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The Discrete Orthonormal Wavelet Transform: An Introduction

Michael Frazier and Arun Kumar

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1. Introduction

In this paper we discuss the wavelet transform in a simple but rigorous manner. Two underlying themes are developed simultaneously. The first is that of signal recovery from a sequence of inner products. The second theme is operational, and concerns the design of filters needed to implement the wavelet transform in hardware or software. Necessary and sufficient conditions for orthonormal decomposition and perfect reconstruction are established. Algorithms for the design of wavelet filters are presented together with illustrative examples. A rudimentary knowledge of z–transforms, vector spaces, and inner–products, is assumed. The exposition is otherwise complete and self–contained.

The theory of wavelets [1]–[7] is related to older ideas in the field of multirate and subband signal processing [8]–[11]. Histories of the development of wavelet theory can be found in [3] and [7]. Our own interest in wavelets came about through a study of the Phi–transform [12]–[18], which is also related to multirate signal processing. Almost all the results derived here are known from subband–filter and wavelet literature, but are presented here in a simple unified manner that underscores the close correspondence between the engineering and the mathematical points of view. Our purpose is to give an elementary exposition of discrete wavelet theory that is accessible to a general audience, and which will provide sufficient background and reference information for non–specialists who would like to implement and use wavelet analysis in their work.

In Section 2 we discuss the construction of "first generation" wavelet bases for discrete finite and infinite–dimensional vector spaces. The first generation basis consists of even translates of two functions f_L and f_H, usually thought of as low and high–pass filters, respectively. The fundamental result here is Theorem 1 which gives a necessary and sufficient condition for the construction of such an orthonormal basis. The decomposition and reconstruction of an input signal in terms of this wavelet basis is best implemented through a one–stage filter–bank structure. For such a structure there is a simple condition relating the "synthesizing filters" g_L and g_H to the "analyzing filters" f_L and f_H, which is necessary and sufficient for the perfect reconstruction of every input signal. This result is also derived in Section 2.

We discuss the practical design of first generation wavelet bases in Section 3. An explicit algorithm is given for obtaining all possible f_L and f_H satisfying the condition of Theorem 1. We give some examples of such wavelet bases, in particular the "Shannon" and the "real Shannon" bases, which are highly localized in the frequency–domain. We also discuss the problem of finding the nearest (in the l^2–sense) first generation wavelet to a given vector. This problem has a simple solution based on our wavelet–construction algorithm.

In Section 4 we discuss the construction of higher–generation wavelet bases. The basic result is Theorem 4 which describes how to pass from a wavelet basis at one generation to the next. In the finite case this requires the construction of new filters; but these can be obtained from the first–generation filters by a simple process described in Lemma 7.

In Section 5 we state and prove the results necessary to extend the development to mutidimensional signals lying in discrete finite and infinite–dimensional vector spaces. Since the main ideas are laid out in Sections 2 to 4, Section 5 is brief. The purpose is merely to provide a complete reference; the new difficulties are mostly notational.

We use the z–transform as a basic tool in our discussion. Among other things, it provides a convenient notation, allowing us to deal with the finite and infinite–dimensional cases simultaneously.

1.1. Notation

By $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Z}$, and $\mathbf{Z}^+$, are meant the set of complex numbers, the set of real numbers, the set of integers (positive, negative, and zero), and the set of positive integers. If $a, b \in \mathbf{Z}$ are integers, then $a | b$ means that a divides b. $\mathbf{Z}_N$ denotes the ring of integers $\{0, \ldots, N-1\}$. For any discrete set U, $l^1(U)$ denotes the vector space of all absolutely summable sequences defined from U to $\mathbf{C}$, and $l^2(U)$ the vector space of all absolutely square summable sequences defined from U to $\mathbf{C}$. If f is a function from a discrete set U to $\mathbf{C}$, then the *support* of f is defined as that subset of U where f does not vanish: $\text{supp}(f) = \{u \in U : f(u) \neq 0\}$.

If A is a set of vectors in some vector space V, then $\text{span}(A)$ is the subspace of V consisting of all finite linear combinations of the elements of A. For any set $A \subseteq V$ the topological closure of A in V will be written $\text{cl}(A)$.

If α is a string over an alphabet A we will write $\alpha \in A^*$ where "$*$", in this context, denotes the Kleene operator. If α and β are strings, then $\alpha\beta$ is their concatention. We write $\alpha = \alpha(1)\alpha(2)\ldots\alpha(n)$ where $\alpha(i)$ is the i-th character in α. By $|\alpha| \in \mathbf{Z}^+ \cup \{0\}$ will be meant the length of the string α. The string α is the *null string* if $|\alpha| = 0$. If $\alpha, \beta, \gamma \in A^*$, and $\alpha = \beta\gamma$, then β is called a *prefix* of α. By $\beta = \mathrm{pfx}(\alpha, i)$, $0 \le i \le |\alpha|$, we will mean that β is a prefix of α and $|\beta| = i$.

2. Two–Band Orthonormal Decomposition And Perfect Reconstruction Of Discrete One–Dimensional Signals

In this section we describe how an arbitrary signal in the discrete infinite–dimensional vector space $l^2(\mathbf{Z})$ or the discrete N–dimensional vector space $\mathbf{C}^N$ can be written as a weighted sum of certain "elementary" synthesizing functions. The expression of a signal x as a weighted sum of certain synthesizing functions is called a *decomposition* of x. In order to set up the problem consider the arrangement of filters, downsamplers, and upsamplers in Figure 1. By "$\downarrow 2$" is meant *downsampling*, or the deletion of every other number from the input sequence. By "$\uparrow 2$" is meant *upsampling*, or the insertion of a zero between every pair of numbers in the input sequence. The symbols $\mathcal{D}$ and $\mathcal{U}$ will also be used to denote the down and upsampling operators. The symbols f_L and f_H denote *analyzing filters*, while g_L and g_H are *synthesizing filters*. The subscripts "H" and "L" could be thought of as abbreviations for "highpass" and "lowpass", respectively; because it is usual, though not necessary, that f_H and g_H are highpass filters, and f_L and g_L lowpass filters.

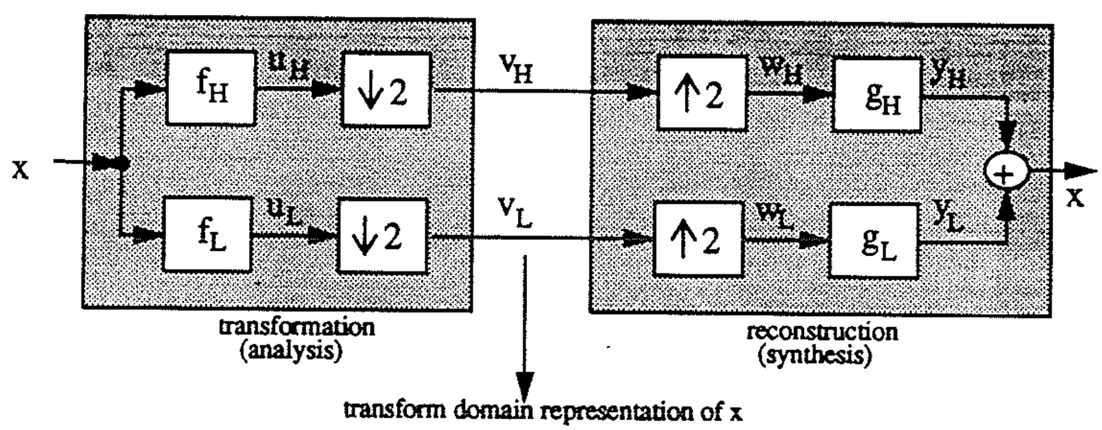

Figure 1: An arrangement of filters, downsamplers, and upsamplers.

If the signal x lies in infinite–dimensional vector space $V = l^2(\mathbf{Z})$, then we will assume that $f_L, f_H, g_L, g_H \in l^1(\mathbf{Z})$. It follows that for $x \in l^2(\mathbf{Z})$, the sequences u_L, v_L, w_L, y_L, u_H, v_H, w_H, and y_H, all belong to $l^2(\mathbf{Z})$. For x in the finite–dimensional signal (vector) space $V = \mathbf{C}^N = l^2(\mathbf{Z}_N)$, we will assume $f_L, f_H, g_L, g_H \in \mathbf{C}^N$. Then for $x \in \mathbf{C}^N$ the sequences u_L, w_L, y_L, u_H, w_H, and y_H, belong to $\mathbf{C}^N$; while the sequences v_L and v_H belong to $\mathbf{C}^{N/2}$.

The sequences v_H and v_L are said to define a discrete orthonormal wavelet transform of the signal x, if the analyzing–filter sequences f_H and f_L are chosen according to rules stated in Subsection 2.2. The synthesizing–filter sequences g_H and g_L will be chosen so as to attain a perfect reconstruction of x. Perfect reconstruction will be seen to define a decomposition of x.

The main results of this section are stated in two theorems that place constraints on the design of filter sequences f_H, f_L, g_H, and g_L. Another goal of this section is the unification of the engineering and the mathematical points of view. While the problem is so far stated as a filter design problem, it could also be studied as a pure vector space problem. We will see that the process of filtering is equivalent to the process of the computation of a series of inner–products, and there exists a close relation between the engineering and the mathematical points of view.

2.1. z–Transform Notation

For an infinite sequence $x = (\ldots, x(-1), x(0), x(1), \ldots) \in l^2(\mathbf{Z})$, $x(i) \in \mathbf{C}$ for all i, the z–transform $\hat{x}(z)$ of x is written:

$$\hat{x}(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}, \tag{1}$$

where the indeterminate z ranges over the unit circle $\mathbf{T}$ in the complex plane. In the standard definition of the z–transform, the indeterminate z ranges over the entire complex plane. Here we will use restrictions of the standard definition. Note that $x(n)$ is the n-th Fourier coefficient of the function that maps $\omega \in [-\pi, \pi]$ to $\hat{x}(e^{-j\omega})$; in particular then,

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(e^{-j\omega})\, e^{-jn\omega}\, d\omega. \tag{2}$$

From (2), if $\hat{x}(z) = \hat{y}(z)$ for all $z \in \mathbf{T}$, then $x = y$; hence the z–transform is one–to–one.

If x is a finite sequence $x = (x(0), \ldots, x(N-1)) \in \mathbf{C}^N$ for some $N \in \mathbf{Z}^+$, then the z–transform $\hat{x}(z)$ of x is written:

$$\hat{x}(z) = \sum_{n=0}^{N-1} x(n) z^{-n}. \tag{3}$$

The function $\hat{x}$ is defined for $z \in \mathbf{W}_N = \{e^{-j2\pi m/N} : m \in \mathbf{Z}_N\}$, the set of all the N-th roots of unity. The complex number $\hat{x}(e^{-j2\pi m/N})$, $m \in \mathbf{Z}_N$, is the m-th Discrete Fourier Transform (DFT) coefficient of x. By DFT inversion,

$$x(n) = \frac{1}{N} \sum_{m=0}^{N-1} \hat{x}(e^{-j2\pi m/N}) e^{-j2\pi mn/N}. \tag{4}$$

From (4), if $\hat{x}(z) = \hat{y}(z)$ for all $z \in \mathbf{W}_N$, then $x = y$.

If x lies in $l^2(\mathbf{Z})$, then by $(\mathcal{F}x)(\omega)$ we will mean the Fourier transform of x evaluated at $\omega \in [-\pi, \pi]$, $(\mathcal{F}x)(\omega) = \hat{x}(e^{-j\omega})$. If x lies in $\mathbf{C}^N$, then by $(\mathcal{F}x)(m)$ we will mean the DFT of x evaluated at $m \in \mathbf{Z}_N$; $(\mathcal{F}x)(m) = \hat{x}(e^{-j2\pi m/N})$.

When the range of the index of summation is not specified in some particular equation, it will mean that the equation holds equally for sums over $\mathbf{Z}$ and $\mathbf{Z}_N$. Similarly, when we make a statement about some property of $\hat{x}(z)$ that holds "for all z" or "$\forall z$" we will mean "$\forall z \in \mathbf{T}$" or "$\forall z \in \mathbf{W}_N$" depending on whether x belongs to the signal space $V = l^2(\mathbf{Z})$ or to $V = \mathbf{C}^N$.

In case of finite sequences x, since $z \in \mathbf{W}_N$, all arithmetic on the powers of the indeterminate z^{-1} in $\hat{x}(z)$ will be done in the ring $\mathbf{Z}_N$. Thus, if $N = 4$, then $az^{-2} + bz^{-5} + cz^{9} = az^{-2} + bz^{-1} + cz^{-3}$. When, for a finite sequence, its z–transform is written only in terms of z^{-1} raised to some number in $\mathbf{Z}_N$, then we will say that the z–transform is written in the *canonical form*. For infinite sequences the z–transform is always in the canonical form. If we assume that all z–transforms are always written in the canonical form then the mathematical development here is valid, simultaneously, for the finite and infinite–dimensional signal spaces $\mathbf{C}^N$ and $l^2(\mathbf{Z})$.

For $x, y \in l^2(\mathbf{Z})$ or $x, y \in \mathbf{C}^N$, their convolution $(x * y)$ is defined by

$$(x * y)(n) = \sum_m x(m) y(n - m). \tag{5}$$

If $x, y \in l^2(\mathbf{Z})$, then n and m range over $\mathbf{Z}$. If $x, y \in \mathbf{C}^N$, them n and m range over $\mathbf{Z}_N$. In this latter case, in (5), the index $(n - m)$ into the sequence y may *appear* to go out of the domain $\mathbf{Z}_N$ of y. However, when dealing with finite convolutions, we will do all arithmetic on sequence indices in the ring $\mathbf{Z}_N$. This is as in the case of the powers of z^{-1} in z–transforms. Then there is no problem with indices going out of range, and all finite convolutions are "circular" convolutions in the standard terminology of digital signal processing.

The z–transform of the convolution $(x * y)$ of x and y is the product of the z–transforms of x and y: $(x * y)\hat{}(z) = \hat{x}(z)\hat{y}(z)$. By $\mathrm{coeff}_k(\hat{x}(z))$ we will mean the coefficient of z^{-k} in the canonical form $\hat{x}(z)$. By (2) or (4), this coefficient is well defined. Define the *inner product* of $f, h \in l^2(\mathbf{Z})$ or $f, h \in \mathbf{C}^N$ by,

$$\langle f, h\rangle = \sum_n f(n)\overline{h}(n)\,; \tag{6}$$

where the index n runs over $\mathbf{Z}$ or $\mathbf{Z}_N$ depending upon the signal space under consideration. This inner product can be written in the z–transform notation.

Lemma 1. *If f and h are complex sequences and $z \in \mathbf{T}$ or $z \in \mathbf{W}_N$, then $\langle f, h\rangle = \mathrm{coeff}_0\left(\hat{f}(z)\overline{\hat{h}}(z)\right)$.*

Proof of Lemma 1.

$$\begin{aligned}\hat{f}(z)\overline{\hat{h}}(z) &= \left(\sum_n f(n)z^{-n}\right)\overline{\left(\sum_m h(m)z^{-m}\right)} &(7)\\ &= \sum_n\sum_m f(n)\overline{h}(m)z^{m-n}. &(8)\end{aligned}$$

In going from (7) to (8) we have used the fact that for $z \in \mathbf{T}$ or $z \in \mathbf{W}_N$, $\overline{z^{-1}} = z$. Equation 8 implies the lemma. □

2.2. Orthonormal Decomposition

Let R denote a *rightshift operator* that acts upon a sequence $x \in l^2(\mathbf{Z})$ or $x \in \mathbf{C}^N$ such that $(Rx)(n) = x(n-1)$, or equivalently,

$$(Rx)\hat{}(z) = z^{-1}\hat{x}(z). \tag{9}$$

In the finite–dimensional signal space $\mathbf{C}^N$, the right shift operator defined in (9) wraps sequences around; i.e. when x is shifted right once, then the number $x(N-1)$ moves to occupy the place where

$x(0)$ was. This is a consequence of using $\mathbf{Z}_N$ arithmetic upon indices for x. The z–transform notation reflects this wrap–around accurately when the canonical form is employed.

Let f be the impulse response of a filter with input x and output y. Define $\tilde{f}(n) = \overline{f}(-n)$. The sequence $\tilde{f}$ is the complex conjugate of the time–reversal of the sequence f. Then,

$$\begin{aligned} y(k) &= \textstyle\sum_n x(n)f(k-n) &&= \sum_n x(n)\,\overline{\tilde{f}}(n-k) = \sum_n x(n)\,\left(R^k\overline{\tilde{f}}\right)(n) && (10)\\ &= \textstyle\sum_n x(n)\,\overline{\left(R^k\tilde{f}\right)}(n) &&= \left\langle x, R^k\tilde{f}\right\rangle. && (11)\end{aligned}$$

Equation (11) follows from (10) because the rightshifted conjugate of the sequence $\tilde{f}$ is the same as the conjugate of the rightshifted sequence $\tilde{f}$. From (10) and (11) it is evident that the process of filtering a signal x with a filter f is *equivalent* to the computation of a sequence of inner products. One inner product is computed for each number $y(k)$ produced by the filter.

A *multiset* (or a *bag*) is a "set" which may contain multiple copies of one or more of its elements. In Figure 1, the sequence u_L is made up of the inner products of x against elements in the *multiset* $\{R^k\tilde{f}_L\}_{k\in\mathbf{Z}\text{ or }k\in\mathbf{Z}_N}$. The reason why $\{R^k\tilde{f}_L\}_k$ is declared to be a multiset, and not a set, is that it is possible that for some $k_1 \neq k_2$, $R^{k_1}\tilde{f}_L = R^{k_2}\tilde{f}_L$. The sequence v_L is obtained from u_L by discarding every other number, and therefore consists of the inner products of x against elements in the multiset

$$\tilde{B}_L = \{R^{2k}\tilde{f}_L\}_k, \tag{12}$$

where $k \in \mathbf{Z}$ or $k \in \mathbf{Z}_{N/2}$ depending upon the signal space.

Define also the multisets

$$\begin{aligned}\tilde{B}_H &= \{R^{2k}\tilde{f}_H\}_k && (13)\\ \tilde{B} &= \tilde{B}_L \cup \tilde{B}_H. && (14)\end{aligned}$$

The union in (14) is a multiset union that preserves the multiplicity of multiset elements.

In orthonormal wavelet analysis, $\tilde{B}$ is required to define an orthonormal basis of the signal space $V = l^2(\mathbf{Z})$ or $V = \mathbf{C}^N$. The multisets $\tilde{B}_L$ and $\tilde{B}_H$ are required to define orthonormal bases of the mutually orthogonal subspaces V_L and V_H of V. Then $\tilde{B}_L$, $\tilde{B}_H$, and $\tilde{B}$, must each be a *set*. We will see that the orthonormality of $\tilde{B}$ will lead to particularly simple conditions for the decomposition and perfect reconstruction of signals.

We next deduce the consequences of the orthonormality of $\tilde{B}$. Before that, however, we state a lemma that will result in somewhat simpler notation.

Lemma 2. *Define the multisets*

$$B_L = \{R^{2k} f_L\}_k \tag{15}$$

$$B_H = \{R^{2k} f_H\}_k \tag{16}$$

$$B = B_L \cup B_H. \tag{17}$$

Then $\tilde{B}_L$ is orthonormal if and only if B_L is; $\tilde{B}_H$ is orthonormal if and only if B_H is; and $\tilde{B}$ is orthonormal if and only if B is.

Proof of Lemma 2. For $f \in l^2(\mathbf{Z})$ or $f \in \mathbf{C}^N$,

$$(\tilde{f})\hat{}(z) = \sum_n \tilde{f}(n) z^{-n} = \sum_n \overline{f}(-n) z^{-n} = \sum_{m=-n} \overline{f}(m) z^m = \overline{\left(\sum_m f(m) z^{-m}\right)} = \overline{\hat{f}}(z). \tag{18}$$

By Lemma 1 and (18),

$$\langle R^{2k}\tilde{f}, R^{2l}\tilde{h}\rangle = \text{coeff}_0\left(z^{-2k}(\tilde{f})\hat{}(z) z^{2l}\overline{(\tilde{h})\hat{}(z)}\right) = \text{coeff}_0\left(z^{-2k}\overline{\hat{f}}(z) z^{2l}\hat{h}(z)\right) = \langle R^{-2l}h, R^{-2k}f\rangle. \tag{19}$$

If $\tilde{B}_L$ is orthonormal, then

$$\langle R^{2k}\tilde{f}_L, R^{2l}\tilde{f}_L\rangle = \delta(k-l); \tag{20}$$

where δ is the Kronecker delta:

$$\delta(i) \triangleq \begin{cases} 1 & , i = 0 \\ 0 & , i \neq 0. \end{cases} \tag{21}$$

Then from (19) and (20), $\langle R^{-2l} f_L, R^{-2k} f_L\rangle = \delta(k-l)$, and B_L is orthonormal. The lemma follows by similar arguments. □

It is easy to see that the multiset $B = \{R^{2k} f_L\}_k \cup \{R^{2k} f_H\}_k$ (and therefore $\tilde{B}$) is orthonormal if and only if the following equations hold $\forall k \in \mathbf{Z}$ or $\forall k \in \mathbf{Z}_{N/2}$:

$$\langle f_L, R^{2k} f_L \rangle = \delta(k) \tag{22}$$

$$\langle f_H, R^{2k} f_H \rangle = \delta(k) \tag{23}$$

$$\langle f_L, R^{2k} f_H \rangle \equiv 0. \tag{24}$$

Equations (22)–(24) will be called the *orthonormality conditions.*

The two lemmas that follow will be used in the proof of the main theorem governing orthonormal decompositions. A definition is necessary before the statement of the lemmas. A function $\hat{h}(z)$ is said to be *odd* in z if and only if $\hat{h}(z) = -\hat{h}(-z)$. We will call a function $\hat{g}(z)$ an *almost-odd* function of z if and only if $\hat{g}(z)$ is the sum of the constant function "1" with an odd function.

Lemma 3. *Let $f \in l^2(\mathbf{Z})$ or $\mathbf{C}^N$. The following are equivalent:*

P1: *For all $k \in \mathbf{Z}$ or $k \in \mathbf{Z}_{N/2}$, $\langle f, R^{2k} f \rangle = \delta(k)$.*

P2: *$\forall z$, $|\hat{f}(z)|^2$ is an almost-odd function of z.*

P3: *$\forall z$, $|\hat{f}(z)|^2 + |\hat{f}(-z)|^2 = 2$.*

Proof of Lemma 3. By Lemma 1,

$$\langle f, R^{2k} f \rangle = \operatorname{coeff}_0 \left(\hat{f}(z)\, \overline{(R^{2k} f)\hat{}\,(z)} \right) = \operatorname{coeff}_0 \left(\hat{f}(z)\, \overline{z^{-2k}}\, \overline{\hat{f}}(z) \right) \tag{25}$$

$$= \operatorname{coeff}_0 \left(z^{2k}\, |\hat{f}(z)|^2 \right) = \operatorname{coeff}_{2k} \left(|\hat{f}(z)|^2 \right). \tag{26}$$

Define

$$\hat{\alpha}(z) = \textstyle\sum_n \alpha(n) z^{-n} = |\hat{f}(z)|^2. \tag{27}$$

By (26), P1 is true if and only if,

$$\hat{\alpha}(z) = 1 + \sum_n \alpha(2n+1) z^{-(2n+1)}. \tag{28}$$

Equation (28) implies that $\hat{\alpha}$ is an *almost–odd* function of z. Hence P1 $\Rightarrow$ P2. Conversely, if P2 holds, then (2) and (4) and a symmetry argument show that $\alpha(2n) = 0$ for $n \neq 0$. Hence (26) shows that P2$\Rightarrow$P1.

By P2, $|\hat{f}(z)|^2 = 1 + \hat{\beta}(z)$, where $\hat{\beta}(z)$ is some function that is odd in z. By direct substitution, P3 is equivalent to $\hat{\beta}(z) + \hat{\beta}(-z) = 0$. Hence P2$\Leftrightarrow$P3. □

Lemma 4. *Let $f, h \in l^2(\mathbf{Z})$ or $\mathbf{C}^N$. The following are equivalent:*

P1: *For all $k \in \mathbf{Z}$ or $k \in \mathbf{Z}_{N/2}$, $\langle f, R^{2k}h \rangle = 0$.*

P2: *$\forall z$, $(\hat{f}\,\overline{\hat{h}})$ is an odd function of z.*

P3: *$\forall z$, $\hat{f}(z)\overline{\hat{h}}(z) + \hat{f}(-z)\overline{\hat{h}}(-z) = 0$.*

Proof of Lemma 4. From Lemma 1,

$$\langle f, R^{2k}h \rangle = \mathrm{coeff}_{2k}\left(\hat{f}(z)\overline{\hat{h}}(z)\right). \tag{29}$$

Arguing as in the proof of Lemma 3, P1 is true if, and only if, $(\hat{f}\,\overline{\hat{h}})$ is an odd function of z. Therefore P1 $\Leftrightarrow$ P2. Also P2$\Leftrightarrow$ P3 by definition. □

In other words, Lemma 3 says that the sets $B_L = \{R^{2k}f_L\}_k$ and $B_H = \{R^{2k}f_H\}_k$ (and therefore $\tilde{B}_L$ and $\tilde{B}_H$) are each orthonormal if, and only if, the polynomials $|\hat{f}_L(z)|^2$ and $|\hat{f}_H(z)|^2$ are almost–odd as functions of $z \in \mathbf{T}$ or $z \in \mathbf{W}_N$. Lemma 4 says that the sets B_L and B_H (and therefore $\tilde{B}_L$ and $\tilde{B}_H$) are mutually orthogonal if, and only if, the polynomial $\hat{f}(z)\,\overline{\hat{g}}(z)$ is an odd function of $z \in \mathbf{T}$ or $z \in \mathbf{W}_N$. These lemmas give us the following interesting result (cf. Meyer[21]):

Theorem 1. *The set $\tilde{B} = \{R^{2k}\tilde{f}_L\}_k \cup \{R^{2k}\tilde{f}_H\}_k$ is orthonormal if and only if the matrix*

$$A(z) = \frac{1}{\sqrt{2}}\begin{pmatrix} \hat{f}_L(z) & \hat{f}_H(z) \\ \hat{f}_L(-z) & \hat{f}_H(-z) \end{pmatrix}. \tag{30}$$

is unitary for all $z \in \mathbf{T}$ or $\mathbf{W}_N$.

Proof of Theorem 1. From the orthonormality conditions (22)–(24), and Lemmas 2–4, we see that the set $\tilde{B}$ is orthonormal if and only if the following equations hold for all $z \in \mathbf{T}$ or $\mathbf{W}_N$.

$$|\hat{f}_L(z)|^2 + |\hat{f}_L(-z)|^2 = 2 \tag{31}$$

$$|\hat{f}_H(z)|^2 + |\hat{f}_H(-z)|^2 = 2 \tag{32}$$

$$\hat{f}_L(z)\overline{\hat{f}_H}(z) + \hat{f}_L(-z)\overline{\hat{f}_H}(-z) = 0. \tag{33}$$

Equations (31)–(33) are equivalent to the unitarity of the matrix (30). □

Equations (31)–(33) may in fact be said to define the unitarity of $A(z)$ in (30). Equations (31) and (32) assert that the two columns of $A(z)$ must each have *norm* (or "size", or "length") of unity in some appropriately defined inner product space W. Equation (33) says that the columns of $A(z)$ regarded as vectors in W must be orthogonal, in that their inner product in W is required to vanish. The matrix $A(z)$ in (30) will be encountered over and over again, and will be called the *system matrix.*

In the finite case (30) is required to be unitary only for $z \in \mathbf{W}_N$. Since $\hat{f}(e^{-j2\pi m/N}) = (\mathcal{F}f)(m)$, Theorem 1 shows that the multiset $\tilde{B}$ is orthonormal if and only if

$$A(m) = \frac{1}{\sqrt{2}} \begin{pmatrix} (\mathcal{F}f_L)(m) & (\mathcal{F}f_H)(m) \\ (\mathcal{F}f_L)(m \pm \frac{N}{2}) & (\mathcal{F}f_H)(m \pm \frac{N}{2}) \end{pmatrix} \tag{34}$$

is unitary for $m = 0, \ldots, (N-1)$; where the sign in (34) is chosen such that $m \pm \frac{N}{2} \in \{0, \ldots, (N-1)\}$. In order to determine the validity of a filter pair (f_L, f_H) for the orthonormal decomposition of an N-dimensional signal space we only need check the unitarity of $N/2$ (2×2)–matrices. The remaining $N/2$ (2×2)–matrices are row–reversed copies of the first $N/2$, and their unitarity is automatic.

2.3. Perfect Reconstruction

We have so far established the necessary and sufficient conditions for orthonormal decomposition. We now establish conditions such that the arrangement in Figure 1 will yield a perfect reconstruction of the input signal. The downsampling operation can result in the violation of the Nyquist sampling criterion in each branch of Figure 1. It is remarkable that these violations do not prohibit the noiseless recovery of a signal. From the lowpass (i.e. lower) branch of Figure 1 we have:

$$\hat{u}_L(z) = \hat{f}_L(z)\hat{x}(z) \tag{35}$$

$$\hat{v}_L(z) = (\mathcal{D}u_L)\hat{\ }(z) = \frac{1}{2}\left(\hat{u}_L(z^{1/2}) + \hat{u}_L(-z^{1/2})\right) \tag{36}$$

$$\hat{w}_L(z) = (\mathcal{U}v_L)\hat{\ }(z) = \hat{v}_L(z^2) \tag{37}$$

$$\hat{y}_L(z) = \hat{g}_L(z)\hat{w}_L(z). \tag{38}$$

Equations (36) and (37) describe the down and upsampling operations, respectively, and may in fact be considered to define those operations. Alternatively, we may define the downsampling operator $\mathcal{D} : l^2(\mathbf{Z}) \to l^2(\mathbf{Z})$ or $\mathcal{D} : \mathbf{C}^N \to \mathbf{C}^{N/2}$ by

$$(\mathcal{D}u)(n) = u(2n), \quad n \in l^2(\mathbf{Z}) \text{ or } n \in \mathbf{C}^{N/2}. \tag{39}$$

The definitions of $\mathcal{D}$ in (36) and (39) are equivalent because

$$\hat{u}(z^{1/2}) + \hat{u}(-z^{1/2}) = \sum_n u(n)z^{-n/2} + \sum_n u(n)(-1)^n z^{-n/2} \tag{40}$$

$$= 2\sum_{n \text{ even}} u(n)z^{-n/2} = 2\sum_m u(2m)z^{-m}. \tag{41}$$

Similarly, the upsampling operator $\mathcal{U} : l^2(\mathbf{Z}) \to l^2(\mathbf{Z})$ or $\mathcal{U} : \mathbf{C}^{N/2} \to \mathbf{C}^N$ may be defined as:

$$(\mathcal{U}v)(n) = \begin{cases} v(n/2), & n \text{ even} \\ 0, & \text{otherwise.} \end{cases} \tag{42}$$

The definitions of $\mathcal{U}$ in (37) and (42) are equivalent.

From (35)–(38),

$$\hat{y}_L(z) = \frac{1}{2}\hat{g}_L(z)\hat{f}_L(z)\hat{x}(z) + \frac{1}{2}\hat{g}_L(z)\hat{f}_L(-z)\hat{x}(-z). \tag{43}$$

Similarly, from the highpass branch in Figure 1,

$$\hat{y}_H(z) = \frac{1}{2}\hat{g}_H(z)\hat{f}_H(z)\hat{x}(z) + \frac{1}{2}\hat{g}_H(z)\hat{f}_H(-z)\hat{x}(-z). \tag{44}$$

Perfect reconstruction holds in Figure 1 if and only if $\hat{x}(z) = \hat{y}_L(z) + \hat{y}_H(z)$; i.e. if and only if

$$\hat{x}(z) = \frac{1}{2}\left(\hat{g}_L(z)\hat{f}_L(z) + \hat{g}_H(z)\hat{f}_H(z)\right)\hat{x}(z) + \frac{1}{2}\left(\hat{g}_L(z)\hat{f}_L(-z) + \hat{g}_H(z)\hat{f}_H(-z)\right)\hat{x}(-z). \tag{45}$$

We also have the following:

Lemma 5. *If $A(z)$, $B(z)$ are fixed polynomials in z, and if for all $\hat{x}(z)$*

$$\hat{x}(z) = A(z)\hat{x}(z) + B(z)\hat{x}(-z), \tag{46}$$

then $A(z) = 1$ and $B(z) = 0$.

Proof of Lemma 5. Substituting $\hat{x}(z) = 1$ and $\hat{x}(z) = z$ into (46) we have

$$z = zA(z) + zB(z) \tag{47}$$

$$z = zA(z) - zB(z). \tag{48}$$

By adding and subtracting (47) and (48) we have the lemma. □

From (45) and Lemma 5 we deduce that perfect reconstruction holds if and only if the following equations are true:

$$\hat{g}_L(z)\hat{f}_L(z) + \hat{g}_H(z)\hat{f}_H(z) = 2 \tag{49}$$

$$\hat{g}_L(z)\hat{f}_L(-z) + \hat{g}_H(z)\hat{f}_H(-z) = 0 \tag{50}$$

The system of equations (49) and (50) can be written as the single matrix equation below.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \hat{f}_L(z) & \hat{f}_H(z) \\ \hat{f}_L(-z) & \hat{f}_H(-z) \end{pmatrix}\begin{pmatrix} \hat{g}_L(z) \\ \hat{g}_H(z) \end{pmatrix} = A(z)\begin{pmatrix} \hat{g}_L(z) \\ \hat{g}_H(z) \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix}. \tag{51}$$

The condition (51) is necessary and sufficient for perfect reconstruction, *whether or not* $\tilde{B}$ is orthonormal. If $\tilde{B}$ *is* orthonormal then by Theorem 1 the system matrix $A(z)$ is unitary and is, therefore, particularly easy to invert: $A^{-1}(z) = A^*(z) = \overline{A}^T(z)$. Hence (51) is easily solved:

$$\begin{pmatrix} \hat{g}_L(z) \\ \hat{g}_H(z) \end{pmatrix} = A^*(z)\begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{\hat{f}_L(z)} \\ \overline{\hat{f}_H(z)} \end{pmatrix} \tag{52}$$

From (52) and (18), when $\tilde{B}$ is orthonormal, perfect reconstruction requires that $g_L = \tilde{f}_L$ and $g_H = \tilde{f}_H$.

This is hardly a surprising result in view of the fact that given an orthonormal basis $\tilde{B}$ of an inner product space V, $x \in V$ can be decomposed as a weighted sum of vectors in $\tilde{B}$, where the weights are given by the inner products of x computed against the basis vectors. Recall from (10)–(11) that $(x * f)(2k) = \langle x, R^{2k}\tilde{f}\rangle$. By perfect reconstruction in Figure 1,

$$x(n) = (g_L * (\mathcal{UD}(x * f_L)))(n) + (g_H * (\mathcal{UD}(x * f_H)))(n) \tag{53}$$

$$= \sum_k g_L(n-2k)\langle x, R^{2k}\tilde{f}_L\rangle + \sum_k g_H(n-2k)\langle x, R^{2k}\tilde{f}_H\rangle \tag{54}$$

$$= \sum_k \langle x, R^{2k}\tilde{f}_L\rangle (R^{2k}g_L)(n) + \sum_k \langle x, R^{2k}\tilde{f}_H\rangle (R^{2k}g_H)(n). \tag{55}$$

In view of the decomposition (55) of x, and in view of the orthonormality of $\tilde{B}$, it is not surprising that $g_L = \tilde{f}_L$ and $g_H = \tilde{f}_H$. Equation (55), along with the equations $g_L = \tilde{f}_L$ and $g_H = \tilde{f}_H$, demonstrates the *completeness* of $\tilde{B}$. These three equations show that every vector $x \in l^2(\mathbf{Z})$ or $\mathbf{C}^N$ can be written as a sum of the elements of $\tilde{B}$. The sequences $\tilde{f}_L$ and $\tilde{f}_H$ are *first-generation wavelets.*

We have proved:

Theorem 2. *The following are equivalent:*

P1: *The set $\tilde{B} = \{R^{2k}\tilde{f}_L\}_k \cup \{R^{2k}\tilde{f}_H\}_k$ is orthonormal, and we have perfect reconstruction in Figure 1.*

P2: *The system matrix $A(z)$ in (30) is unitary for all $z \in \mathbf{T}$ or $z \in \mathbf{W}_N$, $g_L = \tilde{f}_L$, and $g_H = \tilde{f}_H$.*

3. The Design Of Wavelet Filters

In this section we discuss the design of the analysis filters f_L and f_H, and of the synthesis filters g_L and g_H, in the signal spaces $l^2(\mathbf{Z})$ and $\mathbf{C}^N$. The requirements of orthonormality determine the unitarity of the system matrix $A(z)$ by Theorem 1. By the unitarity condition the norm of the first column of the system matrix $A(z)$ must be unity. Therefore,

$$|\hat{f}_L(z)|^2 + |\hat{f}_L(-z)|^2 = 2. \tag{56}$$

From (56) and Lemma 3, $|\hat{f}_L(z)|^2$ is almost–odd. Then $|\hat{f}_L(z)|^2 = 1 + \hat{h}(z)$, where $\hat{h}$ is some odd function of z. Moreover $\hat{h}(z)$ is real–valued because $|\hat{f}_L(z)|^2 - 1$ is. As the squared modulus of a complex number, $|\hat{f}_L(z)|^2$ is bounded below by zero for all $z \in \mathbf{T}$. From (56), $|\hat{f}_L(z)|^2$ is also bounded above: $|\hat{f}_L(z)|^2 \leq 2$, $\forall z \in \mathbf{T}$ or $\mathbf{W}_N$. Then $-1 \leq \hat{h}(z) \leq 1$, $\forall z \in \mathbf{T}$ or $\mathbf{W}_N$. These observations yield a recipe for the construction of $\hat{f}_L(z)$.

Let $\hat{h}(z)$ be any real–valued function defined upon the complex unit circle $\mathbf{T}$ or the roots $\mathbf{W}_N$ of unity, such that $\hat{h}(z) = -\hat{h}(-z)$; and $-1 \leq \hat{h}(z) \leq 1$. Let $\rho(z)$ be another arbitrary real–valued function defined on $\mathbf{T}$ or $\mathbf{W}_N$. Define

$$\hat{f}_L(z) = \sqrt{1 + \hat{h}(z)}\, e^{j\rho(z)}. \tag{57}$$

The form of $\hat{f}_L(z)$ in (56) is the most general possible. Since $\rho(z)$ need not be a polynomial in z, we do not write $\hat{\rho}(z)$.

Because $A(z)$ is unitary, so is $A^T(z)$. Because the norm of the first column of $A^T(z)$ must be unity, $|\hat{f}_L(z)|^2 + |\hat{f}_H(z)|^2 = 2$. Then $\hat{f}_H(z)$ must have the form:

$$\hat{f}_H(z) = \sqrt{1 - \hat{h}(z)}\, e^{j\sigma(z)}, \tag{58}$$

where $\sigma(z)$ is a real–valued function of z.

Substituting (57) and (58) into the system matrix (30), and using the oddness of $\hat{h}(z)$, we have:

$$A(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + \hat{h}(z)}\, e^{j\rho(z)} & \sqrt{1 - \hat{h}(z)}\, e^{j\sigma(z)} \\ \sqrt{1 - \hat{h}(z)}\, e^{j\rho(-z)} & \sqrt{1 + \hat{h}(z)}\, e^{j\sigma(-z)} \end{pmatrix}. \tag{59}$$

Theorem 1 also requires the orthogonality of the two column vectors in $A^T(z)$. In other words, inner product of the column vectors in $A^T(z)$ must vanish, then

$$\sqrt{1-\hat{h}^2(z)}\left(e^{j(\rho(z)-\rho(-z))}+e^{j(\sigma(z)-\sigma(-z))}\right)=0 \tag{60}$$

If $\hat{h}(z) \neq \pm 1$, then (60) requires that:

$$\sigma(z)-\sigma(-z)=(2k+1)\pi+\rho(z)-\rho(-z), \tag{61}$$

for some $k \in \mathbf{Z}$, k depending possibly on z. If $\hat{h}(z) = \pm 1$, then $\sigma(z)$ and $\rho(z)$ are unconstrained.

We have shown that if $A(z)$ is unitary then $A(z)$ has the form (59) for $\hat{h}(z)$, $\sigma(z)$, and $\rho(z)$, as described. Conversely, any such matrix is easily seen to be unitary. Finally, the synthesis filter sequences g_L and g_H must be chosen so as to satisfy Theorem 2. The three steps **S1**, **S2**, and **S3**, in the construction of the filter sequences f_L, f_H, g_L, and g_H, are summarized below.

S1. Construct an arbitrary real-valued function $\hat{h}(z)$, $z \in \mathbf{T}$ or $\mathbf{W}_N$, such that $-1 \leq \hat{h}(z) \leq 1$, and $\hat{h}(z) = -\hat{h}(-z)$. Construct an arbitrary real-valued function $\rho(z)$, $z \in \mathbf{T}$ or $\mathbf{W}_N$. Define $\hat{f}_L(z) = \sqrt{1+\hat{h}(z)}\,e^{j\rho(z)}$, $z \in \mathbf{T}$ or $\mathbf{W}_N$.

S2. Construct a real-valued function $\sigma(z)$, $z \in \mathbf{T}$ or $\mathbf{W}_N$, such that $\sigma(z)-\sigma(-z) = (2k+1)\pi + \rho(z)-\rho(-z)$, if $\hat{h}(z) \neq \pm 1$. Here $k \in \mathbf{Z}$ is an arbitrary integer that possibly depends on z. Define $\hat{f}_H(z) = \sqrt{1-\hat{h}(z)}\,e^{j\sigma(z)}$, $z \in \mathbf{T}$ or $\mathbf{W}_N$.

S3. Define $\hat{g}_L(z) = \overline{\hat{f}_L}(z)$; $\hat{g}_H(z) = \overline{\hat{f}_H}(z)$.

Theorem 2 shows that any and all wavelet filters at the first stage can be constructed with this algorithm.

3.1. Examples Of Wavelet Filter Construction

Let $x \in \mathbf{C}^4$. Let $\omega = e^{-j2\pi/4} = -j$; $\mathbf{W}_4 = \{\omega^0, \omega^1, \omega^2, \omega^3\} = \{1, -j, -1, j\}$. Choose

$$\hat{h}(1)=1,\ \hat{h}(-j)=-1,\ \hat{h}(-1)=-1,\ \hat{h}(j)=1. \tag{62}$$

Choose $\rho(z) \equiv 0$. Then from $\hat{f}_L(z) = \sqrt{1+\hat{h}(z)}$ we have

$$\hat{f}_L(1) = \sqrt{2},\ \hat{f}_L(-j) = 0,\ \hat{f}_L(-1) = 0,\ \hat{f}_L(j) = \sqrt{2}. \tag{63}$$

The filter sequence f_L itself can be computed as the inverse DFT of the vector $\mathcal{F}f_L = (\sqrt{2}, 0, 0, \sqrt{2})$: $f_L = 2^{-3/2}(2, (1+j), 0, (1-j))$. Knowing f_L, we can write the polynomial $\hat{f}_L(z)$:

$$\hat{f}_L(z) = \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}}(1+j)z^{-1} + \frac{1}{2\sqrt{2}}(1-j)z^{-3}. \tag{64}$$

In order to compute $\hat{f}_H(z)$ we need to choose the function $\sigma(z)$. Since $\hat{h}(z) = \pm 1$ for $z = 1, j, -1, -j$, the choice of $\sigma(z)$ is unconstrained by $\rho(z)$. We choose $\sigma(z) \equiv 0$. Then the DFT of f_H, computed from $\hat{f}_H(z) = \sqrt{1-\hat{h}(z)}$ is:

$$\mathcal{F}f_H = (0, \sqrt{2}, \sqrt{2}, 0), \tag{65}$$

and

$$\hat{f}_H(z) = \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}}(1+j)z^{-1} - \frac{1}{2\sqrt{2}}(1-j)z^{-3}. \tag{66}$$

By Step 3 of the filter construction algorithm we have $\hat{g}_L(z) = \overline{\hat{f}_L}(z) = \hat{f}_L(z)$ and $\hat{g}_H(z) = \overline{\hat{f}_H}(z) = \hat{f}_H(z)$.

3.1.1. A Second Example: The Shannon Wavelet Basis. We can generalize the last example for signal spaces $\mathbf{C}^N$, where $4|N$. Let A and B be the sets

$$A = \{0, 1, \ldots, N/4-1\} \cup \{3N/4, \ldots, N-1\} \tag{67}$$

$$B = \{N/4, \ldots, 3N/4-1\}. \tag{68}$$

Define

$$(\mathcal{F}h)(m) = \begin{cases} 1 & , m \in A \\ 0 & , m \in B \end{cases} \tag{69}$$

$$\sigma(z) = \rho(z) \equiv 0. \tag{70}$$

This gives:

$$(\mathcal{F}f_L)(m) = \hat{f}_L(e^{-j2\pi m/N}) = \begin{cases} \sqrt{2} & , m \in A \\ 0 & , m \in B \end{cases} \tag{71}$$

$$(\mathcal{F}f_H)(m) = \hat{f}_H(e^{-j2\pi m/N}) = \begin{cases} 0 & , m \in A \\ \sqrt{2} & , m \in B \end{cases} \tag{72}$$

The filters f_H and f_L have disjoint frequency supports. We call the basis defined by this pair of filters the *Shannon wavelet basis* because the filters are similar to the sinc functions that appear in the Shannon sampling theorem.

3.1.2. A Third Example: The Real Shannon Wavelet Basis. If the signals being dealt with are real–valued, then it is sometimes advantageous to have real–valued filters so that all inner–products are real too. This saves computation time and storage. We can modify the last example to yield a real–valued basis. A function $f \in \mathbf{C}^N$ is real–valued if and only if $(\mathcal{F}f)(m) = \overline{(\mathcal{F}f)}(N - m), \forall m \in \mathbf{Z}_N$. The filter sequences $\mathcal{F}f_L$ and $\mathcal{F}f_H$ in the last example each fail this criterion at $m = N/4$. However, Theorem 1 tells us how to alter f_L and f_H in order to obtain real filters.

If f_L is to be real, we require $(\mathcal{F}f_L)(N/4) = (\overline{\mathcal{F}f_L})(3N/4) \Rightarrow \hat{f}_L(e^{-j\pi/2}) = \overline{\hat{f}_L}(e^{-j3\pi/2}) \Rightarrow \hat{f}_L(-j) = \overline{\hat{f}_L}(j)$. Moreover, by Theorem 1 the system matrix is required to be unitary for all z in $\mathbf{W}_N$. For $z = -j$,

$$A(-j) = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{f}_L(-j) & \hat{f}_H(-j) \\ \hat{f}_L(j) & \hat{f}_H(j) \end{pmatrix}. \tag{73}$$

If we choose, $\hat{f}_L(-j) = j$; $\hat{f}_L(j) = -j$; and $\hat{f}_H(-j) = \hat{f}_H(j) = 1$; then $A(-j)$ is unitary, and $A(j)$ too. For $\pm j \neq z \in \mathbf{W}_N$, $A(z)$ is unitary for $\hat{f}_L(z)$ and $\hat{f}_H(z)$ as defined in the last example. Therefore, the following pair of filters defines a valid wavelet basis:

$$(\mathcal{F}f_L)(m) = \quad \hat{f}_L(e^{-j2\pi m/N}) = \begin{cases} \sqrt{2} & ,0 \le m \le N/4 - 1 \text{ or } 3N/4 + 1 \le m \le N - 1 \\ j & ,m = N/4 \\ -j & ,m = 3N/4 \\ 0 & ,N/4 + 1 \le m \le 3N/4 - 1 \end{cases} \tag{74}$$

$$(\mathcal{F}f_H)(m) = \quad \hat{f}_H(e^{-j2\pi m/N}) = \begin{cases} 0 & ,0 \le m \le N/4 - 1 \text{ or } 3N/4 + 1 \le m \le N - 1 \\ 1 & ,m = N/4 \text{ or } m = 3N/4 \\ \sqrt{2} & ,N/4 + 1 \le m \le 3N/4 - 1 \end{cases} \tag{75}$$

We have $\mathrm{supp}(\mathcal{F}f_L) \cap \mathrm{supp}(\mathcal{F}f_H) = \{N/4, 3N/4\}$, a minimal overlap in support. We call the basis defined by this filter pair the *real Shannon wavelet basis.*

3.2. Time–Frequency Localization

Wavelets and other methods of time–frequency analysis, like the Phi–transform, have many practical applications which require the filter sequences or "analyzing functions" to possess certain specific properties. These properties concern the frequency–localization or the simultaneous time–frequency–localization [15]–[18] of filter sequences. We have seen that it is easy to construct wavelet bases that are well–localized in the frequency domain: viz. the Shannon and the real Shannon bases. Now consider the problem of simultaneous time–frequency localization. In the case of the FJT time–frequency localization is easy [17]. In the case of wavelets it is not, because we are hemmed in by the orthogonality conditions. In what follows we formulate the time–frequency localization problem for wavelets.

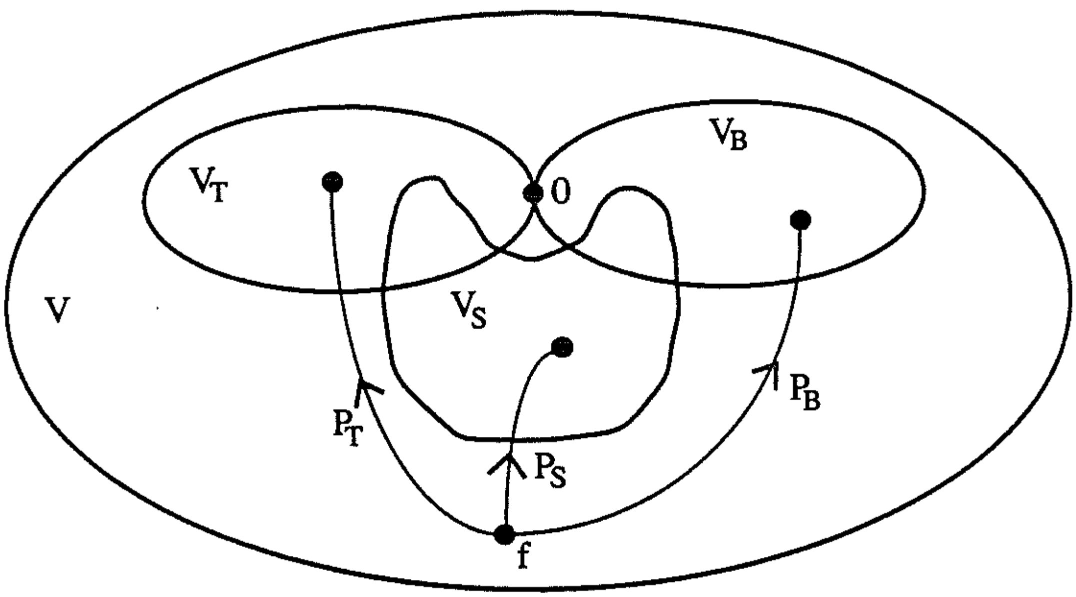

Figure 2: The signal space V, the subspaces V_T and V_B, the manifold V_S, and mappings from V into V_T, V_B, and V_S.

In Figure 2 are drawn the subspaces V_T and V_B of $V = l^2(\mathbf{Z})$ or $V = \mathbf{C}^N$. The subspace V_T consists of all sequences $x \in V$, with a fixed supp(x), which is some proper subset of V. The subspace V_B consists of all sequences $x \in V$, with a fixed support for $\mathcal{F}x$. The only vector common to V_T and V_B is the zero vector. $P_T : V \longrightarrow V_T$ and $P_B : V \longrightarrow V_B$ are projection operators.

Let $V_S \subseteq V$ be the set of all valid wavelets f_L and f_H. It is easy to check that V_S is neither a subspace nor an affine space of V, hence any mapping $P_S : V \longrightarrow V_S$ cannot be a projection. V_S is in fact a manifold that lies embedded in the surface of the unit sphere in $l^2(\mathbf{Z})$ or $\mathbf{C}^N$. Define $P_S : V \longrightarrow V_S$ to be an operator that maps any given $f \in V$ to $\psi = P_S f \in V_S$, such that distance

$\|f-\psi\|_V = \langle f-\psi, f-\psi\rangle^{1/2}$ between f and ψ is a minimum. We now construct this operator for $V = \mathbf{C}^N$.

Given any $f \in \mathbf{C}^N$ we would like to find a wavelet ψ that is closest to f. If ψ is a wavelet that is closest to f, then $\|f-\psi\|^2_{\mathbf{C}^N}$ is a minimum.

$$\|f-\psi\|^2_{\mathbf{C}^N} = \frac{1}{N}\|(\mathcal{F}f)-(\mathcal{F}\psi)\|^2_{\mathbf{C}^N} = \frac{1}{N}\sum_{m=0}^{N-1}|(\mathcal{F}f)(m)-(\mathcal{F}\psi)(m)|^2 \tag{76}$$

$$= \frac{1}{N}\sum_{m=0}^{N/2-1}\left(|(\mathcal{F}f)(m)-(\mathcal{F}\psi)(m)|^2 + |(\mathcal{F}f)(m+N/2)-(\mathcal{F}\psi)(m+N/2)|^2\right). \tag{77}$$

For $m \in \mathbf{Z}_{N/2}$; define

$$A(m) = |(\mathcal{F}f)(m)-(\mathcal{F}\psi)(m)|^2 + |(\mathcal{F}f)(m+N/2)-(\mathcal{F}\psi)(m+N/2)|^2. \tag{78}$$

From (76)–(78),

$$\|f-\psi\|^2_{\mathbf{C}^N} = \frac{1}{N}\sum_{m=0}^{N/2-1} A(m). \tag{79}$$

In order to minimize $\|f-\psi\|^2_{\mathbf{C}^N}$, we minimize $A(m)$ for each $m \in \mathbf{Z}_{N/2}$.

In order for ψ to be a valid wavelet filter f_L or f_H, the following equation must hold for $m \in \mathbf{Z}_N$:

$$(\mathcal{F}\psi)(m) = \sqrt{1+(\mathcal{F}h)(m)}\, e^{j\rho(e^{-j2\pi m/N})}, \tag{80}$$

by the conditions set forth in step S1 of the wavelet construction algorithm. Also it must be that $(\mathcal{F}h)(m+N/2) = -(\mathcal{F}h)(m) \in [-1,1]$, and $\rho(e^{-j2\pi m/N})$ real, for $m \in \mathbf{Z}_N$.

Define $a(m)$, $b(m)$, $\theta(m)$, and $\gamma(m)$ by

$$(\mathcal{F}f)(m) = a(m)e^{j\theta(m)};\ m = 0,\ldots,(N/2-1) \tag{81}$$

$$(\mathcal{F}f)(m+N/2) = b(m)e^{j\gamma(m)};\ m = 0,\ldots,(N/2-1), \tag{82}$$

where $a(m), b(m) \in \mathbf{R}$; $a(m), b(m) \geq 0$; and $\theta(m), \gamma(m) \in [-\pi,\pi]$; for $m \in \mathbf{Z}_{N/2}$.

From (78) and (80)–(82), for $m = 0,\ldots,(N/2-1)$,

$$A(m) = \left|a(m)e^{j\theta(m)} - \sqrt{1+(\mathcal{F}h)(m)}\, e^{j\rho(e^{-j2\pi m/N})}\right|^2 + \left|b(m)e^{j\gamma(m)} - \sqrt{1-(\mathcal{F}h)(m)}\, e^{j\rho(-e^{-j2\pi m/N})}\right|^2 \tag{83}$$

If $a(m) = b(m) = 0$ for some $m = m_0$ then (83) tells us that $A(m_0) = 2$, and we have all the freedom we want in the choice of $(\mathcal{F}h)(m_0) \in [-1,1]$, $\rho(e^{-j2\pi m_0/N}) \in [-\pi,\pi]$, and $\rho(-e^{-j2\pi m_0/N}) \in [-\pi,\pi]$. It follows that there may be infinitely many wavelets ψ that are closest to a given $f \in V$.

Now assume that $a(m)$ and $b(m)$ are not both zero. From (83) we note that in order to minimize $A(m)$ we want $\rho(e^{-j2\pi m/N}) = \theta(m)$ and $\rho(-e^{-j2\pi m/N}) = \gamma(m)$. Then (83) reduces to:

$$A(m) = 2 + a^2(m) + b^2(m) - 2\left(a(m)\sqrt{1+(\mathcal{F}h)(m)} + b(m)\sqrt{1-(\mathcal{F}h)(m)}\right). \tag{84}$$

In (84) $a(m)$ are $b(m)$ are fixed, and we wish to choose $(\mathcal{F}h)(m) \in [-1,1]$ so as to minimize $A(m)$. This will be done if we choose $(\mathcal{F}h)(m)$ so as to maximize $a(m)\sqrt{1+(\mathcal{F}h)(m)} + b(m)\sqrt{1-(\mathcal{F}h)(m)}$.

By calculus, if $a, b \geq 0$ are fixed and are not both zero, then the function $f(x) = a\sqrt{1+x} + b\sqrt{1-x}$ attains a maximum on $[-1,1]$ at $x = (a^2-b^2)/(a^2+b^2)$. Hence, if not both $a(m)$ and $b(m)$ are zero, then $a(m)\sqrt{1+(\mathcal{F}h)(m)} + b(m)\sqrt{1-(\mathcal{F}h)(m)}$ is maximized (and $A(m)$ minimized) by the following choice of $(\mathcal{F}h)(m) \in [-1,1]$, $m \in \mathbf{Z}_{N/2}$:

$$(\mathcal{F}h)(m) = \frac{a^2(m)-b^2(m)}{a^2(m)+b^2(m)} = \frac{|(\mathcal{F}f)(m)|^2 - |(\mathcal{F}f)(m+N/2)|^2}{|(\mathcal{F}f)(m)|^2 + |(\mathcal{F}f)(m+N/2)|^2};\ m \in \mathbf{Z}_{N/2}. \tag{85}$$

Hence the DFT coefficient $(\mathcal{F}\psi)(m)$, $m \in \mathbf{Z}_{N/2}$, of the wavelet ψ that is closest to $f \in V$ is

$$(\mathcal{F}\psi)(m) = \sqrt{1+(\mathcal{F}h)(m)}\, e^{j\theta(m)} = \frac{\sqrt{2}\, a(m) e^{j\theta(m)}}{\sqrt{a^2(m)+b^2(m)}} \tag{86}$$

$$= \frac{\sqrt{2}\,(\mathcal{F}f)(m)}{\sqrt{|(\mathcal{F}f)(m)|^2 + |(\mathcal{F}f)(m+N/2)|^2}};\ m \in \mathbf{Z}_{N/2}. \tag{87}$$

For $N/2 \leq m < N$, step **S1** in the wavelet construction algorithm tell us that $(\mathcal{F}h)(m) = -(\mathcal{F}h)(m-N/2)$. Then, from (85), for $N/2 \leq m < N$,

$$(\mathcal{F}h)(m) = -(\mathcal{F}h)(m-N/2) = -\frac{|(\mathcal{F}f)(m-N/2)|^2 - |(\mathcal{F}f)(m)|^2}{|(\mathcal{F}f)(m-N/2)|^2 + |(\mathcal{F}f)(m)|^2}. \tag{88}$$

By $\mathbf{Z}_N$ arithmetic upon indices into $(\mathcal{F}f)$, $(\mathcal{F}f)(m-N/2) = \mathcal{F}(m+N/2)$, and we note from (85) and (88) that (85) and (87) hold not only for $m \in \mathbf{Z}_{N/2}$ but for the full range $m \in \mathbf{Z}_N$.

In summary, we present the following simple three-step algorithm for the construction of a wavelet ψ that is closest to a given vector $f \in \mathbf{C}^N$, $f \neq 0$:

S1. If $(\mathcal{F}f)(m) = (\mathcal{F}f)(m+N/2) = 0$, then assign any value $C \in [0, \sqrt{2}]$ to $|(\mathcal{F}\psi)(m)|$. To $|(\mathcal{F}\psi)(m+N/2)|$ assign $\sqrt{2-C^2}$. To each of the two phase terms $\rho(e^{-j2\pi m/N})$ and $\rho(-e^{-j2\pi m/N})$ of $(\mathcal{F}\psi)(m)$ and $(\mathcal{F}\psi)(m+N/2)$ assign any value in $[-\pi, \pi]$.

S2. If not both $(\mathcal{F}f)(m)$ and $(\mathcal{F}f)(m+N/2)$ vanish, then assign

$$(\mathcal{F}\psi)(m) = \frac{\sqrt{2}(\mathcal{F}f)(m)}{\sqrt{|(\mathcal{F}f)(m)|^2 + |(\mathcal{F}f)(m+N/2)|^2}}. \tag{89}$$

S3. From $\mathcal{F}\psi$ compute ψ by DFT inversion.

This algorithm is an operational description of the operator $P_S : V \to V_S$, $V = \mathbf{C}^N$. A similar description of P_S is possible for the signal space $l^2(\mathbf{Z})$.

We note from the above algorithm that if $\mathcal{F}f$ is real–valued, then we can always find a wavelet ψ closest to f such that $\mathcal{F}\psi$ is also real–valued.

In case of FJT analyzing functions, time–frequency localization involves the computation of an eigenvector of the double projection operator $(P_B P_T)$ [17]. This eigenvector may not lie in V_S and may not be a valid wavelet. One approach to the construction of a localized wavelet may be to map the eigenvector ϕ of $(P_B P_T)$ into V_S using P_S. Let ψ be the image of ϕ under P_S. If the support of ϕ is not too severely restricted in the frequency domain, i.e. if not both $(\mathcal{F}\phi)(m)$ and $(\mathcal{F}\phi)(m+N/2)$ vanish for any m, then (89) tells us that ψ will preserve, precisely, the frequency localization of f. If the support of $\mathcal{F}\phi$ *is* severely restricted then we can still, to some extent, control the support of $\mathcal{F}\psi$ through a judicious choice of the constants C in S1. It is not, however, possible to say anything about the time–localization of ψ on the basis of the analysis in this section.

It is possible to choose ϕ such that both ϕ and $\mathcal{F}\phi$ are real–valued. Then we can produce a wavelet ψ closest to ϕ that is also real–valued in both the time and the frequency domains.

Define $(\mathcal{F}g)(m) = \sqrt{2}\left(|(\mathcal{F}\phi)(m)|^2 + |(\mathcal{F}\phi)(m+N/2)|^2\right)^{-1/2}$. Then, from (89), $\psi = \phi * g$. It follows that if ϕ and g are well–localized in time, then ψ will be also. The function ϕ is well–localized by design, and that leaves us with questions concerning the time–localization of g. This question remains open.

4. Wavelet Recursion For One–Dimensional Signals

If in Figure 1 we subject the sequence v_L to the same treatment as x was subjected to, then we obtain the situation depicted in Figure 3. If we decompose and reconstruct both v_L and v_H, we have the situation in Figure 4. Like the level–1 filters f_L, f_H, g_L, and g_H; the level–2 filters f_{LL}, f_{LH}, f_{HL}, f_{HH}, g_{LL}, g_{LH}, g_{HL}, and g_{HH} must also obey the requirements set forth in Theorem 2 if we require orthogonal decomposition and perfect reconstruction. Then the matrices

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \hat{f}_{LL}(z) & \hat{f}_{LH}(z) \\ \hat{f}_{LL}(-z) & \hat{f}_{LH}(-z) \end{pmatrix}, \text{ and } \frac{1}{\sqrt{2}}\begin{pmatrix} \hat{f}_{HL}(z) & \hat{f}_{HH}(z) \\ \hat{f}_{HL}(-z) & \hat{f}_{HH}(-z) \end{pmatrix},$$

must be unitary; and the reconstruction filters g_{LL}, g_{LH}, g_{HL}, and g_{HH}, must equal $\tilde{f}_{LL}$, $\tilde{f}_{LH}$, $\tilde{f}_{HL}$, and $\tilde{f}_{HH}$, respectively. The process can be repeated for v_{LL}, v_{LH}, v_{HL}, or v_{HH}.

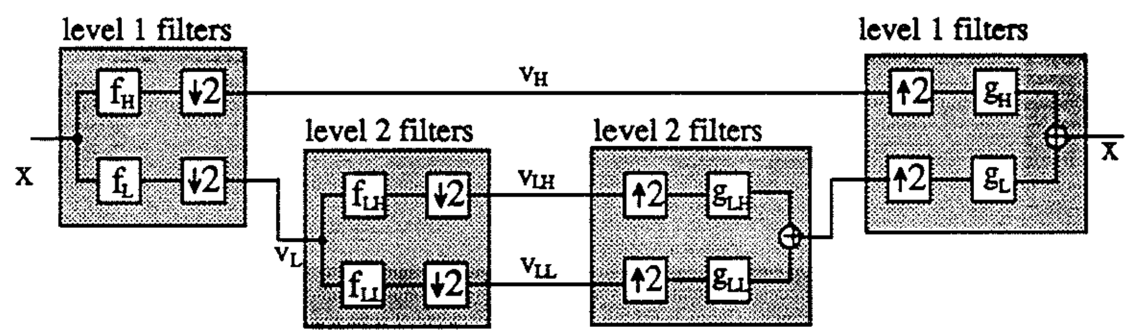

Figure 3: Recursion in the lower branch.

4.1. Wavelets, Filter Banks, And Wavelet Packets

In the classical wavelet analysis of Lemarie, Meyer, Mallat, and Daubechies, [1]–[6] recursion is performed only in the lower–most branch. Then wavelet decomposition generates the sequence of graphs in Figure 5, where each node represents a single filter–pair with their associated up–sample–by–2 or down–sample–by–2 operators. The number of filters at each level of analysis and synthesis is constant.

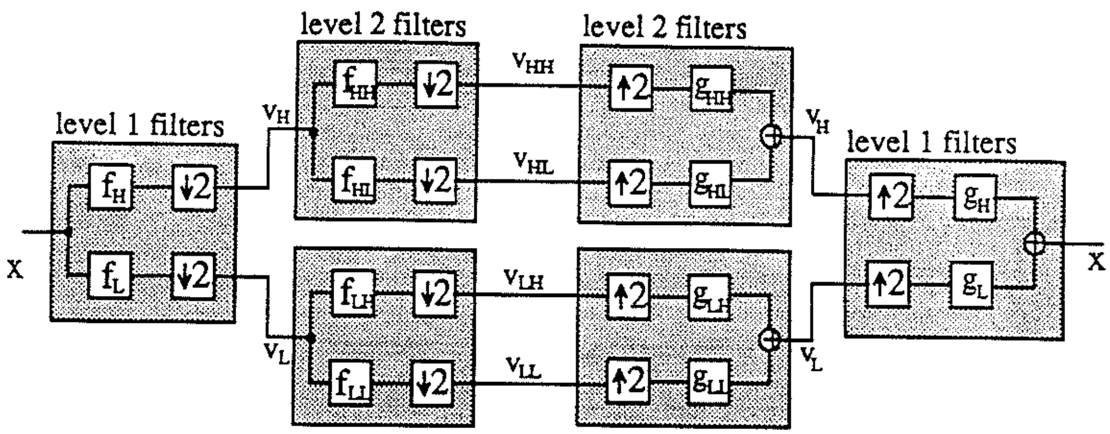

Figure 4: Recursion in both branches.

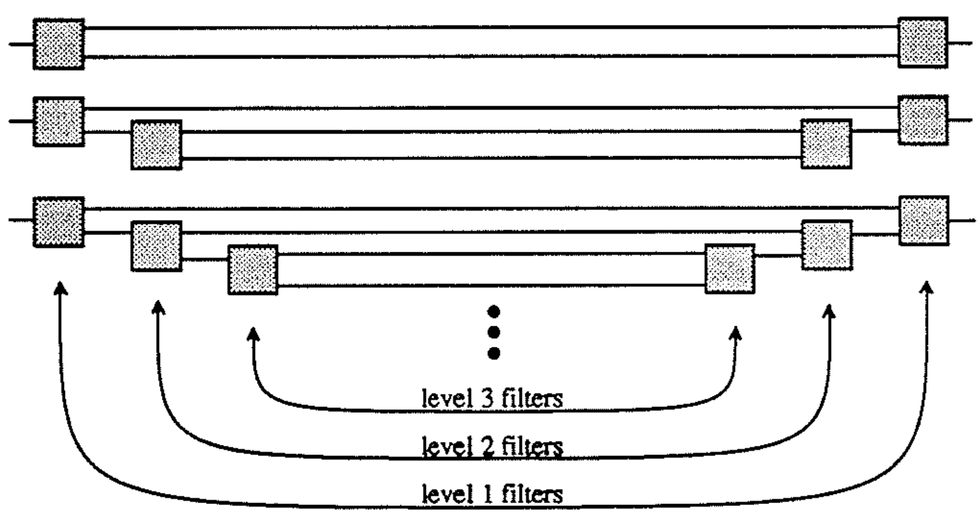

Figure 5: The sequence of graphs generated in classical wavelet analysis.

In case of full recursion at every level, we have the sequence of graphs in Figure 6. The number of filters doubles at each level of analysis and synthesis. This is the approach adopted by the filter bank school [8][9]. Yet a third approach to recursion is that of "best-adapted wavelet-packets" pioneered by Wickerhauser and Coifman et al. [19] [20], and characterized by an arrangement of the sort in Figure 7. In the best-adapted wavelet-packet method recursion is or is not performed at a certain level in the transformation tree depending upon a criterion of the optimality of representation.

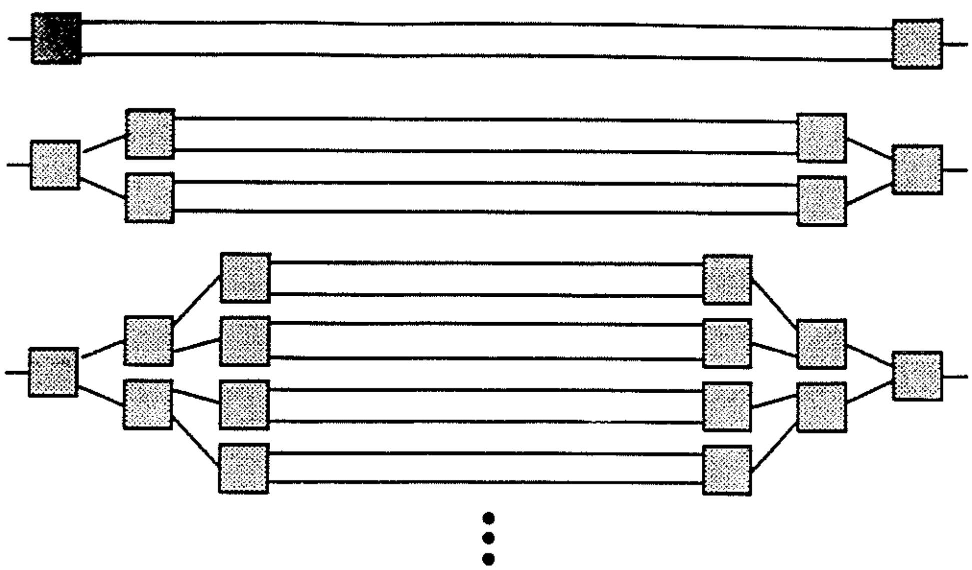

Figure 6: The sequence of graphs generated in classical filter–bank analysis.

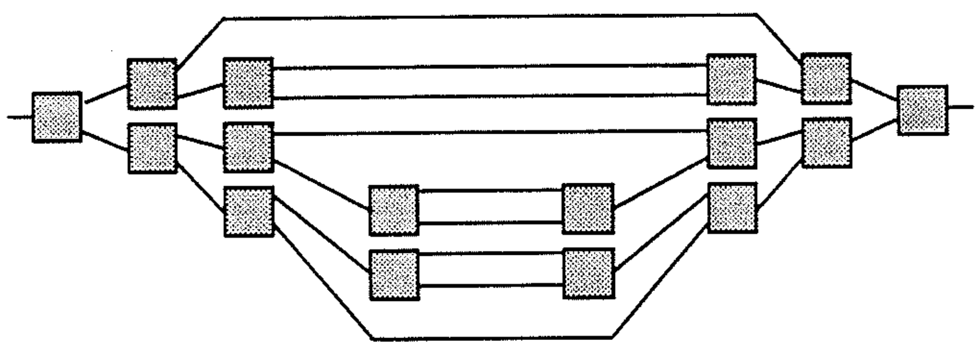

Figure 7: Irregular recursion is characteristic of wavelet–packet representations.

4.2. Equivalent Non-Recursive Structures

The recursive structures in Figures 3 and 4 have non–recursive equivalents. While the non–recursive structures are not as efficient as their recursive equivalents so far as computation is concerned, they do help us understand the nature of signal decomposition. Equivalent non–recursive structures are constructed with the help of the theorem stated and proved in this subsection.

Define the "bigstar" notation $\ast$ to do for convolution what the bigcup notation does for union:

$$\ast_{i=n_1}^{n_2} h_i = h_{n_1} * h_{n_1+1} * \ldots * h_{n_2}. \tag{90}$$

Let $\alpha \in \{L, H\}^*$ be a string. For $|\alpha| > 0$, define

$$F_\alpha = *_{i=1}^{|\alpha|} \mathcal{U}^{i-1} f_{\mathrm{pfx}(\alpha,i)} \tag{91}$$

$$G_\alpha = *_{i=1}^{|\alpha|} \mathcal{U}^{i-1} g_{\mathrm{pfx}(\alpha,i)}. \tag{92}$$

Here is an example to illustrate the notation: $F_{HLL} = f_H * (\mathcal{U} f_{HL}) * (\mathcal{U}^2 f_{HLL})$. The next theorem shows that the filters F_α and G_α can be used in place of a series of filters; so that, for example, the non-recursive structure in Figure 8 is equivalent to the recursive structure in Figure 3. The analyzing filters in the non-recursive structure are F_{LL}, F_{LH} and F_L; while the synthesizing filters are G_{LL}, G_{LH} and G_L.

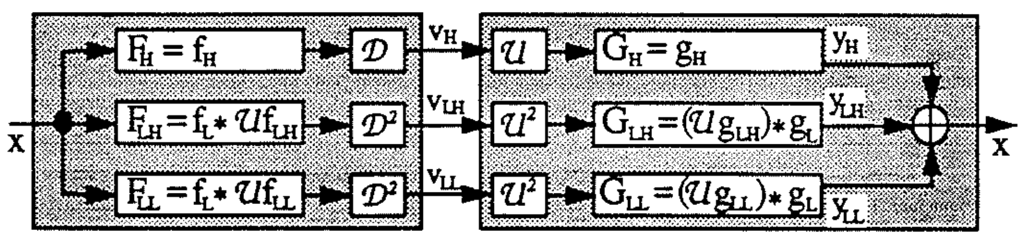

Figure 8: Direct decomposition into three orthogonal subspaces.

For the infinite–dimensional case define $X_i = \mathbf{Z}$, $i \in \mathbf{Z}^+$. For the finite–dimensional case define $X_i = \mathbf{Z}_{N2^{1-i}}$, $i \in \mathbf{Z}^+$. With X_i so defined, the following lemma and theorem hold simultaneously for both finite and infinite–dimensional signal spaces.

Lemma 6. *For the finite–dimensional case assume $2^n | N$. Let $i \in \{1, \ldots, n\}$. Then,*

(A). *For $f \in l^2(X_i)$ and $h \in l^1(X_{i+1})$, $(\mathcal{D}f) * h = \mathcal{D}(f * \mathcal{U}h) \in l^2(X_{i+1})$.*

(B). *For $f \in l^2(X_{i+1})$ and $h \in l^1(X_{i+1})$, $\mathcal{U}(f * h) = (\mathcal{U}f) * (\mathcal{U}h) \in l^2(X_i)$.*

Proof of Lemma 6.

(A). For $n \in X_{i+1}$,

$$(\mathcal{D}(f * \mathcal{U}h))(n) = (f * \mathcal{U}h)(2n) = \sum_{m \in X_i} f(2n-m)\,(\mathcal{U}h)(m) = \sum_{k \in X_{i+1}} f(2n-2k)\,h(k) \quad (93)$$

$$= \sum_{k \in X_{i+1}} (\mathcal{D}f)(n-k)\,h(k) = ((\mathcal{D}f) * h)(n). \quad (94)$$

(B). Suppose $n \in X_i$. If $n = 2m$ for some $m \in X_{i+1}$, then

$$(\mathcal{U}(f * h))(n) = (f * h)(m) = \sum_{k \in X_{i+1}} f(m-k)\,h(k) = \sum_{k \in X_{i+1}} (\mathcal{U}f)(2m-2k)\,(\mathcal{U}h)(2k) \quad (95)$$

$$= \sum_{l \in X_i} (\mathcal{U}f)(n-l)\,(\mathcal{U}h)(l) = ((\mathcal{U}f) * (\mathcal{U}h))(n). \quad (96)$$

Equation (96) follows from (95) because $(\mathcal{U}h)(l) = 0$ unless $l = 2k$ for some $k \in X_{i+1}$.

If n is not of the form $2m$, $m \in X_{i+1}$, then $(\mathcal{U}(f * h))(n) = 0$. However, in this case, $((\mathcal{U}f) * (\mathcal{U}h))(n) = \sum_{l \in X_i} (\mathcal{U}f)(n-l)\,(\mathcal{U}h)(l)$ is zero since $(\mathcal{U}h)(l) \neq 0$ only when $l = 2k$, some $k \in X_{i+1}$, in which case $(\mathcal{U}f)(n-l) = 0$. □

Theorem 3. *Let $\alpha \in \{L, H\}^*$, $|\alpha| = n > 0$, be fixed. Let $2^n | N$. Define $\beta_i = \mathrm{pfx}(\alpha, i)$, $i = 1, \ldots, n$.*

(A). *Suppose $x \in l^2(X_1)$ and $f_{\beta_i} \in l^1(X_i)$. Then*

$$\mathcal{D}(f_{\beta_n} * \mathcal{D}(f_{\beta_{n-1}} * \ldots * \mathcal{D}(f_{\beta_2} * \mathcal{D}(f_{\beta_1} * x))\ldots)) = \mathcal{D}^n\left(x * \left(\mathop{*}_{i=1}^{n} \left(\mathcal{U}^{i-1} f_{\beta_i}\right)\right)\right) \quad (97)$$

$$= \mathcal{D}^n\left(x * F_\alpha\right). \quad (98)$$

(B). *Suppose $v_\alpha \in l^2(X_{n+1})$ and $g_{\beta_i} \in l^1(X_i)$. Then*

$$g_{\beta_1} * (\mathcal{U}(g_{\beta_2} * \ldots * \mathcal{U}(g_{\beta_{n-1}} * (\mathcal{U}(g_{\beta_n} * \mathcal{U}v_\alpha))\ldots)) = (\mathcal{U}^n v_\alpha) * \left(\mathop{*}_{i=1}^{n} \left(\mathcal{U}^{i-1} g_{\beta_i}\right)\right) \quad (99)$$

$$= (\mathcal{U}^n v_\alpha) * G_\alpha. \quad (100)$$

Each part of Theorem 3 follows easily by the repeated use of the corresponding part of Lemma 6. This theorem explains the method for determining the non-recursive equivalent of a recursive structure. In particular, it establishes the equivalence of Figure 3 and Figure 8.

4.3. The Wavelet Basis

The knowledge of equivalent non–recursive filters makes it easy to determine the basis vectors whose inner–products with the signal yield the transform–domain representation of the signal.

Let $\alpha \in \{L, H\}^*$ be a string. Define

$$\tilde{B}_\alpha = \left\{ R^{2^{|\alpha|}k} \tilde{F}_\alpha \right\}_{k \in X_{|\alpha|+1}} \tag{101}$$

$$V_\alpha = \mathrm{cl}\left(\mathrm{span}\left(\tilde{B}_\alpha\right)\right). \tag{102}$$

The definition of $\tilde{B}_\alpha$ in (101) is consistent with the definitions of $\tilde{B}_L$ and $\tilde{B}_H$ in (12) and (13). $\tilde{B}_\alpha$ consists of time–shifted copies of the conjugate time–reversed non–recursive equivalent filter $\tilde{F}_\alpha$. V_α is defined to be the subspace of $V = l^2(\mathbf{Z})$ or $\mathbf{C}^N$ spanned by $\tilde{B}_\alpha$. We have seen in Section 2 that $\tilde{B}_L, \tilde{B}_H$ form the bases for $V_L, V_H \subseteq V$. We now show that recursion splits a subspace V_α into an orthogonal direct sum of subspaces $V_{\alpha L}$ and $V_{\alpha H}$.

Theorem 4. (the iteration theorem) *Let $\alpha \in \{L, H\}^*$, $|\alpha| > 0$, be fixed. In the finite-dimensional case let $2^{(|\alpha|+1)}|N$. Let $\beta \in \{L, H\}$. Suppose that $\tilde{B}_\alpha$ is an orthonormal basis for V_α, $V_\alpha \subseteq l^2(X_1)$. Suppose that $f_{\alpha L}, f_{\alpha H} \in l^1(X_{|\alpha|+1})$ are chosen such that the matrix*

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \hat{f}_{\alpha L}(z) & \hat{f}_{\alpha H}(z) \\ \hat{f}_{\alpha L}(-z) & \hat{f}_{\alpha H}(-z) \end{pmatrix} \tag{103}$$

is unitary for $\forall z \in \mathbf{T}$ or $\mathbf{W}_{N2^{-|\alpha|}}$ (as in the statement of Theorem 1). Let

$$\tilde{F}_{\alpha\beta} = \tilde{F}_\alpha * \mathcal{U}^{|\alpha|} \tilde{f}_{\alpha\beta}. \tag{104}$$

Let $\tilde{B}_{\alpha\beta}$ and $V_{\alpha\beta}$ be as defined in (101) and (102). Then V_α is the orthogonal direct sum of $\{V_{\alpha\beta}\}_\beta$, $V_\alpha = V_{\alpha L} \oplus V_{\alpha H}$. Also $\tilde{B}_{\alpha\beta}$ is an orthonormal basis for $V_{\alpha\beta}$.

In Theorem 4 we may regard $\tilde{F}_\alpha$ as being determined by the filters $\{f_\gamma\}_\gamma$ where γ is some prefix of α; but this is not essential. This theorem will be proved in its multidimensional form as Theorem 8.

In any recursive structure that is characterized by a set of analysis or synthesis branches labelled with distinct strings α, the non-recursive equivalent filters $\{F_\alpha\}_\alpha$, taken together, are said to generate the *wavelet basis*

$$\tilde{B} = \bigcup_\alpha \tilde{B}_\alpha = \bigcup_\alpha \left\{ R^{2^{|\alpha|}k} \tilde{F}_\alpha \right\}_{k \in X_{|\alpha|+1}} \tag{105}$$

of the signal space $V = l^2(\mathbf{Z})$ or $\mathbf{C}^N$. For example, in Figures 3 and 8 the wavelet basis is $\tilde{B} = \tilde{B}_{LL} \cup \tilde{B}_{LH} \cup \tilde{B}_H$. We show below that the sequences v_α consists of numbers called *wavelet transform coefficients* that are the inner-products of the signal x computed against the vectors in $\tilde{B}_\alpha$. The numbers in $\{v_\alpha\}_\alpha$ completely determine the signal x. We show also that y_α's are linear combinations of the vectors in $\tilde{B}_\alpha$, and the weighting coefficients are the elements of v_α. Then, by (102), $y_\alpha \in V_\alpha$.

If x is an input signal, and if the filters along a recursion path are $f_{\beta_1}, \ldots, f_{\beta_{|\alpha|}}$ in the terminology of Theorem 3, then the analysis or transformation part consists of the computation of the following sequences: first $x * f_{\beta_1}$, then $\mathcal{D}(x * f_{\beta_1})$, then $f_{\beta_2} * (\mathcal{D}(x * f_{\beta_1}))$, then $\mathcal{D}(f_{\beta 2} * (\mathcal{D}(x * f_{\beta_1})))$, and so on. The result after $|\alpha|$ filtering and downsampling steps is $\mathcal{D}^{|\alpha|}(x * F_\alpha)$ by (98). Note that for $k \in X_{|\alpha|+1}$,

$$\begin{aligned} v_\alpha(k) &= \left(\mathcal{D}^{|\alpha|}(x * F_\alpha)\right)(k) = (x * F_\alpha)\left(2^{|\alpha|}k\right) = \sum_{l \in X_1} x(l)\, F_\alpha\left(2^{|\alpha|}k - l\right) && (106) \\ &= \sum_{l \in X_1} x(l)\, \overline{\tilde{F}_\alpha}\left(l - 2^{|\alpha|}k\right) = \sum_{l \in X_1} x(l)\, \overline{\left(R^{2^{|\alpha|}k} \tilde{F}_\alpha\right)(l)} && (107) \\ &= \left\langle x, R^{2^{|\alpha|}k} \tilde{F}_\alpha \right\rangle. && (108) \end{aligned}$$

The output v_α of the analysis part is exactly the set of coefficients in the wavelet expansion of x corresponding to the wavelet basis functions $\tilde{B}_\alpha = \{R^{2^{|\alpha|}k} \tilde{F}_\alpha\}_{k \in X_{|\alpha|+1}}$.

Let $v_\alpha = (\mathcal{D}^{|\alpha|}(x * F_\alpha))$ be the output of the analyzing step. Consider the output of the corresponding synthesizing or reconstruction step. This output is the result of the following sequence of computations:

$$v_\alpha \to \mathcal{U} v_\alpha \to g_{\beta_{|\alpha|}} * \mathcal{U} v_\alpha \to \mathcal{U}\left(g_{\beta_{|\alpha|}} * \mathcal{U} v_\alpha\right) \to g_{\beta_{|\alpha|-1}} * \mathcal{U}\left(g_{\beta_{|\alpha|}} * \mathcal{U} v_\alpha\right) \to \ldots \tag{109}$$

By part B of Theorem 3, the final output after $|\alpha|$ steps is $(\mathcal{U}^{|\alpha|}v_\alpha) * G_\alpha$. Note that for $n \in X_1$,

$$\left(\left(\mathcal{U}^{|\alpha|}v_\alpha\right) * G_\alpha\right)(n) = \sum_{l \in X_1} \left(\mathcal{U}^{|\alpha|}v_\alpha\right)(l)\, G_\alpha(n-l) = \sum_{k \in X_{|\alpha|+1}} v_\alpha(k)\, G_\alpha\left(n - 2^{|\alpha|}k\right) \tag{110}$$

$$= \sum_{k \in X_{|\alpha|+1}} v_\alpha(k) \left(R^{2^{|\alpha|}k} G_\alpha\right)(n). \tag{111}$$

That is

$$\left(\left(\mathcal{U}^{|\alpha|}v_\alpha\right) * G_\alpha\right) = \sum_{k \in X_{|\alpha|+1}} v_\alpha(k) \left(R^{2^{|\alpha|}k} G_\alpha\right). \tag{112}$$

If we had selected $f_{\beta_1}, \ldots, f_{\beta_\alpha}$ so as to obtain orthonormal wavelets $\{R^{2^{|\alpha|}k} \tilde{F}_\alpha\}_{k \in X_{|\alpha|+1}}$, and if according to the perfect reconstruction condition we choose $g_{\beta_i} = \tilde{f}_{\beta_i}$ so that $G_\alpha = \tilde{F}_\alpha$, then from (108) and (112) the output of the α branch can be written:

$$y_\alpha = \sum_{k \in X_{|\alpha|+1}} \left\langle x, R^{2^{|\alpha|}k} \tilde{F}_\alpha \right\rangle R^{2^{|\alpha|}k} \tilde{F}_\alpha. \tag{113}$$

4.4. Recursion With Repeated Filters

Consider the choice of the level–2 filters f_{LL}, f_{LH}, f_{HL}, f_{HH}; of the level–3 filters f_{LLL}, f_{LLH}, f_{LHL}, etc.; and so on. It is possible to construct the filters independently and differently at each level acording to the prescription in Theorem 1. It is possible also to derive the filters at levels higher than the first from the level–1 filters f_L and f_H.

In case of filters $f \in l^1(\mathbf{Z})$, we can clearly choose $f_{\alpha L} = f_L$ and $f_{\alpha H} = f_H$, where α is some string over the alphabet $\{L, H\}$. This does not quite make sense in the case of finite signals since, for example, if f_L lies in $\mathbf{C}^N$ then $f_{\alpha L}$ lies in $\mathbf{C}^{N2^{-|\alpha|}}$. If f_L and f_H are such that $A(z)$ in (30) is unitary $\forall z \in \mathbf{W}_N$, then (103) will be unitary $\forall z \in \mathbf{W}_{N2^{-|\alpha|}}$ if we can determine $f_{\alpha L}$ and $f_{\alpha H}$ so that $\hat{f}_{\alpha L}(z) = \hat{f}_L(z)$ and $\hat{f}_{\alpha H}(z) = \hat{f}_H(z)$ for all $z \in \mathbf{W}_{N2^{-|\alpha|}}$. The next two lemmas tells us how to do this.

Lemma 7. (the folding lemma) *Let $f \in \mathbf{C}^N$, $h \in \mathbf{C}^{N/2}$. Then the following are equivalent:*

P1: $\hat{h}(z) = \hat{f}(z)$, $\forall z \in \mathbf{W}_{N/2}$.

P2: $(\mathcal{F}h)(m) = (\mathcal{F}f)(2m)$, $\forall m \in \mathbf{Z}_{N/2}$.

P3: $h(n) = f(n) + f(n + N/2)$, $\forall n \in \mathbf{Z}_{N/2}$.

The folding lemma will be proved in a general form as Lemma 15 when we discuss the wavelet decomposition of multidimensional signals. This lemma indicates that we can obtain $f_{LL} \in \mathbf{C}^{N/2}$ from $f_L \in \mathbf{C}^N$, for example, by "folding" f_L; i.e. by breaking f_L into two halves, placing one half on top of the other, and summing pairwise: $f_{LL}(n) = f_L(n) + f_L(n+N/2)$. Similarly, to obtain f_{LH} we compute $f_H(n) + f_H(n+N/2)$. If $4|N$, then we can continue by defining, for example, $f_{HHL} \in \mathbf{C}^{N/4}$ by $f_{HHL}(n) = f_{HL}(n) + f_{HL}(n + N/4) = [f_L(n) + f_L(n + N/2)] + [f_L(n + N/4) + f_L(n + 3N/4)]$, for $n \in \mathbf{Z}_{N/4}$. The following lemma is immediate from Lemma 7.

Lemma 8. *Let $2^{|\alpha|}|N$, $f \in \mathbf{C}^N$, $h \in \mathbf{C}^{N2^{-|\alpha|}}$. Then the following are equivalent:*

P1: $\hat{h}(z) = \hat{f}(z)$, $\forall z \in \mathbf{W}_{N2^{-|\alpha|}}$.

P2: $(\mathcal{F}h)(m) = (\mathcal{F}f)(2^{|\alpha|}m)$, $\forall m \in \mathbf{Z}_{N2^{-|\alpha|}}$.

P3: $h(n) = \sum_{i=0}^{[2^{|\alpha|}-1]} f(n + iN2^{-|\alpha|})$, $\forall n \in \mathbf{Z}_{N2^{-|\alpha|}}$.

4.5. A Summary Theorem For Classical Wavelet Analysis

The following theorem summarizes the results of this section for the case of *classical* wavelet analysis, where recursion is performed only in the low–pass branch using repeated filters. In this special case the sequence of non–recursive analyzing filters is $F_H, F_{LH}, F_{LLH}, \ldots, F_{\alpha H}, F_{\alpha L}$, where $\alpha \in \{L\}^*$.

Theorem 5. *Suppose $N \in \mathbf{Z}$, $\alpha \in \{L\}^*$, $2^{(|\alpha|+1)}|N$, and β some prefix of α, $0 \leq |\beta| \leq |\alpha|$. Suppose $f_L, f_H \in l^1(\mathbf{Z})$ or $\mathbf{C}^N$ are such that the system matrix in (30) is unitary for all $z \in \mathbf{T}$ or $\mathbf{W}_N$. Suppose that $F_{\alpha L}$ and $F_{\beta H}$ satisfy*

$$\hat{F}_{\alpha L}(z) = \prod_{i=1}^{|\alpha|+1} \hat{f}_L\left(z^{2^{(i-1)}}\right) \tag{114}$$

$$\hat{F}_{\beta H}(z) = \hat{f}_H\left(z^{2^{|\beta|}}\right) \prod_{i=1}^{|\beta|} \hat{f}_L\left(z^{2^{(i-1)}}\right). \tag{115}$$

Then,

$$\tilde{B} = \tilde{B}_{\alpha L} \cup \left(\bigcup_{\substack{\beta \\ 0 \leq |\beta| \leq |\alpha|}} \tilde{B}_{\beta H} \right) \tag{116}$$

is an orthonormal wavelet basis for $V = l^2(\mathbf{Z})$ or $\mathbf{C}^N$.

Proof of Theorem 5. Define $f_{\alpha L} \in l^1(X_{|\alpha|+1})$ and $f_{\beta H} \in l^1(X_{|\beta|+1})$ such that $\hat{f}_{\alpha L}(z) = \hat{f}_L(z)$ and $\hat{f}_{\beta H}(z) = \hat{f}_H(z)$. Then (by Lemma 8 and (37)) (114) and (115) are consistent with (91). The result follows by an induction argument based on the iteration theorem, where we consistently factor the subspace $V_{\beta L}$ through recursion. □

In the finite–dimensional case, the following expressions for the Fourier transforms of $F_{\alpha L}$ and $F_{\beta H}$ are immediate from the equations (114) and (115):

$$(\mathcal{F}F_{\alpha L})(m) = \prod_{i=1}^{|\alpha|+1} (\mathcal{F}f_L)(2^{(i-1)}m) \tag{117}$$

$$(\mathcal{F}F_{\beta H})(m) = (\mathcal{F}F_H)(2^{|\beta|}m) \prod_{i=1}^{|\beta|} (\mathcal{F}F_L)(2^{(i-1)}m). \tag{118}$$

4.6. Shannon And Real Shannon Wavelet Bases In Classical Wavelet Analysis

Let β be any prefix of $\alpha \in \{L\}^*$. For $F_{\beta H}, F_{\alpha L} \in \mathbb{C}^N$, $2^{(|\alpha|+2)}|N$, an induction argument using (117) and (118) gives the following values for the non-recursive Shannon filters:

$$(\mathcal{F}F_{\alpha L})(m) = \begin{cases} 2^{(|\alpha|+1)/2} & ,0 \le m \le \left(N2^{-(|\alpha|+2)} - 1\right) \text{ or} \\ & \left(1 - 2^{-(|\alpha|+2)}\right) N \le m \le (N-1) \\ 0 & ,\text{otherwise} \end{cases} \tag{119}$$

$$(\mathcal{F}F_{\beta H})(m) = \begin{cases} 2^{(|\beta|+1)/2} & ,N2^{-(|\beta|+2)} \le m \le \left(N2^{-(|\beta|+1)} - 1\right) \text{ or} \\ & \left(1 - 2^{-(|\beta|+1)}\right) N \le m \le \left(\left(1 - 2^{-(|\beta|+2)}\right) N - 1\right) \\ 0 & ,\text{otherwise.} \end{cases} \tag{120}$$

Notice that the supports of $\{F_{\alpha L}, F_{\beta H}\}_\beta$ are disjoint in the frequency domain.

Similarly, for the real Shannon filters we have:

$$(\mathcal{F}F_{\alpha L})(m) = \begin{cases} 2^{(|\alpha|+1)/2} & ,0 \le m \le \left(N2^{-(|\alpha|+2)} - 1\right) \text{ or} \\ & \left(\left(1 - 2^{-(|\alpha|+2)}\right) N + 1\right) \le m \le (N-1) \\ j2^{|\alpha|/2} & ,m = N2^{-(|\alpha|+2)} \\ -j2^{|\alpha|/2} & ,m = \left(1 - 2^{-(|\alpha|+2)}\right) N \\ 0 & ,\text{otherwise} \end{cases} \tag{121}$$

$$(\mathcal{F}F_{\beta H})(m) = \begin{cases} 2^{|\beta|/2} & ,m = N2^{-(|\beta|+2)} \text{ or } m = \left(1 - 2^{-(|\beta|+2)}\right) N \\ 2^{(|\beta|+1)/2} & ,\left(N2^{-(|\beta|+2)} + 1\right) \le m \le \left(N2^{-(|\beta|+1)} - 1\right) \text{ or} \\ & \left(\left(1 - 2^{-(|\beta|+1)}\right) N + 1\right) \le m \le \left(\left(1 - 2^{-(|\beta|+2)}\right) N - 1\right) \\ j2^{|\beta|/2} & ,m = N2^{-(|\beta|+1)} \\ -j2^{|\beta|/2} & ,m = \left(1 - 2^{-(|\beta|+1)}\right) N \\ 0 & ,\text{otherwise.} \end{cases} \tag{122}$$

The real Shannon filters $\{F_{\alpha L}, F_{\beta H}\}_\beta$ are real-valued; and if any two overlap in the frequency-domain they do so only at their end-points.

4.7. Computational Complexity

In this subsection we discuss signal spaces $\mathbb{C}^N$. The case of classical wavelet analysis is considered, where recursion is performed only in the low-pass branch. Let M be the number of levels of analysis, e.g. $M = 2$ in Figures 3 and 8.

If a non-recursive structure like that in Figure 8 is employed, and convolution is performed through the computation of dot-products in the frequency-domain, then the computation of the wavelet coefficients in all the sequences v_α (the analysis step) requires

$$2(M+1)N \log N + (M+1)N = O(MN \log N) \tag{123}$$

complex multiplies, and a little fewer number of complex additions.

The recursive structure in Figure 3 is more efficient and requires

$$2\sum_{i=0}^{M-1}\left(\frac{N}{2^i}\log\left(\frac{N}{2^i}\right)\right)+\sum_{i=0}^{M-1}\left(\frac{N}{2^i}\right)+\frac{N}{2^{M+1}}\log\left(\frac{N}{2^{M+1}}\right)+\frac{N}{2^{M+1}} \tag{124}$$

$$=\left(4-2^{1-M}\right)N\log N+2\left(1+2^{1-M}\right)N=\mathrm{O}(N\log N) \tag{125}$$

complex multiplies.

In both cases signal recovery from the transform coefficients in v_α takes a similar number of multiplications and additions.

If the filters selected are short in the time–domain, i.e. if they have a small support in the time–domain, then the operation count can be made linear in N. An example of such a filter–pair is the four–point *Daubechies filter–pair*. The Daubechies filters are defined by equations (129) and (130) below:

$$a=-\sqrt{3} \quad ; \quad c=\frac{1-a}{1+a} \tag{126}$$

$$h_L(n) = \begin{cases} 1 & ,n=0 \\ a & ,n=1 \\ -ac & ,n=2 \\ c & ,n=3 \\ 0 & ,4\leq n<N \end{cases} \tag{127}$$

$$h_H(n) = \begin{cases} 1 & ,n=0 \\ a & ,n=1 \\ a/c & ,n=2 \\ -1/c & ,n=3 \\ 0 & ,4\leq n<N \end{cases} \tag{128}$$

$$f_L = \frac{h_L}{\|h_L\|_{\mathbb{C}^N}} \tag{129}$$

$$f_H = \frac{h_H}{\|h_H\|_{\mathbb{C}^N}}. \tag{130}$$

M–level non–recursive wavelet analysis with filters supported on T points in the time–domain requires MTN complex multiplies when the convolutions are performed directly in the time–domain. The number of multiplies for recursive implementation is $\sum_{i=0}^{M-1}\left(TN2^{-i}\right)+TN2^{1-M}=2TN$.

5. Orthonormal Wavelets For Multidimensional Signals

Let P be a fixed positive integer. Let $x(n)$ be a P–dimensional signal in $l^2(\mathbf{Z}^P)$ or in $\mathbf{C}^{N^P} = l^2(\mathbf{Z}_N^P) = l^2((\mathbf{Z}_N)^P)$, N even. A signal in $\mathbf{C}^{N^2}$, for example, could be thought of as representing a square picture of size N pixels by N pixels. While it is trivial to extend the treatment here to cover rectangular "multidimensional pictures", we will rest content with square pictures in order to keep the notation simple. The dimensionality of a signal, and the dimensionality of the vector space in which the signal lies are two distinct concepts that must be kept apart. For example, a two–dimensional picture in $\mathbf{C}^{N^2}$ lies in an N^2–dimensional space. We will use multiindex notation when convenient, writing

$$\hat{x}(z) = \sum_n x(n) z^{-n} \tag{131}$$

for

$$\hat{x}(z_1, z_2, \ldots, z_P) = \sum_{n_1} \sum_{n_2} \ldots \sum_{n_P} x(n_1, \ldots, n_P) z_1^{-n_1} \ldots z_P^{-n_P}. \tag{132}$$

In (131) z ranges over $\mathbf{T}^P$ or $\mathbf{W}_N^P$, and n ranges over $\mathbf{Z}^P$ or $\mathbf{Z}_N^P$, depending upon whether x belongs to $l^2(\mathbf{Z}^P)$ or to $\mathbf{C}^{N^P}$.

The first lemma below states a result of general utility.

Lemma 9. *For any $n \in \mathbf{Z}^P$ or $n \in \mathbf{Z}_N^P$,*

$$\sum_{k \in \{0,1\}^P} (-1)^{(k_1 n_1 + \ldots + k_P n_P)} = \sum_{k \in \mathbf{Z}_2^P} (-1)^{k \cdot n} = \begin{cases} 0, & \text{if for some } i,\ n_i \text{ is odd} \\ 2^P, & \text{otherwise;} \end{cases} \tag{133}$$

where $k \cdot n$ denotes the dot–product $k \cdot n = k_1 n_1 + \ldots + k_P n_P$.

Proof of Lemma 9. If $n = 2m = (2m_1, \ldots, 2m_P)$ for $m \in \mathbf{Z}^P$ or $m \in \mathbf{Z}_{N/2}^P$, then each term in the sum is 1, and the sum is 2^P. Else, if n_l is odd for some l, write

$$\sum_{k \in \mathbf{Z}_2^P} (-1)^{k \cdot n} = \sum_{k_1, \ldots, k_{l-1}, k_{l+1}, \ldots, k_P \in \mathbf{Z}_2} (-1)^{(k_1 n_1 + \ldots + k_{l-1} n_{l-1} + k_{l+1} n_{l+1} + \ldots + k_P n_P)} \left(\sum_{k_l = 0}^{1} (-1)^{k_l n_l} \right). \tag{134}$$

Because n_l is odd, the last sum and, hence, the entire sum vanishes. □

Define the downsampling operator $\mathcal{D}$ by $(\mathcal{D}f)(n) = f(2n)$, $n \in \mathbf{Z}^P$ or $n \in \mathbf{Z}^P_{N/2}$. $\mathcal{D} : l^2(\mathbf{Z}^P) \to l^2(\mathbf{Z}^P)$ or $\mathcal{D} : \mathbf{C}^{N^P} \to \mathbf{C}^{(N/2)^P}$. By way of example we draw a signal in $\mathbf{C}^{4^2}$ in Figure 9. The points retained by the downsampling operation are dark-circled. The next lemma states a property of the downsampling operator concerning its action upon a filter sequence in $l^2(\mathbf{Z}^P)$ or $\mathbf{C}^{N^P}$.

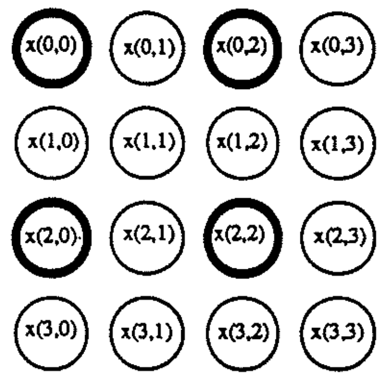

Figure 9: The downsampling operator retains the elements with dark circles.

Lemma 10. *For $z \in \mathbf{T}^P$ or ${\mathbf{W}_N}^P$,*

$$(\mathcal{D}f)\hat{}(z) = 2^{-P} \sum_{k \in \mathbf{Z}_2^P} \hat{f}\left((-1)^{k_1} {z_1}^{1/2}, \dots, (-1)^{k_P} z_P^{1/2}\right) = 2^{-P} \sum_{k \in \mathbf{Z}_2^P} \hat{f}((-1)^k z^{1/2}). \tag{135}$$

Proof of Lemma 10. For $n \in \mathbf{Z}^P$ or $n \in \mathbf{Z}_N^P$,

$$\hat{f}((-1)^k z^{1/2}) = \sum_n f(n)((-1)^k z^{1/2})^{-n} = \sum_n f(n)(-1)^{k \cdot n} z^{-n/2} \tag{136}$$

$$= \sum_n f(n)(-1)^{(k_1 n_1 + \dots + k_P n_P)} z_1^{-n_1/2} \dots z_P^{-n_P/2}. \tag{137}$$

Therefore,

$$2^{-P} \sum_{k \in \mathbf{Z}_2^P} \hat{f}((-1)^k z^{1/2}) = 2^{-P} \sum_{k \in \mathbf{Z}_2^P} \sum_n f(n)(-1)^{k \cdot n} z_1^{-n_1/2} \dots z_P^{-n_P/2}. \tag{138}$$

Interchanging the order of summation in (138), and using Lemma 9, we have

$$2^{-P}\sum_{k\in \mathbf{Z}_2^P}\hat{f}((-1)^k z^{1/2}) = \sum_{n=2m} f(n_1,\dots,n_P)z_1^{-n_1/2}\dots z_P^{-n_P/2} \tag{139}$$

$$= \sum_m f(2m_1,\dots,2m_P)z_1^{-m_1}\dots z_P^{-m_P} \tag{140}$$

$$= \sum_m (\mathcal{D}f)(m_1,\dots,m_P)z_1^{-m_1}\dots z_P^{-m_P} = (\mathcal{D}f)\hat{}(z). \tag{141}$$

□

Define the upsampling operator $\mathcal{U} : l^2(\mathbf{Z}^P) \to l^2(\mathbf{Z}^P)$ or $\mathcal{U} : \mathbf{C}^{N^P} \to \mathbf{C}^{(2N)^P}$ by

$$(\mathcal{U}f)(n) = \begin{cases} f(n/2), & \text{if } n = 2m,\ m \in \mathbf{Z}^P \text{ or } m \in \mathbf{Z}_N^P \\ 0, & \text{otherwise.} \end{cases} \tag{142}$$

It is easy to see that $(\mathcal{U}f)\hat{}(z) = \hat{f}(z^2)$, i.e. $(\mathcal{U}f)\hat{}(z_1,\dots,z_P) = \hat{f}(z_1^2,\dots,z_P^2)$.

Define the shift operator R so that $(R^k f)(n) = f(n-k)$, or

$$(R^{(k_1,\dots,k_P)}f)(n_1,\dots,n_P) = f(n_1-k_1,\dots,n_P-k_P). \tag{143}$$

It is easy to see that $(R^k f)\hat{}(z) = z^{-k}\hat{f}(z) = z_1^{-k_1}\dots z_P^{-k_P}\hat{f}(z_1,\dots,z_P)$. By way of example, Figure 10 shows the effect of the operator $R^{(2,3)}$ upon a sequence in $\mathbf{C}^{4^2}$. $R^{(2,3)}$ shifts the input sequence by two in the "direction" of z_1 (down) and by three in the "direction" of z_2 (right). The choice of the "directions" was dictated by the fact that we would like $x(i,k)$ to be the coefficient of ${z_1}^{-i}{z_2}^{-k}$.

By $\text{coeff}_k\hat{f}(z)$; $k \in \mathbf{Z}^P$ or $k \in \mathbf{Z}_N^P$; will be meant the coefficient of $z^{-k} = z_1^{-k_1}\dots z_P^{-k_P}$ in the canonical form of the polynomial $\hat{f}(z)$. For $f = \{f(n)\}_n$ and $g = \{g(n)\}_n$, $\langle f,g\rangle = \sum_n f(n)\overline{g}(n)$. It is easy to see that $\langle f, R^k g\rangle = \text{coeff}_k(\hat{f}(z)\overline{\hat{g}}(z))$; $z \in \mathbf{T}^P$ or $\mathbf{W}_N^P$.

5.1. The Orthonormality Condition

In this section we present three lemmas that lead to a multidimensional version of Theorem 1.

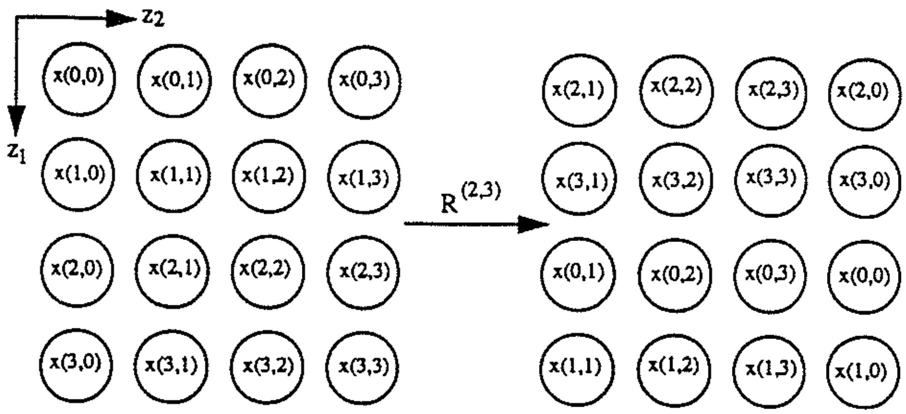

Figure 10: An Example Of The Shift Operation

Lemma 11. *Let $f = f(n)$, $n \in \mathbf{Z}^P$ or $n \in \mathbf{Z}_N^P$. Then the following are equivalent:*

P1: $\langle f, R^{2k} f\rangle = \delta(k) = \delta(k_1)\ldots\delta(k_P)$.

P2: $\sum_{l\in \mathbf{Z}_2^P} |\hat{f}((-1)^l z)|^2 = 2^P$, $\forall z \in \mathbf{T}^P$ *or* $\mathbf{W}_N^P$.

Here $\hat{f}((-1)^l z) = \hat{f}((-1)^{l_1} z_1, \ldots, (-1)^{l_P} z_P)$ *for* $l = (l_1, \ldots, l_P) \in \mathbf{Z}_2^P$.

Proof of Lemma 11. P1 is equivalent to:

$$\mathrm{coeff}_{2k}\left(|\hat{f}(z)|^2\right) = \delta(k). \tag{144}$$

Define

$$\hat{h}(z) = |\hat{f}(z)|^2. \tag{145}$$

For $m \in \mathbf{Z}^P$ or $m \in \mathbf{Z}_{N/2}^P$, (144) is equivalent to:

$$\sum_{n=2m} h(n) z^{-n} = 1. \tag{146}$$

But, $\sum_{n=2m} h(n)z^{-n} = (\mathcal{UD}h)\hat{}(z) = (\mathcal{D}h)\hat{}(z^2)$. Therefore, by Lemma 10,

$$\sum_{n=2m} h(n) z^{-n} = (\mathcal{D}h)\hat{}(z^2) = 2^{-P} \sum_{l\in \mathbf{Z}_2^P} \hat{h}((-1)^l z). \tag{147}$$

From (147), (146) is equivalent to:

$$\sum_{l\in \mathbf{Z}_2^P} \hat{h}((-1)^l z) = 2^P. \tag{148}$$

From (145), $\hat{h}((-1)^l z) = |\hat{f}((-1)^l z)|^2$. Substituting for $\hat{h}((-1)^l z)$ in (148) we have the lemma. □

Lemma 12. *Let $f = f(n)$, $g = g(n)$, $n \in \mathbf{Z}^P$ or $n \in \mathbf{Z}_N^P$. Then the following are equivalent:*

P1: $\langle f, R^{2k} g\rangle = 0$, $\forall k$.

P2: $\sum_{l\in\mathbf{Z}_2^P} \hat{f}((-1)^l z)\overline{\hat{g}}((-1)^l z) = 0$, $\forall z \in \mathbf{T}^P$ *or* $\mathbf{W}_N^P$.

Proof of Lemma 12. The proof is similar to that of Lemma 11. P1 is equivalent to:

$$\operatorname{coeff}_{2k}\left(\hat{f}(z)\overline{\hat{g}}(z)\right) = 0. \tag{149}$$

Define

$$\hat{h}(z) = \hat{f}(z)\overline{\hat{g}}(z). \tag{150}$$

From (150), for $m \in \mathbf{Z}^P$ or $m \in \mathbf{Z}_{N/2}^P$, (149) is equivalent to:

$$\sum_{n=2m} h(n)z^{-n} = 0. \tag{151}$$

By Lemma 10,

$$\sum_{n=2m} h(n)z^{-n} = (\mathcal{UD}h)\hat{}(z) = (\mathcal{D}h)\hat{}(z^2) = 2^{-P}\sum_{l\in\mathbf{Z}_2^P}\hat{h}((-1)^l z). \tag{152}$$

Substituting (150) in (152) we have the lemma. □

The proof of the following lemma is similar to the proof of Lemma 2.

Lemma 13. *For $i \in \mathbf{Z}_{2^P}$ define*

$$\begin{aligned}
\tilde{f}_i(n) &= \overline{f_i}(-n) && (153)\\
\tilde{B} &= \bigcup_{i=0}^{2^P-1}\{R^{2k}\tilde{f}_i : k \in \mathbf{Z}^P \text{ or } k \in \mathbf{Z}_{N/2}^P\} && (154)\\
B &= \bigcup_{i=0}^{2^P-1}\{R^{2k}f_i : k \in \mathbf{Z}^P \text{ or } k \in \mathbf{Z}_{N/2}^P\}. && (155)
\end{aligned}$$

The multiset $\tilde{B}$ is orthonormal if and only if the multiset B is.

Consider now the orthonormal decomposition of the signal x in Figure 11.

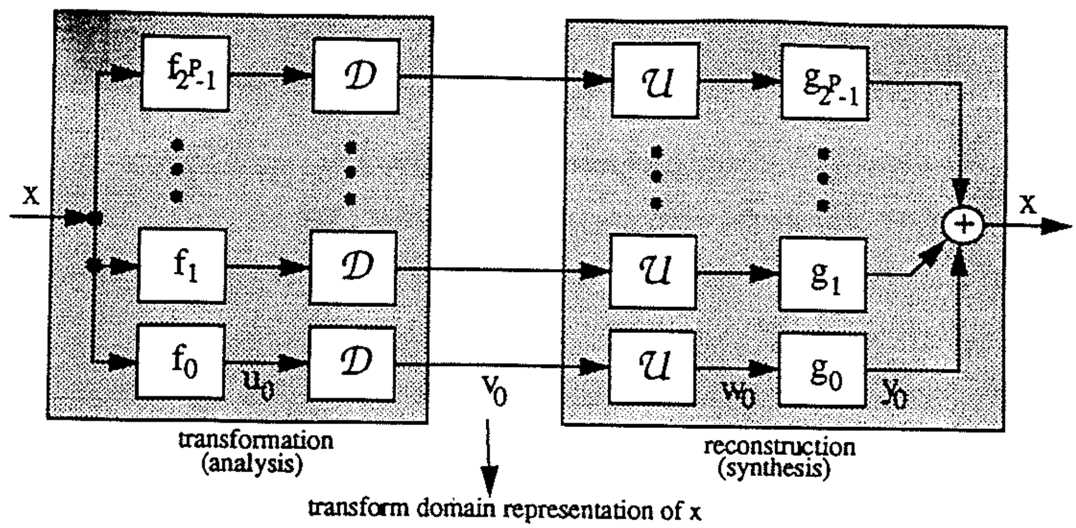

Figure 11: The basic filter–bank for a P-dimensional signal.

We need some new notation for a statement of the main theorem in this subsection. Let $j = (j_1, \ldots, j_P) \in \{0,1\}^P = \mathbf{Z}_2^P$. Since $\mathrm{card}(\{0,1\}^P) = 2^P$, we can enumerate all such j's as $j^{(0)}, \ldots, j^{(2^P-1)}$. Select this enumeration such that $j^{(0)} = (0,0,\ldots,0)$. Then, we write $j^{(i)} = (j_1^{(i)}, \ldots, j_P^{(i)})$, with $j_k^{(i)} \in \mathbf{Z}_2$ for all k. We also write $\hat{f}((-1)^{j^{(i)}}z) = \hat{f}((-1)^{j_1^{(i)}}z_1, \ldots, (-1)^{j_P^{(i)}}z_P)$.

Theorem 6. *Let $f_0(n) = f_0(n_1, \ldots, n_P), \ldots, f_{2^P-1}(n) = f_{2^P-1}(n_1, \ldots, n_P)$ be 2^P sequences, with $n \in \mathbf{Z}^P$ or $n \in \mathbf{Z}_N^P$. Define a $2^P \times 2^P$ matrix called the* system matrix $A(z) = (A_{k,i}(z))_{k,i \in \mathbf{Z}_{2^P}}$, *$z \in \mathbf{T}^P$ or $\mathbf{W}_N^P$, such that $A_{k,i}(z) = 2^{-P/2}\hat{f}_i((-1)^{j^{(k)}}z)$, i.e.,*

$$A(z) = 2^{-P/2}\begin{pmatrix} \hat{f}_0((-1)^{j^{(0)}}z) & \hat{f}_1((-1)^{j^{(0)}}z) & \ldots & \hat{f}_{2^P-1}((-1)^{j^{(0)}}z) \\ \hat{f}_0((-1)^{j^{(1)}}z) & \hat{f}_1((-1)^{j^{(1)}}z) & \ldots & \hat{f}_{2^P-1}((-1)^{j^{(1)}}z) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{f}_0((-1)^{j^{(2^P-1)}}z) & \hat{f}_1((-1)^{j^{(2^P-1)}}z) & \ldots & \hat{f}_{2^P-1}((-1)^{j^{(2^P-1)}}z) \end{pmatrix}. \tag{156}$$

Then $\tilde{B}$ is orthonormal if, and only if, $A(z)$ is unitary for all $z \in \mathbf{T}^P$ or $\mathbf{W}_N^P$.

Proof of Theorem 6. By Lemma 13, the orthonormality of $\tilde{B}$ is equivalent to the orthonormality of B. By Lemma 11 the orthonormality of $\{R^{2k}f_i : k \in \mathbf{Z}^P \text{ or } k \in \mathbf{Z}_{N/2}^P\}$ for every fixed i is equivalent to the i-th column of A having length "1". By Lemma 12 the cross–orthogonality of $\{R^{2k}f_{i_1} : k \in \mathbf{Z}^P \text{ or } k \in \mathbf{Z}_{N/2}^P\}$ and $\{R^{2k}f_{i_2} : k \in \mathbf{Z}^P \text{ or } k \in \mathbf{Z}_{N/2}^P\}$ is equivalent to the orthogonality of the i_1-th and the i_2-th columns of A. □

Theorem 6 is the higher-dimensional analog to Theorem 1 (cf. Meyer[21]). By way of example, the system matrix for the signal space C^{4^2} is as follows:

$$A(z_1,z_2)=\frac{1}{2}\begin{pmatrix} \hat{f}_0(z_1,z_2) & \hat{f}_1(z_1,z_2) & \hat{f}_2(z_1,z_2) & \hat{f}_3(z_1,z_2) \\ \hat{f}_0(-z_1,z_2) & \hat{f}_1(-z_1,z_2) & \hat{f}_2(-z_1,z_2) & \hat{f}_3(-z_1,z_2) \\ \hat{f}_0(z_1,-z_2) & \hat{f}_1(z_1,-z_2) & \hat{f}_2(z_1,-z_2) & \hat{f}_3(z_1,-z_2) \\ \hat{f}_0(-z_1,-z_2) & \hat{f}_1(-z_1,-z_2) & \hat{f}_2(-z_1,-z_2) & \hat{f}_3(-z_1,-z_2) \end{pmatrix} \tag{157}$$

5.2. Perfect Reconstruction

From Figure 11 we can see that perfect-reconstruction occurs if and only if the following equation holds for all input signals x:

$$\sum_{i=0}^{2^P-1}(g_i * \mathcal{U}\mathcal{D}(f_i * x))\hat{}(z)=\hat{x}(z). \tag{158}$$

Equation (158) will be called the *perfect reconstruction condition*.

Lemma 14. *The perfect reconstruction condition is satisfied if, and only if,*

$$\sum_{i=0}^{2^P-1}\hat{f}_i((-1)^k z)\hat{g}_i(z)=2^P\delta(k). \tag{159}$$

Proof of Lemma 14. From Lemma 10,

$$(\mathcal{D}(f_i * x))\hat{}(z)=2^{-P}\sum_{k\in\mathbf{Z}_2^P}\hat{f}_i((-1)^k z^{1/2})\hat{x}((-1)^k z^{1/2}). \tag{160}$$

From (160) it follows that

$$(\mathcal{U}\mathcal{D}(f_i * x))\hat{}(z)=2^{-P}\sum_{k\in\mathbf{Z}_2^P}\hat{f}_i((-1)^k z)\hat{x}((-1)^k z). \tag{161}$$

Using (161), the perfect reconstruction condition (158) can be written as:

$$2^{-P}\sum_{i=0}^{2^P-1}\sum_{k\in\mathbf{Z}_2^P}\hat{f}_i((-1)^k z)\hat{x}((-1)^k z)\hat{g}_i(z)=\hat{x}(z). \tag{162}$$

Define

$$\hat{C}_k(z) = \sum_{i=0}^{2^P-1} \hat{f}_i((-1)^k z)\hat{g}_i(z). \tag{163}$$

The perfect reconstruction condition in (162) now reads:

$$2^{-P} \sum_{k\in \mathbf{Z}_2^P} \hat{C}_k(z)\hat{x}((-1)^k z) = \hat{x}(z). \tag{164}$$

Define $\hat{x}_l(z) = z^l$, for $l \in \mathbf{Z}_2^P$. Since the perfect reconstruction condition must hold for all $\hat{x}(z)$, it must *in particular* hold for all the $\hat{x}_l(z)$, $l \in \mathbf{Z}_2^P$. Then for every l, from (164),

$$\hat{x}_l(z) = z^l = 2^{-P} \sum_{k\in \mathbf{Z}_2^P} ((-1)^k z)^l \hat{C}_k(z) = 2^{-P} \sum_{k\in \mathbf{Z}_2^P} (-1)^{k\cdot l} z^l \hat{C}_k(z). \tag{165}$$

From (165) it follows that, for all $l \in \mathbf{Z}_2^P$,

$$2^{-P} \sum_{k\in \mathbf{Z}_2^P} (-1)^{k\cdot l}\hat{C}_k(z) = 1. \tag{166}$$

Let $K \in \mathbf{Z}_2^P$ be any fixed binary P-tuple. Multiplying both sides in (166) by $(-1)^{K\cdot l}$, and summing over all l we have:

$$\begin{aligned}
\sum_{l\in \mathbf{Z}_2^P} (-1)^{K\cdot l} &= 2^{-P} \sum_{l\in \mathbf{Z}_2^P} \sum_{k\in \mathbf{Z}_2^P} (-1)^{k\cdot l}(-1)^{K\cdot l}\hat{C}_k(z) && (167)\\
&= 2^{-P} \sum_{l\in \mathbf{Z}_2^P} \sum_{k\in \mathbf{Z}_2^P} (-1)^{k\cdot l}(-1)^{-K\cdot l}\hat{C}_k(z) && (168)\\
&= 2^{-P} \sum_{k\in \mathbf{Z}_2^P} \hat{C}_k(z) \sum_{l\in \mathbf{Z}_2^P} (-1)^{(k-K)\cdot l}. && (169)
\end{aligned}$$

By Lemma 9, because $K \in \mathbf{Z}_2^P$,

$$\sum_{l\in \mathbf{Z}_2^P} (-1)^{K\cdot l} = 2^P \delta(K) = 2^P \delta(K_1)\ldots\delta(K_P). \tag{170}$$

Also by Lemma 9,

$$\sum_{l\in \mathbf{Z}_2^P} (-1)^{(k-K)\cdot l} = 2^P \delta(k-K). \tag{171}$$

From (167)–(171),

$$\sum_{k\in \mathbf{Z}_2^P} \hat{C}_k(z)\delta(k-K) = 2^P\delta(K), \tag{172}$$

or

$$\hat{C}_K(z) = 2^P\delta(K). \tag{173}$$

From (163) and (173) we have the lemma. □

The following theorem is immediate from Lemma 14.

Theorem 7. *Let $A(z)$ be the system matrix (156). The system in Figure 11 gives perfect reconstruction if and only if, $\forall z \in \mathbf{T}^P$ or $\mathbf{C}_N^P$,*

$$A(z)\begin{pmatrix} \hat{g}_0(z) \\ \hat{g}_1(z) \\ \vdots \\ \hat{g}_{2^P-1}(z) \end{pmatrix} = \begin{pmatrix} 2^{P/2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{174}$$

In the following corollary we summarize the results of the orthonormal decomposition and perfect reconstruction of signals in $l^2(\mathbf{Z}^P)$ and $\mathbf{C}^{N^P}$.

Corollary 1 *The following are equivalent:*

P1: *The set $\tilde{B} = \bigcup_{i=0}^{2^P-1}\{R^{2k}\tilde{f}_i : k \in \mathbf{Z}^P$ or $k \in \mathbf{Z}_{N/2}^P\}$ is orthonormal, and we have perfect reconstruction in Figure 11.*

P2: *The system matrix $A(z)$ in (156) is unitary, and $\forall i \in \mathbf{Z}_{2^P}$, $g_i = \tilde{f}_i$.*

Proof of Corollary 1 (P1 $\Rightarrow$ P2). The unitarity of $A(z)$ follows from Theorem 6. Hence $A^{-1}(z) = \overline{A}^T(z)$. From this observation and Theorem 7 follows the fact that, $\forall i \in \mathbf{Z}_{2^P}$, $\hat{g}_i(z) = \overline{\hat{\tilde{f}}_i}(z)$; or equivalently $g_i = \tilde{f}_i$.

(P1 $\Leftarrow$ P2). By Theorem 6, B is orthonormal. Since $\hat{g}_i(z) = \overline{\hat{f}_i}(z)$, the equation

$$A(z)\begin{pmatrix} 2^{-P/2}\hat{g}_0(z) \\ 2^{-P/2}\hat{g}_1(z) \\ \vdots \\ 2^{-P/2}\hat{g}_{2^P-1}(z) \end{pmatrix} = A(z)\begin{pmatrix} 2^{-P/2}\overline{\hat{f}_0}(z) \\ 2^{-P/2}\overline{\hat{f}_1}(z) \\ \vdots \\ 2^{-P/2}\overline{\hat{f}_{2^P-1}}(z) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{175}$$

is true, because the rows of $A(z)$ are orthonormal by the unitarity of $A(z)$. Then perfect reconstruction follows by Theorem 7. □

5.3. Multidimensional Product Filters

Suppose $\hat{f}_L$ and $\hat{f}_H$ are one–dimensional filters such that

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \hat{f}_L(z) & \hat{f}_H(z) \\ \hat{f}_L(-z) & \hat{f}_H(-z) \end{pmatrix} \tag{176}$$

is unitary for all $z \in \mathbf{T}$ or $\mathbf{W}_N$. Define

$$f_0(n_1, n_2) = f_L(n_1) f_L(n_2) \tag{177}$$

$$f_1(n_1, n_2) = f_H(n_1) f_L(n_2) \tag{178}$$

$$f_2(n_1, n_2) = f_L(n_1) f_H(n_2) \tag{179}$$

$$f_3(n_1, n_2) = f_H(n_1) f_H(n_2). \tag{180}$$

With this definition, the matrix (157) is a Kronecker product of two matrices of form (176):

$$A(z_1, z_2) = \frac{1}{2}\begin{pmatrix} \hat{f}_0(z_1,z_2) & \hat{f}_1(z_1,z_2) & \hat{f}_2(z_1,z_2) & \hat{f}_3(z_1,z_2) \\ \hat{f}_0(-z_1,z_2) & \hat{f}_1(-z_1,z_2) & \hat{f}_2(-z_1,z_2) & \hat{f}_3(-z_1,z_2) \\ \hat{f}_0(z_1,-z_2) & \hat{f}_1(z_1,-z_2) & \hat{f}_2(z_1,-z_2) & \hat{f}_3(z_1,-z_2) \\ \hat{f}_0(-z_1,-z_2) & \hat{f}_1(-z_1,-z_2) & \hat{f}_2(-z_1,-z_2) & \hat{f}_3(-z_1,-z_2) \end{pmatrix} \tag{181}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} \hat{f}_L(z_2) & \hat{f}_H(z_2) \\ \hat{f}_L(-z_2) & \hat{f}_H(-z_2) \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} \hat{f}_L(z_1) & \hat{f}_H(z_1) \\ \hat{f}_L(-z_1) & \hat{f}_H(-z_1) \end{pmatrix}. \tag{182}$$

It is easy to check directly that the orthonormality of the set $\{R^{2k} f_L\}_k \cup \{R^{2k} f_H\}_k$ implies the orthonormality of the set $\tilde{B}$ in (154), for $f_0, \ldots, f_3$ given by (177)–(180). This is the usual way of constructing two–dimensional wavelets from one–dimensional wavelets. We call $f_0, \ldots, f_3$ *product wavelets*. This is consistent with Theorem 6, since the unitarity of the matrix (181) follows from the unitarity of the matrices in the Kronecker product formula in (182).

Thus, two–dimensional wavelet filters can be constructed as products of one–dimensional filters. However, Theorem 6 shows that there exist two–dimensional wavelet filters that are not product filters. This follows from the existence of the following unitary matrix, which cannot be expressed as a Kronecker product:

$$\frac{1}{4}\begin{pmatrix} \sqrt{2} & 0 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & \sqrt{2} & 0 \\ 0 & \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}. \tag{183}$$

These arguments can be extended to the construction of general multidimensional filters.

5.4. Recursion

Let $V = l^2(\mathbf{Z}^P)$ or C_N^P. Let $\alpha \in \mathbf{Z}_{2^P}^*$, $|\alpha| = n > 0$, be fixed. Define $\beta_i = \mathrm{pfx}(\alpha, i)$, $i = 1, \ldots, n$. Let $F_\alpha, G_\alpha \in V$ be defined as in (91) and (92). For $k = 1, \ldots, n+1$, let $X_k = \mathbf{Z}^P$ in the infinite–dimensional case. In the finite–dimensional case assume $2^n|N$, and let $X_k = \mathbf{Z}^P_{N2^{1-k}}$ for $k = 1, \ldots, n+1$. Then Theorem 3 carries over to the multidimensional case.

Let $\tilde{B}_\alpha$ and V_α be defined exactly as in (101) and (102). We now prove the following analog to Theorem 4:

Theorem 8. (the iteration theorem) *Let $\alpha \in \mathbf{Z}_{2^P}^*$, $|\alpha| > 0$, be fixed. In the finite-dimensional case let $2^{(|\alpha|+1)}|N$. Let $\beta \in \mathbf{Z}_{2^P}$. Suppose that $\tilde{B}_\alpha$ is an orthonormal basis for V_α, $V_\alpha \subseteq l^2(X_1)$. Suppose that $\{f_{\alpha\beta} \in X_{|\alpha|+1}\}_\beta = \{f_{\alpha 0}, f_{\alpha 1}, \ldots, f_{\alpha(2^P-1)}\}$ satisfy Theorem 6. Let*

$$\tilde{F}_{\alpha\beta} = \tilde{F}_\alpha * \mathcal{U}^{|\alpha|}\tilde{f}_{\alpha\beta}. \tag{184}$$

Let $\tilde{B}_{\alpha\beta}$ and $V_{\alpha\beta}$ be as defined in (101) and (102). Then V_α is the orthogonal direct sum of $\{V_{\alpha\beta}\}_\beta$,

$$V_\alpha = \bigoplus\nolimits_\beta V_{\alpha\beta}. \tag{185}$$

Also $\tilde{B}_{\alpha\beta}$ is an orthonormal basis for $V_{\alpha\beta}$.

Proof of Theorem 8. We first demonstrate the orthonormality of $\bigcup_\beta \tilde{B}_{\alpha\beta}$. In order to do that it suffices to prove that

$$\left\langle \tilde{F}_{\alpha\beta_1}, R^{2^{(|\alpha|+1)}k}\tilde{F}_{\alpha\beta_2} \right\rangle = \begin{cases} 1 & ,k = 0 \text{ and } \beta_1 = \beta_2 \\ 0 & ,\text{otherwise.} \end{cases} \tag{186}$$

We have:

$$\begin{aligned}
\left\langle \tilde{F}_{\alpha\beta_1}, R^{2^{(|\alpha|+1)}k}\tilde{F}_{\alpha\beta_2}\right\rangle &= \operatorname{coeff}_0\left(\left(\tilde{F}_{\alpha\beta_1}\right)^{\wedge}(z)\,\overline{\left(R^{2^{(|\alpha|+1)}k}\tilde{F}_{\alpha\beta_2}\right)^{\wedge}(z)}\right) && (187)\\
&= \operatorname{coeff}_0\left(\overline{\hat{F}_{\alpha\beta_1}}(z)\, z^{2^{(|\alpha|+1)}k}\,\hat{F}_{\alpha\beta_2}(z)\right) && (188)\\
&= \operatorname{coeff}_{2^{(|\alpha|+1)}k}\left(\overline{\hat{F}_{\alpha\beta_1}}(z)\,\hat{F}_{\alpha\beta_2}(z)\right). && (189)
\end{aligned}$$

From (184),

$$\hat{F}_{\alpha\beta}(z) = \hat{F}_{\alpha}(z)\,\left(\mathcal{U}^{|\alpha|}f_{\alpha\beta}\right)^{\wedge}(z) = \hat{F}_{\alpha}(z)\,\hat{f}_{\alpha\beta}\left(z^{2^{|\alpha|}}\right). \tag{190}$$

From (189) and (190),

$$\left\langle \tilde{F}_{\alpha\beta_1}, R^{2^{(|\alpha|+1)}k}\tilde{F}_{\alpha\beta_2}\right\rangle = \operatorname{coeff}_{2^{(|\alpha|+1)}k}\left(\left|\hat{F}_{\alpha}\right|^2(z)\,\overline{\hat{f}_{\alpha\beta_1}}\left(z^{2^{|\alpha|}}\right)\,\hat{f}_{\alpha\beta_2}\left(z^{2^{|\alpha|}}\right)\right). \tag{191}$$

By the orthonormality of $\tilde{B}_{\alpha} = \left\{R^{2^{|\alpha|}k}\tilde{F}_{\alpha}\right\}_{k\in X_{|\alpha|+1}}$,

$$\left\langle \tilde{F}_{\alpha}, R^{2^{|\alpha|}k}\tilde{F}_{\alpha}\right\rangle = \operatorname{coeff}_{2^{|\alpha|}k}\left(\left|\hat{F}_{\alpha}\right|^2(z)\right) = \delta(k). \tag{192}$$

From (192) it follows that

$$\left|\hat{F}_{\alpha}\right|^2(z) = 1 + \sum_{\substack{l\in X_1\\ 2^{|\alpha|}\nmid l}} a(l)z^{-l} \tag{193}$$

for some $a(l)$. Here by "$2^{|\alpha|}\nmid l$" we mean that $\exists i, 2^{|\alpha|}\nmid l_i$, where l_i is the ith component of the vector $l \in X_1 = \mathbf{Z}^P$ or $\mathbf{Z}_N^P$.

Similarly, by the orthonormality of $\{f_{\alpha\beta} \in l^2(X_{|\alpha|+1})\}_{\beta\in\mathbf{Z}_{2^P}}$,

$$\left\langle f_{\alpha\beta_1}, R^{2k}f_{\alpha\beta_2}\right\rangle = \operatorname{coeff}_{2k}\left(\hat{f}_{\alpha\beta_1}(z)\overline{\hat{f}_{\alpha\beta_2}}(z)\right) = \delta(k)\delta(\beta_1-\beta_2) \tag{194}$$

From (194) it follows that

$$\hat{f}_{\alpha\beta_1}(z)\overline{\hat{f}_{\alpha\beta_2}}(z) = \delta(\beta_1-\beta_2) + \sum_{\substack{l\in X_{|\alpha|+1}\\ 2\nmid l}} b(l)z^{-l} \tag{195}$$

for some $b(l)$. Again, by "$2\not| l$" we mean that $\exists i, 2\not| l_i$. From (191), (193), and (195) we have

$$\left\langle \tilde{F}_{\alpha\beta_1}, R^{2^{(|\alpha|+1)}k}\tilde{F}_{\alpha\beta_2}\right\rangle = \operatorname{coeff}_{2^{|\alpha|+1}k}\left(\delta(\beta_1-\beta_2)\left(1+\sum_{\substack{l\in X_1\\ 2^{|\alpha|}\not| l}} a(l)z^{-l}\right)+\sum_{\substack{l\in X_{|\alpha|+1}\\ 2\not| l}} b(l)z^{-l2^{|\alpha|}}+\right.$$

$$\left.\sum_{\substack{l\in X_1\\ 2^{|\alpha|}\not| l}}\ \sum_{\substack{m\in X_{|\alpha|+1}\\ 2\not| m}} a(l)b(m)z^{-(l+m2^{|\alpha|})}\right). \tag{196}$$

Equation (186) follows from (196), and $\bigcup_\beta \tilde{B}_{\alpha\beta}$ is orthonormal.

We now show that $\left(\bigcup_\beta \tilde{B}_{\alpha\beta}\right) \subseteq \mathrm{cl}\left(\mathrm{span}\left(\tilde{B}_\alpha\right)\right) = V_\alpha$. For $n \in X_1$,

$$\tilde{F}_{\alpha\beta}(n) = \left(\tilde{F}_\alpha * \mathcal{U}^{|\alpha|}\tilde{f}_{\alpha\beta}\right)(n) = \sum_{m\in X_1} \tilde{F}_\alpha(n-m)\left(\mathcal{U}^{|\alpha|}\tilde{f}_{\alpha\beta}\right)(m) \tag{197}$$

$$= \sum_{k\in X_{|\alpha|+1}} \tilde{F}_\alpha(n-2^{|\alpha|}k)\tilde{f}_{\alpha\beta}(k) = \sum_{k\in X_{|\alpha|+1}} \tilde{f}_{\alpha\beta}(k)\left(R^{2^{|\alpha|}k}\tilde{F}_\alpha\right)(n). \tag{198}$$

In (198) we have $\tilde{F}_{\alpha\beta}$ expressed as a linear combination of the elements of $\tilde{B}_\alpha$. Hence $\tilde{F}_{\alpha\beta} \in V_\alpha$. Since $\tilde{B}_\alpha$ is invariant under the operator $R^{2^{|\alpha|}k}$, for all $k \in X_{|\alpha|+1}$, we obtain $\left(\bigcup_\beta \tilde{B}_{\alpha\beta}\right) \subseteq V_\alpha$. Hence $\mathrm{cl}\left(\mathrm{span}\left(\bigcup_\beta \tilde{B}_{\alpha\beta}\right)\right) = \bigoplus_\beta V_{\alpha\beta} \subseteq \mathrm{cl}\left(\mathrm{span}\left(\tilde{B}_\alpha\right)\right)$.

In the finite–dimensional case the proof is complete, since the cardinalities of the orthonormal sets $\tilde{B}_\alpha$ and $\bigcup_\beta \tilde{B}_{\alpha\beta}$ are equal, so that $\mathrm{span}\left(\tilde{B}_\alpha\right) = \mathrm{span}\left(\bigcup_\beta \tilde{B}_{\alpha\beta}\right)$. For the infinite–dimensional case, however, the inclusion $\tilde{B}_\alpha \subseteq \mathrm{cl}\left(\mathrm{span}\left(\bigcup_\beta \tilde{B}_{\alpha\beta}\right)\right) = \bigoplus_\beta V_{\alpha\beta}$ remains to be established. Using (198) note that for all $n, k \in \mathbf{Z}^P$ and $\beta \in \mathbf{Z}_{2^P}$,

$$\left(R^{2^{|\alpha|+1}k}\tilde{F}_{\alpha\beta}\right)(n) = \sum_{m\in\mathbf{Z}^P} \tilde{f}_{\alpha\beta}(m)\,\tilde{F}_\alpha\left(n-2^{|\alpha|}(m+2k)\right) \tag{199}$$

$$= \sum_{m\in\mathbf{Z}^P} \tilde{F}_\alpha\left(n-2^{|\alpha|}m\right)\,\tilde{f}_{\alpha\beta}(m-2k) \tag{200}$$

$$= \sum_{m\in\mathbf{Z}^P} \left(R^{2k}\tilde{f}_{\alpha\beta}\right)(m)\left(R^{2^{|\alpha|}m}\tilde{F}_\alpha\right)(n). \tag{201}$$

For $p \in \mathbf{Z}^P$ fixed, and $n \in \mathbf{Z}^P$, define δ_p by $\delta_p(n) = 1$ if $p = n$, $\delta_p(n) = 0$ otherwise. Expanding δ_p with respect to the basis $\left\{R^{2k}\tilde{f}_{\alpha\beta} : k \in \mathbf{Z}^P, \beta \in \mathbf{Z}_{2^P}\right\}$ gives

$$\delta_p = \sum_{\beta\in\mathbf{Z}_{2^P}}\sum_{k\in\mathbf{Z}^P} \overline{\tilde{f}_{\alpha\beta}}(p-2k)\,R^{2k}\tilde{f}_{\alpha\beta}. \tag{202}$$

Hence, using (201), for each $p \in \mathbf{Z}^P$ we have

$$\sum_{\beta \in \mathbf{Z}_{2^P}} \sum_{k \in \mathbf{Z}^P} \overline{\tilde{f}_{\alpha\beta}}(p-2k)\, R^{2^{|\alpha|+1}k} \left(\tilde{F}_\alpha * \mathcal{U}^{|\alpha|} \tilde{f}_{\alpha\beta}\right) \tag{203}$$

$$= \sum_{n \in \mathbf{Z}^P} \sum_{\beta \in \mathbf{Z}_{2^P}} \sum_{k \in \mathbf{Z}^P} \overline{\tilde{f}_{\alpha\beta}}(p-2k) \left(R^{2k} \tilde{f}_{\alpha\beta}\right)(n) \left(R^{2^{|\alpha|}n} \tilde{F}_\alpha\right) \tag{204}$$

$$= \sum_{n \in \mathbf{Z}^P} \delta_p(n) \left(R^{2^{|\alpha|}n} \tilde{F}_\alpha\right) = R^{2^{|\alpha|}p} \tilde{F}_\alpha. \tag{205}$$

This expresses each element of $\tilde{B}_\alpha$ as a linear combination (converging in $l^2(\mathbf{Z}^P)$ since $\tilde{f}_{\alpha\beta} \in l^1(\mathbf{Z}^P)$) of the elements of $\bigcup_\beta \tilde{B}_{\alpha\beta}$. This completes the proof. □

In like manner to (108), (113) it is easy to see that for any signal $x \in X_1$, and for $k \in X_{|\alpha|+1}$,

$$v_\alpha(k) = \left\langle x, R^{2^{|\alpha|}k} \tilde{F}_\alpha \right\rangle \tag{206}$$

$$y_\alpha = \sum_{k \in X_{|\alpha|+1}} \left\langle x, R^{2^{|\alpha|}k} \tilde{F}_\alpha \right\rangle R^{2^{|\alpha|}k} \tilde{F}_\alpha. \tag{207}$$

The sum of y_α for different α yields a decomposition of x in terms of the wavelet bases.

Recursion can be performed with "repeated" filters, as discussed in the single–dimensional case in Subsection 4.4. We now prove the following folding lemma that allows us to construct filters at higher levels by "folding" the filters at the previous level. This lemma is analogous to Lemma 7.

Lemma 15. (the folding lemma) *Suppose $M \in \mathbf{Z}^+$, $f = \{f(n)\}_{n \in \mathbf{Z}^P_{2M}}$, and $h = \{h(n)\}_{n \in \mathbf{Z}^P_M}$. Then the following are equivalent:*

P1: $\hat{h}(z) = \hat{f}(z)$, $\forall z \in \mathbf{W}^P_M$.

P2: $(\mathcal{F}h)(m) = (\mathcal{F}f)(2m)$, $\forall m \in \mathbf{Z}^P_M$.

P3: $h(n) = \sum_{k \in \mathbf{Z}^P_2} f(n + Mk)$, $\forall n \in \mathbf{Z}^P_M$.

Proof of Lemma 15. We have:

$$(\mathcal{F}h)(n) = \hat{h}(e^{-j2\pi n/M}), \qquad n \in \mathbf{Z}^P_M \tag{208}$$

$$(\mathcal{F}f)(2n) = \hat{f}\left(e^{-j2\pi 2n/2M}\right) = \hat{f}\left(e^{-j2\pi n/M}\right), \quad n \in \mathbf{Z}^P_M. \tag{209}$$

Hence P1⇔P2. Also,

$$\hat{f}(z) = \sum_{n \in \mathbf{Z}_{2M}^P} f(n) z^{-n} = \sum_{n \in \mathbf{Z}_M^P} \sum_{k \in \mathbf{Z}_2^P} f(n + Mk) z^{-n-Mk}. \tag{210}$$

Therefore, for $z \in \mathbf{W}_M^P$,

$$\hat{f}(z) = \sum_{n \in \mathbf{Z}_M^P} \sum_{k \in \mathbf{Z}_2^P} f(n + Mk) z^{-n}. \tag{211}$$

Thus P3⇒P1, and P1⇒P3 since the z–transform is one–to–one on $\mathbf{Z}_M^P$. □

6. Conclusions and Discussion

We can summarize the results presented here in the following way. Theorems 1 and 6 give the necessary and sufficient conditions for the construction of first generation wavelets. Lemmas 7 and 15 show how to obtain higher level auxiliary filters needed for the iteration step. Theorems 4 and 8 describe this iteration, showing how to obtain higher–generation wavelet bases. From a strictly mathematical point of view these results are all that is necessary to create discrete wavelet bases. However, the decomposition and reconstruction of a signal by a direct implementation of such a basis is inefficient as evidenced by (123)–(125). Much faster implementation is obtained through a recursive filter–bank setup. Theorems 2 and 7 give the necessary and sufficient conditions for perfect reconstruction in this setup. Theorem 3 and (206)–(207) establish that this filter–bank arrangement does in fact implement the decomposition and reconstruction of a signal in terms of the orthonormal wavelet bases we have developed.

An algorithm is given for the construction of any and all wavelet filters for the decomposition of one–dimensional signals. Filters for higher–dimensional signal spaces can be realized as tensor products of one–dimensional filters, but the direct design of multidimensional filters with desirable properties remains a problem for further research.

The problems of frequency localization and simultaneous time–frequency localization of one–dimensional wavelet filters are discussed because of their importance to data compression and coding. While the mapping of a filter in the signal space to a closest wavelet is seen to preserve its frequency–localization, the mapping destroys time–localization. The problem of simultaneous time–frequency localization is an important open problem so far as wavelets are concerned.

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