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Castelnuovo-Mumford Regularity of General Rational Curves on Hypersurfaces.

Sara Gharahbeigi
Washington University in St. Louis

WASHINGTON UNIVERSITY IN ST. LOUIS

Department of Mathematics

Dissertation Examination Committee:
N. Mohan Kumar, Chair
Roya Beheshti, Chair
Ram Cowsik
Matthew Kerr
Prabhakar Rao
David Wright

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by

Sara Gharahbeigi

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ABSTRACT

We show that for a general smooth rational curve on a general hypersurface of degree $d \leq N$ in $\mathbb{P}^N$, $N \geq 4$, the restriction map of global sections is of maximal rank, and therefore the regularity index of such curves is as small as possible.

ACKNOWLEDGEMENTS

To my parents and my lovely hometown Shiraz.

Contents

Chapter 1

Normality of curves

1.1 Introduction

Recall that a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^N$ is called $m-regular$ if $H^i(\mathbb{P}^N, \mathcal{F}(m-i)) = 0$ for all $i \geq 1$. A closed subvariety $Z \subset \mathbb{P}^N$ is called $m-regular$ if the corresponding ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_{\mathbb{P}^N}$ is $m-$regular. By [11] (lecture 14), for an $m-$regular sheaf $\mathcal{F}$, $\mathcal{F}(m)$ is generated by global sections.

Several bounds are known for regularity the of smooth varieties. In the case of smooth curves, the following is proven in [5]:

Theorem 1.1. *Let $C \subset \mathbb{P}^N$ be a (reduced and irreducible) non-degenerate curve of degree e. Then C is $(e+2-N)$-regular.*

For a smooth rational curve C in $\mathbb{P}^N$, consider the restriction map

$$r_C(n) : H^0(\mathbb{P}^N, \mathcal{O}_{P^N}(n)) \longrightarrow H^0(C, \mathcal{O}_C(n)).$$

By the proposition below, to compute the regularity of a smooth rational curve, we only need to verify the vanishing of the first cohomology group.

Proposition 1.2. *For a smooth rational curve C in $\mathbb{P}^N$, if the map $r_C(n)$ is surjective*

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for some $n \geq 1$, then C is $m-$regular for all $m \geq n + 1$.

Proof. For a smooth rational curve if the map $r_C(n)$ is surjective for some $n \geq 1$, then $H^1(\mathcal{O}_{\mathbb{P}^N}, \mathcal{I}_C(n)) = 0$. But we also have $H^1(C, \mathcal{O}_C(m)) = 0$ for all $m \geq 0$, as C is a smooth rational curve. From this and the long exact sequence of cohomology

$$... \to H^1(C, \mathcal{O}_C(m)) \to H^2(\mathbb{P}^N, \mathcal{I}_C(m)) \to H^2(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \to ...$$

we get $H^2(\mathbb{P}^N, \mathcal{I}_C(m)) = 0$ for all $m \geq 0$. Therefore $H^1(\mathbb{P}^N, \mathcal{I}_C(n)) = H^2(\mathbb{P}^N, \mathcal{I}_C(n - 1)) = 0$, so C is $(n+1)$-regular. One can then use [11] (lecture 14) to show that C is $m-$regular for all $m \geq n + 1$. □

Corollary 1.3. *The surjectivity of $r_C(n)$ for some $n \geq 1$ implies the surjectivity of $r_C(m)$ for all $m \geq n$.*

The map $r_C(n)$ is said to be of *maximal rank* if it is either injective or surjective. The curve C is said to be of *maximal rank* if for any integer $n \geq 1$, $r_C(n)$ is either injective or surjective.

It was conjectured by J.Harris [6] that for a *general* smooth rational curve of degree e, the map $r_C(n)$ is of maximal rank for all n. It is shown by Hartshorne [9] that a union of general lines in projective space is of maximal rank, using induction both on n and N. By similar methods Hartshorne and Hirschowitz [7] (and Ballico and Ellia [1]) proved that the union of a general rational curve and lines in $\mathbb{P}^3$ (union of general rational curves in $\mathbb{P}^N$, $N \geq 4$, resp.) is of maximal rank.

The method used in all these cases is the degeneration method. First, one degenerates a curve into a nodal rational curve (with embedded points) which satisfies the conjecture. Then one uses deformation theory to say there is a smoothing of such nodal curve, and hence conclude the conjecture for a general member of the smoothing family.

The main result of this note is a similar statement for rational curves on a general hypersurface of degree $d \leq N$. For numerical reasons, degenerating the curve alone, does not work for the case of curves on a hypersurface. What we do to resolve the issue is to degenerate the hypersurface as well as the curve and show the theorem for a rational tree on a union of two smooth hypersurfaces (therefore we induct not only on n but also on d, the degree of the hypersurface). Of course, our construction should be done in such a way that allows us to deform the curve and the hypersurface. We prove the following:

Theorem 1.4. *For $N \geq 4$, let $Y \subset \mathbb{P}^N$ be a general hypersurface of degree $d \leq N$. If for some positive integers e and n*

$$1 + ne \leq \binom{N+n}{N} - \binom{N+n-d}{N} \tag{1.1}$$

then for a general rational curve C of degree e on Y, the map $r_C(n)$ is surjective.

Note that equation (1) is the numerical condition $h^0(Y, \mathcal{O}_Y(n)) \geq h^0(C, \mathcal{O}_C(n))$. For any given degree e we show that there is a rational tree (without embedded points, unlike previous works) of degree e on a singular hypersurface of degree d satisfying the statement of the theorem and then we use some deformation theory to conclude the same result for a general curve of degree e on a general hypersurface of degree d. We will prove several numerical lemmas which are needed throughout the arguments. These will be shown in the appendix.

Our theorem has the following nice application. Recall that a projective variety Y of dimension m is called *uniruled* if there is a variety Z of dimension $m-1$ and a dominant rational map

$$Z \times \mathbb{P}^1 - -- > Y.$$

For a general hypersurface $X \subset \mathbb{P}^N$ of degree d, let $R_e(X)$ be the parametrizing space of all smooth irreducible rational curves of degree e on X. In [2], R. Beheshti proves

that :

Theorem 1.5. *Let $X \subset \mathbb{P}^N$ ($N \geq 12$) be a general hypersurface of degree d. If a general smooth rational curve $C \subset X$ of degree e is $m-normal$ (i.e $r_C(m)$ is surjective) and if*

$$d^2 + (2m+1)d \geq (m+1)(m+2)N + 2$$

then $R_e(X)$ is not uniruled.

Our theorem improves the range of integers m that satisfy the above theorem. Note that the right and left hand side of the numerical inequality in the theorem above, are degree two and degree one polynomials in terms of m, respectively. Thus finding smaller m's for which a curve is $m-$normal, makes this inequality more possible to hold.

1.2 Definitions

We make a few definitions which will be used throughout.

Definition 1.1. A rational tree is a connected projective curve C whose singular points are nodes and $\chi(C, \mathcal{O}_C) = 1$.

The above definition is equivalent to saying that irreducible components of C are smooth rational curves and there are $Card(Sing(C)) + 1$ of them. A smoothing of a rational tree C consists of a smooth pointed 1-dimensional scheme $(T, 0)$ and a flat relative projective curve $\mathcal{C} \to T$ whose fiber over 0 is C and and all other fibers are smooth rational curves.

Theorem 1.6. *A rational tree in $\mathbb{P}^N$ is smoothable, i.e it has a smoothing.*

Proof. See [4], page 101. □

Definition 1.2. An admissible curve $X \subset \mathbb{P}^N$ of type (e, k) is defined as a disjoint union of a rational tree of degree e and k lines.

Definition 1.3. A projective scheme $X \subset \mathbb{P}^r$ is $n-$normal if the restriction map

$$r_X(n) : H^0(\mathbb{P}^N, \mathcal{O}_{P^N}(n)) \longrightarrow H^0(X, \mathcal{O}_X(n)).$$

is surjective.

Definition 1.4. A vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ is called balanced, if it's splitting as sums of the line bundles is in the following form:

$$\mathcal{E} \cong \bigoplus \mathcal{O}(a_i)$$

and $|a_i - a_j| \leq 1$ for all i, j.

Chapter 2

Rational curves in projective space

In this chapter we investigate the maximal rank property of smooth rational curves in a projective space. It is known by Hartshorne and Hirschowitz [9] that disjoint union of general lines in a projective space is of maximal rank. In [7] they prove that general disjoint union of one smooth rational curve and several lines in $\mathbb{P}^3$ is of maximal rank. This result was later generalized by Ballico and Ellia in [**?**] for disjoint union of general rational curves in $\mathbb{P}^N$ for $N \geq 4$. Here we reprove a similar result without using embedded points.

2.1 Preliminaries

Fix a hyperplane $H \subset \mathbb{P}^N$. For integers $t \geq 0$, $k' \geq 0$ and $b \geq 1$, let $\mathcal{Y}$ be the parametrizing space of ordered tuples $(C, l_1, \cdots, l_t, L_1, \cdots, L_{k'})$, where C is a non-degenerate [1]rational curve of degree b in $\mathbb{P}^N$ and $l_1, \cdots l_t, L_1 \cdots, L_{k'}$ are disjoint lines and

$$X := C \cup (\cup_{i=1}^t l_i) \cup (\cup_{j=1}^{k'} L_j)$$

is a *smooth* admissible curve of type $(b, t + k')$ intersecting H transversely.

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[1]By non-degenerate we mean the linear span of the curve has maximum possible dimension.

Lemma 2.1. *Let $\mathcal{T}$ be the parametrizing space of $t+1$ distinct ordered points in H. We have the incidence correspondence*

$$\mathcal{I} := \{(C, l_1, \cdots, l_t, L_1, \cdots, L_{k'}, q, p_1, ..., p_t) | q \in C \cap H, p_r \in l_r \cap H$$

$$(C, l_1, \cdots, l_t, L_1, \cdots, L_{k'}) \in \mathcal{Y}\}$$

and the two projections π_1 and π_2 from $\mathcal{I}$ to its factors
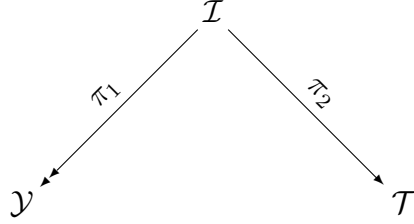
Then π_1 is surjective and π_2 is dominant.

Proof. Under the assumptions, $\mathcal{Y}$ is irreducible and smooth. It is clear that π_1 is surjective. First we calculate $\dim \mathcal{I}$. For $(C, l_1, \cdots, l_t, L_1, \cdots, L_{k'}) \in \mathcal{Y}$ and $X := C \cup (\cup_{i=1}^{t} l_i) \cup (\cup_{j=1}^{k'} L_j)$ we have

$$h^0(N_{X/\mathbb{P}^N}) = h^0(N_{C/\mathbb{P}^N}) + (t+k')h^0(N_{L_i}) = (b+1)(N+1) - 4 + 2(t+k')(N-1).$$

Thus $\dim \mathcal{Y} = (b+1)(N+1) - 4 + 2(t+k')(N-1)$. Note that fibers of π_1 are zero dimensional, hence $\dim \mathcal{I} = \dim \mathcal{Y}$.

For $p := (q, p_1, \cdots, p_t) \in \pi_2(\mathcal{I})$,

$$\dim \pi_2^{-1}(p) = h^0(N_{C/\mathbb{P}^N}(-q)) + t[h^0(N_{L_1/\mathbb{P}^N}(-p_1))] + 2k'(N-1)$$

$$= (N+1)b - 2 + t(N-1) + 2k'(N-1)$$

$$= (N+1)b + (2k'+t)(N-1) - 2$$

But the fiber dimension also satisfies:

$$\dim \mathcal{T} \geq \dim \pi_2(\mathcal{I}) \geq \dim \mathcal{I} - \dim {\pi_2}^{-1}(p)$$

$$= (b+1)(N+1) - 4 + 2(t+k')(N-1)$$

$$- [(N+1)b + (2k'+t)(N-1) - 2]$$

$$= \dim \mathcal{T}. \tag{2.1}$$

Hence $\dim \pi_2(\mathcal{I}) = \dim \mathcal{T}$ and therefore π_2 is dominant. $\qquad \square$

Corollary 2.2. *Keeping the notation of the above lemma, assume that there exists a smooth admissible curve X_0 parametrized by $\mathcal{Y}$ of type $(b, t+k')$ which is $n-$normal for some integer $n \geq 1$. Let*

$$\mathcal{M} = \left\{ \pi_1^{-1}(X) | [X] \in \mathcal{Y} \text{ and is } n\text{-normal} \right\}$$

Then $\mathcal{M} \subset \mathcal{I}$ is dense and open, and the restriction of π_2 to $\mathcal{M}$ is dominant.

In other words, over a dense non-empty set in $\mathcal{T}$, the fibers of π_2 are non-empty and contain a smooth admissible curve of type $(b, t+k')$ which intersects H transversely and is $n-$normal.

Proof. Let $\mathcal{Z}$ be the parametrizing space of all ordered tuples $(C, l_1, \cdots, l_t, L_1, \cdots, L_{k'})$, where $X := C \cup (\cup_{i=1}^t l_i) \cup (\cup_{j=1}^{k'} L_j)$ is a *smooth* admissible curve of type $(b, t+k')$ and C is non-degenerate. Note that $\mathcal{Z}$ is irreducible because the parametrizing space of non-degenerate smooth rational curves is irreducible, so is the parametrizing space of $t + k'$ disjoint lines in $\mathbb{P}^N$. Note that $\mathcal{Y} \subset \mathcal{Z}$ is open (the curves intersecting H non-transversely form a closed set), and therefore dense since $\mathcal{Z}$ is irreducible.

The existence of such $X_0 \subset \mathbb{P}^N$ (by our assumption) implies that there exists a non-empty open set $\mathcal{U} \subset \mathcal{Z}$ of smooth elements which are $n-$normal and of type

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$(b, t + k')$. Hence $\mathcal{U} \cap \mathcal{Y} \subset \mathcal{Z}$ is open and non-empty. But π_1 is surjective, therefore $\pi_1^{-1}(\mathcal{U} \cap \mathcal{Y}) \subset \mathcal{I}$ is open and $\pi_2|_{\pi_1^{-1}(\mathcal{U} \cap \mathcal{Y})}$ is dominant. $\qquad \square$

Lemma 2.3. *Let t, H and $\mathcal{T}$ be as in the previous lemma and let $\mathcal{V}$ be the nonempty open subset of Hilbert scheme of smooth non-degenerate rational curves of degree b in H. Furthermore let*

$$\mathcal{J} := \{(C, p_1, ..., p_t, p_{t+1} | p_i \in C \in \mathcal{V}\}.$$

If $b \geq t+1$, then the projection $\pi_1 : \mathcal{J} \to \mathcal{V}$ is surjective and $\pi_2 : \mathcal{J} \to \mathcal{T}$ is dominant.

Proof. With a similar calculation as in the previous lemma (except that the fibers of π_1 are $t + 1$ dimensional) we get $\dim \mathcal{J} = (b+1)N - 4 + (t+1)$. For a general point $p := [p_1, ..., p_{t+1}] \in \pi_2(\mathcal{J})$, $\dim \pi_2^{-1}(p) = h^0(N_{C/H}(-p_1 - ... - p_{t+1}))$.

By [12], we know for a general rational curve C in the projective space $\mathbb{P}^{N-1} = H$, the vector bundle $N_{C/H}$ is balanced. We first show that $H^1(N_{C/H}(-p_1 - ... - p_{t+1})) = 0$. Note that $N_{C/H}$ is a balanced vector bundle of degree $Nb-2$ and rank $N-2$; therefore, to show $H^1(N_{C/H}(-p_1-...-p_{t+1})) = 0$, it suffices to check $\lfloor \frac{Nb-2}{N-2} \rfloor - (t+1) \geq -1$, which is true because $t+1 \leq b$. Hence we get $h^0(N_{C/H}(-p_1 - ... - p_{t+1})) = N(b-t) + 2t - 2$, which is the fiber dimension of π_2. Hence

$$\begin{aligned}
\dim \pi_2(\mathcal{J}) &= \dim \mathcal{J} - \dim \pi_2^{-1}(p) \\
&= (b+1)N + t - 3 - N(b-t) - 2t + 2 \\
&= (N-1)(t+1) = \dim \mathcal{T}.
\end{aligned} \tag{2.2}$$

Hence π_2 is dominant. $\qquad \square$

Corollary 2.4. *With the same notation as in the last lemma, suppose that there exists $[C_0] \in \mathcal{V}$ such that C_0 is a degree b smooth, irreducible and $n-$normal for some*

$n \geq 1$. Let

$$\mathcal{M} = \left\{ \pi_1^{-1}(C) \mid [C] \in \mathcal{V} \text{ and } C \text{ is } n\text{-normal} \right\}.$$

Then $\mathcal{M} \subset \mathcal{I}$ is dense and open and the restriction of π_2 to $\mathcal{M}$ is dominant.

In other words, there exists an open dense subset of $\mathcal{T}$ over which the fibers of π_2 are non-empty and contain an element which is $n-$normal.

Lemma 2.5. *Suppose* $X \subset \mathbb{P}^N$ *is a smooth irreducible rational curve such that* $r_X(n) :$ $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) \to H^0(X, \mathcal{O}_X(n))$ *is surjective for some positive integer* n. *For a non-negative integer* δ, *let* X_δ *denote the union of* X *and* δ *general points in* $\mathbb{P}^N$. *Then for* $0 \leq \delta \leq h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) - h^0(X, \mathcal{O}_X(n))$, *the map*

$$r_{X_\delta}(n) : H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) \to H^0(X_\delta, \mathcal{O}_{X_\delta}(n))$$

is surjective.

Proof. We use induction on δ. The statement is correct when $\delta = 0$. Assuming the lemma for $\delta - 1$, we prove it for δ. Note that surjectivity of $r_{X_{\delta-1}}(n)$ implies $h^0(I_{X_{\delta-1}}(n)) = h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) - h^0(X_{\delta-1}, \mathcal{O}_{X_{\delta-1}}(n))$, where $I_{X_\delta}(n)$ is the ideal sheaf of X_δ in $\mathbb{P}^N$ twisted by n. So

$$\begin{aligned} h^0(I_{X_{\delta-1}}(n)) &= h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) - h^0(X_{\delta-1}, \mathcal{O}_{X_{\delta-1}}) \\ &= h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) - h^0(X, \mathcal{O}_X(n)) - (\delta - 1) > 0. \end{aligned}$$

Now let $Y \subset \mathbb{P}^N$ be a hypersurface of degree n containing $X_{\delta-1}$ and let $p_\delta \notin Y$ be a general point. Therefore if $Y = \{f = 0\}$, we have that $Y \in H^0(I_{X_{\delta-1}}(n))$, but $Y \notin H^0(I_{X_\delta}(n))$ for this p_δ. Hence $h^0(I_{X_\delta \cup p_\delta}(n)) = h^0(I_{X_{\delta-1}}(n)) - 1$. Hence the map $r_{X_\delta}(n)$ is surjective. $\qquad\square$

2.2 Induction Argument

Let $N \geq 4$, $k \geq 0$ and $e \geq 1$ be integers. Set

$$\nu(e, k, N) = \min \left\{ m \geq 1 \mid 1 + me + k(m+1) \leq \binom{N+m}{N} \right\}.$$

We call $\nu(e, k, N)$ the *value of* (e, k) *with respect to* N.

Note that if X is an admissible curve of type (e, k) and m is a positive integer, then

$$1 + me + k(m+1) = \dim H^0(\mathcal{O}_X(m)).$$

If $H^0(\mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(\mathcal{O}_C(m))$ is surjective, then $m \geq \nu(e, k, N)$.

For integers $n \geq 1$ and $N \geq 3$, let $H(n, N)$ be the following statement:

H(n,N): For all choices of $k \geq 0$ and $e \geq 1$ such that $\nu(e, k, N) \leq n$, and $k \leq n$ there exists an admissible curve X of type (e, k) such that the restriction map $r_X(n) : H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) \to H^0(X, \mathcal{O}_X(n))$ is surjective.

The main goal is to prove $H(n, N)$, using induction on both n and N.

Theorem 2.6. $H(n, N)$ *holds for all* $n \geq 1$, $N \geq 3$.

An important consequence of this theorem, we get the following corollary regarding the regularity of a general rational curve.

Theorem 2.7. *A general rational curve of degree e in $\mathbb{P}^N$, $N \geq 3$, is $[\nu(e, 0, N) + 1]-regular.*

Proof. By the last theorem above for a given $e \geq 1$, there exists a rational tree of degree e which is $\nu(e, 0, N)-$normal. By Theorem (1.6) a rational tree is smoothable. Normality is an open property, hence the lemma follows by the semi-continuity theorem. $\square$

2.2.1 The Base case of the induction

Proof. By [7], we know $H(n, 3)$ holds for any n. It is proved in [7] that a general disjoint union of a rational curve and lines in $\mathbb{P}^3$ is of maximal rank. So we can take the rational tree in the definition of $H(n, 3)$ to be a smooth irreducible rational curve.

$H(1, N)$ is a statement about linear normality. $H(1, N)$ states that if for some integers $e \geq 1$ and $0 \leq k \leq 1$, $\nu(e, k, N) = 1$ (i.e if $1 + e + 2k \leq \binom{N+1}{N} = N + 1$), then there exists an admissible curve $X \subset \mathbb{P}^N$ of type (e, k), such that $r_X(1)$ is surjective.

First assume that $k = 0$ and let $e \leq N$ be some integer. Let $C \subset \mathbb{P}^N$ be a non-degenerate rational normal curve of degree e. Hence $r_C(1)$ is surjective. Now if $e \leq N - 2$, we can always find a line L such that it does not intersect the linear span of a non-degenerate rational normal curve C of degree e. Therefore, $r_{C \cup L}(1)$ remains surjective. $\qquad\square$

2.2.2 The Induction Step

Proof. We want to show $H(n, N)$ for $n \geq 2$ and $N \geq 4$. Assume $H(i, N)$ for $1 \leq i \leq n - 1$ and $H(n, N - 1)$. Note that when $e = 1$, $H(n, N)$ is a statement about a disjoint union of lines. But by [9], we know that a disjoint union of general lines is of maximal rank, so we may assume $e \geq 2$. To prove $H(n, N)$ assume $e \geq 2$ and $0 \leq k \leq n$ are integers satisfying $\nu(e, k, N) \leq n$. But the cases where $\nu(e, k, N) \leq n - 1$ and $k \leq n - 1$ are already known by our assumption $H(n - 1, N)$, so we only need to consider the cases $\nu(e, k, N) = n$ and $\nu(e, n, N) = n - 1$.

Consider the case $\nu(e, k, N) = n$. Let $1 \leq a < e$ be the largest integer such that $1 + (n - 1)a \leq \binom{N+n-1}{N}$ (note that $e \geq 2$ and therefore such a exists). Having fixed a, let $0 \leq k' \leq k$ be the largest integers satisfying $1 + (n - 1)a + k'n \leq \binom{N+n-1}{N}$.

So by definition $\nu(a, k', N) \le n - 1$ and hence by the induction hypothesis $H(n - 1, N)$ and Theorem(1.6), there exists a smooth admissible curve $X_1 := C_1 \cup L_1 \cup ... \cup L_{k'}$, of type (a, k') such that $r_{X_1}(n - 1)$ is surjective.

The argument in our proof below depends on whether $\nu(e, k, N) = \nu(e + 1, k, N)$ or not. So here we discuss the two cases:

Case(A): $\nu(e, k, N) = \nu(e + 1, k, N)$

Let a and k' be as defined above, which implies $\nu(a, k', N) \le n - 1$ and therefore we can apply $H(n - 1, N)$. Also by the remark after Lemma (4.1), we have $\nu(e - a, k - k', N - 1) \le n$, so we can use $H(n, N - 1)$. Now we apply Lemma (2.1) and its corollary in the case $t = 0$, $b = a$, also Lemma (2.3) and its corollary in the case $b = e - a$ and $t = 0$. So for a hyperplane $H \subset \mathbb{P}^N$ we can find a point $p \in H$ so that there exists a smooth admissible curve $X_1 := C_1 \cup L_1 \cup ... \cup L_{k'} \subset \mathbb{P}^N$ of type (a, k') which intersects H transversely and $r_{X_1}(n - 1)$ is surjective, and a smooth admissible curve $X_2 = C_2 \cup l_1 \cup \cdots l_{k-k'} \subset H$ of type $(e - a, k - k')$ such that $r_{X_2}(n)$ is surjective and $X_1 \cap X_2 = C_1 \cap C_2 = p_1$. We take X_1 so that $X_1 \cap H$ are in general position in H.

Let $X = X_1 \cup X_2$. The inclusion $X \cap H \subset X$ implies we have an exact sequence

$$\mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_{X \cap H} \to 0$$

and since $X_2 \subset H$, the above sequence reduces to the short exact sequence

$$0 \to \mathcal{O}_{X_1}(-H) \to \mathcal{O}_X \to \mathcal{O}_{X \cap H} \to 0$$

Twisting by $\mathcal{O}_{\mathbb{P}^N}(n)$ and taking cohomology, we get
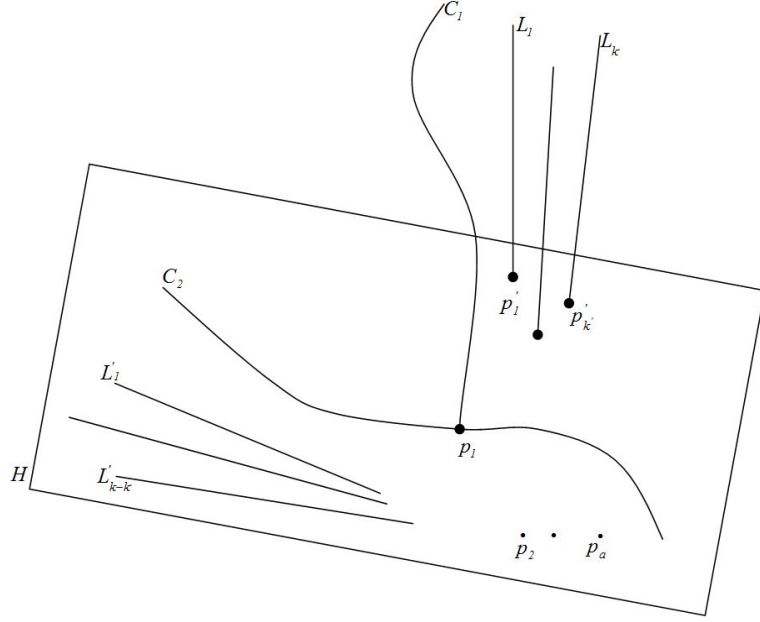
Figure 2.1: Case(A)

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^N}(n-1)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^N}(n)) \longrightarrow H^0(\mathcal{O}_H(n)) \longrightarrow 0$$
$$r_{X_1}(n-1)\Big\downarrow \qquad\qquad r_X(n)\Big\downarrow \qquad\qquad r_{X\cap H}(n)\Big\downarrow$$
$$0 \longrightarrow H^0(\mathcal{O}_{X_1}(n-1)) \longrightarrow H^0(\mathcal{O}_X(n)) \longrightarrow H^0(\mathcal{O}_{X\cap H}(n)) \longrightarrow 0$$

The left vertical map is surjective by $H(n-1, N)$. For the vertical map on the right, note that we chose X_1 so that the points in $X_1 \cap H$ are in independent positions.

Lemma (4.1) provides the dimension requirement for the right map to be surjective. Then one can use the smoothing theorem (1.6) along with $H(n, N-1)$ to conclude the right vertical map is surjective. Note that $X \cap H$ contains the points $X_1 \cap H$ which are chosen to be in independent position. Thus by Lemma (2.5), the surjectivity of $r_{X_2}(n)$ is preserved after adding this point.

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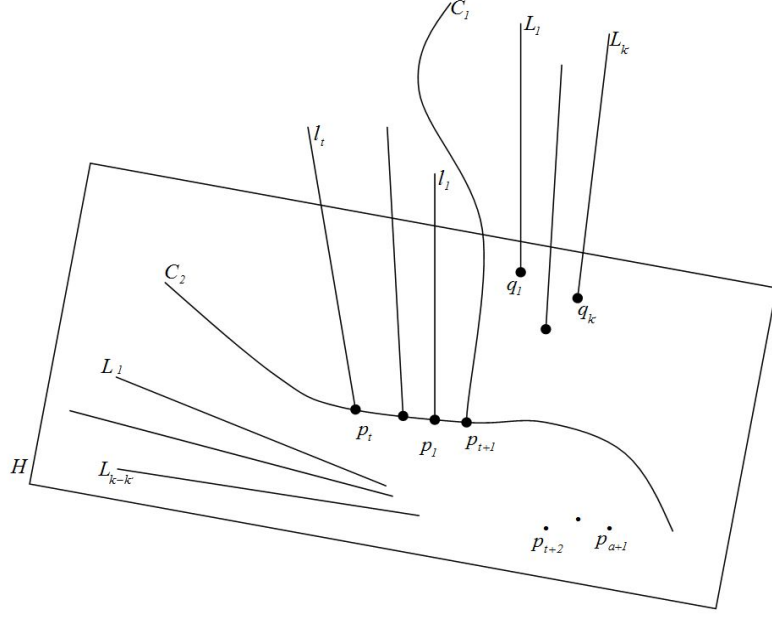
Figure 2.2: Case B

Case(B): $\nu(e, k, N) = n < \nu(e + 1, k, N)$

In this case, one can easily find examples of such values e such that the right map in the diagram above can not be surjective by comparing the dimensions of the two spaces. To remedy the problem we first modify our choice of the curve X_1. Define a, k' and H as before, and let $\mathcal{T}$ be the parametrizing space of ordered $t + 1$ points in H, where $t := \binom{N+n-1}{N} - 1 - (n - 1)a$.

By the Lemma (4.3) in the appendix, in this case $k' = 0$ and $t \leq n - 1$. Hence $\nu(a - t, t, N) \leq n - 1$ while $t \leq n - 1$, so we can apply $H(n - 1, N)$ and Theorem 1.6 to conclude that there exists a smooth admissible curve of type $(a - t, t)$ which is $n - 1$-normal.

Similar to the previous case, we have $\nu(e - a, k - k', N - 1) = \nu(e - a, k, N - 1) \leq n$ by Corollary (4.5). Hence we may apply $H(n, N-1)$ and Theorem 1.6 to get a smooth admissible curve $X_2 \subset H$ of type $(e - a, k)$, so that X_2 is n−normal.

Let $P = (p_1, \cdots, p_{t+1}) \in \mathcal{T}$, and use Corollary (2.2) and Corollary (4.6) in the

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case $b = a - t$, and Corollary (2.4) in the case $b = e - a$ to get that there exists a smooth $n - 1$-normal admissible curve $X_1 := C_1 \cup l_1 \cup ... \cup l_t \subset \mathbb{P}^N$ of type $(a - t, t)$, and a smooth $n-$normal admissible curve $X_2 := C_2 \cup L_1' \cup ... \cup L_{k-k'} \subset H$ so that $P \subset X_1 \cap H$.

Note that C_2 intersects C_1 and l_i's each at a single point, and these are the only points in $X_1 \cap X_2$.

Therefore $X_1 \cup X_2$ is an admissible curve of type (e, k) with $t + 1$ nodal singularities. We consider the previous diagram in this case and make a similar argument to get that the middle map is surjective. The left vertical map is surjective by $H(n - 1, N)$ and Theorem (1.6). Lemma (4.4) implies that we have the numerical requirement for the right vertical map to be surjective. Then $H(n, N - 1)$ along with Lemma (2.5) imply that it is indeed surjective.

We are left to consider the case $\nu(e, n, N) = n - 1$, which is a possible situation for $H(n, N)$. By $H(n - 1, N)$ and the smoothing theorem (1.6), we know there exists a smooth admissible curve X_1 of type $(e, n - 1)$ which is $(n - 1)-$normal, that is because $\nu(e, n - 1) \leq n - 1$. Now consider a line L on a hyperplane H. To get the surjectivity of $r_{X_1 \cup L}(n)$ (which is a smooth admissible curve of type (e, n) we argue as in the above cases. The only thing to check is the dimension requirement for surjectivity of $r_{L \cup X_1 \cap H}(n)$. That is to check

$$2n + e - 1 \leq \binom{N + n - 1}{N - 1}.$$

But this holds because the assumption $\nu(e, n, N) = n - 1$ implies

$$e \leq \frac{1}{n - 1}\binom{N + n - 1}{N} - \frac{n^2 + 1}{n - 1}.$$

$\square$

Chapter 3

Rational curves on hypersurfaces

The main result of this chapter is to prove a similar theorem to the previous chapter, for general rational curves on general hypersurfaces. We basically follow the same idea as in the case of curves in projective space. For numerical reason we will need to degenerate the curve along with the hypersurface. This will naturally add some complications to our arguments.

3.1 Preliminaries

First we review some known facts about rational curves on Fano hypersurfaces. For the next two definitions we assume Y is a smooth projective variety.

Definition 3.1. Let r be a nonnegative integer. A rational curve $f : \mathbb{P}^1 \to Y$ on a smooth variety is $r-$free if $f^* T_Y \otimes \mathcal{O}_{\mathbb{P}^1}(-r)$ is generated by its global sections. We will say free instead of $0-$free.

Recall that a variety Y of dimension n is called *uniruled* if there exists a variety of dimension $n - 1$ and a dominant rational map $\mathbb{P}^1 \times Z - -- > Y$. In other words a variety is uniruled if every closed point $p \in Y$ is contained in the image of a finite map $f : \mathbb{P}^1 \to Y$.

Theorem 3.1. *(Mori) Every Fano variety of positive dimension is uniruled.*

This fact and Proposition (4.9) in [4] imply that:

Proposition 3.2 ([4], corollary 4.11)**.** *If the characteristic is zero and Y is uniruled and projective, there exist a free rational curve through a general point of Y.*

Therefore when $Y \subset \mathbb{P}^N$ is a smooth irreducible hypersurface of degree $d \leq N$ (i.e a Fano hypersurface), through a general point $p \in Y$ there exists a free rational C. Hence

$$T_Y|_C = \mathcal{O}(a_1) \oplus \cdots \mathcal{O}(a_{N-1})$$

where $a_i \geq 0$, $i = 1, ..., N-1$. So $H^1(T_Y|_C(-1)) = 0$, i.e $T_Y|_C$ is semi-positive. From this and the short exact sequence

$$0 \to T_C \to T_Y|_C \to N_{C/Y} \to 0$$

we get that $H^1(N_{C/Y}(-1) = 0$.

Corollary 3.3. *Let $Y \subset \mathbb{P}^N$ be a smooth irreducible hypersurface of degree $d \leq N$. Through a general point of Y there exists a rational curve C such that $N_{C/Y}$ is semi-positive.*

For $N \geq 4$, let $Y \subset \mathbb{P}^N$ be a smooth irreducible hypersurface of degree $d-1 \leq N-1$ and $H \subset \mathbb{P}^N$ a hyperplane intersecting Y transversely. For integers $t \geq 0$ and $b \geq 1$, let $\mathcal{Y}$ be the parametrizing space of ordered tuples $(C, L_1, ..., L_t)$, where $C \cup_{i=1}^t L_i \subset Y$ is a smooth admissible curve of type (b, t) intersecting H properly and C is a non-degenerate rational curve of degree b. Let $\mathcal{T}$ be the parametrizing space of $t + 1$ distinct ordered points in $H \cap Y$. We have the incidence correspondence:

$$\mathcal{I} := \{(C, L_1, ..., L_t, q, p_1, ..., p_t | q \in C, p_r \in L_r \cap H, (C, L_1, ..., L_t) \in \mathcal{Y}\}.$$

Lemma 3.4. *If π_1 and π_2 denote the projections from $\mathcal{I}$ to $\mathcal{Y}$ and $\mathcal{T}$ respectively, then π_1 is surjective and the restriction of π_2 to every irreducible component of $\mathcal{I}$ is dominant.*

Proof. First we calculate $\dim \mathcal{I}$. We know by Lemma (3.2) and its corollary that for general C on Y, the normal bundle $N_{C/Y}$ is semi-positive, i.e $\dim H^1(N_{C/Y}(-1)) = 0$. Hence for $[X] \in \mathcal{Y}$ where $X = C \cup L_1 \cup \cdots L_t$, we have $H^1(N_{X/Y}(-1)) = 0$ and

$$h^0(N_{X/Y}) = h^0(T_Y|_X) - h^0(T_X)$$
$$= b(N - d + 2) + N - 4 + t(2N - d - 2)$$

implies that there is an irreducible component $\mathcal{U}$ of $\mathcal{Y}$ of dimension at least $b(N - d + 2) + N - 4 + t(2N - d - 2)$. So we replace $\mathcal{Y}$ by $\mathcal{U}$. The fibers of π_1 are zero dimensional, hence

$$\dim \mathcal{I} = \dim \mathcal{U} \geq b(N - d + 2) + N - 4 + t(2N - d - 2).$$

For $p := (q, p_1, ..., p_t) \in \pi_2(\mathcal{I})$,

$$\dim {\pi_2}^{-1}(p) \leq h^0(N_{C/Y}(-q)) + h^0(N_{L_1/Y}(-p_1)) + ... + h^0(N_{L_t/Y}(-p_t))$$
$$= t(N - d) + b(N - d + 2) - 2$$
$$= (b + t)(N - d + 2) - 2(t + 1),$$

which is strictly positive by the assumption $d \leq N$. Thus

$$\dim \pi_2(\mathcal{I}) \geq \dim \mathcal{I} - \dim {\pi_2}^{-1}(p)$$

$$\geq b(N - d + 2) + N - 4 + t(2N - d - 2)$$

$$- [(b + t)(N - d + 2) - 2(t + 1)]$$

$$= (N - 2)(t + 1) = \dim \mathcal{T}.$$

Hence π_2 is dominant. $\qquad\square$

Corollary 3.5. *Using the notation of the above lemma, assume that there exists an* $[X_0] \in \mathcal{Y}$ *where* X_0 *is a smooth admissible curve of type* (b, t) *which is* $n-$*normal for some integer* $n \geq 1$ *and its relative normal bundle* $N_{X_0/Y}$ *is semi-positive. Let*

$$\mathcal{M} := \left\{ \pi_1^{-1}(X) | [X] \in \mathcal{Y} \text{ and } X \text{ is } n\text{-normal} \right\}.$$

Then $\mathcal{M} \subset \mathcal{I}$ *is dense and open and the restriction of* π_2 *to* $\mathcal{M}$ *is dominant.*

In other words, over a dense non-empty set in $\mathcal{T}$*, the fibers of* π_2 *are non-empty and contain a smooth admissible curve of type* (b, t) *which intersects* H *transversely and is* $n-$*normal.*

Proof. Let $\mathcal{Z}$ be the parametrizing space of all ordered tuples $(C, l_1, \cdots, l_t)$ where $X := C \cup (\cup_{i=1}^{t} l_i)) \subset \mathcal{Y}$ is a *smooth* admissible curve of type (b, t) and C is non-degenerate. Suppose such an $X_0 \in \mathcal{Y}$ (by our assumption) exists. This implies that there is an irreducible component of $X_0 \in \mathcal{U} \subset \mathcal{Z}$ of smooth elements which are $n-$normal and of type (b, t). Note that by [3], the parametrizing space of smooth rational curves of a given degree on Y is irreducible when $\deg Y \leq N/2$, so is the parametrizing space of t disjoint lines in Y. Note that $\mathcal{Y} \subset \mathcal{Z}$ is open (the curves intersecting H non-transversely form a closed set) and $X_0 \in \mathcal{Y} \cap \mathcal{U}$.

Hence $\mathcal{U} \cap \mathcal{Y} \subset \mathcal{Z}$ is open and non-empty. But π_1 is surjective, therefore $\pi_1^{-1}(\mathcal{U} \cap \mathcal{Y}) \subset \mathcal{I}$ is open and $\pi_2|_{\pi_1^{-1}(\mathcal{U} \cap \mathcal{Y})}$ is dominant. $\square$

Lemma 3.6. *Let t, b and H be as in the previous lemma. Furthermore let*

$$\mathcal{J} := \{(C, p_1, ..., p_t, p_{t+1}) | p_i \in C, [C] \in \mathcal{V}\},$$

where $\mathcal{V}$ is the parametrizing space of smooth rational curves of degree b on the hyperplane H. Then the projection $\pi_1 : \mathcal{J} \to \mathcal{V}$ is surjective and if $\lfloor \frac{Nb-2}{N-2} \rfloor - t \geq 0$, then $\pi_2 : \mathcal{J} \to \mathcal{T}$ is dominant.

Proof. With the same calculation as in the previous lemma (except that there are no L_i's here) we get $\dim \mathcal{J} = (b+1)N - 4$. For $p := (p_1, ..., p_{t+1}) \in \pi_2(\mathcal{J})$, $\dim \pi_2^{-1}(p) \leq h^0(N_{C/H}(-p_1 - ... - p_{t+1}))$.

By [12], for a general curve C in the projective space $\mathbf{P}^{N-1} = H$, the vector bundle $N_{C/H}$ is balanced. We first show that $H^1(N_{C/H}(-p_1 - ... - p_{t+1})) = 0$. Note that $N_{C/H}$ is a balanced vector bundle of degree $Nb-2$ and rank $N-2$. Therefore, to show $H^1(N_{C/H}(-p_1 - ... - p_{t+1})) = 0$, it suffices to check $\lfloor \frac{Nb-2}{N-2} \rfloor - (t+1) \geq -1$, which is part of the assumption. Hence by Riemann-Roch we get $h^0(N_{C/H}(-p_1 - ... - p_{t+1})) = N(b-t) + 2t - 2$ which is the fiber dimension of π_2. Thus,

$$\dim \pi_2(\mathcal{J}) \geq \dim \mathcal{J} - \dim \pi_2^{-1}(p)$$
$$\geq (b+1)N - 4 - [N(b-t) + 2t - 2]$$
$$= (N-2)(t+1) = \dim \mathcal{T}.$$

Hence π_2 is dominant. $\square$

Corollary 3.7. *With notation of the previous lemma, let $\mathcal{M}$ be the subscheme of $\mathcal{V}$ parametrizing smooth irreducible $n-$normal curves with semi-positive relative normal*

bundle. Then $\pi_1^{-1}(\mathcal{M})$ is open and hence if it is non-empty the restriction of π_2 to $\pi_1^{-1}(\mathcal{M})$ is dominant.

In other words, there exists an open dense subset of $\mathcal{T}$ over which the fibers of π_2 are non-empty and contain an element which is $n-$normal and its relative normal bundle is sami-positive.

Lemma 3.8. *Suppose $C \subset \mathbb{P}^N$ is a smooth curve (possibly not connected) such that $r_C(n) : H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) \to H^0(C, \mathcal{O}_C(n))$ is surjective for some $n \geq 1$. Let $V_C := H^0(I_C(n))$ and for any hypersurface $Y \subset \mathbb{P}^N$ of degree r, let $V_Y := H^0(I_Y(n))$. Then for a hypersurface Y of degree r not containing any component of C:*

1. *$V_C \cap V_Y = H^0(I_C(n-r))$ and hence $\dim(V_C \cap V_Y) = h^0(I_C(n-r))$.*

2. *For an integer $\delta \geq 0$, let C_δ denote the union of C and δ general points in Y. Then for*
$$0 \leq \delta \leq h^0(I_C(n)) - h^0(I_C(n-r)),$$
   the map $r_{C_\delta}(n) : H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) \to H^0(C_\delta, \mathcal{O}_{C_\delta}(n))$ is surjective.

Proof. Let $f \in V_C \cap V_Y$. Then f is a polynomial of degree n vanishing on C and divisible by the equation defining Y, call it g_Y. But g_Y does not vanish on any component of C, therefore $\frac{f}{g}$ (which is a polynomial of degree $n-r$) has to vanish on C. This gives a bijection between $V_C \cap V_Y$ and $H^0(I_C(n-r))$.

For (2) we do induction on δ. The statement is correct when $\delta = 0$. Assume the lemma holds for $\delta - 1$. To prove it for δ note that surjectivity of $r_{C_{\delta-1}}(n)$ implies

$$h^0(I_{C_{\delta-1}}(n)) = h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) - h^0(C_{\delta-1}, \mathcal{O}_{C_{\delta-1}}(n))$$
$$= h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(n)) - h^0(C, \mathcal{O}_C(n)) - (\delta - 1) > 0.$$

Now let $Z \subset \mathbb{P}^N$ be a hypersurface of degree n such that $Z \supset C_{\delta-1}$ and let $p_\delta \notin Z$ be a general point. Let $Z = \{f = 0\}$, then $f \in H^0(I_{C_{\delta-1}}(n))$ but $f \notin H^0(I_{C_\delta}(n))$. Hence

$h^0(I_{C_\delta}(n)) = h^0(I_{C_{\delta-1}}(n)) - 1$. Thus the map is surjective. $\qquad\qquad\square$

3.2 Deformation of nested schemes

The aim in this section is to show that rational trees on certain singular hypersurface are smoothable along with the hypersurface. We first discuss some general facts about deformation of nested(flag) schemes. More details on nested (flag) Hilbert schemes can be found in [13] and [8].

Let $X \subset Y \subset \mathbb{P}^N$ be two closed subschemes of the projective space $\mathbb{P}^N$, with Hilbert polynomials P and Q respectively. Then there exists a projective scheme $\mathcal{HF} := Hilb^{P,Q}$ called the Hilbert-Flag scheme of the pair $X \subset Y$, parametrizing all such pair of closed subschemes of $\mathbb{P}^N$ having Hilbert polynomial P and Q.

Definition 3.2. ([13], D.1) An embedding of schemes $j : X \subset Y$ is a regular embedding of codimension n at the point $x \in X$ if $j(x)$ has an affine open neighborhood $Spec(R)$ in Y such that the ideal of $j(X) \cap Spec(R)$ in R can be generated by a regular sequence of length n. If this happens at every point of X we say that j is a regular embedding of codimension n.

Remark: A flag $X \subset Y$ of closed subschemes of a projective scheme Z is said to be *regularly embedded in Z* if both $X \subset Y$ and $Y \subset Z$ are regular embeddings.

Remark: If $X \subset Y$ is a regular embedding of codimension n, then $\mathcal{I}/\mathcal{I}^2$ and $N_{X/Y}$ are both locally free of rank n.

For closed subschemes $X \subset Y \subset \mathbb{P}^N$, let $N_X := \mathcal{H}om_{\mathcal{O}_X}(I_X/I_X{}^2, \mathcal{O}_X)$ and $N_{X/Y} := (I_{X/Y}/I_{X/Y}{}^2)^\curlyvee$ be the normal sheaves of X in $\mathbb{P}^N$ and relative normal sheaf of X in Y respectively. When both X and Y are smooth, we have the short exact sequence

of locally free sheaves:

$$0 \to N_{X/Y} \to N_X \to N_Y \otimes \mathcal{O}_X \to 0$$

Let $\mathcal{N}$ be the pullback in the following diagram:

$$
\begin{array}{ccc}
\mathcal{N} & \longrightarrow & N_Y \\
\downarrow & & \downarrow \\
N_X & \longrightarrow & N_Y \otimes \mathcal{O}_X
\end{array}
\qquad (3.1)
$$

We will need both parts of the following proposition ([13], prop 4.5.3):

Proposition 3.9. *Let $X \subset Y$ be a flag in $\mathbb{P}^N$. Then:*

(I) *There is a natural identification:* $\mathrm{T}_{[X \subset Y], \mathcal{HF}} = H^0(\mathbb{P}^N, \mathcal{N})$. *Hence with $\mathcal{N}$ defined as the pullback in the above diagram, we get*
$$\mathrm{T}_{\{X \subset Y\}, \mathcal{HF}} = H^0(N_X) \times_{H^0(N_Y \otimes \mathcal{O}_X)} H^0(N_Y).$$

(II) *If $X \subset Y$ and $Y \subset \mathbb{P}^N$ both are regular embeddings, then the obstruction space of the local ring $\mathcal{O}_{\mathcal{HF}, [X \subset Y]}$ is contained in $H^1(\mathbb{P}^N, \mathcal{N})$.*

With the same assumptions as in Proposition (3.9), let $T_{[X \subset Y]}$ be the tangent space of Flag-Hilbert scheme at the point $[X \subset Y]$. From the two projections of this tangent space to its factors along with the long exact sequence of cohomology induced by the above short exact sequence of normal bundles, we get

$$
\begin{array}{ccc}
T_{[X \subset Y]} & \longrightarrow & H^0(N_Y) \\
\downarrow & & \downarrow \\
\end{array}
$$
$$0 \longrightarrow H^0(N_{X/Y}) \longrightarrow H^0(N_X) \xrightarrow{\psi} H^0(N_Y \otimes \mathcal{O}_X) \longrightarrow H^1(N_{X/Y})$$

where $H^0(N_Y)$ is the space of infinitestimal deformations of Y. To show X can be smoothed along with Y, it suffices to show the map $\mathcal{HF} \longrightarrow \mathcal{Y}$ is dominant,

where $\mathcal{Y}$ is the Hilbert scheme of $Y \subset \mathbb{P}^N$. For that we will show the corresponding map of tangent spaces i.e $T_{[X \subset Y]} \longrightarrow H^0(N_Y)$ is surjective at some smooth point $[X \subset Y] \in \mathcal{HF}$. But by the above diagram it is equivalent to show

$$\psi : H^0(N_X) \to H^0(N_Y \otimes O_X)$$

is surjective, i.e we need to show $H^1(N_{X/Y})$ vanishes.

In order to compute the above cohomology, we first show the relative ideal sheaf $I_{X/Y}/I^2{}_{X/Y}$ is locally free and the sequence of normal sheaves is exact. Then we prove

$$H^1(N_{X/Y}) = 0,$$

which we need to get the surjectivity of ψ, also holds.

Lemma 3.10. *Let $Y := Y_1 \cup Y_2$ be the union of two smooth irreducible hypersurfaces in $\mathbb{P}^N$ meeting transversally. Suppose $X \subset Y$ is a rational tree so that the singular points of X lie on $Y_1 \cap Y_2$, but none of the irreducible components of X lie on $Y_1 \cap Y_2$. Then the relative ideal sheaf $I_{X/Y}/I_{X/Y}{}^2$ is locally free.*

Proof. The statement is local and correct away from singular points of the curve. Hence we only have to verify the lemma at singular points. Let p be the nodal singular point of the smooth irreducible curves $C_1 \subset Y_1$ and $C_2 \subset H$ meeting at p. Let the regular sequence $(x_1, ..., x_N)$ be the maximal ideal of $p \in P^N$. We may assume $I_{C_1,p} = (x_2, ..., x_N)$, where $I_{Y_1,p} = (x_2)$. Now suppose $I_{C_2,p} = (y_2, ..., y_N)$ where $I_{Y_2,p} = (y_2)$. By the assumption of $C_1 \nsubseteq Y_2$ and $C_2 \nsubseteq Y_1$, we have $x_2 \notin (y_2, ..., y_N)$ and $y_2 \notin (x_2, ..., x_N)$. Using the transversality of C_1 and C_2, we get $(x_1, ..., x_N) = (y_2, x_2, ..., x_N)$ and therefore can write:

$$y_2 = a_1 x_1 + ... + a_N x_N,$$

$$x_1 = by_2 + b_2 x_2 + ... + b_N x_N.$$

Hence, modulo $(x_2, ..., x_N)$, we have $x_1 = y_2$. Therefore, $I_{C_2,p} = (x_1, y_3, ..., y_N)$ where $I_{Y_2,p} = (x_1)$. Using the transversality one more time, we get $(x_1, x_2, ..., x_N) = (x_1, x_2, y_3, ..., y_N)$ which implies $(x_3, ..., x_N) = (y_3, ..., y_N)$ modulo (x_1, x_2). Hence $I_{X=C_1 \cup C_2} = (x_2 y_2, x_3, ..., x_N) = (x_1 x_2, x_3, ..., x_N)$ and $I_{X/Y} = (x_3, ..., x_N)$. Thus, $I_{X/Y}/I_{X/Y}^2$ is locally free. $\square$

Corollary 3.11. *With the same assumptions of the lemma above, we have the sequence of locally free sheaves:*

$$0 \to N_{X/Y} \to N_X \to N_Y \otimes \mathcal{O}_X \to 0.$$

Proof. Y is a hypersurface of degree d, so $N_Y = \mathcal{O}_Y(d)$. The first term in the sequence below is a line bundle. A non-zero map from a line bundle to a locally free sheaf is injective, so the first map in this sequence is an injective map of sheaves:

$$0 \to I_Y/I_Y^2 \otimes \mathcal{O}_X \to I_X/I_X^2 \to I_{X/Y}/I_{X/Y}^2 \to 0.$$

However, we already showed that the cokernel is a vector bundle, which makes this an exact sequence of vector bundles. $\square$

For the next lemma, we need the following form of Grothendieck duality:

Proposition 3.12. *If $X \subset Z$ is a closed subscheme of codimension d, and if both X and Z are Cohen-Macauly, then $\mathcal{E}xt^d(\mathcal{O}_X, \omega_Z) = \omega_X$.*

Corollary 3.13. *For Z a rational tree and $X \subset Z$ an irreducible subcurve, we have $\mathcal{H}om(\mathcal{O}_X, \omega_Z) = \omega_X$.*

Lemma 3.14. *Keeping the hypothesis of Lemma (3.10), further assume $\deg Y_2 = 1$, $\deg Y_1 = d - 1 \leq N - 1$, and that $X := C_1 \cup C_2 \cup ... \cup C_{t+2}$ is a rational tree with*

$t + 1$ *singular points. Assume* $C_i \subset Y_1$, $1 \le i \le t + 1$, *are disjoint union of general smooth rational curves, and* $C_{t+2} \subset Y_2$ *is a general smooth smooth rational curve meeting every other* C_i *at exactly one point. If* $\lfloor \frac{Nb-2}{N-2} \rfloor \ge t$ *where* $b = \deg C_{t+2}$, *then* $H^1(N_{X/Y}) = 0$.

Proof. We already showed that $I_{X/Y}/I_{X/Y}{}^2$ a locally free sheaf. Hence

$$H^1(N_{X/Y}) = H^0(I_{X/Y}/I_{X/Y}{}^2 \otimes \omega_X).$$

To show it vanishes, tensor the inclusion below by ω_X:

$$I_{X/Y}/I_{X/Y}{}^2 \hookrightarrow I_{C_1/Y_1}/I_{C_1/Y_1}{}^2 \oplus I_{C_1/Y_1}/I_{C_1/Y_1}{}^2 \oplus$$

$$\cdots \oplus I_{C_{t+1}/Y_1}/I_{C_{t+1}/Y_1}{}^2 \oplus I_{C_{t+2}/Y_2}/I_{C_{t+2}/Y_2}{}^2.$$

It suffices to show $H^0(I_{C_{t+2}/Y_2}/I_{C_{t+2}/Y_2}{}^2 \otimes \omega_X) = H^0(I_{C_i/Y_1}/I_{C_i/Y_1}{}^2 \otimes \omega_X) = 0$ for all $1 \le i \le t + 1$. Let $(C_1 \cup ... \cup C_{t+1}) \cap C_{t+2} = \{p_1, ..., p_{t+1}\}$ where $\{p_i\} = C_i \cap C_{t+2}$. Consider the following exact sequence

$$0 \to \mathcal{O}_{C_1 \cup ... \cup C_{t+1}}(-p_1 - ... - p_{t+1}) \to \mathcal{O}_X \to \mathcal{O}_{C_{t+2}} \to 0.$$

Taking $\mathcal{H}om(-, \omega_X)$ of this exact sequence we get

$$0 \to \omega_{C_{t+2}} \to \omega_X \to \omega_{C_1 \cup ... \cup C_{t+1}}(p_1 + ... + p_{t+1}) \to 0.$$

Restricting this exact sequence to C_1 and the fact that restriction is right exact, we get $\omega_X|_{C_1} \simeq \omega_{C_1}(p_1)$. Hence

$$H^0(I_{C_1/Y_1}/I_{C_1/Y_1}{}^2 \otimes \omega_X \,|_{C_1}) \simeq H^0(I_{C_1/Y_1}/I_{C_1/Y_1}{}^2 \otimes \omega_{C_1}(p_1)) \simeq H^1(N_{C_1/Y_1}(-p_1)).$$

But $\deg(N_{C_1/Y_1}) = e_1(N - d + 2) - 2 \geq 0$ where e_1 and $d - 1$ are the degrees of C_1 and Y_1 respectively. Because $\deg Y_1 = d - 1 < N$, by Corollary (3.3) N_{C_1/Y_1} is semipositive and therefore $H^1(N_{C_1/Y_1}(-p_1)) = 0$. Similarly, $H^1(N_{C_i/Y_1}(-p_i)) = 0$ for all $1 \leq i \leq t$.

To show the vanishing of $H^0(I_{X_{C_{t+2}}/Y_2}/I_{X_{C_{t+2}}/Y_2}{}^2 \otimes \omega_X)$, recall that $C_{t+2} \subset Y_2$. Restricting the above sequence of canonical sheaves to C_{t+2}, we get:

$$H^0(I_{C_{t+2}/Y_2}/I^2{}_{C_{t+2}/Y_2} \otimes \omega_X \,|_{C_2}) = H^0(I_{C_{t+2}/Y_2}/I^2{}_{C_{t+2}/Y_2} \otimes \omega_{C_{t+2}(p_1 + \ldots + p_{t+1})})$$

$$= H^1(N_{C_{t+2}/H}(p_1 + \ldots + p_{t+1})).$$

By the main result of [12], we know that the normal bundle of a general smooth rational curve is balanced. Therefore Y_2 being a hyperplane and $C_{t+2} \subset Y_2$ a general rational curve, it follows that N_{C_{t+2}/Y_2} is balanced. But $deg(N_{C_{t+2}/Y_2}) = bN - 2$ where $\deg C_{t+2} = b$ and the rank of N_{C_{t+2}/Y_2} is $N - 2$. Therefore the degrees of the line bundles appearing in the decomposition of $N_{C_{t+2}/H}$ are at least $\lfloor \frac{bN-2}{N-2} \rfloor$. So, in order to get the vanishing of $H^1(N_{C_{t+2}/Y_2}(-p_1 - \ldots - P_{t+1}))$, we require $\lfloor \frac{bN-2}{N-2} \rfloor - (t+1) \geq -1$. But this inequality holds by the assumption. $\qquad \square$

Proposition 3.15. *For a rational tree X, $H^1(N_X) = 0$.*

Proof. By the following two short sequences, it suffices to show $H^1(X, \mathcal{O}_X) = 0$.

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1) \oplus \ldots \oplus \mathcal{O}_X(1) \longrightarrow T_{\mathbb{P}^N}|_X \longrightarrow 0,$$

$$0 \longrightarrow T_X \longrightarrow T_{\mathbb{P}^N}|_X \longrightarrow N_X \longrightarrow 0.$$

We show $H^1(O_X) = 0$ by induction on r, the number of irreducible components. When $r = 1$, X is a smooth irreducible rational curve and $H^1(N_X) = 0$ by Riemman-Roch. Suppose the lemma holds for trees with $r - 1$ components and let $X =$

$C_1 \cup ... \cup C_r$ be a rational tree with r components. Consider the following sequence

$$0 \longrightarrow \mathcal{O}_{C_r}(-p) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{C_1 \cup ... \cup C_{r-1}} \longrightarrow 0,$$

and note that $H^1(\mathcal{O}_{C_1 \cup ... \cup C_{r-1}}) = 0$ by induction and $H^1(\mathcal{O}_{C_r}(-p)) = 0$ because C_r is a smooth irreducible rational curve. Hence, the first cohomology of the middle sheaf is zero. $\square$

Proposition 3.16. *Let $X \subset Y$ be as in Lemma (3.10). If $H^1(N_{X/Y}) = 0$, then the Hilbert-flag scheme $\mathcal{HF}$ is smooth at the point $[X \subset Y]$.*

Proof. By Proposition (3.9), to show the smoothness of $\mathcal{HF}$ at the point $[X \subset Y]$, it suffices to show $H^1(\mathbb{P}^N, \mathcal{N}) = 0$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I_X N_Y & \longrightarrow & \mathcal{N} & \longrightarrow & N_X & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_X N_Y & \longrightarrow & N_Y & \longrightarrow & N_Y|_X & \longrightarrow & 0
\end{array}
$$

Therefore we get the long exact sequence of cohomology:

$$
\begin{array}{ccccccccc}
0 \to H^0(I_X N_Y) & \to H^0(\mathcal{N}) & \to H^0(N_X) & \to H^1(I_X N_Y) & \to H^1(\mathcal{N}) \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 \to H^0(I_X N_Y) & \to H^0(N_Y) & \to H^0(N_Y|_X) & \to H^1(I_X N_Y) & \to H^1(N_Y) = 0
\end{array}
$$

So an element $\alpha \in H^1(\mathcal{N})$ corresponds to some element in $H^1(I_X N_Y)$, which by surjectivity of the maps it finally corresponds to an element in $H^0(N_X)$. But by exactness of the sequence, an element in $H^0(N_X)$ is mapped to zero in $H^1(\mathcal{N})$, hence $\alpha = 0$. So $H^1(\mathcal{N}) = 0$. Noticed that we used the fact $H^1(N_X) = 0$ which was proved in the previous proposition. $\square$

Corollary 3.17. *Under the assumption as in the proposition above, the map $\psi : H^0(N_X) \to H^0(N_Y \otimes \mathcal{O}_X)$ is surjective and the therefore X can be smoothed along*

with Y.

3.3 Induction Argument

Fix integers $N \geq 4$, $1 \leq d \leq N$ and $k \geq 0$, $e \geq 1$. Let $n := \nu(e, k, d, N)$, which we call the *value of (e, k) with respect to N and d*, is defined as the minimum over all positive integers m satisfying the following:

$$n := \min \left\{ m \geq 1 \mid 1 + me + k(m+1) \leq \binom{N+m}{N} - \binom{N+m-d}{N} \right\}.$$

For integers $n \geq 1$ and $1 \leq d \leq N$, let $H(n, d)$ be the following statement:

H(n,d) : For all choices of integers $k \geq 0$ and $e \geq 1$ such that $\nu(e, k, d, N) \leq n$, if $k \leq n$ then there exists a possibly singular hypersurface $Y \subset \mathbb{P}^N$ of degree d and an admissible curve $X \subset Y$ of type (e, k) such that the relative normal bundle of all its irreducible component in Y are semi-positive and X is $n-$normal. We require that if Y is singular, then Y should be the union of two smooth irreducible hypersurfaces meeting transversely. Also the number of singular points of X should be less than n.

We will use the following proposition throughout the proof of the main theorem.

Proposition 3.18. *Suppose X is a rational tree and Y is a hypersurface satisfying $H(n, d)$. Then X is smoothable along with the hypersurface Y.*

Proof. The proposition follows by the definition of a curve satisfying $H(n, d)$, along with corollary (3.17). $\qquad\square$

We prove $H(n, d)$ by induction on both n and d. We fix $N \geq 4$.

Theorem 3.19. *$H(n, d)$ holds for any $n \geq 1$, $1 \leq d \leq N$.*

Corollary 3.20. *Let* $n = \nu(e, k, d, N)$. *Let* C *be a general rational curve of degree* e *on a general hypersurface of degree* $d \leq N$ *in* $\mathbb{P}^N$. *Then the regularity index of* C *is* $\nu(e, 0, d, N) + 1$.

3.3.1 Base case of the induction

Proof. As a result of [1], $H(n, 1)$ holds for any n. It is proved in [?] that a disjoint union of general rational curves in $\mathbb{P}^N$ for $N \geq 4$ is of maximal rank.

$H(1, d)$: The case $d = 1$ has already been considered above. So we show $H(1, d)$ for $2 \leq d \leq N$. By definition of $H(n, d)$ and the fact that $n - d = 1 - d < 0$ and hence $\binom{N+n-d}{N} = 0$, we need to prove that for a fixed N, if $e + 2k \leq N$ then there is a hypersurface $Y \subset \mathbb{P}^N$ of degree d and $X \subset Y$ where X is a disjoint union of a rational tree of degree e and k lines and is of maximal rank, meaning:

$$r_X(1) : H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \twoheadrightarrow H^0(X, \mathcal{O}_X(1)).$$

So fix $2 \leq d \leq N$ and $e \leq N$. Let $C \subset \mathbb{P}^N$ be a smooth rational normal curve of degree e. Its ideal is generated by quadric terms, and therefore there are always hypersurfaces of any degree $d \geq 2$ containing it.

Now if $e + 2k \leq N$ for some $k \geq 1$, let $C \subset Y$ be a normal rational curve of degree e as above (i.e. so that $r_C(1)$ is surjective). Let $L_1 \subset Y$ be a line so that

$$L_1 \cap \operatorname{linspan}(C) = \emptyset$$

Note that this is possible since $k \geq 1$ and hence $\dim \operatorname{linspan}(C) = e \leq N - 2$. Thus

$r_{C \cup L_1}(1)$ is surjective. We can inductively find a line $L_r \subset Y$ such that

$$L_r \cap \text{linspan}(C \cup L_1 \cup ... \cup L_{r-1}) = \emptyset$$

as long as $2r \leq N - e$. Hence we get $H(1, d)$ for $d \leq N$. $\qquad \square$

3.3.2 The Induction Step

The aim of this section is to prove $H(n, d)$ for $n \geq 2$ and $d \geq 2$. Before we proceed let's note that if $n \geq 2$ and $\nu(e, k, d, N) = n$ for some $k \leq n$, then $e \geq 2$ except when $N = 4$ and $n = k = 2$, by (4.7). We first prove $H(2, d)$ when $e = 1$ and $k = 2$, and then prove $H(n, d)$ for all other cases, for which the assumption $e \geq 2$ will be essential.

When $e = 1$ and $k = n = 2$, $H(2, 4)$ is a statement about a disjoint union of three lines. We prove the following:

Proposition 3.21. *Let $\mathcal{L}$ be the parametrizing space of disjoint union of three lines in $\mathbb{P}^4$ and $\mathcal{H}$ be the set of all hypersurfaces $X \subset \mathbb{P}^4$ of degree d, $2 \leq d \leq 4$.*

$$\mathcal{I}_d := \{(L_1, L_2, L_3, X) | L_i \subset X \in \mathcal{H}\}$$

$$\begin{array}{ccc} & \mathcal{I}_d & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathcal{L} & & \mathcal{H} \end{array}$$

Let π_1 and π_2 be the projections from $\mathcal{I}_d$ to $\mathcal{L}$ and $\mathcal{H}$. The map π_2 is dominant.

Proof. Note that for a line $L \subset \mathbb{P}^N$, the normal bundle is of the form

$$\mathcal{N}_L = \bigoplus \mathcal{O}^{N-1}(1).$$

Hence when $N = 4$, $\dim \mathcal{L} = 3h^0(\mathcal{N}_L) = 18$. Note that for $(L_1, L_2, L_3) \in \mathcal{L}$,

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$\pi_1^{-1}(L_1, L_2, L_3) \in \mathcal{L}$ is the space of hypersurfaces of degree d in $\mathbb{P}^4$ containing (L_1, L_2, L_3).

So

$$\dim \pi_1^{-1}(L_1, L_2, L_3) = h^0(\mathcal{O}_{\mathbb{P}^4}(d)) - h^0(\mathcal{O}_{L_1 \cup L_2 \cup L_3}(d))$$
$$= \binom{4+d}{4} - 3d - 3,$$

is a positive number for values $d = 2, 3, 4$. Hence π_1 is surjective. Therefore we can compute the dimension of $\mathcal{I}_d$ as

$$\dim \mathcal{I}_d = \dim \mathcal{L} + \binom{4+d}{4} - 3d - 3$$
$$= \binom{4+d}{4} - 3d + 16.$$

Now if π_2 is not dominant, then $\dim Im(\pi_2) \leq \dim \mathcal{H} - 1$. Hence by a fiber dimension consideration we get that for $X \in \mathcal{H}$

$$3h^0(\mathcal{N}_{L/X}) = \dim \pi_2^{-1}(X) \geq \dim \mathcal{I}_d - [\dim \mathcal{H} - 2]$$
$$= 16 - 3d. \tag{3.2}$$

But for $L \subset X \subset \mathbb{P}^N$ where L and X are general, we have (see [10], p. 269)

$$\mathcal{N}_{L/X} = \mathcal{O}^{d-1} \oplus \mathcal{O}^{N-1-d}(1)$$

when $d \leq N - 1$, and

$$\mathcal{N}_{L/X} = \mathcal{O}^{2n-3-d} \oplus \mathcal{O}^{d-n+1}(-1)$$

for $d \geq N - 1$. Therefore for $d = 2, 3, 4$ the inequality (3.2) is reduced to $9 \geq 10$, $6 \geq 7$ and $3 \geq 4$, respectively, contradiction in all cases. Therefore π_2 is dominant.

$\square$

Proof. Assume $H(i,j)$ for $i \leq n-1$ and $j \leq d$. To show $H(n,d)$, note that we already discussed the cases $d = 1$ and $n = 1$, and so we may assume from now on that $n \geq 2$ and $d \geq 2$. Suppose $e \geq 1$ and $0 \leq k \leq n$ are integers satisfying $\nu(e,k,d,N) \leq n$. But the cases where $\nu(e,k,d,N) \leq n-1$ and $k \leq n-1$ are already known by our assumption $H(n-1,d)$. So we may only consider the cases $\nu(e,k,d,N) = n$ and $\nu(e,n,d,N) = n-1$.

Suppose $\nu(e,k,d,N) = n$. Let $a < e$ be the largest integer such that

$$1 + (n-1)a \leq \binom{N+n-1}{N} - \binom{N+n-d}{N}.$$

Having fixed a, let $k' < k$ be the largest integer satisfying

$$1 + (n-1)a + k'n \leq \binom{N+n-1}{N} - \binom{N+n-d}{N}.$$

Then by the induction hypothesis $H(n-1,d-1)$ and proposition (3.18), on a general hypersurface $Y_1 \subset \mathbb{P}^N$ of degree $d-1$, there exists a smooth admissible curve

$$X_1 = C_1 \cup L_1 \cup ... \cup L_{k'} \subset Y_1$$

of type of type (a,k') which is $(n-1)-$normal and N_{C_1/Y_1} is semi-positive.

Now let H be a general hyperplane containing neither X_1 nor Y_1, but such that they both intersect H transversely. Suppose $H \cap C_1 = \{p_1, p_2, ..., p_a\}$ and $H \cap L_i = p_i'$.

Now the idea is to attach to our curve X_1, a rational curve (or a disjoint union of them) lying on H and passing through some p_i's. The choice of what curves to add in H dependents on whether or not the critical value jumps when e is increased by one. That is, depends on whether $\nu(e,k,d,N) = \nu(e+1,k,d,N)$ or not. We discuss the two cases:
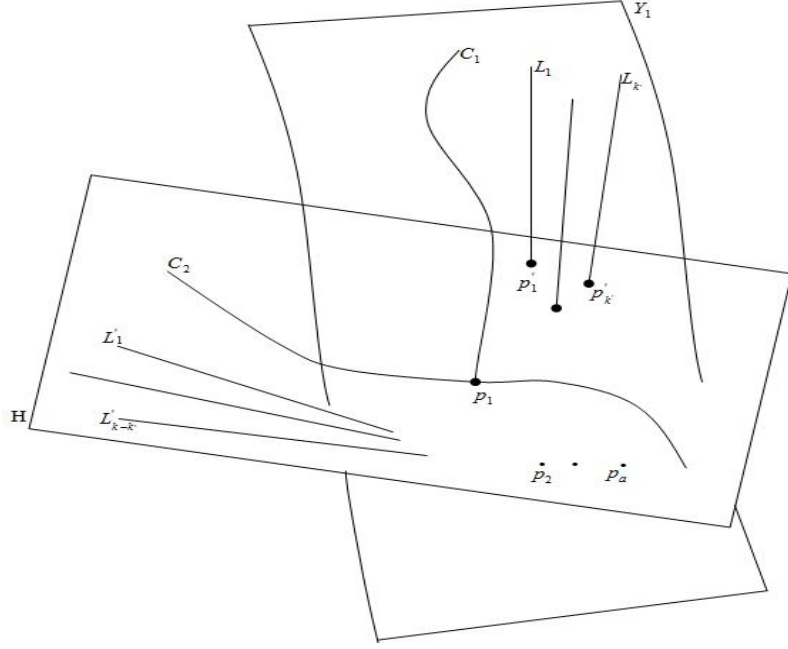
Figure 3.1: Case(A)

Case(A): $\nu(e, k, d, N) = \nu(e+1, k, d, N)$.

By the first remark after Lemma (4.9) in the appendix, we have $\nu(e - a, k - k', 1, N) \leq n$. Thus we can use $H(n, 1)$ for $\bar{e} = e - a$ and $\bar{k} = k - k'$. So by $H(n, 1)$ on a general hyperplane H (with the properties described in the paragraph above) there exists a smooth admissible curve

$$X_2 = C_2 \cup L_{k'+1} \cup ... \cup L_k \subset H,$$

of type $(e - a, k - k')$ such that $N_{C_2/H}$ is semi-positive, the L_i's are disjoint lines and $r_{X_2}(n)$ is surjective. We may assume C_2 passes only through the point p_1 but the L_i's do not pass through any of the points in $H \cap X_1$. Note that here we require the curve C_2 to pass through only one point. That such C_2 exists is easy to verify. So $X = X_1 \cup X_2$ is a disjoint union of a reducible connected rational curve of degree e (with a single nodal singularity) and k disjoint lines.

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The inclusion $X \cap H \subset X$ implies we have an exact sequence

$$\mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_{X \cap H} \to 0$$

and since $X_2 \subset H$, the above sequence reduces to the short exact sequence

$$0 \to \mathcal{O}_{X_1}(-H) \to \mathcal{O}_X \to \mathcal{O}_{X \cap H} \to 0$$

Twisting by $\mathcal{O}_{\mathbb{P}^N}(n)$ and taking cohomologies we get,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^N}(n-1)) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^N}(n)) & \longrightarrow & H^0(\mathcal{O}_H(n)) & \longrightarrow & 0 \\
& & \Big\downarrow r_{X_1}(n-1) & & \Big\downarrow r_X(n) & & \Big\downarrow r_{X \cap H}(n) & & \\
0 & \longrightarrow & H^0(\mathcal{O}_{X_1}(n-1)) & \longrightarrow & H^0(\mathcal{O}_X(n)) & \longrightarrow & H^0(\mathcal{O}_{X \cap H}(n)) & \longrightarrow & 0
\end{array}
$$

The map on the right is surjective by $H(n,1)$ and Proposition (3.18). By Corollary (4.9) and its following remark the right vertical map has the dimensional condition that is required for it to be surjective. Also, Lemma (3.8) ensure to us that adding the points $p_2, ..., p_a, p'_1, ..., p'_{k'}$ to the right vertical map does not affect the surjectivity. The map on the left is surjective by $H(n-1, d-1)$ and Proposition (3.18).

Case(B): $\nu(e, k, d, N) = n < \nu(e+1, k, d, N)$ and $(e, k, d, N) \neq (7, 3, 4, 4)$.

One can easily check that for such values of e, the right map in the diagram above cannot be surjective by comparing the dimensions of the two spaces. To remedy the problem we modify our choice of the curve X_1. Recall that $Y_1 \subset \mathbb{P}^N$ is a smooth irreducible hyersurface of degree $d - 1$ and $H \subset \mathbb{P}^N$ is a hyperplane meeting Y_1 transversely. Set

$$t = \binom{N+n-1}{N} - \binom{N+n-d}{N} - 1 - (n-1)a - k'n.$$
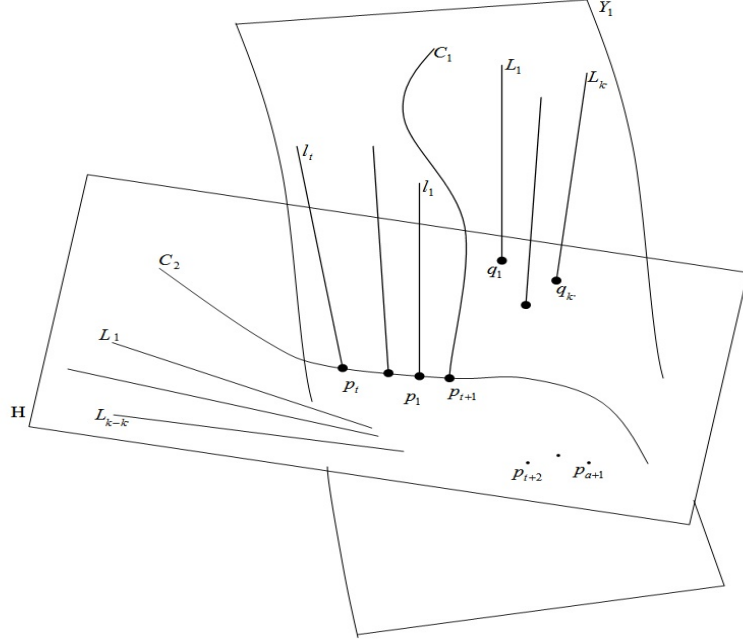
Figure 3.2: Case B

Let $\mathcal{T}$ be the parametrizing space of $t+1$ ordered points in $H \cap Y_1$. Note that by Lemma (4.12) we have $\nu(a-t, t+k', d-1, N) \leq n-1$ and by Lemma (4.11) we have $t + k' \leq n-1$. Hence we may use $H(n-1, d-1)$ and Proposition (3.18)to conclude that there exists a $(n-1)$-normal, smooth admissible curve $X_1 \subset Y_1$ of type $(a-t, t+k')$.

Also by Lemma (4.14), $\nu(e-a, k-k', 1, N) \leq n$. Therefore we may apply $H(n, 1)$ and Proposition (3.18) to conclude the existence of a $n-$normal, smooth irreducible curve $X_2 \subset H$ of type $(e-a, k-k')$.

Now we apply Lemma (3.4) and its corollary in the case $b = a - t$, and Lemma (3.6) and its corollary in the case $b = e - a$, to conclude the following.

There exists a point $P = (p_1, \cdots, p_{t+1}) \in \mathcal{T}$ and a smooth $(n-1)-$normal curve

$$X_1 := C_1 \cup l_1 \cup ... \cup l_t \cup L_1 \cup ... \cup L_{k'} \subset Y_1$$

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of type $(a - t, k' + t)$ such that N_{C_1/Y_1} is semi-positive and a smooth $n-$normal curve

$$X_2 := C_2 \cup L_1' \cup ... \cup L_{k-k'} \subset H$$

of type $(e - a, k - k')$ (that meets both Y_1 and X_1 transversely) such that $N_{C_2/H}$ is semi-positive and $X_1 \cap X_2 = \{p_1, \cdots p_{t+1}\}$, where

$$q_i = H \cap L_i, \ p_i = H \cap l_i, \ H \cap C_1 = \{p_{t+1}, ..., p_a\}.$$

By our construction, $C_2 \cap X_1 = \{p_1, \cdots, p_{t+1}\}$. Therefore $X_1 \cup X_2$ is a disjoint union of k disjoint lines and a rational tree of degree e with $t+1$ singular points. Note that the curve $X_1 \cup X_2$ satisfies the assumptions of Lemma (3.14), hence $H^1(N_{X_1 \cup X_2/H \cup Y_1}) = 0$. Thus by Corollary (3.17), $X_1 \cup X_2$ is smoothable along with $Y_1 \cup H$.

To show $X_1 \cup X_2$ is $n-$normal, consider the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^N}(n-1)) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^N}(n)) & \longrightarrow & H^0(\mathcal{O}_H(n)) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle r_{X_1}(n-1)} & & \downarrow{\scriptstyle r_X(n)} & & \downarrow{\scriptstyle r_{X \cap H}(n)} & & \\
0 & \longrightarrow & H^0(\mathcal{O}_{X_1}(n-1)) & \longrightarrow & H^0(\mathcal{O}_X(n)) & \longrightarrow & H^0(\mathcal{O}_{X \cap H}(n)) & \longrightarrow & 0
\end{array}
$$

That $r_{X_1 \cup X_2}(n)$ is surjective is equivalent to the left and the right vertical maps being surjective. The argument is similar to the the previous case. Indeed, the left vertical map is surjective by $H(n-1, d-1)$ and Proposition (3.18). The numerical condition for the right vertical map to be surjective is provided by Lemma (4.13). Then Proposition (3.18) and $H(n, 1)$ imply the surjectivity.

Recall that the case $(e, k, d, N) = (7, 3, 4, 4)$ was excluded from the above construction. Note that $\nu(7, 3, 4, 4) = 3 < \nu(8, 3, 4, 4)$ and correspond to these values we have $a = 6, k' = 0, t = 2$. Therefore Lemma (4.15) does not hold and the above

construction we made for Case(B) does not hold. We modify our construction for the case $(e, k, d, N) = (7, 3, 4, 4)$ by letting

$$X_1 = C_1 \cup L_1 \cup L_2 \cup L_3 \subset Y_1$$

to be a smooth curve of type $(4, 2)$, and

$$X_2 = C_2 \cup L_4 \subset H$$

be a smooth curve of type $(3, 1)$ such that $X_1 \cap X_2 = C_1 \cap C_2$ and it is a single point. Then the rest of the argument above holds to conclude $X_1 \cup X_2$ is 3-normal and smoothable.

Now we consider the case $\nu(e, n, d, N) = n-1$ (that by definition, this is a possible case in $H(n, d)$). Note that $\nu(e, n, d, N) = n - 1$ implies $\nu(e, n - 1, d, N) \leq n - 1$. Now we apply to $\nu(e, n - 1, d, N) = n - 1$ the same construction as in case (A) or (B) and define X_1 and X_2 correspondingly, depending on whether $\nu(e + 1, n - 1, d, N) = \nu(e, n - 1, d, N)$ or not. Note that in both cases (A) and (B) (applied to an element in $H(n - 1, d)$) we had

$$h^0(\mathcal{O}_{X_2}(n - 1)) + \mathrm{Card}\{X_1 \cap H\} \leq \binom{N + n - 2}{N - 1}. \tag{3.3}$$

Now to recover the case $\nu(e, n, d, N) = n - 1$, we need to add a line L to X_2 and show that $r_{X_2 \cup L}(n)$ is surjective (we are proving a case in $H(n, d)$, hence we want to show n-normality). The requirement for this to happen is to have

$$h^0(\mathcal{O}_{X_2 \cup L}(n)) + \mathrm{Card}\{X_1 \cap H\} \leq \binom{N + n - 1}{N - 1},$$

which holds by (3.3). □

Chapter 4

Appendix

4.1 Appendix A

In this section we prove the numerical lemmas which are required through out the second chapter. Fix integers $N \geq 4$, $k \geq 0$ and $e \geq 1$. Let $n := \nu(N, e, k)$ (see 2.2 for definition).

Lemma 4.1. *Suppose* $\nu(e, k, N) = \nu(e + 1, k, N) = n$ *for some fixed* $N \geq 4$, $n \geq 2$, $k \leq n$ *and* $e \geq 2$. *Let* $a < e$ *be the largest integer such that*

$$1 + (n - 1)a \leq \binom{N + n - 1}{N}.$$

Now having fixed a, *let* $0 \leq k' < k$ *be the largest integer such that*

$$1 + (n - 1)a + k'n \leq \binom{N + n - 1}{N}.$$

Then $1 + n(e - a) + (k - k')(n + 1) + (a - 1) + k' \leq \binom{N+n-1}{N-1}$.

Proof. The assumption $\nu(e + 1, k, N) = n$ implies

$$1 + n(e + 1) + k(n + 1) \leq \binom{N + n}{N}. \tag{4.1}$$

Note that a is the largest number which is strictly smaller than e and $1 + (n-1)a \le \binom{N+n-1}{N}$. This implies that if we increase a by one, then either $a + 1 = e$ or the last inequality does not hold if we replace a by $a + 1$. That is $(n-1)(a+1) \ge \binom{N+n-1}{N}$. Now if $(n-1)(a+1) \ge \binom{N+n-1}{N}$ then $\binom{N+n-1}{N} - (n-1)a - 1 \le n - 2$, and so $k' = 0$ by definition. Then by this inequality and (4.1) we get:

$$1 + n(e - a) + k(n + 1) + a + 1 \le \binom{N+n-1}{N-1},$$

which is stronger than the desired inequality. Hence the lemma follows.

Suppose $a = e - 1$ (which is the case for the rest of the proof). By definition $k' < k$ is the largest integer with

$$1 + (n-1)(e-1) + k'n \le \binom{N+n-1}{N}. \tag{4.2}$$

We will have the following cases:

(I) $k' = 0$, which happens if $k \le 1$ or if $\binom{N+n-1}{N} - (n-1)(e-1) - 1 \le n - 1$

(II) $k' = k - 1$

(III) $k \ge 2, k' \le k - 2$. Then because k' is maximal, increasing k' by one implies:

$$(n-1)(e-1) + n(k'+1) \ge \binom{N+n-1}{N}. \tag{4.3}$$

In case (III), we need to show

$$1 + n + (n+1)(k - k') + k' + e - 2 \le \binom{N+n-1}{N-1}.$$

Note that multiplying (4.3) by minus one and adding it to (4.1) implies:

$$1 + n + (n+1)(k - k') + k' + e - 1 \leq \binom{N+n-1}{N-1},$$

which is stronger than the desired inequality and the lemma follows.

In case (I) when $k \leq 1$ (and therefore $k' = 0$), the lemma asks to prove $1 + n + (n+1)k + e - 2 \leq \binom{N+n-1}{N-1}$. Because $k \leq 1$, so it's enough to show $2n + e \leq \binom{N+n-1}{N-1}$. By (4.2), we have:

$$2n + e \leq 2n + \frac{1}{n-1}\binom{N+n-1}{N} - \frac{1}{n-1} + 1.$$

Rewriting $\binom{N+n-1}{N} = \frac{n}{N}\binom{N+n-1}{N-1}$, it suffices to prove:

$$2n - \frac{1}{n-1} \leq \left[1 - \frac{n}{(n-1)N}\right]\binom{N+n-1}{N-1} - 1. \tag{4.4}$$

Note that for a fixed n, the right hand side of the last inequality is an increasing function of N. Therefore it is enough to verify this for $N = 4$. For $N = 4$, (4.4) reduces to:

$$\frac{2n(n-1)-1}{n-1} \leq \frac{3n-4}{4(n-1)}\binom{n+3}{3}$$
$$= \frac{3n-4}{4(n-1)}\frac{(n+3)(n+2)(n+1)}{6}, \tag{4.5}$$

which reduces to showing $0 \leq (3n-4)(n+3)(n+2)(n+1) - 24\left[2n(n-1)-1\right]$. This holds for $n \geq 0$.

Suppose $k' = 0$ and $1 + (n-1)e \geq \binom{N+n-1}{N}$. Multiplying this last inequality by -1 and adding with (4.1) implies $(n+1)k + n + e \leq \binom{N+n-1}{N-1}$, which is stronger than what we wanted. We are left with the case(II), i.e when $a = e - 1$ and $k' = k - 1$. We

want to show $2n + k + e - 1 \le \binom{N+n-1}{N-1}$. Note that (4.2) along with our assumption $k' = k - 1$, implies

$$2n + k + e - 1 \le 2n + k + \frac{1}{n-1}\binom{N+n-1}{N} - \frac{n}{n-1}(k-1) - \frac{1}{n-1}.$$

Hence it is enough to show the right hand side is less than $\binom{N+n-1}{N-1}$. Rewriting $\binom{N+n-1}{N} = \frac{n}{N}\binom{N+n-1}{N-1}$, we need to show

$$(2n+1) - \frac{k}{n-1} \le [1 - \frac{n}{N(n-1)}]\binom{N+n-1}{N-1}.$$

For this we show

$$1 + 2n \le \left[1 - \frac{n}{N(n-1)}\right]\binom{N+n-1}{N-1}.$$

Similar to the previous case, this has to be verified only for $N = 4$ and $n \ge 2$, in which case we need to show:

$$1 + 2n \le \frac{(3n-4)}{4(n-1)}\frac{(n+3)(n+2)(n+1)}{6}$$

Again one verifies that $0 \le (3n-4)(n+3)(n+2)(n+1) - 24(n-1)(1+2n)$ for $n \ge 2$. $\qquad \square$

Remark(1): The above lemma implies $\nu(e - a, k - k', N - 1) \le n$, which enables us to use $H(n, N-1)$ in Case(A) of the proof of theorem.

Remark(2): The way we define a and k' implies $\nu(a, k', N) \le n - 1$. Hence we could use $H(n-1, N)$ in proof of theorem, Case(A). In fact one can show that $\nu(a, k', N) = n - 1$, but we did not need this fact.

Lemma 4.2. *Suppose $\nu(e, k, N) = n < \nu(e+1, k, N)$ for fixed integers $e \ge 1$, $N \ge 4$, where $0 \le k \le n$. Then $\nu(e, 0, N) = n$ if $n \ge 5$.*

Proof. Suppose on the contrary that $\nu(e, 0, N) \le n - 1$. Then by the definition

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of the critical value $1 + e(n-1) \le \binom{N+n-1}{N}$. So $e \le \frac{1}{(n-1)}\binom{N+n-1}{N}$. Also, from $n < \nu(e+1, k, N)$ we get

$$\binom{N+n}{N} \le (e+1)n + k(n+1) \le (e+1)n + n(n+1),$$

where the last inequality is because $k \le n$. From this we get

$$e \ge 1/n\binom{N+n}{N} - n - 2.$$

But then it forces

$$\frac{1}{n}\binom{N+n}{N} - n - 2 \le \frac{1}{n-1}\binom{N+n-1}{N}.$$

Hence by simplifying further

$$[(nN - N - n)/n]\binom{N+n-1}{N} \le n(n-1)(n+2), \qquad (4.6)$$

which is a contradiction. Indeed the left side of the above inequality is an increasing function of N. So to show (4.6) cannot hold it suffices to check the case $N = 4$. That is,

$$\frac{(3n-4)}{n}\binom{n+3}{4} = \frac{(3n-4)(n+3)(n+2)(n+1)}{4!} > n(n-1)(n+2),$$

which holds when $n \ge 5$.

$\square$

Lemma 4.3. *Suppose $\nu(e, k, N) = n < \nu(e+1, k, N)$ where $k \le n$. Let $1 \le a < e$ be the largest integer such that $1 + (n-1)a \le \binom{N+n-1}{N}$. Now let $0 \le k' < k$ be the largest integer so that*

$$1 + (n-1)a + nk' \le \binom{N+n-1}{N}.$$

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Then $k' = 0$ and

(1): $a > n$

(2): $0 \leq t \leq n - 1$ *where* $t := \binom{N+n-1}{N} - 1 - (n-1)a$.

Proof. We prove this for the following cases separately:

Case: $N \geq 4$, $n \geq 5$:

Let $1 \leq a < e$ be as defined in the lemma, which implies $\nu(a, 0, N) \leq n - 1$. Note that by the previous lemma, $\nu(e, 0, N) = n$ and therefore such a would be the largest integer such that $\nu(a, 0, N) = n - 1$ and therefore in this case $t \leq n - 1$ by maximality of a, where by definition $t := \binom{N+n-1}{N} - 1 - (n-1)a$. To show $a > n$ i.e $a \geq n + 1$, by maximality of a it is enough to show $1 + (n+1)(n-1) \leq \binom{N+n-1}{N}$. It suffices to check it when $N = 4$, i.e to verify $n^2 \leq \binom{n+3}{4} = \frac{(n+3)(n+2)(n+1)n}{4!}$. This reduces to showing $n^4 + 6n^3 - 13n^2 + 6n \geq 0$, which holds for all n. Hence $a > n$.

Case: $N = 4$, $n \leq 4$:

We need to check all the remaining cases one by one, so need to check the three parts of the lemma for the cases: $(N, n) = (4, 2), (4, 3), (4, 4)$.

Case $(N, n) = (4, 4)$. By the assumptions $\nu(e, k, N) = n < \nu(e+1, k, N)$, it is forced that

$$1 + 4e + 5k \leq \binom{8}{4} = 70 \leq (e+1)4 + 5k,$$

which implies that we can only have the pairs $(e, k) = (12, 4), (13, 3), (14, 2), (16, 1), (17, 0)$. Now we find the largest integer a in each case such that it satisfy both parts of the lemma, i.e the largest $a < e$ so that $1 + 3a \leq \binom{7}{4} = 35$ and we see that for all pairs of (e,k) we have $a = 11 > 4 = n$, $t = 1$, $k' = 0$ and so $t + k' = 1 \leq 3 = n - 1$.

Case $(N, n) = (4, 3)$. Then we are looking for the pairs (e, k) where $k \leq n = 3$ and

$$1 + 3e + 4k \leq \binom{7}{4} \leq 3(e+1) + 4k$$

. Possibilities are $(e, k) = (8, 2), (10, 1), (11, 0)$, and corresponding to all these pairs

we get $a = 7 > 2 = n - 1$. Also $k' = t = 0$.

Case $(N, n) = (4, 2)$. Then the pairs (e, k) should satisfy $k \leq 2$ and

$$1 + 2e + 3k \leq \binom{6}{4} \leq (e + 1)2 + 3k.$$

The only possibilities are $(e, k) = (7, 0), (5, 1), (4, 2)$. Then correspond to $(e, k) = (7, 0), (5, 1)$ we get $a = 4 \geq 1 = n - 1$ and $t = k' = 0$.

Correspond to $(e, k) = (4, 2)$, we get $a = 3, k' = 0, t = 1$.

$\square$

Lemma 4.4. *Suppose* $\nu(e, k, N) = n < \nu(e + 1, k, N)$ *where* $k \leq n$. *With* a, k' *and* t *as in lemma (4.3), then*

$$1 + n(e - a) + (k - k')(n + 1) + (a - t - 1) + k' \leq \binom{N + n - 1}{N - 1}. \qquad (4.7)$$

Proof. Use the definition of t as in lemma (4.3) and rewrite the left side of (4.7) as below:

$$
\begin{aligned}
&1 + n(e - a) + (k - k')(n + 1) + (a - t - 1) + k' \\
&= 1 + n(e - a) + (k - k')(n + 1) + a - 1 + k' \\
&\quad - \binom{N + n - 1}{N} + 1 + (n - 1)a + nk' \\
&= 1 + ne + k(n + 1) - \binom{N + n - 1}{N} \\
&\leq \binom{N + n}{N} - \binom{N + n - 1}{N} = \binom{N + n - 1}{N - 1}.
\end{aligned}
$$

Thus the lemma follows.

$\square$

Corollary 4.5. *Suppose $\nu(e, k, N) = n < \nu(e+1, k, N)$ where $k \le n$. Let a, k' and t be as in lemma(4.3), then $\nu(e - a, k - k', N - 1) \le n$.*

Proof. We need to show

$$1 + n(e - a) + (k - k')(n + 1) \le \binom{N + n - 1}{N - 1}.$$

But (4.7) is stronger and would imply this one, because $(a - t - 1) + k' \ge 0$ by lemma (4.3). □

Lemma 4.6. *Suppose $\nu(e, k, d, N) = n < \nu(e+1, k, d, N)$ where $k \le n$. Let a and t be as in lemma (6), then we always have $e - a \ge t + 1$.*

Proof. First note that by definition $t \le n - 1$, so it suffices to show $n \le e - a$. By the proof of lemma (4.3), a is the maximal integer so that $\nu(a, 0, N - 1) = n - 1$. In order to show $e - a \ge n$ (i.e $e - n + 1 > a$), it suffices to show $\nu(e - n + 1, 0, N - 1) > n - 1$. Hence we need to show that $(n - 1)(e - n + 1) \ge \binom{N+n-1}{N}$, i.e:

$$(e - n + 1) \ge \frac{1}{n - 1} \binom{N + n - 1}{N} \tag{4.8}$$

Note that the assumptions $n < \nu(e + 1, k, N)$ and $k \le n$ imply

$$\binom{N + n}{N} \le n(e + 1) + k(n + 1)$$

$$\le n(e + 1) + n(n + 1)$$

$$= n(e + n - 1).$$

Hence

$$\frac{1}{n} \binom{N + n}{N} - \frac{1}{n} - 2n + 2 \le e - n + 1. \tag{4.9}$$

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By (4.9), to show (4.8) it suffices to prove:

$$\frac{1}{n}\binom{N+n}{N} - \frac{1}{n} - 2n + 2 \geq \frac{1}{n-1}\binom{N+n-1}{N}.$$

Rewrite $\binom{N+n}{N} = \frac{N+n}{n}\binom{N+n-1}{N}$, we want to show

$$
\begin{aligned}
2n + \frac{1}{n} - 2 &\leq \left[\frac{N+n}{n^2} - \frac{1}{n-1}\right]\binom{N+n-1}{N} \\
&= \left[\frac{Nn - N - n}{n^2(n-1)}\right]\binom{N+n-1}{N}.
\end{aligned}
\tag{4.10}
$$

The right hand side of (4.10) is an increasing function of N. Thus it suffices to verify (4.10) only for $N = 4$. That is

$$2n + \frac{1}{n} - 2 \leq \frac{3n-4}{n^2(n-1)}\binom{n+3}{4} = \frac{(3n-4)(n+3)(n+2)(n+1)}{24n(n-1)},$$

which holds for $n \geq 2$. $\qquad\qquad\square$

4.2   Appendix B

In this appendix we prove the numerical lemmas which are required in the third chapter.

Lemma 4.7. *If $\nu(1, k, d, N) = n \geq 2$ for some $N \geq 4$ and $2 \leq d \leq N$, then $k \geq n+1$ unless when $n = k = 2$ and $N = 4$.*

Proof. If $\nu(1, k, d, N) = n \geq 2$, then by definition of ν we get

$$
\begin{aligned}
(n-1)e + nk = n - 1 + nk &\geq \binom{N+n-1}{N} - \binom{N+n-1-d}{N} \\
&\geq \binom{N+n-1}{N} - \binom{N+n-3}{N}
\end{aligned}
$$

where the last inequality is because $d \geq 2$. Now if $k \leq n$, by above inequality we get

$$n^2 + n - 1 \geq \binom{N + n - 1}{N} - \binom{N + n - 3}{N}. \tag{4.11}$$

Note that the right hand side of (4.11) is an increasing function of N, so suffices to check it does not hold for $N = 4$. When $N = 4$ it reduces to

$$n^2 + n - 1 \geq \binom{3 + n}{4} - \binom{1 + n}{4},$$

which only holds when $n = 1, 2$. Thus the lemma follows. $\qquad \square$

Lemma 4.8. *Suppose $\nu(e, k, d, N) = \nu(e+1, k, d, N) = n$ for some $N \geq 4$, $n \geq d \geq 2$. Let $1 \leq a < e$ be the largest integer such that*

$$1 + (n - 1)a \leq \binom{N + n - 1}{N} - \binom{N + n - d}{N}.$$

Let $0 \leq k' < k$ be the largest integer such that

$$1 + (n - 1)a + k'n \leq \binom{N + n - 1}{N} - \binom{N + n - d}{N}.$$

Then

$$a - 1 + k' \leq \binom{N + n - 1}{N - 1} - \binom{N + n - d}{N - 1} - (d - 1)(e - a + k - k').$$

Proof. We need to prove

$$a(2 - d) + (e + k - k')(d - 1) - 1 \leq \binom{N + n - 1}{N - 1} - \binom{N + n - d}{N - 1}. \tag{4.12}$$

The assumption $\nu(e + 1, k, d, N) = n$ implies

$$1 + n(e + 1) + k(n + 1) \leq \binom{N + n}{N} - \binom{N + n - d}{N} \tag{4.13}$$

and hence

$$e + k \leq \frac{1}{n}\left[\binom{N + n}{N} - \binom{N + n - d}{N}\right] - \frac{k + 1}{n} - 1 \tag{4.14}$$

The maximality of a implies that if we increase a by one, then either $a + 1 = e$ or the last inequality does not hold if we replace a by $a + 1$, i.e

$$(n - 1)(a + 1) \geq \binom{N + n - 1}{N} - \binom{N + n - d}{N} \tag{4.15}$$

But this last inequality implies

$$\binom{N + n - 1}{N} - \binom{N + n - d}{N} - (n - 1)a - 1 \leq n - 2$$

and hence $k' = 0$ by definition of k'. Also (4.15) implies

$$-a \leq -\frac{1}{n - 1}\left[\binom{N + n - 1}{N} - \binom{N + n - d}{N}\right] + 1. \tag{4.16}$$

Therefore by (4.16), (4.14) and the fact that $2 - d \leq 0$ we get

$$a(2 - d) + (e + k)(d - 1) - 1 \leq -\frac{d - 2}{n - 1}\left[\binom{N + n - 1}{N} - \binom{N + n - d}{N}\right] + (d - 2)$$

$$+ \frac{d - 1}{n}\left[\binom{N + n}{N} - \binom{N + n - d}{N}\right] - \frac{(k + 1)(d - 1)}{n} - (d - 1)$$

$$= \left[\frac{(d - 1)(N + n)}{n^2} - \frac{d - 2}{N - 1}\right]\binom{N + n - 1}{N} + \left[\frac{d - 2}{n - 1} - \frac{d - 1}{n}\right]\binom{N + n - d}{N}$$

$$- \frac{(k + 1)(d - 1)}{n} - 1. \tag{4.17}$$

Therefore to prove the lemma it suffices to show

$$\left[\frac{(d-1)(N+n)}{n^2} - \frac{d-2}{N-1}\right]\binom{N+n-1}{N} + \left[\frac{d-2}{n-1} - \frac{d-1}{n}\right]\binom{N+n-d}{N}$$
$$- \frac{(k+1)(d-1)}{n} - 1$$
$$\leq \binom{N+n-1}{N-1} - \binom{N+n-d}{N-1},$$

which simplifies to showing

$$\left[\frac{(d-1)(N+n)}{n^2} - \frac{d-2}{n-1} - \frac{N}{n}\right]\binom{N+n-1}{N}$$
$$+ \left[\frac{d-2}{n-1} - \frac{d-1}{n} + \frac{N}{n-d+1}\right]\binom{N+n-d}{N}$$
$$\leq \frac{(k+1)(d-1)}{n} + 1 \qquad (4.18)$$

Let $x_N := \frac{(d-1)(N+n)}{n^2} - \frac{d-2}{n-1} - \frac{N}{n}$ and $y_N := \frac{d-2}{n-1} - \frac{d-1}{n} + \frac{N}{n-d+1}$. Therefore (4.18)

simplifies to

$$F(N)\binom{N+n-d}{N} \leq \frac{(k+1)(d-1)}{n} + 1 \qquad (4.19)$$

where

$$F(N) := x_N\frac{(N+n-1)\cdots(N+n-d+1)}{(n-1)\cdots(n-d+1)} + y_N \qquad (4.20)$$

First note that (4.19) holds when $d = n = 2$. So from now we exclude this case. To prove (4.19), it suffices to show it's left hand side is non-positive, i.e we show $F(N) \leq 0$. We do this by showing that it's derivative with respect to N is negative

and that $F(4) \leq 0$. For now assume that $x_N \leq 0$ for all N.

$$F'(N) = x'_N \frac{(N+n-1)\cdots(N+n-d+1)}{(n-1)\cdots(n-d+1)}$$
$$+ x_N \frac{(N+n-1)\cdots(N+n-d+1)}{(n-1)\cdots(n-d+1)} \left(\sum_{i=1}^{i=d-1} \frac{1}{N+n-i} \right) + \frac{1}{n-d+1}$$
$$\leq x'_N \frac{(N+n-1)\cdots(N+n-d+1)}{(n-1)\cdots(n-d+1)} + \frac{1}{n-d+1}$$
$$= \frac{(d-1-n)}{(n^2)} \frac{(N+n-1)\cdots(N+n-d+1)}{(n-1)\cdots(n-d+1)} + \frac{1}{n-d+1}$$

First note that for $d = 2$ we get the valid inequality below:

$$F'(N) = \frac{-(N+n+1)}{n^2} + \frac{1}{n-1} \leq 0$$

(note that $N \geq 4$ and $n \geq 2$). For $3 \leq d \leq n$, to say that $F'(N)$ above is non-positive

is to say

$$\frac{(d-1-n)(N+n-1)\cdots(N+n-d+1)}{n^2(n-1)\cdots(n-d+2)} \leq -1$$

Because $d - 1 - n < 0$ the left hand side above, is a decreasing function of N, and

therefore the last inequality only needs to be verified at $N = 4$, which is to check

$$\frac{(d-1-n)(n+3)\cdots(n-d+5)}{n^2(n-1)\cdots(n-d+2)} \leq -1 \tag{4.21}$$

which holds for all $3 \leq d \leq n$. Therefore $F(N)$ is a decreasing function of N. We

now check our claim that $x_N \leq 0$. By definition

$$x_N = \frac{(d-1)(N+n)}{n^2} - \frac{d-2}{n-1} - \frac{N}{n}$$
$$= \frac{(n-1)(d-1)(N+n) - n^2(d-2) - n(n-1)N}{n^2(n-1)}$$
$$= \frac{N[n(d-n) + (1-d)] + n + n(n-d)}{n^2(n-1)} \tag{4.22}$$

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Because $3 \leq d \leq n$, so $d - n \leq 0$ and $1 - d \leq 0$ and so for fixed n and d, x is a decreasing function of N. So suffices to check $x_N \leq 0$ when $N = 4$, in which case we are reduced to show

$$-3n^2 + 3nd - 4d + 4 + n = n(-3n + 3d + 1) + 4(1 - d) \leq 0$$

which holds since $3 \leq d \leq n$ (we only needed to check the numerator in x in non positive).

So far we showed $F(N)$ in a decreasing function of N, and so to prove (4.19), it suffices to show $F(4) \leq 0$, because the right hand side in (4.19) is always greater than one.

For the rest of the proof we consider the case $a = e - 1$.

Let as defined above, $k' < k$ be the largest integer with

$$1 + (n - 1)(e - 1) + k'n \leq \binom{N + n - 1}{N} - \binom{N + n - d}{N} \tag{4.23}$$

We will have the following cases:

(I) $k' = 0$ which happens if $k \leq 1$ or if
$$\binom{N+n-1}{N} - \binom{N+n-d}{N} - (n - 1)(e - 1) - 1 \leq n - 1$$

(II) $k' = k - 1$

(III) $k \geq 2, k' \leq k - 2$. Then because k' is maximal, increasing k' by one implies:

$$(n - 1)(e - 1) + n(k' + 1) \geq \binom{N + n - 1}{N} - \binom{N + n - d}{N} \tag{4.24}$$

Case (II), where $a = e - 1$ and $k' = k - 1$. So the lemma asks to prove

$$e + k + 2d - 5 \leq \binom{N + n - 1}{N - 1} - \binom{N + n - d}{N - 1} \tag{4.25}$$

But the assumption $\nu(e + 1, k, d, N) = n$ implies $1 + n(e + 1) + k(n + 1) \leq \binom{N+n}{N} - \binom{N+n-d}{N}$ and hence

$$e + k \leq \frac{1}{n} \left[\binom{N + n}{N} - \binom{N + n - d}{N} \right] - \frac{k + 1}{n} - 1 \tag{4.26}$$

Therefore to show (16) it suffices

$$\frac{1}{n} \left[\binom{N + n}{N} - \binom{N + n - d}{N} \right] - \frac{k + 1}{n} - 1 + (2d - 5) \leq \binom{N + n - 1}{N - 1} - \binom{N + n - d}{N - 1}$$

Which simplifies to showing

$$\left[\frac{1}{n} - \frac{N}{N + n} \right] \binom{N + n}{N} + \left[\frac{N}{n - d + 1} - \frac{1}{n} \right] \binom{N + n - d}{N}$$

$$= \frac{N + n - nN}{n(N + n)} \binom{N + n}{N} + \frac{nN - n + d - 1}{n(n - d + 1)} \binom{N + n - d}{N}$$

$$= \left[\frac{(N + n - nN)}{n(n + N)} \frac{(N + n) \cdots (N + n - d + 1)}{n \cdots (n - d + 1)} + \frac{nN - n + d - 1}{n(n - d + 1)} \right] \binom{N + n - d}{N}$$

$$\leq \frac{k + 1}{n} - 2d + 6 \tag{4.27}$$

Similar to the previous case, we make the following definitions

$x_N = \frac{(N+n-nN)}{n(n+N)}$, $y_N = \frac{nN-n+d-1}{n(n-d+1)}$ and $F(N) = x_N \frac{(N+n)\cdots(N+n-d+1)}{n\cdots(n-d+1)} + y_N$. So the inequality above can be written as

$$F(N) \binom{N + n - d}{N} \leq \frac{k + 1}{n} - 2d + 6 \tag{4.28}$$

which holds when $d = 2$, the case which we exclude from now on. To prove the last inequality above we first will show $F(N)$ is a decreasing function of N (by showing

that it's derivative with respect to N is negative) and then show that $F(4) \leq -2d$.

$$
\begin{aligned}
F'(N) &= x'_N \frac{(N+n)\cdots(N+n-d+1)}{n\cdots(n-d+1)} + \\
&\quad + x_N \frac{(N+n-1)\cdots(N+n-d+1)}{(n-1)\cdots(n-d+1)} \left(\sum_{i=0}^{i=d-1} \frac{1}{N+n-i} \right) + \frac{1}{n-d+1} \\
&\leq x'_N \frac{(N+n)\cdots(N+n-d+1)}{(n)\cdots(n-d+1)} + \frac{1}{n-d+1} \\
&= \frac{(-n)}{(n+N)^2} \frac{(N+n)\cdots(N+n-d+1)}{(n)\cdots(n-d+1)} + \frac{1}{n-d+1}
\end{aligned} \tag{4.29}
$$

To say that the equation above is non-positive is to say

$$
\frac{-(N+n-1)\cdots(N+n-d+1)}{(n+N)(n-1)\cdots(n-d+2)} \leq -1 \tag{4.30}
$$

the left hand side above, is a decreasing function of N, and therefore the last inequality only needs to be verified at $N = 4$, which is to check

$$
\frac{-(n+3)\cdots(n-d+5)}{(n+4)(n-1)\cdots(n-d+2)} \leq -1 \tag{4.31}
$$

which holds for all $3 \leq d \leq n$.

Hence we got that $F(N)\binom{N+n-d}{N}$ is a decreasing function of N. Again by Mathematica we show $F(4)\binom{4+n-d}{4} \leq -2d$ and hence $F(N)\binom{N+n-d}{N} \leq -2d$ for all $N \geq 4$ (this suffices to prove (4.28), as the right hand side in (4.28) is larger than $-2d$ because $k \geq 0$).

We now consider case (III), i.e when $a = e - 1$ and

$$
(n-1)(e-1) + n(k'+1) \geq \binom{N+n-1}{N} - \binom{N+n-d}{N}.
$$

By the last inequality

$$-k' \leq -\frac{1}{n}\left[\binom{N+n-1}{N} - \binom{N+n-d}{N}\right] + \frac{(n-1)(e-1)}{n} + 1 \qquad (4.32)$$

Also by (4.14) we have

$$e \leq \frac{1}{n}\left[\binom{N+n}{N} - \binom{N+n-d}{N}\right] - (k+1) - \frac{k+1}{n} - 1 \qquad (4.33)$$

Note that the lemma asks to prove that

$$e - 2 + (d-1)(k+1) - k'(d-2) \leq \binom{N+n-1}{N-1} - \binom{N+n-d}{N-1} \qquad (4.34)$$

But by (4.32) and (4.33) we get

$$e - 2 + (d-1)(k+1) - k'(d-2) \leq \frac{1}{n}\left[\binom{N+n}{N} - \binom{N+n-d}{N}\right]$$
$$- (k+1) - \frac{k+1}{n} - 3 + (d-1)(k+1) + \frac{(n-1)(e-1)(d-2)}{n} + (d-2)$$
$$- \frac{d-2}{n}\left[\binom{N+n-1}{N} - \binom{N+n-d}{N}\right]$$
$$= \left[\frac{N+n}{n^2} - \frac{d-2}{n}\right]\binom{N+n-1}{N} + \left[\frac{d-2}{n} - \frac{1}{n}\right]\binom{N+n-d}{N}$$
$$+ (d-2)(k+2) - \frac{k+1}{n} - 3 \qquad (4.35)$$

Hence to prove (4.34) it suffices to prove the right hand side of (4.35) is less than $\binom{N+n-1}{N-1} - \binom{N+n-d}{N-1}$, i.e

$$\left[\frac{N+n}{n^2} - \frac{d-2}{n} - \frac{N}{n}\right]\binom{N+n-1}{N} + \left[\frac{d-2}{n} - \frac{1}{n} + \frac{N}{n-d+1}\right]\binom{N+n-d}{N}$$
$$\leq -(d-2)(k+2) + \frac{k+1}{n} + 3 \qquad (4.36)$$

As before let $x_N = \frac{N+n}{n^2} - \frac{d-2}{n} - \frac{N}{n}$ and $y_n = \frac{d-2}{n} - \frac{1}{n} + \frac{N}{n-d+1}$, therefore (4.36) can be

written as

$$F(N)\binom{N+n-d}{N} \leq -(d-2)(k+2) + \frac{k+1}{n} + 3 \qquad (4.37)$$

where

$$F(N) := x_N \frac{(N+n-1)\cdots(N+n-d+1)}{(n-1)\cdots(n-d+1)} + y_N \qquad (4.38)$$

Note that $x_N = \frac{n(3-d)+N(1-n)}{n^2} < 0$, because $n \geq d \geq 2$ and $N \geq 4$. Also $x_{N+1} - x_N = \frac{1-n}{n^2} < 0$ and $y_{N+1} - y_N = \frac{1}{n-d+1}$. Therefore

$$F(N+1) - F(N) = x_{N+1}\frac{(N+n)\cdots(N+n-d+2)}{(n-1)\cdots(n-d+1)} - x_N \frac{(N+n-1)\cdots(N+n-d+1)}{(n-1)\cdots(n-d+1)}$$

$$+ (y_{N+1} - y_N)$$

$$= \frac{(N+n-1)\cdots(N+n-d+2)}{(n-1)\cdots(n-d+1)}[(N+n)x_{N+1} - (N+n-d+1)x_N]$$

$$+ \frac{1}{n-d+1}$$

$$= \frac{(N+n-1)\cdots(N+n-d+2)}{(n-1)\cdots(n-d+1)}\left[\underbrace{(N+n)(x_{N+1}-x_N)}_{\leq -1} + \underbrace{(d-1)x_N}_{\leq 0}\right]$$

$$+ \frac{1}{n-d+1} < 0 \qquad (4.39)$$

The above expression is negative because the term inside the bracket is less than -1 and $\frac{(N+n-1)\cdots(N+n-d+2)}{(n-1)\cdots(n-d+1)} > \frac{1}{n-d+1}$. Therefore $F(N)$ is a decreasing function of N. We check by Mathematica that $F(4) < 0$ for $n \geq d \geq 2$ (except for $d = n = 2$ in which case $F(4) = 1$). Therefore $F(N)\binom{N+n-d}{N} < 0$ and is a decreasing function. Again by Mathematica we check that $F(4)\binom{4+n-d}{4} \leq -(d-2)(n+2) + \frac{1}{n} + 3$. Which implies (using the fact that $k \leq n$)

$$F(N)\binom{N+n-d}{N} \leq -(d-2)(n+2) + \frac{1}{n} + 3 \leq -(d-2)(k+2) + \frac{k+1}{n} + 3$$

for all $N \geq 4$ and $n \geq d \geq 2$ except when $n = d = 2$. When $n = d = 2$, (4.36) is

reduced to

$$\frac{(N+1)(2-N)}{2} + N - \frac{1}{2} \le (k+1)/2 + 3$$

which holds for all $N \ge 4$ because $k \ge 0$ (the left hand side above is always negative for $N \ge 4$). So the lemma holds for all $N \ge 4$ and $n \ge d \ge 2$. $\qquad \square$

Corollary 4.9. *Under the assumption of the previous lemma,*

$$1 + n(e - a) + (k - k')(n + 1) + (a - 1) + k' \le \binom{N + n - 1}{N - 1}$$

.

Remark(1): The above lemma implies $\nu(e - a, k - k', 1, N) \le n$, which enables us to use $H(n, 1)$ in Case(A) of the proof of theorem one.

Remark(2):The way we define a and k' implies $\nu(a, k', d - 1, N) \le n - 1$, and hence we could use $H(n - 1, d - 1)$ in proof of theorem one, Case(A) (in fact one can show that $\nu(a, k', d - 1, N) = n - 1$, but we did not need this fact)

Lemma 4.10. *Suppose $\nu(e, k, d, N) = n < \nu(e + 1, k, d, N)$ for fixed integers $e \ge 1$, $N \ge d \ge 2$, $N \ge 4$, where $0 \le k \le n$. Then $\nu(e, 0, d - 1, N) = n$ if $n \ge 5$.*

Proof. Suppose on the contrary that $\nu(e, 0, d - 1, N) \le n - 1$, then by definition of the critical value

$$1 + e(n - 1) \le \binom{N + n - 1}{N} - \binom{N + n - d}{N}.$$

And therefore

$$e \le 1/(n - 1)\left[\binom{N + n - 1}{N} - \binom{N + n - d}{N}\right] \qquad (4.40)$$

Also from $n < \nu(e + 1, k, d, N)$ we get

$$\binom{N+n}{N} - \binom{N+n-d}{N} \leq (e+1)n + k(n+1)$$

$$\leq (e+1)n + n(n+1)$$

$$= n(e + n + 2)$$

where the last inequality is because $k \leq n$. From this we get

$$e \geq 1/n \left[\binom{N+n}{N} - \binom{N+n-d}{N} \right] - n - 2 \tag{4.41}$$

But then (4.40) and (4.41) forces

$$1/n \left[\binom{N+n}{N} - \binom{N+n-d}{N} \right] - n - 2 \leq 1/(n-1) \left[\binom{N+n-1}{N} - \binom{N+n-d}{N} \right]$$

Rewrite $\binom{N+n}{N} = \frac{N+n}{n} \binom{N+n-1}{N}$, the last inequality simplifies to

$$\left[\frac{N+n}{n^2} - \frac{1}{n-1} \right] \binom{N+n-1}{N} - n - 2 \leq \left[\frac{1}{n} - \frac{1}{n-1} \right] \binom{N+n-d}{N} \tag{4.42}$$

But the right hand side in the last inequality is non positive and hence it forces the left hand side to be non positive, i.e

$$\frac{nN - N - n)}{n^2(n-1)} \binom{N+n-1}{N} \leq n+2$$

Therefore we should have

$$(nN - N - n) \binom{N+n-1}{N} \leq n^2(n-1)(n+2) \tag{4.43}$$

Note that the lest hand side of (4.43) is an increasing function of N, therefore if (4.43) does not hold for some N_1 it holds for no $N \geq N_1$. First let $N = 5$, then (4.43)

simplifies to

$$(4n - 5)\binom{n + 4}{5} \leq n^2(n - 1)(n + 2)$$

which further implies

$$(4n - 5)(n + 4)(n + 3)(n + 2)(n + 1)n \leq 120n^2(n - 1)(n + 2) \tag{4.44}$$

Using WolframAlpha we can check the last inequality is invalid for $n \geq 2$. Therefore (??) gives a contradiction for all $N \geq 5$ and $n \geq 2$. Now let $N = 4$, then (4.43) simplifies to

$$(3n - 4)\binom{3 + n}{4} \leq n^2(n - 1)(n + 2)$$

which is equivalent to

$$(3n - 4)(n + 3)(n + 2)(n + 1)n \leq 24n^2(n - 1)(n + 2)$$

We check by WolframAlpha that the last inequality is invalid for $n \geq 5$. $\qquad\square$

Lemma 4.11. *Suppose* $\nu(e, k, d, N) = n < \nu(e + 1, k, d, N)$ *where* $k \leq n$. *Then there exists* $a \leq e$ *and* $0 \leq k' \leq n - 1$ *such that:*

(1): $a > n$

(2): $0 \leq t \leq n - 1$ *where* $t := \binom{N+n-1}{N} - \binom{N+n-d}{N} - 1 - (n - 1)a - nk'$

(3): $t + k' \leq n - 1$

Proof. We prove this for two different cases separately:

Case: $N \geq 4$, $n \geq 5$:

Choose the maximum $a < e$ such that

$$t := \binom{N + n - 1}{N} - \binom{N + n - d}{N} - (n - 1)a - 1 \geq 0$$

Note that by lemma(7), $\nu(e, 0, d - 1, N) = n$ and therefore such a would be the largest

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integer such that $\nu(a, 0, d-1, N) = n-1$ and therefore in this case $t \le n-1$ by maximality of a, and hence $k' = 0$, hence we get (2) and (3). Note that lemma(4.10) is essential to get the maximality of a.

Now to show $a > n$ i.e $a \ge n+1$, (again by maximality of a) it is enough to show $1 + (n+1)(n-1) \le \binom{N+n-1}{N} - \binom{N+n-d}{N}$. Note that

$$1 + (n+1)(n-1) \le \binom{N+n-1}{N} - \binom{N+n-2}{N} \le \binom{N+n-1}{N} - \binom{N+n-d}{N}$$

where the left inequality holds for $N \ge 4$ and $n \ge 2$. So we get $a > n$.

Case: $N = 4$, $n \le 4$:

We need to check all remaining cases one by one, so need to check the three parts of the lemma for the cases: (N,n)= (4,2),(4,3),(4,4) and d can take values 2, 3, 4 we check the lemma for (N,n)=(4,4) and the rest can be checked similarly.

Case (N,n,d)=(4,4,4) by the assumptions $\nu(e, k, d, N) = n$ and $n < \nu(e+1, k, d, N)$, it is forced that

$$1 + 4e + 5k \le \binom{8}{4} - \binom{4}{4} = 69 \le (e+1)4 + 5k$$

which implies that we can only have the pairs (e,k)=(12,4),(13,3),(14,2),(15,1),(17,0). Now we find the largest integer a in each case such that it satisfy both parts of the lemma, i.e the largest $a < e$ so that $1 + 3a \le \binom{7}{4} - \binom{4}{4} = 34$ and we see that for all pairs of (e,k) we have $a = 11 > 4 = n$,$k' = 0$ and $t + k' = 0 \le 3 = n-1$

In the case (N,n,d)=(4,4,3), we have (e,k)=(11,4),(12,3),(13,2),(14,1),(16,0) In all cases we get $a = 9$, $k' = 0$ and $t = 2$, and the lemma holds.

In the case (N,n,d)=(4,4,2), we have (e,k)=(8,4),(9,3),(11,2),(12,1),(13,0) and for all these pairs we get $a = 6$,$t = 1$ and $k' = 0$ and again the lemma holds.

Below we write the result of computation for the remaining cases and see that lemma holds in all of them:

$(n, d) = (3, 4)$ then either $(e, k) = (7, 3)$ which implies $a = 6, k' = 0, t = 2$ or $e > 7$ in which case $a = 7, k' = t = 0$

$(n, d) = (3, 3)$ then $11 \geq e \geq 7$ and $a = 6, k' = 0, t = 1$

$(n, d) = (3, 2)$ then $9 \geq e \geq 5$ and $a = 4, k' = 0, t = 1$

$(n, d) = (2, 4), (2, 3)$ then $7 \geq e \geq 4$. For $e = 4$, we get $a = 3, k' = 0, t = 1$ and for $e = 5, 6, 7$ we get $a = 4, k' = t = 0$

$(n, d) = (2, 2)$ then either $e = 3$ in which case $a = 2, k' = 0, t = 1$ or $e = 4, 5, 6$ and so $a = 3, t = k' = 0$.

Corollary 4.12. *By the assumption of the previous lemma, we have $\nu(a - t, t + k', d - 1, N) \leq n - 1$.*

Proof. We already have $\nu(a, k', d - 1, N) \leq n - 1$. The lemmas follows by the definition of t. $\square$

Remark: One can see that in cases that $\nu(e, 0, N, d) = \nu(e, k, N, d)$ $< \nu(e + 1, k, n, d)$, we get $k' = 0$ and $t \leq n - 1$. The t here is going to be the number of lines we need to add to the curve X_1 in Case(B) of proof of theorem one. $\square$

Lemma 4.13. *Suppose $\nu(e, k, d, N) = n < \nu(e + 1, k, d, N)$ where $k \leq n$. Let a, k' and t be as in lemma(4.11), then*

$$1 + n(e - a) + (k - k')(n + 1) + (a - t - 1) + k' \leq \binom{N + n - 1}{N} - \binom{N + n - d}{N} \quad (4.45)$$

Proof. Use the definition of t as in Lemma (4.11) and rewrite the left side of (4.45) as below:

$$1 + n(e - a) + (k - k')(n + 1) + (a - t - 1) + k'$$

$$= 1 + n(e - a) + (k - k')(n + 1) + a - 1 + k'$$

$$- \binom{N + n - 1}{N} + \binom{N + n - d}{N} + 1 + (n - 1)a + nk'$$

$$= 1 + ne + k(n + 1) - \binom{N + n - 1}{N} + \binom{N + n - d}{N}$$

$$\leq \binom{N + n}{N} - \binom{N + n - 1}{N} = \binom{N + n - 1}{N - 1} \tag{4.46}$$

Note that to get the last inequality the fact that $\nu(e, k, d, N) = n$ is used. $\qquad\square$

Corollary 4.14. *Suppose* $\nu(e, k, d, N) = n < \nu(e + 1, k, d, N)$ *where* $k \leq n$. *With* a, k' *and* t *as in lemma(4.11), then* $\nu(e - a, k - k', 1, N - 1) \leq n$

Proof. To prove $\nu(e - a, k - k', 1, N) \leq n$ we need to show

$$1 + n(e - a) + (k - k')(n + 1) \leq \binom{N + n - 1}{N - 1} - \binom{N + n - d}{N}$$

But the inequality (4.45) in the previous lemma is stronger and would imply this one, because $(a - t - 1) + k' \geq 0$ by the first and the third part of lemma (4.11). $\qquad\square$

Lemma 4.15. *Suppose* $\nu(e, k, d, N) = n < \nu(e + 1, k, d, N)$ *where* $k \leq n$. *Let* a *and* t *be as in lemma(4.11), then except when* $(e, k, d, N, n) = (7, 3, 4, 4, 3)$, *we always have* $\left\lfloor \frac{(e - a)N - 2}{N - 2} \right\rfloor - (t + 1) \geq -1$.

Proof. The lemma holds for $N = 4$ and $2 \leq n \leq 4$ except when $(e, k, d, N, n) = (7, 3, 4, 4, 3)$, as in the proof of lemma (4.11), we computed all possible e, a, k' in this range. Assume $n \geq 5$. To prove the lemma is for $n \geq 5$, we prove $e - a \geq t + 1$ (from which lemma follows). By the proof of Lemma (4.11),

$$(n - 1)(a + 1) \geq \binom{N + n - 1}{N} - \binom{N + n - d}{N} \tag{4.47}$$

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By definition $t = \binom{N+n-1}{N} - \binom{N+n-d}{N} - 1 - (n-1)a$ so the lemma asks to show

$$\binom{N+n-1}{N} - \binom{N+n-d}{N} \leq e - a + (n-1)a$$

$$= e + (n-2)a \qquad (4.48)$$

But $\nu(e+1, k, d, N) > n$, therefore $(e+k+1)n + k \geq \binom{N+n}{N} - \binom{N+n-d}{N}$. Hence by this last inequality and (4.47) which give us an inequality for e and a respectively, we get

$$e + (n-2)a \geq \frac{1}{n}\left[\binom{N+n}{N} - \binom{N+n-d}{N}\right] - \frac{k}{n} - (k+1)$$
$$+ \frac{n-2}{n-1}\left[\binom{N+n-1}{N} - \binom{N+n-d}{N}\right]$$
$$= \left[\frac{N+n}{n^2} + \frac{n-2}{n-1}\right]\binom{N+n-1}{N} + \left[-\frac{n-2}{n-1} - \frac{1}{n}\right]\binom{N+n-d}{N}$$
$$- \frac{k}{n} - (k+1)$$

Therefore to show (refcorrectlines3) it suffices to prove that the right hand side of the last inequality is greater than $\binom{N+n-1}{N} - \binom{N+n-d}{N}$, i.e to prove

$$\left[\frac{N+n}{n^2} - 1 + \frac{n-2}{n-1}\right]\binom{N+n-1}{N} + \left[1 - \frac{n-2}{n-1} - \frac{1}{n}\right]\binom{N+n-d}{N} \geq \frac{k}{n} + (k+1)$$
$$(4.49)$$

Because $k \leq n$, we get $\frac{k}{n} + (k+1) \leq n+2$. So to prove the last inequality it suffices to show

$$\left[\frac{N+n}{n^2} - 1 + \frac{n-2}{n-1}\right]\binom{N+n-1}{N} = \frac{Nn - N - n}{n^2(n-1)}\binom{N+n-1}{N} \geq n+2 \quad (4.50)$$

Note that the left hand side of last inequality is an increasing function of N, therefore

it suffices to check the inequality for $N = 4$, in which case it reduces to showing

$$\frac{3n - 4}{n^2(n - 1)} \binom{n + 3}{4} = \frac{(3n - 4)(n + 3)(n + 2)(n + 1)}{24n(n - 1)}$$

$$\geq n + 2 \qquad (4.51)$$

But that is to show $(3n - 4)(n + 3)(n + 1) \geq 24n(n - 1)$, which holds for $n \geq 5$ and we are done (recall that we already discussed cases $n = 2, 3, 4$ at the beginning of the proof).

$\square$

Lemma 4.16. *Suppose* $\nu(e, k, d, N) = n < \nu(e + 1, k, d, N)$ *where* $k \leq n$, $N \geq 4$ *and* $2 \leq d \leq n$. *Let* a *and* t *be as before. Then*

$$\binom{N + n - d}{N - 1} \leq 1 + (e - a)(n - d + 1) + k(n - d + 2)$$

Proof. By assumptions $\nu(e + 1, k, d, N) > n$ and the definition of a we get

$$n(e + 1) + k(n + 1) \geq \binom{N + n}{N} - \binom{N + n - d}{N}$$

$$1 + (n - 1)a \leq \binom{N + n - 1}{N} - \binom{N + n - d}{N}$$

respectively. Therefore

$$e \geq \frac{1}{n} \left[\binom{N + n}{N} - \binom{N + n - d}{N} \right] - \frac{k}{n} - k - 1$$

$$-a \geq -\frac{1}{n - 1} \left[\binom{N + n - 1}{N} - \binom{N + n - d}{N} \right] + \frac{1}{n - 1}$$

So to show the lemma it suffices to prove the last inequality in (4.52) below

$$e - a \geq \frac{1}{n}\left[\binom{N+n}{N} - \binom{N+n-d}{N}\right] - \frac{k}{n} - k - 1 - \frac{1}{n-1}\left[\binom{N+n-1}{N} - \binom{N+n-d}{N}\right] + \frac{1}{n-}$$

$$\geq \frac{1}{n-d+1}\binom{N+n-d}{N-1} - \frac{1}{n-d+1} - k - \frac{k}{n-d+1} \tag{4.52}$$

But the last inequality in (4.52) simplifies to showing

$$\left[\frac{N+n}{n^2} - \frac{1}{n-1}\right]\binom{N+n-1}{N} + \left[-\frac{1}{n} + \frac{1}{n-1} - \frac{N}{(n-d+1)^2}\right]\binom{N+n-d}{N}$$

$$\geq -\frac{1}{n-1} - \frac{1}{n-d+1} - \frac{k(d-1)}{n(n-d+1)} + 1 \tag{4.53}$$

Because $k \geq 0$, to prove (4.53) it suffices to show

$$G(N,d) := \left[\frac{N+n}{n^2} - \frac{1}{n-1}\right]\binom{N+n-1}{N}$$

$$+ \left[-\frac{1}{n} + \frac{1}{n-1} - \frac{N}{(n-d+1)^2}\right]\binom{N+n-d}{N}$$

$$+ \frac{1}{n-1} + \frac{1}{n-d+1} - 1 \geq 0 \tag{4.54}$$

First consider the case $d = n$, where we are reduced to show

$$\left[\frac{N+n}{n^2} - \frac{1}{n-1}\right]\binom{N+n-1}{N} + \frac{n+1}{n(n-1)} - N \geq 0$$

which is an increasing function of N and therefore suffices to be verified only when $N = 4$, in which case we need to show

$$\frac{3n-4}{n^2(n-1)}\binom{n+3}{4} + \frac{n+1}{n(n-1)} - 4 \geq 0$$

But that simplifies to the inequality below

$$\frac{(3n-4)(n+3)(n+2)(n+1)}{24n(n-1)} + \frac{n+1}{n(n-1)} - 4 \geq 0$$

which is valid for $n \geq 2$, hence the lemma holds when $d = n$. From now on we assume $n > d \geq 2$.

Now we show that for fixed N and n, $G(n,d)$ is an increasing function of d.

$$\begin{aligned}
G(N, d+1) - G(N, d) &= \left[\frac{1}{n(n-1)} - \frac{N}{(n-d)^2}\right]\binom{N+n-d-1}{N} \\
&\quad - \left[\frac{1}{n(n-1)} - \frac{N}{(n-d+1)^2}\right]\binom{N+n-d}{N} \\
&\quad + \frac{1}{n-d} - \frac{1}{n-d+1}
\end{aligned} \tag{4.55}$$

Rewrite $\binom{N+n-d}{N} = \frac{N+n-d}{n-d}\binom{N+n-d-1}{N}$ and note that $n > d$, so (4.55) simplifies to

$$\begin{aligned}
&\left[\frac{1}{n(n-1)} - \frac{N}{(n-d)^2} - \frac{N+n-d}{n(n-1)(n-d)} + \frac{N(N+n-d)}{(n-d)(n-d+1)^2}\right]\binom{N+n-d-1}{N} \\
&+ \frac{1}{(n-d)(n-d+1)} \\
&= \left[\frac{-N}{n(n-1)(n-d)} + \frac{N[(n-d)(N-2)-1]}{(n-d)^2(n-d+1)^2}\right]\binom{N+n-d-1}{N} \\
&+ \frac{1}{(n-d)(n-d+1)}
\end{aligned} \tag{4.56}$$

Note that the last term above is positive. Therefore to show (4.56) is positive, it suffices to show the coefficient of $\binom{N+n-d-1}{N}$ in (4.56) is non-negative, that is to show

$$\frac{1}{n(n-1)} \leq \frac{(n-d)(N-2)-1}{(n-d)(n-d+1)^2} \tag{4.57}$$

The coefficient of N in the inequality above is positive, so it is an increasing function of N and suffices to check the last inequality only when $N = 4$, in which case it

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simplifies to showing

$$n(n-1)[2(n-d)-1] - (n-d)(n-d+1)^2 \geq 0$$

which holds because $n(n-1) \geq (n-d+1)^2$ and $2(n-d)-1 \geq n-d$.

So far we have shown that for a fixed N, $G(N,d)$ is an increasing function of d and we need to check (4.54) only when $d = 2$, in which case we need to prove

$$\left[\frac{N+n}{n^2} - \frac{1}{n-1}\right]\binom{N+n-1}{N} + \left[\frac{1}{n(n-1)} - \frac{N}{(n-1)^2}\right]\binom{N+n-2}{N}$$
$$+ \frac{2}{n-1} - 1 \geq 0 \tag{4.58}$$

which simplifies to showing

$$\left[\frac{(Nn-n-N)(N+n-1)}{n^2(n-1)^2} - \frac{n-1-Nn}{n(n-1)^2}\right]\binom{N+n-2}{N}$$
$$+ \frac{2}{n-1} - 1$$
$$= \frac{(N-1)(2n+N)}{n^2(n-1)}\binom{N+n-2}{N} + \frac{2}{n-1} - 1 \geq 0 \tag{4.59}$$

which is again an increasing function of N and needs to be verified only when $N = 4$, in which case it is easy to see it holds for all $n \geq 2$. $\qquad \square$

Lemma 4.17. *If for some $e \geq 2$ and $2 \leq d \leq N$, $\nu(e,k,d,N) = n$ and $1 \leq a < e$ is the largest integer such that*

$$1 + (n-1)a \leq \binom{N+n-1}{N} - \binom{N+n-d}{N} \tag{4.60}$$

then $a + n - 2 \leq \binom{N+n-1}{N-1} - \binom{N+n-d}{N-1}$.

Proof. By (4.60) we get $a \leq \frac{1}{n-1}\left[\binom{N+n-1}{N} - \binom{N+n-d}{N}\right]$. So it suffices to show

$$\frac{1}{n-1}\left[\binom{N+n-1}{N} - \binom{N+n-d}{N}\right] + n - 2 \leq \binom{N+n-1}{N-1} - \binom{N+n-d}{N-1} \quad (4.61)$$

Rewrite $\binom{N+n-1}{N} = \frac{n}{N}\binom{N+n-1}{N-1}$ and $\binom{N+n-d}{N} = \frac{n-d+1}{N}\binom{N+n-d}{N-1}$, then (4.61) simplifies to

$$\left[1 - \frac{n-d+1}{N(n-1)}\right]\binom{N+n-d}{N-1} + n - 2 \leq \left[1 - \frac{n}{N(n-1)}\right]\binom{N+n-1}{N-1}$$

But $2 \leq d \leq N$ and therefore the left hand side of the last inequality is less than

$$\left[1 - \frac{n-N+1}{N(n-1)}\right]\binom{N+n-2}{N-1} + n - 2$$

Therefore to show (4.62), it suffices to show

$$\left[1 - \frac{n-N+1}{N(n-1)}\right]\binom{N+n-2}{N-1} + n - 2 \leq \left[1 - \frac{n}{N(n-1)}\right]\binom{N+n-1}{N-1} \quad (4.62)$$

Again write $\binom{N+n-1}{N-1} = \frac{N+n-1}{n}\binom{N+n-2}{N-1}$ and the last inequality simplifies to showing

$$\begin{aligned}
n - 2 &\leq \left[\left(1 - \frac{n}{N(n-1)}\right)\frac{N+n-1}{n} - 1 + \frac{n-N+1}{N(n-1)}\right]\binom{N+n-2}{N-1} \\
&= \left[\frac{N+n-1}{n} - 1 + \frac{n(n-N+1) - n(N+n-1)}{n(n-1)N}\right]\binom{N+n-2}{N-1} \\
&= \left[\frac{N+n-1}{n} - 1 + \frac{2(1-N)}{(n-1)N}\right]\binom{N+n-2}{N-1} \quad (4.63)
\end{aligned}$$

For a fixed n, $\frac{N+n-1}{n} - 1 + \frac{2(1-N)}{(n-1)N}$ is an increasing function of N (for that we need to check $\frac{N+n-1}{n} + \frac{2(1-N)}{(n-1)N} \leq \frac{N+n}{n} + \frac{-2N}{(n-1)(N+1)}$, i.e to check $\frac{-1}{n} \leq \frac{-2}{N(N+1)(n-1)}$. But $2n \leq N(N+1)(n-1)$ because $N \geq 4$ and $n \geq 2$). Hence to prove (4.63) it suffices

to prove it only when $N = 4$, in which case we need to show

$$
\begin{aligned}
n - 2 &\leq \left[\frac{n+3}{n} - 1 + \frac{-6}{4(n-1)} \right] \binom{n+2}{3} \\
&= \frac{6(n-2)}{4n(n-1)} \binom{n+2}{3} \\
&= \frac{(n-2)(n+2)(n+1)}{4(n-1)}
\end{aligned}
\tag{4.64}
$$

Hence we are reduced to showing $4(n-1) \leq (n+2)(n+1)$ which holds for $n \geq 2$. $\quad \square$

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