5-24-2012

Quotients of Subgroup Lattices of Finite Abelian p-groups

Marina Dombrovskaya
Washington University in St. Louis

WASHINGTON UNIVERSITY IN ST. LOUIS

Department of Mathematics

Dissertation Examination Committee:
John Shareshian, Chair
Gary Jensen
Robert Pless
Rachel Roberts
Cynthia Traub
David Wright

Quotients of Subgroup Lattices of Finite Abelian p-groups

by

Marina Dombrovskaya

A dissertation presented to the
Graduate School of Arts and Sciences
of Washington University in
partial fulfillment of the
requirements for the degree
of Doctor of Philosophy

May 2012

Saint Louis, Missouri

Acknowledgements

I would like to thank my advisor Dr. John Shareshian for his immense contribution to my mathematical growth. Thank you for sharing your love of mathematics with me, for being an inspiration, for your limitless patience and kindness. I would not be where I am now without your guidance and support. Thank you from the bottom of my heart.

Dr. Gary Jensen, Dr. David Wright, Dr. Mohan Kumar and Dr. Guido Weiss thank you for your encouragement and for insightful conversations. Dr. Russ Woodroofe, thank you for discussions of Combinatorics and valuable comments.

Thank you to all my friend in the Mathematics department for stimulating discussions and for making my graduate school experience unforgettable.

To my friend Angela, thank you for always being there and for believing in me.

A special thank you to my Mom and Dad for their love, support and for giving me an opportunity to pursue my dreams.

And last but not least I would like to express my infinite gratitude to Larry. Words cannot express how much your love, support and encouragement means to me. I can never thank you enough for everything you have done for me.

Table of Contents

List of Figures

Chapter 1

Introduction

Let G be a finite abelian p-group of type $\lambda = (\lambda_1, ..., \lambda_n)$, where λ is a partition, that is $\lambda_i \geq \lambda_{i+1} > 0$. By the Fundamental Theorem of Finitely Generated Abelian Groups [7] we can write $G = \mathbb{Z}_{p^{\lambda_1}} \times \cdots \times \mathbb{Z}_{p^{\lambda_n}}$, where $\mathbb{Z}_{p^{\lambda_i}}$ is a cyclic group of order p^{λ_i}. Let $L_\lambda(p)$ be the subgroup lattice of G. We define $[0, \lambda_i]$ to be a totally ordered set (or a chain) of integers from 0 to λ_i, and $[0, \lambda] = [0, \lambda_1] \times \cdots \times [0, \lambda_n]$ to be a lattice of n-tuples $(a_1, ..., a_n)$, where $a_i \in [0, \lambda_i]$, with an order relation given by $(a_1, ..., a_n) \leq (b_1, ..., b_n)$ if $a_i \leq b_i$ for all i. We call $[0, \lambda]$ a product of chains of length λ_i for $1 \leq i \leq n$.

In [1] and [2] L. Butler categorized $L_\lambda(p)$ as an order-theoretic p-analogue of $[0, \lambda] = [0, \lambda_1] \times \cdots \times [0, \lambda_n]$. That is, she defined an order preserving map $\varphi : L_\lambda(p) \to [0, \lambda]$ such that there are a power of p subgroups of G corresponding to each element of $[0, \lambda]$. The map φ satisfies certain technical properties that make it very useful in attacking combinatorial problems concerning

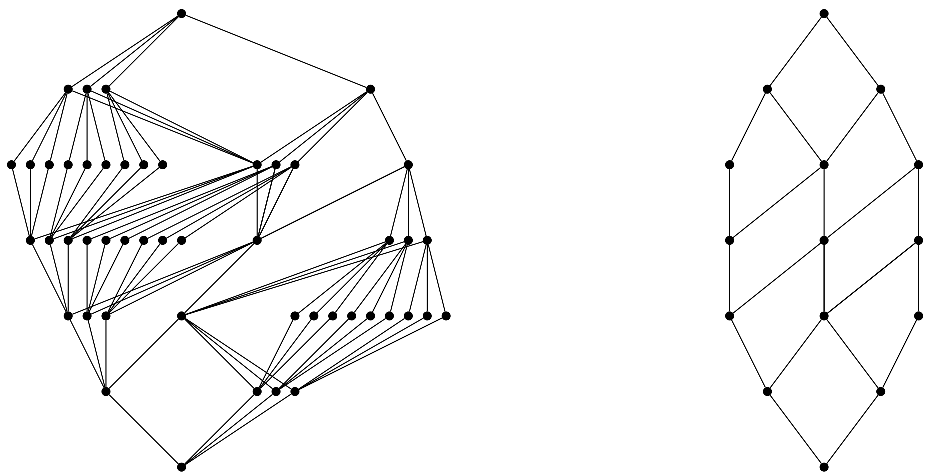

Figure 1.1: Subgroup lattice of $G = \mathbb{Z}_{3^4} \times \mathbb{Z}_{3^2}$ on the left and the product of chains $[0,4] \times [0,2]$ on the right.

$L_\lambda(p)$. Figure 1.1 represents the correspondence of the lattice of subgroups of $G = \mathbb{Z}_{3^4} \times \mathbb{Z}_{3^2}$ with the product of chains $[0,4] \times [0,2]$. Subgroups in the lattice of subgroups of $\mathbb{Z}_{3^4} \times \mathbb{Z}_{3^2}$ are clustered together if they correspond to the same element of the product of chains. As described in [1] and [2], $L_\lambda(p)$ has many very attractive enumerative properties. In [2] L. Butler defined a set of Hall generators for a subgroup of a finite abelian p-group, which are defined in Chapter 2. Hall generators proved to be extremely useful tools that we use extensively throughout the thesis.

We observe that for a finite abelian p-group G of type λ such that $\lambda_i = 1$ for all $1 \leq i \leq n$ the quotient, $\overline{L_\lambda(p)}$, of the lattice of subgroups under the action of a Sylow p-subgroup, S^p, of the group of automorphisms of G is equal to the product of chains $[0, \lambda]$. We became interested in the question of how the action of S^p on $L_\lambda(p)$ is related to map φ described by Butler, which will be discussed in Chapter 3. These are our main results related to this question. In Theorem 3.1 we show that each orbit of the action of S^p is contained in a corresponding fiber of φ. In Theorem 3.2 we address the question for which finite abelian p-groups $\overline{L_\lambda(p)} = [0, \lambda]$. We show that if G is a finite abelian p-group of type λ such that $\lambda_1 = ... = \lambda_n$ or of type λ such that $\lambda_i - \lambda_t \leq 1$ for all $1 \leq i < t \leq n$, then $\overline{L_\lambda(p)} = [0, \lambda]$. We observed that whenever G is a finite abelian p-group of type λ such that $\lambda_i - \lambda_j \geq 2$ for some i and j, some fibers of φ split into orbits of S^p whose size is equal to the same power of p. In Theorem 3.3, we assume that $G \cong Z_{p^m} \times \mathbb{Z}_{p^n}$ such that $m - n \geq 2$ and describe conditions under which the orbits of S^p spilt the corresponding fiber of φ into smaller parts of the same size.

In Chapter 4 we will discuss the relationship between the quotient of $L_\lambda(p)$ under the action of the group of lattice automorphisms of G and the quotient of $L_\lambda(p)$ under the action of the group of lattice automorphisms of G induced by group automorphisms.

For elementary abelian groups $G \cong (\mathbb{Z}_p)^n$ such that $n \geq 3$ the Fundamental Theorem of Projective Geometry implies that every lattice automorphism is induced by a group automorphism. Moreover, Baer's Theorem [8] (1939) states that for every finite abelian p-group of type λ such that $\lambda_1 = \lambda_3$ we also have that every lattice automorphism is induced by a group automorphism. However, this is not the case for every finite abelian p-group. It is not difficult to see that when $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for $p \geq 5$ there are lattice automorphisms that are not induced by group automorphism. Our main result is stated in Theorem 4.5 that for $G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ the quotient of $L_\lambda(p)$ under the action of the group of lattice automorphisms of G is equal to the quotient of $L_\lambda(p)$ under the action of the group of lattice automorphisms of G induced by group automorphisms. This result is particularly striking because the group of lattice automorphisms of G is often much larger then the group of lattice automorphisms of G induced by group automorphisms. The same result also holds for some larger finite abelian p-groups as described in Theorem 4.8. However, we conjecture that this is not true in general and present a potential counterexample at the end of Chapter 4.

Many projects related to the material discussed here remain to be explored. We will discuss them in Chapter 5.

Chapter 2

Background and Definitions

2.1 Automorphisms of finite abelian p-groups

Since we will be interested in examining the actions of group automorphisms on the lattice of subgroups of a finite abelian p-group, we begin by discussing group automorphisms of finite abelian p-groups. Let G be an abelian p-group of type λ and let g_i be an additive generator for the group $\mathbb{Z}_{p^{\lambda_i}}$. (Although g_i is an equivalence class in $\mathbb{Z}$ modulo p^{λ_i}, we will abuse notation slightly by identifying g_i with its representative and using the same notation for both.) We may assume that $g_i \equiv 1 \bmod p^{\lambda_i}$ for each i. Thus, we can write $G \cong < g_1 > \times \cdots \times < g_n >$. So, an element of g of G can be represented as a row vector $(a_1 g_1, \ldots, a_n g_n)$ with $a_i \in \mathbb{Z}$ and $a_i g_i$ taken modulo p^{λ_i}.

2.1.1 Extending endomorphisms to automorphisms

First, we will describe $E = \mathrm{End}(G)$, the endomorphism ring of G. The elements of E are group homomorphisms from G into itself and it is clear that E is a ring under function addition and composition. It is important to note that we will consider homomorphisms in E to be acting on G on the right. Each homomorphism in E is determined by the images of generators $g_1, \ldots, g_n$ of G. Since we can represent elements of G in vector form and since G has n generators, we can think of elements of E as $n \times n$ matrices.

In order to describe all matrices in $\mathrm{End}(G)$, we define $R = \{(a_{ij}) \in \mathbb{Z}^{n \times n} : p^{\lambda_j - \lambda_i} \mid a_{ij} \text{ for } 1 \leq j \leq i \leq n\}$. By noting that every element $A \in R$ can be written in $\mathbb{Q}^{n \times n}$ as $A = PA'P^{-1}$, where $A' \in Z^{n \times n}$ and $P = \mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_n})$, it is straightforward to see (Lemma 3.2, [5]) that R is a ring under matrix multiplication. Now consider the mapping $\psi : R \to \mathrm{End}(G)$ defined by

$$(\bar{h}_1, \ldots, \bar{h}_n)\psi(A) = \pi((h_1, \ldots, h_n)A),$$

where $(h_1, \ldots, h_n) \in Z^n$, $(\bar{h}_1, \ldots, \bar{h}_n) \in G$ such that $\bar{h}_i \in \mathbb{Z}_{p^{\lambda_i}}$ and $h_i \in \mathbb{Z}$ is an integral representative of $\bar{h}_i$ ($h_i \equiv \bar{h}_i \bmod p^{\lambda_i}$), $A \in R$, and π is the canonical projection from Z^n onto G. By Theorem 3.3 in [5] ψ is a surjective ring homomorphism. The proof of this theorem is based on the fact that for $A \in End(G)$ and generators $w_i = (0, \ldots, g_i, \ldots, 0)$, $w_i A = (\bar{h}_{i1}, \ldots, \bar{h}_{in})$ and $0 = (p^{\lambda_i} w_i)A = (\overline{p^{\lambda_i} h_{i1}}, \ldots, \overline{p^{\lambda_i} h_{in}})$ imply that $p^{\lambda_j} \mid p^{\lambda_i} h_{ij}$ for all i, j and thus

$p^{\lambda_j - \lambda_i} \mid h_{ij}$ when $j \leq i$. By Lemma 3.4 in [5] the kernel of ψ is equal to the set of matrices such that $p^{\lambda_j} | a_{ij}$ for all i, j. Therefore, $R/\ker\,\psi \cong \mathrm{End}(G)$. Knowing the structure of End(G), we can describe automorphisms of G, the units in End(G).

Theorem 2.1. *(Theorem 3.6, [5]) An endomorphism $M = \psi(A)$ of G is an automorphism if and only if $A \pmod p \in GL_n(\mathbb{F}_p)$, where $\mathbb{F}_p$ is a finite field with p elements.*

The proof of this theorem is also straightforward once a fact from elementary linear algebra is invoked: for an $n \times n$ integer nonsingular matrix A there exists a unique $n \times n$ integer matrix B such that $AB = BA = \det(A)I$. Now we will look at some examples of group automorphisms of finite abelian p-groups.

Example 2.1. Suppose $\lambda_i = 1$ for all i. Then $G \cong (\mathbb{Z}_p)^n$, that is we can think of G as a vector space. It is clear that End(G) is isomorphic to the ring of all $n \times n$ matrices with entries in $\mathbb{F}_p$. Then Theorem 2.1 implies (as expected) that $\mathrm{Aut}(G) \cong GL_n(\mathbb{F}_p)$.

Example 2.2. The situation is similar to that of the previous example whenever $G \cong (\mathbb{Z}_{p^m})^n$, that is $\lambda_i = m$ for all i, where $m \geq 1$. Although we can no longer think of G as a vector space, we have that $\mathrm{End}(G) \cong M_n(\mathbb{Z}_{p^m})$, the set of all $n \times n$ matrices with entries in $\mathbb{Z}_{p^m}$, and $\mathrm{Aut}(G) \cong GL_n(\mathbb{Z}_{p^m})$.

Example 2.3. Suppose $n = 3$, $\lambda_1 = 5$, $\lambda_2 = 3$, and $\lambda_3 = 2$. Then

$$\mathrm{End}(G) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}p^2 & a_{22} & a_{23} \\ a_{31}p^3 & a_{32}p & a_{33} \end{pmatrix} : a_{ij} \in \mathbb{Z}_{p^{\lambda_j}} \right\}.$$

By Theorem 2.1 every automorphism $A \in \mathrm{End}(G)$ of G has the form $A(\bmod p) \in GL_n(\mathbb{F}_p)$. But $A(\bmod p)$ is an upper-triangular matrix in $M_n(\mathbb{F}_p)$. Since $\det A = \Pi_i a_{ii}$, for A to be in $GL_n(\mathbb{F}_p)$ we must have that a_{ii} is not divisible by p for every i, that is each $a_{ii} \in (\mathbb{Z}_{p^{\lambda_i}})^*$, where $(\mathbb{Z}_{p^{\lambda_i}})^*$ is the group of multiplicative units of $\mathbb{Z}_{p^{\lambda_i}}$. Thus

$$\mathrm{Aut}(G) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}p^2 & a_{22} & a_{23} \\ a_{31}p^3 & a_{32}p & a_{33} \end{pmatrix} : a_{ij} \in \mathbb{Z}_{p^{\lambda_j}},\ p \nmid a_{ii} \right\}.$$

It is clear from above that a matrix corresponding to an automorphism of G has entries that lie in different rings. However, the product of two such matrices is well-defined and corresponds to the composition of two automorphisms of G, which is also an automorphism of G. Given $A, B \in \mathrm{Aut}(G)$, consider $C = AB$. Then $c_{ii} = a_{i1}b_{1i}p^{\lambda_1-\lambda_i} + a_{i2}b_{2i}p^{|\lambda_2-\lambda_i|} + a_{i3}b_{3i}p^{\lambda_i-\lambda_3}$, where the sum is taken modulo p^{λ_i}. Then $c_{ii} \equiv a_{ii}b_{ii} \bmod p \not\equiv 0 \bmod p$, so $p \nmid c_{ii}$. Also, for $j > i$ we have $c_{ji} = a_{j1}b_{1i}p^{\lambda_1-\lambda_j} + a_{j2}b_{2i}p^{\lambda_2-\lambda_j+\lambda_i-\lambda_2} + a_{j3}b_{3i}p^{\lambda_i-\lambda_3} = p^{\lambda_i-\lambda_j}(a_{j1}b_{1i}p^{\lambda_1-\lambda_i} + a_{j2}b_{2i} + a_{j3}b_{3i}p^{\lambda_j-\lambda_3})$ since $j > i$ implies that $j \geq 2$ and

$i \leq 2$ and where the sum is taken modulo p^{λ_i}. So, $C \in \mathrm{Aut}(G)$.

Example 2.4. Suppose $\lambda_1 > \lambda_1 > \cdots > \lambda_n$. Then similarly to the previous example

$$\mathrm{Aut}(G) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21}p^{\lambda_1-\lambda_2} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}p^{\lambda_1-\lambda_n} & a_{n2}p^{\lambda_2-\lambda_n} & \dots & a_{nn} \end{pmatrix} : a_{ij} \in \mathbb{Z}_{p^{\lambda_j}}, a_{ii} \in (\mathbb{Z}_{p^{\lambda_i}})^* \right\}.$$

Note that the product of two automorphisms of G is again an automorphism of G which can be seen by performing calculations similar to the ones done in the previous example.

Example 2.5. In general, suppose G has type $\lambda = (\underbrace{\lambda_1, ..., \lambda_1}_{m_1}, \dots, \underbrace{\lambda_k, \dots, \lambda_k}_{m_k})$, where $\lambda_1 > \cdots > \lambda_k \geq 1$ and $t_1 + \cdots + t_k = n$. Then an endomorphism $A = (a_{ij})$ of G is an $n \times n$ block matrix of the form

$$\begin{pmatrix} M_1 & * & \dots & * \\ * & M_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & M_k \end{pmatrix},$$

where M_j is a $t_j \times t_j$ matrix with entries in $\mathbb{Z}_{p^{\lambda_j}}$ for $1 \leq j \leq k$ and all entries below the block diagonal are divisible by p. Therefore, A is an automorphism of G, that is A is invertible modulo p, if and only if $\det(A) \bmod p =$

$\Pi_j \det(M_j) \bmod p \neq 0 \bmod p$. Also, notice that $\det(M_j) \neq 0 \bmod p$ for every $j = 1, \ldots, k$, so each M_j is an automorphism of $(\mathbb{Z}_{p^{\lambda_j}})^{t_j}$.

2.1.2 Counting automorphisms

Using Theorem 2.1, we start with a matrix $M \in GL_n(\mathbb{F}_p)$ and extend M to an automorphism of G. First, we define the upper and lower bounds for the number of identical parts in the partition λ

$$d_k = \max\{l : \lambda_k = \lambda_l\}, \;\; c_k = \min\{l : \lambda_k = \lambda_l\}.$$

Note that $d_k \geq k$ and $c_k \leq k$. We represent the matrix M in terms of d_k's and c_k's as follows

$$M = \begin{pmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ \vdots & \vdots & \ldots & \vdots \\ m_{d_1 1} & & & \\ & m_{d_2 2} & & \\ & & \ddots & \\ 0 & & & m_{d_n n} \end{pmatrix} = \begin{pmatrix} m_{1c_1} & & & & * \\ & m_{2c_2} & & & \\ & & \ddots & & \\ 0 & & & m_{nc_n} \;\; \ldots & m_{nn} \end{pmatrix}.$$

Then, as described in [5], we count the number of possible entries in M. Since columns of M are linearly independent, there are

$$(p^{d_1} - 1)(p^{d_2} - p) \ldots (p^{d_n} - p^{n-1})$$

possible entries for M. Extending M to an automorphism A of G, we see that in A lower triangular zero entries in M can be any element of $p^{\lambda_j-\lambda_i}\mathbb{Z}_{p^{\lambda_j}}$ for $j < i$ in A. Thus, there are $\Pi_{i=1}^{n}(p^{\lambda_i})^{c_i-1}$ ways to extend the necessary zeros in M to A. Also, we can extend the not necessary zero entries in M: we want all $a_{ij} \in \mathbb{Z}_{p^{\lambda_j}}$ such that $a_{ij} \equiv m_{ij} (\operatorname{mod} p)$. For each m_{ij} there are p^{λ_j-1} such a_{ij}'s. Thus

$$|\mathrm{Aut}(G)| = \Pi_{k=1}^{n}(p^{d_k} - p^{k-1})\Pi_{i=1}^{n}(p^{\lambda_i})^{c_i-1}\Pi_{j=1}^{n}(p^{\lambda_j-1})^{d_j}.$$

Example 2.6. Let G be a finite abelian p-group of type $\lambda = (m, n)$, where $m > n$. Then $d_1 = 1$, $d_2 = 2$, $c_1 = 1$, and $c_2 = 2$. So

$$\begin{aligned}|\mathrm{Aut}(G)| &= (p-1)(p^2-p)(p^m)^{1-1}(p^n)^{2-1}(p^{m-1})^1(p^{n-1})^2 \\ &= (p-1)^2p^{m+3n-2}.\end{aligned}$$

2.1.3 Sylow p-subgroups of Aut(G)

Now we will discuss Sylow p-subgroups of Aut(G). We begin by examining the 2-dimensional case $G = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, where $m > n \geq 1$. Note that from examples above we know that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an automorphism of G if and only if $a \in (\mathbb{Z}_{p^m})^*$, $b \in \mathbb{Z}_{p^n}$, $c = kp^{m-n} \in \mathbb{Z}_{p^m}$, and $d \in (\mathbb{Z}_{p^n})^*$. First, we assume that p is an odd prime. As described in [4], for an odd prime p we have that $(\mathbb{Z}_{p^t})^* \cong \mathbb{Z}_{p^{t-1}} \times \mathbb{Z}_{(p-1)}$. Note that we will be thinking of $\mathbb{Z}_{p^{t-1}}$ and

$\mathbb{Z}_{(p-1)}$ as subgroups of $(\mathbb{Z}_{p^t})^*$. Then we can write $a = a'u$, where $a' \in \mathbb{Z}_{p^{m-1}}$ and $u \in \mathbb{Z}_{(p-1)}$, and $d = d'v$, where $d' \in \mathbb{Z}_{p^{n-1}}$ and $v \in \mathbb{Z}_{(p-1)}$. Then we can decompose A into a product of two matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix},$$

where $b' = bv^{-1}$ and $c' = cu^{-1}$, where we think of v^{-1} and u^{-1} as inverses of respectively v and u in $(\mathbb{Z}_{p^t})^*$. It is crucial to observe (as was done in [4]) that matrices $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ and $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ correspond to automorphisms of G and sets of all such matrices form subgroups S_p and N respectively of $\mathrm{Aut}(G)$ with the following properties:

(1) $S_p \cap N = \{e\}$, the identity of G;

(2) $S_pN = \mathrm{Aut}(G)$;

(3) $S_p \lhd \mathrm{Aut}(G)$.

Note that the properties (1) and (2) are clear from definitions of S_p, N, and $\mathrm{Aut}(G)$. By Example 2.6, $|\mathrm{Aut}(G)| = (p-1)^2 p^{m+3n-2}$ and by construction S_p has exactly $p^{m-1+n+n+n-1} = p^{m+3n-2}$ elements (p^{m-1} choices for a', p^n choices for b', p^n choices for c', and p^{n-1} choices for d'). Thus, S_p is a Sylow p-subgroup of $\mathrm{Aut}(G)$. Since any $A \in \mathrm{Aut}(G)$ can be written as $A = PQ$, where $P \in S_p$ and $Q \in N$, we have $A^{-1}S_pA = (PQ)^{-1}S_p(PQ) = Q^{-1}S_pQ$. If

$Q = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_p$, then

$$\begin{pmatrix} u^{-1} & 0 \\ 0 & v^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} a & bu^{-1}v \\ cv^{-1}u & d \end{pmatrix} \in S_p.$$

Therefore, S_p is a normal subgroup of $\mathrm{Aut}(G)$ and thus the unique Sylow p-subgroup of $\mathrm{Aut}(G)$. We take another look at the structure of S_p. Each element of S_p is of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a \in (\mathbb{Z}_{p^m})^*$, $b \in \mathbb{Z}_{p^n}$, $c \in p^{m-n}\mathbb{Z}_{p^m}$, and $d \in (\mathbb{Z}_{p^n})^*$. Since a subgroup of $\mathbb{Z}_{p^m}$ isomorphic to $(\mathbb{Z}_{p^m})^*$ is generated by $1+p$, we can write an element of S_p as $\begin{pmatrix} (1+p)^k & b \\ cp^{m-n} & (1+p)^l \end{pmatrix}$, where $0 \leq k < p^{m-1}$, $b \in \mathbb{Z}_{p^n}$, $0 \leq c < p^n$, and $0 \leq l < p^{n-1}$.

Let us consider $G = \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n}$, where $m > n \geq 1$. From [4], for $t \geq 3$ we have $(\mathbb{Z}_{2^t})^* \cong \mathbb{Z}_{2^{t-2}} \times \mathbb{Z}_2$ and for $t \leq 2$ we have that $(\mathbb{Z}_{2^t})^* \cong \mathbb{Z}_{2^{t-1}}$. If $n \geq 3$, then for an automorphism $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we can write $a = a'u$, where $a' \in \mathbb{Z}_{2^{m-2}}$ and $u \in \mathbb{Z}_2$, and $d = d'v$, where $d' \in \mathbb{Z}_{2^{n-2}}$, $v \in \mathbb{Z}_2$. If $n \leq 3$, then in an automorphism A as above, we can write $d = d'v$, where $d' \in \mathbb{Z}_{2^{n-1}}$ and $v = 1$ and if $m \leq 3$, then we can write $a = a'u$, where $a' \in \mathbb{Z}_{2^{m-1}}$ and $u = 1$. We write $A = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$, where $b' = bv^{-1}$ and $c' = cu^{-1}$. Again, matrices of the form $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ and $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ correspond to automorphisms of G and form subgroups S_2 and N of $\mathrm{Aut}(G)$ such that

(1) $S_2 \cap N = \{e\}$;

(2) $S_2 N = \mathrm{Aut}(G)$;

(3) $S_2 \triangleleft \mathrm{Aut}(G)$.

Similarly to the argument above, S_2 is the unique Sylow 2-subgroup of $\mathrm{Aut}(G)$. A typical element of S_2 is of the form $\begin{pmatrix} 3^k & b \\ 2^{m-n}c & 3^l \end{pmatrix}$, where $0 \leq k < m-2$, $b \in \mathbb{Z}/2^n\mathbb{Z}$, $0 \leq c < p^n$, and $0 \leq l < p^{n-2}$.

Consider $G = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m}$ for some integer $m > 0$. Since $\mathrm{Aut}(G) = GL_2(\mathbb{F}_p)$ and $|\mathrm{Aut}(G)| = p^{4m-3}(p^2-1)(p-2)$, a Sylow p-subgroup of $\mathrm{Aut}(G)$ is a subgroup of $GL_2(\mathbb{F}_p)$ and has order p^{4m-3}. Notice that $\mathrm{Aut}(G)$ contains multiple Sylow p-subgroups. For instance, the subgroup consisting of elements of the form

$$\begin{pmatrix} (1+p)^k & bp \\ c & (1+p)^l \end{pmatrix},$$

where $0 \leq k, l < p^{m-1}$, $c \in \mathbb{Z}_{p^m}$, and $0 \leq b < p^{m-1}$, is a Sylow p-subgroup of G since the determinant $(1+p)^k(1+p)^l - bcp$ of every element in the subgroup is not a a multiple of p and the order of the subgroup is p^{4m-3} (there are p^{m-1} elements of the form $(1+p)^t$ for $0 \leq t < p^{m-1}$, p^m choices for b and p^{m-1} choices for c). Similarly, subgroup that contains elements of the form

$$\begin{pmatrix} b & (1+p)^k \\ (1+p)^l & cp \end{pmatrix},$$

where $0 \le k, l < p^{m-1}$, $b \in \mathbb{Z}_{p^m}$, and $0 \le c < p^{m-1}$, is also a Sylow p-subgroup of G. Thus, Sylow p-subgroups of $\mathrm{Aut}(G)$ are not normal.

Notice that techniques described above can be extended in a straightforward manner to an arbitrary finite abelian p-group. For example, for $G = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}$, where $m > n$, $|\mathrm{Aut}(G)| = p^{8n+m-4}(p-1)^2(p^2-1)$. Then a subgroup containing elements of the form

$$\begin{pmatrix} (1+p)^{b_1} & a_{12} & a_{13} \\ a_{21}p^{m-n} & (1+p)^{b_2} & a_{23}p \\ a_{31}p^{m-n} & a_{32} & (1+p)^{b_3} \end{pmatrix},$$

where $0 \le b_1 < p^{m-1}$, $0 \le b_2, b_3 < p^{n-1}$, $a_{ij} \in \mathbb{Z}_{p^n}$ for $i < j$, $0 \le a_{21}, a_{31} < p^n$, and $0 \le a_{32} < p^{n-1}$, is a Sylow p-subgroup of G. Notice that the block matrix in the matrix above

$$\begin{pmatrix} (1+p)^{b_2} & a_{23}p \\ a_{32} & (1+p)^{b_3} \end{pmatrix}$$

corresponds to a Sylow p-subgroup of $\mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}$.

For convenience, whenever $\lambda_i = \lambda_{i+k-1}$ for some i and $k-1$, we choose the $k \times k$ block matrix corresponding to a Sylow p-subgroup of $\Pi_{j=1}^{k}\mathbb{Z}_{p^{\lambda_i}}$ to be of the form

$$\begin{pmatrix} a_{11} & a_{12}p & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k}p \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix},$$

where $a_{ii} \equiv 1 \bmod p$.

For a finite abelian p-group of type λ such that $\lambda_i > \lambda_{i+1}$ for $1 \leq i \leq n-1$ a Sylow p-subgroup of $\mathrm{Aut}(G)$ is of the form

$$\begin{pmatrix} (1+p)^{b_1} & a_{12} & \dots & a_{1n} \\ a_{21}p^{\lambda_1-\lambda_2} & (1+p)^{b_2} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}p^{\lambda_1-\lambda_n} & a_{n2}p^{\lambda_2-\lambda_n} & \dots & (1+p)^{b_n} \end{pmatrix},$$

where $0 \leq b_i < p^{\lambda_i-1}$, $a_{ij} \in \mathbb{Z}_{p^{\lambda_j}}$ for $i < j$, and $0 \leq a_{ij} < p^{\lambda_i}$ for $i > j$. This is a unique Sylow p-subgroup of $\mathrm{Aut}(G)$ and therefore normal. Notice that a Sylow p-subgroup of $\mathrm{Aut}(G)$ is unique if and only if $\lambda_i > \lambda_{i+1}$ for all i.

2.2 Hall Generators

Let G be a finite abelian p-group of type $\lambda = (\lambda_1, ..., \lambda_n)$ such that $\lambda_i \geq \lambda_{i+1}$. In [2] L. Butler defined a set of generators for a subgroup H of G, called a set of Hall generators, that will be used extensively in the chapters ahead. Sets of Hall generators have very nice properties and provide extremely useful tools for working with subgroups of G in a systematic way.

Definition 2.1. Let H be a subgroup of G of isomorphism type $\mu = (\mu_1, ..., \mu_k)$. We call an ordered set $\{h^1, ..., h^k\}$ of generators of H a *set of Hall generators* for H if it satisfies the following conditions:

1. The order of $h^i = (h^i_1, ..., h^i_n)$ is p^{μ_i}.

   Given i, let I be the largest j such that order$(h^i_j) = p^{\mu_i}$.

2. If $j > i$, then $h^j_I = 0$.

3. If $j > i$ and $\mu_j = \mu_i$, then $J < I$.

Notice that I is the position of the right most component of h^i that has order p^{μ_i}, the order of h^i. It is always possible to find Hall generators of a finite abelian p-group and although the set of Hall generators of G is not unique in general, when we impose restrictions that $h^i_I = p^{\lambda_I - \mu_i}$ and $h^j_I < h^i_I \in \mathbb{Z}_{p^{\lambda_I}}$ for $j < i$ we fix exactly one set of Hall generators of G.

The following are the assumptions we will be making from now about a set of Hall generators of a subgroup H of G.

Assumption 1: A set of Hall generators will have restrictions $h^i_I = p^{\lambda_I - \mu_i}$ and $h^j_I < h^i_I$ for all $j < i$.

Assumption 2: If one of the entries of a Hall generator has the form xp^k for some $p \nmid x$ and $k \leq 0$, we assume that $xp^k = x$, that is $p^k = 1$.

Now we will present a few examples to clarify the definition of a set of Hall generators for a subgroup H of G.

Example 2.7. Let $G = \mathbb{Z}_{3^4} \times \mathbb{Z}_{3^2}$. Let H be a subgroup of isomorphism type $(2, 1)$. Then a possible set of Hall generators for H is $h^1 = (3^{4-2}, x3^{2-2+1}) = (3^2, 3x)$ and $h^2 = (0, 3^{2-1}) = (0, 3)$. Notice that the first component of h^1 has order 3^2 and the second component of h^2 has order 3. So $I = 1$ when $i = 1$ and $I = 2$ when $i = 2$. Since $h_I^j < h_I^i$ for $j < i$, $x = 0$. Thus, $h^1 = (9, 0)$, $h^2 = (0, 3)$. Another possible set of Hall generators for H is $h^1 = (x3^{4-2}, 3^{2-2}) = (3^2x, 1)$ and $h^2 = (3^{4-1}, 0) = (3^3, 0)$. Notice that the second component of h^1 has order 3^2 and the first component of h^2 has order 3. Again under the restriction $h_I^j < h_I^i$ for $j < i$, we have that the possible values of x are $0, 1, 2$.

Suppose H has isomorphism type $(2, 2)$. Then $h^1 = (x3^{4-2}, 3^{2-2}) = (3^2x, 1)$ and $h^2 = (3^{4-2}, 0) = (3^2, 0)$. Since $3^2x < 3^2$, $x = 0$. Thus the set of Hall generators of H is $\{(0, 1), (9, 0)\}$.

Suppose H is a cyclic subgroup of isomorphism type $(2, 0)$. Then possible Hall generators of H are $h^1 = (3^2, x3^{2-2+1}) = (9, 3x)$, where $x = 0, 1, 2$. Other possible Hall generators of H are $h^1 = (y3^{4-2}, 3^{2-2}) = (9y, 1)$, where $y = 0, ..., 8$.

Example 2.8. Let $G = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^s}$. Let H be a subgroup of G of isomorphism type (μ_1, μ_2, μ_3) with $\mu_1 > \mu_2 > \mu_3$. Then one possible set of Hall generators is $h^1 = (p^{m-\mu_1}, xp^{n-\mu_1+1}, yp^{s-\mu_1+1})$, $h^2 = (0, p^{n-\mu_2}, zp^{s-\mu_2+1})$, and $h^3 = (0, 0, p^{s-\mu_3})$, where $xp^{n-\mu_1+1} < p^{n-\mu_2}$ and $yp^{s-\mu_1+1}, zp^{s-\mu_2+1} < p^{s-\mu_3}$.

Suppose H has isomorphism type (μ_1, μ_2) such that $\mu_1 \leq \lambda_2$ and $\mu_2 \leq \lambda_3$. Then a possible set of Hall generators of H is $h^1 = (xp^{m-\mu_1}, p^{n-\mu_1}, yp^{s-\mu_1})$ and $h^2 = (zp^{m-\mu_2}, 0, p^{s-\mu_2})$, where $yp^{s-\mu_1} \leq p^{s-\mu_2}$.

Let H be a subgroup of G of type $\mu = (\mu_1, ..., \mu_k)$ and $\{h^1, ..., h^k\}$ be the set of Hall generators of H. Let e_I be the n-tuple that has 1 in the I'th component and 0's everywhere else.

Definition 2.2. The *Hall type* of H is an n-tuple $\oplus_i \mu_i e_I$, where I is defined in Definition 2.1.

The Hall type of a subgroup H of G is a permutation of the isomorphism type μ of H according to the placement of I. Notice that if two subgroups H, K of G are isomorphic, then their Hall types are the same when reordered as partitions. Also notice that a subgroup of Hall type $\oplus_i \mu_i e_I$ is isomorphic to the direct product of $\mathbb{Z}_{p^{\mu_i}}$ for $1 \leq i \leq k$, with the convention $\mathbb{Z}_{p^{\mu_j}} = \{1\}$ if $\mu_j = 0$, where the position of $\mathbb{Z}_{p^{\mu_i}}$ in the direct product is determined by the position of μ_i in $\oplus_i \mu_i e_I$.

Example 2.9. In Example 2.7, the Hall type of the subgroup generated by $\{(9, 0), (0, 3)\}$ is $(2, 1)$ since the first component of h^1 has order 3^2, while the Hall type of subgroups generated by $\{(9x, 1), (27, 0)\}$ is $(1, 2)$ since the second component of h^1 has order 3^2. The Hall type of a subgroup generated by

$\{(0,1),(9,0)\}$ is $(2,2)$. Cyclic subgroups of isomorphism type $(2,0)$ generated by $(9,3x)$ have Hall type $(2,0)$ and cyclic subgroups generated by $(9y,1)$ have Hall type $(0,2)$.

In Example 2.8, the Hall type of the first subgroup is (μ_1,μ_2,μ_3) and the Hall type of the second subgroup is $(0,\mu_1,\mu_2)$ since the right most component of order p^{μ_1} in h^1 is the second component and the right most component of order p^{μ_2} in h^2 is the third component.

Chapter 3

Sylow p-subgroups of Aut(G) and subgroup lattices of finite abelian p-groups

Let G be a finite abelian p-group of type $\lambda = (\lambda_1, ..., \lambda_n)$. Let $L_\lambda(p)$ be the lattice of subgroups of G. It is well-known [1] that there exists a correspondence between $L_\lambda(p)$ and the product of chains $[0, \lambda] = [0, \lambda_1] \times \cdots \times [0, \lambda_n]$. The correspondence between $L_\lambda(p)$ and $[0, \lambda]$ is defined in [2] as follows: for $\varphi : L_\lambda(p) \to [0, \lambda]$ let H be a subgroup of type $\mu = (\mu_1, ..., \mu_k)$ in G and $\{h^1, ..., h^k\}$ is the set of Hall generators of H, then $\varphi(H) =$ Hall type of H. Using enumerative properties of φ presented in [1], L. Butler showed in [2] that $L_\lambda(p)$ is an order-theoretic p-analogue of $[0, \lambda]$. We use the following definition of an *order-theoretic p-analogue* as defined in [2].

Definition 3.1. The graded, rank n poset $L(p)$ in a family indexed by an infinite set of positive integers is called an *order-theoretic p-analogue* of a graded, rank n poset L if there is a surjection $\phi : L(p) \to L$ such that

(a) If $H < K$ in $L(p)$, the $\phi(H) < \phi(K)$ in L.

(b) If $\alpha \leq \phi(K)$, then the cardinality of $\{H \mid H \leq K \text{ and } \phi(H) = \alpha\}$ is a power of p determined by α and $\phi(K)$.

(c) If $\{\alpha \mid \alpha \leq \phi(K)\}$ is a chain in L, then $\{H \mid H \leq K\}$ is a chain in $L(p)$.

Let S^p be a Sylow p-subgroup of the group of automorphisms of G. Let $\overline{L_\lambda(p)}$ be the quotient of the lattice of subgroups of G under the action of S^p. We refer the reader to Section 2.1.3 for a review of Sylow p-subgroups of $\mathrm{Aut}(G)$ in terms of matrices. Naturally, the orbits of the action of S^p in $L_\lambda(p)$ have size equal to a power of p. Let H be a subgroup of G. We denote the orbit of H under the action of S^p in $L_\lambda(p)$ by $S^p(H) = \{f(H) \mid f \in S^p\}$.

We begin with several examples.

Example 3.1. Let $G \cong \mathbb{Z}_{3^4} \times \mathbb{Z}_3$. So, $p = 3$ and $\lambda = (4, 1)$. Figure 3.1 shows the lattice of subgroups of G represented by Hall generators. Notice that subgroups inside boxes with thick border surrounding have the same Hall type and thus correspond to the same element of $[0, 4] \times [0, 2]$ under the correspondence φ defined above. Subgroups that are not inside a box correspond to exactly one element of $[0, 4] \times [0, 2]$. By discussion in Section 2.1.3 S^p is a unique Sylow p-subgroup of $\mathrm{Aut}(G)$. A typical element of S^p has

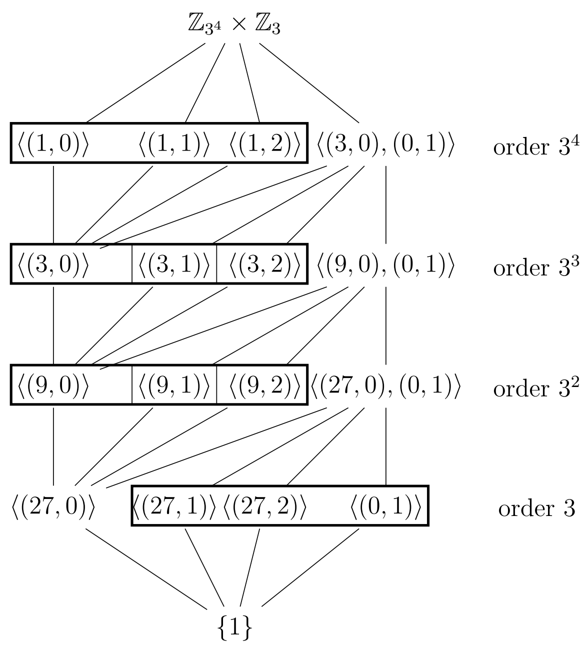

Figure 3.1: Subgroup lattice of $G = \mathbb{Z}_{3^4} \times \mathbb{Z}_3$.

the form $\begin{pmatrix} (1+3)^k & b \\ 3^{4-1}c & 1 \end{pmatrix}$, where $0 \leq k < 3^3$, $b \in \mathbb{Z}_3$, and $0 \leq c < 3$. Then the subgroups $\langle(3^a, 0)\rangle$ for $a \geq 1$ are fixed by every element of S^p. Also, $(9,1) \to (9((1+3)^k + 3c), 1) \in \langle(9,1)\rangle$. Thus, $\langle(9,1)\rangle$ forms its own orbit under the action of S^p. Similarly, we can see that subgroups in subdivided smaller boxes form individual orbits of S^p in $L_\lambda(p)$. Note that the fibers of φ in $L_\lambda(p)$ containing $\langle(3,0)\rangle$ and $\langle(9,0)\rangle$ each of size 3 split into the orbits of S^p that have size 1. Figure 3.2 compares the the quotient $\overline{L_\lambda(p)}$ and the product of chains $[0,4] \times [0,1]$. Notice that we can think of $\overline{L_\lambda(p)}$ as splitting of certain points and edges of $[0, \lambda]$.

We saw in the beginning of the chapter that the correspondence φ between

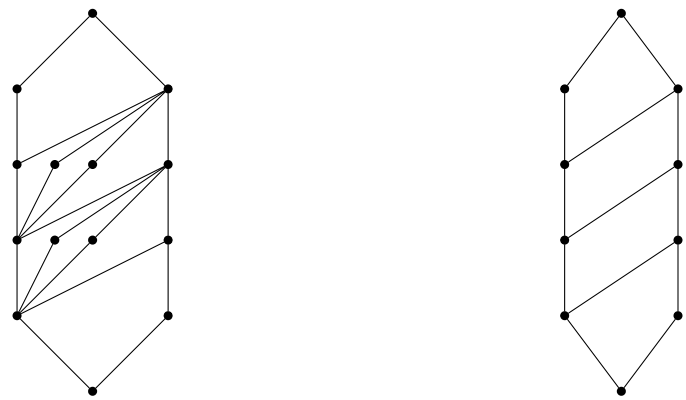

Figure 3.2: Quotient of the subgroup lattice of $G = \mathbb{Z}_{3^4} \times \mathbb{Z}_3$ under the action of S^p on the left and the product of chains $[0,4] \times [0,1]$ on the right.

$L_\lambda(p)$ and $[0,\lambda]$ is determined by the Hall type of a subgroup. Although the process of of determining the Hall type of a subgroup is reminiscent of matrix row reduction and thus should be close to being a group automorphism, the correspondence φ does not respect group automorphisms as we will see in the following example. Thus, the advantage of classifying finite abelian p-groups via the quotients $\overline{L_\lambda(p)}$ is that this action respects group automorphisms.

Example 3.2. Let $G \cong \mathbb{Z}_{3^4} \times \mathbb{Z}_{3^2}$. Figure 3.3 illustrates the lattice of subgroups of G and Figure 3.4 contrasts the quotient of $L_\lambda(p)$ under the action of S^p and the product of chains $[0,4] \times [0,2]$. Note that subgroups $\langle(3,0),(0,3)\rangle$, $\langle(3,1),(0,3)\rangle$, and $\langle(3,2),(0,3)\rangle$ have Hall type $(3,1)$, thus these subgroups lie in the fiber of $(3,1)$ under the correspondence φ. However, the subgroup $H = \langle(3,0),(0,3)\rangle$ is fixed by every lattice automorphism, and thus it is invariant under every group automorphism.

First, we make an important observation that the orbits of the action of

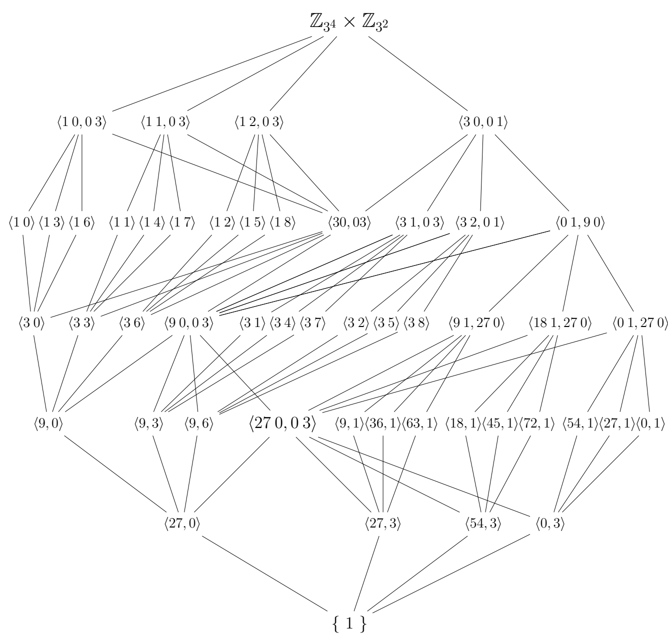

Figure 3.3: $G = \mathbb{Z}_{3^4} \times \mathbb{Z}_{3^2}$.

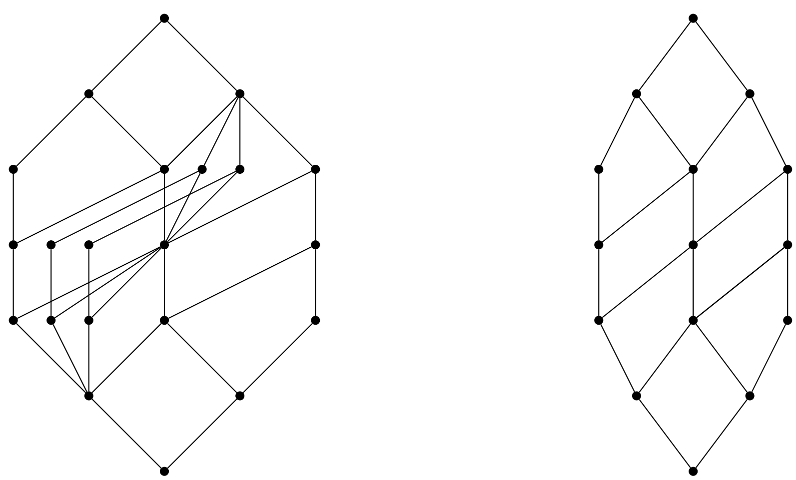

Figure 3.4: Quotient of the subgroup lattice of $G = \mathbb{Z}_{3^4} \times \mathbb{Z}_{3^2}$ under the action of S^p on the left and the product of chains $[0, 4] \times [0, 2]$ on the right.

S^p on $L_\lambda(p)$ are contained within subgroups of the same Hall type.

Theorem 3.1. *Let G be a finite abelian p-group of type λ. Let H be a subgroup of G of isomorphism type $\mu = (\mu_1, ..., \mu_k)$ and Hall type $\overline{\mu}$. Then the orbit of the action of S^p on $L_\lambda(p)$ containing H is contained in the fiber of $\overline{\mu}$ under φ, that is $S^p(H) \subset \varphi^{-1}(\overline{\mu})$.*

Proof. First we note that $\overline{\mu}$ is a permutation of entries of partition μ, that is $\overline{\mu}$ is not necessarily a partition. Let (a_{ij}) be the matrix associated with a Sylow p-subgroup of $\mathrm{Aut}(G)$ as described in Section 2.1.3. Let $\{h^1, ..., h^k\}$ be a set of Hall generators of H and I_i be the index of the right most component of h^i that has order p^{μ_i} as described in Section 2.2. Let (h_{ij}) be $k \times n$ matrix representing the set of Hall generators of H chosen above. Then $(h_{ij})(a_{ij}) = (b_{ij})$ is a $k \times n$ matrix. Since $a_{ii} \equiv 1 \bmod p^{\lambda_i}$ for all i, $b_{iI_i} = c_{iI_i}p^{\lambda_{I_i}-\mu_i}$, where $c_{iI_i} \equiv 1 \bmod p^{\lambda_{I_i}}$. Also for all other j if $h_{ij} = d_{ij}p^{k_{ij}}$, then $b_{ij} = e_{ij}p^{k_{ij}}$. Thus, we can row reduce (b_{ij}) so that $b_{ij} = 0$ for $j > I_i$ and $b_{iI_i} = p^{\lambda_{I_i}-\mu_i}$. Therefore, we can reduce matrix (b_{ij}) to represent Hall generators. Since the location of pivot points in (b_{ij}) is the same as in (h_{ij}), subgroup represented by (b_{ij}) has the same Hall type as H. Therefore, the orbit of the action of S^p containing H, $S^p(H)$, is contained in the fiber of $\overline{\mu}$, the Hall type of H, under the correspondence φ defined at the beginning of this chapter. $\square$

For groups in Examples 3.1 and 3.2 we saw that $\overline{L_\lambda(p)}$ was not equal to the product of chains $[0, \lambda]$. However, there are certain finite abelian p-groups for which $\overline{L_\lambda(p)}$ is equal to $[0, \lambda]$. We will discuss such subgroups next.

Theorem 3.2. *Let G be a finite abelian p-group of type λ such that $\lambda_1 = ... = \lambda_n$ or of type λ such that either $\lambda_i - \lambda_t = 0$ or $\lambda_i - \lambda_t = 1$ for all $1 \leq i < t \leq n$. Then $\overline{L_\lambda(p)} = [0, \lambda]$.*

Proof. First, we let $G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m}$. Let H be a subgroup of isomorphism $\mu = (\mu_1, \mu_2)$, where $\mu_1 \geq \mu_2$. Suppose H has Hall type (μ_1, μ_2), where $\mu_1 > \mu_2$. Then a set of Hall generators for H is $h^1 = (p^{m-\mu_1}, xp^{m-\mu_1+1})$ and $h^2 = (0, p^{m-\mu_2})$. An element of a Sylow p-subgroup of $\mathrm{Aut}(G)$ has the form $\begin{pmatrix} a & bp \\ c & d \end{pmatrix}$, where $c \in \mathbb{Z}_{p^m}$, $a = (1+p)^k$, and $d = (1+p)^l$. Let $a = 1 = d$ and $c = 0$. Then

$$\begin{pmatrix} p^{m-\mu_1} & xp^{m-\mu_1+1} \\ 0 & p^{m-\mu_2} \end{pmatrix} \begin{pmatrix} 1 & bp \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^{m-\mu_1} & p^{m-\mu_1+1}(b+x) \\ 0 & p^{m-\mu_2} \end{pmatrix}.$$

Thus, S^p maps H to a subgroup $\langle (p^{m-\mu_1}, p^{m-\mu_1+1}(b+x)), (0, p^{m-\mu_2})$. Since b is free, H can be mapped to every subgroup of the same Hall type. Thus, $\varphi^{-1}((\mu_1, \mu_2)) = S^p(H)$, where $S^p(H)$ is the orbit of H in $L_\lambda(p)$ under the action of S^p.

If H has Hall type (μ_2, μ_1), where $\mu_1 > \mu_2$. Then a set of Hall generators for H is $h^1 = (xp^{m-\mu_1}, p^{m-\mu_1})$ and $h^2 = (p^{m-\mu_2}, 0)$. Then choosing $a = 1 = d$ and $b = 0$, we see that H maps to a subgroup $\langle (p^{m-\mu_1}(x + c), p^{m-\mu_1}), (p^{m-\mu_2}, 0) \rangle$. Since c is free, H is mapped to every subgroup of Hall type (μ_2, μ_1).

If H has Hall type (μ_1, μ_1), then the set of Hall generators of H is $h^1 =$

$(0, p^{m-\mu_1})$ and $h^2 = (p^{m-\mu_1}, 0)$. Thus there is only one subgroup of such Hall type and it is fixed by every element of S^p. Therefore, $\overline{L_\lambda(p)} = [0, \lambda]$.

Similar calculations although cumbersome extend to the general case when G has type λ and H is a subgroup of G of type $\mu = (\mu_1, ..., \mu_k)$ under the action of Sylow p-subgroup of the form (a_{ij}), where $a_{ii} \equiv 1 \bmod p^m$, $a_{ij} = pb_{ij}$ for $i < j$ and $a_{ij} \in \mathbb{Z}_{p^m}$ for $j < i$.

Now suppose $G \cong \mathbb{Z}_{p^{m+1}} \times \mathbb{Z}_{p^m}$. Let H be a subgroup of G of Hall type (μ_1, μ_2), where $\mu_1 > \mu_2$. The set of Hall generators of H is $h^1 = (p^{m-\mu_1+1}, xp^{m-\mu_1+1})$ and $h^2 = (0, p^{m-\mu_2})$. A Sylow p-subgroup of Aut(G) has the form $\begin{pmatrix} a & b \\ cp & d \end{pmatrix}$, where $b \in \mathbb{Z}_{p^m}$, $a = (1+p)^k$, and $d = (1+p)^l$. Choosing $a = d = 1$ and $c = 0$, we have that H maps to a subgroup $K = \langle (p^{m-\mu_1+1}, p^{m-\mu_1+1}(b+x)), (0, p^{m-\mu_2})$. Since b is free, k is an arbitrary subgroup of Hall type (μ_1, μ_2). Thus, the orbit $S^p(H)$ is equal to $\varphi^{-1}((\mu_1, \mu_2))$.

Let H be a subgroup of G of Hall type (μ_2, μ_1) with $\mu_1 > \mu_2$. Then a set of Hall generators for H is $h^1 = (xp^{m-\mu_1+1}, p^{m-\mu_1})$ and $h^2 = (p^{m-\mu_2+1}, 0)$. Letting $a = d = 1$ and $b = 0$, H maps to $K = \langle (p^{m-\mu_1+1}(x+c), p^{m-\mu_1}),$ $(p^{m-\mu_2+1}, 0)\rangle$. Since c was arbitrary, K is an arbitrary subgroup of Hall type (μ_2, μ_1).

Similarly to above G contains a unique subgroup of type (μ_1, μ_1), which is fixed by every element of S^p. Thus, $\overline{L_\lambda(p)} = [0, \lambda]$.

For a general group G of type λ such that $\lambda_i - \lambda_t = 0$ or 1 for all

$1 \leq i < t \leq n$, let k be such that $\lambda_{k-1} - \lambda_k = 1$. A Sylow p-subgroup of $\mathrm{Aut}(G)$ is of the form (a_{ij}) such that

- $a_{ii} = (1+p)^{b_i}$ for $0 \leq b_i < p^{\lambda_1 - 1}$ for $i < k$ and $0 \leq b_i < p^{\lambda_k - 1}$ for $i \geq k$,
- $a_{ij} \in \mathbb{Z}_{p^{\lambda_j}}$ for $i < j$, $j \geq k$, and $i < k$,
- $a_{ij} = c_{ij}p$ for all other $i < j$,
- $a_{ij} = c_{ij}p$ for $i > j$, $j < k$, and $i \geq k$,
- $a_{ij} \in \mathbb{Z}_{p^{\lambda_j}}$ for all other $i > j$.

Similar computation as above can be done with this matrix to show that a subgroup of Hall type $\mu = (\mu_1, ..., \mu_k)$ can be mapped by an element of a S^p to an arbitrary subgroup of Hall type μ. Thus, $\overline{L_\lambda(p)} = [0, \lambda]$. $\square$

If we examine Examples 3.1 and 3.2, we notice that orbits of S^p split up fibers of the correspondence φ in a systematic manner. In the next Theorem we classify quotients of subgroup lattices of $G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ under the action of S^p. Notice that the cases when $m = n$ and $m = n + 1$ are covered in Theorem 3.2.

Theorem 3.3. *Let $G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, where $m - n \geq 2$. Let H be a subgroup of G of isomorphism type $\mu = (\mu_1, \mu_2)$. If the Hall type of H is (μ_1, μ_2), where $m > \mu_1 > \mu_2$, then $S^p(H)$ is strictly contained in the fiber $\varphi^{-1}([\mu_1, \mu_2])$, which splits into smaller orbits of S^p of equal size. Otherwise, $\varphi(\mu) = S^p(H)$.*

Proof. Suppose the Hall type of H is (μ_1, μ_2), where $\mu_1 > \mu_2$, then a set of Hall generators of H has the form $h^1 = (p^{m-\mu_1}, xp^{n-\mu_1+1})$ and $h^2 = (0, p^{n-\mu_2})$. Then multiplying the matrix representation of Hall generators of H on the left by a matrix representation of an element of the Sylow p-subgroup of H we get the following

$$\begin{pmatrix} p^{m-\mu_1} & xp^{n-\mu_1+1} \\ 0 & p^{n-\mu_2} \end{pmatrix} \begin{pmatrix} a & b \\ cp^{m-n} & d \end{pmatrix} =$$

$$\begin{pmatrix} p^{m-\mu_1}(a+cxp) & p^{n-\mu_1+1}(bp^{m-n-1}+xd) \\ p^{m-\mu_2}c & p^{n-\mu_2}d \end{pmatrix},$$

where $a \equiv 1 \bmod p$ and $d \equiv 1 \bmod p$. Since $a + cxp \equiv 1 \bmod p$, it is a unit, call it u. Notice that u^{-1} is also of the form $1 + \alpha p$ for some $\alpha \in \mathbb{Z}_{p^m}$. By multiplying the first row by $-cu^{-1}p^{\mu_1-\mu_2}$ and adding it to the second, we see that the coefficient of $p^{n-\mu_2}$ in the lower right entry is a unit since d was a unit. Thus, after row reduction the second row has the form $(0, p^{n-\mu_2})$. Multiplying the first row by u^{-1}, we have $u^{-1}(bp^{m-n-1} + xd)p^{n-\mu_1+1}$. Since b is arbitrary and $d \equiv 1 \bmod p$, the upper right entry of this matrix does not depend on u^{-1}. Thus, subgroups that H maps to under S^p have generators of the form

$$\begin{pmatrix} p^{m-\mu_1} & p^{n-\mu_1+1}(bp^{m-n-1}+xd) \\ 0 & p^{n-\mu_2} \end{pmatrix}.$$

Whenever $\mu_1 = m$, $p^{n-\mu_1+1}(bp^{m-n-1} + xd) = b + xd$ in the matrix above.

Since b is arbitrary, H maps to an arbitrary subgroup of the same Hall type as H. Therefore, in this case the orbit of S^p is equal to $\varphi^{-1}([\mu_1, \mu_2])$.

Suppose that $\mu_1 < m$. Let $d = 1 + \alpha p$, where α is arbitrary, and $x = \beta p^l$ for some $0 \leq l$ and $p \nmid \beta$. Then

$$bp^{m-n-1} + xd = bp^{m-n-1} + x(1 + \alpha p) \tag{3.1}$$

$$= p(bp^{m-n-2} + \beta\alpha p^l) + x \tag{3.2}$$

$$= p^{t+1}(bp^{m-n-2-t} + \beta\alpha p^{l-t}) + x, \tag{3.3}$$

where $t = \min\{m - n - 2, l\}$. Since both b and α are arbitrary, the orbit of H consists of those subgroups K that have Hall generators $k^1 = (p^{m-\mu_1}, yp^{n-\mu_1+1})$ and $k^2 = (0, p^{n-\mu_2})$ with $y \equiv x \bmod p^{t+1}$. Notice that the orbit of H depends completely on the selection of h^1. Notice that the size of the orbit of H is equal to p^{t+1}.

Suppose H has Hall type (μ_2, μ_1) such that $\mu_1 > \mu_2$. Then a set of Hall generators of H is $h^1 = (xp^{m-\mu 1}, p^{n-\mu_1})$ and $h^2 = (p^{m-\mu_2}, 0)$. Then applying matrix $\begin{pmatrix} 1 & 0 \\ cp^{m-n} & 1 \end{pmatrix}$ to H gives the matrix $\begin{pmatrix} p^{m-\mu_1}(x+c) & p^{n-\mu_1} \\ p^{n-\mu_2} & 0 \end{pmatrix}$, which corresponds to an arbitrary subgroup of Hall type (μ_2, μ_1) since c is arbitrary. Therefore, $S^p(H) = \varphi^{-1}([\mu_2, \mu_1])$.

Let H be a subgroup of G of Hall type (μ_1, μ_1). Then it is unique and thus $S^p(H) = \varphi^{-1}([\mu_1, \mu_1])$.

$\square$

Although computations are significantly more cumbersome in a general case the computations done in the previous Theorem could be extended to an arbitrary finite abelian p-group.

Corollary 3.1. *Let G be a finite abelian p-group of type λ and let H be a subgroup of G of isomorphism type $\mu = (\mu_1, ..., \mu_k)$. Suppose that H has Hall type $\overline{\mu} = (\mu_k, ..., \mu_1)$ or Hall type $\overline{\mu}$ such that $mu_i = \lambda_i$ for $1 \leq i \leq k-1$ or $\mu_1 = \mu_j$ for all $1 \leq j \leq k$, then $S^p(H) = \varphi^{-1}(\overline{\mu})$. Otherwise, the orbit of S^p is strictly contained in the inverse image of the correspondence function of $\overline{\mu}$, $\varphi^{-1}(\overline{\mu})$ (unless $|\varphi^{-1}(\overline{\mu})| = 1$).*

It is well-known that for a partition $\lambda = (\lambda_1, ..., \lambda_n)$ the product of chains $[0, \lambda]$ is self-dual lattice, that is there exists an order-reversing mapping from $[0, \lambda]$ to itself. From Figure 3.2 and Figure 3.4 we see that $\overline{L_\lambda(p)}$ is a self-dual lattice. For $G = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ where $m \geq n+2$ we have seen many examples that strongly suggest that $\overline{L_\lambda(p)}$ is a self-dual lattice.

Conjecture 3.1. *For $G = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, where $m \geq n+2$, $\overline{L_\lambda(p)}$ is a self-dual lattice.*

The idea for the possible proof was to find an order-reversing involution that normalizes orbits of S^p in the lattice of subgroups of G. However, we first needed to examine the relationship between the orbits of the actions of the group of lattice automorphisms and the subgroup of lattice automorphisms induced by group automorphism. This topic will be examined in the next chapter.

Chapter 4

Orbits of Autoprojectivities in Subgroup Lattices of Finite Abelian p-groups

Let G be a finite abelian p-group of type $\lambda = (\lambda_1, ..., \lambda_n)$. Let $L(G)$ be the lattice of subgroups of G. We call an automorphism of $L(G)$ an autoprojectivity of G. Let $P(G)$ be the group of all autoprojectivities of G and $PA(G)$ be the group of all autoprojectivities of G induced by automorphisms of G. Notice that $PA(G)$ is the quotient of the automorphism group of G, $\mathrm{Aut}(G)$, by the subgroup of all automorphisms fixing every subgroup of G, that is

$$PA(G) \cong \mathrm{Aut}(G)/\{\varphi \in \mathrm{Aut}(G) \mid \varphi(H) = H \ \forall\ H \leq G\}.$$

We say that two subgroups H, K in $L(G)$ are in the same *orbit* of $P(G)$ (or $PA(G)$) if there exists $\varphi \in P(G)$ (or $\varphi \in PA(G)$) such that $\varphi(H) = K$. It is clear that if H and K are in the same orbit of $PA(G)$, then H is isomorphic to K. Since every subgroup of an abelian p-group is an abelian p-group and since an abelian p-group is distinguished among abelian p-groups by its subgroup lattice, we have that if subgroups H and K of G are in the same orbit of $P(G)$, they are isomorphic. We will be interested in comparing the orbits of actions of $P(G)$ and $PA(G)$ on the subgroup lattice $L(G)$.

We will explore these ideas in an example below.

Example 4.1. Let $G = \mathbb{Z}_{3^3} \times \mathbb{Z}_3$. So, $p = 3$ and $\lambda = (3, 1)$. In the Figure 4.1, subgroups of G are represented by Hall generators, which will be defined later. In Figure 4.1 subgroups enclosed in a box represent subgroups in the same orbit of $P(G)$. Subgroups not enclosed in a box represent the orbits of $P(G)$ that contain exactly one element. These subgroups are fixed by every autoprojectivity of G. For instance, subgroup generated by (9, 0) cannot be moved by any autoprojectivity because there is no other subgroups order 3 that is contained in four subgroups of order 3^2. In this case there are three nontrivial orbits of $P(G)$.

Now we will calculate the orbits of $PA(G)$. We refer the reader to Section 2.1 for a review of group automorphisms of a finite abelian p-group. In our case, a group automorphism of G looks like $\begin{pmatrix} a & b \\ 3^2c & d \end{pmatrix}$, where $a, b \in \mathbb{Z}_{3^3}$, $c, d \in \mathbb{Z}_3$, and a, d are not divisible by p. We can map the subgroup generated

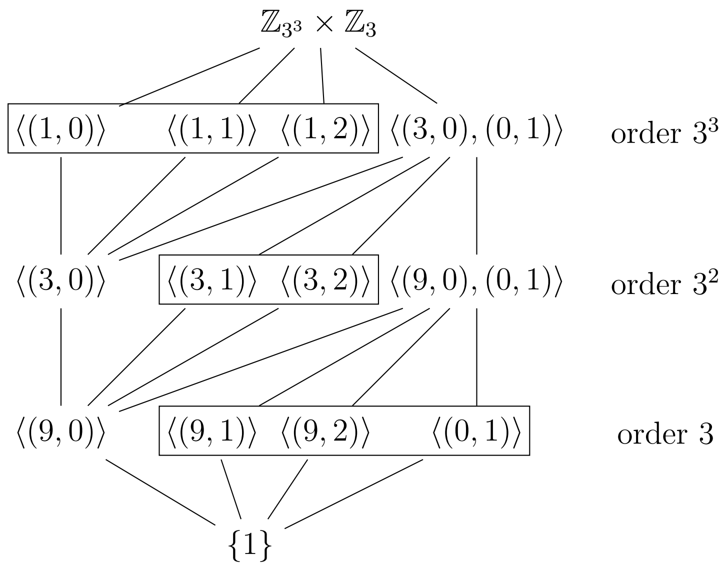

Figure 4.1: $G = \mathbb{Z}_{3^3} \times \mathbb{Z}_3$.

by $(0,1)$ to subgroups generated by $(9,1)$ and $(9,2)$ via the group automorphism $\begin{pmatrix} 1 & 0 \\ 3^2 & d \end{pmatrix}$, where $d = 1,2$. Also, we can map the subgroup generated by $(3,1)$ to the subgroup generated by $(3,2)$ via the group automorphism $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Finally, the subgroup generated by $(1,1)$ can be mapped to subgroups generated by $(1,1)$ and $(1,2)$ via $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$, where $d = 1,2$. Thus, the orbits of $PA(G)$ are the same as the orbits of $P(G)$.

4.1 Elementary Abelian Groups

First, we consider the most basic of finite abelian p-groups: elementary abelian groups. Let G be an elementary abelian group of order p^n, where $n \geq 1$. Then $G = (\mathbb{Z}_p)^n$. We can think of G as an n-dimensional vector space over the field $\mathbb{F}_p$ with p elements. The lattice of subgroups of G corresponds to the projective geometry of G as a vector space, where subspaces are subgroups of G. For reference check Figure 4.2 where a subgroup lattice of $G = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ is pictured. A projectivity of a projective geometry of a vector space is a bijection from one projective space to another that preserves the ordering of subspaces under inclusion. Then an autoprojectivity of G is a projectivity in the sense of projective geometry. Since the group of automorphisms of G, $\mathrm{Aut}(G)$, is isomorphic to the general linear group of $n \times n$ invertible matrices over $\mathbb{F}_p$, $GL(n, p)$, we have that $PA(G) \cong PGL(n, p)$, the projective linear group of $n \times n$ matrices over $\mathbb{F}_p$.

Suppose $n \geq 3$. The Fundamental Theorem of Projective Geometry ([8], p. 25) states that every autoprojectivity of G is induced by a semilinear transformation $f : G \to G$, where a semilinear transformation f is a mapping such that given a field automorphism θ of $\mathbb{F}_p$ for all $x, y \in G$ and $k \in \mathbb{F}_p$ we have $f(x + y) = f(x) + f(y)$ and $f(kx) = \theta(k)f(x)$. Since the only field automorphism of $\mathbb{F}_p$ is the trivial automorphism, we have that every autoprojectivity of G is induced by a linear transformation of G. Thus, every autoprojectivity of G is induced by a group automorphism, that is

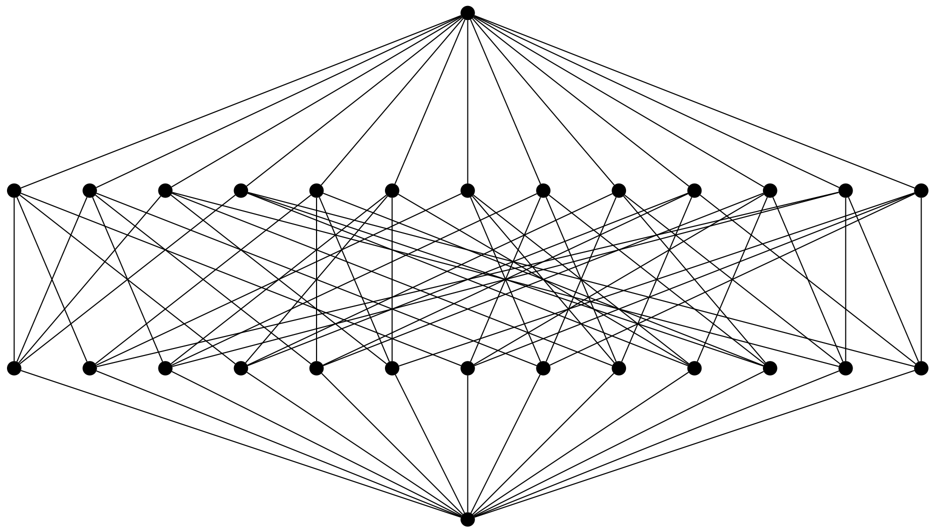

Figure 4.2: $G = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

$P(G) = PA(G)$. Therefore, trivially the orbits of $P(G)$ are equal to the orbits of $PA(G)$.

Suppose $n = 2$. Then the only nontrivial subgroups of G are cyclic subgroups of order p. There are $p+1$ cyclic subgroups of G which form an antichain in $L(G)/\{G, \{1\}\}$. Figure 4.3 provides an example of the subgroup lattice of $G = \mathbb{Z}_3 \times \mathbb{Z}_3$. Notice that an autoprojectivity of G can be described as a permutation of these $p+1$ subgroups. Thus, $P(G) \cong S_{p+1}$, the symmetric group on $p+1$ symbols. Notice that $|PGL(2,p)| = (p^2-1)(p^2-p)/(p-1) = (p-1)p(p+1)$. It is well-known that the action of $PGL(2,p)$ on the set of projective lines $\mathbb{P}_p^1$ induces an injection from $PGL(2,p)$ to S_{p+1}. By comparing the sizes of groups we have that $PGL(2,2) \cong S_3$ and $PGL(2,3) \cong S_4$. Thus, for $n = 2$ and $p = 2, 3$ we have that $PA(G) = P(G)$. For $p \geq 5$, $(p-1)p(p+1) = |PA(G)| < |P(G)| = (p+1)!$. Therefore, $PA(G)$ is a proper

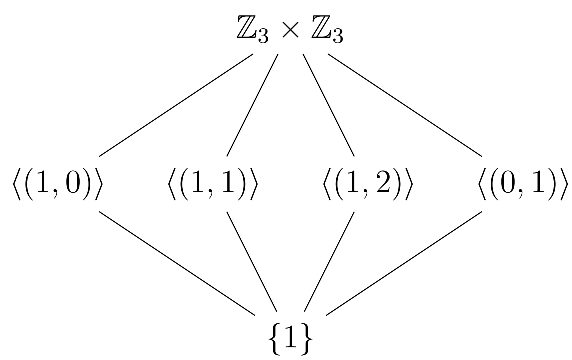

Figure 4.3: $G = \mathbb{Z}_3 \times \mathbb{Z}_3$.

subgroup of $P(G)$. Since $PA(G)$ is much smaller than $P(G)$ it is not clear whether the action of $PA(G)$ on $L(G)$ would create smaller orbits than the action of $P(G)$. Further analysis is needed to determine whether there exist subgroup H, K in $L(G)$ such that H and K are in the same orbit of $P(G)$ but it is not possible to find a group automorphism mapping H to K.

We collect results discussed above in the following Theorem.

Theorem 4.1. *Let G be an elementary abelian group of order p^n with $n \geq 1$. Then for $n \geq 3$ and for $n = 2$ and $p = 2, 3$ the orbits of $PA(G)$ are equal to the orbits of $P(G)$.*

In 1939 R. Baer generalized Theorem 4.1 to a much larger class of finite abelian p-groups than elementary abelian groups with $n \geq 3$ without the use of Fundamental Theorem of Projective Geometry.

Theorem 4.2. *Baer's Theorem (Theorem 2.6.7, [8]) Let G be a finite abelian p-group of type λ such that G contains at least 3 independent elements of or-*

der p^{λ_1}, then every autoprojectivity of G is induced by a group automorphism of G.

In other words, if a finite abelian p-group G has type λ such that $\lambda_1 = \lambda_3$, then $P(G) \cong PA(G)$. For such a finite abelian p-group, the orbits of $P(G)$ and $PA(G)$ in $L(G)$ are equal trivially. Notice that elementary abelian groups with $n \geq 3$ form a special case of Baer's Theorem.

We will examine the rest of finite abelian p-groups and their subgroup lattices under the actions of $P(G)$ and $PA(G)$. We begin with finite abelian p-groups of the form $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, where $m \geq n$.

4.2 Autoprojectivities of Subgroup Lattices of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$.

Let $G \cong \mathbb{Z}_{p^m} \times Z_{p^n}$, where p is prime and $m \geq n$. The structure of $P(G)$ is well-known and was described by C. Holmes in [6]. Let $SL(G)$ be the meet semi-lattice of cyclic subgroups of G and $\mathrm{Aut}(SL(G))$ be the group of all automorphisms of $SL(G)$. Holmes showed that the following result holds.

Theorem 4.3. *(Theorems 1, 2, [6]) Suppose $G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$. Then $P(G) =$ Aut($SL(G)$). If $G = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m}$, then $P(G) = (S_p)^{n-1} \wr S_{p+1}$, where $(S_p)^n$ is the n-fold wreath product. If $G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ with $m > n$, then $P(G) \cong S_p^n \times (S_p^{n-1} \wr S_{p-1}) \times \cdots \times (S_p^{n-1} \wr S_{p-1}) \times S_p^n$, where there are $m-n+1$ factors in the direct product.*

It is particularly interesting that by Theorem 4.3 we see that $P(G)$ is completely determined by $\mathrm{Aut}(SL(G))$. It is true because, as Holmes showed in [6], every automorphism of $SL(G)$ can be extended to an automorphism of $L(G)$. In particular, if H is a subgroup of G of isomorphism type (a, r) with $a \leq m$, $r \leq n$, and $a > r$, that is $H \cong \mathbb{Z}_{p^a} \times \mathbb{Z}_{p^r}$, then H can be written as $H = C \vee G(r)$, where C is a cyclic subgroup of H of order p^a and $G(r)$ is the unique subgroup of H isomorphic to $\mathbb{Z}_{p^r} \times \mathbb{Z}_{p^r}$. If f is an automorphism of $SL(G)$ then f extends to $\hat{f}$, a map on $L(G)$, as follows $\hat{f}(H) = \hat{f}(C \vee G(r)) = f(C) \vee G(r)$.

Notice that although a restriction of an autoprojectivity of G to the meet semi-lattice of cyclic subgroups is an automorphism of $SL(G)$, it is not always possible to extend an automorphism of $SL(G)$ to an automorphism of $L(G)$. For instance, if $G = (\mathbb{Z}_p)^n$ for $n \geq 3$ and H and K are non-trivial cyclic subgroups of G of the same order, then since G is vector space over the finite field $\mathbb{F}_p$ a permutation (H, K) is an automorphism of $SL(G)$. This automorphism is not linear, but as we discussed in Section 4.1 every autoprojectivity of G is linear as a consequence of the Fundamental Theorem of Projective Geometry. Therefore, this automorphism of $SL(G)$ cannot be extended to an automorphism of the subgroup lattice of G.

The explicit structure of $P(G)$ in terms of wreath products comes from the structure of $SL(G)$ which is straightforward to understand and will be discussed later.

We use the explicit structure of $P(G)$ described in Theorem 4.3 to gain insight into the relationship of orbits in $L(G)$ under the actions of $P(G)$ and $PA(G)$. We begin with a definition.

Definition 4.1. Let SL be a meet semi-lattice such that each element of dimension k contains exactly one element of dimension $k-1$. We define S to be of *type* $(n(1), ..., n(k))$ if every element of dimension $j-1$ is contained in exactly $n(j)$ elements of dimension j.

First, we consider the case when $G = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m}$. The identity subgroup is contained in $p+1$ cyclic subgroups of order p. Let H be a cyclic subgroup of G of order p^k for $1 \leq k < n$. Then H is generated by a Hall generator $h = (xp^{m-k}, yp^{m-k})$, where either $x \equiv 1 \bmod p$ or $y \equiv 1 \bmod p$. Then H is contained in cyclic subgroups of order p^{k+1} that are generated by Hall generator of the form $(xp^{m-k} + ap^{m-1}, yp^{m-k} + bp^{m-1})$, where $a \equiv 0 \bmod p^m$ if $x \equiv 1 \bmod p$ and $b \equiv 0 \bmod p^m$ if $y \equiv 1 \bmod p$. Thus, there are p cyclic subgroups of order p^{k+1} that contain H. Therefore, $SL(G)$ is of type $(p+1, p, ..., p)$. Lemma 5 in [6] states that if SL is a semi-lattice of type $(n(1), ..., n(k))$, then $\mathrm{Aut}(SL) = (S_{n(k)} \wr \cdots \wr S_{n(2)}) \wr S_{n(1)}$. Thus, $\mathrm{Aut}(SL(G))$ $= (S_p)^{n-1} \wr S_{p+1}$.

The structure of $P(G)$ implies that for every of $p+1$ cyclic subgroups of order p there exists an autoprojectivity mapping it to another cyclic subgroup of order p. The $(n-1)$-fold wreath product of S_p in $P(G)$ implies that there exists an autoprojectivity among every of p cyclic subgroups of order $p^2 \leq p^k \leq p^m$ lying above a certain cyclic subgroup of order p^{k-1}. Thus, the structure of $P(G)$ implies that there exists a lattice automorphism

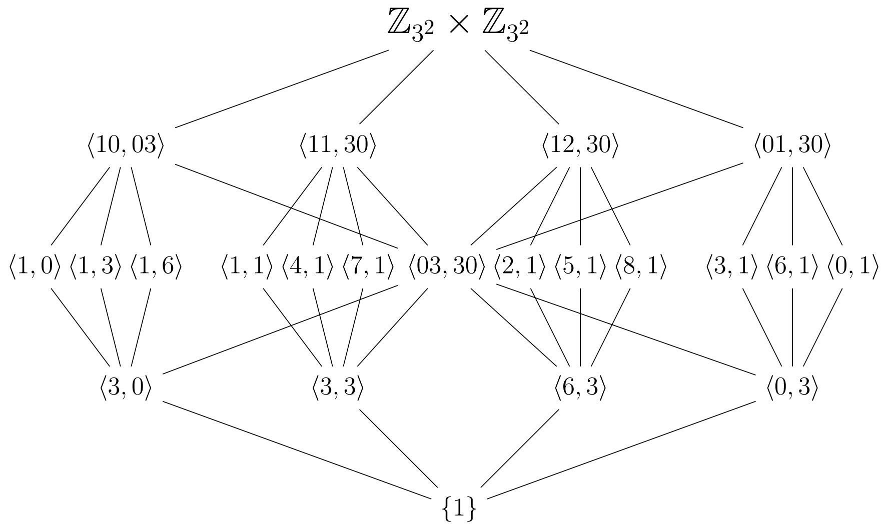

Figure 4.4: Lattice of subgroups of $G = \mathbb{Z}_{3^2} \times \mathbb{Z}_{3^2}$.

between any two cyclic subgroups of the same order. Since by results in [6] every lattice automorphism on cyclic subgroups can be extended to a lattice automorphism of $L(G)$, there exists an autoprojectivity between every two isomorphic subgroups of G. We refer to Figure 4.4 for an example of a subgroup lattice of $G = \mathbb{Z}_{3^2} \times \mathbb{Z}_{3^2}$.

Notice that although the orbits of $P(G)$ are relatively easy to see by looking at the shape of the subgroup lattice, the same is not true for the orbits of $PA(G)$ since group automorphisms are more restrictive than lattice automorphisms. Thus, just by looking at the subgroup lattice of G it is very difficult to understand how group automorphism act on the subgroup lattice and get our hands on the orbits of $PA(G)$. To describe group automorphism explicitly we need to work with some specific information that comes from subgroups. We discovered that a set of generators of a subgroup of a finite abelian p-group described by L. Butler in [2] has a particularly nice structure and provide a convenient way for describing orbits of $P(G)$ and $PA(G)$ in $L(G)$ in a systematic way. We invite the reader to review the definition of Hall generators in Section 2.2.

Suppose there exists an autoprojectivity between subgroups H, K of G. Then as noted above H and K have the same isomorphism type $\mu = (\mu_1, \mu_2)$, where $\mu_1 \geq \mu_2$. Then either H and K have the same Hall type or the Hall type of K is a nontrivial permutation of the Hall type of H. Notice that there is only one subgroup of type (μ_1, μ_2) when $\mu_1 = \mu_2$. Thus, without loss of generality, we can assume that $\mu_1 > \mu_2$ and we have to consider three distinct cases:

Case 1: H and K have Hall type (μ_1, μ_2).

Then we can write Hall generators $\{h^1, h^2\}$ of H and $\{g^1, g^2\}$ of K as $h^1 = (p^{m-\mu_1}, xp^{m-\mu_1+1})$, $h^2 = (0, p^{m-\mu_2})$ and $g^1 = (p^{m-\mu_1}, yp^{m-\mu_1+1})$, $g^2 = (0, p^{m-\mu_2})$, where $x, y \in \mathbb{Z}_{p^m}$. Then a group automorphism φ defined by the matrix $\begin{pmatrix} 1 & bp \\ 0 & 1 \end{pmatrix}$, where $b, d \in \mathbb{Z}_{p^m}$, takes h^1 to $(p^{m-\mu_1}, bp^{m-\mu_1+1} + xp^{m-\mu_1+1}) = (p^{m-\mu_1}, p^{m-\mu_1+1}(b + x))$. Since b is arbitrary, $b + x = y$ has a solution for any $y \in \mathbb{Z}_{p^m}$. Moreover, φ takes h^2 to g^2. Thus, H and K are in the same orbit of $PA(G)$.

Case 2: H and K have Hall type (μ_2, μ_1).

Then we can write Hall generators $\{h^1, h^2\}$ of H and $\{g^1, g^2\}$ of K as $h^1 = (xp^{m-\mu_1}, p^{m-\mu_1})$, $h^2 = (p^{m-\mu_2}, 0)$ and $g^1 = (yp^{m-\mu_1}, p^{m-\mu_1})$,

$g^2 = (p^{m-\mu_2}, 0)$, where $x, y \in \mathbb{Z}_{p^m}$. Then a group automorphism φ defined by the matrix $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$, where $c \in \mathbb{Z}_{p^m}$, takes h^1 to $(xp^{m-\mu_1} + cp^{m-\mu_1}, p^{m-\mu_1}) = (p^{m-\mu_1}(x+c), p^{m-\mu_1})$. Since c is arbitrary, $x + c = y$ has a solution for every $y \in \mathbb{Z}_{p^m}$. Also, φ takes h^2 to g^2. Thus, H and K are in the same orbit of $PA(G)$.

Case 3: H has Hall type (μ_1, μ_2) and K has Hall type (μ_2, μ_1).

Then we can write Hall generators $\{h^1, h^2\}$ of H as $h^1 = (p^{m-\mu_1}, xp^{m-\mu_1+1})$, $h^2 = (0, p^{m-\mu_2})$ and Hall generators $\{g^1, g^2\}$ of K as $g^1 = (yp^{m-\mu_1}, p^{m-\mu_1})$, $g^2 = (p^{m-\mu_2}, 0)$, where $x, y \in \mathbb{Z}_{p^m}$. Then a group automorphism φ defined by the matrix $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$, where $a \in \mathbb{Z}_{p^m}$, takes h^1 to $(ap^{m-\mu_1} + xp^{m-\mu_1+1}, p^{m-\mu_1}) = (p^{m-\mu_1}(a+cp), p^{m-\mu_1})$. Since a is arbitrary, $a + cp = y$ has a solution for every $y \in \mathbb{Z}_{p^m}$. Moreover, it is clear that φ takes h^2 to g^2. Thus, H and K are in the same orbit of $PA(G)$.

All of three cases above are still valid for cyclic groups, when $\mu_2 = 0$. Thus, we have shown that

Theorem 4.4. *When $G = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m}$, the orbits of $P(G)$ and $PA(G)$ on $L(G)$ coincide.*

Now, suppose $G \cong \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, where $m > n$. Let F be the intersection of all cyclic subgroups of order p^m in G. Notice that p^n subgroups of order p^m

in G have Hall generators $(1, x)$, where $x \in \mathbb{Z}_{p^n}$. So, F is a cyclic subgroup of order p^{m-n} with Hall generator $(p^n, 0)$. The chain of subgroups of F, including F, has $m - n + 1$ element. Every non-trivial subgroups of F has a Hall generator of the form $(p^{m-k}, 0)$, where $0 < k \leq m - n$. Since no other subgroup of order p^{m-n} is contained in every subgroup of order p^m, F and the chain of its subgroups is fixed by every autoprojectivity of G.

We can describe the meet semi-lattice of cyclic subgroups $SL(G)$ in terms of semi-lattices of cyclic subgroups of various types attached to the chain of subgroups of F. We attach semi-lattices of cyclic subgroups of type $(p, ..., p)$ to F and the identity subgroup and to all other $m - n - 1$ subgroups in the chain of subgroups of F we attach semi-lattices cyclic subgroups of type $(p - 1, p, ..., p)$. We say that two cyclic subgroups H and K are in the same *branch collection* if $H \cap F = K \cap F$. We associate a branch collection with the corresponding subgroup of F and call it a *k-branch collection*, where $0 \leq k \leq n - m$ and p^k is the order of the corresponding subgroup of F. We clarify these ideas in the following example.

Example 4.2. Figure 4.5 illustrates the meet semi-lattice of cyclic subgroups of $G = \mathbb{Z}_{3^4} \times \mathbb{Z}_{3^2}$. The filled-in circles represent cyclic subgroups of G of order 3^4. The filled in diamond node represents the intersection of all cyclic subgroups of order 3^4 and the thick line segments and white diamond nodes represent the chain of cyclic subgroups of F. Notice that F and the chain of its subgroups cannot be moved by any automorphism of $SL(G)$ and also by any automorphism of $L(G)$. The bottom node is the identity subgroup. Notice

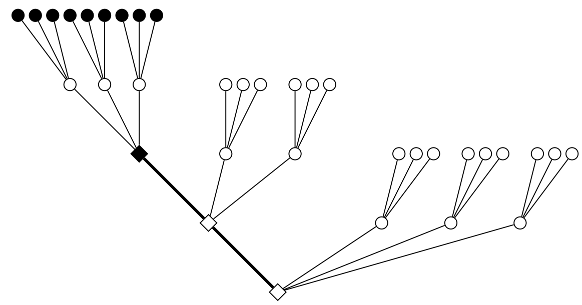

Figure 4.5: Meet semi-lattice of cyclic subgroups of $G = \mathbb{Z}_{3^4} \times \mathbb{Z}_{3^2}$.

that 3 branches of cyclic subgroups are attached the identity. These branches can be permuted among themselves via automorphisms of $SL(G)$. Also, two branches of cyclic subgroups are attached at a cyclic non-trivial subgroup of F and can be permuted among themselves. Lastly, three branches of cyclic subgroups are attached at F and also can be permuted among themselves.

We can extend the characterization of $SL(G)$ in terms of the chain of subgroups of F to the entire subgroup lattice of G as follows: we say that a subgroup H of G with isomorphism type (μ_1, μ_2), where $\mu_1 \geq \mu_2$, is *contained in the k-branch collection* if the Hall generator of H of order p^{μ_1} is contained in the k-branch collection. In other words, if $\{h^1, h^2\}$ is a set of Hall generators of H, then H belongs to the same branch collection as h^1.

We first consider the 0-branch collection, that is all subgroups of G that intersect the chain of subgroups of F at the identity subgroup. Notice that subgroups of order p are all cyclic and either have Hall type $(1, 0)$ or $(0, 1)$. Subgroups of Hall type $(1, 0)$ are cyclic subgroups that are generated by a Hall generator of the form $(p^{m-1}, xp^{n-1+1}) = (p^{m-1}, xp^n) = (p^{m-1}, 0)$. Thus, there is only one subgroup of Hall type $(1,0)$ and it is a subgroup of F. Thus, all of subgroups of order p that are contained in the 0-branch collection have Hall type $(0,1)$. Then a cyclic subgroup of order p^k, where $1 \le k \le n$, containing a subgroup of Hall type $(0,1)$ has Hall type $(0,k)$. Therefore, cyclic subgroups contained in the 0-branch collection have Hall type $(0,k)$. We will show that all subgroups contained in the 0-branch collection have Hall type (μ_2, μ_1), where $\mu_1 \ge \mu_2$.

Proposition 4.1. *If subgroups H and K are contained in the same branch collection and have the same isomorphism type, then they have the same Hall type.*

Proof. Suppose H is a subgroup of G of Hall type (μ_1, μ_2), where $\mu_1 > \mu_2$. Then the set of Hall generators for H is $h^1 = (p^{m-\mu_1}, xp^{n-\mu_1+1})$ and $h^2 = (0, p^{n-\mu_2})$, where $xp^{n-\mu_1+1} < p^{n-\mu_2} \in \mathbb{Z}_{p^n}$. By definition, H is contained in the branch collection of h^1, which has Hall type $(\mu_1, 0)$. In fact, we can calculate the exact branch collection of H. Suppose $x = \alpha p^a$, where p does not divide α and $0 \le a < \mu_1 - 1$. Then $p^{\mu_1-1-a}h^1 = (p^{m-a-1}, 0)$. So H is contained in the $(m-a-1)$-branch collection. Since $\mu_1 \le m$ and $a < \mu_1 - 1$, $m - a - 1 \ne 0$. So, H is not contained in the 0-branch collection. Therefore, if a subgroup is contained in the 0-branch collection, it has Hall type (μ_2, μ_1), where $\mu_1 \ge \mu_2$.

Suppose subgroups H and K have the same isomorphism type (μ_1, μ_2), where $\mu_1 > \mu_2$, but different Hall types. Suppose H has Hall type (μ_1, μ_2) and K has Hall type (μ_2, μ_1). Then K is contained in the 0-branch collection and H is not by the argument above. Therefore, H and K are not contained in the same branch collection. $\square$

Now we will examine the relationship between orbits of $P(G)$ and branch collections.

Proposition 4.2. *Subgroups H and K of G are in the same orbit of $P(G)$ if and only if H and K have the same isomorphism type and they are contained in the same branch collection.*

Proof. Suppose H and K are in the same orbit of $P(G)$. Then there exists an autoprojectivity φ between H and K. Then H and K are finite abelian p-groups and since finite abelian p-groups are completely distinguished by their subgroup lattices among other finite abelian p-groups, H and K have the same isomorphism type (μ_1, μ_2), where $\mu_1 \geq \mu_2 > 0$ (if $\mu_2 = 0$, then we have a cyclic subgroup and from the discussion prior to Example 2 H and K have belong to the same branch collection). Let $\{h^1, h^2\}$ and $\{k^1, k^2\}$ be sets of Hall generators for H and K respectively. Then φ restricted to the meet semi-lattice of cyclic subgroups belongs to $\mathrm{Aut}(SL(G))$. Thus, $\varphi(\langle h^1 \rangle) \in K$ is a cyclic subgroup of K of order p^{μ_1} that belongs to the same branch collection as $\langle h^1 \rangle$. Suppose $\varphi(\langle h^1 \rangle)$ and $\langle k^1 \rangle$ are cyclic subgroups that belong to different branch collections.

Case 1: Suppose the Hall type of H is (μ_1, μ_2), where $\mu_1 > \mu_2 > 0$. Then $h^1 = (p^{m-\mu_1}, xp^{n-\mu_1+1})$, where $x = ap^k$ for some $0 \leq k$ and $p \nmid a$, and $h^2 = (0, p^{n-\mu_2})$. Then $\langle \varphi(\langle h^1 \rangle) \rangle = \langle (p^{m-\mu_1}, yp^{n-\mu_1+1}) \rangle$ for some $y = bp^d$

such that b such that $p \nmid b$. Also, $\langle k^1 \rangle$ has Hall type $(\mu_1, 0)$ or $(0, \mu_1)$. If $\langle k^1 \rangle$ has Hall type $(\mu_1, 0)$, then $\langle k^1 \rangle = (p^{m-\mu_1}, zp^{n-\mu_1+1})$ for some $z = cp^t$ for some $t \neq d$ and $p \nmid c$ and $k^2 = (0, p^{n-\mu_2})$. By exchanging H and K and replacing φ by φ^{-1}, without loss of generality we may assume that $t < k$. Since we are assuming that $\langle k^1 \rangle$ belongs to a different branch collection than $\langle h^1 \rangle$, we assume that $\mu_1 \leq n$. Since K is generated by k^1 and k^2 and contains $\langle (p^{m-\mu_1}, bp^d p^{n-\mu_1+1}) \rangle$, we can write $(p^{m-\mu_1}, bp^d p^{n-\mu_1+1}) = \alpha(p^{m-\mu_1}, cp^t p^{n-\mu_1+1}) + \beta(0, p^{n-\mu_2})$, which implies that $\alpha = 1$. Then we can write $bp^d p^{n-\mu_1+1} = cp^t p^{n-\mu_1+1} + \beta p^{n-\mu_2} = p^{n-\mu_1+1+t}(c + \beta p^{\mu_1-\mu_2-t-1})$, since by definition of Hall generators $n - \mu_1 + 1 + t < n - \mu_2$, thus $\mu_2 + 1 + t < \mu_1$. Thus, since $t < d$ we have $c + \beta p^{\mu_1-\mu_2-t-1}$ is a power p, which implies that $p^{\mu_1-\mu_2-t-1} = 1$. Thus, $\mu_1 - \mu_2 - t - 1 \leq 0 \Rightarrow \mu_1 \leq \mu_2 + t + 1$, which is a contradiction. Therefore, $z = 0$. Since cyclic subgroup $\langle (p^{m-\mu_1}, 0) \rangle$ is invariant with respect to lattice automorphisms of G, H would have to be equal to K.

If K has Hall type (μ_2, μ_1). Then $k^1 = (zp^{m-\mu_1}, p^{n-\mu_1})$ for some $z = cp^t$ for some $t \geq 0$ and $p \nmid c$ and $k^2 = (p^{m-\mu_2}, 0)$. Then the subgroup $\langle (p^{m-\mu_1}, yp^{n-\mu_1+1}) \rangle$ cannot be contained in K since $(p^{m-\mu_1}, bp^k p^{n-\mu_1+1})$ $= \alpha(cp^t p^{m-\mu_1}, p^{n-\mu_1}) + \beta(p^{m-\mu_2}, 0)$ since $m - \mu_2 > m - \mu_1$ implies that $\beta = 0$, $t = 0$ and $\alpha = c^{-1}$. This is a contradiction since $bp^k p^{n-\mu_1+1} \neq c^{-1} p^{n-\mu_1}$.

Therefore, we can assume that $\varphi(\langle h^1 \rangle)$ belongs to the same branch collection as k^1. Thus, H and K belong to the same branch collection.

Case 2: Suppose the Hall type of H is (μ_2, μ_1), where $\mu_1 > \mu_2 > 0$. Then $h^1 = (xp^{m-\mu_1}, p^{n-\mu_1})$ and $h^2 = (p^{m-\mu_2}, 0)$. Notice that h^1 and thus H belongs to the 0-branch collection, as does any subgroup of Hall type (μ_2, μ_1). Thus, $\varphi(\langle h^1 \rangle) = \langle (yp^{m-\mu_1}, p^{n-\mu_1}) \rangle$ belongs to the 0-branch collection. If k^1 belongs to a different branch collection than $\langle h^1 \rangle$, then K has Hall type (μ_1, μ_2) with Hall generators $k^1 = (p^{m-\mu_1}, zp^{n-\mu_1} + 1)$ and $p \nmid c$ and $k^2 = (0, p^{m-\mu_2})$. Then since $p^{n-\mu_1} + 1$ and $p^{m-\mu_2}$ are strictly greater than $p^{n-\mu_1}$, we cannot write $(yp^{m-\mu_1}, p^{n-\mu_1})$ as a linear combination of k^1 and k^2.

Therefore, we can assume that k^1 belongs to the 0-branch collection and thus H and K are in the same branch collection.

Case 3: Say H has Hall type (μ_1, μ_1). There is a unique subgroup of Hall type (μ_1, μ_1), thus if H has Hall type (μ_1, μ_1), then $H = K$ and H and K are contained in the same branch collection trivially.

By results in [6], since every automorphism on the semi-lattice of cyclic subgroups could be extended to an automorphism of $L(G)$, every autoprojectivity is completely described by autprojectivities of its cyclic subgroups. Since by the structure of $P(G)$ in Theorem 4.3 a lattice automorphism among cyclic subgroups exists only if they are in the same branch collection. Thus, H and K are in the same branch collection.

Suppose H and K have the same isomorphism type and belong to the same k-branch collection. We will show in Theorem 4.5 that H and K belong to the same orbit of $PA(G)$. Since $PA(G)$ is a subgroup of $P(G)$, H and K also lie in the same orbit of $P(G)$.

$\square$

We would like to establish the relationship between orbits of lattice automorphisms and orbits of lattice automorphisms induced by group automorphisms but first we will examine an example.

Example 4.3. Let $G = \mathbb{Z}_{3^4} \times \mathbb{Z}_{3^2}$. Figure 4.6 contains the lattice of subgroups of G and orbits of $P(G)$ and $PA(G)$. Cyclic subgroups of order 3^4 are of the form $\langle(1,x)\rangle$, where $x = 0,...,8$. The intersection of all cyclic subgroups of order 3^4 is the cyclic subgroup $F = \langle(9,0)\rangle$, which has order 3^2. Notice that $\langle(9,0)\rangle$ and its subgroups are stable under the action of lattice automorphisms and therefore under the action of group automorphisms, since a group automorphism $\begin{pmatrix} a & b \\ 3^{4-2}c & d \end{pmatrix}$, where $a, c \in \mathbb{Z}_{3^4}$, $b, d \in \mathbb{Z}_{3^2}$ and a, d not divisible by p, maps $(3^k, 0)$, where $2 \leq k \leq 4$, to $(a3^k, 0)$.

As described in Example 4.2, cyclic subgroups H,K of G are in the same $P(G)$-orbit if and only if $|H| = |K|$ and $H \cap F = K \cap F$. Cyclic subgroups in the same orbit of $P(G)$ are enclosed in boxes. Cyclic subgroups that are not enclosed in a box, are fixed by every autoprojectivity of G and form orbits of $P(G)$ that contain exactly one element. It is straightforward to calculate that orbits of $P(G)$ containing cyclic subgroups are also orbits of $PA(G)$. For instance, $(0,1)$ is mapped to $(x,1)$ for $0 \leq x \leq 8$ via group automorphisms

$$\begin{pmatrix} 1 & 0 \\ x3^2 & 1 \end{pmatrix}.$$

Notice that only cyclic subgroups in the branch collection that attaches at the identity have Hall type $(0, \mu_1)$. Also, notice that although cyclic subgroups $\langle(9,3)\rangle$ and $\langle(9,0)\rangle$ have the same Hall type $(2,0)$, they do not belong to the same branch collection since $\langle(9,3)\rangle$ is intersects the chain of subgroups of F at $\langle(27,0)\rangle$.

Recall that a noncyclic subgroup H of G with Hall generators h^1, h^2 is said to be in the same branch collection as the subgroup generated by h^1. For example, $H = \langle(0,1),(27,0)\rangle$ is in the branch collection of the identity since $h^1 = \langle(0,1)\rangle$. An important observation is that subgroups that belong to the same branch collection have the same Hall type as subgroups $\langle(3,1),(0,3)\rangle$ and $\langle(3,2),(0,1)\rangle$ have Hall type $(3,1)$. However, it is also possible for subgroups to have the same Hall type, but belong to different branch collections as both $\langle(3,0),(0,3)\rangle$ and $\langle(3,1),(0,3)\rangle$ have Hall type $(3,1)$ but $\langle(3,0),(0,3)\rangle$ belongs to the branch collection of $\langle(9,0)\rangle$ and $\langle(3,1),(0,3)\rangle$ belongs to the branch collection of $\langle(27,0)\rangle$.

Let H be a subgroup of G of isomorphism type (μ_1, μ_2). Since every automorphism of the meet semi-lattice of cyclic subgroups extends to a lattice automorphism of $L(G)$ and by definition the extension depends only on cyclic subgroups of H of order p^{μ_1}, we see that subgroups that are of the same isomorphism type and are in the same branch collection lie in the same orbit of $P(G)$. The boxes surrounding two-generator subgroups represent orbits of

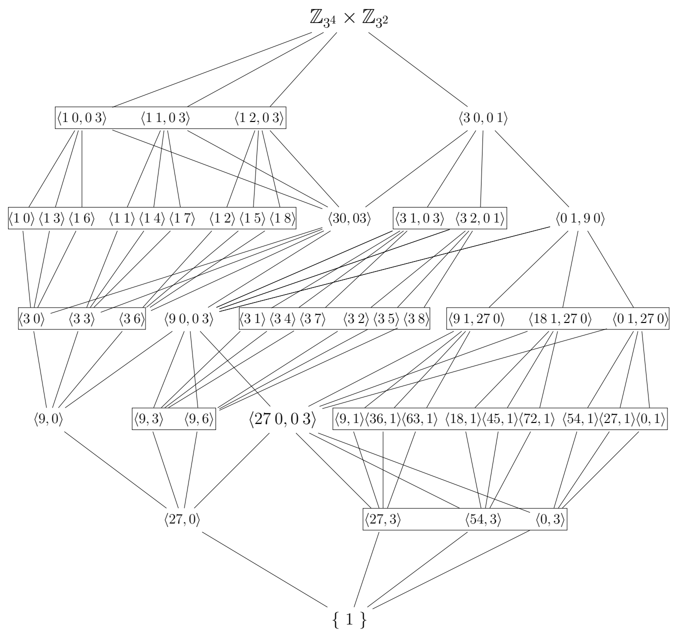

Figure 4.6: Orbits of $P(G)$ and $PA(G)$ for $G = \mathbb{Z}_{3^4} \times \mathbb{Z}_{3^2}$.

$P(G)$. We can also show that these orbits are also orbits of $PA(G)$. In other words, for every subgroup in the box there exists a group automorphism that box moves one subgroup in the box to another. For instance, $\langle (0,1),(27,0)\rangle$ could be mapped to $\langle (k,1),(27,0)\rangle$, where $k = 9, 18$ by group automorphisms $\begin{pmatrix} 1 & 0 \\ 3^2c & 1 \end{pmatrix}$, where $c = 1, 2$, respectively.

It is interesting to notice that this example is the smallest group such that $p > 2$ and the chain of subgroups of F includes a non-trivial subgroup. Also, this is the smallest group that contains orbits of $P(G)$ whose size is a multiple of p, but not a power of p. The orbits whose size is not a power of p come from the branch collections that are attached at non-trivial cyclic subgroups of F.

Observations presented in the example above are generalized in the following Theorem.

Theorem 4.5. *Orbits of $P(G)$ in $L(G)$ coincide with orbits of $PA(G)$.*

Proof. Suppose subgroups H and K of G are contained in the same orbit of $P(G)$. Then by Proposition 4.2, H and K have the same isomorphism type and belong to the same branch collections. By Proposition 4.1, H and K have the same Hall type.

Case 1: H and K have Hall type (μ_1, μ_2), where $\mu_1 > \mu_2$. Then sets of Hall generators $\{h^1, h^2\}$ for H and $\{g^1, g^2\}$ for K are $h^1 = (p^{m-\mu_1}, xp^{n-\mu_1+1})$, $h^2 = (0, p^{n-\mu_2})$ and $g^1 = (p^{m-\mu_1}, yp^{n-\mu_1+1})$, $g^2 = (0, p^{n-\mu_2})$, where $x, y \in \mathbb{Z}_{p^n}$, $x = \alpha p^a$, $y = \beta p^b$, with α, β not divisible by p and $0 \leq a, b < \mu_1 - 1$, and $xp^{n-\mu_1+1}, yp^{n-\mu_1+1} < p^{n-\mu_2}$. Since H and K are in the same branch collection, h^1 and g^1 are in the same branch collection. Since the branch collection depends exclusively on the power of p in the second component of h^1, we have that h^1 is in the $(m-a-1)$-branch collection. Since g^1 is in the same branch collection as h^1, $b = a$. Since both α and β are units in $\mathbb{Z}_{p^n}$, there exists $d \in (\mathbb{Z}_{p^n})^*$ such that $\alpha = d\beta$. Then the automorphism $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$ maps h^1 to g^1 and h^2 to dh^2. Since d is a unit in $\mathbb{Z}_{p^n}$, dh^2 is also a generator for K. Therefore, H and K are in the same $PA(G)$ orbit.

Case 2: H and K have Hall type (μ_2, μ_1), where $\mu_1 > \mu_2$. We can assume that $\mu_1 \leq n$. The Hall generators $\{h^1, h^2\}$ of H and $\{g^1, g^2\}$ of K can be written as $h^1 = (xp^{m-\mu_1}, p^{n-\mu_1})$, $h^2 = (p^{m-\mu_2}, 0)$ and $g^1 = (yp^{m-\mu_1}, p^{n-\mu_1})$, $g^2 = (p^{m-\mu_2}, 0)$, where $x, y \in \mathbb{Z}_{p^m}$, $x = \alpha p^a$, $y = \beta p^b$, with α, β not divisible by p and $0 \leq a, b < \mu_1 - 1$, and $xp^{m-\mu_1}$, $yp^{m-\mu_1+1} < p^{m-\mu_2}$. By Proposition 4.1, H and K belong to the 0-branch collection. Since α and β are units in $\mathbb{Z}_{p^m}$, there exists $c \in (\mathbb{Z}_{p^m})^*$ such that $\beta = c\alpha$.

Suppose $a > b$ or $a < b$, then the automorphism $\begin{pmatrix} c & 0 \\ (y-cx)p^{m-n} & 1 \end{pmatrix}$ maps h^1 to g^1 and h^2 to ch^2. Since c is a unit in $\mathbb{Z}_{p^m}$, ch^2 is also a generator for K.

Suppose $a = b$, the the automorphism $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$ maps h^1 to g^1 and h^2 to ch^2. Since c is a unit in $\mathbb{Z}_{p^m}$, ch^2 is also a generator for K.

Therefore, H and K are in the same $PA(G)$ orbit.

$\square$

4.3 Autoprojectivities of Subgroup Lattices of Finite Abelian p-groups of type λ such that $\lambda_1 > \lambda_3$ and $\lambda_3 \neq 0$.

Let G be a finite abelian p-group of type $\lambda = (\lambda_1, ..., \lambda_n)$, where $\lambda_1 > \lambda_3$ and $\lambda_3 \neq 0$.

First, we will consider finite abelian p-groups of type λ such that $\lambda_1 > \lambda_2 > ... > \lambda_n$. While the situation becomes more complicated in the case when $n \geq 3$ than it is when $n = 2$, we can gain some insight through the examination of the meet semi-lattice of cyclic subgroups

Example 4.4. Let $G = \mathbb{Z}_{3^4} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_3$. Figure 4.7 represents the meet semi-lattice of cyclic subgroups of G. Notice that this meet semi-lattice of cyclic subgroups looks quite different than the meet semi-lattice of $\mathbb{Z}_{3^4} \times \mathbb{Z}_{3^2}$ illustrated in Figure 4.5. Subgroups on the same level are of the same order. Cyclic subgroups at the top level are subgroups of order 3^4. The intersection of all cyclic subgroups of order 3^4 is a cyclic subgroup of order $3^{4-2} = 3^2$. We call this subgroup F and it is represented by the top-most diamond node. The thick black line and diamond nodes represent the chain of cyclic subgroups of F. Notice that every element of the chain of subgroups of F is stabilized by every autoprojectivity of G.

We can think about the meet semi-lattice on Figure 4.7 in terms of subtrees being attached at the identity subgroup in the following way: we group

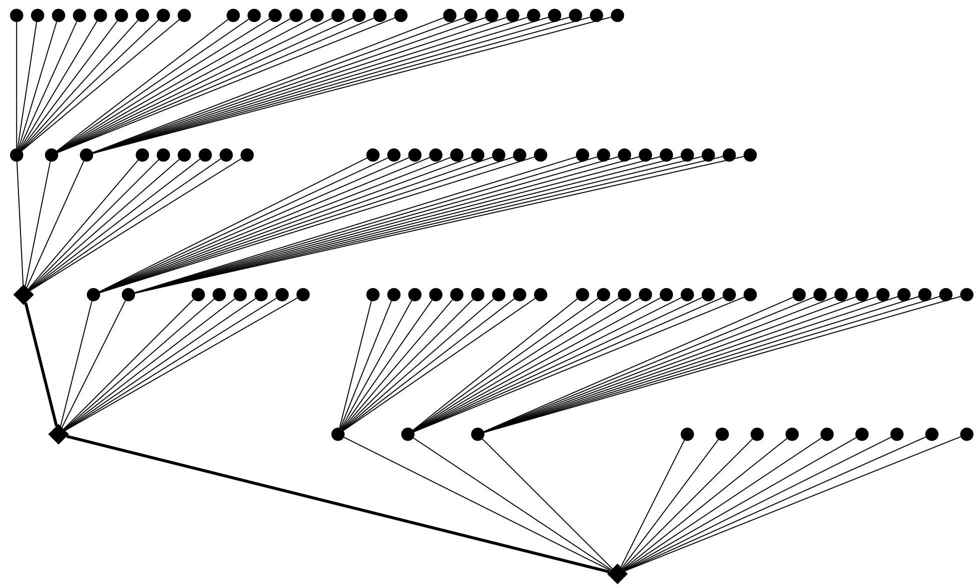

Figure 4.7: Meet semi-lattice of cyclic subgroups of $G = \mathbb{Z}_{3^4} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_3$.

cyclic subgroups into a subtree if their Hall types have non-zero entries in the same position. For example, all cyclic subgroups that have Hall type $(\mu, 0, 0)$, where $1 \leq \mu \leq 4$, belong to the left most subtree on Figure 4.7 which ends on the diamond node on the second from the bottom level. We label subtrees by the corresponding Hall types $(0, ..., 0, \mu, 0, ..., 0)$ with $1 \leq \mu \leq \lambda_i$ in the ith position. Note that the height of a subtree labeled $(0, ..., 0, \mu, 0, ..., 0)$ is equal to λ_i, where p^{λ_i} is the order of the largest cyclic subgroup contained in that subtree. The height of the subtree $(\mu, 0, 0)$ in the example above is 4. Subtree $(0, \mu, 0)$ has height 2 and consists of three points on the second from the bottom level with nine subgroups attached to each point. Notice that G on Figure 4.7 has three distinct subtrees attached at the identity with corresponding heights 4, 2, and 1.

One of the main difference between the meet semi-lattice of a finite abelian p-group containing three components in the direct product and two components in the direct product is that a subtree of subgroups that lies above a subgroup may not be symmetric. As we can see in Figure 4.7 F is contained in three subgroups that have subtrees of cyclic subgroups above them and six subgroups that are not contained in any cyclic subgroup of higher order. Notice that $F = \langle (p^2, 0, 0) \rangle$ and three subgroups with subtrees above them are $\langle (p, ap, 0) \rangle$ with $a = 0, 1, 2$ and six subgroups that contain F have the the form $\langle (p, b, c) \rangle$, where $b \in (\mathbb{Z}_{p^2})^*$ and $c \in \mathbb{Z}_p$. Notice that if a cyclic subgroup has Hall generators that are all powers of p and it is not the top level subgroup in its subtree that attaches at the identity subgroup, then there is a subtree of cyclic subgroups above it. Also, if such a subgroup is contained in a cyclic subgroups where one of the Hall generators is not a power of p, then the subtree of cyclic subgroups above it is asymmetric.

In general, if G is a finite abelian p-group of type λ such that $\lambda_1 > \lambda_2 > ... > \lambda_n$. We would like to describe the structure of $L(G)$. Let F be the intersection of all cyclic subgroups of order p^{λ_1}, which have the form $\langle (1, x_2, ..., x_n) \rangle$, where $x_i \in \mathbb{Z}_{p^{\lambda_i}}$ for $2 \leq i \leq n$. There are $p^{\lambda_2 + \cdots + \lambda_n}$ of cyclic subgroups of order p^{λ_1}. Then $F = \langle (p^{\lambda_2}, 0, ..., 0) \rangle$ and is a cyclic subgroup of order $p^{\lambda_1 - \lambda_2}$ and Hall type $(\lambda_1 - \lambda_2, 0, ..., 0)$. The chain of subgroups of F, including F, is fixed by every autoprojectivity of G. There are p^{n-1} cyclic subgroups of order $p^{\lambda_1 - \lambda_2 + 1}$ containing F. These subgroups have the form $\langle (p^{\lambda_1 - 1}, p^{\lambda_2 - 1} x_2, ..., p^{\lambda_n - 1} x_n) \rangle$, where $x_i \in \mathbb{Z}_{p^{\lambda_i}}$. Notice that if a cyclic subgroup of G has at least one of the components in its Hall generator that is not a multiple of p, then it is not contained in any cyclic subgroups. A cyclic subgroup H of G of Hall type $(0, ..., 0, \mu_i, 0, ..., 0)$ for $1 \leq i \leq n$ such that all components of its Hall generator are multiples of p is contained in p^{n-1} of cyclic subgroups.

Notice that if G is a finite abelian p-groups of type λ such that $\lambda_1 > \lambda_2 > ... > \lambda_n$, the meet semi-lattice of cyclic subgroups of G has n distinct subtrees that attach at the identity of respective heights λ_i as described in the example above. If H and K are subgroups of G that lie in the same orbit of $P(G)$, then cyclic subgroups of H have to be mapped to cyclic subgroups of K via an autoprojectivity. Since an autoprojectivity of G restricted to the meet semi-lattice of cyclic subgroups is an automorphism on the meet semi-lattice of cyclic subgroups. Clearly, an autoprojectivity maps cyclic subgroups of to cyclic subgroups of the same order. Notice that cyclic subgroups of the same order that belong to different cyclic subgroup subtrees that attach at the identity cannot be mapped to each other via an autoprojectivity because of the different height of subtrees they belong to.

Now we describe the dependency of subgroups of in the same orbit of $P(G)$ and their Hall types.

Theorem 4.6. *Let G be a finite abelian p-groups of type λ such that $\lambda_1 > \lambda_2 > ... > \lambda_n$. Suppose subgroups H and K of G are in the same orbit of $P(G)$. Then H and K have the same Hall type.*

Proof. Since H and K are in the same orbit of $P(G)$, they have the same isomorphism type $\mu = (\mu_1, ..., \mu_l)$. If all μ_i are equal, then the subgroup of this isomorphism type is unique. So, we may assume that not all of μ_i are equal. Let $\{h^1, ..., h^l\}$ and $\{k^1, ..., k^l\}$ be sets of Hall generators for H and K, respectively. Suppose H and K have different Hall types. Since the Hall type of a subgroup is a permutation of μ, the subgroup's isomorphism type, there exists $1 \leq i \leq l$ such that the ith position in the Hall type of H is the first position where Hall types of H and K differ. Without loss of generality the entry in the ith component of the Hall type of H is greater than the entry in the ith component of the Hall type of K. Suppose μ_j is in the ith component in the Hall type of H. Then there exists $i < t \leq l$ such that μ_j is the tth component in the Hall type of K. Note that $\mu_j \leq \lambda_i$ and $\mu_j \leq \lambda_t$.

By definition of a set of Hall generators, Hall generators h^j and k^j have order p^{μ_j} in H and K respectively. A cyclic subgroup generated by h^j, $\langle h^j \rangle$, has Hall type $(0, ..., 0, \mu_j, 0, ..., 0)$, where μ_j is in the ith position. A cyclic subgroup generated by k^j, $\langle k^j \rangle$, has Hall type $(0, ..., 0, \mu_j, 0, ..., 0)$, where μ_j is in the tth position. If H and K belong to the same orbit of $P(G)$, then there exists an autoprojectivity of G that maps the subgroup lattice of H onto the subgroup lattice of K. Therefore, $\langle h^j \rangle$ has to be mapped onto a cyclic subgroup of K of order p^{μ_j}. However, since $\lambda_i > \lambda_t$, $\langle h^j \rangle$ and $\langle k^j \rangle$ are contained in subtrees of different height in the meet semi-lattice of cyclic subgroups of G, $\langle h^j \rangle$ could not be mapped by an autoprojectivity of G to $\langle k^j \rangle$. But since H maps to K via an autoprojectivity, $\langle h^j \rangle$ has to map to a cyclic subgroup of K that is in the same subtree in the meet semi-lattice of cyclic subgroups as $\langle h^j \rangle$, which means that some cyclic subgroup of K has to be of Hall type $(0, ..., 0, \mu_j, 0, ..., 0)$, where μ_j is in the ith position. Suppose K contains a subgroup of Hall type $(0, ..., 0, \mu_j, 0, ..., 0)$ with μ_j is in the ith position. Then it contains a cyclic subgroup of the form $\langle (a_1 p^{\lambda_1 - \mu_j}, ..., a_{i-1} p^{\lambda_{i-1} - \mu_j}, p^{\lambda_i - \mu_j}, a_{i+1} p^{\lambda_{i+1} - \mu_j + 1}, ..., a_l p^{\lambda_l - \mu_j + 1}) \rangle$ which could be chosen to be one of Hall generators for K, which would change the Hall type of K. But this is a contradiction since the Hall type of a subgroup is well-defined (In other words, if K contains both

$$\langle g \rangle = \langle (a_1 p^{\lambda_1 - \mu_j}, ..., a_{i-1} p^{\lambda_{i-1} - \mu_j}, p^{\lambda_i - \mu_j}, a_{i+1} p^{\lambda_{i+1} - \mu_j + 1}, ..., a_l p^{\lambda_l - \mu_j + 1}) \rangle$$

and

$$\langle k^j \rangle = \langle (b_1 p^{\lambda_1 - \mu_j}, ..., b_{t-1} p^{\lambda_{t-1} - \mu_j}, p^{\lambda_t - \mu_j}, b_{t+1} p^{\lambda_{t+1} - \mu_j + 1}, ..., b_l p^{\lambda_l - \mu_j + 1}) \rangle,$$

where some of b_i could be equal to 0, then $\langle g \rangle \vee \langle k^j \rangle$ is a subgroup generated by k^j and

$$(1 - a_t b_i)^{-1}(-a_t k^j + g) =$$

$$(c_1 p^{\lambda_1 - \mu_j}, ..., c_{i-1} p^{\lambda_{i-1} - \mu_j}, p^{\lambda_i - \mu_j}, c_{i+1} p^{\lambda_{i+1} - \mu_j + 1}, ..., 0, ... c_l p^{\lambda_l - \mu_j + 1}),$$

where 0 is in the tth position. Therefore, this subgroup is isomorphic to the direct product of $\mathbb{Z}_{p^{\mu_j}}$ in the ith and tth positions and $\{1\}$ everywhere else, which cannot be contained in K since by assumption K is isomorphic to a direct product of $\mathbb{Z}_{p^{\mu_r}}$'s with $\mathbb{Z}_{p^{\mu_s}}$ for some $s < j$ in the ith component of the direct product.) Therefore, H and K cannot be in the same orbit of $P(G)$. $\square$

We explore application of Theorem 4.6 in an example.

Example 4.5. Suppose $G = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^s}$, where $m \geq n \geq s$. Let H and K be subgroups of G of isomorphism type $\mu = (\mu_1, \mu_2, \mu_3)$ such that $\mu_1 \neq \mu_2$, that is $\mu_1 > \mu_2$. Suppose H has Hall type (μ_1, μ_2, μ_3) and K has Hall type (μ_2, μ_1, μ_3). Then H has a Hall generator h^1 of order p^{μ_1} and $\langle h^1 \rangle$, the cyclic subgroup generated by h^1, has Hall type $(\mu_1, 0, 0)$. Also, K has a Hall generator k^1 of order p^{μ_1} and $\langle k^1 \rangle$ has Hall type $(0, \mu_1, 0)$. Since $\langle h^1 \rangle$ and $\langle k^1 \rangle$ belong to subtrees in the meet semi-lattice of cyclic subgroups of G that have different height, $\langle h^1 \rangle$ cannot be mapped to $\langle k^1 \rangle$ via an autoprojectivity of G. If H and K are in the same orbit of $P(G)$, then K should contain a cyclic subgroup that $\langle h^1 \rangle$ could be mapped to and that cyclic subgroup should be contained in the same subtree as $\langle h^1 \rangle$ and have Hall type $(\mu_1, 0, 0)$. But if K contains a cyclic subgroup of Hall type $(\mu_1, 0, 0)$ then it is of the form $(p^{m-\mu_1}, ap^{n-\mu_1+1}, bp^{s-\mu_1+1})$ could be written as a linear combination of $k^1 = (xp^{m-\mu_1}, p^{n-\mu_1}, yp^{s-\mu_1+1})$, $k^2 = (p^{m-\mu_2}, 0, zp^{s-\mu_2+1})$, and $k^3 = (0, 0, p^{s-\mu_3})$. The only way to get $ap^{n-\mu_1+1}$ is to multiply k^1 by ap, but then there exists c such that $p^{m-\mu_1} = axp^{m-\mu_1+1} + cp^{m-\mu_2} = p^{m-\mu_1+1}(ax + cp^{\mu_1-\mu_2-1})$ since $\mu_1 > \mu_2$. This is a contradiction. Therefore, H cannot be mapped to K by a lattice automorphism.

When G is of type λ such that $\lambda_i = \lambda_i + 1$ for some i, then it is clear that in the meet semi-lattice of cyclic subgroups of G has subtrees attached at the identity of the same height, that is subtrees of type $(0, ..., 0, \lambda_i, 0, ..., 0)$ and $(0, ..., 0, \lambda_{i+1}, 0, ..., 0)$ have the same height λ_i. Then it is possible for two subgroups of different Hall type to be in the same orbit of $P(G)$ and $PA(G)$. For instance, if $G = \mathbb{Z}_{3^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3$, then subgroups $H = \langle(0, 1, 0)\rangle$ and $K = \langle(0, 0, 1)\rangle$ have different Hall types $(0, 1, 0)$ and $(0, 0, 1)$ respectively, but the group automorphism of G represented by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ maps H to K. Thus, H and K belong to the same orbit of $PA(G)$ and the same orbit of $P(G)$. It is interesting to notice that if $\lambda_1 = \lambda_k$ for some $k \geq 2$, then there is no unique subgroup that is an intersection of all subgroups of order p^{λ_1} that is fixed by every automorphism. We called such a subgroup F above. Notice that there are k F's in such a G that could be permuted among themselves. From proof of Theorem 4.6 and discussion above we get the following Corollary.

Corollary 4.1. *Let G be a finite abelian p-group of type λ. Suppose subgroups H and K of G are in the same orbit of $P(G)$ and have isomorphism type μ. Then either H and K have the same Hall type or H and K have the same Hall type up to permuting $\mu_i = \mu_j$ for some i and j whenever $\lambda_i = \lambda_j$.*

Let G be a finite abelian p-group of exponent p^n such that $G = H \oplus C$, where $H = \langle a\rangle \oplus \langle b\rangle$ and $|a| = p^n \geq |b| = p^m \geq \exp C = p^s \neq 0$ such that either $|a| > |b|$ or $|a| = |b|$ and $s < n$.

First, we define some useful subgroups of $P(G)$. Let

$$R_s(G) = \{\rho \in P(G) \mid \Omega_s(G) = 1\},\ R(G) = \{\rho \in R_s(G) \mid H^\rho = H\}.$$

Since (a, b) is a basis of H, we call $\mathcal{A} = (\langle a \rangle, \langle b \rangle)$ the frame associated to (a, b) and $u = \langle p^{n-m}a + b \rangle$ a unit point. Also, we define

$$R_{\mathcal{A}}(G) = \{\rho \in R(G) \mid \mathcal{A}^\rho = \mathcal{A}\},\ \ R_{\mathcal{A},u}(G) = \{\rho \in R_{\mathcal{A}}(G) \mid u^\rho = u\}.$$

Let $\phi \in P(G)$. Then we define $\mathcal{A}_\varphi = (\langle a \rangle^\varphi, \langle b \rangle^\varphi)$ and $u_\varphi = u^\varphi$. Similarly to above, we define

$$R_{\mathcal{A}_\varphi}(G) = \{\rho \in R(G) \mid \mathcal{A}_\varphi^\rho = \mathcal{A}_\varphi\},\ \ R_{\mathcal{A}_\varphi,u_\varphi}(G) = \{\rho \in R_{\mathcal{A}_\varphi}(G) \mid u_\varphi^\rho = u_\varphi\}.$$

In 1998 Constantini, Holmes and Zacher proved the following theorem defining the structure of $P(G)$.

Theorem 4.7. *(Theorem 1.1 [3])* $P(G) = R_{\mathcal{A},u}(G)PA(G)$, *where* $R_{\mathcal{A},u}(G) \cap PA(G) = 1$.

In lattice theoretic terms a subgroup basis is a set of all subgroups of G such that the join of all basis elements is equal to G and and the meet of a basis element with the join of all other basis elements is the identity of G. Notice that $\mathcal{A}$ is a lattice theoretic basis for G. Since any $\varphi \in P(G)$ preserves the group structure of G, φ takes lattice basis elements of G to basis elements, thus $\mathcal{A}_\varphi$ is also a lattice theoretic basis for G. We would like to extend Theorem 4.7 to $R_{\mathcal{A}_\varphi,u_\varphi}(G)$ for any $\varphi \in P(G)$.

First, we need a Lemma.

Lemma 4.1. *Let $\varphi \in P(G)$. Then $\varphi^{-1}R_{\mathcal{A},u}\varphi = R_{\mathcal{A}_\varphi,u_\varphi}(G)$.*

Proof. Let $\rho \in R_{\mathcal{A},u}(G)$. Then applying $\varphi^{-1}\rho\varphi$ to $\langle a\rangle^\varphi$ we get $(\langle a\rangle^\varphi)^{\varphi^{-1}\rho\varphi} = \langle a\rangle^{\rho\varphi} = \langle a\rangle^\varphi$ since $\rho \in R_{\mathcal{A},u}(G)$ implies that $\rho \in R_{\mathcal{A}}(G)$ and thus $\langle a\rangle^\rho = \langle a\rangle$. Similarly, $(\langle b\rangle^\varphi)^{\varphi^{-1}\rho\varphi} = \langle b\rangle^\varphi$ and $(u^\varphi)^{\varphi^{-1}\rho\varphi} = u^\varphi$. Therefore, $\mathcal{A}_\varphi^{\varphi^{-1}\rho\varphi} = \mathcal{A}_\varphi$ and $u_\varphi^{\varphi^{-1}\rho\varphi} = u_\varphi$. So, $\varphi^{-1}\rho\varphi \in R_{\mathcal{A}_\varphi,u_\varphi}(G)$.

Now, let $\tau \in R_{\mathcal{A}_\varphi,u_\varphi}(G)$. Then applying $\varphi\tau\varphi^{-1}$ to $\langle a\rangle$ we get $\langle a\rangle^{\varphi\tau\varphi^{-1}} = (\langle a\rangle^\varphi)^{\tau\varphi^{-1}} = \langle a\rangle^{\varphi\varphi^{-1}} = \langle a\rangle$ since τ fixes $\langle a\rangle^\varphi$. Similarly, since τ fixes $\langle b\rangle^\varphi$ and u^φ, $\langle b\rangle^{\varphi\tau\varphi^{-1}} = \langle b\rangle$ and $u^{\varphi\tau\varphi^{-1}} = u$. So, $\varphi\tau\varphi^{-1} \in R_{\mathcal{A},u}(G)$. Thus, $\tau = \varphi^{-1}(\varphi\tau\varphi^{-1})\varphi \in \varphi^{-1}R_{\mathcal{A},u}\varphi$ and $\varphi^{-1}R_{\mathcal{A},u}\varphi = R_{\mathcal{A}_\varphi,u_\varphi}(G)$. □

We also need the fact from [3]: If $G = A + B$ an abelian p-group of finite exponent such that exp $B =$ exp $(A \cap B) = p^s$ and $p^{s-1}A$ is not cyclic, then for $\alpha, \beta \in PA(G)$ $\alpha = \beta$ if and only if $\alpha|A = \beta|A$ and $\alpha|B = \beta|B$.

The proof of the following Corollary will closely follow the proof of Theorem 4.7 in [3]. As a reminder $\Omega_s(G) = \langle\{g \in G \mid g^{p^s} = 1\}\rangle$.

Corollary 4.2. *Let $\varphi \in P(G)$. Then $P(G) = \varphi^{-1}R_{\mathcal{A},u}(G)\varphi PA(G)$, where $\varphi^{-1}R_{\mathcal{A},u}(G)\varphi \cap PA(G) = 1$.*

Proof. Let (c_i) be a basis of C and let $\psi \in P(G)$. Since ψ preserves the group structure of G, there exists $\alpha \in PA(G)$ such that for $\tau = \psi\alpha$ we have $\mathcal{A}_\varphi^\tau = \mathcal{A}_\varphi$, $u_\varphi^\tau = u_\varphi$, and $(\langle c_i\rangle^\varphi)^\tau = \langle c_i\rangle^\varphi$. Now, consider $\bar{\tau} = \tau|\Omega_s(G)$. Since $\Omega_s(G)$ has at least 3 elements of order p^s, by Baer's Theorem ([8]) we have that $\bar{\tau}$ is induced by a group automorphism on $\Omega_s(G)$, that is $\bar{\tau} \in PA(\Omega_s(G))$. Since $C \subset \Omega_s(G)$, $\bar{\tau}$ is induced by a group automorphism of the form $1 \oplus \gamma$, where 1 is the identity map on H and $\gamma \in \mathrm{Aut}(C)$. Let $\beta = 1 \oplus \gamma^{-1}$. Clearly, β is a group automorphism of G. Consider $\tau\beta$. Then by definition $\tau\beta \in R_{\mathcal{A}_\varphi, u_\varphi}(G) = \varphi^{-1}R_{\mathcal{A},u}(G)\varphi$ by Lemma 4.1. Then since $\tau\beta = \psi\alpha\beta$ and $\alpha\beta \in PA(G)$, we have $\psi \in \varphi^{-1}R_{\mathcal{A},u}(G)\varphi PA(G)$. Now, let $\rho \in \varphi^{-1}R_{\mathcal{A},u}(G)\varphi \cap PA(G)$. Since by Lemma $\varphi^{-1}R_{\mathcal{A},u}\varphi = R_{\mathcal{A}_\varphi,u_\varphi}(G)$, we have $\rho|H = 1$ and $\rho|\Omega_s(G) = 1$ Thus, by the fact mentioned above $\rho = 1$. $\square$

Now we would like to discuss orbits of $P(G)$ and $PA(G)$ in

Theorem 4.8. *Let G be a finite abelian p-group of type $\lambda = (\lambda_1, ..., \lambda_n)$ such that $\lambda_2 \leq 2$ and $\lambda_3 = 1$. Then orbits of $PA(G)$ are equal to the orbits of $P(G)$.*

Proof. Suppose subgroups H and K of G are in the same orbit of $P(G)$. By Theorem 4.7 we have that $P(G) = R_{\mathcal{A},u}(G)PA(G)$. Then there exists $\varphi = \tau\alpha \in P(G)$ with $\tau \in R_{\mathcal{A},u}(G)$ and $\alpha \in PA(G)$ such that $\varphi(H) = K$. Then H and $\alpha(H)$ belong to the same orbit of $PA(G)$ and $\alpha(H)$ and K are in the same orbit of $R_{\mathcal{A},u}(G)$ and thus in the same orbit of $P(G)$. Thus, without loss of generality we may assume that H and K are in the same orbit of $R_{\mathcal{A},u}(G)$. By Theorem 4.6 H and K have the same Hall type.

Case 1: Suppose $\lambda_2 = 2$ and H has Hall type $(\mu_1, ..., \mu_k)$ such that $\mu_1 > \lambda_2$,

$\mu_2 = 2$, and $k \geq 3$. Notice that $\mu_3 = ... = \mu_k = 1$. Then Hall generators for H $h^1 = (p^{\lambda_1-\mu_1}, a_1^1 p^{\lambda_2-\mu_1+1}, ..., a_{n-1}^1 p^{\lambda_n-\mu_1+1}) = (p^{\lambda_1-\mu_1}, a_1^1, ..., a_{n-1}^1)$ since $\mu_1 > \lambda_2$, $h^2 = (0, p^{\lambda_2-\mu_2}, a_1^2 p^{\lambda_3-\mu_2+1}, ..., a_{n-2}^2 p^{\lambda_n-\mu_2+1}) = (0, 1, a_1^2, ..., a_{n-2}^2)$ and $h^i = (0, 0, ..., a_j^i, 0, 1, 0, ..., 0)$, where $3 \leq i \leq k$ and 1 is in the same position as μ_i for $3 \leq i \leq k$ in the Hall type of H and $a_j^i \in \mathbb{Z}_p$, where $3 \leq j \leq k-1$ and a_j^i is to the left of 1. Also, $a_2^1 = 0$ and $a_j^t = 0$ for $t = 1, 2$ in the same position as 1 in h^i for $3 \leq i \leq k$. By Corollary 4.1, K either has the same Hall type as H or K has Hall type μ such that μ_1 and μ_2 are in the first and second positions respectively and μ_3 through μ_k are permuted among positions 3 through n.

A group automorphism of G looks like a matrix $A = (a_{ij})$ such that $a_{ii} \in (\mathbb{Z}_{p^{\lambda_i}})^*$ for $i = 1, 2$, $a_{ii} \in \mathbb{Z}_p$ for $i \geq 3$, $a_{ij} \in \mathbb{Z}_{p^{\lambda_j}}$ for $i < j$, $a_{ij} = b_{ij} p^{\lambda_j - \lambda_i} \in \mathbb{Z}_{p^{\lambda_j}}$ for $i > j$, and the $n-2 \times n-2$ matrix (a_{ij}) for $3 \leq i, j \leq n$ has determinant not equal to a multiple of p. Then there are enough free variable in the matrix A to map H to K via a group automorphism.

(For instance, if $G = \mathbb{Z}_{p^{\lambda_1}} \times \mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p$ and H has Hall type $(\mu_1, 2, 0, 1)$ with $\mu_1 > 2$, then K could have the same Hall type as H or $(\mu_1, 2, 1, 0)$. Then $h^1 = (p^{\lambda_1-\mu_1}, 0, a_2^1, 0)$, $h^2 = (0, 1, a_1^2, 0)$, and $h^3 = (0, 0, a_1^3, 1)$. Then either $k^1 = (p^{\lambda_1-\mu_1}, 0, b_2^1, 0)$, $k^2 = (0, 1, b_1^2, 0)$, and $k^3 = (0, 0, b_1^3, 1)$ or $k^1 = (p^{\lambda_1-\mu_1}, 0, 0, b_3^1)$, $k^2 = (0, 1, 0, b_2^2)$, and $k^3 = (0, 0, 1, 0)$. Then the group automorphism $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a_{23} & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & a_{43} & 1 \end{pmatrix}$, where $a_2^1 a_{33} = b_2^1$, $a_{23} + a_1^2 a_{33} = b_1^2$, and $a_1^3 a_{33} + a_{43} = b_1^3$, maps H to K if K has the same Hall type as H. Moreover, the group automorphism $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 1 & a_{44} \end{pmatrix}$, where $a_2^1 a_{34} = b_3^1$, $a_{24} + a_1^2 a_{34} = b_2^2$, and $a_1^3 a_{34} + a_{44} = 0$, maps H to K if K has Hall type $(\mu_1, 2, 1)$).

Suppose $\lambda_2 = 1$ and H has Hall type $\mu = (\mu_1, ..., \mu_k)$ such that $\mu_1 > 1$ and $k \geq 2$. By Corollary 4.1 K either has the same Hall type as H or a permutation of μ_2 through μ_k in positions 2 through n. Then $h^1 = (p^{\lambda_1 - \mu_1}, a_1^1, ..., a_{n-1}^1)$ and $h^i = (0, 0, ..., a_j^i, 0, 1, 0, ..., 0)$, where $2 \leq i \leq k$ and 1 is in the same position as μ_i for $2 \leq i \leq k$ in the Hall type of H and $a_j^i \in \mathbb{Z}_p$, where $2 \leq j \leq k-1$ and a_j^i is to the left of 1. Also, $a_j^1 = 0$ in the same position as 1 in h^i for $2 \leq i \leq k$. We can define the Hall generators k^i similarly to above except maybe for a permutation of the position of 1's in k^j for $2 \leq j \leq k$.

A group automorphism of G looks like a matrix $A = (a_{ij})$ such that $a_{11} \in (\mathbb{Z}_{p^{\lambda_1}})^*$, $a_{ii} \in \mathbb{Z}_p$ for $i \geq 2$, $a_{ij} \in \mathbb{Z}_{p^{\lambda_j}}$ for $i < j$, $a_{ij} = b_{ij} p^{\lambda_j - \lambda_i} \in \mathbb{Z}_{p^{\lambda_j}}$ for $i > j$, and the $n-3 \times n-3$ matrix (a_{ij}) for $2 \leq i, j \leq n$

has determinant not equal to a multiple of p. Counting the number of a_l^t that are not equal to 0 and considering the entries of A we can see that there are enough variables in the group automorphism matrix A so that H maps to K.

Case 2: Suppose $\mu_1 = 2$, $\mu_2 = 1$, and H has Hall type $\mu = (\mu_1, ..., \mu_k)$. Then the Hall generators of H $h^1 = (p^{\lambda_1-\mu_1}, a_1^1 p^{\lambda_2-\mu_1+1}, ..., a_{n-1}^1 p^{\lambda_n-\mu_1+1}) = (p^{\lambda_1-2}, a_1^1 p^{\lambda_2-1}, a_2^1, ..., a_{n-1}^1)$, $h^2 = (0, p^{\lambda_2-\mu_2}, 0, ..., 0)$, and $h^i = (0, ..., 0, ..., a_j^i, ..., 1, 0, ..., 0)$, where $3 \leq i \leq k$ and 1 is in the same position as μ_i for $2 \leq i \leq k$ in the Hall type of H and $a_j^i \in \mathbb{Z}_p$, where $3 \leq j \leq k-1$ and a_j^i is to the left of 1. Also, $a_j^1 = 0$ in the same position as 1 in h^i for $2 \leq i \leq k$. Hall generators of K can be described similarly to the case above. Also, similarly to the case above we have enough free variables in A to map H to K.

Suppose $\mu_1 = 2$, $\mu_2 = 2$, and Hall type of H is $\mu = (\mu_1, \mu_2, ..., \mu_k)$. Then $h^1 = (0, 1, a_1^1, ..., a_{n-2}^1)$, $h^2 = (p^{\lambda_1-2}, 0, a_1^2, ..., a_{n-2}^2)$, and $h^i = (0, ..., 0, ..., a_j^i, ..., 1, 0, ..., 0)$ as above. Also, similarly to the case above we have enough free variables in A to map H to K.

Case 3: Suppose the Hall type of H is $\mu = (\mu_2, \mu_1, ..., \mu_k)$ such that $\mu_1 = 2$ and $\mu_2 = 1$ or 0. Then $h^1 = (a_1^1 p^{\lambda_1-2}, 1, a_2^1, ..., a_{n-1}^1)$, $h^i = (0, ..., 0, ..., a_j^i, ..., 1, 0, ..., 0)$ such that $2 \leq i \leq k$ and 1 is in the same position as μ_i for $2 \leq i \leq k$ in the Hall type of H and $a_j^i \in \mathbb{Z}_p$, where $2 \leq j \leq k-1$ and a_j^i is to the left of 1.

Suppose the Hall type of H is μ such that $\mu_i = 1$ for all $1 \leq i \leq k$. Then $h^i = (0, ..., 0, ..., a^i_j, ..., 1, 0, ..., 0)$ such that $1 \leq i \leq k$ and 1 is in the same position as μ_i for $2 \leq i \leq k$ in the Hall type of H and $a^i_j \in \mathbb{Z}_p$, where $1 \leq j \leq k-1$ and a^i_j is to the left of 1.

For both options in Case 3 there are enough free variables in A that map H to K.

Thus, H and K are contained in the same orbit of $PA(G)$.

□

Theorem 4.6 is a generalization of Proposition 4.1, however, we conjecture that conclusions of Theorem 4.4 do not necessarily extend to finite abelian p-groups of type λ such that $\lambda_3 \geq 1$ as we will see in the discussion that follows.

Consider $G = \mathbb{Z}_{p^7} \times \mathbb{Z}_{p^5} \times \mathbb{Z}_{p^2}$, where $p \geq 3$. Let $H = \langle (p^2, p, 2), (0, p^2, 1), (0, 0, p) \rangle$ and $K = \langle (p^2, p, 1), (0, p^2, 2), (0, 0, p) \rangle$, represented by sets of Hall generators. Then H and K have order p^9 and Hall type $(5, 3, 1)$. If two subgroups of G belong to the same orbit of $P(G)$, then by Theorem 4.6 they have the same Hall type. Moreover, in order to for an autoprojectivity of G to map one subgroup of G to another, subgroups lattices of both subgroups have to be of the same shape as well as shapes of subgroup trees above these subgroups. We would like to show that H and K belong to the same orbit of $P(G)$. A subgroup of Hall type $(5, 3, 1)$ could be contained in groups of order p^{10} of Hall types $(6, 3, 1)$, $(5, 4, 1)$, and $(5, 3, 2)$.

1. Subgroups of Hall type $(6,3,1)$ have Hall generators of the form

   $g^1 = (p^{7-6}, a_1p^{5-6+1}, a_2p^{2-6+1}) = (p, a_1, a_2)$,

   $g^2 = (0, p^{5-3}, a_3p^{2-3+1}) = (0, p^2, a_3)$,

   $g^3 = (0, 0, p^{2-1}) = (0, 0, p)$, where $a_1 < p^2$, $a_2, a_3 < p$.

2. Subgroups of Hall type $(5,4,1)$ have Hall generators of the form

   $g^1 = (p^{7-5}, b_1p^{5-5+1}, b_2p^{2-5+1}) = (p^2, b_1p, b_2)$,

   $g^2 = (0, p^{5-4}, b_3p^{2-4+1}) = (0, p, b_3)$,

   $g^3 = (0, 0, p^{2-1}) = (0, 0, p)$, where $b_1p < p$, $b_2, b_3 < p$.

   Thus $g^1 = (p^2, 0, b_2)$, $g^2 = (0, p, b_3)$, $g^3 = (0, 0, p)$.

3. Subgroups of Hall type $(5,3,2)$ have Hall generators of the form

   $g^1 = (p^{7-5}, c_1p^{5-5+1}, c_2p^{2-5+1}) = (p^2, c_1p, c_2)$,

   $g^2 = (0, p^{5-3}, c_3p^{2-3+1}) = (0, p^2, c_3)$,

   $g^3 = (0, 0, p^{2-2}) = (0, 0, 1)$, where $c_1 < p$, $c_2, c_3 < 1$.

   Thus, $g^1 = (p^2, c_1p, 0)$, $g^2 = (0, p^2, 0)$, $g^3 = (0, 0, 1)$.

By inspecting Hall generators of subgroups of Hall type $(6,3,1)$, we see that $pg^1 + ag^2 \neq (p^2, p, 2)$ or $(p^2, p, 1)$ for any $a \in \mathbb{Z}$. Thus, subgroups of Hall type $(6,3,1)$ cannot contain either H or K. Also similarly, subgroups of Hall type $(5,4,1)$ cannot contain either H or K since they cannot contain cyclic subgroups $\langle 0, p^2, 1\rangle$ or $\langle 0, p^2, 2\rangle$ (since $pg^2 + bg^3 \neq (0, p^2, 1)$ or $(0, p^2, 2)$).

Both H and K are contained in exactly one subgroup of order p^{10} of Hall type $(5,3,2)$ with Hall generators $h^1 = (p^2, p, 0)$, $h^2 = (0, p^2, 0)$, and $h^3 = (0, 0, 1)$.

Notice that cyclic subgroups of G generated by Hall generators of H, $\langle h^i \rangle$, could be mapped to cyclic subgroups generated by Hall generators of K of the same order $\langle k^i \rangle$ by a lattice automorphism since for instance h^1 and k^1 are of the same Hall type, attach at the same point to the chain of subgroups of F and don't have subgroup trees above them. Since subgroups H and K and their subgroup trees are contained in the same types of subgroups, pending a computation in GAP we conjecture that H and K belong to the same orbit of $P(G)$.

Suppose H and K are in the same orbit of $PA(G)$. Then there exists a group automorphism φ, represented by a matrix (a_{ij}), such that $\varphi(H) = K$.

$$\begin{pmatrix} p^2 & p & 2 \\ 0 & p^2 & 1 \\ 0 & 0 & p \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}p^2 & a_{22} & a_{23} \\ a_{31}p^5 & a_{32}p^3 & a_{33} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}p^2 + a_{21}p^3 + 2a_{31}p^5 & a_{12}p^2 + a_{22}p + 2a_{32}p^3 & a_{23}p + 2a_{33} \\ a_{21}p^4 + a_{31}p^5 & a_{22}p^2 + a_{32}p^3 & a_{33} \\ a_{31}p^6 & a_{32}p^4 & a_{33}p \end{pmatrix} =$$

$$\begin{pmatrix} p^2(a_{11} + a_{21}p + 2a_{31}p^3) & p(a_{12}p + a_{22} + 2a_{32}p^2) & a_{23}p + 2a_{33} \\ p^4(a_{21} + a_{31}p) & p^2(a_{22} + a_{32}p) & a_{33} \\ a_{31}p^6 & a_{32}p^4 & a_{33}p \end{pmatrix} =$$

$$\begin{pmatrix} p^2 & p & 1 \\ 0 & p^2 & 2 \\ 0 & 0 & p \end{pmatrix}$$

where $a_{ii} \in (\mathbb{Z}_{p^{\lambda_i}})^*$ for $i = 1, 2, 3$ and $a_{ij} \in \mathbb{Z}_{p^{\lambda_j}}$ for $i < j$.

Notice that $a_{11} + a_{21}p + 2a_{31}p^3 \in (\mathbb{Z}_{p^7})^*$ and $a_{22} + a_{32}p \in (\mathbb{Z}_{p^5})^*$ since a_{11} and a_{22} are units. We start row reduce the resulting matrix. By multiplying the first row by $(a_{11}+a_{21}p+2a_{31}p^3)^{-1}$ and equating it with the matrix entries for k^1, we see that $a_{12}p + a_{22} + 2a_{32}p^2 = a_{11} + a_{21}p + 2a_{31}p^3 \Rightarrow a_{11} = a_{22}$. Also, we have $a_{23}p + 2a_{33} = a_{11} + a_{21}p + 2a_{31}p^3 = a_{11} + a_{21}p \in \mathbb{Z}_{p^2}$. Thus, $2a_{33} = a_{11}$ and $a_{23} = a_{21}$. Multiplying the first row by $-(a_{11} + a_{21}p + 2a_{31}p^3)^{-1}(a_{21} + a_{31}p)p^2$ and adding it to the second row we have $a_{33} - (a_{11} + a_{21}p + 2a_{31}p^3)^{-1}(a_{21} + a_{31}p)p^2(a_{23}p + 2a_{33}) = a_{33} \in \mathbb{Z}_{p^2}$. Then $a_{33} = 2$. Also, $p^2(a_{22} + a_{32}p - p(a_{11} + a_{21}p + 2a_{31}p^3)^{-1}(a_{21} + a_{31}p)(a_{12}p + a_{22} + 2a_{32}p^2) = p^2$ $\Rightarrow a_{22} = 1$. Since $2a_{33} = a_{11} = a_{22}$, $a_{33} = 2^{-1}$. But above we determined that $a_{33} = 2$, which is a contradiction. Therefore, there is no group automorphism between H and K.

Chapter 5

Future Work

Many future projects related to the questions discussed in the chapters above remain to be completed. In Theorem 4.5 we have shown that for $G = \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ such that $m \geq n$ the orbits of $P(G)$ and $PA(G)$ in the subgroup lattice of G are equal. We would like to analyze the relationship between Sylow p-subgroup of $P(G)$ and $\mathrm{Aut}(G)$, which we hope would shed light on the proof of our conjecture that the quotient of the lattice of subgroups of G under the action of a Sylow p-subgroup of $\mathrm{Aut}(G)$. It seems less likely that for a finite abelian p-group of type $\lambda = (\lambda_1, ..., \lambda_n)$ such that $n \geq 3$ and $\lambda_3 \geq 2$ $\overline{L_\lambda(p)}$ is a self-dual lattice. We would like to further explore combinatorial and enumerative properties of $\overline{L_\lambda(p)}$. Another goal is to understand whether the action of S^p on subgroup lattice of finite non-abelian p-groups would be useful in classifying certain non-abelian p-groups in combinatorial terms.

Bibliography

[1] L. M. Butler, *Subgroup lattices and symmetric functions*, Mem. Amer. Math. Soc. **112** (1994), no. 539.

[2] ———, *Order analogues and Betti polynomials*, Adv. Math. **121** (1996), no. 1, 62 – 79.

[3] M. Constantini, C.S. Holmes, and G. Zacher, *A representation theorem for the group of autoprojectivities of an abelian p-group of finite exponent*, Ann. Mat. Pura Appl. **175** (1998), no. 4, 119 – 140.

[4] M. Golasiński and D. L. Gonçlaves, *On automorphisms of finite abelian p-groups*, Math. Slovaca **58** (2008), no. 4, 405 – 412.

[5] C. J. Hillar and D. L. Rhea, *Automorphisms of finite abelian groups*, Amer. Math. Monthly **114** (2007), no. 10, 917 – 923.

[6] C. Holmes, *Automorphisms of the lattice of subgroups of* $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, Arch. Math. **51** (1988), 491 – 495.

[7] S. Lang, *Algebra*, 3 ed., Addison-Wesley Publishing Company, New York, 1993.

[8] R. Schmidt, *Subgroup lattices of groups*, Walter de Gruyter, Berlin, 1994.