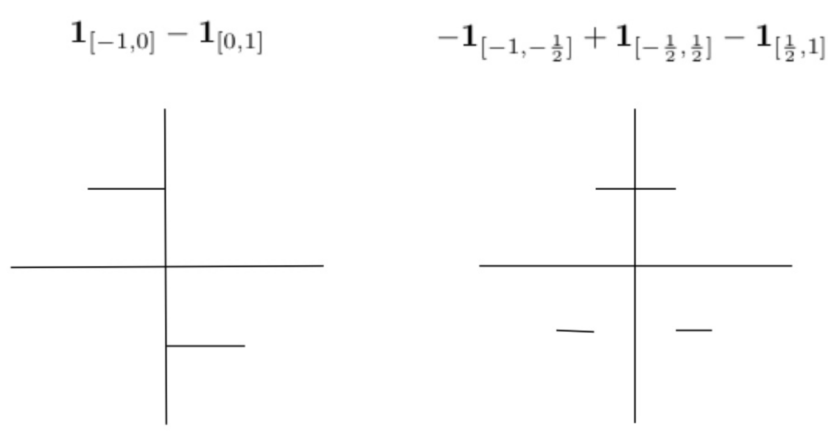

FIGURE 5.1. One-dimensional input data for Example 5.8.

Let u and v solve

$$-u'' = \mathbf{1}_{[-1,0]} - \mathbf{1}_{[0,1]} \text{ in } (-1,1), \qquad -v'' = -\mathbf{1}_{[-1,-\frac{1}{2}]} + \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]} - \mathbf{1}_{[\frac{1}{2},1]} \text{ in } (-1,1),$$
$$u'(-1) = u'(1) = 0, \qquad v'(-1) = v'(1) = 0.$$

We show that $\operatorname*{osc}_{[-1,1]} u > \operatorname*{osc}_{[-1,1]} v$.

It is straightforward to check that the solutions u and v are given by the formulas

$$u(x) = \begin{cases} -\frac{1}{2}x^2 - x - \frac{1}{2} & \text{if } -1 \le x \le 0 \\ \frac{1}{2}x^2 - x - \frac{1}{2} & \text{if } 0 \le x \le 1, \end{cases}$$

and

$$v(x) = \begin{cases} \frac{1}{2}x^2 + x + \frac{1}{2} & \text{if } -1 \le x \le -\frac{1}{2} \\ -\frac{1}{2}x^2 + \frac{1}{4} & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \frac{1}{2}x^2 - x + \frac{1}{2} & \text{if } \frac{1}{2} \le x \le 1, \end{cases}$$

so $\operatorname*{osc}_{[-1,1]} v = v(0) - v(1) = \frac{1}{4}$ while $\operatorname*{osc}_{[-1,1]} u = u(-1) - u(1) = 1$.

Example 5.9 (Two Dimensions)**.** Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk, $\mathbb{D}^+ = \{z \in \mathbb{D} : \text{Re } z > 0\}$ denote the right half of the unit disk, $\mathbb{D}^- = \{z \in \mathbb{D} : \text{Re } z < 0\}$ denote the left half, $B = \{z \in \mathbb{C} : |z| < \frac{1}{\sqrt{2}}\}$ be the disk centered at the origin of radius $\frac{1}{\sqrt{2}}$, and $A = A(\frac{1}{\sqrt{2}}, 1) = \{z \in \mathbb{C} : \frac{1}{\sqrt{2}} < |z| < 1\}$ be the annulus centered at the origin with inner radius $\frac{1}{\sqrt{2}}$ and outer radius 1. Notice A and B each have area $\frac{\pi}{2}$. Hence $(\mathbf{1}_{\mathbb{D}^+} - \mathbf{1}_{\mathbb{D}^-})^{\#} = \mathbf{1}_B - 1_A$ where # denotes the Schwarz rearrangement. Let u and v be weak solutions to

$$\begin{aligned} -\Delta u &= \mathbf{1}_{\mathbb{D}^+} - \mathbf{1}_{\mathbb{D}^-} \quad \text{in} \quad \mathbb{D}, & \qquad -\Delta v &= \mathbf{1}_B - 1_A \quad \text{in} \quad \mathbb{D}, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}, & \frac{\partial v}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}, \end{aligned}$$

We will show $\operatorname*{osc}_{\mathbb{D}} u > \operatorname*{osc}_{\mathbb{D}} v$.

One readily checks that a solution v is given radially by

$$v(r) = \begin{cases} -\frac{1}{4}r^2 & \text{if } 0 < r < \frac{1}{\sqrt{2}} \\ \frac{1}{4}r^2 - \frac{1}{2}\log r - \frac{1}{4}(1+\log 2) & \text{if } \frac{1}{\sqrt{2}} < r < 1, \end{cases}$$

and it is readily checked that v is radially decreasing. Thus $\underset{D}{\operatorname{osc}}\, v = v(0) - v(1) = -\frac{1}{4} + \frac{1}{4}(1+\log 2) \approx .1733$.

Calculating the oscillation of u is a little trickier. We will use the Green's representation for u as in Corollary A.2 of Appendix A. We then have

$$\begin{aligned} u(e^{i\phi}) &= -\frac{1}{2\pi}\int_{\mathbb{D}} \log\left(\left|e^{i\phi} - w\right|\left|1 - e^{-i\phi}w\right|\right)(\mathbf{1}_{D^+}(w) - \mathbf{1}_{D^-}(w))\ \mathrm{d}m(w) \\ &= -\frac{1}{\pi}\int_{\mathbb{D}^+} \log\left|1 - e^{-i\phi}w\right|\ \mathrm{d}m(w) \\ &\quad +\frac{1}{\pi}\int_{\mathbb{D}^-} \log\left|1 - e^{-i\phi}w\right|\ \mathrm{d}m(w) \\ &= \frac{1}{\pi}\int_{\mathbb{D}^+} \log\frac{|1 + e^{-i\phi}w|}{|1 - e^{-i\phi}w|}\ \mathrm{d}m(w). \end{aligned}$$

From here we use the series representation

$$\log\frac{1+z}{1-z} = \sum_{n \text{ odd}} \frac{2}{n} z^n$$

to compute

$$\begin{aligned} \frac{1}{\pi}\int_{\mathbb{D}^+} \log\frac{|1 + e^{-i\phi}w|}{|1 - e^{-i\phi}w|} &= \frac{2}{\pi}\mathrm{Re}\Big[\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 (re^{i\theta}e^{-i\phi})^{2n+1} r\ \mathrm{d}r\ \mathrm{d}\theta\Big] \\ &= \frac{4}{\pi}\mathrm{Re}\Big[\sum_{n=0}^{\infty}\frac{(-1)^n e^{-i(2n+1)\phi}}{(2n+1)^2(2n+3)}\Big] \\ &= \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n \cos\big((2n+1)\phi\big)}{(2n+1)^2(2n+3)}. \end{aligned}$$

With $\phi = 0$, we see $u(1) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2(2n+3)} \approx .4014$. When $\phi = \pi$, we see $u(-1) = -u(1)$. Hence $\operatorname*{osc}_{\mathbb{D}} u > .8 > \operatorname*{osc}_{\mathbb{D}} v$.

CHAPTER 6

Unweighted Conjectures in the Unit Disk

The final section in Chapter 5 contains non-examples. They show that the Schwarz rearrangement does not give any reasonable comparison theorem for the unit disk using homogeneous Neumann boundary conditions. Theorem 5.7 gives a perfectly reasonable comparison result, but it deals with cap symmetrization, a "partial" symmetrization. In this chapter we discuss a "total" symmetrization in the unit disk that could potentially play the role for homogeneous Neumann boundary conditions that the Schwarz rearrangement does for homogeneous Dirichlet boundary conditions.

We begin with the following problem.

Problem 6.1. Find a "total" rearrangement # in the unit disk $\mathbb{D} \subset \mathbb{R}^2$ so that when $f \in L^2(\mathbb{D})$ with $\int_{\mathbb{D}} f \, \mathrm{d}m = 0$ and u and v are weak solutions to

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \mathbb{D}, & \qquad -\Delta v &= f^{\#} \quad \text{in} \quad \mathbb{D}, \\ \tfrac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}, & \qquad \tfrac{\partial v}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}, \end{aligned}$$

then $\|u\|_{L^p(\mathbb{D})} \leq \|v\|_{L^p(\mathbb{D})}$ for $1 \leq p \leq \infty$ and $\operatorname*{osc}_{\mathbb{D}} u \leq \operatorname*{osc}_{\mathbb{D}} v$.

We emphasize the desire for a total rearrangement. If we only seek a partial rearrangement, then Problem 6.1 is solved by Theorem 5.7, taking # to be $(1, 2)$ cap symmetrization.

Problem 6.1 poses a difficult question indeed. Let us work with a special class of functions f that assume only the values 1 and -1, each on half of $\mathbb{D}$.

Problem 6.2. Let $E \subset \mathbb{D}$ be a subset of area $\frac{\pi}{2}$ and assume u is a weak solution to

$$\begin{aligned} -\Delta u &= \mathbf{1}_E - \mathbf{1}_{\mathbb{D}\setminus E} \quad \text{in} \quad \mathbb{D}, \\ \tfrac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}, \end{aligned}$$

and satisfies the additive normalization $\int_{\mathbb{D}} u \,\mathrm{d}m = 0$. Identify the set E so that the solution u has maximal L^p norms and oscillation.

The figure below shows several possible arrangements. The white area represents the set E and the black area represents $\mathbb{D}\setminus E$.

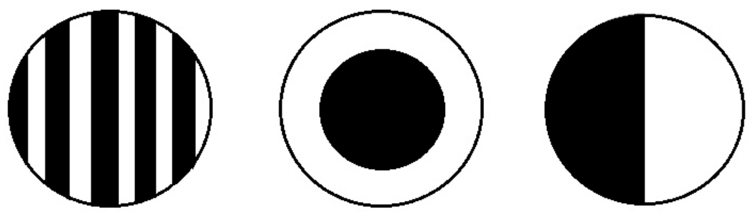

FIGURE 6.1. Several possible arrangements of heat sources and sinks. White areas represent heat sources and black areas represent heat sinks.

If the data assumes the values 1 and -1, each on half the disk with respect to stereographic measure (rather than Lebesgue measure), then we have the following result.

Corollary 6.3 (Weighted Solution to Problem 6.2)**.** *Let $E \subseteq \mathbb{D}$ be a subset with $\mu(E) = \frac{1}{2}\mu(\mathbb{D})$. Then the weak solution u to*

$$\begin{aligned} -\tfrac{1}{\rho}\Delta u &= \mathbf{1}_E - \mathbf{1}_{\mathbb{D}\setminus E} \quad \text{in} \quad \mathbb{D}, \\ \tfrac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}, \end{aligned}$$

normalized with $\int_{\mathbb{D}} u \,\mathrm{d}\mu = 0$ has maximal L^p norms and oscillation precisely when E is a half-disk.

It is natural, then, to make the following conjecture.

Conjecture 6.4. *Problem 6.2 is solved by taking E to be a half-disk.*

Let us compare with the analogous problem in the Dirichlet situation, where the solution u is maximized by taking the input function to be radially decreasing. What is special in the Dirichlet case is that on the level sets of u, ∇u is constant in magnitude. The same phenomena holds true when we consider the weighted problem in the disk, taking the input data to be stereographically rearranged. When the input data is 1 on the right half-disk and -1 on the left half-disk in the unweighted problem, the solution's gradient no longer has constant magnitude on the level sets.

To formulate a conjecture for Problem 6.1, we first discuss a rearrangement in the unit disk that is similar in appearance to the stereographic rearrangement, but which takes place with respect to Lebesgue measure rather than stereographic measure.

For $\theta \in (0, \pi)$, let $C(\theta)$ denote the circular arc, symmetric with respect to the real axis, that meets $\partial\mathbb{D}$ orthogonally at the points $e^{i\theta}$ and $e^{-i\theta}$ as shown in the figure below.

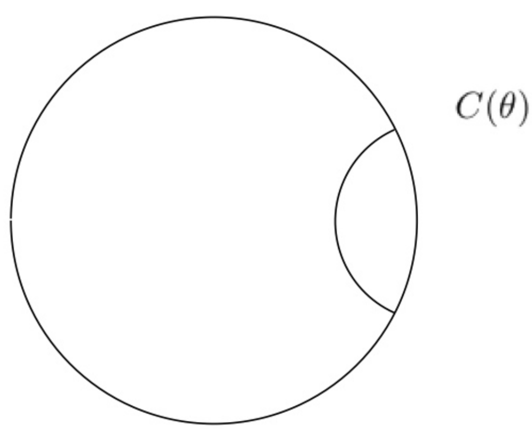

FIGURE 6.2. A picture of the arc $C(\theta)$.

Let $A(\theta)$ denote the region inside $\mathbb{D}$ to the right of $C(\theta)$ as shown in the figure below.

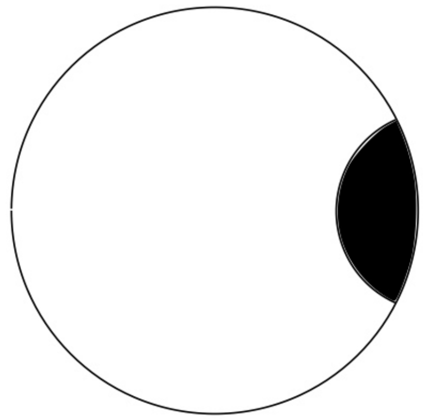

FIGURE 6.3. A picture of the region $A(\theta)$.

Given a function $f : \mathbb{D} \to \mathbb{R}$, we define a rearrangement $f^{\#} : \mathbb{D} \to \mathbb{R}$ by the formula

$$f^{\#}(x) \quad = \quad f^*(|A(\theta)|),$$

where f^* is the decreasing rearrangement of f, $x \in C(\theta)$, and $|A(\theta)|$ is the area of $A(\theta)$.

This rearrangement is discussed by Leckband in [**Le**], and accordingly, we refer to $f^{\#}$ as the *Leckband rearrangement.* Leckband shows, among other properties, that $f^{\#}$ is a rearrangement of f, and he uses this rearrangement to prove a sharp version of Moser's inequality on the unit ball.

Note that if $f = \mathbf{1}_E - \mathbf{1}_{\mathbb{D}\setminus E}$ where E has half the area of the unit disk, then the Leckband rearrangement is $f^{\#} = \mathbf{1}_{\mathbb{D}^+} - \mathbf{1}_{\mathbb{D}^-}$ where $\mathbb{D}^+$ is the right half of the unit disk and $\mathbb{D}^-$ is the left half.

Conjecture 6.5. *Taking $\#$ to be the Leckband rearrangement solves Problem 6.1.*

Before we prove a partial result, we need the following version of the Hardy-Littlewood inequality for Leckband's rearrangement.

Theorem 6.6 (Hardy-Littlewood for Leckband's Rearrangement)**.** *If $f, g \in L^1(\mathbb{D}, \mathrm{d}m)$, then*

$$\int_{\mathbb{D}} fg \ \mathrm{d}m \ \leq \ \int_{\mathbb{D}} f^{\#} g^{\#} \ \mathrm{d}m,$$

where $f^{\#}$ and $g^{\#}$ are the Leckband rearrangements of f and g respectively.

Proof. By the standard Hardy-Littlewood inequality, Theorem 1.2.2 of [**Ke**], we have

$$\int_{\mathbb{D}} fg \ \mathrm{d}m \ \leq \ \int_0^{\pi} f^*(t) g^*(t) \ \mathrm{d}t, \tag{6.1}$$

where f^* and g^* are the decreasing rearrangements of f and g respectively. Now make the change of variable $t = |A(\theta)|$. We get

$$\begin{aligned} \int_0^{\pi} f^*(t) g^*(t) \ \mathrm{d}t &= \int_0^{\pi} f^*(|A(\theta)|) g^*(|A(\theta)|) \ \frac{d}{d\theta} |A(\theta)| \ \mathrm{d}\theta \\ &= \int_{\mathbb{D}} f^{\#} g^{\#} \ \mathrm{d}m, \end{aligned}$$

where the last equality follows from Lemma 2.1 of [**Le**]. Combining the above equality with inequality (6.1) gives the result. □

We have the following partial result in the direction of Conjecture 6.5.

Theorem 6.7 (Oscillation Along Axis)**.** *Let $f \in L^\infty(\mathbb{D})$ with $\int_{\mathbb{D}} f \ \mathrm{d}m = 0$. Suppose u and v are weak solutions to*

$$\begin{aligned} -\Delta u &= f \quad in \quad \mathbb{D}, & \qquad -\Delta v &= f^{\#} \quad in \quad \mathbb{D}, \\ \tfrac{\partial u}{\partial n} &= 0 \quad on \quad \partial\mathbb{D}, & \tfrac{\partial v}{\partial n} &= 0 \quad on \quad \partial\mathbb{D}, \end{aligned}$$

where $f^{\#}$ is the Leckband rearrangement of f.

If u and v are additively normalized so that $\int_{\mathbb{D}} u \ \mathrm{d}m = \int_{\mathbb{D}} v \ \mathrm{d}m = 0$, then

$$u(1) - u(-1) \quad \leq \quad v(1) - v(-1).$$

PROOF. Consider the Möbius transformation $T(z) = \frac{1+z}{1-z}$, a conformal mapping from the unit disk $\mathbb{D}$ onto the right half plane $\mathbb{H}^+ = \{z \in \mathbb{C} : \text{Re } z > 0\}$. T takes the boundary of the unit disk to the imaginary axis (together with the point at ∞). We now find the image of $C(\theta)$ under T. Since Möbius transformations take circles in the extended complex plane $\mathbb{C} \cup \{\infty\}$ to circles in $\mathbb{C} \cup \{\infty\}$, $T(C(\theta))$ will either be the arc of a circle or part of a line. But since $T(\overline{z}) = \overline{T(z)}$ and T is conformal, $T(C(\theta))$ must be symmetric with respect to the real axis and meet the imaginary axis orthogonally, because $C(\theta)$ meets $\partial\mathbb{D}$ orthogonally. Hence, $T(C(\theta))$ is a half-circle in $\mathbb{H}^+$ centered at the origin as shown in the figure below.

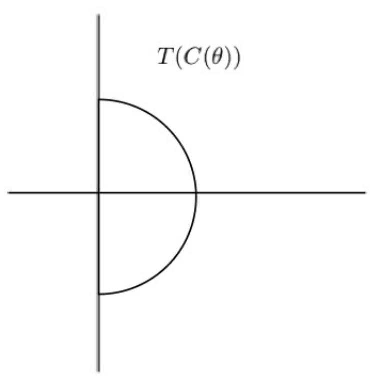

FIGURE 6.4. A graph of the circular arc $T(C(\theta))$.

Consequently, $|T|$ is constant on $C(\theta)$. The function $\frac{1}{\pi}\log|T(z)|$ is also constant on $C(\theta)$, and as θ increases from 0 to π, $\frac{1}{\pi}\log|T(z)|$ decreases. Thus, we have shown

$$\big(\frac{1}{\pi}\log|T|\big)^{\#} \;=\; \frac{1}{\pi}\log|T|.$$

Next, we use the Neumann Green's function for $\mathbb{D}$ as in Corollary A.2 of Appendix A to compute

$$\begin{aligned} u(1)-u(-1) &= \frac{1}{\pi}\int_{\mathbb{D}}\log\frac{|1+z|}{|1-z|}f(z)\;\mathrm{d}m(z) \\ &\le \int_{\mathbb{D}}\big(\frac{1}{\pi}\log\frac{|1+z|}{|1-z|}\big)^{\#}f^{\#}(z)\;\mathrm{d}m(z) \\ &= \frac{1}{\pi}\int_{\mathbb{D}}\log\frac{|1+z|}{|1-z|}f^{\#}(z)\;\mathrm{d}m(z) \\ &= v(1)-v(-1), \end{aligned}$$

where the inequality follows from Theorem 6.6. □

APPENDIX A

Neumann Green's Function for the Unit Disk

Let $\mathbb{D} = \{z \in \mathbb{C} : \big|z\big| < 1\}$ denote the unit disk. The Dirichlet Green's function for the unit disk is given by

$$g(z,w) \;=\; \frac{1}{2\pi}\log\Big|\frac{1-z\overline{w}}{z-w}\Big|.$$

In other words, holding $w \in \mathbb{D}$ fixed, we have

$$\begin{aligned} -\Delta_z g(z,w) &= \delta_w(z) \quad \text{for} \quad z\in \mathbb{D}, \\ g(z,w) &= 0 \quad \text{for} \quad z \in \partial\mathbb{D}, \end{aligned} \tag{A.1}$$

where $\delta_w(z)$ denotes unit point-mass at w.

Consequently, given $f \in L^\infty(\mathbb{D})$, in order to solve

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \mathbb{D}, \\ u &= 0 \quad \text{on} \quad \partial\mathbb{D}, \end{aligned}$$

we can take

$$u(z) \;=\; \int_{\mathbb{D}} g(z,w)f(w)\;\mathrm{d}m(w),$$

where dm denotes standard Lebesgue measure. Our first task is to construct the Neumann analogue of g, which we call the Neumann Green's function. It appears in the literature, but is not widely known. Define

$$G(z,w) \;=\; \frac{1}{2\pi}\log\frac{1}{|z-w|}+\frac{1}{2\pi}\log\frac{1}{|1-z\overline{w}|}+\frac{|z|^2+|w|^2}{4\pi}-\frac{3}{8\pi}$$

for $z, w \in \mathbb{D}$ and $z \neq w$. Our next theorem shows that G satisfies the properties desired of the Neumann Green's function.

Theorem A.1 (Neumann Green's Function for the Unit Disk)**.** *For $z, w \in \mathbb{D}$, let G be defined as above. Holding $w \in \mathbb{D}$ fixed, G satisfies the following three properties:*

1. $-\Delta_z G(z,w) = \delta_w(z) - \frac{1}{\pi}$ *for* $z \in \mathbb{D}$.

2. $\frac{\partial G(z,w)}{\partial n_z} = 0$ *for* $z \in \partial\mathbb{D}$.

3. $\int_{\mathbb{D}} G(z,w)\ \mathrm{d}m(z) = 0$.

We call G is the Neumann Green's function in the unit disk $\mathbb{D}$.

Proof. To prove property 1, let g denote the Dirichlet Green's function for the unit disk. Then

$$G(z,w) \;=\; g(z,w)+\frac{1}{\pi}\log\frac{1}{|1-z\overline{w}|}+\frac{|z|^2+|w|^2}{4\pi}-\frac{3}{8\pi}.$$

Holding $w \in \mathbb{D}$ fixed in the above equation and applying $-\Delta_z$ to both sides, and using equation (A.1), we get

$$-\Delta_z G(z,w) \;=\; \delta_w(z)-\frac{1}{\pi}$$

as desired.

For property 2, fix $w \in \mathbb{D}$ and write $z = re^{i\theta}$ and compute

$$\begin{aligned}\frac{\partial G(z,w)}{\partial r} &= -\frac{1}{4\pi}\frac{1}{|z-w|^2}\frac{\partial}{\partial r}(|z|^2 - 2\mathrm{Re}(z\overline{w}) + |w|^2) \\ &\quad -\frac{1}{4\pi}\frac{1}{|1-z\bar{w}|^2}\frac{\partial}{\partial r}(1 - 2\mathrm{Re}(z\overline{w}) + |w|^2|z|^2) + \frac{|z|}{2\pi}.\end{aligned}$$

Taking $r = 1$, we get

$$\begin{aligned}\frac{\partial G(z,w)}{\partial n_z} &= -\frac{1}{4\pi}\frac{1}{|e^{i\theta}-w|^2}(2 - 4\mathrm{Re}(e^{i\theta}\overline{w}) + 2|w|^2) + \frac{1}{2\pi} \\ &= -\frac{1}{2\pi} + \frac{1}{2\pi} \\ &= 0\end{aligned}$$

which establishes property 2.

To establish property 3, we split up the integral as follows:

$$\begin{aligned}\int_{\mathbb{D}} G(z,w)\ \mathrm{d}m(z) &= \frac{1}{2\pi}\int_{\mathbb{D}} \log\frac{1}{|z-w|}\ \mathrm{d}m(z) + \frac{1}{2\pi}\int_{\mathbb{D}} \log\frac{1}{|1-z\overline{w}|}\ \mathrm{d}m(z) \\ &+ \int_{\mathbb{D}} \Big(\frac{|z|^2+|w|^2}{4\pi} - \frac{3}{8\pi}\Big)\ \mathrm{d}m(z) \\ &= (1) + (2) + (3).\end{aligned}$$

Now

$$\begin{aligned}(1) &= -\int_0^{|w|}\frac{1}{2\pi}\int_0^{2\pi}\log|re^{i\theta} - w|\ \mathrm{d}\theta\ r\mathrm{d}r \\ &\quad -\int_{|w|}^1\frac{1}{2\pi}\int_0^{2\pi}\log|re^{i\theta} - w|\ \mathrm{d}\theta\ r\mathrm{d}r \\ &= -\frac{|w|^2}{2}\log|w| - \int_{|w|}^1 \big(\log r\big) r\ \mathrm{d}r \\ &= \frac{1}{4}(1 - |w|^2),\end{aligned}$$

where the second to last equality follows since $\log|z|$ is harmonic for $z \neq 0$ and by use of Jensen's formula. We next observe that (2) equals 0 because the function $\log \frac{1}{|1-z\overline{w}|}$ is harmonic for $z \in \mathbb{D}$ and equals 0 at the origin. Finally, we compute

$$\begin{aligned}(3) &= \frac{1}{2}\int_0^1 r^3 \,\mathrm{d}r + (\frac{|w|^2}{4\pi} - \frac{3}{8\pi})\pi \\ &= \frac{1}{4}(|w|^2 - 1).\end{aligned}$$

Combining these calculations, we have that

$$\int_{\mathbb{D}} G(z,w)\,\mathrm{d}m(z) = (1) + (2) + (3) = 0$$

giving property 3. □

From Theorem A.1 it follows that we can solve Poisson's equation in the unit disk with homogeneous Neumann boundary conditions by integrating the data against the Neumann Green's function.

Corollary A.2 (Green's Representation for Solutions)**.** *Suppose $f \in L^\infty(\mathbb{D})$ with $\int_{\mathbb{D}} f\ \mathrm{d}m = 0$ and suppose u is the weak solution of*

$$\begin{aligned}-\Delta u &= f \quad \text{in} \quad \mathbb{D}, \\ \tfrac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D},\end{aligned}$$

with the additive normalization $\int_{\mathbb{D}} u\ \mathrm{d}m = 0$. Then,

$$u(z) = \int_{\mathbb{D}} G(z,w) f(w)\,\mathrm{d}m(w),$$

where G is the Neumann Green's function as in Theorem A.1.

Proof. First, assume f is Lipschitz continuous on $\overline{\mathbb{D}}$. By Theorem 3.2 of [**LU**], the solution u belongs to $C^2(\overline{\mathbb{D}})$ and so $\frac{\partial u}{\partial n} = 0$ classically on $\partial\mathbb{D}$. Fix $z \in \mathbb{D}$ and let $\epsilon > 0$ be small enough so that $B(z,\epsilon) = \{w \in \mathbb{C} : |w - z| < \epsilon\} \subset \mathbb{D}$ and let $V_\epsilon = \mathbb{D} \backslash \overline{B(z,\epsilon)}$. By Green's second identity,

$$\int_{V_\epsilon} u(w)\Delta_w G(z,w)\ \mathrm{d}m(w) - \int_{V_\epsilon} G(z,w)\Delta u(w)\ \mathrm{d}m(w) \tag{A.2}$$

$$= \int_{\partial V_\epsilon} u(w)\frac{\partial G(z,w)}{\partial n_w}\ \mathrm{d}S(w) - \int_{\partial V_\epsilon} G(z,w)\frac{\partial u}{\partial n}(w)\ \mathrm{d}S(w). \tag{A.3}$$

By Theorem A.1, $\Delta_w G(z,w) = \frac{1}{\pi}$ in V_ϵ and by assumption $-\Delta u = f$. We therefore have

$$\begin{aligned} (A.2) &= \frac{1}{\pi}\int_{V_\epsilon} u(w)\ \mathrm{d}m(w) + \int_{V_\epsilon} G(z,w)f(w)\ \mathrm{d}m(w) \\ &\to \int_{\mathbb{D}} G(z,w)f(w)\ \mathrm{d}m(w) \end{aligned}$$

as $\epsilon \to 0$ since $G(z,\cdot)$ is integrable over the unit disk $\mathbb{D}$ and $\int_{\mathbb{D}} u(w)\ \mathrm{d}m(w) = 0$.

On the other hand, by Theorem A.1, $\frac{\partial G(z,w)}{\partial n_w} = 0$ for $w \in \partial\mathbb{D}$ and by assumption $\frac{\partial u}{\partial n} = 0$ on $\partial\mathbb{D}$. Hence,

$$(A.3) = -\int_{\partial B(z,\epsilon)} u(w)\frac{\partial G(z,w)}{\partial n_w}\ \mathrm{d}S(w) + \int_{\partial B(z,\epsilon)} G(z,w)\frac{\partial u}{\partial n}(w)\ \mathrm{d}S(w).$$

As $\epsilon \to 0$,

$$\int_{\partial B(z,\epsilon)} G(z,w)\frac{\partial u}{\partial n}(w)\ \mathrm{d}S(w) \to 0$$

and also

$$\begin{aligned} \lim_{\epsilon\to 0} -\int_{\partial B(z,\epsilon)} u(w)\frac{\partial G(z,w)}{\partial n_w}\ \mathrm{d}S(w) &= \lim_{\epsilon\to 0}\frac{1}{2\pi\epsilon}\int_{\partial B(z,\epsilon)} u(w)\ \mathrm{d}S(w) \\ &= u(z). \end{aligned}$$

Hence, letting $\epsilon \to 0$ in (A.2) and (A.3) gives

$$u(z) = \int_{\mathbb{D}} G(z,w) f(w) \, \mathrm{d}m(w).$$

Now let $f \in L^\infty(\mathbb{D})$ with mean zero. Choose a sequence of test functions $f_k \in C_c^\infty(\mathbb{D})$ each with mean zero and where $f_k \to f$ in $L^2(\mathbb{D})$. Let u_k solve

$$\begin{aligned} -\Delta u_k &= f_k \quad \text{in} \quad \mathbb{D}, \\ \frac{\partial u_k}{\partial n} &= 0 \quad \text{on} \quad \partial\mathbb{D}, \end{aligned}$$

and assume each u_k is normalized to have mean zero. By Corollary 1.29, $u_k \to u$ in $L^2(\mathbb{D})$. Hence by passing to a subsequence of the original f_k, we may assume that $f_k \to f$ and $u_k \to u$ pointwise a.e. Additionally, by truncating if necessary, we may assume the f_k are uniformly bounded.

Since each f_k is Lipschitz continuous on $\overline{\mathbb{D}}$, the work above gives

$$u_k(z) = \int_{\mathbb{D}} G(z,w) f_k(w) \, \mathrm{d}m(w).$$

Thus,

$$u_k(z) = \int_{\mathbb{D}} G(z,w) f_k(w) \, \mathrm{d}m(w) \quad \to \quad \int_{\mathbb{D}} G(z,w) f(w) \, \mathrm{d}m(w)$$

for $z \in \mathbb{D}$. On the other hand, since $u_k \to u$ a.e. we have

$$u(z) = \int_{\mathbb{D}} G(z,w) f(w) \, \mathrm{d}m(w).$$

□

APPENDIX B

A Formula for the Stereographic Operator

Let $\Phi : \mathbb{S}^n \to \mathbb{R}^n \cup \{\infty\}$ denote stereographic projection where the north pole $e_{n+1} = (0, \ldots, 0, 1)$ corresponds to the origin and the south pole $-e_{n+1}$ corresponds to ∞. To be precise, given $(\xi_1, \ldots, \xi_{n+1}) \in \mathbb{S}^n$, define

$$\Phi(\xi_1, \ldots, \xi_{n+1}) \quad = \quad (x_1, \ldots, x_n),$$

where

$$x_i \quad = \quad \frac{\xi_i}{1 + \xi_{n+1}} \quad \text{for } 1 \leq i \leq n.$$

Given a function $u \in C^2(\mathbb{R}^n)$, define

$$Lu(x) \quad = \quad \Delta_{\mathbb{S}}(u \circ \Phi)(\Phi^{-1}(x)).$$

We call L the *stereographic operator.*

Define functions ρ and ψ on $\mathbb{R}^n$ by the formulas

$$\rho(x) \quad = \quad \big(\frac{2}{1 + |x|^2}\big)^2,$$
$$\psi(x) \quad = \quad \log\big(\frac{2}{1 + |x|^2}\big).$$

Theorem B.1 (Formula for Stereographic Operator)**.** *The stereographic operator L has the form*

$$Lu = \frac{1}{\rho}\big(\Delta u + (n-2)\nabla\psi \cdot \nabla u\big).$$

Proof. The proof will be accomplished in several steps.

Step 1: Preliminary calculations.

Extend Φ to a spherical shell about $\mathbb{S}^n$ by homogeneity. That is, define $\Phi(y) = (x_1, \ldots, x_n)$ where

$$x_i = \frac{y_i}{|y| + y_{n+1}}.$$

We then have

$$\frac{\partial \Phi_i}{\partial y_j} = \frac{\delta_{i,j}}{|y| + y_{n+1}} - \frac{y_i}{(|y| + y_{n+1})^2}\big(\frac{y_j}{|y|} + \delta_{j,n+1}\big),$$

where $\delta_{i,j}$ denotes the Kronecker delta function. Thus, for $\xi \in \mathbb{S}^n$,

$$\frac{\partial \Phi_i}{\partial y_j}(\xi) = \frac{\delta_{i,j}}{1 + \xi_{n+1}} - \frac{\xi_i\big(\xi_j + \delta_{j,n+1}\big)}{(1 + \xi_{n+1})^2}.$$

Hence, for $x \in \mathbb{R}^n$,

$$\frac{\partial \Phi_i}{\partial y_j}\big(\Phi^{-1}(x)\big) = \begin{cases} \frac{\delta_{i,j}}{\rho(x)^{\frac{1}{2}}} - x_i x_j & \text{if } j \leq n \\ x_i & \text{if } j = n+1. \end{cases}$$

Step 2: Computation of the matrix product $\big((D\Phi)(\Phi^{-1}(x))\big)\big((D\Phi)(\Phi^{-1}(x))\big)^T$.

We write I_n for the $n \times n$ identity matrix and $\begin{bmatrix} I_n & 0 \end{bmatrix}$ for the $n \times (n+1)$ matrix with the identity matrix in the first n columns and all zeros in the last column. We then have

$$(D\Phi)\big(\Phi^{-1}(x)\big) \;=\; \frac{1}{\rho^{\frac{1}{2}}(x)}\begin{bmatrix} I_n & 0 \end{bmatrix} - \begin{bmatrix} a_{ij} \end{bmatrix},$$

where $\begin{bmatrix} a_{ij} \end{bmatrix}$ is the $n \times (n+1)$ matrix whose ij^{th} entry is given by

$$a_{ij} \;=\; \begin{cases} x_i x_j & \text{if } j \leq n \\ x_i & \text{if } j = n+1. \end{cases}$$

We then calculate

$$\begin{aligned} \big((D\Phi)(\Phi^{-1}(x))\big)\big((D\Phi)\big(\Phi^{-1}(x)\big)\big)^T &= \Big(\frac{1}{\rho^{\frac{1}{2}}(x)}\begin{bmatrix} I_n & 0 \end{bmatrix} - \begin{bmatrix} a_{ij} \end{bmatrix}\Big)\Big(\frac{1}{\rho^{\frac{1}{2}}(x)}\begin{bmatrix} I_n \\ 0 \end{bmatrix} - \begin{bmatrix} a_{ji} \end{bmatrix}\Big) \\ &= \frac{1}{\rho(x)} I_n - \frac{2}{\rho^{\frac{1}{2}}(x)}\begin{bmatrix} x_i x_j \end{bmatrix} + (1+|x|^2)\begin{bmatrix} x_i x_j \end{bmatrix} \\ &= \frac{1}{\rho(x)} I_n. \end{aligned}$$

Step 3: Show $\int_{\mathbb{R}^n}(Lu)(x)v(x)\,\mathrm{d}x = \int_{\mathbb{R}^n}\frac{1}{\rho}\big(\Delta u + (n-2)\nabla\psi\cdot\nabla u\big)v\,\mathrm{d}x$ for each test function v.

Letting $v \in C_c^\infty(\mathbb{R}^n)$, we compute

$$\begin{aligned}
& \int_{\mathbb{R}^n} (Lu)(x)v(x)\ \mathrm{d}x \\
= \quad & \int_{\mathbb{R}^n} \Delta_{\mathbb{S}}(u \circ \Phi)\big(\Phi^{-1}(x)\big)v(x)\ \mathrm{d}x \\
= \quad & \int_{\mathbb{S}^n} \Delta_{\mathbb{S}}(u \circ \Phi)(\xi)(v\rho^{-\frac{n}{2}})\big(\Phi(\xi)\big)\ \mathrm{d}\sigma_n(\xi) \\
= \quad & -\int_{\mathbb{S}^n} \nabla_{\mathbb{S}}(u \circ \Phi)(\xi) \cdot \nabla_{\mathbb{S}}\big((v\rho^{-\frac{n}{2}})(\Phi(\xi)\big)\ \mathrm{d}\sigma_n(\xi) \\
= \quad & -\int_{\mathbb{S}^n} (\nabla u)(\Phi(\xi))\big((D\Phi)(\xi)\big)\big((D\Phi)(\xi)\big)^T\big(\nabla(v\rho^{-\frac{n}{2}})(\Phi(\xi))\big)^T\ \mathrm{d}\sigma_n(\xi) \\
= \quad & -\int_{\mathbb{R}^n} \nabla u(x)\big((D\Phi)(\Phi^{-1}(x))\big)\big(D\Phi(\Phi^{-1}(x))\big)^T\big(\nabla(v\rho^{-\frac{n}{2}})(x)\big)^T\rho^{\frac{n}{2}}(x)\ \mathrm{d}x.
\end{aligned}$$

Using the result of step 2, we have

$$\begin{aligned}
& -\int_{\mathbb{R}^n} \nabla u(x)\big((D\Phi)(\Phi^{-1}(x))\big)\big(D\Phi(\Phi^{-1}(x))\big)^T\big(\nabla(v\rho^{-\frac{n}{2}})(x)\big)^T\rho^{\frac{n}{2}}(x)\ \mathrm{d}x \\
= \quad & -\int_{\mathbb{R}^n} \nabla u(x) \cdot \big(\nabla(v\rho^{-\frac{n}{2}})(x)\big)\rho^{\frac{n}{2}-1}(x)\ \mathrm{d}x \\
= \quad & -\int_{\mathbb{R}^n} \frac{1}{\rho}\nabla u(x) \cdot \big(\nabla v(x) - nv(x)\nabla\psi(x)\big)\ \mathrm{d}x \\
= \quad & \int_{\mathbb{R}^n} \frac{1}{\rho}\big(\Delta u + (n-2)\nabla\psi \cdot \nabla u\big)v\ \mathrm{d}x.
\end{aligned}$$

□

Bibliography

[ALT] A. Alvino, P. L. Lions, G. Trombetti, *Comparison results for elliptic and parabolic equations via symmetrization: a new approach,* Differential Integral Equations 4 (1991), 25–50.

[AMT] A. Alvino, S. Matarasso, G. Trombetti, *Elliptic boundary value problems: comparison results via symmetrization*, Ricerche de matematica, Volume 51, Issue 2, (2002), 341–355.

[Ba1] A. Baernstein II, *Proof of Edrei's spread conjecture*, Proc. London Math. Soc. (3) 26 (1973), 418–434.

[Ba2] A. Baernstein II, *Integral means, univalent functions and circular symmetrization*, Acta Math. 133 (1974), 139–169.

[Ba3] A. Baernstein II, *How the $*$-function solves extremal problems*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), Acad. Sci. Fennica, Helsinki, 1980, 639–644.

[Ba4] A. Baernstein II, *Convolution and rearrangement on the circle*, Complex Variables 12 (1989), 33-37.

[Ba5] A. Baernstein II, *A unified approach to symmetrization*, Partial differential equations of elliptic type (Cortona, 1992), 47–91, Sympos. Math., XXXV, Cambridge Univ. Press, Cambridge, 1994.

[Ba6] A. Baernstein II, *The $*$-function in complex analysis*, Handbook of Complex Analysis, Vol. I: Geometric Function Theory, edited by R. Kühnau, Elsevier Science, 2002, 229–227.

[BT] A. Baernstein II and B.A. Taylor, *Spherical rearrangements, subharmonic functions, and *-functions in n-space*, Duke Math. J. 43 (1976), 245–268.

[BB] R. Bañuelos, K. Burdzy, *On the "hot spots" conjecture of J. Rauch,* J. Funct. Anal. 164 (1999), no. 1, 1–33.

[BPP] R. Bañuelos, M. Pang, and M. Pascu, *Brownian motion with killing and reflection and the "hot-spots" problem*, Probab. Theory Related Fields 130 (2004), no. 1, 56–68.

[BW] K. Burdzy and W. Werner, *A counterexample to the "hot spots" conjecture*, Ann. of Math. (2) 149 (1999), no. 1, 309–317.

[Ev] L. C. Evans, *Partial Differential Equations*, Second edition, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010.

[EG] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[FM] V. Ferone and A. Mercaldo, *Neumann problems and Steiner symmetrization*, Communications in Partial Differential Equations, Volume 30, Issue 10, 2005, 1537–1553.

[Fo] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, Second edition. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999.

[Fra] L. E. Fraenkel, *An Introduction to Maximum Principles and Symmetry in Elliptic Problems*, Cambridge Tracts in Mathematics, 128. Cambridge University Press, Cambridge, 2000.

[Fry] A. Fryszkowski, *Fixed Point Theory for Decomposable Sets: Topological Fixed Point Theory and Its Applications*, Kluwer Academic Publishers, Dordrecht, 2004.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 2001.

[G] R. C. Graham, *Conformal powers of the Laplacian via stereographic projection*, SIGMA Symmetry Integrability Geom. Methods Appl. 3 (2007) Paper 121, 4 pp.

[HLP] G. H. Hardy, J. Littlewood, and G. Pólya, *Inequalities*, Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988.

[Ka] B. Kawohl, *Rearrangements and Convexity of Level Sets in PDE*, Lecture Notes in Math. 1150, Springer, Berlin, 1985.

[Ke] S. Kesavan, *Symmetrization and Its Applications,* Series in Analysis, 3. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

[LU] O. A. Ladyzenskaja and N. N. Ural'ceva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York-London 1968.

[Le] M. Leckband, *Moser's inequality on the ball $\mathbb{B}^n$ for functions with mean value zero*, Comm. Pure Appl. Math. 58 (2005), no. 6, 789–798.

[LL] E. H. Lieb and M. Loss, *Analysis*, Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.

[Li] W. Littman, *A strong maximum principle for weakly L-subharmonic functions*, J. Math. Mech. 8 1959 761–770.

[MS] C. Maderna and S. Salsa, *Symmetrization in Neumann Problems,* Appl. Annl., no. 9, (1979) 247–256.

[R] K. Richardson, *Critical points of the determinant of the Laplace operator*, J. Funct. Anal. 122 (1994), no. 1, 52–83.

[Sa] S. Salsa, *Partial Differential Equations in Action: From Modeling to Theory*, Universitext. Springer-Verlag Italia, Milan, 2008.

[T] G. Talenti, *Elliptic equations and rearrangements,* Ann. Scuola Norm. Sup. Pisa Cl. Sci (4) 3 (1976), 697–718.