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A New Transform For Time-Frequency Analysis

Arun Kumar, Daniel R. Fuhrmann, Michael Frazier, and Bjorn Jawerth

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1. Introduction

The field of time-frequency analysis is currently enjoying a great deal of activity in the signal processing community. Time–frequency analysis has many applications in signal processing and data compression [1]–[5].

From a mathematical viewpoint, two approaches to time-frequency analysis have been proposed in the literature. The first is the class of linear transforms, which attempt to directly decompose a signal into components which are simultaneously localized in time and frequency. The second approach involves nonlinear operations that attempt to produce an energy distribution in the time–frequency domain.

Linear transforms for time-frequency analysis were first proposed by Gabor [1] in 1946. Gabor suggested that a time–frequency description of a signal could be obtained by performing Fourier analysis on the signal as it appears when seen through a set of identical windows that are translated with respect to each other in time. Gabor suggested the use of Gaussian windows, because they are simultaneously well–localized in the time *and* the frequency domains. Alternatively, we can think of the Gabor method as involving the computation of the projections of a signal upon a set of analyzing functions; where these analyzing functions are complex exponentials with Gaussian envelopes. Gabor's method has been extended to a set of methods known collectively as Short–Time Fourier Analysis (see e.g. [6]). Different methods in this body of work employ differently shaped windows.

Both the Gabor transform and the Short-time Fourier Transform have the property that the bandwidth of the analyzing functions is a constant independent of center frequency; likewise the timeduration of the analyzing functions is constant. In some applications it is felt that the analyzing functions should have a constant bandwidth-to-center-frequency ratio. The Wavelet Transform, first introduced in 1986 by Lemarié and Meyer [7], and receiving considerable attention in the mathematical and engineering communities [2]–[4],[8]–[15] does indeed have this property. The analyzing, or the basis, functions for the Wavelet Transform are generated from a single "mother function" by the process of translation and dilation. The almost magical quality of the wavelet basis functions is that they can be made orthogonal, and yield an orthonormal decomposition for L^2 and other function spaces. Wavelets play a key role in the multiresolution analysis of Mallat [3],[4].

The distributional approach to time-frequency analysis is discussed at length in the review article by Cohen [5]. At the center of this methodology is the Wigner–Ville distribution, first proposed by Wigner [16] in 1933 for the characterization of phase–space uncertainty in quantum mechanics. Since phase–space uncertainty and time–frequency uncertainty are analogous phenomena [17], Wigner's technique was adopted by Ville [18] for time-frequency analysis. The Wigner-Ville distribution can be thought of as a time-varying power spectral density computed as the Fourier transform of a time-varying, instantaneous estimate of an autocorrelation function. This distribution exhibits a number of artifacts, including non-positivity and beat frequencies [5]. Much of the literature surrounding the Wigner-Ville distribution is concerned with ways of dealing with these artifacts. Direct comparisons of the characteristics of linear transforms and time-frequency distributions are rare.

The purpose of this paper is to introduce to the engineering community a linear transform for time-frequency analysis which we call the Phi–transform. This transform was developed by Frazier and Jawerth in 1985 for the characterization of function and distribution spaces [19]–[21]. The Phi–transform was developed independently of the Wavelet transform, but the two have much in common. The Phi–transform, like the Wavelet Transform, relies upon a set of translated–and–dilated versions of two generating functions. It differs in that this set is not orthogonal, and that the analyzing functions (which generate the transform coefficients) are not necessarily the same as the synthesizing functions (used in signal reconstruction). Because there is no orthogonality restriction, the derivation of the Phi–transform requires an entirely different and conceptually simpler approach. Although orthogonality may be desirable in some circumstances, it is not essential in many others

[22]. Like the Wavelet Transform, the Phi–transform is not a single linear operator but rather an entire methodology for linear time-frequency analysis. We intend to show that examples of the Phi–transform are easy to construct, and believe that this flexibility in their construction makes possible the selection of analyzing/synthesizing functions with high resolution in the time-frequency plane.

2. Notation

Throughout, by the constant n we mean the dimension of the Euclidean vector space $\mathbf{R}^n$. $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$, and C^∞, denote the set of all integers (positive, negative, and zero), the set of real numbers, the set of complex numbers, and the set of infinitely–differentiable functions, respectively. $\mathbf{R}^+$ denotes the set of positive real numbers. L^2 and l^2 denote the Hilbert spaces of absolutely square–integrable functions, and of absolutely square–summable sequences, respectively.

If $a, b \in \mathbf{R}^n$ are vectors, then by $a \cdot b$ we mean their scalar (or dot) product. The Euclidean norm of a vector $a = (a_1, a_2, \ldots, a_n)$ will be written $\|a\| = (\sum_{i=1}^n |a_i|^2)^{1/2}$. For $c, d \in \mathbf{R}$, $[c, d]$ is the closed interval from c to d, and $[c,d]^n \triangleq \prod_{i=1}^n [c, d]$. Also,

$$\int_{[c,d]^n} f(x)\,dx \triangleq \int_c^d \ldots \int_c^d f(x_1, \ldots, x_n)\,dx_1 \ldots dx_n. \tag{1}$$

By the *support* of a function $f : \mathbf{R}^n \to \mathbf{C}$ we will mean the topological closure of the set of those points $x \in \mathbf{R}^n$ for which $f(x) \neq 0$. We will write *supp* f for this set. The set *supp* f is *compact* if, for some $r > 0$, *supp* $f \subseteq \{x \in \mathbf{R}^n : \|x\| \leq r\}$. A bar drawn above a complex constant, or a complex–valued function, will denote the complex–conjugate of the constant, or the function. The function $\tilde{f}\,(t)$ is defined to be $\overline{f(-t)}$. If f and g are functions from $\mathbf{R}^n$ to $\mathbf{C}$, then $\langle f, g\rangle$ denotes the inner–product of f and g:

$$\langle f, g\rangle \;\triangleq \int_{\mathbf{R}^n} f(x)\,\overline{g(x)}\,dx = \;\overline{\langle g, f\rangle}. \tag{2}$$

By "$\hat{}$" we denote the forward, by "$^\vee$" the inverse, Fourier transform: $\hat{f}(\omega) = \langle f(t), e^{j\omega\cdot t}\rangle$; $f(t) = (2\pi)^{-n}\langle \hat{f}(\omega), e^{-j\omega\cdot t}\rangle$. The convolution of the functions $f(t)$ and $g(t)$ will be written $(f * g)(t)$.

In what follows we will be working with sets of functions that are all obtained from a single parent function through a process of dilation and translation. We now establish a notation for such functions. Define $\hat{\phi}_\nu(\omega) \triangleq \hat{\phi}(2^{-\nu}\omega)$, $\nu \in \mathbf{R}$, $\omega \in \mathbf{R}^n$, as a dilation of ϕ in the frequency–domain. Define $\phi_\nu(t) \triangleq 2^{n\nu}\phi(2^\nu t)$, $\nu \in \mathbf{R}$, $t \in \mathbf{R}^n$, as a dilation of ϕ in the time–domain. Then $(\phi_\nu)\hat{} = \hat{\phi}_\nu$, and $(\hat{\phi}_\nu)^\vee = \phi_\nu$. Also define $\phi_{\nu k}(t) \triangleq 2^{n\nu/2}\phi(2^\nu t - k)$ as a dilation–and–translation of ϕ in the time domain. Please note: $\phi_{\nu 0}(t) \neq \phi_\nu(t)$. We will call ν the *dilation parameter*, and k the *translation parameter*. This subscript notation for ϕ will also be used for the functions Φ, ψ, Ψ, and θ.

The transform and decomposition discussed here apply to signals that belong to $\mathcal{S}'$, the space of "tempered distributions" (see e.g. [23]). $\mathcal{S}'$ is the dual space of $\mathcal{S}$, the Schwartz space of "smooth and rapidly–decreasing functions". A function f is *rapidly–decreasing* if f and all its partial derivatives decay at infinity at a rate faster than any polynomial.

$\mathcal{S}'$ is a very large space that properly includes L^2, and distributions such as the Dirac–delta and its derivatives. Here, in the interest of simplicity, we will state the results only for $f \in L^2$. We will not consider issues relating to convergence; except to say that when using equations with infinite sums, we will implicitly mean the equality in the special sense of L^2–convergence. That is, if $f_N \triangleq \sum_{i=0}^{N} g_i$, then by $f = \sum_{i=0}^{\infty} g_i$ we mean

$$lim_{N\to\infty} \|f - f_N\|_{L^2}^2 = lim_{N\to\infty} \int_{\mathbf{R}^n} |f - f_N|^2 \ dt = 0. \tag{3}$$

For a detailed study of convergence in $\mathcal{S}'$ and other spaces, of the decomposition discussed in this paper, see [19]–[21].

3. The Psi–decomposition of a Function

Let $\mathrm{S} \triangleq \{\Psi_{mk}, \psi_{\nu k}\}_{\nu,k}$; $m, \nu \in \mathrm{Z}$; m fixed; $\nu > m$; $k \in \mathrm{Z}^n$; be called a set of *synthesizing functions.* The functions Ψ_{mk} in S are the translates (in R^n) of the single function Ψ_{m0}; and the $\psi_{\nu k}$ are all translated–and–dilated versions of a single function ψ. Similarly, let $\mathbf{A} \triangleq \{\Phi_{mk}, \phi_{\nu k}\}_{\nu,k}$ be called a set of *analyzing functions.* All functions in $\mathrm{S} \cup \mathrm{A}$ are defined from R^n to C. We will show that if S and A are appropriately chosen, then any given signal or function, $f : \mathrm{R}^n \to \mathrm{C}$, $f \in L^2$, can be written as follows:

$$f(t) = \sum_{k \in \mathrm{Z}^n} \langle f, \Phi_{mk} \rangle \, \Psi_{mk}(t) \; + \sum_{\nu=m+1}^{\infty} \sum_{k \in \mathrm{Z}^n} \langle f, \phi_{\nu k} \rangle \, \psi_{\nu k}(t). \tag{4}$$

We will call (4) a *Psi–decomposition* of the function *f.* The symbol k in (4) denotes a vector $(k_1, \ldots, k_n) \in \mathrm{Z}^n$. Likewise, the arguments of the functions f, Φ_{mk}, Ψ_{mk}, $\phi_{\nu k}$, and $\psi_{\nu k}$, are vectors in R^n.

The expression (4) is not unlike the Fourier series or the Fourier transform decomposition of f. Since $\hat{f}(\omega) = \langle f, e^{j\omega \cdot t} \rangle$, we can write:

$$f(t) = \int_{\mathrm{R}^n} \langle f(t), e^{j\omega \cdot t} \rangle \, \frac{e^{j\omega \cdot t}}{(2\pi)^n} \, d\omega. \tag{5}$$

In (5) we have the function f written as a weighted sum of certain "elementary" functions — the complex exponentials. The complex exponentials are perfectly localized in the frequency domain, while they range everywhere in the time domain. The weights $\langle f(t), e^{j\omega \cdot t} \rangle$ in (5) are the Fourier coefficients of f, and are related to the projections of the signal f onto the complex exponentials $e^{j\omega \cdot t}$.

In the Psi–decomposition (4) of f, f is written as a weighted sum of the synthesizing functions in S. The weights in (4) constitute a countable sequence $\mathbf{T}f = (\langle f, \Phi_{mk} \rangle, \langle f, \phi_{\nu k} \rangle)$ of complex constants or coefficients. We will call the sequence $\mathbf{T}f$ the *Phi–transform* of *f.* The numbers in the sequence $\mathbf{T}f$ are related to the projections of f upon vectors in the set A of analyzing functions. The complex numbers $\langle f, \phi_{\nu k} \rangle = (2\pi)^{-n} \langle \hat{f}, \hat{\phi}_{\nu k} \rangle$ depend upon the behaviour of f only in those regions of the time and frequency domains where the functions $\phi_{\nu k}$ and $\hat{\phi}_{\nu k}$ are appreciably non–zero. If the functions in **A** are chosen such that they are simultaneously localized in both the time and the frequency domains, then $\mathbf{T}f$ will yield a time–frequency representation of f [24],[25].

The importance of the Psi–decomposition in (4) to applications in signal processing is this: The Psi–decomposition defines a linear, continuous, and invertible transformation that maps a signal f into its Phi–transform $\mathbf{T}f$. The Phi–transform provides a time–frequency representation of a signal. A note about the two terms in (4): we will see that the first term results from a lowpass filter operation, while the second term results from a series of bandpass filter operations.

The theorem at the close of this section establishes the Psi–decomposition of a function, and is proved with the help of two lemmas. The theorem states our main result that if the sets **S** and **A** are chosen as prescribed in Lemma 1, then (4) is true for all functions $f \in L^2$.

Lemma 1. *Given an $m \in \mathbf{Z}$; given $\hat{\phi}(\omega)$ such that the properties* P1($\hat{\phi}$), P2($\hat{\phi}$), *and* P3($\hat{\phi}$), *below are true; given $\hat{\Phi}(\omega)$ such that* P1($\hat{\Phi}$), P4($\hat{\Phi}$), *and* P5($\hat{\Phi}$), *are true; $\exists\hat{\psi}(\omega)$ satisfying* P1($\hat{\psi}$) *and* P2($\hat{\psi}$); *and $\exists\hat{\Psi}(\omega)$ satisfying* P1($\hat{\Psi}$) *and* P4($\hat{\Psi}$); *such that $\forall\omega \in \mathbf{R}^n$,*

$$\overline{\hat{\Phi}_m(\omega)}\,\hat{\Psi}_m(\omega) + \sum_{\nu=m+1}^{\infty} \overline{\hat{\phi}_\nu(\omega)}\,\hat{\psi}_\nu(\omega) \equiv 1. \tag{6}$$

where, for some constant $c \in \mathbf{R}^+$,

P1($\hat{\phi}$): $\hat{\phi} \in C^\infty$.

P2($\hat{\phi}$): supp $\hat{\phi}(\omega) \subseteq \{\omega : \pi/4 \leq \|\omega\| \leq \pi\}$.

P3($\hat{\phi}$): $|\hat{\phi}(\omega)| \geq c$, *for* $\omega \in \{\omega : (3\pi/8) - \epsilon \leq \|\omega\| \leq (3\pi/4) + \epsilon\}$; *some* $\epsilon \in \mathbf{R}^+$.

P4($\hat{\Phi}$): supp $\hat{\Phi}(\omega) \subseteq \{\omega : \|\omega\| \leq \pi\}$.

P5($\hat{\Phi}$): $|\hat{\Phi}(\omega)| \geq c$, *for* $\omega \in \{\omega : \|\omega\| \leq (3\pi/4) + \epsilon\}$; *some* $\epsilon \in \mathbf{R}^+$.

Moreover, the functions Ψ and ψ are rapidly–decreasing.

Before we prove Lemma 1, we would like to explain its statement, informally, as follows: this lemma affirms the existence of functions $\hat{\psi}$ and $\hat{\Psi}$ satisfying (6), when given $\hat{\phi}$ and $\hat{\Phi}$ that cover the frequency–domain as in Figure 1. The property P2($\hat{\phi}$) is designed to ensure a compact support for $\hat{\phi}(\omega)$; P4($\hat{\Phi}$) to do the same for $\hat{\Phi}(\omega)$. P3($\hat{\phi}$) is designed to ensure that the frequency space is nicely covered—that there is no "bald patch" between $\hat{\phi}_\nu$ and $\hat{\phi}_{\nu+1}$, for any ν. P5($\hat{\Phi}$) ensures that $\hat{\Phi}_m$ properly caps the space left uncovered by all the $\hat{\phi}_\nu$, $\nu \in \{m+1, m+2, \ldots\}$. The functions $\hat{\phi}$ and $\hat{\Phi}$ are not required to be radially–symmetric. Figure 1 should be seen as a representative radial slice across the frequency space. Because $\hat{\phi}$ and $\hat{\Phi}$ are C^∞ and have compact support, ϕ and Φ are C^∞ and rapidly–decreasing [23]. This rapidly–decreasing character of ϕ and Φ, together with the compact support of $\hat{\phi}$ and $\hat{\Phi}$, provides us with the means for determining the time–frequency behavior of a signal.

Proof of Lemma 1. Let $\hat{\theta}(\omega)$ be a C^∞ function from R^n to R^+ satisfying P3($\hat{\theta}$); and satisfying *supp* $\hat{\theta}(\omega) \subseteq \{\omega\colon |\hat{\phi}(\omega)| > c/2\} \bigcap \{\omega\colon |\hat{\Phi}(\omega)| > c/2\}$. Because $\hat{\theta}(\omega)$ satisfies P3($\hat{\theta}$), and because its values lie in R^+, we have $\sum_{j=-\infty}^{\infty} \hat{\theta}_j(\omega) \geq c > 0$, $\forall \omega \neq 0$. Define

$$\hat{\psi}(\omega) \triangleq \begin{cases} \hat{\theta}(\omega) \,/\, (\overline{\hat{\phi}(\omega)} \sum_{j=-\infty}^{\infty} \hat{\theta}_j(\omega)) & , \omega \in supp\ \hat{\phi}(\omega) \\ 0 & , \text{otherwise.} \end{cases} \tag{7}$$

Because *supp* $\hat{\theta}(\omega) \subseteq \{\omega\colon |\hat{\phi}(\omega)| > c/2\}$, the ratio in (7) is well–defined. Because $\hat{\theta}$ is compactly supported, and because $\hat{\phi}$ and $\hat{\theta}$ are C^∞, $\hat{\psi}$ is C^∞ with compact support. Hence $\hat{\psi}$ is the Fourier transform of some ψ which is C^∞ and rapidly–decreasing. The function $\hat{\psi}(\omega)$ satisfies P2($\hat{\psi}$) since,

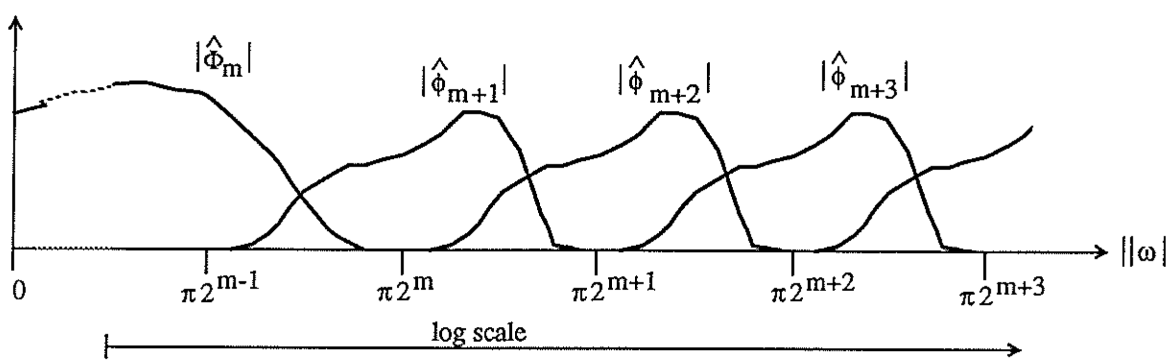

Figure 1: We want the Fourier transforms of the analyzing functions in A to cover the frequency space thus.

by the definition in (7), $supp\,\hat{\psi}(\omega) \subseteq supp\,\hat{\theta}(\omega)$. Define

$$\hat{\Psi}_m(\omega) \triangleq \begin{cases} \left(\sum_{\nu=-\infty}^{m} \hat{\theta}_\nu(\omega)\right) / \left(\overline{\hat{\Phi}_m(\omega)} \sum_{j=-\infty}^{\infty} \hat{\theta}_j(\omega)\right) & , \omega \neq 0, \text{and } \omega \in supp\,\hat{\Phi}_m \\ 0 & , \omega \neq 0, \text{and } \omega \notin supp\,\hat{\Phi}_m \\ 1/\overline{\hat{\Phi}(0)} & , \omega = 0. \end{cases} \tag{8}$$

Because $supp\,\hat{\theta}(\omega) \subseteq \{\omega : |\hat{\Phi}(\omega)| > c/2\}$, the first ratio in (8) is well-defined. Moreover, $\hat{\Psi}(\omega)$ satisfies P4($\hat{\Psi}$), since $supp\,\hat{\Psi}_m(\omega) \subseteq supp\,\hat{\Phi}_m(\omega)$ by the definition in (8). Like ψ, Ψ is C^∞ and rapidly–decreasing. From (7), we have $\forall\omega \neq 0$,

$$\sum_{\nu=m+1}^{\infty} \overline{\hat{\phi}_\nu(\omega)}\hat{\psi}_\nu(\omega) = \sum_{\nu=m+1}^{\infty} \left(\frac{\hat{\theta}_\nu(\omega)}{\sum_{j=-\infty}^{\infty} \hat{\theta}_{j+\nu}(\omega)} \right) = \frac{\sum_{\nu=m+1}^{\infty} \hat{\theta}_\nu(\omega)}{\sum_{j=-\infty}^{\infty} \hat{\theta}_j(\omega)}. \tag{9}$$

From (8), we have $\forall\omega \neq 0$,

$$\overline{\hat{\Phi}_m(\omega)}\hat{\Psi}_m(\omega) = \frac{\sum_{\nu=-\infty}^{m} \hat{\theta}_\nu(\omega)}{\sum_{j=-\infty}^{\infty} \hat{\theta}_j(\omega)}. \tag{10}$$

For $\omega = 0$ the left–hand sides in the equations (9) and (10) are 0 and 1, respectively. Summing these two equations we find that (6) is true for all ω. □

This proof is constructive. Having decided upon a suitable set of analyzing functions, we can build a set of synthesizing functions by following the recipe in the proof. Alternatively, we may find it convenient to choose the analyzing and synthesizing functions directly so as to satisfy (6), without going through the explicit construction suggested in the proof. We will give an example of such a direct choice later in Section 5.

Lemma 2 below establishes a result used in the proof of the main theorem. It uses a technique similar to that used by Shannon [26] in his proof of the sampling theorem.

Lemma 2. *Let* supp $\hat{g}$, supp $\hat{h} \subseteq \{\omega : \|\omega\| \leq \pi 2^\nu\}$, $\nu \in \mathbf{Z}$; $\hat{g} \in L^2$; $\hat{h} \in C^\infty$. *Then for* $s, t \in \mathbf{R}^n$,

$$(g * h)(t) = \int_{\mathbf{R}^n} g(s)\, h(t-s)\, ds = \sum_{k \in \mathbf{Z}^n} 2^{-n\nu} g(k2^{-\nu})\, h(t - k2^{-\nu}). \tag{11}$$

Proof of Lemma 2. By the statement of the lemma, $supp\,\hat{g} \subseteq [-\pi 2^\nu, \pi 2^\nu]^n$. Let $\hat{g}^c$ be a periodic continuation of $\hat{g}$, so that

$$\hat{g}^c(\omega) = \sum_{k \in \mathbf{Z}^n} \hat{g}(\omega - k2\pi 2^\nu). \tag{12}$$

Then $\hat{g}^c(\omega + \Omega) = \hat{g}^c(\omega)$ for $\Omega = \pi 2^{\nu+1} W$; any $W \in \mathbf{Z}^n$. We can now expand $\hat{g}^c(\omega)$ in a Fourier series:

$$\hat{g}^c(\omega) = \sum_{k \in \mathbf{Z}^n} a_k \left(\prod_{i=1}^{n} e^{-j 2\pi k_i \omega_i / \pi 2^{\nu+1}} \right) = \sum_{k \in \mathbf{Z}^n} a_k e^{-j\omega \cdot k 2^{-\nu}}, \tag{13}$$

where

$$a_k = \pi^{-n} 2^{-n(\nu+1)} \int_{[-\pi 2^\nu, \pi 2^\nu]^n} \hat{g}^c(\omega)\, e^{j\omega \cdot k 2^{-\nu}}\, d\omega. \tag{14}$$

Since $\hat{g}(\omega) = \hat{g}^c(\omega)$ within the interval of integration in (14), we can replace $\hat{g}^c$ by $\hat{g}$ in (14) above. Further, since *supp* $\hat{g}(\omega) \subseteq [-\pi 2^\nu, \pi 2^\nu]^n$, we can write (14) as:

$$a_k = \pi^{-n} 2^{-n(\nu+1)} \int_{\mathbf{R}^n} \hat{g}(\omega)\, e^{j\omega \cdot k 2^{-\nu}}\, d\omega = 2^{-n\nu} g(k 2^{-\nu}). \tag{15}$$

Substituting (15) into (13),

$$\hat{g}^c(\omega) = \sum_{k \in \mathbf{Z}^n} 2^{-n\nu} g(k 2^{-\nu}) e^{-j\omega \cdot k 2^{-\nu}}. \tag{16}$$

Since *supp* $\hat{h}(\omega) \subseteq \{\omega : \|\omega\| \leq \pi 2^\nu\}$,

$$(g * h)(t) = (\hat{g}\hat{h})^\vee(t) = (\hat{g}^c \hat{h})^\vee(t). \tag{17}$$

From (16) and (17) we get the desired result:

$$\begin{aligned}
(g * h)(t) &= \left(\sum_{k \in \mathbf{Z}^n} 2^{-n\nu} g(k 2^{-\nu}) e^{-j\omega \cdot k 2^{-\nu}} \hat{h}(\omega) \right)^\vee (t) && (18)\\
&= \sum_{k \in \mathbf{Z}^n} 2^{-n\nu} g(k 2^{-\nu}) \left(\hat{h}(\omega) e^{-j\omega \cdot k 2^{-\nu}} \right)^\vee (t) && (19)\\
&= \sum_{k \in \mathbf{Z}^n} 2^{-n\nu} g(k 2^{-\nu}) h(t - k 2^{-\nu}). && (20)
\end{aligned}$$

$\square$

The assumption $\hat{h} \in \mathcal{C}^\infty$ in the statement of Lemma 2 guarantees, for example, that the convolution $g * h$ is well–defined. If stronger assumptions are made on $\hat{g}$, the same result can be obtained under assumptions on $\hat{h}$ that are weaker than $\hat{h} \in \mathcal{C}^\infty$. Some technicalities of the proof have been omitted. These involve approximating $\hat{g}$ by a $\mathcal{C}^\infty$ function to obtain the convergence of (16), and passing to the limit. For details see [19].

Theorem 1 (see [19], p. 780). *Given $\hat{\phi}$, $\hat{\Phi}$, $\hat{\psi}$, and $\hat{\Psi}$, satisfying all conditions in the statement of Lemma 1; and given a function $f \in L^2$, $f : \mathrm{R}^n \to \mathrm{C}$; we can write:*

$$f = \sum_{k \in \mathrm{Z}^n} \langle f, \Phi_{mk} \rangle \Psi_{mk} \ + \sum_{\nu=m+1}^{\infty} \sum_{k \in \mathrm{Z}^n} \langle f, \phi_{\nu k} \rangle \psi_{\nu k}. \tag{21}$$

Proof of Theorem 1. From Lemma 1, $\overline{\hat{\Phi}_m}\, \hat{\Psi}_m + \sum_{\nu=m+1}^{\infty} \overline{\hat{\phi}_\nu}\, \hat{\psi}_\nu \equiv 1$. Taking the inverse Fourier transform,

$$\tilde{\Phi}_m * \Psi_m + \sum_{\nu=m+1}^{\infty} \tilde{\phi}_\nu * \psi_\nu = \delta, \tag{22}$$

where $\tilde{g}\,(t) = \overline{g(-t)}$. Convolving both sides with f,

$$f = f * \tilde{\Phi}_m * \Psi_m + \sum_{\nu=m+1}^{\infty} f * \tilde{\phi}_\nu * \psi_\nu. \tag{23}$$

Since $supp\ (f * \tilde{\Phi}_m)\hat{}\ \subseteq\ supp\ \hat{\Phi}_m;\ supp\ \hat{\Phi}_m, supp\ \hat{\Psi}_m \subseteq \{\omega : \|\omega\| \le \pi 2^m\}$; $(f * \tilde{\Phi}_m)\hat{} \in L^2$; and $\hat{\Psi}_m \in \mathcal{C}^\infty$; we have (using Lemma 2):

$$(f * \tilde{\Phi}_m) * \Psi_m = \sum_{k \in \mathrm{Z}^n} 2^{-nm} (f * \tilde{\Phi}_m)(k2^{-m})\ \Psi_m(t - k2^{-m}). \tag{24}$$

Simplifying the convolution within the summation above,

$$\begin{aligned} (f * \tilde{\Phi}_m)(k2^{-m}) &= (f * \tilde{\Phi}_m)(t)\Big|_{t=k2^{-m}} && (25) \\ &= \int_{\mathrm{R}^n} f(t')\, \overline{\Phi_m(t' - k2^{-m})}\, dt' = 2^{nm/2} \langle f, \Phi_{mk} \rangle. && (26) \end{aligned}$$

Substituting (26) into (24),

$$f * \tilde{\Phi}_m * \Psi_m = \sum_{k \in \mathrm{Z}^n} 2^{-nm/2} \langle f, \Phi_{mk} \rangle\, \Psi_m(t - k2^{-m}) = \sum_{k \in \mathrm{Z}^n} \langle f, \Phi_{mk} \rangle\, \Psi_{mk}. \tag{27}$$

Similarly,

$$f * \tilde{\phi}_\nu * \psi_\nu = \sum_{k \in \mathrm{Z}^n} \langle f, \phi_{\nu k} \rangle\, \psi_{\nu k}. \tag{28}$$

Substituting (27) and (28) into (23) we get the desired result. □

The equality in the Psi–decomposition of (21) is meant in the sense of (3) for $f \in L^2$, where

$$f_N \triangleq \sum_{k \in \mathbf{Z}^n} \langle f, \Phi_{mk} \rangle \Psi_{mk} + \sum_{\nu=m+1}^{N} \sum_{k \in \mathbf{Z}^n} \langle f, \phi_{\nu k} \rangle \psi_{\nu k}. \tag{29}$$

In fact, as in the case of the Fourier transform [23], if it is assumed that $\int_{\mathbf{R}^n} |f(x)|\, dx < \infty$ and $\int_{\mathbf{R}^n} |\hat{f}(\omega)|\, d\omega < \infty$, then the representation of f in (21) holds pointwise: $f(x) = lim_{N \to \infty} f_N(x)$, for every $x \in \mathbf{R}^n$. If f is not continuous, the convergence of f_N to f cannot be uniform. In this case the representation (29) will exhibit Gibbs phenomenon.

4. Some Properties of the Psi–decomposition

In this section we discuss a few interesting properties of the Psi–decomposition. These properties concern the non–orthogonality of the Psi–decomposition, a Parseval–like theorem for the Phi–transform representation, and the possible real–valuedness of the Phi–transform coefficients.

4.1. Non–orthogonality

Theorem 2 . *If the set* A *of analyzing functions and the set* S *of synthesizing functions are defined according to the conditions in Lemma 1, then neither* A *nor* S *is orthogonal.*

Proof of Theorem 2. Assume $\mathbf{A}$ is orthogonal. We will derive a contradiction. Define

$$\phi^* \triangleq \frac{\phi}{||\phi||} \tag{30}$$

$$\psi^* \triangleq ||\phi||\,\psi. \tag{31}$$

Let the set $\mathbf{A}^*$ be obtained from $\mathbf{A}$ by replacing the functions $\phi_{\nu k}$ with $\phi^*_{\nu k}$; $\nu \in (m+1),\ldots$; $k \in \mathbf{Z}^n$. Similarly, let $\mathbf{S}^*$ be obtained from $\mathbf{S}$ by replacing the functions $\psi_{\nu k}$ with $\psi^*_{\nu k}$. The sets $\mathbf{A}^*$ and $\mathbf{S}^*$ also obey the conditions in Lemma 1. Moreover, $\mathbf{A}^*$ is orthogonal. By Theorem 1, for any $f \in L^2(\mathbf{R}^n)$,

$$f = \sum_{k\in\mathbf{Z}^n} \langle f, \Phi_{mk}\rangle \Psi_{mk} + \sum_{\nu=m+1}^{\infty} \sum_{k\in\mathbf{Z}^n} \langle f, \phi_{\nu k}\rangle \psi_{\nu k}. \tag{32}$$

Substituting $f = \phi_{\nu_0 k_0}$, $\nu_0 \in \{(m+1),\ldots\}$, $k_0 \in \mathbf{Z}$, in (32); and by the orthogonality of $\mathbf{A}$; we have

$$\phi_{\nu_0 k_0} = ||\phi_{\nu_0 k_0}||^2\, \psi_{\nu_0 k_0} = ||\phi||^2\, \psi_{\nu_0 k_0}. \tag{33}$$

From (30), (31), and (33) it follows that

$$\phi^* = \psi^*, \text{ and} \tag{34}$$

$$||\phi^*|| = 1. \tag{35}$$

Applying equation (6) in Lemma 1 to the sets $\mathbf{A}^*$ and $\mathbf{S}^*$ we have, $\forall\omega$,

$$\overline{\hat{\Phi}_m(\omega)}\,\hat{\Psi}_m(\omega) + \sum_{\nu=m+1}^{\infty} |\hat{\phi}^*_\nu(\omega)|^2 = 1. \tag{36}$$

Substituting $f = \Phi_{m_0 0}$ in (32), $\Phi_{m_0 0} = ||\Phi||^2 \Psi_{m_0 0}$ by the orthogonality of **A**. Hence the first term in (36) is greater than or equal to zero for all ω; which implies that for all ν and ω,

$$|\hat{\phi}^*_\nu(\omega)| \leq 1. \tag{37}$$

By condition P2($\hat{\phi}^*$) of Lemma 1,

$$supp\, \hat{\phi}^* \subseteq \{\omega : \pi/4 \leq ||\omega|| \leq \pi\}. \tag{38}$$

The volume of an n–dimensional sphere of radius r is (see e.g. [27], p. 620)

$$\frac{\sqrt{\pi^n}\, r^n}{\Gamma(n/2+1)}, \tag{39}$$

where Γ is the gamma function. From (37), (38), and (39), it follows that

$$||\hat{\phi}^*||^2 \leq \frac{\sqrt{\pi^n}}{\Gamma(n/2+1)} (\pi^n - (\pi/4)^n) \qquad \Rightarrow \qquad ||\phi^*||^2 \leq \frac{\pi^{3n/2}\,(1-4^{-n})}{(2\pi)^n\, \Gamma(n/2+1)} < 1. \tag{40}$$

The inequality (40) contradicts equation (35), hence **A** cannot be orthogonal.

Assume now that **S** is orthogonal. Again, we will derive a contradiction. From (32), by the orthogonality of **S**,

$$\langle f, \psi_{\nu_1 k_1}\rangle = \langle f, \phi_{\nu_1 k_1}\rangle ||\psi_{\nu_1 k_1}||^2 = \langle f, \phi_{\nu_1 k_1}\rangle ||\psi||^2; \text{ and} \tag{41}$$

$$\langle f, \psi_{\nu_2 k_2}\rangle = \langle f, \phi_{\nu_2 k_2}\rangle ||\psi||^2. \tag{42}$$

Substituting $f = \psi_{\nu_2 k_2}$ in (41), and $f = \phi_{\nu_1 k_1}$ in (42):

$$\langle \psi_{\nu_2 k_2}, \psi_{\nu_1 k_1}\rangle = ||\psi||^2 \langle \psi_{\nu_2 k_2}, \phi_{\nu_1 k_1}\rangle; \text{ and} \tag{43}$$

$$\langle \phi_{\nu_1 k_1}, \psi_{\nu_2 k_2}\rangle = ||\psi||^2 \langle \phi_{\nu_1 k_1}, \phi_{\nu_2 k_2}\rangle. \tag{44}$$

From (43) and (44), for $(\nu_1, k_1) \neq (\nu_2, k_2)$,

$$\langle \phi_{\nu_1 k_1}, \phi_{\nu_2 k_2}\rangle = ||\psi||^{-2} \langle \phi_{\nu_1 k_1}, \psi_{\nu_2 k_2}\rangle = ||\psi||^{-2}\, \overline{\langle \psi_{\nu_2 k_2}, \phi_{\nu_1 k_1}\rangle} \tag{45}$$

$$= ||\psi||^{-4}\, \overline{\langle \psi_{\nu_2 k_2}, \psi_{\nu_1 k_1}\rangle} = ||\psi||^{-4} \langle \psi_{\nu_1 k_1}, \psi_{\nu_2 k_2}\rangle = 0. \tag{46}$$

It follows that the orthogonality of **S** requires the orthogonality of **A** which has been shown to be impossible. □

Theorem 1 says that the set $S = \{\Psi_{mk}, \psi_{\nu k}\}_{\nu,k}$ is *complete* in L^2, i.e. any function in L^2 can be written as a sum of elements in S. However, S is not a basis for L^2, and every function can be written as a sum of the elements of S in an infinity of different ways. Yet Theorem 1 defines a single-valued function from L^2 to l^2, where the image of f is determined uniquely by the inner-products in (21).

4.2. Possible Equality of Analyzing and Synthesizing Functions

While neither of the sets A or S can be orthogonal as constructed in Lemma 1, these sets could be designed to satisfy the condition A = S. To do this, let $\hat{\Phi}(\omega)$ satisfy the following properties (see Fig. 2):

Q1: $\hat{\Phi}(\omega) \in \mathrm{R}$, for $\omega \in \mathrm{R}^n$.

Q2: *supp* $\hat{\Phi}(\omega) \subseteq \{\omega\colon \|\omega\| \leq \omega_1\}$; some $\omega_1 \in \mathrm{R}^+$.

Q3: $\hat{\Phi}(\omega) \equiv 1$, for $\omega \in \{\omega\colon \|\omega\| \leq \omega_2\}$; $\omega_1 > \omega_2 \in \mathrm{R}^+$.

Q4: $\|\omega'\| < \|\omega''\| \Rightarrow \hat{\Phi}(\omega'') \leq \hat{\Phi}(\omega')$.

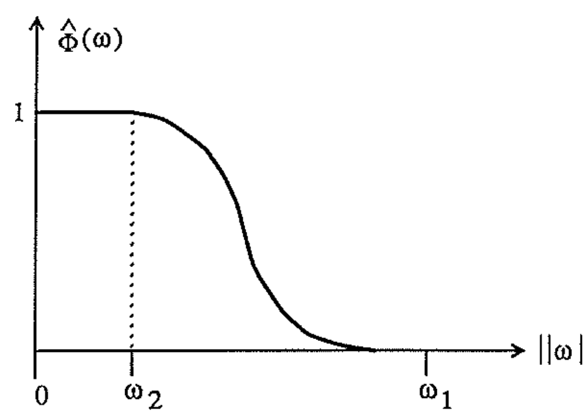

Figure 2: A $\hat{\Phi}(\omega)$ that satisfies properties Q1–Q4.

Define $\hat{\phi}^2(\omega) \triangleq \hat{\Phi}^2(2^{-1}\omega) - \hat{\Phi}^2(\omega)$, for $\omega \in \mathrm{R}^n$. By the monotonicity property Q4, we know that $\hat{\phi}(\omega)$ is real. Then,

$$\sum_{\nu=0}^{N} |\hat{\phi}_\nu(\omega)|^2 = \sum_{\nu=0}^{N} \hat{\phi}_\nu^2(\omega) = \sum_{\nu=0}^{N} \left[\hat{\Phi}^2(2^{-(\nu+1)}\omega) - \hat{\Phi}^2(2^{-\nu}\omega)\right] \tag{47}$$

$$= \hat{\Phi}^2(2^{-(N+1)}\omega) - \hat{\Phi}^2(\omega). \tag{48}$$

Define $K_N(\omega) \triangleq \hat{\Phi}^2(\omega) + \sum_{\nu=0}^{N} |\hat{\phi}_\nu(\omega)|^2$. Then by (48), $K_N(\omega) = \hat{\Phi}^2(2^{-(N+1)}\omega)$. By property Q3, $K_N(\omega) = \hat{\Phi}^2(2^{-(N+1)}\omega) \equiv 1$, for $\omega \in \{\omega \colon ||\omega|| \leq 2^{N+1}\omega_2\}$. Therefore,

$$K_\infty(\omega) = |\hat{\Phi}(\omega)|^2 + \sum_{\nu=0}^{\infty} |\hat{\phi}_\nu(\omega)|^2 \equiv 1, \text{ for } \omega \in \mathbf{R}^n. \tag{49}$$

Equation (49) is just like equation (6) with $\Phi = \Psi$, and $\phi = \psi$.

4.3. Conservation of Energy

We now consider an energy–conservation property of the Phi–transform.

Theorem 3 (Parseval–like) . *If* A *and* S *are constructed as in the statement of Lemma 1; if, further,* $\Phi = \Psi$ *and* $\phi = \psi$*; and if* Tf *denotes the transform sequence* $(\langle f, \Phi_{mk}\rangle, \langle f, \phi_{\nu k}\rangle)$ *of a function* f*; then* $||\mathbf{T}f||_{l^2} = ||f||_{L^2}$.

Proof of Theorem 3. From (21),

$$||f||_{L^2}^2 = \langle f, f\rangle = \left\langle \left(\sum_{k \in \mathbf{Z}^n} \langle f, \Phi_{mk}\rangle \Psi_{mk} + \sum_{\nu=m+1}^{\infty} \sum_{k \in \mathbf{Z}^n} \langle f, \phi_{\nu k}\rangle \psi_{\nu k} \right), f \right\rangle \tag{50}$$

$$= \sum_{k \in \mathbf{Z}^n} \langle f, \Phi_{mk}\rangle\langle \Phi_{mk}, f\rangle + \sum_{\nu=m+1}^{\infty} \sum_{k \in \mathbf{Z}^n} \langle f, \phi_{\nu k}\rangle\langle \phi_{\nu k}, f\rangle \tag{51}$$

$$= \sum_{k \in \mathbf{Z}^n} |\langle f, \Phi_{mk}\rangle|^2 + \sum_{\nu=m+1}^{\infty} \sum_{k \in \mathbf{Z}^n} |\langle f, \phi_{\nu k}\rangle|^2 \tag{52}$$

$$= \langle \mathbf{T}f, \mathbf{T}f\rangle_{l^2} = ||\mathbf{T}f||_{l^2}^2. \tag{53}$$

□

In general (i.e. for $\Phi \neq \Psi$, $\phi \neq \psi$) it can be shown that the Phi–transform of a signal is *norm–equivalent* to the signal; i.e. $||\mathbf{T}f||_{l^2} \approx ||f||_{L^2}$. By this it is meant that there exist constants C_1 and C_2 in $\mathbf{R}$, such that for all $f \in L^2$,

$$||f||_{L^2} \leq C_1||\mathbf{T}f||_{l^2}, \text{ and } ||\mathbf{T}f||_{l^2} \leq C_2||f||_{L^2}. \tag{54}$$

4.4. Real–valued Phi–transforms

Lastly we note that if the function f is real–valued, and if the set A consists of real–valued functions, then the Phi–transform sequence Tf of f consists of real numbers.

5. Examples of the Phi–transform

Lemma 1 permits considerable flexibility in the choice of the sets **A** and **S** of analyzing and synthesizing functions. This means that the Phi–transform is actually an entire family of transforms. In this section we give specific examples of the sets **A** and **S**, and of the transformation and reconstruction of one–dimensional continuous–time signals. Through these examples, we intend to demonstrate the simplicity of the Phi–transform. This is in contrast to the Wavelet transform where the orthogonality conditions place severe restrictions upon on the allowable synthesizing functions, and make them difficult to construct.

In our first example we choose the sets **A** and **S** to satisfy (6) *a priori*; instead of constructing them according to the prescription in Lemma 1. If we choose the "window–function" $\hat{\theta}(\omega)$ such that it is a raised cosine pulse whose argument is the logarithm of the frequency variable,

$$\hat{\theta}(\omega) \triangleq \begin{cases} 1/2(1-\cos(\pi \log_2 \|\omega\|)) & , \pi/4 \leq \|\omega\| \leq \pi \\ 0 & , \text{otherwise} \end{cases} \tag{55}$$

then the sum of all the dilations of $\hat{\theta}$ is 1, for all $\omega \neq 0$. Each of the functions $\hat{\phi}$ and $\hat{\psi}$ can now be chosen to be the square–root of $\hat{\theta}$; and $\hat{\phi} = \hat{\psi}$. The DC cap functions $\hat{\Phi}$ and $\hat{\Psi}$ can be designed to satisfy (6), and again we choose $\hat{\Phi} = \hat{\Psi}$. We should point out that the functions $\hat{\phi}$, $\hat{\Phi}$, $\hat{\psi}$, and $\hat{\Psi}$ in this first example do not belong to C^∞ as would functions chosen according to the prior conditions in the statement of Lemma 1, and those constructed by the prescription in its proof. Indeed, $\hat{\phi}$, $\hat{\Phi}$, $\hat{\psi}$, and $\hat{\Psi}$ as designed here have discontinuous first derivatives. However, for well–behaved functions f, the C^∞ conditions on analyzing and synthesizing functions can be weakened.

The sets **S**=**A** in this first example are constructed first by choosing an integer m; then by computing the translates $\{\Phi_{mk} : k \in \mathbf{Z}\}$; then by computing the translates and dilates $\{\phi_{\nu k} : \nu = m+1, m+2, \ldots;$ and $k \in \mathbf{Z}\}$. This completes the construction of a set of analyzing and synthesizing functions. In Figures 3 and 4 we plot the functions $\Phi_m, \phi_{m+1}, \phi_{m+2}, \ldots, \phi_1$, for the choice $m = -5$ in the frequency and time domains respectively.

In our second example we look at the problem of deriving a set of analyzing and synthesizing functions which are optimal in some sense. We have noted that the Phi–transform is important in that it can provide a time–frequency description of a signal. Better simultaneous localization of ϕ and Φ in the frequency and the time domains would yield a better time–frequency description. As is well–known, a function cannot be compactly supported in both domains. However, it is possible to pose an optimization problem in which the function is compactly supported in one domain and is "concentrated" in the other. Examples of this kind of function are the prolate spheroidal wave function [28], [29], which are strictly bandlimited on the frequency interval $[-B, B]$, and have the maximum fraction of their energy in the time interval $[-T/2, T/2]$.

In this example, we have constructed analyzing functions which are solutions to the eigenvalue problem [24]:

$$P_B P_T \phi = \lambda \phi. \tag{56}$$

Here P_B is a projection operator onto the space of bandpass bandlimited functions, and P_T is a projection operator onto the space of time–limited functions. It can be seen that such a function solves the following problem: find a bandlimited function with given spectral support with the maximum fraction of energy in a given time interval. An earlier and more detailed account of the construction and the properties of such concentrated functions and of equations like (56) can be found in [30].

If P_B and P_T describe projections onto vectors that are characteristic functions of an annulus and a disk, respectively, then the functions ϕ and Φ generated as solutions to (56) are not continuous in the frequency domain, and hence not well–localized in the time domain. We can smooth $\hat{\phi}$ and $\hat{\Phi}$ by convolving them with a compactly supported pulse in the class C^∞, of the form

$$g(\omega) = \begin{cases} ce^{-(\omega+k)^{-2}} e^{-(\omega-k)^{-2}} & , -k \le \omega \le k \\ 0 & , \text{otherwise} \end{cases} \tag{57}$$

where $c, k \in \mathbf{R}^+$ are some constants. This smoothing operation yields new functions $\hat{\phi}$ and $\hat{\Phi}$ which are C^∞, yet closely approximate the original, optimally–concentrated, functions. The functions $\Phi_{-5}, \phi_{-4}, \ldots, \phi_1$ derived from smoothed solutions to (56) are plotted in the frequency and the time domains in Figures 5 and 6. The set $\mathbf{S}$ of synthesizing functions for this second example can be derived from the set $\mathbf{A}$ defined by ϕ and Φ by following the procedure outlined in the proof of Lemma 1.

In both the above constructions the analyzing functions are real and symmetric in both frequency and time domains. The localization of the analyzing functions in the time domain is comparable in both of the examples above. In Figures 7–10 we show the analysis and synthesis of two signals, carried out with the cosine–of–log analyzing and synthesizing functions. The time axis k points to the right, while the frequency axis ν points down. In Figures 7 and 9, the number of Phi–transform coefficients at frequency–level ν is half that at level $(\nu + 1)$. The Phi–transform coefficients are all real since f and the elements of $\mathbf{A}$ are real. In the reconstruction pictures in Figures 8 and 10 we plot the partial sums (29) for $m = -5$, and $N = -5$(top) to $N = 1$(bottom).

In Figure 10, notice that the high–frequency information required to pass from the $N = -2$ reconstruction to the $N = -1$ reconstruction is contained in a small number of numerically significant coefficients. This is in contrast to the usual Fourier reconstruction methods, where in general twice as many sample values are required to double the frequency range. This indicates that for signals with localized high–frequency components, it is reasonable to expect the Phi–transform to yield data compression.

Lastly, in Figure 11, we plot the difference between the Phi–transform coefficients of the two signals at the top. The signals differ at a very few points, and their Phi-transform differs at a few points too. This behavior of the Phi–transform is in sharp contrast to Fourier analysis, where change at a single point in a signal reverberates across the entire spectrum. By linearity, the Phi–transform of the difference of two signals equals the difference of the Phi–transforms of the two signals.

Because we are discussing the continuous Psi–decomposition in the space $L^2(\mathrm{R}^n)$ in this paper, we have done all computations with Reimann sum approximations to integral equations. This is not a practical method of computation. For discrete–time signal processing, a formalism akin to the continuous Phi–transform discussed here is required that works for signals in $l^2(\mathbf{Z}^n)$ and $\mathbf{C}^n$. Preliminary work in this area is described in [25]. A full treatment is the subject of forthcoming work.

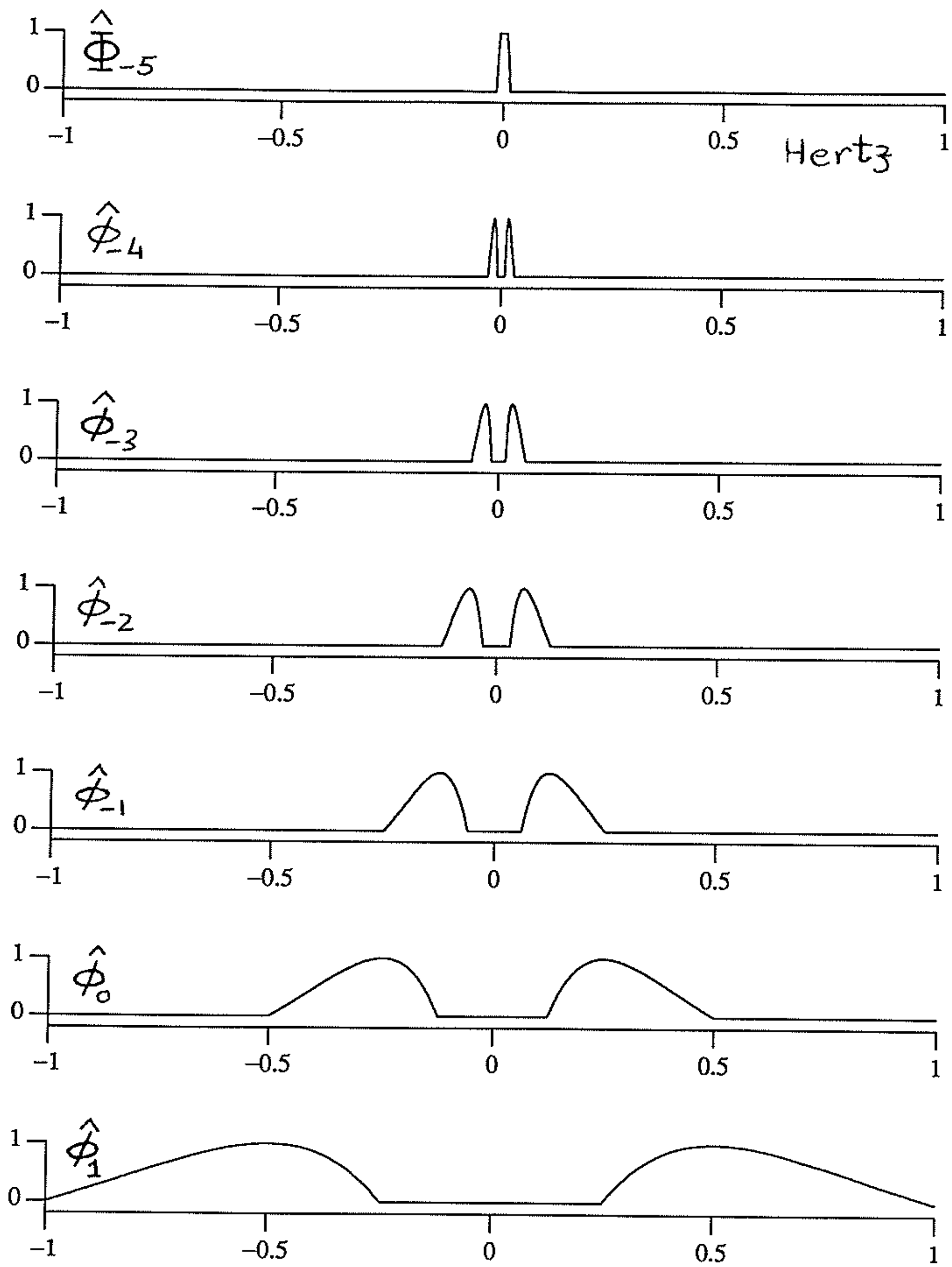

Figure 3: The analyzing/synthesizing functions generated as the cosines–of–logs are plotted here in the frequency domain.

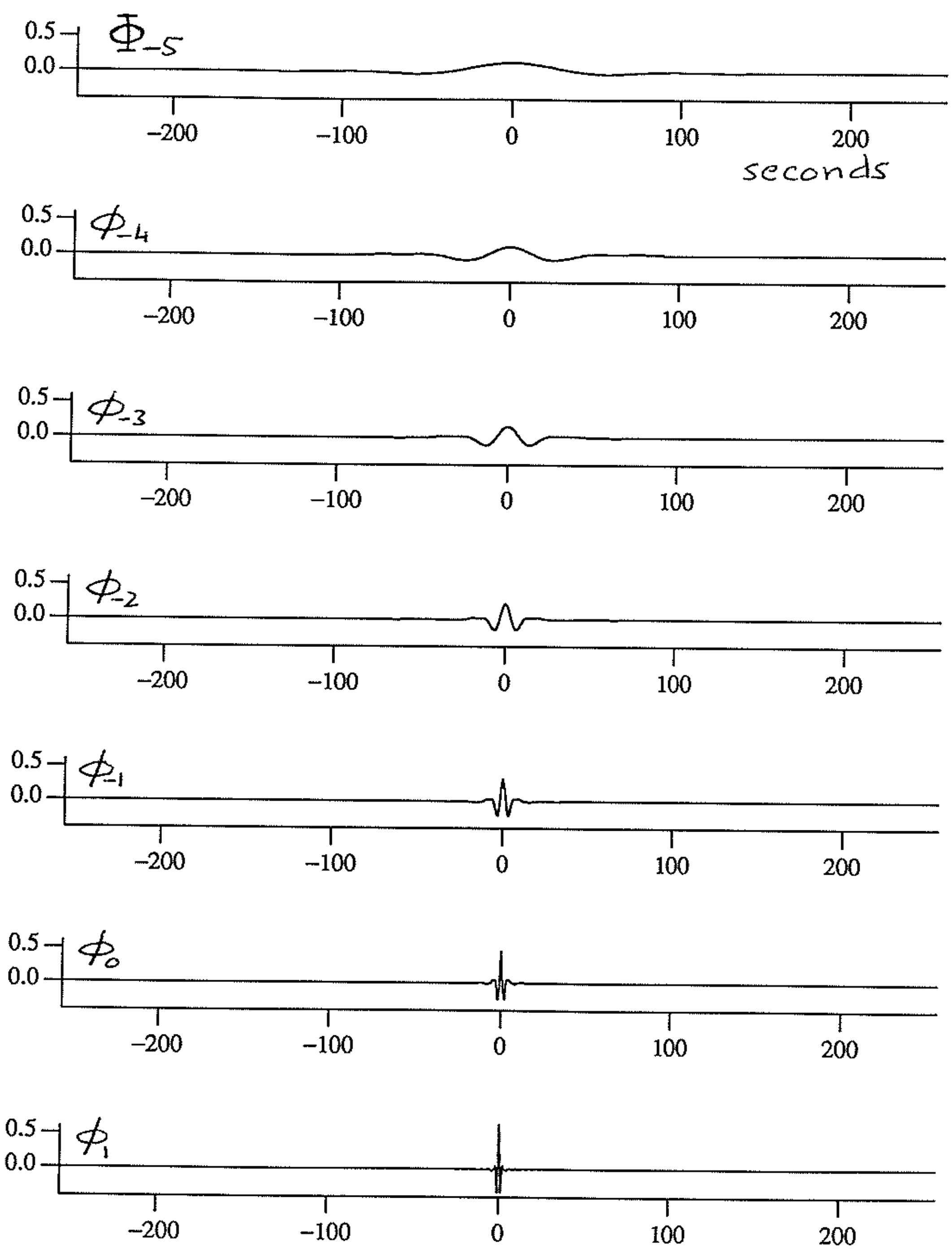

Figure 4: Cosine–of–log analyzing/synthesizing functions in the time domain.

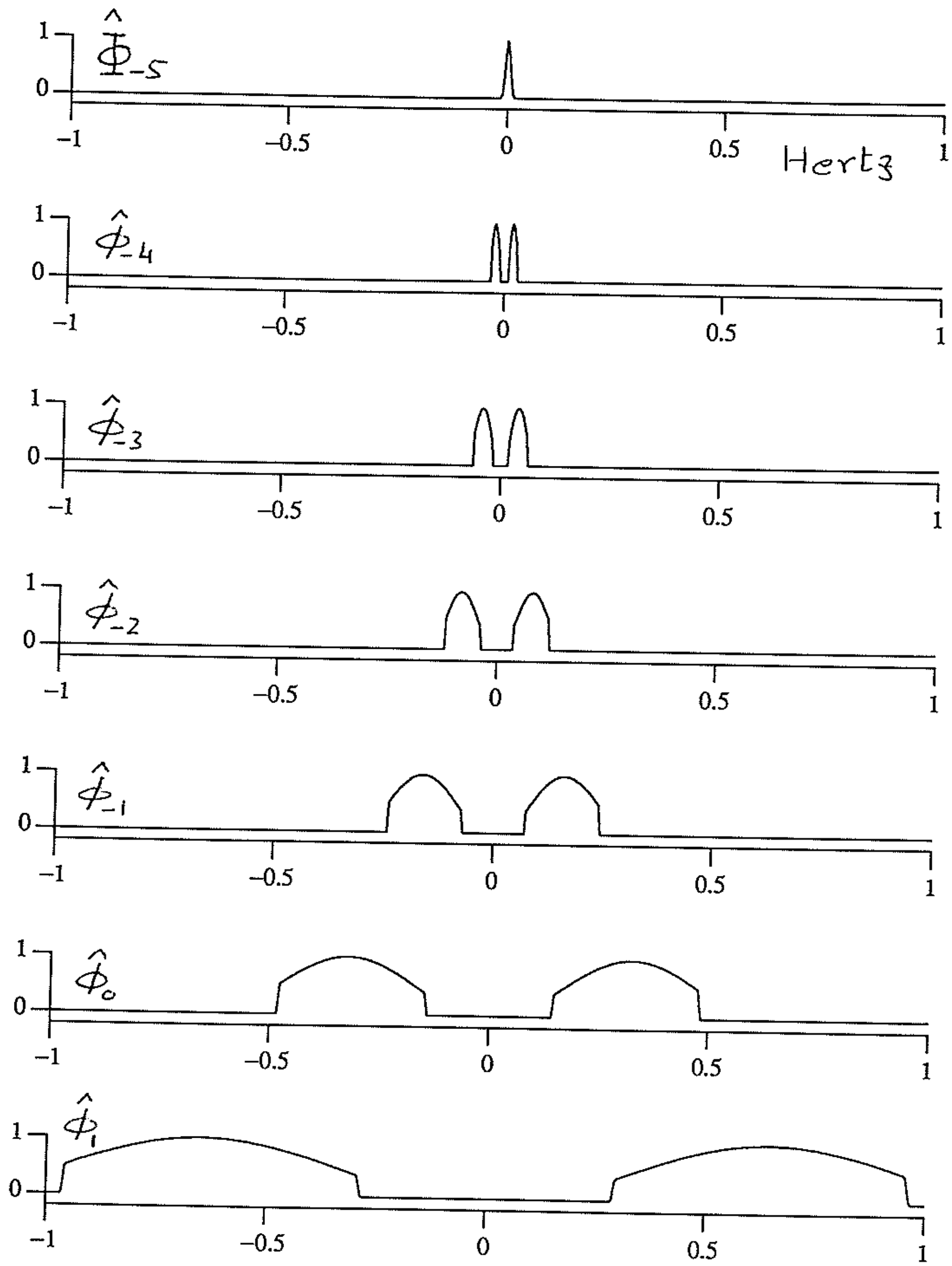

Figure 5: The smoothed–eigenfunction analyzing functions in the frequency domain.

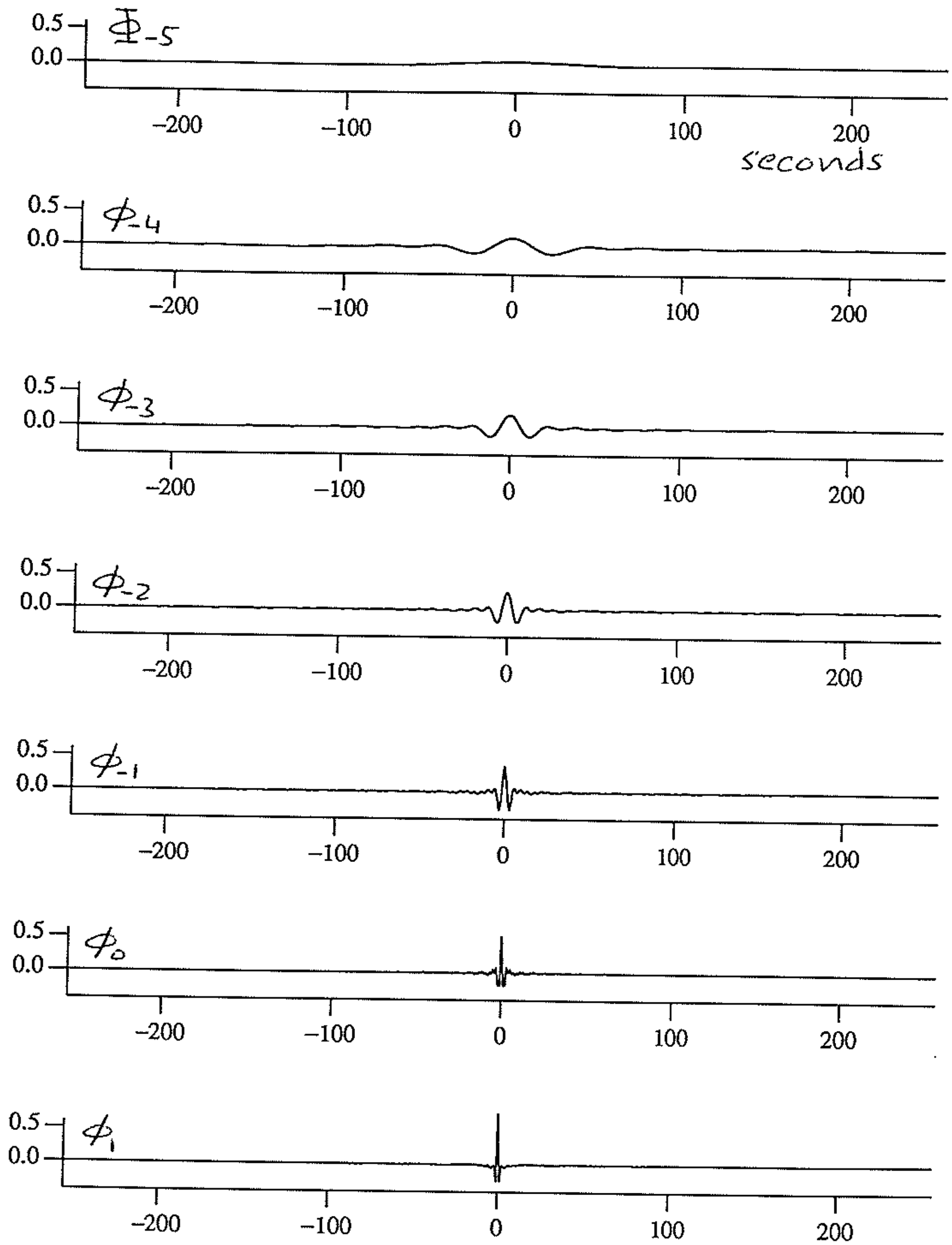

Figure 6: The smoothed–eigenfunction analyzing functions in the time domain.

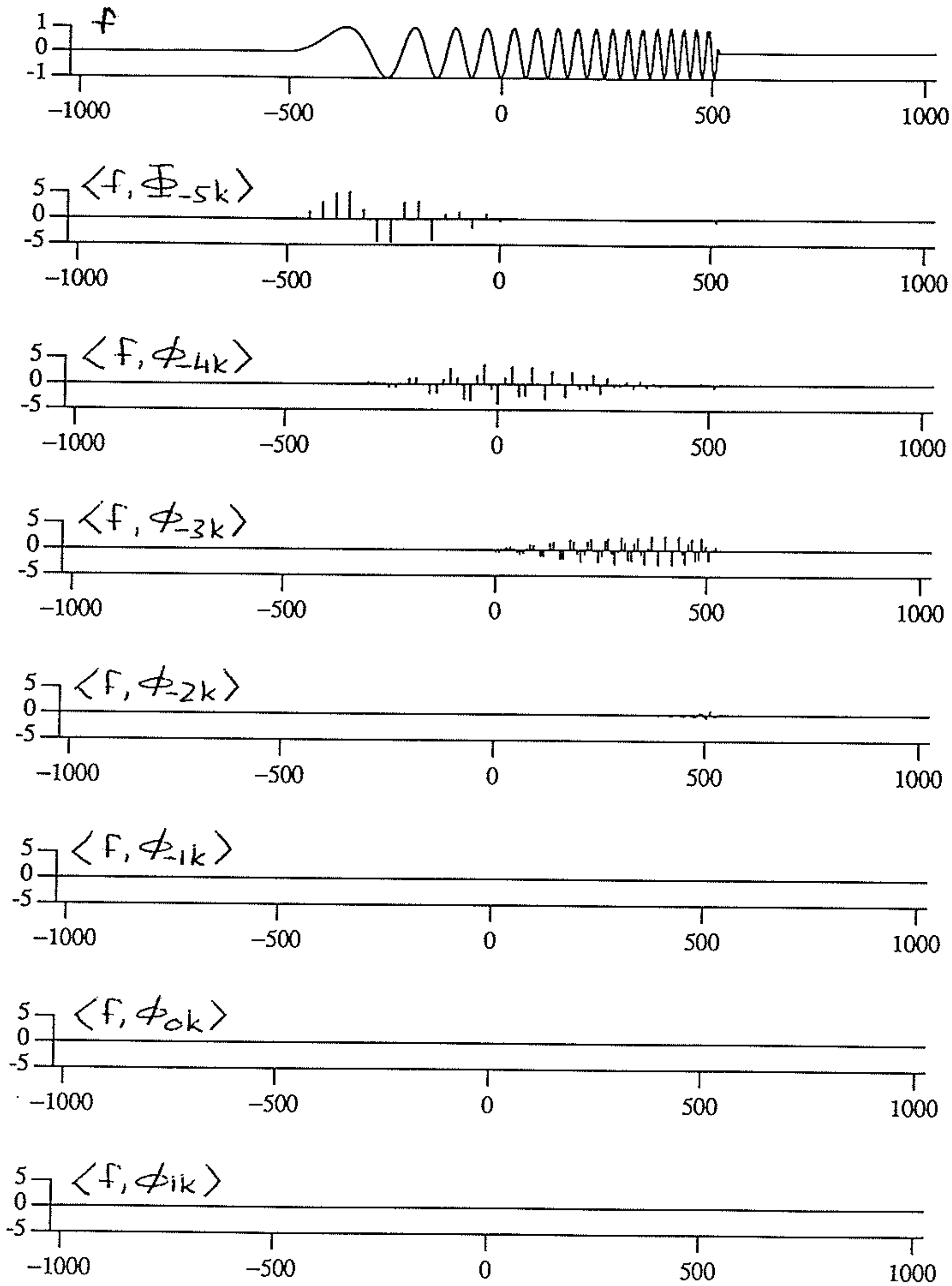

Figure 7: The Phi–transform coefficients of a chirp are computed as the inner–products of the chirp with the analyzing functions.

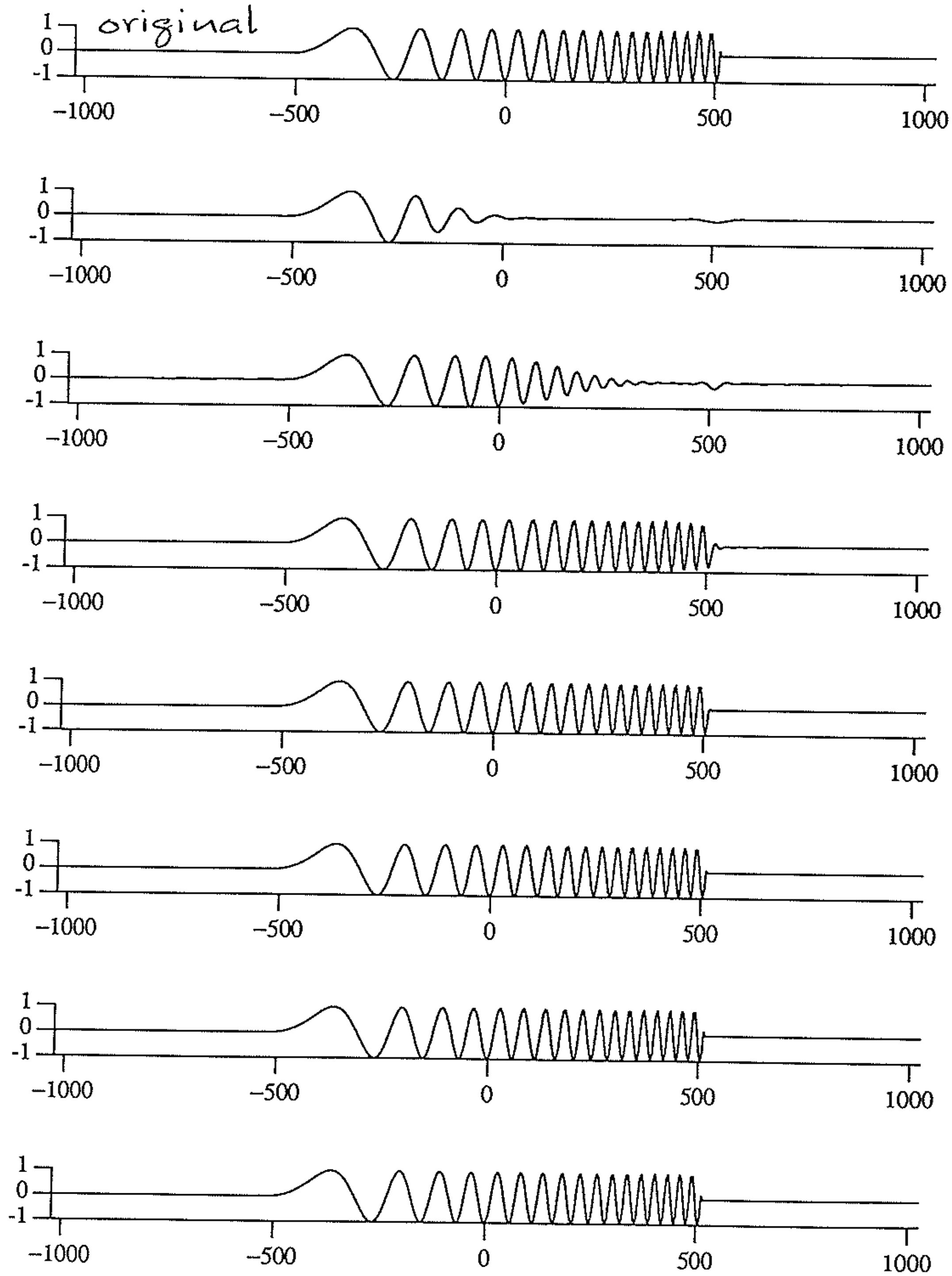

Figure 8: The reconstruction of the chirp from its Phi–transform coefficients in Figure 7. The plot at the top is the original signal. Partial reconstructions are plotted below the original signal.

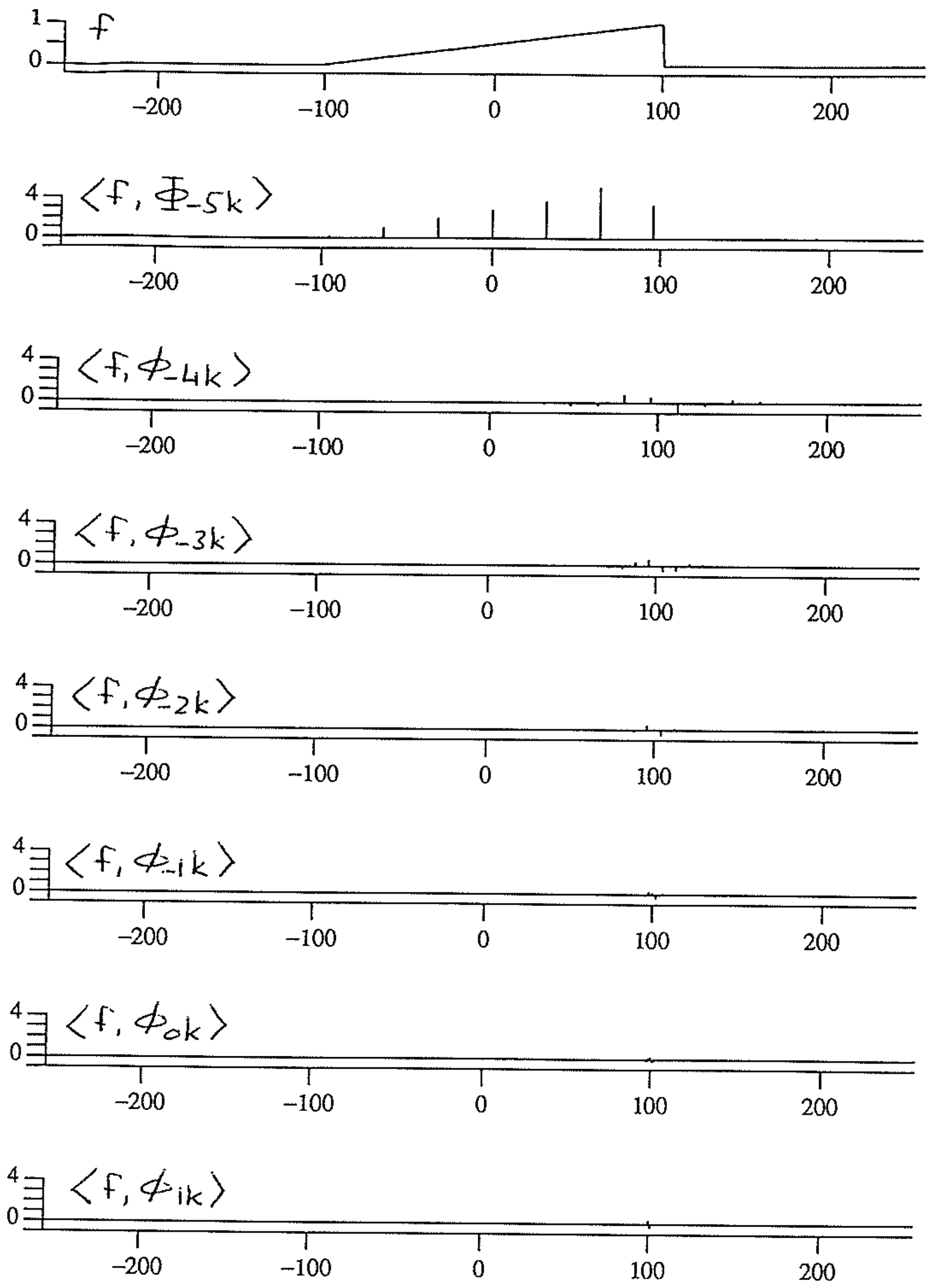

Figure 9: The Phi–transform coefficients of a ramp.

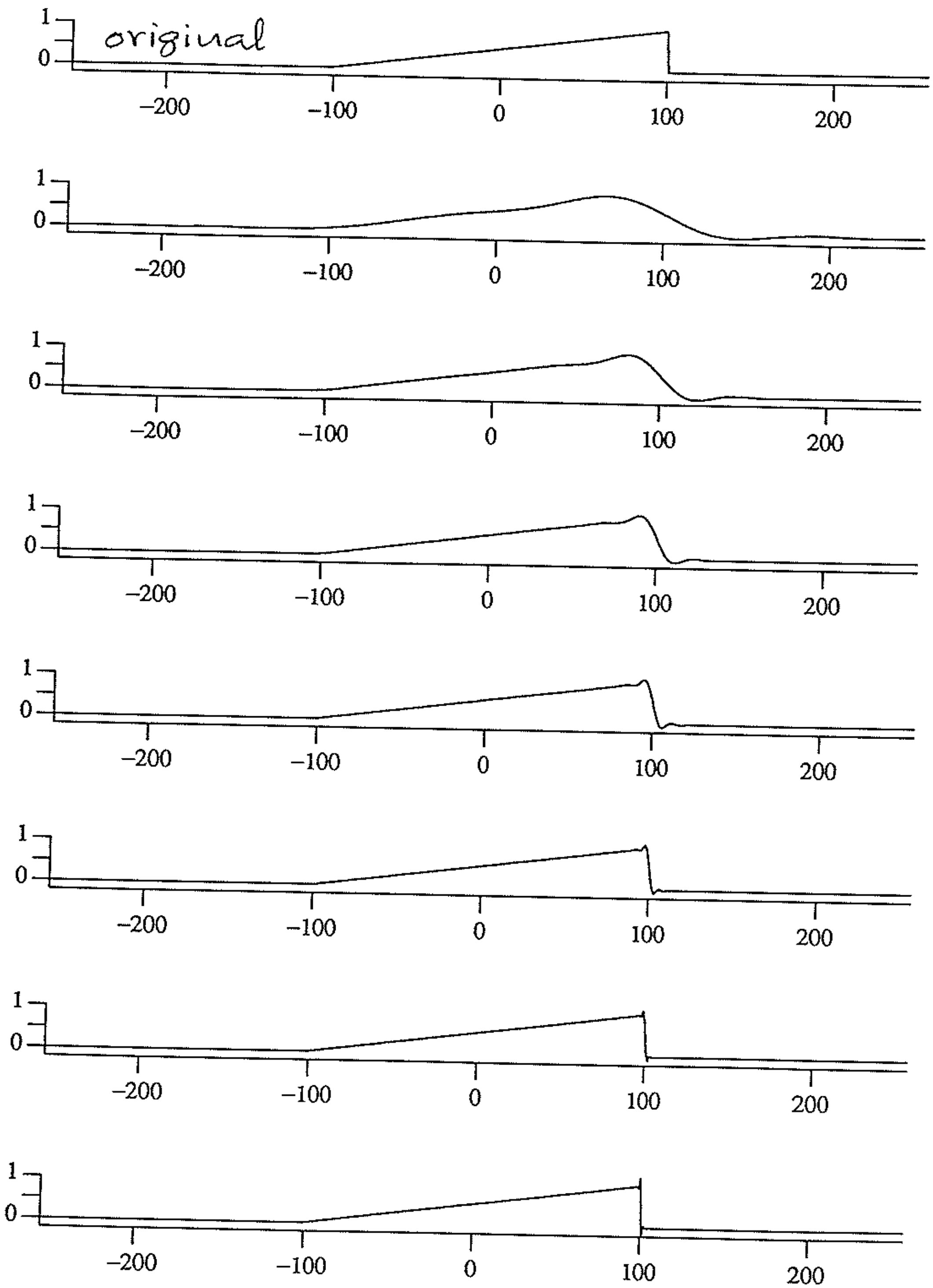

Figure 10: The reconstruction of the ramp from its Phi–transform coefficients in Figure 9. Gibbs phenomenon is visible at the point of discontinuity.

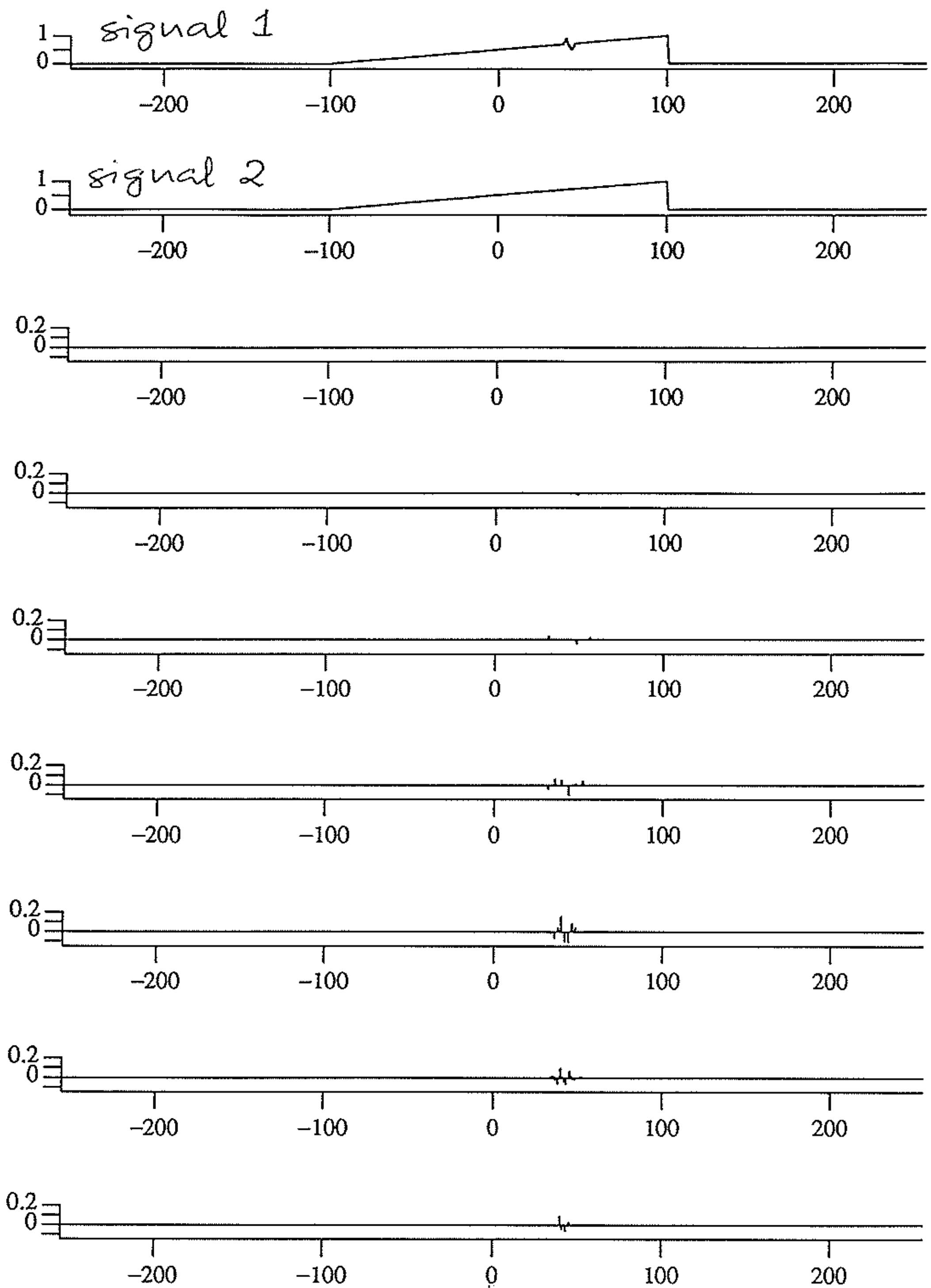

Figure 11: The difference of the Phi–transform coefficients of the two signals (at the top) is plotted. A local difference in the two signals is seen to produce local differences in the Phi–transform coefficients.

6. Conclusions

We have established the Psi–decomposition (4) for functions in $L^2(\mathbf{R}^n)$. We have noted that the Psi–decomposition defines the Phi–transform, and that the Phi–transform of a function gives us a time–frequency representation if the analyzing functions are well-localized in the time *and* the frequency domains. The Phi–transform is linear, continuous, and invertible. If the analyzing functions are chosen to be real in the time domain, and if the signal being analyzed is real as well, then the Phi–transform consists of real numbers.

There are more general decompositions [21], similar to the Psi–decomposition, that are not treated here. Methods completely different from those used in Lemma 1 could be used to generate the sets **A** and **S** of analyzing and synthesizing functions [21]. We need not identify the supports of the functions in **A** and **S**. We might require that the elements of **A**, or **S**, or both, be compactly–supported in the time–domain and not in the frequency–domain. Or we may forego compact support in both the time and the frequency domains, requiring only that our functions be small and rapidly decaying outside appropriate compact intervals.

We have given examples of the construction of the sets **S** and **A** of synthesizing and analyzing functions, and of Phi–transform plots. From these examples we see that the Phi–transform does indeed give us a time–frequency representation.

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References

[1] D. Gabor, "Theory of Communication," *J. IEE*, vol. 93, pp. 429-457, 1946.

[2] Y. Meyer, S. Jaffard, and O. Rioul, "L'Analyse Par Ondelettes," *Pour la Science (édition Français de Scientific American)*, no. 119, pp. 28–37, Sept. 1987.

[3] S. G. Mallat, "Multiresolution Representation and Wavelets," Ph.D. dissertation, Elect. Engg. Dept., Univ. of Pennsylvania, 1988.

[4] S. G. Mallat, "A Theory For Multiresolution Signal Decomposition: The Wavelet Representation," *IEEE Trans. Pattern Analysis and Machine Intelligence*, vol. 11, no. 7, pp. 674–693, July 1989.

[5] L. Cohen, "Time–Frequency Distributions — A Review," *Proceedings of the IEEE*, vol. 77, no. 7, pp. 941-981, July 1989.

[6] J. B. Allen, and L. R. Rabiner, "A Unified Approach To Short–Time Fourier Analysis And Synthesis," *Proc. IEEE*, vol. 65, no. 11, pp. 1558–1564, Nov. 1977.

[7] P. G. Lemarié, and Y. Meyer, "Ondelettes et bases Hilbertiennes," *Revista Matemática Iberoamericana*, vol. 2, nos. 1 and 2, pp. 1–18, 1986.

[8] I. Daubechies, "The Wavelet Transform, Time–Frequency Localization, and Signal Analaysis," to appear in *IEEE Trans. Information Theory*.

[9] A. Grossmann, J. Morlet, and T. Paul, "Transforms Associated to Square Integrable Group Representations I. General Results," *Journal of Mathematical Physics*, vol. 26, pp. 2473–2479, 1985.

[10] A. Grossmann, M. Holschneider, R. Kronland-Martinet, and J. Morlet, "Detection of Abrupt Changes in Sound Signals With the Help of Wavelet Transforms," in *Inverse Problems*, Academic Press, pp. 289–306, 1987.

[11] P. Federbush, "Quantum Field Theory in Ninety Minutes," *Bulletin (New Series) of the American Mathematical Society*, vol. 17, no. 1, pp. 93–103, July 1987.

[12] I. Daubechies, "Orthonormal Bases of Compactly Supported Wavelets," *Communications of Pure and Applied Mathematics*, vol. 41, pp. 909–996, 1988.

[13] R. Kronland-Martinet, "The Wavelet Transform for Analysis, Synthesis, and Processing of Speech and Music Sounds," *Computer Music Journal*, MIT, vol. 12, no. 4, Winter pp. 11–20, 1988.

[14] M. Holschneider, R. Kronland-Martinet, J. Morlet, and P. Tchamitchian, "A Real-Time Algorithm for Signal Analysis With the Help of the Wavelet Transform," preprint.

[15] C. E. Heil, and D. F. Walnut, "Continuous And Discrete Wavelet Transforms," *SIAM Review*, vol. 31, no. 4, pp. 628–666, Dec. 1989.

[16] E. Wigner, "On the Quantum Correction for Thermodynamic Equilibrium," *Physical Review*, vol. 40, pp. 749–759, June 1932.

[17] N. G. De Bruijn, "Uncertainty Principles in Fourier Analysis," in *Inequalities*, O. Shisha (ed.), Academic Press, New York, pp. 57–71, 1967.

[18] J. Ville, "Theorie et applications de la notion de signal analytique," *Cables et Transmission*, vol. 2A, pp. 61–74, 1948.

[19] M. Frazier, and B. Jawerth, "Decomposition of Besov Spaces," *Indiana University Mathematics Journal*, vol. 34, no. 3, pp. 777–799, 1985.

[20] M. Frazier, and B. Jawerth, "The ϕ–Transform and Applications to Distribution Spaces," in *Function Spaces and Applications*, M. Cwikel et al. (eds.), Springer Lecture Notes in Mathematics no. 1302, pp. 223–246, 1988.

[21] M. Frazier, and B. Jawerth, "A Discrete Transform and Decompositions of Distribution Spaces", to appear in the *Journal of Functional Analysis.*

[22] I. Daubechies, A. Grossman, and Y. Meyer, "Painless Nonorthogonal Expansions," *Journal of Mathematical Physics*, vol. 27, no. 5, pp. 1271–1283, May 1986.

[23] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications,* John Wiley and Sons, New York, pp. 258-267, 1984.

[24] D. R. Fuhrmann, A. Kumar, and J. R. Cox, Jr., "Compact Functions And The Frazier–Jawerth Transform," *6–th IEEE Multidimensional Signal Processing Workshop*, Pacific Grove, CA, Sept. 1989.

[25] A. Kumar, and D. R. Fuhrmann, "The Frazier–Jawerth Transform," *Proc. International Conf. on Acoust., Speech, and Signal Processing,* Albuquerque, New Mexico, vol. 5, pp. 2483–2486, Apr. 1990.

[26] C. E. Shannon, "Communication in the Presence of Noise," *Proceedings of the IRE*, vol. 37, pp. 10–21, Jan. 1949.

[27] I. S. Gradshteyn, and I. M. Ryzhik, *Tables Of Integrals, Series, And Products,* Academic Press, New York, 1980.

[28] D. Slepian, and H. O. Pollak, "Prolate Spheroidal Wave Functions, Fourier Analysis, and Uncertainty—I," *Bell Systems Technical Journal,* pp. 43–63, Jan. 1961.

[29] H. J. Landau, and H. O. Pollak, "Prolate Spheroidal Wave Functions, Fourier Analysis, and Uncertainty — II," *Bell Systems Technical Journal,* pp. 65–84, Jan. 1961.

[30] I. Daubechies, "Time–Frequency Localization Operators: A Geometric Phase Space Approach," *IEEE Trans. Inform. Theory*, vol. 34, no. 4, pp. 605–612, July 1988.